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Continuity Equations for the Classical Euclidean Two-Dimensional Non-Linear σ - Models

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Abstract:
We derive an infinite set of independent covariant local non-polynomial continuity equations for the classical non-linear chiral O_n-models in two-dimensional Euclidean space.

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The two-dimensional non-linear chiral O_3 model is the closest known tractable analogue model for the four-dimensional pure Yang-Mills theory[1)]. Like the latter it defines an asymptotically free theory and enjoys instanton solutions[2,3)]. Because of this analogy, the model - which under the name "continuum isotropic planar spin one Heisenberg ferromagnet" always had attracted considerable interest in mathematical physics - came under particularly intense examination recently.

The classical non-linear chiral O_n models in two-dimensional Minkowski space possess an infinite number of conservation laws[4)]. In this note we want to point out that the corresponding classical models in two-dimensional Euclidean space (entering the Euclidean functional integrals of the corresponding quantized models via the extrema of the Euclidean actions) possess an infinite number of continuity equations.

We shall follow the same line of arguments as given in reference [4)] with the only deviation that at an intermediate stage we shall also consider complex solutions.
Towards the end of this note we shall present an alternative, more direct derivationof the continuity equations for the instanton solutions of the O_3-model.

The two-dimensional non-linear chiral O_n model involves a real n-component unit vector field $q = q(x,y) = (q_1(x,y), \ldots, q_n(x,y))$,
Its Euclidean action is given by

$$S = \frac{1}{2}\int_{\mathbb{R}^2} dx\,dy\,(q_x^2 + q_y^2).$$

Here we use the short hand notation

$$p\cdot q = \sum_j p_j q_j \,, \quad q^2 = \sum q_i^2 \,, \quad q_x = \frac{\partial}{\partial x} q \,, \quad q_y = \frac{\partial}{\partial y} q \,, \ \ldots$$

The action density is invariant under global internal O_n transformations and under conformal transformations.
We introduce the complex-plane notation

$$z = \frac{x+iy}{2} \,, \quad \bar z = \frac{x-iy}{2}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \,, \quad \frac{\partial}{\partial \bar z} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

$$q = q(z,\bar z) \,, \quad q_z = \frac{\partial}{\partial z} q \,, \quad q_{\bar z} = \frac{\partial}{\partial \bar z} q \,, \ \ldots$$

Complex conjugation is denoted by a bar.

The extrema of the Euclidean action satisfy the equations

$$q_{z\bar{z}} + (q_z \cdot q_{\bar{z}})\, q = 0 \quad , \quad q^2 = 1 \; .$$

(Moreoever, $q_z^2 = 0$ and $q_{\bar{z}}^2 = 0$ are valid for the instanton solutions of the O_3-model.)

In the following we assume that $n \geq 3$, the cases n = 1 and 2 being trivial and explicitly soluble, respectively.

<u>Lemma 1:</u> Along with every (not necessarily real) vector field q, a whole two-parameter family of vector fields $q^{(\xi)}$, $\xi \in \mathbb{C}$ with

$$q_z^{(\xi)\,2} = \xi^{-2} q_z^2 \; , \quad q_{\bar{z}}^{(\xi)\,2} = \xi^2 q_{\bar{z}}^2 \; , \quad (q_z^{(\xi)} \cdot q_{\bar{z}}^{(\xi)}) = (q_z \cdot q_{\bar{z}})$$

satisfies the extremum equations

$$q_{z\bar{z}} + (q_z \cdot q_{\bar{z}})\, q = 0 \quad , \quad q^2 = 1 \; .$$

The vector field $q^{(\xi)}$ is obtained from the vector field q with the help of a $SL(n,\mathbb{C})$ matrix $\mathcal{R}^{(\xi)} = \mathcal{R}^{(\xi)}(z,\bar{z};q)$ with

$$\mathcal{R}^{(\xi)} \mathcal{R}^{(\xi)\,tr} = \mathcal{R}^{(\xi)\,tr} \mathcal{R}^{(\xi)} = \mathbb{1} \quad , \quad \mathcal{R}^{(1)} = \mathbb{1} \; :$$

$$q^{(\xi)} = \mathcal{R}^{(\xi)} q \; , \quad q_z^{(\xi)} = \xi^{-1} \mathcal{R}^{(\xi)} q_z \; , \quad q_{\bar{z}}^{(\xi)} = \xi\, \mathcal{R}^{(\xi)} q_{\bar{z}} \; .$$

Real solution vectors q go over into real solution vectors $q^{(\xi)}$ if and only if $|\xi| = 1$.

