

## EQUIVARIANCE IN HIGHER GEOMETRY

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Bundesstraße 55, D-20146 Hamburg***Abstract**

We study (pre-)sheaves in bicategories on geometric categories: smooth manifolds, manifolds with a Lie group action and Lie groupoids. We present three main results: we describe equivariant descent, we generalize the plus construction to our setting and show that the plus construction yields a 2-stackification for 2-prestacks. Finally we show that, for a 2-stack, the pullback functor along a Morita-equivalence of Lie groupoids is an equivalence of bicategories.

Our results have direct applications to gerbes and 2-vector bundles. For instance, they allow to construct equivariant gerbes from local data and can be used to simplify the description of the local data. We illustrate the usefulness of our results in a systematic discussion of holonomies for unoriented surfaces.

**KEYWORDS:** 2-stacks, equivariant descent, Morita equivalence of Lie groupoids, bundle gerbes, 2-vector bundles

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# 1 Introduction

In a typical geometric situation, one selects a category of geometric spaces, e.g. smooth manifolds, and then considers for every geometric space  $M$  a category  $\mathfrak{X}(M)$  of geometric objects on  $M$ , e.g. complex line bundles or principal  $G$ -bundles, with  $G$  a Lie group. The categories for different geometric spaces are related by pullback functors: they form a presheaf in categories. In this paper, the category of geometric spaces we consider is the category  $\text{LieGrpd}$  of Lie groupoids. This category has crucial advantages: it contains Čech groupoids and thus provides a convenient setting to discuss local data. Moreover, it contains action groupoids and thus allows us to deal with equivariant geometric objects as well.

We show that any presheaf  $\mathfrak{X}$  on manifolds can be naturally extended to a presheaf on Lie groupoids. We also generalize the structure we associate to a geometric space  $M$  by considering a bicategory  $\mathfrak{X}(M)$ . This choice is motivated by the fact that bundle gerbes and bundle gerbes with connection on a given manifold have the structure of a bicategory [Ste00, Wal07]. Hence we will work with a presheaf in bicategories on the geometric category  $\text{LieGrpd}$  of Lie groupoids. Our theory extends the theory for (pre-)sheaves in categories on smooth manifolds presented in [Met03, Hei05].

A hallmark of any geometric theory is a procedure to obtain global objects from locally defined objects by a gluing procedure. To this end, one considers open covers which are, in the category of smooth manifolds, just a special class  $\tau_{open}$  of morphisms. More generally, we endow the category of manifolds with a Grothendieck topology, although we will not directly use this language to keep this article at a more elementary level. The two prime examples for choices of  $\tau$  for the category of smooth manifolds are  $\tau_{open}$ , i.e. open covers, and  $\tau_{sub}$ , i.e. surjective submersions.

Having fixed a choice for  $\tau$ , we get a notion of  $\tau$ -essential surjectivity of Lie functors and of  $\tau$ -equivalence of Lie groupoids  $\Gamma$  and  $\Lambda$ . ( $\tau_{sub}$ -equivalent Lie groupoids are also called *Morita equivalent*; some authors also call a  $\tau_{sub}$ -equivalence a Morita equivalence.) Imposing different gluing conditions on the presheaf  $\mathfrak{X}$  on  $\text{LieGrpd}$  for morphisms in  $\tau$ , we get the notion of a  $\tau$ -2-prestack on  $\text{LieGrpd}$  and of a  $\tau$ -2-stack on  $\text{LieGrpd}$ , respectively. To simplify the notation, we refer to a 2-prestack as a prestack and to a 2-stack as a stack.

These basic definitions are the subject of section 2. At the end of this section, we can state our first main theorem 2.16:

**Theorem 2.16.**

*Suppose,  $\Gamma$  and  $\Lambda$  are Lie groupoids and  $\Gamma \rightarrow \Lambda$  is a  $\tau$ -equivalence of Lie groupoids.*

1. *Let  $\mathfrak{X}$  be a  $\tau$ -prestack on  $\text{LieGrpd}$ . Then the functor*

$$\mathfrak{X}(\Lambda) \rightarrow \mathfrak{X}(\Gamma)$$

*given by pullback is fully faithful, i.e. an equivalence on the Hom categories.*

2. *Let  $\mathfrak{X}$  be a  $\tau$ -stack on  $\text{LieGrpd}$ . Then the functor*

$$\mathfrak{X}(\Lambda) \rightarrow \mathfrak{X}(\Gamma)$$

*given by pullback is an equivalence of bicategories.*

This theorem, or more precisely its first assertion, is a central ingredient for our second main result which we explain in section 3. In analogy to the sheafification of a presheaf, we associate to any prestack  $\mathfrak{X}$  a presheaf in bicategories  $\mathfrak{X}^+$  where the objects of the bicategory  $\mathfrak{X}^+(M)$  consist of a cover  $Y \rightarrow M$  and an object in the descent bicategory  $\mathcal{D}esc_{\mathfrak{X}}(Y \rightarrow M)$ . We call this construction the plus construction. We then state theorem 3.3:

**Theorem 3.3.**

*Let  $\mathfrak{X}$  be a prestack on  $\text{Man}$ . Then the presheaf in bicategories  $\mathfrak{X}^+$  on  $\text{Man}$  obtained by the plus construction is a stack. Furthermore the canonical embedding  $\mathfrak{X}(M) \rightarrow \mathfrak{X}^+(M)$  is fully faithful for each manifold  $M$ .*

The plus construction is a powerful tool to construct geometric objects. In section 4, we show this in the example of bundle gerbes with connection: we introduce a bicategory  $\mathcal{G}rbtriv^{\nabla}$  of trivial bundle gerbes with connection whose objects are given by 2-forms. A brief check reveals that the plus construction yields bundle gerbes,

$$\mathcal{G}rb^{\nabla} = (\mathcal{G}rbtriv^{\nabla})^+ .$$

Theorem 3.3 then immediately implies that bundle gerbes form a stack.

Bundle gerbes give rise to a notion of surface holonomy. We then apply the reasoning leading to the definition of bundle gerbes to the definition of surface holonomy for unoriented surfaces and find the notion of a Jandl gerbe. In appendix A.2, we also compare this notion to the notion of a Jandl structure on a gerbe that has been introduced earlier [SSW07]. Based on the notion of Jandl gerbe, we introduce in appendix A.2 the notion of an orientifold background on a Lie groupoid  $\Lambda$ . Theorem 2.16 allows us to define a surface holonomy for any Hilsun-Skandalis morphism [Met03, definition 62] from the unoriented worldsheet  $\Sigma$  to  $\Lambda$ .

It should be stressed that our results apply to general higher geometric objects, in particular to non-abelian gerbes and 2-vector bundles. To illustrate this point, subsection 4.3 contains a short discussion of 2-vector bundles. In all cases, theorem 3.3 immediately ensures that these higher geometric objects form a stack over the category of manifolds (and even of Lie groupoids).

Together, these results provide us with tools to construct concrete geometric objects: theorem 3.3 allows us to glue together geometric objects like e.g. gerbes from locally defined geometric object. Applications frequently require not only gerbes, but equivariant gerbes. Here, it pays off that our approach is set off for Lie groupoids rather than for manifolds only, since the latter combine equivariance and local data on the same footing. In particular, we are able to formulate in this framework theorem 7.5 on *equivariant* descent. One application of this theorem is to obtain equivariant gerbes from locally defined equivariant gerbes.

Theorem 2.16 and theorem 7.5 can then be combined with standard results on the action of Lie groups or Lie groupoids [DK00, Wei02] to obtain a simplified description of the local situation in terms of stabilizer groups. This strategy provides, in particular, an elegant understanding of equivariant higher categorical geometric objects, see e.g. [Nik09] for the construction of gerbes on compact Lie groups [Mei03, GR04] that are equivariant under the adjoint action.

We have collected the proofs of the theorems in the second part of this paper in sections 5 – 9. In an appendix, we discuss applications to surface holonomies and systematically introduce a notion of holonomy for unoriented surfaces.

**Acknowledgements.** We thank Till Barmeier, Urs Schreiber and Konrad Waldorf for helpful discussions and Konrad Waldorf for comments on the draft. TN and CS are partially supported by the Collaborative Research Centre 676 “Particles, Strings and the Early Universe - the Structure of Matter and Space-Time”.

## 2 Sheaves on Lie groupoids

### 2.1 Lie groupoids

We start our discussion with an introduction to Lie groupoids. Groupoids are categories in which all morphisms are isomorphisms. A small groupoid, more specifically, consists of a set  $\Gamma_0$  of objects and a set  $\Gamma_1$  of morphisms, together with maps  $s, t : \Gamma_1 \rightarrow \Gamma_0$ ,  $\iota : \Gamma_0 \rightarrow \Gamma_1$  that associate to a morphism  $f \in \Gamma_1$  its source  $s(f) \in \Gamma_0$  and its target  $t(f) \in \Gamma_0$  and to an object  $m \in \Gamma_0$  the identity  $\text{id}_m \in \Gamma_1$ . Finally, there is an involution  $in : \Gamma_1 \rightarrow \Gamma_1$  that obeys the axioms of an inverse. Concatenation is a map  $\circ : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \rightarrow \Gamma_1$  where it should be appreciated that in the category of sets the pullback  $\Gamma_1 \times_{\Gamma_0} \Gamma_1 = \{(f_1, f_2) \in \Gamma_1 \times \Gamma_1 | t(f_1) = s(f_2)\}$  exists. It is straightforward to translate the usual axioms of a category into commuting diagrams.

A Lie groupoid is groupoid object in the category of smooth manifolds:

**Definition 2.1**

*A groupoid in the category  $\text{Man}$  or a Lie-groupoid consists of two smooth manifolds  $\Gamma_0$  and  $\Gamma_1$  together with the following collection of smooth maps:*

- *Source and target maps  $s, t : \Gamma_1 \rightarrow \Gamma_0$ .*

*To be able to define compositions, we need the existence of the pullback  $\Gamma_1 \times_{\Gamma_0} \Gamma_1$ . To ensure its existence, we require  $s$  and  $t$  to be surjective submersions.*

The other structural maps are:

- A composition map  $\circ : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \rightarrow \Gamma_1$
- A neutral map  $\iota : \Gamma_0 \rightarrow \Gamma_1$  providing identities
- A map  $in : \Gamma_1 \rightarrow \Gamma_1$  giving inverses

such that the usual diagrams commute.

### Examples 2.2.

1. For any manifold, we have the trivial Lie groupoid  $M \rightrightarrows M$  in which all structure maps are identities. We use this to embed  $\text{Man}$  into  $\text{LieGrpd}$ .
2. Given any Lie group  $G$ , we consider the Lie groupoid  $BG$  with structure maps  $G \rightrightarrows pt$  with  $pt$  the smooth zero-dimensional manifold consisting of a single point. The neutral map  $pt \rightarrow G$  is given by the map to the neutral element and composition  $G \times G \rightarrow G$  is group multiplication. Hence Lie groupoids are also a generalization of Lie groups.
3. More generally, if a Lie group  $G$  is acting smoothly on a smooth manifold  $M$ , the action groupoid  $M//G$  has  $\Gamma_0 := M$  as objects and the manifold  $\Gamma_1 := G \times M$  as morphisms. The source map  $s$  is projection to  $M$ , the target map  $t$  is given by the action  $t(g, m) := g \cdot m$ . The neutral map is the injection  $m \mapsto (1, m)$  and composition is given by the group product,  $(g, m) \circ (h, n) := (gh, n)$ . Action Lie groupoids frequently are the appropriate generalizations of quotient spaces.
4. For any covering  $(U_i)_{i \in I}$  of a manifold  $M$  by open sets  $U_i \subset M$ , we consider the disjoint union  $Y := \sqcup_{i \in I} U_i$  with the natural local homeomorphism  $\pi : Y \rightarrow M$ . Consider the two natural projections  $Y \times_M Y \rightrightarrows Y$  with the composition map  $(Y \times_M Y) \times_Y (Y \times_M Y) \cong Y^{[3]} \rightarrow Y^{[2]}$  given by omission of the second element. The neutral map is the diagonal map  $Y \rightarrow Y \times_M Y$ . This defines a groupoid  $\check{C}(Y)$ , the Čech-groupoid.

The last two examples show that Lie groupoids provide a convenient framework to unify “local data” and equivariant objects.

We next need to introduce morphisms of Lie groupoids.

### Definition 2.3

A morphism of Lie groupoids or Lie functor  $F : (\Gamma_1 \rightrightarrows \Gamma_0) \rightarrow (\Omega_1 \rightrightarrows \Omega_0)$  consists of smooth maps  $F_0 : \Gamma_0 \rightarrow \Omega_0$  and  $F_1 : \Gamma_1 \rightarrow \Omega_1$  that are required to commute with the structure maps. For example, for the source map  $s$ , we have the commuting diagram

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{F_1} & \Omega_1 \\ s \downarrow & & \downarrow s \\ \Gamma_0 & \xrightarrow{F_0} & \Omega_0 \end{array}$$

### Examples 2.4.

1. Given two smooth manifolds  $M, N$ , every Lie functor  $F : (M \rightrightarrows M) \rightarrow (N \rightrightarrows N)$  is given by a smooth map  $f : M \rightarrow N$  with  $F_0 = F_1 = f$ . Hence  $M \mapsto (M \rightrightarrows M)$  is a fully faithful embedding and we identify the manifold  $M$  with the Lie groupoid  $M \rightrightarrows M$ .

2. Given two Lie groups  $G$  and  $H$ , the Lie functors  $F : BG \rightarrow BH$  between the corresponding Lie groupoids are given by smooth group homomorphisms  $f : G \rightarrow H$ . Thus the functor  $G \mapsto BG$  is a fully faithful embedding of Lie groups into Lie groupoids.
3. For any two action groupoids  $M//G$  and  $N//G$ , a  $G$ -equivariant map  $f : M \rightarrow N$  provides a Lie functor via  $F_0 := f$  and  $F_1 := f \times \text{id} : M \times G \rightarrow N \times G$ . The previous example with  $M = N = pt$  shows that not all Lie functors between action groupoids are of this form.
4. Consider a refinement  $Z \twoheadrightarrow M$  of a covering  $Y \twoheadrightarrow M$  together with the refinement map  $s : Z \rightarrow Y$ . This provides a Lie functor  $S : \check{C}(Z) \rightarrow \check{C}(Y)$  of Čech groupoids which acts on objects by  $S_0 := s : Z \rightarrow Y$  and on morphisms  $S_1 : Z \times_M Z \rightarrow Y \times_M Y$  by  $S_1(z_1, z_2) := (s(z_1), s(z_2)) \in Y \times_M Y$ .
5. As a special case, any covering  $Y \twoheadrightarrow M$  is a refinement of the trivial covering  $\text{id} : M \twoheadrightarrow M$  and we obtain a Lie functor  $\Pi^Y : \check{C}(Y) \rightarrow M$ .

## 2.2 Presheaves in bicategories on Lie groupoids

A presheaf in bicategories  $\mathfrak{X}$  on the category  $\mathcal{M}an$  of manifolds consist of a bicategory  $\mathfrak{X}(M)$  for each manifold  $M$ , a pullback functor  $f^* : \mathfrak{X}(N) \rightarrow \mathfrak{X}(M)$  for each smooth map  $f : M \rightarrow N$  and natural isomorphisms  $f^* \circ g^* \cong (g \circ f)^*$  for composable smooth maps  $f$  and  $g$ . Moreover, we need higher coherence isomorphisms satisfying the obvious, but lengthy conditions. More precisely,  $\mathfrak{X}$  is a weak functor

$$\mathfrak{X} : \mathcal{M}an^{op} \rightarrow \mathcal{B}iCat.$$

Furthermore we impose the technical condition that  $\mathfrak{X}$  preserves products, i.e. for a disjoint union  $M = \bigsqcup_{i \in I} M_i$  of manifolds indexed by a set  $I$  the following equivalence holds:

$$\mathfrak{X}(M) \cong \prod_{i \in I} \mathfrak{X}(M_i). \quad (2.1)$$

Our next step is to extend such a presheaf in bicategories on  $\mathcal{M}an$  to a presheaf in bicategories on Lie groupoids. For a Lie groupoid  $\Gamma$  finite fiber products  $\Gamma_1 \times_{\Gamma_0} \cdots \times_{\Gamma_0} \Gamma_1$  exist in  $\mathcal{M}an$  and we introduce the notation  $\Gamma_2 = \Gamma_1 \times_{\Gamma_0} \Gamma_1$  and  $\Gamma_n$  analogously.

