HOPF ALGEBRAS AND FINITE TENSOR CATEGORIES
IN CONFORMAL FIELD THEORY

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Abstract

In conformal field theory the understanding of correlation functions can be divided into two distinct conceptual levels: The analytic properties of the correlators endow the representation categories of the underlying chiral symmetry algebras with additional structure, which in suitable cases is the one of a finite tensor category. The problem of specifying the correlators can then be encoded in algebraic structure internal to those categories.

After reviewing results for conformal field theories for which these representation categories are semisimple, we explain what is known about representation categories of chiral symmetry algebras that are not semisimple. We focus on generalizations of the Verlinde formula, for which certain finite-dimensional complex Hopf algebras are used as a tool, and on the structural importance of the presence of a Hopf algebra internal to finite tensor categories.

1 Introduction

For at least twenty-five years, two-dimensional conformal quantum field theory – or CFT, for short – has engaged physicists (working for instance on critical systems in statistical mechanics, on quasi one-dimensional condensed matter systems, or on string theory) and mathematicians (concerned with e.g. infinite-dimensional algebra, operator algebras, topology, the theory of modular forms, or algebraic geometry) alike [FRS8].

A crucial technical ingredient in the algebraic study of CFT is a certain finiteness property which distinguishes among the various models those which are easiest to understand. More specifically, conserved quantities lead to an algebraic structure $\mathcal{V}$, called the *chiral symmetry algebra*, and much information about a CFT model is encoded in the representation category $\mathcal{C} \simeq \mathcal{R}ep(\mathcal{V})$ of $\mathcal{V}$, called the category of chiral data. The relevant finiteness property is then that $\mathcal{C}$ has the structure of a *finite tensor category* in the sense of [EO]. This class of categories contains in particular the *modular tensor categories*, which are semisimple. Another class of examples of (braided) finite tensor categories is provided by the categories of modules over finite-dimensional Hopf algebras, while Hopf algebras internal to finite tensor categories allow for a generalization of the notion of modular tensor category that encompasses also non-semisimple categories. In this report we present an overview of some pertinent aspects of CFT related to finite tensor categories; semisimple categories, categories of modules over Hopf algebras, and general (not necessarily semisimple) modular categories are considered in sections 2, 3 and 4, respectively.

Of central interest in CFT (or, for that matter, in any quantum field theory) are the correlation functions. A correlation function is, roughly, a linear functional on an appropriate tensor product of state spaces that depends on geometric data, compatible with various structures on those data. Correlation functions are often of direct relevance in applications. The relevant state spaces are related, by some state-field correspondence, to field operators which, in turn, can describe observable quantities like for instance quasi-particle excitations in condensed matter physics.

In CFT, the geometric data for a correlation function include a two-dimensional manifold Y (with additional structure) called the *world sheet*. They include in particular the genus and the moduli of a conformal structure on Y as well as the location of field insertion points on Y. To each insertion point there is associated an appropriate state space depending on the type of field insertion. These state spaces, as well as further aspects of the world sheet like e.g. boundary conditions, are specified with the help of certain *decoration data*, in addition to the chiral data that are given by the category $\mathcal{C}$.

Essential information about CFT correlation functions can already be obtained when one knows the category $\mathcal{C}$ just as an abstract category (with additional properties, inherited from the chiral symmetry algebra $\mathcal{V}$) rather than concretely realized as a representation category. In particular, the problem of distinguishing between different CFT models that possess the same chiral symmetries, and thus the same chiral data, can be addressed on this purely categorical level. Indeed, a complete solution of this problem has been given in the case that $\mathcal{C}$ is a (semisimple) modular tensor category. The corresponding CFT models are called (semisimple) *rational* CFTs, or RCFTs.

For a rational CFT, a construction of all correlation functions can be achieved by expressing them in terms of invariants of suitable three-manifolds (with boundary). Apart from the

modular tensor category $\mathcal{C}$ this construction requires as one additional input datum a certain Frobenius algebra internal to $\mathcal{C}$, and it makes heavy use of a three-dimensional topological field theory that is associated to $\mathcal{C}$. This construction is outlined in sections $2.4-2.7$. Prior to that we provide, in sections $2.1-2.3$, some relevant details about CFT correlation functions, and in particular about the consistency conditions they are required to satisfy.

The analysis of non-rational CFTs turns out to be much harder, and so far no construction of correlation functions similar to the RCFT case is available. We discuss two issues which are expected to be relevant for gaining a better understanding of non-rational CFTs. First, section 3 is devoted to relations between fusion rules and modular transformations of characters. In section 3.3 we present some Verlinde-like relations that have been observed for a specific class of models, while in section 3.4 certain finite-dimensional Hopf algebras related to these models are described; a Verlinde-like formula for the Higman ideal of any factorizable ribbon Hopf algebra is quoted in section 3.6.

The second issue, a generalization of the notion of modular tensor category that includes non-semisimple finite tensor categories, is the subject of section 4. This generalization, given in section 4.5, makes use of a categorical Hopf algebra that is defined as the coend of a functor related to rigidity. Remarkably, this Hopf algebra gives rise both to three-manifold invariants (section 4.4) and to representations of mapping class groups (section 4.5), albeit they do not quite fit together. One may thus suspect that in the study of conformal field theory, methods from three-dimensional topological field theory can still be relevant also for non-rational CFTs.