<u>Proof:</u> The first part of the lemma follows from the compatibility of the equations

$$\mathcal{R}_z^{(\xi)} = (1-\xi^{-1})\, \mathcal{R}^{(\xi)} \cdot (q \otimes q_z - q_z \otimes q)$$

$$\mathcal{R}_{\bar{z}}^{(\xi)} = (1-\xi)\, \mathcal{R}^{(\xi)} \cdot (q \otimes q_{\bar{z}} - q_{\bar{z}} \otimes q)$$

and

$$\mathcal{R}^{(\xi)} \mathcal{R}^{(\xi)\,tr} = \mathcal{R}^{(\xi)\,tr} \mathcal{R}^{(\xi)} = \mathbb{1} \; .$$

The second part of the lemma follows from the fact that for a real solution vector q and for $|\xi| = 1$ along with $\mathcal{R}^{(\xi)}$ also $\overline{\mathcal{R}^{(\xi)}}$ satisfies the last three equations.

<u>Lemma 2:</u> Along with every (not necessarily real) vector field q, the vector fields q^+ and q^- - defined (up to some coordinate independent rotations) by the four complex compatible equations

$$(q^{\overset{+}{(-)}} \overset{+}{\underset{(-)}{}} q)_z = \overset{+}{\underset{(-)}{}} \frac{(q^{\overset{+}{(-)}} \cdot q_z) - (q_z^{\overset{+}{(-)}} \cdot q)}{2} \, (q^{\overset{+}{(-)}} \overset{-}{\underset{(+)}{}} q)$$

$$(q^{\overset{+}{(-)}} \overset{-}{\underset{(+)}{}} q)_{\bar{z}} = \overset{+}{\underset{(-)}{}} \frac{(q_{\bar{z}}^{\overset{+}{(-)}} \cdot q) - (q^{\overset{+}{(-)}} \cdot q_{\bar{z}})}{2} \, (q^{\overset{+}{(-)}} \overset{+}{\underset{(-)}{}} q)$$

$$q^{\overset{+}{(-)}})^2 = 1 \quad , \quad (q^{\overset{+}{(-)}} \cdot q) = 0$$

satisfy the extremum equations

$$q_{z\bar{z}} + (q_z \cdot q_{\bar{z}})\, q = 0 \quad , \quad q^2 = 1 \; .$$

<u>Proof:</u> The relevant compatibility equations are just

$$q_{z\bar{z}}^{\overset{+}{(-)}} + (q_z^{\overset{+}{(-)}} \cdot q_{\bar{z}}^{\overset{+}{(-)}})\, q^{\overset{+}{(-)}} = 0 \quad \text{and} \quad q_{z\bar{z}} + (q_z \cdot q_{\bar{z}})\, q = 0 \; .$$

<u>Lemma 3:</u> For every $\xi \in \mathbb{C}$, the following continuity equations are valid:

$$0 = \tfrac{\xi}{2} (q^{(\xi)\overset{+}{(-)}} \cdot q_z^{(\xi)})_{\bar{z}} + \tfrac{\xi}{2} (q^{(\xi)\overset{+}{(-)}} \cdot q_{\bar{z}}^{(\xi)})_z$$

$$= \tfrac{\xi}{2} \{ (q^{(\xi)\overset{+}{(-)}} \cdot q_z^{(\xi)}) + (q^{(\xi)\overset{+}{(-)}} \cdot q_{\bar{z}}^{(\xi)}) \}_x - \tfrac{1}{2i} \{ (q^{(\xi)\overset{+}{(-)}} \cdot q_z^{(\xi)}) - (q^{(\xi)\overset{+}{(-)}}\, q_{\bar{z}}^{(\xi)}) \}_y$$

The lemma is proved by carrying out the differentiations of the brackets and by inserting the defining equations of $q^{(\xi)\overset{+}{(-)}}$.

Next we expand $q^{(\xi)+}$ near the asymptote of $(q_z^2)^{-\frac{1}{2}} \mathcal{R}^{(\xi)} q_z$ ($q^{(\xi^{-1})-}$ near the asymptote of $(q_{\bar{z}}^2)^{-\frac{1}{2}} \mathcal{R}^{(\xi^{-1})} q_{\bar{z}}$) for $\xi \sim 0$ into a formal power series in ξ, insert this expansion into the above continuity equations, collect all terms of the same order in ξ and set the resulting coefficients of the various powers of ξ separately equal to zero. In this way we obtain an infinite number of independent covariant local non-polynomial continuity equations.