We can then use the nerve construction to associate to a Lie groupoid a simplicial manifold

$$\left( \begin{array}{c} \cdots \\ \xrightarrow{\partial_0} \\ \xrightarrow{\partial_0} \\ \xrightarrow{\partial_0} \\ \xrightarrow{\partial_3} \end{array} \Gamma_2 \xrightarrow{\partial_0} \Gamma_1 \xrightarrow{\partial_0} \Gamma_0 \right) =: \Gamma_\bullet .$$

We can think of  $\Gamma_n$  as  $n$ -tuples of morphisms in  $\Gamma_1$  that can be concatenated. The map  $\partial_i : \Gamma_n \rightarrow \Gamma_{n-1}$  is given by composition of the  $i$ -th and  $i+1$ -th morphism. Thus

$$\begin{aligned} \partial_i(f_1, \dots, f_n) &:= (f_1, \dots, f_i \circ f_{i+1}, \dots, f_n) \\ \partial_0(f_1, \dots, f_n) &:= (f_2, \dots, f_n) \\ \partial_n(f_1, \dots, f_n) &:= (f_1, \dots, f_{n-1}). \end{aligned}$$

In particular,  $\partial_1, \partial_0 : \Gamma_1 \rightarrow \Gamma_0$  are the source and target map of the groupoid. One easily verifies the simplicial identities  $\partial_i \partial_{j+1} = \partial_j \partial_i$  for  $i \leq j$ . (We suppress the discussion of the

degeneracy maps  $\sigma_i : \Gamma_n \rightarrow \Gamma_{n+1}$  which are given by insertion of an identity morphism at the  $i$ -the position.)

The nerve construction can also be applied to Lie functors and provides an embedding of Lie groupoids into simplicial manifolds. Suppose we are given a Lie functor  $F : (\Gamma_1 \rightrightarrows \Gamma_0) \rightarrow (\Omega \rightrightarrows \Omega_0)$ . Consider the nerves  $\Gamma_\bullet$  and  $\Omega_\bullet$  and define a family  $F_\bullet = (F_i)$  of maps, a *simplicial map*

$$F_i : \Gamma_i \rightarrow \Omega_i$$

for all  $i = 0, 1, 2, \dots$  with  $F_0, F_1$  given by the Lie functor and maps given for  $i > 1$  by

$$\begin{aligned} F_i : \Gamma_1 \times_{\Gamma_0} \dots \times_{\Gamma_0} \Gamma_1 &\rightarrow \Omega_1 \times_{\Omega_0} \dots \times_{\Omega_0} \Omega_1 \\ (f_1, \dots, f_n) &\mapsto (F_1(f_1), \dots, F_1(f_n)) . \end{aligned}$$

By definition, the maps  $F_i$  commute with the maps  $\partial_j$  and  $\sigma_k$  that are part of the simplicial object. We summarize this in the following diagram:

$$\begin{array}{ccccc} \cdots & \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} & \Gamma_2 & \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} & \Gamma_1 & \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} & \Gamma_0 \\ & & \downarrow F_2 & & \downarrow F_1 & & \downarrow F_0 \\ \cdots & \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} & \Omega_2 & \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} & \Omega_1 & \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} & \Omega_0 \end{array}$$

### Definition 2.5

Let  $\mathfrak{X}$  be a presheaf in bicategories on  $\mathcal{M}an$  and  $\Gamma$  a Lie groupoid or, more generally, a simplicial manifold. A  $\Gamma$ -equivariant object of  $\mathfrak{X}$  consists of

(O1) an object  $\mathcal{G}$  of  $\mathfrak{X}(\Gamma_0)$ ;

(O2) a 1-isomorphism

$$P : \partial_0^* \mathcal{G} \rightarrow \partial_1^* \mathcal{G}$$

in  $\mathfrak{X}(\Gamma_1)$ ;

(O3) a 2-isomorphism

$$\mu : \partial_2^* P \otimes \partial_0^* P \rightrightarrows \partial_1^* P$$

in  $\mathfrak{X}(\Gamma_2)$ , where we denote the horizontal product by  $\otimes$ ;

(O4) a coherence condition

$$\partial_2^* \mu \circ (\text{id} \otimes \partial_0^* \mu) = \partial_1^* \mu \circ (\partial_3^* \mu \otimes \text{id})$$

on 2-morphisms in  $\mathfrak{X}(\Gamma_3)$ .

We next introduce 1-morphisms and 2-morphisms of  $\Gamma$ -equivariant objects:

### Definition 2.6

1. A 1-morphism between  $\Gamma$ -equivariant objects  $(\mathcal{G}, P, \mu)$  and  $(\mathcal{G}', P', \mu')$  in  $\mathfrak{X}$  consists of the following data on the simplicial manifold

$$\left( \cdots \Gamma_4 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} \Gamma_2 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} \Gamma_1 \begin{array}{c} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{array} \Gamma_0 \right) = \Gamma_\bullet$$

- (1M1) A 1-morphism  $A: \mathcal{G} \rightarrow \mathcal{G}'$  in  $\mathfrak{X}(\Gamma_0)$ ;
- (1M2) A 2-isomorphism  $\alpha: P' \otimes \partial_0^* A \Rightarrow \partial_1^* A \otimes P$  in  $\mathfrak{X}(\Gamma_1)$ ;
- (1M3) A commutative diagram

$$(\text{id} \otimes \mu') \circ (\partial_2^* \alpha \otimes \text{id}) \circ (\text{id} \otimes \partial_0^* \alpha) = \partial_1^* \alpha \circ (\mu \otimes \text{id})$$

of 2-morphisms in  $\mathfrak{X}(\Gamma_2)$ .

- 2. A 2-morphism between two such 1-morphisms  $(A, \alpha)$  and  $(A', \alpha')$  consists of

- (2M1) A 2-morphism  $\beta: A \Rightarrow A'$  in  $\mathfrak{X}(\Gamma_0)$ ;
- (2M2) a commutative diagram

$$\alpha' \circ (\text{id} \otimes \partial_0^* \beta) = (\partial_1^* \beta \otimes \text{id}) \circ \alpha$$

of 2-morphisms in  $\mathfrak{X}(\Gamma_1)$ .

We define the composition of morphisms using simplicial identities and composition in the bicategories  $\mathfrak{X}(\Gamma_i)$ , see e.g. [Wal07]. The relevant definitions are lengthy but straightforward, and we refrain from giving details.

One can check that in this way, one obtains the structure of a bicategory.

**Remarks 2.7.**

1. *Similar descent bicategories have been introduced in [Bre94] and [Dus89]. For a related discussion of equivariance in presheaves in bicategories, see also [Sko09].*
2. *We have defined  $\Gamma$ -equivariant objects for a presheaf  $\mathfrak{X}$  in bicategories. Any presheaf  $\mathfrak{X}$  in categories can be considered as a presheaf in bicategories with trivial 2-morphisms. We thus obtain a definition for  $\mathfrak{X}(\Gamma)$  for presheaves in categories as well, where the 2-morphisms in (03) on  $\Gamma_3$  become identities and the condition (O4) is trivially fulfilled. Similar remarks apply to morphisms. All 2-morphisms are identities, hence  $\mathfrak{X}(\Gamma_\bullet)$  can be identified with a category. This allows us to deal with presheaves in categories as special cases of our more general results on presheaves in bicategories and to recover part of the results of [Met03, Hei05].*

One can check that the following proposition holds:

**Proposition 2.8.**

*Our construction provides for any Lie groupoid  $\Gamma$  a bicategory  $\mathfrak{X}(\Gamma)$ . The bicategories form a presheaf in bicategories on the category  $\text{LieGrpd}$  of Lie groupoids.*

To make contact with existing literature, we introduce for the special case of an action groupoid  $N//G$  as in example 2.2.3 the alternative notation

$$\mathfrak{X}_G(N) := \mathfrak{X}(N//G).$$

**Remarks 2.9.**

1. *For the convenience of the reader, we spell out the definition of a  $G$ -equivariant object of a presheaf in bicategories  $\mathfrak{X}$  for the special case of a discrete group  $G$ . A  $G$ -equivariant object on a  $G$ -manifold  $M$  consists of*



- An object  $\mathcal{G} \in \mathfrak{X}(N)$ .
- For every group element  $g \in G$  a morphism  $g^*\mathcal{G} \xrightarrow{\varphi_g} \mathcal{G}$ .
- A coherence 2-isomorphism for every pair of group elements  $g, h \in G$ ,

$$\begin{array}{ccc}
g^*h^*\mathcal{G} & \xrightarrow{g^*\varphi_h} & g^*\mathcal{G} \\
& \searrow \varphi_{hg} & \swarrow \cong \\
& & \mathcal{G} \\
& & \downarrow \varphi_g
\end{array}$$

- A coherence condition.

2. We also show how to obtain the usual definition of equivariant bundles on a  $G$ -manifold  $N$ , where  $G$  is a Lie group. We denote the action by  $w : M \times G \rightarrow M$ . An equivariant bundle on  $N$  consists of the following data: a bundle  $\pi : P \rightarrow N$  on  $N$ . The simplicial map  $\partial_0 : N \times G \rightarrow N$  is projection,  $\partial_1 = w$  is the action. Hence  $\partial_0^*P = P \times G$  and  $\partial_1^*P = w^*P$ . The second data is a morphism  $P \times G \rightarrow w^*P = (N \times G) \times_N P$ . A morphism to a fibre product is a commuting diagram

$$\begin{array}{ccc}
P \times G & \longrightarrow & P \\
\downarrow & & \downarrow \pi \\
N \times G & \xrightarrow{w} & N
\end{array}$$

The left vertical map is bound to be  $\pi \times \text{id}_G$ . The coherence condition of the equivariant object tells us that  $\tilde{w} : P \times G \rightarrow P$  is in fact a  $G$ -action that covers the  $G$ -action on  $N$ .

### Corollary 2.10.

Let  $G$  be a Lie group. Then the functor  $\mathfrak{X}_G$  forms a presheaf in bicategories on the category  $\text{Man}_G$  of smooth manifolds with  $G$  action.

By abuse of notation, we denote the presheaf in bicategories on  $\text{LieGrpd}$  introduced in proposition 2.8 by  $\mathfrak{X}$ . This is justified by the fact that for a constant Lie groupoid  $M \rightrightarrows M$  one has the equivalence  $\mathfrak{X}(M \rightrightarrows M) \cong \mathfrak{X}(M)$ .

We next wish to impose generalizations of the sheaf conditions on a presheaf. To this end, we have to single out a collection  $\tau$  of morphisms in  $\text{Man}$ . Technically, such a collection should form a Grothendieck (pre-)topology. This means essentially that the collection  $\tau$  of morphisms is closed under compositions, pullbacks and contains all identities. See [Met03] for a detailed introduction. For our purposes, two families are important:

- The family  $\tau_{sub}$  of surjective submersions.
- The family  $\tau_{open}$  that consists of morphisms obtained from an open covering  $(U_i)_{i \in I}$  of a manifold  $M$  by taking the local homeomorphism  $\pi : Y \rightarrow M$  with  $Y := \sqcup_{i \in I} U_i$ .

From now on, two-headed arrows will be reserved for morphisms in the relevant topology  $\rho$ . Whenever, in the sequel, no explicit topology is mentioned, we refer to  $\tau_{sub}$  as our standard (pre-)topology.

### Remarks 2.11.

Let  $\rho$  be a topology on  $\text{Man}$ .

- For any morphism  $\pi : Y \twoheadrightarrow M$  in  $\rho$ , we can form a Čech groupoid  $\check{C}(Y)$  as in example 2.2.4 which we again call the Čech groupoid.
- Given a morphism  $\pi : Y \twoheadrightarrow M$  of  $\rho$ , we define the descent bicategory by

$$\mathcal{D}esc_{\mathfrak{X}}(Y \twoheadrightarrow M) := \mathfrak{X}(\check{C}(Y)) .$$

Recall the Lie functor  $\Pi^Y : \check{C}(Y) \rightarrow M$  for the Čech cover  $Y \twoheadrightarrow M$  introduced in example 2.4.5. Applying the presheaf functor  $\mathfrak{X}$  to this Lie functor, gives the functor of bicategories

$$\tau_Y : \mathfrak{X}(M) \rightarrow \mathfrak{X}(\check{C}(Y)) = \mathcal{D}esc_{\mathfrak{X}}(Y \twoheadrightarrow M) \quad (2.2)$$

We are now ready for two definitions:

**Definition 2.12**

Let  $\mathfrak{X}$  be a presheaf in bicategories on  $\mathcal{M}an$  and  $\tau$  a topology on  $\mathcal{M}an$ .

1. A presheaf  $\mathfrak{X}$  is called a  $\tau$ -prestack, if for every covering  $Y \twoheadrightarrow M$  in  $\tau$  the functor  $\tau_Y$  of bicategories in (2.2) is fully faithful. (A functor of bicategories is called fully faithful, if all functors on Hom categories are equivalences of categories.)
2. A presheaf  $\mathfrak{X}$  is called a  $\tau$ -stack, if for every covering  $Y \twoheadrightarrow M$  in  $\tau$  the functor  $\tau_Y$  of bicategories is an equivalence of bicategories.

We also use the topology  $\tau$  to single out certain morphisms of Lie groupoids that we call equivalences of Lie groupoids. To motivate our definition, we discuss equivalences of small categories  $\mathcal{C}, \mathcal{D}$ . A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence, if it is fully faithful and essentially surjective. The latter condition means that for any object  $d \in \mathcal{D}$ , there exists an object  $c \in \mathcal{C}$  and a isomorphism  $F(c) \xrightarrow{f} d$  in  $\mathcal{D}$ . If the category  $\mathcal{D}$  is a groupoid, this amounts to the requirement that the map from

$$\mathcal{C}_0 \times_{\mathcal{D}_0} \mathcal{D}_1 = \{(c, f) | c \in \mathcal{C}_0 = Ob(\mathcal{C}), f \in \mathcal{D}_1 = Mor(\mathcal{D}) \text{ with } F(c) = s(f)\}$$

to  $\mathcal{D}_1$  induced by the target map is surjective. In the context of Lie groupoids, we will require this map to be in  $\tau$ .

**Definition 2.13**

1. A morphism of Lie groupoids  $\Gamma \rightarrow \Lambda$  is called fully faithful, if the diagram

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{F_1} & \Lambda_1 \\ s \times t \downarrow & & s \times t \downarrow \\ \Gamma_0 \times \Gamma_0 & \xrightarrow{F_0 \times F_0} & \Lambda_0 \times \Lambda_0 \end{array}$$

is a pull back diagram.

2. A morphism of Lie groupoids  $\Gamma \rightarrow \Lambda$  is called  $\tau$ -essentially surjective, if the smooth map

$$\Gamma_0 \times_{\Lambda_0} \Lambda_1 \rightarrow \Lambda_1$$

induced by the target map in  $\Lambda$  is in  $\tau$ .

3. A Lie functor is called a  $\tau$ -equivalence of Lie groupoids, if it is fully faithful and  $\tau$ -essentially surjective. If we omit the prefix  $\tau$ , we always refer to  $\tau_{sub}$ -equivalences.

**Remark 2.14.**

*Lie functors which are  $\tau_{sub}$ -equivalences are called Morita equivalences by some authors. Since the use of the word Morita equivalence is not uniform we refrain from using it. Morita equivalent Lie groupoids naturally arise in the local description of actions of Lie groups on manifolds.*

**Example 2.15.**

*The Lie functor  $\Pi^Y : \check{C}(Y) \rightarrow M$  is a  $\tau$ -equivalence for all  $\tau$ -covers.*

The stack axiom just asserts that for all  $\tau$ -equivalences of this type, the induced functor on bicategories  $\tau_Y : \mathfrak{X}(M) \rightarrow \mathfrak{X}(\check{C}(Y))$  is an equivalence of bicategories. The first theorem of this paper generalizes this statement to all  $\tau$ -equivalences of Lie groupoids:

**Theorem 2.16.**

*Suppose,  $\Gamma$  and  $\Lambda$  are Lie groupoids and  $\Gamma \rightarrow \Lambda$  is a  $\tau$ -equivalence of Lie groupoids.*

1. *Let  $\mathfrak{X}$  be a  $\tau$ -prestack on LieGrpd. Then the functor*

$$\mathfrak{X}(\Lambda) \rightarrow \mathfrak{X}(\Gamma)$$

*given by pullback is fully faithful.*

2. *Let  $\mathfrak{X}$  be a  $\tau$ -stack on LieGrpd. Then the functor*

$$\mathfrak{X}(\Lambda) \rightarrow \mathfrak{X}(\Gamma)$$

*given by pullback is an equivalence of bicategories.*

Roughly speaking,  $\tau$ -covers of manifolds can be thought of as being dense enough in  $\tau$ -equivalences of Lie groupoids to allow an extension of the (pre-)stack condition.

We defer the proof of the theorem to section 5 - 8 and first present some applications.

## 2.3 Open coverings versus surjective submersions

We have already introduced two Grothendieck (pre-)topologies  $\tau_{open}$  and  $\tau_{sub}$  on the category of smooth manifolds. Since open covers are special examples of surjective submersions, any  $\tau_{sub}$ -(pre)stack is obviously a  $\tau_{open}$ -(pre)stack. From theorem 2.16, we deduce the converse:

**Proposition 2.17.**

*A presheaf in bicategories on LieGrpd is a  $\tau_{open}$ -(pre)stack if and only if it is a  $\tau_{sub}$ -(pre)stack.*

The proposition implies in particular that it is enough to check the stack condition on open covers.

**Proof:**

It remains to be shown that any  $\tau_{open}$ -stack  $\mathfrak{X}$  is also a  $\tau_{sub}$ -stack. We fix a surjective submersion  $\pi : Y \twoheadrightarrow M$  and obtain a functor

$$\tau_Y : \mathfrak{X}(M) \rightarrow \mathcal{D}esc_{\mathfrak{X}}(Y \twoheadrightarrow M) = \mathfrak{X}(\check{C}(Y)) .$$

For the surjective submersion  $\pi$ , we can find local sections

$$s_i : U_i \rightarrow Y$$

for an open cover  $(U_i)_{i \in I}$  of  $M$ . We glue together these sections to a map  $s$  on the disjoint union of the open subsets. Then the diagram

$$\begin{array}{ccc} \sqcup_{i \in I} U_i & \xrightarrow{s} & Y \\ & \searrow & \downarrow \pi \\ & & M \end{array}$$

commutes. Here the unlabeled arrow is the inclusion of open subsets. This diagram induces a commuting diagram of Lie groupoids

$$\begin{array}{ccc} \check{C}(\sqcup_{i \in I} U_i) & \xrightarrow{s} & \check{C}(Y) \\ & \searrow & \downarrow \pi \\ & & M \end{array}$$

in which  $s$  is an  $\tau_{open}$ -equivalence of Lie groupoids. Since  $\mathfrak{X}$  is a  $\tau_{open}$ -stack, the application of  $\mathfrak{X}$  yields a diagram that commutes up to a 2-cell,

$$\begin{array}{ccc} \mathcal{D}esc_{\mathfrak{X}}(\sqcup_{i \in I} U_i) & \xleftarrow{s^*} & \mathcal{D}esc_{\mathfrak{X}}(Y) \\ & \searrow & \uparrow \pi^* \\ & & M \end{array}$$

We wish to show that the vertical arrow is an equivalence of bicategories. The lower left arrow is an equivalence of bicategories, since  $\mathfrak{X}$  is assumed to be a  $\tau_{open}$ -stack. Since  $s$  is a  $\tau_{open}$ -equivalence of Lie groupoids, theorem 2.16 implies that  $s^*$  is an equivalence of bicategories and the assertion follows.  $\square$

Since presheaves in categories are particular examples, an immediate corollary is:

**Corollary 2.18.**

*A presheaf in categories on  $\text{LieGrpd}$  is a  $\tau_{open}$ -(pre)stack if and only if it is a  $\tau_{sub}$ -(pre)stack.*

After one further decategorification, we also obtain

**Corollary 2.19.**

*A presheaf on  $\text{LieGrpd}$  is a  $\tau_{open}$ -separated presheaf if and only if it is a  $\tau_{sub}$ -separated presheaf. A presheaf on  $\text{LieGrpd}$  is a  $\tau_{open}$ -sheaf if and only if it is a  $\tau_{sub}$ -sheaf.*

Let us discuss an application of this result:  $U(1)$  principal bundles form a stack on  $\text{Man}$  with respect to the open topology  $\tau_{open}$ , see e.g. [FNSW08]. As a consequence of corollary 2.18,  $U(1)$  bundles also form a stack with respect to surjective submersions. Hence we can glue bundles also with respect of surjective submersions. In this way, we recover the following well-known

**Proposition 2.20.**

*Consider a free action groupoid  $M//G$  so that the quotient space  $M/G$  has a natural structure of a smooth manifold and the canonical projection is a submersion. (This is, e.g., the case if the action of  $G$  on  $M$  is proper and discontinuous.) Then the category of smooth  $U(1)$ -bundles on  $M/G$  is equivalent to the category of  $G$ -equivariant  $U(1)$ -bundles on  $M$ .*

**Proof:**

Since the action is free, the canonical projection  $\pi : M \rightarrow M/G$  is a submersion that induces a  $\tau_{sub}$ -equivalence of Lie groupoids. We have seen that  $U(1)$ -bundles form a  $\tau_{sub}$ -stack, and hence by theorem 2.16 the canonical projection  $\pi$  induces an equivalence of categories.  $\square$

We have formulated this result for the special case of  $U(1)$  bundles. Obviously, the same argument applies to any stack on  $\mathcal{M}an$ , and we obtain similar equivalences of categories for  $G$ -equivariant principal bundles, and associated bundles for any structure group.

### 3 The plus construction

In this section we describe a general procedure for 2-stackification. More precisely, we show how to obtain a 2-stack  $\mathfrak{X}^+$  on  $\mathcal{M}an$  starting from 2-prestack  $\mathfrak{X}$  on  $\mathcal{M}an$ . In analogy to the case of sheaves, we call this construction the plus construction. The idea is to complement the bicategories  $\mathfrak{X}(M)$  by adding objects in descent bicategories. The main result is then that the 2-presheaf in bicategories obtained in this way is closed under descent.

We first describe the bicategory  $\mathfrak{X}^+(M)$  for a manifold  $M$ .

**Definition 3.1**

An object of  $\mathfrak{X}^+(M)$  consists of a covering  $Y \twoheadrightarrow M$  and an object  $G$  in the descent bicategory  $\mathcal{D}esc_{\mathfrak{X}}(Y)$ .

In order to define 1-morphisms and 2-morphisms between objects with possibly different coverings  $\pi : Y \twoheadrightarrow M$  and  $\pi' : Y' \twoheadrightarrow M$ , we pull all the data back to a common refinement of these coverings and compare them there. We call a covering  $\zeta : Z \twoheadrightarrow M$  a *common refinement* of  $\pi$  and  $\pi'$  iff there exist coverings  $s : Z \twoheadrightarrow Y$  and  $s' : Z \twoheadrightarrow Y'$  such that the diagram

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow^s & & \searrow^{s'} & \\
 Y & & & & Y' \\
 & \searrow^{\pi} & \downarrow^{\zeta} & \swarrow^{\pi'} & \\
 & & M & & 
 \end{array} \tag{3.1}$$

commutes. An example of such a common refinement is the fibre product  $Z := Y \times_M Y' \twoheadrightarrow M$ , with the maps  $Z \twoheadrightarrow Y$  and  $Z \twoheadrightarrow Y'$  given by the projections. We call this the *canonical common refinement*. The maps  $s$  and  $s'$  of a common refinement  $Z \twoheadrightarrow M$  induce Lie functors on the Lie groupoids

$$\check{C}(Y) \longleftarrow \check{C}(Z) \longrightarrow \check{C}(Y').$$

Hence we have *refinement functors*  $s^*$  and  $s'^*$ :

$$\mathcal{D}esc_{\mathfrak{X}}(Y) \xleftarrow{s^*} \mathcal{D}esc_{\mathfrak{X}}(Z) \xrightarrow{(s')^*} \mathcal{D}esc_{\mathfrak{X}}(Y').$$

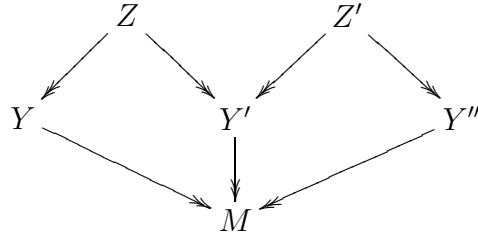
For an object  $G$  in  $\mathcal{D}esc_{\mathfrak{X}}(Y)$  we denote the refinement  $s^*(G)$  by  $G_Z$ .

**Definition 3.2**

- A 1-morphism between objects  $\mathcal{G} = (Y, G)$  and  $\mathcal{G}' = (Y', G')$  of  $\mathfrak{X}^+(M)$  consists of a common refinement  $Z \twoheadrightarrow M$  of the coverings  $Y \twoheadrightarrow M$  and  $Y' \twoheadrightarrow M$  and a 1-morphism  $A : G_Z \rightarrow G_{Z'}$  of the two refinements in  $\mathcal{D}esc_{\mathfrak{X}}(Z)$ .

- A 2-morphism between 1-morphisms  $\mathbf{m} = (Z, A)$  and  $\mathbf{m}' = (Z', A')$  consists of a common refinement  $W \twoheadrightarrow M$  of the coverings  $Z \twoheadrightarrow M$  and  $Z' \twoheadrightarrow M$  (respecting the projections to  $Y$  and  $Y'$ , respectively) and a 2-morphism  $\beta : \mathbf{m}_W \Rightarrow \mathbf{m}'_W$  of the refined morphisms in  $\mathcal{D}esc_{\mathfrak{X}}(W)$ . In addition two such 2-morphisms  $(W, \beta)$  and  $(W', \beta')$  must be identified iff there exists a further common refinement  $V \twoheadrightarrow M$  of  $W \twoheadrightarrow M$  and  $W' \twoheadrightarrow M$ , compatible with the other projections, such that the refined 2-morphisms agree on  $V$ .

Now that we have defined objects, morphisms and 2-morphisms in  $\mathfrak{X}^+(M)$  it remains to define compositions and identities. We will just indicate how this is done. For example let  $\mathcal{G} = (Y, G)$ ,  $\mathcal{G}' = (Y', G')$  and  $\mathcal{G}'' = (Y'', G'')$  be objects and  $\mathbf{m} = (Z, A) : \mathcal{G} \rightarrow \mathcal{G}'$  and  $\mathbf{m}' = (Z', A') : \mathcal{G}' \rightarrow \mathcal{G}''$  be morphisms. The covers can then be arranged to the diagram



Now let  $Z'' := Z \times_{Y'} Z'$  be the pullback of the upper diagram. This exists in  $\mathcal{M}an$  and is evidently a common refinement of  $Y$  and  $Y''$ . The composition  $\mathbf{m}' \circ \mathbf{m}$  is then defined to be the tuple  $(Z'', A'_{Z''} \circ A_{Z''})$  where  $A'_{Z''} \circ A_{Z''}$  denotes the composition of the refined morphisms in  $\mathcal{D}esc_{\mathfrak{X}}(Z'')$ .

Finally one can check that this defines the structure of a bicategory  $\mathfrak{X}^+(M)$ . See [Wal07] for a very detailed treatment of a related bicategory. In order to turn the bicategories  $\mathfrak{X}^+(M)$  into a stack we have to define the pullback functors

$$f^* : \mathfrak{X}^+(N) \rightarrow \mathfrak{X}^+(M)$$

for all smooth maps  $f : M \rightarrow N$ . This is done in the obvious way using the pullback of covers and the pullback functors of the prestack  $\mathfrak{X}$ .

**Theorem 3.3.**

*If  $\mathfrak{X}$  is a prestack, then  $\mathfrak{X}^+$  is a stack. Furthermore the canonical embedding  $\mathfrak{X}(M) \rightarrow \mathfrak{X}^+(M)$  is fully faithful for each  $M$ .*

We relegate the proof of this theorem to section 9.

**Remark 3.4.**

1. If we choose the covers in definition 3.1 and 3.2 to be in the topology  $\tau_{open}$  we obtain a slightly different stack  $\mathfrak{X}_{open}^+$ . Argument similar to the ones used in section 2.3 show that  $\mathfrak{X}^+(M) \cong \mathfrak{X}_{open}^+(M)$  for each smooth manifold  $M$ .
2. As in remark 2.7.2, one can specialize to presheaves in categories and obtains the stackification process for 1-prestacks.

## 4 Applications of the plus construction

### 4.1 Bundle gerbes

We next present several applications of the plus construction. The input for this construction is a presheaf in bicategories on  $\mathcal{Man}$ . In the same way a monoid is the simplest example of a category (with one object), any monoidal category gives a bicategory with a single object. An example for a bicategory can thus be obtained from the monoidal category of principal  $A$ -bundles, where  $A$  is any abelian Lie group. This way, we get a presheaf  $\mathcal{Grbtriv}_A$  of trivial  $A$ -gerbes. Since bundles can be glued together, the homomorphism categories are closed under descent. The presheaf  $\mathcal{Grbtriv}_A$  is thus a prestack. The plus construction yields the stack

$$\mathcal{Grb}_A := (\mathcal{Grbtriv}_A)^+$$

of gerbes (without connection). Our general result implies that gerbes form a sheaf on  $\mathcal{Man}$ . Together with theorem 2.16 and theorem 7.5 of this paper, this provides a local construction of gerbes and the definition of equivariant gerbes.

Let us next construct gerbes with connection; for simplicity, we restrict to the abelian group  $A = U(1)$  and suppress the index  $A$ . The guiding principle for our construction is the requirement that gerbes should lead to a notion of surface holonomy (for a review, see e.g. [FNSW08] and appendix A.1). Hence we consider a bicategory whose objects are two-forms. These two-forms are intended to be integrated; hence it is not natural to require them to be equal, but only to require them to be equal up to a total derivative or, in physical terms, up to a gauge transformation. We are thus lead to consider for every manifold  $M$  the following bicategory

- Objects are in bijection with 2-forms  $\omega \in \Omega^2(M)$  and denoted by  $\mathcal{I}_\omega$ .
- A 1-morphism  $\mathcal{I}_\omega \rightarrow \mathcal{I}'_\omega$  is a 1-form  $\lambda$  such that  $d\lambda = \omega' - \omega$ .
- A 2-morphism  $\lambda \rightarrow \lambda'$  is a  $U(1)$ -valued function  $g$  on  $M$  such that  $d\log g = \lambda' - \lambda$ .

The last two items define the homomorphism categories. It is convenient to close them first under descent. This way, we obtain the prestack  $\mathcal{Grbtriv}^\nabla$  of trivial bundle gerbes with connection where the bicategories  $\mathcal{Grbtriv}^\nabla(M)$  are defined by:

- An object is a 2-form  $\omega \in \Omega^2(M)$ , called a *trivial bundle gerbe with connection* and denoted by  $\mathcal{I}_\omega$ .
- A 1-morphism  $\mathcal{I}_\omega \rightarrow \mathcal{I}'_\omega$  is a  $U(1)$  bundle  $L$  with connection of curvature  $\omega' - \omega$ .
- A 2-morphism  $\phi : L \rightarrow L'$  is a morphism of bundles with connection.

There is also a natural pullback operation along maps, induced by pullback on differential forms and pullback on  $U(1)$ -bundles. One checks that  $\mathcal{Grbtriv}^\nabla$  is a prestack. By theorem 3.3, the plus construction yields a stack

$$\mathcal{Grb}^\nabla := (\mathcal{Grbtriv}^\nabla)^+$$

on  $\mathcal{Man}$  and even a stack on the category of Lie groupoids. In particular, definition 2.5 provides a natural notion of an equivariant gerbe. Theorem 2.16 now implies:

**Corollary 4.1.**

For an equivalence  $F : \Gamma \rightarrow \Lambda$  of Lie groupoids, the pullback functor

$$F^* : \mathcal{G}rb(\Lambda) \rightarrow \mathcal{G}rb(\Gamma) \quad \mathcal{G}rb^\nabla(\Lambda) \rightarrow \mathcal{G}rb^\nabla(\Gamma)$$

is an equivalence of bicategories. In particular, for a free, proper and discontinuous action of a Lie group  $G$  on a smooth manifold  $M$  we have the following equivalences of bicategories

$$\mathcal{G}rb_G(M) \cong \mathcal{G}rb(M/G) \quad \text{respectively} \quad \mathcal{G}rb_G^\nabla(M) \cong \mathcal{G}rb^\nabla(M/G) .$$

We compare the stack  $\mathcal{G}rb^\nabla$  with objects introduced in the literature. An object in  $\mathcal{G}rb^\nabla(M)$  consists by definition of a covering  $Y \rightarrow M$  and an object  $G$  in  $\mathcal{D}esc_{\mathcal{G}rbtriv^\nabla}(Y)$ . Spelling out the data explicitly, one verifies that objects are just bundle gerbes in the sense of [Mur96] and [Ste00]. For the special case of an open cover  $Y := \bigsqcup U_i$ , an object in  $\mathcal{D}esc_{\mathcal{G}rbtriv^\nabla}(Y)$  is an Chaterjee-Hitchin gerbe, see [Cha98].

To compare different morphisms introduced in the literature, we first need a definition:

**Definition 4.2**

i) A morphism  $\mathcal{A} : (Y, G) \rightarrow (Y', G')$  in  $\mathfrak{X}^+(M)$  is called a stable isomorphism, if it is defined on the canonical common refinement

$$Z := Y \times_M Y' .$$

- ii) A stable 2-isomorphism in  $\mathfrak{X}(M)$  between stable isomorphisms  $(Z, A)$  and  $(Z, A')$  is a morphism in  $\mathcal{D}esc_{\mathfrak{X}}(Z \rightarrow M)$ , i.e. a morphism on the canonical common refinement  $Z = Y \times_M Y'$ .
- iii) Two objects  $(Y, \mathcal{G})$  and  $(Y', \mathcal{G}')$  are called stably isomorphic if there is a stable isomorphism  $(Y, \mathcal{G}) \rightarrow (Y', \mathcal{G}')$ .