The first three pairs of complex continuity equations

$$U^{(i)}_{\bar{z}} + V^{(i)}_z = 0 \quad , \quad \tilde{U}^{(i)}_z + \tilde{V}^{(i)}_{\bar{z}} = 0$$

are given explicitly by

$$U^{(1)} = \frac{1}{2} q_z^2 \quad , \quad V^{(1)} = 0 \, ,$$

$$U^{(2)} = \frac{1}{2\sqrt{q_z^2}} \left[\left(\frac{q_z}{\sqrt{q_z^2}} \right)_z \right]^2 \quad , \quad V^{(2)} = - \frac{(q_z \cdot q_{\bar z})}{\sqrt{q_z^2}} \, ,$$

$$U^{(3)} = \frac{1}{2\sqrt{q_z^2}} \left[\left(\frac{\left(\frac{q_z}{\sqrt{q_z^2}} \right)_z}{\sqrt{q_z^2}} \right)_z \right]^2 - \frac{5}{8\sqrt{q_z^2}^{\,3}} \left[\left(\left(\frac{q_z}{\sqrt{q_z^2}} \right)_z \right)^2 \right]^2 , \quad V^{(3)} = \frac{(q_z \cdot q_{\bar z})}{2\sqrt{q_z^2}^{\,3}} \left[\left(\frac{q_z}{\sqrt{q_z^2}} \right)_z \right]^2$$

$$\left(\tilde U^{(i)} , \tilde V^{(i)} \right) = \left(U^{(i)} , V^{(i)} \right)_{(z \leftrightarrow \bar z)} .$$

Now we specialize to real solution vectors for which the second set of continuity equations apart from complex conjugation is identical with the first one. The first three pairs of real continuity equations are explicitly given by

$$\left\{ \mathrm{Re} \left(U^{(i)} + V^{(i)} \right) \right\}_x + \left\{ \mathrm{Im} \left(V^{(i)} - U^{(i)} \right) \right\}_y = 0$$

$$\left\{ \mathrm{Im} \left(U^{(i)} + V^{(i)} \right) \right\}_x + \left\{ \mathrm{Re} \left(U^{(i)} - V^{(i)} \right) \right\}_y = 0$$

with the above expressions for $U^{(i)} , V^{(i)} , \; i = 1, 2, 3 \ldots$

It is possible, though cumbersome, to derive the continuity equations for the instanton solutions of the non-linear chiral O_3 model, the only finite extrema of the corresponding Euclidean action[5], by a limiting procedure ($q_z^2 \equiv 0 \equiv q_{\bar z}^2$!) from the above coninuity equations. Instead, we shall present a simpler and more direct approach.

Set the O_3 action density $\frac{1}{2}(q_z \cdot q_{\bar z})$ equal to $\frac{1}{2} e^{\vartheta}$. It is an easy matter to show that ϑ satisfies the Liouville equation

$$\vartheta_{z\bar z} + e^{\vartheta} = 0 .$$

Real solutions of the Liouville equation are mapped into complex solutions of the same differential equation by the following family of transformations $T_\xi , \; \xi \in \mathbb{C}$

$$T_\xi : \left(\frac{\vartheta' + \vartheta}{2} \right)_z = \xi^{-1} \sinh \left(\frac{\vartheta' - \vartheta}{2} \right) , \left(\frac{\vartheta' - \vartheta}{2} \right)_{\bar z} = - \xi e^{\left(\frac{\vartheta' + \vartheta}{2} \right)}$$

$$\left(\bar T_\xi : \left(\frac{\vartheta' + \vartheta}{2} \right)_{\bar z} = \xi^{-1} \sinh \left(\frac{\vartheta' - \vartheta}{2} \right) , \left(\frac{\vartheta' - \vartheta}{2} \right)_z = - \xi e^{\left(\frac{\vartheta' + \vartheta}{2} \right)} \right) .$$

The compatibility equations are just

$$\vartheta_{z\bar z} + e^{\vartheta} = 0 \quad , \quad \vartheta'_{z\bar z} + e^{\vartheta'} = 0 .$$

Along with the transformations T_ξ go the coninuity equations

$$\xi \left\{ e^{\left(\frac{\vartheta' + \vartheta}{2} \right)} \right\}_z + \xi^{-1} \left\{ \cosh \left(\frac{\vartheta' - \vartheta}{2} \right) \right\}_{\bar z} = 0 .$$