For bundle gerbes  $(Y, G)$  and  $(Y', G')$ , stable morphisms are a subcategory,

$$\text{Hom}_{\text{Stab}}\left((Y, G), (Y', G')\right) \subset \text{Hom}_{\mathcal{G}rb^\nabla}\left((Y, G), (Y', G')\right) .$$

We next show that the these categories are in fact equivalent. We start with the following observation:

**Lemma 4.3.**

Let  $Z \twoheadrightarrow M$  be a common refinement of  $Y \twoheadrightarrow M$  and  $Y' \twoheadrightarrow M$  with morphisms  $s : Z \twoheadrightarrow Y$  and  $s' : Z \twoheadrightarrow Y'$  as in (3.1). Then the morphism  $s \times_M s' : Z \rightarrow Y \times_M Y'$  induces a  $\tau$ -equivalence of Čech groupoids

$$\check{C}(Z) \xrightarrow{\sim} \check{C}(Y \times_M Y') .$$

**Proof:**

Spelling out the definition of  $\tau$ -essential surjectivity for the Lie functor  $\check{C}(Z) \rightarrow \check{C}(Y \times_M Y')$ , we see that we have to show that the smooth map

$$Z \times_{Y \times_M Y'} (Y \times_M Y' \times_M Y \times_M Y') \cong Z \times_M Y \times_M Y' \rightarrow Y \times_M Y'$$



is in  $\tau$ . This follows at once from the pullback diagram:

$$\begin{array}{ccc} Z \times_M Y \times_M Y' & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y \times_M Y & \longrightarrow & M \end{array}$$

It remains to show that the Lie functor is fully faithful. From example 2.15 we know that the vertical morphisms in the diagram

$$\begin{array}{ccc} \check{C}(Z) & \longrightarrow & \check{C}(Y \times_M Y') \\ & \searrow & \swarrow \\ & M & \end{array}$$

are  $\tau$ -equivalences, and thus in particular fully faithful. Elementary properties of pullback diagrams then imply that the horizontal morphism is fully faithful as well.  $\square$

Hence the induced morphism of Lie groupoids

$$\check{C}(Z) \rightarrow \check{C}(Y \times_M Y')$$

is fully faithful and  $\tau$ -essentially surjective. Since  $\mathfrak{X}$  is a prestack, we deduce from the first assertion of theorem 2.16

**Proposition 4.4.**

*For any two objects  $\mathcal{O} = (Y, \mathcal{G})$  and  $\mathcal{O}' = (Y', \mathcal{G}')$  in  $\mathfrak{X}^+(M)$ , the 1-category  $\text{Hom}(\mathcal{O}, \mathcal{O}')$  is equivalent to the subcategory of stable isomorphisms and stable 2-isomorphisms.*

*In particular, two objects are isomorphic in  $\mathfrak{X}^+(M)$ , if and only if they are stably isomorphic.*

**Remark 4.5.**

- *Stable isomorphisms have been introduced in [Ste00, MS01]; proposition 4.4 shows that our bicategory is equivalent to the one in that paper. With our definition of morphisms, composition has a much simpler structure.*
- *In [Wal07] a further different choice of common refinement was made. The bicategory introduced in [Wal07] has as morphisms categories that are contained in our morphism categories and contain the morphism categories of [MS01]. Hence all three bicategories are equivalent.*

## 4.2 Jandl gerbes

It is instructive to apply the same reasoning to the construction of Jandl gerbes. The slightly less general notion of gerbes with a Jandl structure has been introduced in [SSW07] to obtain a notion of surface holonomy for unoriented surfaces. In this subsection, we introduce the more general notion of a Jandl gerbe. To this end, we follow the general pattern from section 4.1 and first define Jandl bundles:

**Definition 4.6**

A Jandl bundle over  $M$  is a pair, consisting of a  $U(1)$ -bundle  $P$  with connection over  $M$  and

a smooth map  $\sigma : M \rightarrow \mathbb{Z}/2 = \{1, -1\}$ . Morphisms of Jandl bundles  $(P, \sigma) \rightarrow (Q, \mu)$  only exist if  $\sigma = \mu$ . In this case they are morphisms  $P \rightarrow Q$  of bundles with connection. We denote the category of Jandl bundles by  $\mathcal{JBun}^\nabla(M)$

We need the covariant functor

$$(?)^{-1} : \mathcal{Bun}^\nabla(M) \rightarrow \mathcal{Bun}^\nabla(M)$$

which sends a bundle  $P$  to its dual bundle  $P^*$ . A morphism  $f : P \rightarrow Q$  is sent to  $(f^*)^{-1} : P^* \rightarrow Q^*$ . This functor is well defined since all morphisms in  $\mathcal{Bun}^\nabla(M)$  are isomorphisms. It squares to the identity and thus defines an  $\mathbb{Z}/2$  action on the category  $\mathcal{Bun}^\nabla(M)$ .

Smooth maps  $\sigma : M \rightarrow \mathbb{Z}/2$  are constant on connected components of  $M$ . For each such map  $\sigma$ , we get a functor by letting  $(?)^{-1}$  acting on each connected component by the power given by the value of  $\sigma$  on that connected component. For each map  $\sigma$  we thus have a functor

$$(?)^\sigma : \mathcal{Bun}^\nabla(M) \rightarrow \mathcal{Bun}^\nabla(M).$$

For our construction, we need a monoidal category of morphisms of trivial objects. Hence we endow  $\mathcal{JBun}^\nabla(M)$  with a monoidal structure;

$$(P, \sigma) \otimes (Q, \mu) := (P \otimes Q^\sigma, \sigma\mu) .$$

Now we are ready to define the prestack  $\mathcal{JGrbtriv}^\nabla$  of trivial Jandl gerbes. Again the guiding principle is the definition of holonomies, this time for unoriented surfaces (for more details, see appendix A.2).

- An object is a 2-form  $\omega \in \Omega^2(M)$ , called a *trivial Jandl gerbe with connection* and denoted by  $\mathcal{I}_\omega$ .
- A 1-morphism  $\mathcal{I}_\omega \rightarrow \mathcal{I}_{\omega'}$  is a Jandl bundle  $(P, \sigma)$  of curvature  $\text{curv}P = \sigma \cdot \omega' - \omega$ .
- A 2-morphism  $\phi : (P, \sigma) \rightarrow (Q, \mu)$  is a morphism of Jandl bundles with connection.

Composition of morphisms is defined as the tensor product of Jandl bundles. It is easy to see that  $\mathcal{JGrbtriv}^\nabla$  is a prestack. We define Jandl gerbes by applying the plus construction:

$$\mathcal{JGrb}^\nabla := (\mathcal{JGrbtriv}^\nabla)^+.$$

By theorem 3.3, this defines a stack.

**Remark 4.7.**

1. We relegate the discussion of the relation between Jandl gerbes and gerbes with a Jandl structure introduced in [SSW07] to appendix A.2, see proposition A.2.3. In the same appendix, we discuss holonomy for unoriented surfaces.
2. In terms of descent data, we can describe a Jandl gerbe on  $M$  by a cover  $Y \rightrightarrows M$ , a two-form  $\omega \in \Omega^2(Y)$ , a Jandl bundle  $(P, \sigma)$  on  $Y^{[2]}$  such that  $\sigma\partial_1^*\omega - \partial_0^* = \text{curv}(P)$  and a 2-morphism

$$\mu : \partial_2^*(P, \sigma) \otimes \partial_0^*(P, \sigma) \rightrightarrows \partial_1^*(P, \sigma)$$

of Jandl bundles on  $Y^{[3]}$ . The definition of morphisms of Jandl bundles implies that such a morphism only exists, if the identity

$$\partial_2^* \sigma \cdot \partial_0^* \sigma = \partial_1^* \sigma \quad (4.1)$$

holds. Under this conditions, the data on  $Y^{[3]}$  reduce to a morphism of  $U(1)$ -bundles

$$\mu : \partial_2^* P \otimes \partial_0^* P \Rightarrow \partial_1^* P$$

that obeys the same associativity condition on  $Y^{[4]}$  as ordinary gerbes.

3. Both trivial Jandl gerbes and trivial bundle gerbes are given, as objects, by 2-forms; hence they are locally the same. The crucial difference between Jandl gerbes and bundle gerbes is the fact that there are more 1-morphisms between Jandl gerbes: apart from the morphisms  $(P, 1)$ , we also have “odd” morphisms  $(P, -1)$ .

We have the inclusion  $j : \mathcal{B}un(M) \rightarrow \mathcal{J}\mathcal{B}un(M)$  where we identify a bundle  $P \in \mathcal{B}un(M)$  with a Jandl bundle  $(P, 1) \in \mathcal{J}\mathcal{B}un(M)$ . Here  $1 : M \rightarrow \mathbb{Z}/2$  is the constant function to the neutral element. The category  $\mathcal{B}un(M)$  is thus a full subcategory of  $\mathcal{J}\mathcal{B}un(M)$ . The inclusion functor is clearly monoidal and thus yields an inclusion  $\mathcal{G}rbtriv^\nabla(M) \rightarrow \mathcal{J}\mathcal{G}rbtriv^\nabla(M)$  of bicategories. Finally this induces an inclusion functor

$$\mathcal{J} : \mathcal{G}rb^\nabla(M) \rightarrow \mathcal{J}\mathcal{G}rb^\nabla(M) .$$

In terms of descent data, the functor  $\mathcal{J}$  maps

$$(Y, \omega, P, \mu) \mapsto (Y, \omega, (P, 1), \mu) .$$

The inclusion functor  $\mathcal{J}$  is faithful, but neither full nor essentially surjective. Hence we have to understand its essential image.

Given a Jandl bundle  $(P, \sigma)$ , we can forget  $P$  and just keep the smooth map  $\sigma$ . Since morphisms in  $\mathcal{J}\mathcal{B}un(M)$  preserve  $\sigma$  by definition, this yields a functor

$$o : \mathcal{J}\mathcal{B}un^\nabla(M) \rightarrow C^\infty(M, \mathbb{Z}/2) \quad (4.2)$$

where the category on the right hand side has  $\mathbb{Z}/2$ -valued smooth functions as objects and only identities as morphisms.

The functor  $o$  is monoidal, i.e.  $(P, \sigma) \otimes (Q, \mu) \mapsto \sigma \cdot \mu$ . We denote the category of  $\mathbb{Z}/2$  bundles on  $M$  by  $\mathcal{B}un_{\mathbb{Z}/2}(M)$ . It contains the full subcategory  $\mathcal{B}untriv_{\mathbb{Z}/2}(M)$  of trivial  $\mathbb{Z}/2$ -bundles:

- The category  $\mathcal{B}untriv_{\mathbb{Z}/2}(M)$  has exactly one object, the trivial  $\mathbb{Z}/2$  bundle  $M \times \mathbb{Z}/2 \rightarrow M$ .
- The endomorphisms of  $M \times \mathbb{Z}/2$  are given by elements in  $C^\infty(M, \mathbb{Z}/2)$ .
- Composition of endomorphisms is pointwise multiplication of smooth maps  $M \rightarrow \mathbb{Z}/2$

Together with this observation the functor (4.2) yields a functor

$$\mathcal{J}\mathcal{G}rbtriv^\nabla(M) \rightarrow \mathcal{B}untriv_{\mathbb{Z}/2}(M).$$

Applying the plus construction, that functor induces a functor

$$\mathcal{O} : \mathcal{JGrb}^\nabla(M) \rightarrow \mathcal{Bun}_{\mathbb{Z}/2}(M).$$

In terms of descent data, the functor  $\mathcal{O}$  maps

$$(Y, \omega, (P, \sigma), \mu) \mapsto (Y, \sigma) .$$

Equation (4.1) implies that the cocycle condition holds on  $Y^{[3]}$  so that the pair  $(Y, \sigma)$  indeed describes a  $\mathbb{Z}/2$ -bundle in terms of local data. For later use, we note that a section of the bundle  $(Y, \sigma)$  is described in local data by a function  $s : Y \rightarrow \mathbb{Z}/2$  such that the identity  $\sigma = \partial_0^* s \partial_1^* s$  holds on  $Y^{[2]}$ .

We are now ready for the next definition:

**Definition 4.8**

1. We call  $\mathcal{O}(\mathcal{G})$  the orientation bundle of the Jandl gerbe  $\mathcal{G}$ .
2. A global section  $s$  of the orientation bundle  $\mathcal{O}(\mathcal{G})$  is called an orientation of the Jandl gerbe  $\mathcal{G}$ .
3. A morphism  $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$  of oriented Jandl gerbes is called orientation preserving, if the morphism  $\mathcal{O}(\varphi)$  of  $\mathbb{Z}/2$ -covers preserves the global sections,  $\mathcal{O}(\varphi) \circ s = s'$ .
4. Together with all 2-morphism of Jandl gerbes, we obtain the bicategory  $\mathcal{JGrb}_{or}^\nabla(M)$  of oriented Jandl gerbes.

**Proposition 4.9.**

1. For any gerbe  $\mathcal{G}$ , the induced Jandl gerbe  $\mathcal{J}(\mathcal{G})$  is canonically oriented. For any morphism  $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$  of gerbes, the induced morphism  $\mathcal{J}(\varphi) : \mathcal{J}(\mathcal{G}) \rightarrow \mathcal{J}(\mathcal{G}')$  is orientation preserving.
2. The functor  $\mathcal{J}$  induces an equivalence of bicategories

$$\mathcal{Grb}^\nabla(M) \rightarrow \mathcal{JGrb}_{or}^\nabla(M) .$$

Hence the choice of an orientation reduces a Jandl gerbe to a gerbe.

**Proof:**

1. Let  $\mathcal{G}$  be an ordinary gerbe with connection in  $\mathcal{Grb}^\nabla(M)$ . By definition of the functors  $\mathcal{J}$  and  $\mathcal{O}$ , the bundle  $\mathcal{O}(\mathcal{J}(\mathcal{G}))$  is given by the trivial  $\mathbb{Z}/2$  cocycle on the covering of  $\mathcal{G}$ . Hence it admits a canonical section  $s_{\mathcal{G}}$ . This section is preserved by  $\mathcal{O}(\mathcal{J}(\varphi))$  for any morphism  $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$  of gerbes. This shows part 1 of the claim.

2. By looking at the local data, we find that data and conditions of a Jandl gerbe  $(Y, \mathcal{I}_\omega, (P, \sigma), \mu)$  with  $\sigma : Y^{[2]} \rightarrow \mathbb{Z}/2$  the constant map to 1 are precisely the local data of a gerbe with connection. Since the orientation bundle  $(Y, 1)$  of such a Jandl gerbe is trivial, we choose the trivial section  $1 : Y \rightarrow \mathbb{Z}/2$  as the canonical orientation. Similarly, one sees that morphisms of such Jandl gerbes preserving the canonical orientation are described by exactly the same local data as morphisms of gerbes with connection. The 2-morphisms between two such morphisms are the same anyway. Hence, the functor  $\mathcal{J}$  is an isomorphism from the bicategory  $\mathcal{G}rb^\nabla(M)$  to the full subcategory of  $\mathcal{J}\mathcal{G}rb_{\text{or}}^\nabla(M)$  with trivial map  $\sigma$ .

It remains to show that any oriented Jandl gerbe with connection is isomorphic within  $\mathcal{J}\mathcal{G}rb_{\text{or}}^\nabla(M)$  to an object in the full subcategory with trivial map  $\sigma$ . To this end, we apply to a general Jandl gerbe  $(Y, \mathcal{I}_\omega, (P, \sigma), \mu)$  with orientation  $s : Y \rightarrow \mathbb{Z}/2$  the isomorphism  $m = (Y, (\text{triv}, s), \text{id})$ . Here  $\text{triv}$  is the trivial  $U(1)$ -bundle on  $Y$ . The target of this isomorphism is a trivially oriented Jandl gerbe of the form  $(Y, \mathcal{I}_{s\omega}, (P^{\partial_0^* s}, 1), \tilde{\mu})$  and thus in the full subcategory of  $\mathcal{J}\mathcal{G}rb_{\text{or}}^\nabla(M)$  described in the preceding paragraph. □

The last assertion crucially enters in the discussion of unoriented surface holonomy in appendix A.2.

### 4.3 Kapranov-Voevodsky 2-vector bundles

As a further application of the plus construction, we investigate 2-vector bundles, more precisely 2-vector bundles modeled on the notion of Kapranov-Voevodsky 2-vector spaces [KV94]. The bicategory of complex KV 2-vector spaces is (equivalent to) the following bicategory:

- Objects are given by non negative integers  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ . This is a shortcut for the category  $\text{Vect}_{\mathbb{C}}^n = \text{Vect}_{\mathbb{C}} \times \dots \times \text{Vect}_{\mathbb{C}}$ , where we have the product of categories.
- 1-morphisms  $n \rightarrow m$  are given by  $m \times n$  matrices  $(V_{ij})_{i,j}$  of complex vector spaces. This encodes an exact functor  $\text{Vect}_{\mathbb{C}}^n \rightarrow \text{Vect}_{\mathbb{C}}^m$ .
- 2-morphisms  $(V_{ij})_{i,j} \Rightarrow (W_{ij})_{i,j}$  are given by families  $(\varphi_{ij})_{i,j}$  of linear maps. This encodes a natural transformation between functors  $\text{Vect}_{\mathbb{C}}^n \rightarrow \text{Vect}_{\mathbb{C}}^m$ .

The 1-isomorphisms in this bicategory are exactly those  $n \times n$  square matrices  $(V_{ij})$  for which the  $n \times n$  matrix with non-negative integral entries  $(\dim_{\mathbb{C}} V_{ij})$  is invertible in the ring  $M(n \times n, \mathbb{Z})$  of matrices with integral entries.

Based on this bicategory we define for a smooth manifold  $M$  the bicategories  $\mathcal{V}ect_{2\text{triv}}(M)$  of trivial Kapranov-Voevodsky 2-vector bundles:

- Objects are given by non negative integers  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ .
- 1-morphisms  $n \rightarrow m$  are given by  $m \times n$  matrices  $(E_{ij})_{i,j}$  of complex vector bundles over  $M$ .
- 2-morphisms  $(E_{ij})_{i,j} \Rightarrow (F_{ij})_{i,j}$  are given by families  $\phi_{ij} : E_{ij} \rightarrow F_{ij}$  of vector bundle morphisms.

The pullback of vector bundles turns this into a presheaf in bicategories. Since vector bundles can be glued together, the presheaf  $\mathcal{Vect}_2triv$  is even a prestack. Hence we can apply the plus construction:

$$\mathcal{Vect}_2 := \left( \mathcal{Vect}_2triv \right)^+.$$

By theorem 3.3, we obtain a stack, the stack of 2-vector bundles. Thus we have properly defined bicategories of  $\mathcal{Vect}_2(M)$  of 2-vector bundles over a manifold  $M$  and even over Lie groupoids and thus obtained a notion of equivariant 2-vector bundles.

In [BDR04] a similar notion of 2-vector bundles on the basis of Kapranov-Voevodsky 2-vector spaces has been introduced under the name of charted 2-vector bundles. They are defined on ordered open covers. Spelling out our definition of objects and 1-morphisms explicitly it is easy to see that the definitions are essentially the same. The focus in [BDR04] is more on the classification up to 1-isomorphisms and thus no 2-morphisms are introduced.

## 5 Proof of theorem 2.16, part 1: Factorizing morphisms

Sections 5–8 are devoted to the proof of theorem 2.16. For this proof, we factor any fully faithful and  $\tau$ -essentially surjective Lie functor  $F : \Gamma \rightarrow \Omega$  into two morphisms of Lie groupoids belonging to special classes of morphisms of Lie groupoids:  $\tau$ -surjective equivalences and internal equivalences. We first discuss these two classes of morphisms.

### 5.1 Internal equivalences

We start with the definition of internal equivalences. To this end, we introduce natural transformations of Lie groupoids: Consider the free groupoid on a single morphism, the interval groupoid:

$$\mathbf{I} := (\mathbf{I}_1 \rightrightarrows \mathbf{I}_0)$$

It has two objects  $\mathbf{I}_0 := \{a, b\}$  and the four isomorphisms  $\mathbf{I}_1 := \{id_a, id_b, \ell, \ell^{-1}\}$  with  $s(\ell) = a, t(\ell) = b$ . Consider two functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  for two categories  $\mathcal{C}, \mathcal{D}$ . For any category, we consider the cylinder category  $\Gamma \times \mathbf{I}$  with the canonical inclusion functors  $i_0, i_1 : \Gamma \rightarrow \Gamma \times \mathbf{I}$ .

It is an easy observation that natural isomorphisms  $\eta : F \Rightarrow G$  are in bijection to functors  $\tilde{\eta} : \mathcal{C} \times \mathbf{I} \rightarrow \mathcal{D}$  with  $\tilde{\eta} \circ i_0 = F$  and  $\tilde{\eta} \circ i_1 = G$ . (The bijection maps  $\eta_c : F(c) \rightarrow G(c)$  to  $\tilde{\eta}(\text{id}_c \times \ell)$ .)

This observation allows us to reduce smoothness conditions on natural transformations to smoothness conditions on functors. Hence, we consider the interval groupoid  $\mathbf{I}$  as a discrete Lie groupoid and obtain for any Lie groupoid  $\Gamma$  the structure of a Lie groupoid on the cylinder groupoid  $\Gamma \times \mathbf{I}$ .

#### **Definition 5.1**

1. A Lie transformation  $\eta$  between two Lie functors  $F, G : \Gamma \rightarrow \Omega$  is a Lie functor  $\eta : \Gamma \times \mathbf{I} \rightarrow \Omega$  with  $\eta \circ i_0 = F$  and  $\eta \circ i_1 = G$ .
2. Two Lie functors  $F$  and  $G$  are called naturally isomorphic,  $F \simeq G$ , if there exists a Lie transformation between  $F$  and  $G$ .

3. A Lie functor  $F : \Gamma \rightarrow \Omega$  is called an internal equivalence, if there exists a Lie functor  $G : \Omega \rightarrow \Gamma$  such that  $G \circ F \simeq \text{id}_\Gamma$  and  $F \circ G \simeq \text{id}_\Omega$ .

We need the following characterization of internal equivalences, which is completely analogous to a well-known statement from category theory:

**Proposition 5.2.**

A Lie functor  $F : \Gamma \rightarrow \Omega$  is an internal equivalence if and only if it is fully faithful and split essential surjective. The latter means that the map in definition 2.13.2

$$\Gamma_0 \times_{\Omega_0} \Omega_1 \rightarrow \Omega_0$$

induced by the target map has a section.

**Proof:**

The proof is roughly the same as in classical category theory c.f. [Kas95] Prop. XI.1.5. We only have to write down everything in diagrams, e.g. the condition fully faithful in terms of pullback diagram as in definition 2.13. Note that the proof in [Kas95] needs the axiom of choice; in our context, we need a section of the map  $\Gamma_0 \times_{\Omega_0} \Omega_1 \rightarrow \Omega_0$ .  $\square$

**Lemma 5.3.**

If a Lie functor  $F : \Gamma \rightarrow \Omega$  admits a fully faithful retract, i.e. a fully faithful left inverse, it is an internal equivalence.

**Proof:**

Let  $P$  be the fully faithful left inverse of  $F$ , hence

$$P \circ F = \text{id}_\Gamma \quad .$$

It remains to find a Lie transformation

$$\eta : F \circ P \implies \text{id}_\Gamma \quad .$$

Since the functor  $P$  is fully faithful, the diagram

$$\begin{array}{ccc} \Omega_1 & \xrightarrow{P_1} & \Gamma_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ \Omega_0 \times \Omega_0 & \xrightarrow{P_0 \times P_0} & \Gamma_0 \times \Gamma_0 \end{array}$$

is by definition 2.13 a pullback diagram. Define  $\eta : \Omega_0 \rightarrow \Omega_1 \cong \Omega_0 \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} \Omega_0$  by

$$\eta(\omega) = (F_0 P_0(\omega), \text{id}_{P_0(\omega)}, \omega) \quad .$$

The identities  $P_0(w) = s(\text{id}_{P_0(w)})$  and  $t(\text{id}_{P_0(w)}) = P_0(w) = P_0 F_0 P_0(\omega)$  imply that this is well-defined; one also checks naturality. The two identities

$$s\eta(\omega) = F_0 P_0(\omega) \quad \text{and} \quad t\eta(\omega) = \omega$$

imply that  $\eta$  is indeed a Lie transformation  $F \circ P \implies \text{id}_\Gamma$ . One verifies that it has also the other properties we were looking for.  $\square$

## 5.2 $\tau$ -surjective equivalences

For any choice of topology  $\tau$ , we introduce the notion of  $\tau$ -surjective equivalence. In contrast to  $\tau$ -equivalences,  $\tau$ -surjective equivalences are required to be  $\tau$ -surjective, rather than only  $\tau$ -essentially surjective, as in definition 2.13.

### Definition 5.4

A  $\tau$ -surjective equivalence is a fully faithful Lie functor  $F : \Lambda \rightarrow \Gamma$  such that  $F_0 : \Lambda_0 \rightarrow \Gamma_0$  is a morphism in  $\tau$ .

### Proposition 5.5.

Let  $F : \Lambda \rightarrow \Gamma$  be a fully faithful Lie functor and  $F_\bullet : \Lambda_\bullet \rightarrow \Gamma_\bullet$  the associated simplicial map. Then  $F$  is a  $\tau$ -surjective equivalence, if and only if all maps  $F_i : \Lambda_i \rightarrow \Gamma_i$  are in  $\tau$ .

The proof is based on

### Lemma 5.6.

For any two  $\tau$ -covers  $\pi : Y \twoheadrightarrow M$  and  $\pi' : Y' \twoheadrightarrow M'$  in  $\mathcal{M}an$ , the product  $\pi \times \pi' : Y \times Y' \twoheadrightarrow M \times M'$  is in  $\tau$  as well.

### Proof:

Writing  $\pi \times \pi' = (\pi \times id) \circ (id \times \pi')$  and using the fact that the composition of  $\tau$ -covers is a  $\tau$ -cover, we can assume that  $\pi' = id : M' \twoheadrightarrow M'$ . The assertion then follows from the observation that the diagram

$$\begin{array}{ccc} Y \times M' & \longrightarrow & Y \\ \downarrow & & \downarrow \pi \\ M \times M' & \longrightarrow & M \end{array}$$

is a pullback diagram and that  $\tau$  is closed under pullbacks. □

### Proof:

of proposition 5.5. Since  $F$  is fully faithful, all diagrams

$$\begin{array}{ccc} \Lambda_n & \xrightarrow{F_n} & \Gamma_n \\ \downarrow & & \downarrow \\ \underbrace{\Lambda_0 \times \cdots \times \Lambda_0}_{n+1} & \xrightarrow{F_0 \times \cdots \times F_0} & \underbrace{\Gamma_0 \times \cdots \times \Gamma_0}_{n+1} \end{array}$$

are pullback diagrams. Then  $F_n$  is a  $\tau$ -cover since  $F_0 \times \cdots \times F_0$  is, by lemma 5.6 a  $\tau$ -cover. □

## 5.3 Factorization

### Proposition 5.7 (Factorization of Lie functors).

Let  $\Gamma$  and  $\Omega$  be Lie groupoids. Every fully faithful and  $\tau$ -essentially surjective Lie functor



$F : \Gamma \rightarrow \Omega$  factors as

$$\begin{array}{ccc} & \Lambda & \\ G \nearrow & & \searrow H \\ \Gamma & \xrightarrow{F} & \Omega \end{array}$$

where  $H$  is a  $\tau$ -surjective equivalence and  $G$  an internal equivalence.

**Proof:**

We ensure the surjectivity of  $H$  by defining

$$\Lambda_0 := \Gamma_0 \times_{F_0} \Omega_1.$$

Then  $H_0 : \Lambda_0 \rightarrow \Omega_0$  is given on objects by the target map of  $\Omega$ . This is a  $\tau$ -covering by the definition of  $\tau$ -essential surjectivity. On objects, we define  $G_0 : \Gamma_0 \rightarrow \Lambda_0$  by  $\gamma \mapsto (\gamma, \text{id}_{F_0(\gamma)})$ . This gives the commutative diagram

$$\begin{array}{ccc} & \Lambda_0 & \\ G_0 \nearrow & & \searrow H_0 \\ \Gamma_0 & \xrightarrow{F_0} & \Omega_0 \end{array}$$

on the level of objects. We combine the maps in the diagram

$$\begin{array}{ccccc} \Gamma_1 & \xrightarrow{F_1} & \Omega_1 & & \\ (s,t) \downarrow & & \downarrow (s,t) & & \\ \Gamma_0 \times \Gamma_0 & \xrightarrow{G_0 \times G_0} & \Lambda_0 \times \Lambda_0 & \xrightarrow{H_0 \times H_0} & \Omega_0 \times \Omega_0 \end{array}$$

which is a pull back diagram by definition 2.13, since  $F$  is fully faithful. To define the Lie functor  $H$  such that it is fully faithful, we have to define  $\Lambda_1$  as the pull back of the right half of the diagram, i.e.  $\Lambda_1 := \Lambda_0 \times_{\Omega_0} \Omega_1 \times_{\Omega_0} \Lambda_0$ . The universal property of pull backs yields a diagram

$$\begin{array}{ccccc} \Gamma_1 & \xrightarrow{G_1} & \Lambda_1 & \xrightarrow{H_1} & \Omega_1 & (5.1) \\ (s,t) \downarrow & & \downarrow (s,t) & & \downarrow (s,t) & \\ \Gamma_0 \times \Gamma_0 & \xrightarrow{G_0 \times G_0} & \Lambda_0 \times \Lambda_0 & \xrightarrow{H_0 \times H_0} & \Omega_0 \times \Omega_0 \end{array}$$

in which all squares are pullbacks. The groupoid structure on  $\Omega = (\Omega_1 \rightrightarrows \Omega_0)$  induces a groupoid structure on  $\Lambda = (\Lambda_1 \rightrightarrows \Lambda_0)$  in such a way that  $G$  and  $H$  become Lie functors.

By construction of this factorization,  $H$  is a  $\tau$ -surjective equivalence. It remains to be shown that  $G$  is an internal equivalence. According to proposition 5.2, it suffices to show that  $G$  is fully faithful and split essential surjective. The left diagram in (5.1) is a pullback diagram. Hence  $G$  is fully faithful. It remains to give a section of the map

$$\Gamma_0 \times_{\Lambda_0} \Lambda_1 \rightarrow \Lambda_0 \quad (5.2)$$

Since we have defined  $\Lambda_1 = \Lambda_0 \times_{\Omega_0} \Omega_1 \times_{\Omega_0} \Lambda_0$ , we have

$$\Gamma_0 \times_{\Lambda_0} \Lambda_1 \cong \Gamma_0 \times_{\Omega_0} \Omega_1 \times_{\Omega_0} \Lambda_0.$$

Thus a section of (5.2) is given by three maps

$$\Lambda_0 \rightarrow \Gamma_0 \quad \Lambda_0 \rightarrow \Omega_1 \quad \Lambda_0 \rightarrow \Lambda_0$$

that agree on  $\Omega_0$ , when composed with the source and the target map of  $\Omega_0$ , respectively. By definition  $\Lambda_0 = \Gamma_0 \times_{F_0} \times_s \Omega_1$ , and we can define the three maps by projection to the first factor, projection to the second factor and the identity.  $\square$

The factorization lemma allows to isolate the violation of  $\tau$ -surjectivity in an internal equivalence and to work with  $\tau$ -surjective equivalences rather than only  $\tau$ -essentially surjective equivalences. Hence it suffices to prove theorem 2.16 for  $\tau$ -surjective equivalences and for internal equivalences. This will be done in sections 6 and 8, respectively.

## 6 Proof of theorem 2.16, part 2: Sheaves and internal equivalences

### Lemma 6.1.

Let  $\mathfrak{X}$  be a presheaf that preserves products, cf. equation (2.1). Let  $\Gamma$  be a Lie groupoid and  $D$  be a discrete Lie groupoids i.e.  $D_0$  and  $D_1$  are discrete manifolds. Then  $D$  can also be regarded as a bicategory and we have natural equivalences

$$\mathfrak{X}(\Gamma \times D) \cong [D, \mathfrak{X}(\Gamma)]$$

where  $[D, \mathfrak{X}(\Gamma)]$  denotes the bicategory of functors  $D \rightarrow \mathfrak{X}(\Gamma)$ .

### Proof:

The claim is merely a consequence the requirement (2.1) that  $\mathfrak{X}$  preserves products: In the case that  $\Gamma$  is a manifold  $M$  considered as a Lie groupoid and  $D$  a set  $I$  considered as a discrete groupoid we have  $M \times I = \bigsqcup_{i \in I} M$ . Thus the left hand is equal to  $\mathfrak{X}(\bigsqcup M)$  and the right hand side to  $\mathfrak{X}(M)^I = \prod_{i \in I} \mathfrak{X}(M)$ . In this case (2.1) directly implies the equivalence.

In the case of a general Lie groupoid, the product  $\Gamma \times D$  decomposes levelwise into a disjoint union. Using this fact and explicitly spelling out  $\mathfrak{X}(\Gamma \times D)$  and  $[D, \mathfrak{X}(\Gamma)]$  according to definition, 2.5 it is straightforward to see that the two bicategories are equivalent.  $\square$

### Proposition 6.2.

Let  $\mathfrak{X}$  be a presheaf in bicategories. Any Lie transformation  $\eta : F \Rightarrow G$  of Lie functors  $F, G : \Gamma \rightarrow \Omega$  induces a natural isomorphism of the pullback functors  $F^*, G^* : \mathfrak{X}(\Omega) \rightarrow \mathfrak{X}(\Gamma)$ .