We expand ϑ' near ϑ for $\xi \sim 0$ into a formal power series in ξ, insert this expansion into the last equation, collect all terms of the same order in ξ and set the resulting coefficients separately equal to zero. Finally we express ϑ in terms of the Euclidean action density. In this way we obtain the desired infinite number of independent covariant non-polynomial local conservation laws for the instanton solutions

$$\left\{ \mathrm{Re} \left(U^{(i)} + V^{(i)} \right) \right\}_x + \left\{ \mathrm{Im} \left(V^{(i)} - U^{(i)} \right) \right\}_y = 0$$

$$\left\{ \mathrm{Im} \left(U^{(i)} + V^{(i)} \right) \right\}_x + \left\{ \mathrm{Re} \left(U^{(i)} - V^{(i)} \right) \right\}_y = 0 .$$

with

$$U^{(1)} = \frac{1}{2} \frac{(q_{zz} \cdot q_{\bar z})^2}{(q_z \cdot q_{\bar z})^2} \quad , \quad V^{(1)} = (q_z \cdot q_{\bar z})$$

$$U^{(2)} = \frac{1}{2} \left[\frac{(q_{zzz} \cdot q_{\bar z})}{(q_z \cdot q_{\bar z})} - \frac{(q_{zz} \cdot q_{\bar z})^2}{(q_z \cdot q_{\bar z})^2} \right]^2 + \frac{1}{8} \frac{(q_{zz} \cdot q_{\bar z})^4}{(q_z \cdot q_{\bar z})^4} , \quad V^{(2)} = \frac{1}{2} \frac{(q_{zz} \cdot q_{\bar z})^2}{(q_z \cdot q_{\bar z})}$$

$$U^{(3)} = \frac{1}{2} \left[\left(\frac{(q_{zz} \cdot q_{\bar z})}{(q_z \cdot q_{\bar z})} \right)_{zz} \right]^2 + \frac{5}{4} \left[\left(\frac{(q_{zz} \cdot q_{\bar z})}{(q_z \cdot q_{\bar z})} \right)_z \right]^2 \frac{(q_{zz} \cdot q_{\bar z})^2}{(q_z \cdot q_{\bar z})^2} + \frac{1}{16} \frac{(q_{zz} \cdot q_{\bar z})^6}{(q_z \cdot q_{\bar z})^6}$$

$$V^{(3)} = \frac{1}{2} \frac{(q_{zzz} \cdot q_{\bar z})^2}{(q_z \cdot q_{\bar z})} - \frac{1}{8} \frac{(q_{zz} \cdot q_{\bar z})^4}{(q_z \cdot q_{\bar z})^3} .$$

In conclusion we remark that - similarly as in the hyperbolic case - the Euclidean extremum equations of the non-linear chiral O_3 model for solutions other than instanton solutions can be locally reduced to the following equations for $q_z^2, q_{\bar z}^2$ and

$$u = \operatorname{arccosh}\left(\frac{(q_z \cdot q_{\bar z})}{\sqrt{q_z^2 \cdot q_{\bar z}^2}}\right) ;$$

$$\{q_z^2\}_{\bar z} = 0 = \{q_{\bar z}^2\}_z \, , \quad u_{z\bar z} + \sqrt{q_z^2 \cdot q_{\bar z}^2}\, \sinh u = 0 .$$

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References:

1. Migdal, A.A.: Sov. Phys. JETP 42, 413, 743 (1975)
Witten, E.: Phys. Rev. Lett. 38, 121 (1977)
Meetz, K.: Forthcoming DESY-report

2. Belavin, A.A., Polyakov, A.M. Schwartz, A.S., and Tyupkin, Yu.,S.: Phys. Lett. B59, 85 (1975)
Gross, D.J., and Wilczek, F.: Phys. Rev. Lett. 30, 1343 (1973)
Politzer, H.D.: Phys. Rev. Lett. 30, 1346 (1973)

3. Belavin, A.A., and Polyakov, A.M.: JETP Lett. 22, 245 (1975)
see e.g. Brezin, E., and Zinn-Justin, J.: Phys. Rev. B14, 3110 (1976)

4. Pohlmeyer, K.: Commun. math. Phys. 46, 2o7 (1976)

5. Woo, G.: J. Math. Phys. 18, 1264 (1977)