### Proof:

Recall from definition 5.1 that the Lie transformation  $\eta$  is by definition a Lie functor

$$\Gamma \times I \rightarrow \Omega ,$$

where  $I$  is the interval groupoid. Applying the presheaf  $\mathfrak{X}$  to this functor yields a functor

$$\mathfrak{X}(\Omega) \rightarrow \mathfrak{X}(\Gamma \times I).$$

Since  $I$  is discrete the preceding lemma 6.1 shows that this is a functor

$$\mathfrak{X}(\Omega) \rightarrow [I, \mathfrak{X}(\Gamma)].$$

That is the same as a functor

$$\mathfrak{X}(\Omega) \times I \rightarrow \mathfrak{X}(\Gamma)$$

i.e. a natural isomorphism of bifunctors. □

**Corollary 6.3.**

*For any presheaf  $\mathfrak{X}$  in bicategories, the pull back along an internal equivalence  $\Gamma \rightarrow \Omega$  induces an equivalence  $\mathfrak{X}(\Omega) \rightarrow \mathfrak{X}(\Gamma)$  of bicategories.*

## 7 Proof of theorem 2.16, part 3: Equivariant descent

To deal with  $\tau$ -surjective equivalences, we need to consider simplicial objects in the category of simplicial objects, i.e. bisimplicial objects. In the course of our investigations, we obtain results about bisimplicial objects that are of independent interest, in particular theorem 7.5 and corollary 7.6 on equivariant descent.

We first generalize the definition of equivariant objects as follows: If we evaluate a presheaf in bicategories  $\mathfrak{X}$  on a simplicial object  $\Gamma_\bullet$ , we obtain the following diagram in  $\mathcal{B}iCat$ :

$$\mathfrak{X}(\Gamma_0) \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_1^*} \end{array} \mathfrak{X}(\Gamma_1) \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_2^*} \end{array} \mathfrak{X}(\Gamma_2) \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_3^*} \\ \xrightarrow{\partial_3^*} \end{array} \cdots$$

in which the cosimplicial identities are obeyed up to natural isomorphism,

$$\partial_j^* \partial_i^* \cong \partial_i^* \partial_{j-1}^* \quad \text{for } i < j .$$

The coherence cells turn this into a weak functor  $\Delta \rightarrow \mathcal{B}iCat$  from the simplicial category  $\Delta$  to  $\mathcal{B}iCat$ . Such a functor will be called a (weak) cosimplicial bicategory.

The equivariant objects can be constructed in this framework by selecting objects in  $\mathfrak{X}(\Gamma_0)$ , 1-morphisms in  $\mathfrak{X}(\Gamma_1)$  and so on. This leads us to the following definition:

**Definition 7.1**

*Given a cosimplicial bicategory  $C_\bullet$ , we introduce the category*

$$\text{holim}_{i \in \Delta} C_i \equiv \text{holim} \left( C_0 \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_1^*} \end{array} C_1 \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_2^*} \end{array} C_2 \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_3^*} \\ \xrightarrow{\partial_3^*} \end{array} \cdots \right)$$

*with objects given by the following data:*

- (O1) An object  $\mathcal{G}$  in the bicategory  $C_0$ ;
- (O2) A 1-isomorphism in the bicategory  $C_1$ ;

$$P: \partial_0^* \mathcal{G} \rightarrow \partial_1^* \mathcal{G}$$

(O3) A 2-isomorphism in the bicategory  $C_2$ ;

$$\mu: \partial_2^* P \otimes \partial_0^* P \Rightarrow \partial_1^* P$$

(O4) A coherence condition of 2-morphisms in the bicategory  $C_3$ :

$$\partial_2^* \mu \circ (\text{id} \otimes \partial_0^* \mu) = \partial_1^* \mu \circ (\partial_3^* \mu \otimes \text{id})$$

Morphisms and 2-morphisms are defined as in definition 2.6.

In this notation, the extension of a prestack  $\mathfrak{X}$  to an equivariant object  $\Gamma_\bullet$  described in definition 2.5 is given by

$$\mathfrak{X}(\Gamma_\bullet) = \text{holim}_{i \in \Delta} \mathfrak{X}(\Gamma_i) . \quad (7.1)$$

In the special case of a  $\tau$ -covering  $Y \rightarrow M$ , we can write the descent object as

$$\mathcal{D}esc_{\mathfrak{X}}(Y \rightarrow M) = \text{holim}_{i \in \Delta} X(Y^{[i+1]}).$$

For the constant simplicial bicategory  $C_\bullet$ , with  $C_i = C$  for all  $i$ , one checks that  $\text{holim}_{i \in \Delta} C_i = C$ .

We next need to extend the notion of a  $\tau$ -covering to a simplicial object:

**Definition 7.2**

1. Let  $\Lambda_\bullet$  and  $\Gamma_\bullet$  be simplicial manifolds and  $\Pi_\bullet : \Lambda_\bullet \rightarrow \Gamma_\bullet$  a simplicial map. Then  $\Pi_\bullet$  is called a  $\tau$ -cover, if all maps  $\Pi_i : \Lambda_i \rightarrow \Gamma_i$  are  $\tau$ -covers.
2. A Lie functor  $\Pi : (\Lambda_1 \rightrightarrows \Lambda_0) \rightarrow (\Gamma_1 \rightrightarrows \Gamma_0)$  is called a  $\tau$ -cover, if the associated simplicial map  $\Pi_\bullet : \Lambda_\bullet \rightarrow \Gamma_\bullet$  of the nerves is a  $\tau$ -cover of simplicial manifolds.

**Remark 7.3.**

1. Proposition 5.5 shows that for a  $\tau$ -surjective equivalence the associated simplicial map is  $\tau$ -cover.
2. For any  $\tau$ -covering  $\pi : Y \rightarrow M$ , the simplicial map induced by the Lie functor  $\check{C}(Y) \rightarrow M$  is an example of a  $\tau$ -cover of simplicial manifolds.

Given a  $\tau$ -cover  $\Pi_\bullet : \Lambda_\bullet \rightarrow \Gamma_\bullet$  of simplicial manifolds, we can construct the simplicial manifold

$$\Lambda_\bullet^{[2]} := \Lambda_\bullet \times_{\Gamma_\bullet} \Lambda_\bullet := \left( \cdots \begin{array}{c} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_0} \\ \xrightarrow{\partial_0} \\ \xrightarrow{\partial_3} \end{array} \Lambda_2 \times_{\Gamma_2} \Lambda_2 \begin{array}{c} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_0} \\ \xrightarrow{\partial_0} \\ \xrightarrow{\partial_2} \end{array} \Lambda_1 \times_{\Gamma_1} \Lambda_1 \begin{array}{c} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_0} \\ \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{array} \Lambda_0 \times_{\Gamma_0} \Lambda_0 \right)$$

with obvious maps  $\partial_i$ . One verifies that the two projections  $\delta_0, \delta_1 : \Lambda_\bullet^{[2]} \rightarrow \Lambda_\bullet$  are simplicial maps. Similarly, we form simplicial manifolds

$$\Lambda_\bullet^{[n]} := \underbrace{\Lambda_\bullet \times_{\Gamma_\bullet} \cdots \times_{\Gamma_\bullet} \Lambda_\bullet}_n$$

and simplicial maps  $\delta_i : \Lambda_\bullet^{[n]} \rightarrow \Lambda_\bullet^{[n-1]}$  with  $i = 0, \dots, n$ . We thus obtain an (augmented) simplicial object

$$(\Lambda_\bullet)^{[\bullet]} := \left( \cdots \begin{array}{c} \xrightarrow{\delta_0} \\ \xrightarrow{\delta_0} \\ \xrightarrow{\delta_0} \\ \xrightarrow{\delta_3} \end{array} \Lambda_\bullet^{[3]} \begin{array}{c} \xrightarrow{\delta_0} \\ \xrightarrow{\delta_0} \\ \xrightarrow{\delta_0} \\ \xrightarrow{\delta_2} \end{array} \Lambda_\bullet^{[2]} \begin{array}{c} \xrightarrow{\delta_0} \\ \xrightarrow{\delta_0} \\ \xrightarrow{\delta_0} \\ \xrightarrow{\delta_1} \end{array} \Lambda_\bullet \end{array} \right) \longrightarrow \Gamma_\bullet$$

in the category of simplicial manifolds. A simplicial object in the category of simplicial manifolds will also be called a *bisimplicial manifold*. In full detail, a bisimplicial manifold consists of the following data:

$$\begin{array}{ccccccc} & & \begin{array}{c} \cdots \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \cdots \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \cdots \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \cdots \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} \\ \cdots & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Lambda_2^{[3]} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Lambda_2^{[2]} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Lambda_2 & \xrightarrow{\quad} & \Gamma_2 \\ & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} \\ \cdots & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Lambda_1^{[3]} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Lambda_1^{[2]} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Lambda_1 & \xrightarrow{\quad} & \Gamma_1 \\ & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} \\ \cdots & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Lambda_0^{[3]} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Lambda_0^{[2]} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Lambda_0 & \xrightarrow{\quad} & \Gamma_0 \end{array}$$

The rows are, by construction, nerves of Čech groupoids. This fact will enter crucially in the proof of our main result on equivariant descent. Before turning to this, we need the following

**Proposition 7.4.**

Let  $\mathfrak{X}$  be a presheaf in bicategories and  $\Omega_{\bullet\bullet}$  a bisimplicial manifold. Then

$$\text{holim}_{i \in \Delta} \text{holim}_{j \in \Delta} \mathfrak{X}(\Omega_{ij}) = \text{holim}_{j \in \Delta} \text{holim}_{i \in \Delta} \mathfrak{X}(\Omega_{ij})$$

**Proof:**

We first discuss what data of the bisimplicial manifold

$$\begin{array}{ccccccc} & & \begin{array}{c} \cdots \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \cdots \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \cdots \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} \\ \cdots & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Omega_{22} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Omega_{21} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Omega_{20} \\ & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} \\ \cdots & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Omega_{12} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Omega_{11} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Omega_{10} \\ & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} & & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \cdots \end{array} \\ \cdots & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Omega_{02} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Omega_{01} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \Omega_{00} \end{array}$$

enter in an object of  $\text{holim}_{i \in \Delta} \text{holim}_{j \in \Delta} \mathfrak{X}(\Omega_{ij})$ . To this end, we denote horizontal boundary maps by  $\delta$  and vertical boundary maps by  $\partial$ . Then such an object is given by

- An object in  $\text{holim}_{j \in \Delta} \mathfrak{X}(\Omega_{0j})$  which in turn consists of
  - An object  $\mathcal{G}$  on  $\Omega_{00}$

- An isomorphism  $A_{01} : \delta_0^* \mathcal{G} \rightarrow \delta_1^* \mathcal{G}$  on  $\Omega_{01}$
- A 2-isomorphism  $\mu_{02} : \delta_2^* A_{01} \otimes \delta_0^* A_{01} \Rightarrow \delta_1^* A_{01}$  on  $\Omega_{02}$
- A coherence condition on  $\Omega_{03}$
- A morphism  $\partial_0^*(\mathcal{G}, A_{0,1}, \mu_{0,2}) \rightarrow \partial_1^*(\mathcal{G}, A_{0,1}, \mu_{0,2})$  in  $\text{holim}_{j \in \Delta} \mathfrak{X}(\Omega_{1j})$  which in turn consists of
  - An isomorphism  $A_{10} : \partial_0^* \mathcal{G} \rightarrow \partial_1^* \mathcal{G}$  on  $\Omega_{1,0}$
  - A 2-isomorphism  $\mu_{11} : \partial_1^* A_{01} \otimes \delta_0^* A_{10} \Rightarrow \delta_1^* A_{10} \otimes \partial_0^* A_{01}$  on  $\Omega_{11}$ .
  - A coherence condition on  $\Omega_{12}$ .
- A 2-isomorphism  $\partial_2^*(A_{10}, \mu_{11}) \otimes \partial_0^*(A_{10}, \mu_{11}) \Rightarrow \partial_1^*(A_{10}, \mu_{11})$  in  $\text{holim}_{j \in \Delta} \mathfrak{X}(\Omega_{2j})$ :
  - A 2-isomorphism  $\mu_{20} : \partial_2^* A_{10} \otimes \partial_0^* A_{10} \Rightarrow \partial_1^* A_{10}$  on  $\Omega_{20}$ .
  - A coherence condition on  $\Omega_{21}$ .
- A condition on the 2-morphisms in  $\text{holim}_{j \in \Delta} \mathfrak{X}(\Omega_{3j})$  which is just
  - A coherence condition on  $\Omega_{30}$ .

To summarize, we get an object  $\mathcal{G} \in \mathfrak{X}(\Omega_{00})$  in the lower right corner of the diagram, two isomorphisms  $A_{01} \in \mathfrak{X}(\Omega_{01})$ ,  $A_{10} \in \mathfrak{X}(\Omega_{01})$  on the diagonal, three 2-isomorphisms  $\mu_{02} \in \mathfrak{X}(\Omega_{02})$ ,  $\mu_{11} \in \mathfrak{X}(\Omega_{11})$ ,  $\mu_{20} \in \mathfrak{X}(\Omega_{20})$  on the first translate of the diagonal and four conditions on the second translate of the diagonal.

For an object in  $\text{holim}_{j \in \Delta} \text{holim}_{i \in \Delta} \mathfrak{X}(\Omega_{ij})$ , we get the same data, as can be seen by exchanging the roles of  $i$  and  $j$ . Since we interchange the roles of  $\partial$  and  $\delta$ , we have to replace the 2-isomorphism  $\mu_{11} : \partial_1^* A_{01} \otimes \delta_0^* A_{10} \Rightarrow \delta_1^* A_{10} \otimes \partial_0^* A_{01}$  by its inverse. For all other isomorphisms and conditions, the objects remain unchanged.

By analogous considerations, one also checks that the morphisms and 2-morphisms in both bicategories coincide.  $\square$

**Theorem 7.5** (Equivariant descent).

Let  $\Pi : \Lambda_\bullet \rightarrow \Gamma_\bullet$  be a  $\tau$ -covering of simplicial manifolds.

1. Let  $\mathfrak{X}$  be a  $\tau$ -stack on  $\text{Man}$ . Then we have the following equivalence of bicategories:

$$\mathfrak{X}(\Gamma_\bullet) \xrightarrow{\sim} \text{holim} \left( \mathfrak{X}(\Lambda_\bullet) \begin{array}{c} \xrightarrow{\delta_0^*} \\ \xrightarrow{\delta_1^*} \end{array} \mathfrak{X}(\Lambda_\bullet^{[2]}) \begin{array}{c} \xrightarrow{\delta_0^*} \\ \xrightarrow{\delta_2^*} \end{array} \mathfrak{X}(\Lambda_\bullet^{[3]}) \begin{array}{c} \xrightarrow{\delta_0^*} \\ \xrightarrow{\delta_3^*} \end{array} \dots \right)$$

In other words, we have extended  $\mathfrak{X}$  to a  $\tau$ -stack on the category of simplicial manifolds.

2. If  $\mathfrak{X}$  is a  $\tau$ -prestack on  $\text{Man}$ , this functor is still fully faithful, i.e. an equivalence of the Hom-categories.

**Proof:**

By definition, we have  $\mathfrak{X}(\Gamma_\bullet) = \text{holim}_{i \in \Delta} \mathfrak{X}(\Gamma_i)$ . Since  $\mathfrak{X}$  is supposed to be a  $\tau$ -stack and since all  $\Pi_i : \Lambda_i \rightarrow \Gamma_i$  are  $\tau$ -covers, we have the following equivalence of bicategories:

$$\mathfrak{X}(\Gamma_i) \xrightarrow{\sim} \mathcal{D}esc_{\mathfrak{X}}(\Lambda_i \rightarrow \Gamma_i) = \text{holim}_{j \in \Delta} \mathfrak{X}(\Lambda_i^{[j]}) .$$

Altogether, we have the equivalence of bicategories

$$\mathfrak{X}(\Gamma_\bullet) \xrightarrow{\sim} \text{holim}_{i \in \Delta} \text{holim}_{j \in \Delta} \mathfrak{X}(\Lambda_i^{[j]})$$

By proposition 7.4, we can exchange the homotopy limits and get

$$\begin{aligned} \text{holim}_{i \in \Delta} \text{holim}_{j \in \Delta} \mathfrak{X}(\Lambda_i^{[j]}) &= \text{holim}_{j \in \Delta} \text{holim}_{i \in \Delta} \mathfrak{X}(\Lambda_i^{[j]}) \\ &\stackrel{(7.1)}{=} \text{holim}_{j \in \Delta} \mathfrak{X}(\Lambda_\bullet^{[j]}) \end{aligned}$$

and thus the assertion for stacks. The assertion in the case when  $\mathfrak{X}$  is a prestack follows by an analogous argument.  $\square$

By restriction, we obtain a  $\tau$ -stack on the full subcategory of Lie groupoids. By a further restriction, we get a  $\tau$ -stack on the full subcategory of  $G$ -manifolds. For convenience, we state our result in the special case of  $G$ -manifolds:

**Corollary 7.6.**

Let  $M$  be a  $G$ -manifold and  $\{U_i\}_{i \in I}$  be a  $G$ -invariant covering. Denote, as usual  $Y := \sqcup_{i \in I} U_i$ . Then we have:

$$\mathfrak{X}_G(M) \xrightarrow{\sim} \text{holim} \left( \mathfrak{X}_G(Y) \begin{array}{c} \xrightarrow{\delta_0^*} \\ \xrightarrow{\delta_1^*} \end{array} \mathfrak{X}_G(Y^{[2]}) \begin{array}{c} \xrightarrow{\delta_0^*} \\ \xrightarrow{\delta_2^*} \end{array} \mathfrak{X}_G(Y^{[3]}) \begin{array}{c} \xrightarrow{\delta_0^*} \\ \xrightarrow{\delta_3^*} \\ \xrightarrow{\delta_3^*} \\ \xrightarrow{\delta_3^*} \end{array} \dots \right)$$

## 8 Proof of theorem 2.16, part 4: Sheaves and $\tau$ -surjective equivalences

We are now ready to prove theorem 2.16 in the special case of  $\tau$ -surjective equivalences. This actually finishes the proof of theorem 2.16, since by the factorization lemma 5.7 we have to consider only the two cases of  $\tau$ -surjective equivalences and internal equivalences. The latter case has already been settled with corollary 6.3. We start with the following

**Lemma 8.1.**

Let  $F : \Gamma \rightarrow \Lambda$  be a  $\tau$ -surjective equivalence of Lie groupoids. By remark 7.3.1, the functor  $F$  induces a  $\tau$ -cover of Lie groupoids.

- (i) For any  $n$ , we have a canonical functor  $M^n : \Gamma^{[n]} \rightarrow \Lambda$  which is given by arbitrary compositions in the augmented simplicial manifold

$$\dots \begin{array}{c} \xrightarrow{\delta_0} \\ \xrightarrow{\delta_3} \\ \xrightarrow{\delta_3} \\ \xrightarrow{\delta_3} \end{array} \Gamma^{[2]} \begin{array}{c} \xrightarrow{\delta_0} \\ \xrightarrow{\delta_2} \end{array} \Gamma^{[1]} \begin{array}{c} \xrightarrow{\delta_0} \\ \xrightarrow{\delta_1} \end{array} \Gamma \xrightarrow{F} \Lambda$$

Then the functor  $M^n$  is a  $\tau$ -surjective equivalence.

(ii) The diagonal functors  $\Gamma \rightarrow \Gamma^{[n]}$  are internal equivalences.

**Proof:**

(i) As compositions of  $\tau$ -coverings, all functors  $M^n$  are  $\tau$ -coverings. The functor  $F : \Gamma \rightarrow \Lambda$  is in particular fully faithful. Hence,

$$\Gamma_1 = \Gamma_0 \times_{\Lambda_0} \Lambda_1 \times_{\Lambda_0} \Gamma_0 .$$

We now calculate

$$\begin{aligned} \Gamma_1^{[n]} &= \Gamma_1 \times_{\Lambda_1} \cdots \times_{\Lambda_1} \Gamma_1 \\ &= \left( \Gamma_0 \times_{\Lambda_0} \Lambda_1 \times_{\Lambda_0} \Gamma_0 \right) \times_{\Lambda_1} \cdots \times_{\Lambda_1} \left( \Gamma_0 \times_{\Lambda_0} \Lambda_1 \times_{\Lambda_0} \Gamma_0 \right) \end{aligned}$$

and find by reordering that this equals

$$\left( \Gamma_0 \times_{\Lambda_1} \cdots \times_{\Lambda_1} \Gamma_0 \right) \times_{\Lambda_0} \Lambda_1 \times_{\Lambda_0} \left( \Gamma_0 \times_{\Lambda_1} \cdots \times_{\Lambda_1} \Gamma_0 \right) .$$

Hence the diagram

$$\begin{array}{ccc} \Gamma_1^{[n]} & \xrightarrow{M_1^n} & \Lambda_1 \\ \downarrow & & \downarrow \\ \Gamma_0^{[n]} \times \Gamma_0^{[n]} & \xrightarrow{M_0^n} & \Lambda_0 \times \Lambda_0 \end{array} \quad (8.1)$$

is a pullback diagram and thus the functor  $M^n$  is fully faithful.

(ii) Take any of the  $n$  possible projection functors  $P^n : \Gamma^{[n]} \rightarrow \Gamma$  and consider the diagram

$$\begin{array}{ccccc} \Gamma_1^{[n]} & \xrightarrow{P_1^n} & \Gamma_1 & \xrightarrow{F_1} & \Lambda_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_0^{[n]} \times \Gamma_0^{[n]} & \xrightarrow{P_0^n \times P_0^n} & \Gamma_0 \times \Gamma_0 & \xrightarrow{F_0 \times F_0} & \Lambda_0 \times \Lambda_0 \end{array}$$

The right diagram is by our assumptions on  $F$  a pullback diagram. The external diagram is just the diagram (8.1) considered in part (i) of the lemma and thus fully faithful, as well. Hence also the left part of the diagram is a pullback diagram and thus the functor  $P^n$  is fully faithful. The functor  $P^n$  is a left inverse of of the diagonal functor  $\Lambda \rightarrow \Lambda^{[n]}$ . Lemma 5.3 now implies that the diagonal functors are internal equivalences.  $\square$

**Proposition 8.2.**

Let  $\mathfrak{X}$  be a presheaf in bicategories and  $\Gamma \rightarrow \Lambda$  be a  $\tau$ -surjective equivalence. Then we have the following equivalences of bicategories

$$\begin{aligned} \mathfrak{X}(\Gamma_\bullet) &\cong \text{holim} \left( \mathfrak{X}(\Gamma_\bullet) \begin{array}{c} \xrightarrow{\delta_0^*} \\ \xrightarrow{\delta_1^*} \end{array} \mathfrak{X}(\Gamma_\bullet^{[2]}) \begin{array}{c} \xrightarrow{\delta_0^*} \\ \xrightarrow{\delta_2^*} \end{array} \mathfrak{X}(\Gamma_\bullet^{[3]}) \begin{array}{c} \xrightarrow{\delta_0^*} \\ \xrightarrow{\delta_3^*} \\ \xrightarrow{\delta_3^*} \end{array} \cdots \right) \\ &\cong \text{holim} \left( \text{Desc}_{\mathfrak{X}}(\Gamma_0 \rightarrow \Lambda_0) \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_1^*} \end{array} \text{Desc}_{\mathfrak{X}}(\Gamma_1 \rightarrow \Lambda_1) \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_2^*} \\ \xrightarrow{\partial_2^*} \end{array} \text{Desc}_{\mathfrak{X}}(\Gamma_2 \rightarrow \Lambda_2) \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_3^*} \\ \xrightarrow{\partial_3^*} \\ \xrightarrow{\partial_3^*} \end{array} \cdots \right) \end{aligned}$$



**Proof:**

The diagonal functors  $\Gamma \rightarrow \Gamma^n$  give a morphism of simplicial manifolds

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{\delta_0} & \Gamma^{[2]} & \xrightarrow{\delta_0} & \Gamma^{[1]} & \xrightarrow{\delta_0} & \Gamma \xrightarrow{F} \Lambda \\
 & \xrightarrow{\delta_3} & \uparrow & \xrightarrow{\delta_2} & \uparrow & \xrightarrow{\delta_1} & \uparrow \\
 \cdots & \xrightarrow{\delta_0} & \Gamma & \xrightarrow{\delta_0} & \Gamma & \xrightarrow{\delta_0} & \Gamma \xrightarrow{F} \Lambda \\
 & \xrightarrow{\delta_3} & & \xrightarrow{\delta_2} & & \xrightarrow{\delta_1} & 
 \end{array}$$

which is by lemma 8.1(ii) in each level an internal equivalence. Using corollary 6.3, we get the following equivalence of bicategories

$$\begin{aligned}
 \text{holim} \left( \mathfrak{X}(\Gamma_\bullet) \xrightarrow[\delta_1^*]{\delta_0^*} \mathfrak{X}(\Gamma_\bullet^{[2]}) \xrightarrow[\delta_2^*]{\delta_0^*} \mathfrak{X}(\Gamma_\bullet^{[3]}) \xrightarrow[\delta_3^*]{\delta_0^*} \cdots \right) \\
 \xrightarrow{\sim} \text{holim} \left( \mathfrak{X}(\Gamma_\bullet) \xrightarrow[\delta_1^*]{\delta_0^*} \mathfrak{X}(\Gamma_\bullet) \xrightarrow[\delta_2^*]{\delta_0^*} \mathfrak{X}(\Gamma_\bullet) \xrightarrow[\delta_3^*]{\delta_0^*} \cdots \right) \cong \mathfrak{X}(\Gamma_\bullet)
 \end{aligned}$$

The second equivalence is now a direct consequence of Proposition 7.4.  $\square$

We are now ready to take the final step and prove theorem 2.16 for  $\tau$ -surjective equivalences:

**Proposition 8.3.**

Let  $F : \Gamma \rightarrow \Lambda$  be a  $\tau$ -surjective equivalence of Lie groupoids.

1. If  $\mathfrak{X}$  is stack, then the functor  $F^* : \mathfrak{X}(\Lambda) \rightarrow \mathfrak{X}(\Gamma)$  is an equivalence of bicategories.
2. If  $\mathfrak{X}$  is a prestack, then the functor  $F^* : \mathfrak{X}(\Lambda) \rightarrow \mathfrak{X}(\Gamma)$  is fully faithful.

**Proof:**

Theorem 7.5 about equivariant descent implies

$$\mathfrak{X}(\Lambda_\bullet) \xrightarrow{\sim} \text{holim} \left( \mathfrak{X}(\Gamma_\bullet) \xrightarrow[\delta_1^*]{\delta_0^*} \mathfrak{X}(\Gamma_\bullet^{[2]}) \xrightarrow[\delta_2^*]{\delta_0^*} \mathfrak{X}(\Gamma_\bullet^{[3]}) \xrightarrow[\delta_3^*]{\delta_0^*} \cdots \right)$$

The preceding proposition 8.2 implies that this bicategory is equivalent to  $\mathfrak{X}(\Gamma_\bullet)$ , which shows part (i). The second statement is proven by a similar argument, using part (ii) of theorem 7.5.  $\square$

## 9 Proof of theorem 3.3

The central ingredient in the proof of theorem 3.3 is an explicit description of descent objects

$$\text{Desc}_{\mathfrak{X}^+}(Y \twoheadrightarrow M) = \mathfrak{X}^+(\check{C}(Y)) .$$

Instead of specializing to the Čech groupoid  $\check{C}(Y)$ , we rather describe  $\mathfrak{X}^+(\Gamma)$  for a general groupoid  $\Gamma$ . The plus construction involves the choice of a cover of  $\Gamma_0$  and a descent object for that cover. For a cover  $Y \twoheadrightarrow \Gamma_0$ , we consider the *covering groupoid*  $\Gamma^Y$  which is defined by

$$\Gamma_0^Y := Y \quad \text{and} \quad \Gamma_1^Y := Y \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} Y .$$

By definition, the diagram

$$\begin{array}{ccc} \Gamma_1^Y & \longrightarrow & \Gamma_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ \Gamma_0^Y \times \Gamma_0^Y & \xrightarrow{\pi \times \pi} & \Gamma_0 \times \Gamma_0 \end{array}$$

is a pullback diagram; hence the map  $\Pi : \Gamma^Y \rightarrow \Gamma$  is a  $\tau$ -equivalence and thus in particular a  $\tau$ -equivalence. All other structure on  $\Gamma^Y$  is induced from the groupoid structure on  $\Gamma$ . We thus have:

**Proposition 9.1.**

*Let  $\mathfrak{X}$  be a prestack and  $\Gamma$  be a groupoid. Then the bicategory  $\mathfrak{X}^+(\Gamma)$  is equivalent to the following bicategory:*

- *Objects are pairs, consisting of a covering  $Y \twoheadrightarrow \Gamma_0$  and an object  $\mathcal{G}$  in  $\mathfrak{X}(\Gamma^Y)$ .*
- *Morphisms between  $(Y, \mathcal{G})$  and  $(Y', \mathcal{G}')$  consist of a common refinement  $Z \twoheadrightarrow \Gamma_0$  of  $Y \twoheadrightarrow \Gamma_0$  and  $Y' \twoheadrightarrow \Gamma_0$  and a morphism  $A$  between the refined objects  $\mathcal{G}_Z$  and  $\mathcal{G}_{Z'}$  in  $\mathfrak{X}(\Gamma^Z)$ .*
- *2-Morphisms between one-morphisms  $(Z, A)$  and  $(Z', A')$  are described by pairs consisting of a common refinement  $W \twoheadrightarrow \Gamma_0$  of  $Z$  and  $Z'$  that is compatible with all projections and a morphism of the refinements  $A_Z$  and  $A_{Z'}$  in  $\mathfrak{X}(\Gamma^W)$ .*
- *We identify 2-morphisms  $(W, g)$  and  $(W', g')$ , if there exists a common refinement  $V \twoheadrightarrow \Gamma_0$  such that the refined 2-morphisms  $g_V$  and  $g'_V$  in  $\mathfrak{X}(\Gamma^V)$  are equal.*

**Proof:**

We describe explicitly an object of the bicategory  $\mathfrak{X}^+(\Gamma)$ : the first piece of data is an object in  $\mathfrak{X}^+(\Gamma_0)$ . This is just a covering  $Y \twoheadrightarrow M$  and

- an object in the descent bicategory  $\mathcal{D}esc_{\mathfrak{X}}(Y \twoheadrightarrow \Gamma_0)$ .

The second piece of data is a morphism that relates the two pullbacks to  $\mathfrak{X}^+(\Gamma_1)$ . Such a morphism contains the coverings  $Y \times_{\Gamma_0} \Gamma_1 \twoheadrightarrow \Gamma_1$  and  $\Gamma_1 \times_{\Gamma_0} Y \twoheadrightarrow \Gamma_1$  where one pullback is along the source map and one pullback along the target map of  $\Gamma$ .

Proposition 4.4 allows us to describe this morphism as a stable morphism on the canonical common refinement

$$Y \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} Y \twoheadrightarrow \Gamma_1 ,$$

i.e.

- A morphism of pullbacks in  $\mathcal{D}esc_{\mathfrak{X}}(\Gamma_1^Y \twoheadrightarrow \Gamma_1)$ .

Further data and axioms can be transported to the canonical common refinement:

- A 2-morphism of pullbacks in  $\mathcal{D}esc_{\mathfrak{X}}(\Gamma_2^Y \rightarrow \Gamma_2)$ .
- A condition on the pullbacks in  $\mathcal{D}esc_{\mathfrak{X}}(\Gamma_3^Y \rightarrow \Gamma_3)$ .

Altogether, we have an object in

$$\text{holim} \left( \mathcal{D}esc_{\mathfrak{X}}(\Gamma_0^Y \rightarrow \Gamma_0) \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_1^*} \end{array} \mathcal{D}esc_{\mathfrak{X}}(\Gamma_1^Y \rightarrow \Gamma_1) \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_2^*} \end{array} \mathcal{D}esc_{\mathfrak{X}}(\Gamma_2^Y \rightarrow \Gamma_2) \begin{array}{c} \xrightarrow{\partial_0^*} \\ \xrightarrow{\partial_3^*} \end{array} \cdots \right)$$

This bicategory is, according to proposition 8.2 equivalent to  $\mathfrak{X}(\Gamma^Y)$ . This shows our assertion for objects; the argument for morphisms and 2-morphisms closely parallels the argument for objects.  $\square$

**Remark 9.2.**

We comment on the relation of the three equivalent descriptions of  $G$ -equivariant objects like e.g. bundle gerbes to objects described in the literature:

1. Definition 2.5, which has the advantage of being conceptually simple. This definition is used for action groupoids of finite groups in [GR04].
2. The definition as a  $G$ -equivariant descent objects, using a  $G$ -equivariant open covering, cf. corollary 7.6. This definition is used in the construction [Mei03] of gerbes on compact Lie groups.
3. The characterization in proposition 9.1, which has the advantage that invariance under  $\tau$ -equivalences is almost immediate from the definition. Such a definition is used in [BX06].

We are now ready for the proof of theorem 3.3:

**Proof:**

We have to show that the presheaf  $\mathfrak{X}^+$  in bicategories is a stack. We thus consider for any cover  $Z \rightarrow M$  the bicategory

$$\mathcal{D}esc_{\mathfrak{X}^+}(Z \rightarrow M) = \mathfrak{X}^+(\check{C}(Z)) .$$

By proposition 9.1, this bicategory is given by objects, morphisms and 2-morphisms on covering groupoids  $\check{C}(Z)^Y$  for covering  $Y \rightarrow Z$ . We write out such a groupoid explicitly:

$$\begin{aligned} \check{C}(Z)_0^Y &= Y = \check{C}(Y)_0 \\ \check{C}(Z)_1^Y &= Y \times_Z (Z \times_M Z) \times_Z Y \\ &= Y \times_M Y = \check{C}(Y)_1 . \end{aligned}$$

We find  $\check{C}(Z)^Y = \check{C}(Y)$ . Thus  $\mathcal{D}esc_{\mathfrak{X}^+}(Z \rightarrow M) = \mathfrak{X}^+(\check{C}(Z))$  is equivalent to the subcategory of objects of  $\mathfrak{X}^+(M)$  which are defined on coverings  $Y \rightarrow Z \rightarrow M$ . This subcategory is obviously equivalent to the bicategory  $\mathfrak{X}^+(M)$ .  $\square$

# A Appendix: surface holonomy

## A.1 Oriented surface holonomy

To prepare the discussion of holonomy for unoriented surfaces, we briefly review the definition of holonomy for oriented surfaces. The holonomy of a trivial bundle gerbe  $\mathcal{I}_\omega$  with  $\omega \in \Omega^2(\Sigma)$  over a closed oriented surface  $\Sigma$  is by definition

$$\text{Hol}_{\mathcal{I}_\omega} := \exp\left(2\pi i \int_{\Sigma} \omega\right) \in \text{U}(1). \quad (\text{A.1})$$

If  $\mathcal{I}_\omega$  and  $\mathcal{I}_{\omega'}$  are two trivial bundle gerbes over  $\Sigma$  such that there exists a 1-isomorphism  $\mathcal{I}_\omega \rightarrow \mathcal{I}_{\omega'}$ , i.e. a  $\text{U}(1)$  bundle  $L$  of rank one, we have

$$\int_{\Sigma} \omega' - \int_{\Sigma} \omega = \int_{\Sigma} \text{curv}(L) \in \mathbb{Z} \quad (\text{A.2})$$

and thus the equality  $\text{Hol}_{\mathcal{I}_\omega} = \text{Hol}_{\mathcal{I}_{\omega'}}$ .

More generally, consider a bundle gerbe  $\mathcal{G}$  with connection over a smooth oriented manifold  $M$ , and a smooth map

$$\Phi : \Sigma \rightarrow M \quad (\text{A.3})$$

defined on a closed oriented surface  $\Sigma$ . Since  $H^3(\Sigma, \mathbb{Z}) = 0$  and since gerbes are classified by this cohomology group, the pullback gerbe  $\Phi^*\mathcal{G}$  is isomorphic to a trivial bundle gerbe  $\mathcal{I}_\omega$ . Hence one can choose a trivialization, i.e. a 1-isomorphism

$$\mathcal{T} : \Phi^*\mathcal{G} \xrightarrow{\sim} \mathcal{I}_\omega \quad (\text{A.4})$$

and define the holonomy of  $\mathcal{G}$  around  $\Phi$  by

$$\text{Hol}_{\mathcal{G}}(\Phi) := \text{Hol}_{\mathcal{I}_\omega}. \quad (\text{A.5})$$

This definition is independent of the choice of the 1-isomorphism  $\mathcal{T}$ : if  $\mathcal{T}' : \Phi^*\mathcal{G} \xrightarrow{\sim} \mathcal{I}_{\omega'}$  is another trivialization, we have a transition isomorphism of gerbes on  $\Sigma$

$$L := \mathcal{T}' \circ \mathcal{T}^{-1} : \mathcal{I}_\omega \xrightarrow{\sim} \mathcal{I}_{\omega'}. \quad (\text{A.6})$$

The independence of the holonomy on the choice of trivialization then follows by the argument given in equation (A.2).

## A.2 Unoriented surface holonomy

Let  $M$  be a smooth manifold and  $\mathcal{J}$  a Jandl gerbe on  $M$ . In this appendix, we discuss the definition of a holonomy for  $\mathcal{J}$  around an unoriented, possibly even unorientable, closed surface  $\Sigma$ . Such a definition is in particular needed to write down Wess-Zumino terms for two-dimensional field theories on unoriented surfaces which arise, e.g. as worldsheets in type I string theories.

We will define surface holonomy for any pair consisting of a smooth map  $\varphi : \Sigma \rightarrow M$  and an isomorphism of  $\mathbb{Z}/2$ -bundles

$$\begin{array}{ccc} \mathcal{O}(\varphi^*\mathcal{J}) & \xrightarrow{\sim} & \hat{\Sigma} \\ & \searrow & \swarrow \\ & \Sigma & \end{array} \quad (\text{A.7})$$

where we denote the orientation bundle of  $\Sigma$  by  $\hat{\Sigma}$ . This is a canonically oriented two-dimensional manifold [BG88]. In particular, the orientation bundle introduced in definition 4.8.1 of the pulled back gerbe  $\varphi^*\mathcal{J}$  must be isomorphic to the orientation bundle of the surface.

Let us first check that this setting allows us to recover the notion of holonomy from appendix A.1 if the surface  $\Sigma$  is oriented. An orientation of  $\Sigma$  is just a global section of the orientation bundle  $\hat{\Sigma} \rightarrow \Sigma$ . Due to the isomorphism (A.7), such a global section gives a global section  $\Sigma \rightarrow \mathcal{O}(\varphi^*\mathcal{J})$ , i.e. an orientation of the Jandl gerbe  $\varphi^*\mathcal{J}$ . By proposition 4.9.2 an oriented Jandl gerbe amounts to a gerbe on  $\Sigma$ , for which we can define a holonomy as in appendix A.1. We will see that the isomorphism in (A.7) is the correct weakening of the choice of an orientation of a Jandl gerbe to the case of unoriented surfaces.

Our first goal is to relate this discussion to the one in [SSW07]. In that paper, a smooth manifold  $N$  together with an involution  $k$  was considered. This involution was not required to act freely, hence we describe the situation by looking at the action groupoid  $N//(\mathbb{Z}/2)$ . Since Jandl gerbes define a stack on  $\mathcal{M}an$  and since any stack on  $\mathcal{M}an$  can be extended by definition 2.5 to a stack on Lie groupoids, the definition of a Jandl gerbe on the Lie groupoid  $N//(\mathbb{Z}/2)$  is clear.

We now need a few facts about  $\mathbb{Z}/2$ -bundles on quotients. For transparency, we formulate them for the action of an arbitrary Lie group  $G$ . Consider a free  $G$ -action on a smooth manifold  $N$  such that  $N/G$  is a smooth manifold and such that the canonical projection  $N \rightarrow N/G$  is a surjective submersion. (This is, e.g., the case if the action of  $G$  on  $M$  is proper and discontinuous.) It is an important fact that then  $N \rightarrow N/G$  is a smooth  $G$ -bundle.

If we wish to generalize this situation to the case where the action of  $G$  is not free any longer, we have to replace the quotient  $N/G$  by the Lie groupoid  $N//G$ . This Lie groupoid can be considered for a free action as well, and then the Lie groupoids  $N/G$  and  $N//G$  are  $\tau$ -equivalent. By theorem 2.16, the categories of  $G$ -bundles over  $N/G$  and  $N//G$  are equivalent. This raises the question whether there is a natural  $G$ -bundle on the Lie groupoid  $N//G$  generalizing the  $G$ -bundle  $N \rightarrow N/G$ . In fact, any action Lie groupoid  $N//G$  comes with a canonical  $G$ -bundle  $\mathfrak{C}an_G$  over  $N//G$  which we describe as in remark 2.9. As a bundle over  $N$ , it is the trivial bundle  $N \times G$ , but it carries a non-trivial  $G$ -equivariant structure. Namely  $g \in G$  acts on  $N \times G$  by diagonal multiplication, i.e.

$$g \cdot (n, h) := (gn, gh) .$$

The following lemma shows that the  $G$ -bundle  $\mathfrak{C}an_G$  has the desired property:

**Lemma A.2.1.**

*Consider a smooth  $G$ -manifold with a free  $G$ -action such that  $N/G$  is a smooth manifold and such that the canonical projection  $N \rightarrow N/G$  is a surjective submersion. Then the pullback of the  $G$ -bundle  $N \rightarrow N/G$  to the action Lie groupoid  $N//G$  is just  $\mathfrak{C}an_G$ .*

**Proof:**

The proof of the lemma consists of a careful unwinding of the definitions. The most subtle aspect concerns the  $G$ -bundle over  $N$  contained in the pullback: this bundle is  $N \times_{N/G} N \rightarrow N$  which has the diagonal as a canonical section.  $\square$

We are now ready to define the target space structure corresponding to (A.7).

**Definition A.2.2**

An orientifold background consists of an action groupoid  $N//(\mathbb{Z}/2)$ , a Jandl gerbe  $\mathcal{J}$  on  $N//(\mathbb{Z}/2)$  and an isomorphism of equivariant  $\mathbb{Z}/2$ -bundles

$$\begin{array}{ccc} \mathcal{O}(\mathcal{J}) & \xrightarrow{\sim} & \mathfrak{C}an_{\mathbb{Z}/2} \\ & \searrow & \swarrow \\ & N//(\mathbb{Z}/2) & \end{array} \tag{A.8}$$

**Proposition A.2.3.**

An orientifold background is the same as a gerbe with Jandl structure from [SSW07, Definition 5]. More precisely we have an equivalence of bicategories between the bicategory of orientifold backgrounds over the Lie groupoid  $N//(\mathbb{Z}/2)$  and the bicategory of gerbes over the manifold  $N$  with Jandl structure with involution  $k : N \rightarrow N$  given by the action of  $-1 \in \mathbb{Z}/2$ .

**Proof:**

We concentrate on how to extract a gerbe with a Jandl structure from the orientifold background. Let us first express from remark 2.9 the data of a Jandl gerbe on the Lie groupoid  $N//(\mathbb{Z}/2)$  in terms of data on the manifold  $N$ . We have just to keep one isomorphism  $\varphi = \varphi_k$  and a single coherence 2-isomorphism, for the non-trivial element  $-1 \in \mathbb{Z}/2$ . We thus get:

- A Jandl gerbe  $\mathcal{J}_N$  on  $N$ .
- A morphism  $\varphi : k^* \mathcal{J}_N \rightarrow \mathcal{J}_N$  of Jandl gerbes.
- A coherence 2-isomorphism  $c$  in the diagram

$$\begin{array}{ccc} \mathcal{J}_N & \xrightarrow{k^* \varphi} & k^* \mathcal{J}_N \\ & \searrow & \swarrow \downarrow \varphi \\ & & \mathcal{J}_N \end{array} \quad \begin{array}{c} c \\ \swarrow \searrow \end{array}$$

- A coherence condition on the 2-isomorphism  $c$ .

Similarly, we extract the data in the isomorphism

$$\mathcal{O}(\mathcal{J}_N) \rightarrow \mathfrak{C}an_{\mathbb{Z}/2}$$

of  $\mathbb{Z}/2$ -bundles over the Lie groupoid  $N//(\mathbb{Z}/2)$  that is the second piece of data in an orientifold background. It consists of

- (i) An isomorphism

$$\mathcal{O}(\mathcal{J}_N) \xrightarrow{\sim} N \times \mathbb{Z}/2$$

of  $\mathbb{Z}/2$ -bundles over the smooth manifold  $N$ .

(ii) A commuting diagram

$$\begin{array}{ccc}
\mathcal{O}(k^* \mathcal{J}_N) & \xrightarrow{\mathcal{O}(\varphi)} & \mathcal{O}(\mathcal{J}_N) \\
k^* s \downarrow & & \downarrow s \\
N \times \mathbb{Z}/2 & \xrightarrow{\text{id}_N \times m_{-1}} & N \times \mathbb{Z}/2
\end{array}$$

where  $m_{-1}$  is multiplication by  $-1 \in \mathbb{Z}/2$ .

Now the data in part (i) are equivalent to a section of the orientation bundle  $\mathcal{O}(\mathcal{J}_N)$ , i.e. an orientation of the Jandl gerbe  $\mathcal{J}_N$ . By proposition 4.9.2, our Jandl gerbe is thus equivalent to an ordinary gerbe  $\mathcal{G}$  on  $N$ . Part (ii) expresses the condition that  $\varphi$  is an orientation reversing morphism of Jandl gerbes. We summarize the data: we get

- A bundle gerbe  $\mathcal{G}$  on  $N$ .
- The odd morphism  $\varphi$  gives, in the language of [SSW07], a morphism  $A : k^* \mathcal{G} \rightarrow \mathcal{G}^*$  of bundle gerbes.
- Similarly, the coherence isomorphism

$$c : \varphi \circ k^* \varphi \Rightarrow \text{id}$$

is in that language a 2-isomorphism

$$A \otimes (k^* A)^* \Rightarrow \text{id}$$

of gerbes which is expressed in [SSW07] by a  $\mathbb{Z}/2$ -equivariant structure on  $A$ .

- Finally, one gets the coherence conditions of [SSW07].

We have thus recovered all data of [SSW07, definition 5]. □

### Corollary A.2.4.

*The bicategory of Jandl gerbes  $\mathcal{J}$  over  $\Sigma$  together with an isomorphism  $f : \mathcal{O}(\mathcal{J}) \xrightarrow{\sim} \hat{\Sigma}$  is equivalent to the bicategory of orientifold backgrounds over  $\hat{\Sigma} // (\mathbb{Z}/2)$ .*

### Proof:

Pull back along the  $\tau$ -equivalence  $\hat{\Sigma} // (\mathbb{Z}/2) \rightarrow \Sigma$  gives by theorem 2.16 an equivalence of bicategories

$$\mathcal{JGrb}^{\nabla}(\Sigma) \xrightarrow{\sim} \mathcal{JGrb}^{\nabla}(\hat{\Sigma} // (\mathbb{Z}/2)).$$

Concatenating  $f$  with the isomorphism  $\hat{\Sigma} \rightarrow \mathfrak{Can}_{\mathbb{Z}/2}$  from lemma A.2.1 provides the second data in the definition A.2.2 of an orientifold background. □

The formula for the holonomy  $\text{Hol}_{\mathcal{J}}(f)$  of such an orientifold background over  $\hat{\Sigma} // (\mathbb{Z}/2)$  is given in [SSW07] and [FNSW08, (5.9)] along the lines of section A.1. We refrain from giving details here. We then define

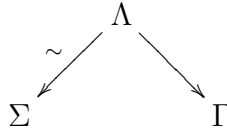
### Definition A.2.5

Let  $M$  be smooth manifold and  $\mathcal{J}$  a Jandl gerbe on  $M$ . Let  $\Sigma$  be an unoriented closed surface. Given a smooth map  $\varphi : \Sigma \rightarrow M$  and a morphism  $f : \mathcal{O}(\varphi^* \mathcal{J}) \rightarrow \hat{\Sigma}$  of  $\mathbb{Z}/2$ -bundles over  $\Sigma$ , we define the surface holonomy to be

$$\text{Hol}_{\mathcal{J}}(\varphi, f) := \text{Hol}_{(\varphi^* \mathcal{J})}(f) .$$

**Remarks A.2.6.**

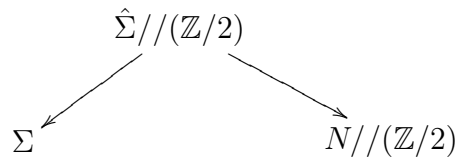
1. This holonomy enters as the exponentiated Wess-Zumino term in a Lagrangian description of two-dimensional sigma models on unoriented surfaces with target space  $M$  which are relevant e.g. for type I string theories.
2. More generally, one considers target spaces which are Lie groupoids. If the target is a Lie groupoid  $\Gamma$ , the smooth map  $\varphi$  has to be replaced by a Hilsum-Skandalis morphism  $\Phi : \Sigma \rightarrow \Lambda$  which is a special span of Lie groupoids



where  $\Lambda \rightarrow \Sigma$  is a  $\tau$ -equivalence. (For a definition and discussion, see [Met03, definition 62].)

Theorem 2.16 ensures that the pullback along  $\Lambda \rightarrow \Gamma$  is an equivalence of bicategories. Using its inverse, we can pull back a Jandl gerbe over  $\Gamma$  along  $\Phi$  to  $\Sigma$ .

3. In particular, we get in this situation a notion of holonomy  $\text{Hol}_{\mathcal{J}}(\Phi, f)$  for a Hilsum-Skandalis morphism  $\Phi$  and an isomorphism  $f$  of  $\mathbb{Z}/2$ -bundles over  $\Sigma$  as before.
4. Consider an orientifold background,  $\Gamma = N//(\mathbb{Z}/2)$ . Then each  $\mathbb{Z}/2$ -equivariant map  $\tilde{\varphi} : \hat{\Sigma} \rightarrow N$  provides a special Hilsum-Skandalis morphism



The pullback of  $\mathfrak{C}_{\text{an}_{\mathbb{Z}/2}}$  on  $N//(\mathbb{Z}/2)$  to  $\hat{\Sigma} // (\mathbb{Z}/2)$  gives again the canonical bundle which by Lemma A.2.1 is mapped to the  $\mathbb{Z}/2$ -bundle  $\hat{\Sigma} \rightarrow \Sigma$ . Thus pulling back the isomorphism of  $\mathbb{Z}/2$ -bundles in the orientifold background to an isomorphism of bundles on  $\Sigma$  gives us just the data needed in definition A.2.5 to define holonomy.

This way, we obtain holonomies  $\text{Hol}_{\mathcal{J}}(\tilde{\varphi})$  which have been introduced in [SSW07] and enter e.g. in orientifolds of the WZW models, see [GSW08].



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