2 Correlators and conformal blocks

2.1 CFT correlation functions

A world sheet Y is characterized by its topology, i.e. the genus and the number of boundary components, and the number m of marked points on Y (further aspects of Y will be given below). Denote by $\mathfrak{M}_Y$ the moduli space of conformal structures and locations of the marked points. As already indicated in the introduction, a correlation function, or *correlator*, of a CFT is a function

$$Cor(Y): \quad \mathfrak{M}_Y \times \vec{\mathcal{H}} \to \mathbb{C}, \tag{2.1}$$

with $\vec{\mathcal{H}}$ an m-tuple of state spaces

$$\vec{\mathcal{H}} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \tag{2.2}$$

assigned to the marked points. The state spaces are (infinite-dimensional) complex vector spaces, and $Cor(Y)$ is multilinear in $\vec{\mathcal{H}}$. For our purposes we do not need to spell out all details of this description, but the following information concerning the world sheet data and the vector spaces $\mathcal{H}_\alpha$ for $\alpha = 1, 2, ..., m$ will be instrumental:

- Each of the state spaces $\mathcal{H}_\alpha$ is a representation space of either the chiral symmetry algebra $\mathcal{V}$ or of the tensor product $\mathcal{V} \otimes_{\mathbb{C}} \mathcal{V}$ of two copies of $\mathcal{V}$, depending on whether the corresponding marked point lies on the boundary ot in the interior of the world sheet.

- One framework for describing the chiral symmetry algebra $\mathcal{V}$ in technical terms is by the structure of a conformal vertex algebra, that is, an infinite-dimensional $\mathbb{Z}_{\geq 0}$-graded complex vector space endowed with a product that depends on a formal parameter and with various additional structure. The precise form of this algebraic structure will, however, not concern us here; rather, it will be the properties of $\mathcal{C}$, rather than $\mathcal{V}$ itself, that are relevant to us.[1] The crucial aspect for us is that there are notions of $\mathcal{V}$-module and of $\mathcal{V}$-module map (or intertwiner), which furnish the objects and morphisms, respectively, of the representation category $\mathcal{R}ep(\mathcal{V})$ of $\mathcal{V}$, and that one can define a tensor product of $\mathcal{V}$-modules [HL], such that $\mathcal{R}ep(\mathcal{V})$ is a monoidal category. Below it will be convenient to work with a strict monoidal category equivalent to $\mathcal{R}ep(\mathcal{V})$; we denote this category by $\mathcal{C}$.

- The *world sheet* $Y \equiv (Y; \boldsymbol{\tau}, (\vec{p}, \vec{V}))$ is a compact surface Y endowed with a conformal structure $\boldsymbol{\tau}$, i.e. an equivalence class, with respect to conformal rescalings, of (Euclidean) metrics, and with a finite collection $\vec{p} = \{p_\alpha \in Y \,|\, \alpha = 1, 2, ..., m\}$ of pairwise distinct points with labeling $\vec{V} = \{V_\alpha \,|\, \alpha = 1, 2, ..., m\}$. Y can in particular have a non-empty boundary ∂Y, and it need not be orientable. The points p_α are referred to as *insertion points*. Their labeling V_α is given by either an object of $\mathcal{C}$ – if p_α lies on the boundary ∂Y – or by a pair (V'_α, V''_α) of objects of $\mathcal{C}$ – if p_α lies in the bulk $Y \backslash \partial Y$.

- Insertion points on ∂Y partition the boundary ∂Y into segments. Each of these segments carries a decoration, which is called a *boundary condition*. This decoration is again an object of a category, namely of the category of A-modules for some algebra A in $\mathcal{C}$ (see the dictionary at the end of section 2.5).

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[1] In fact there is a different approach to CFT, based on nets of von Neumann algebras instead of conformal vertex algebras, which is essentially equivalent at the categorical level. For some details and references see e.g. [KR2].

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- Recalling that $\mathcal{C}$ is the representation category of $\mathcal{V}$, we see that for $p_\alpha \in \partial Y$ the vector space $\mathcal{H}_\alpha$ is just the object V_α of $\mathcal{C}$ regarded concretely as a $\mathcal{V}$-module, while for $p_\alpha \in Y \backslash \partial Y$ it is the space $V'_\alpha \otimes_\mathbb{C} V''_\alpha$, regarded as a module over $\mathcal{V} \otimes_\mathbb{C} \mathcal{V}$. The construction of CFT correlators is compatible with direct sums; hence for rational CFTs, for which $\mathcal{C}$ is semisimple, one may without loss of generality assume that V_α, respectively V'_α and V''_α, are simple.

In physics terminology, an insertion point corresponds to the presence of a *field insertion*. When the insertion point lies on the boundary of the world sheet, $p_\alpha \in \partial Y$, one says that the field insertion ia a *boundary field*, while for $p_\alpha \in Y \backslash \partial Y$ one speaks of a *bulk field*. The insertion points actually come with additional structure, which can be encoded in the choice of a local coordinate system at each point. These data will be largely suppressed in the sequel, except in some of the pictures below, in which their presence is indicated either by drawing small oriented arcs passing through the points, or by replacing them by small oriented intervals or circles (for boundary and bulk insertions, respectively). Another structure that can be present on Y, and which for brevity will below be suppressed as well, are *defect lines* which partition the world sheet in different regions that support different *phases* of the theory. Accordingly there is also another type of field insertions: *defect fields*, which can change the type of defect line. For these, and for various other details, too, we refer the reader to e.g. [SF], sections 2 and 4 of [SFR] and section 1 of [FrFRS2].

As an illustration, the following pictures show two examples of world sheets without defect lines: a closed one of genus two with two bulk insertion points, and one of genus one with three boundary components as well as with two bulk (at p_1 and p_5) and three boundary (at p_2, p_3, , p_4) insertions.
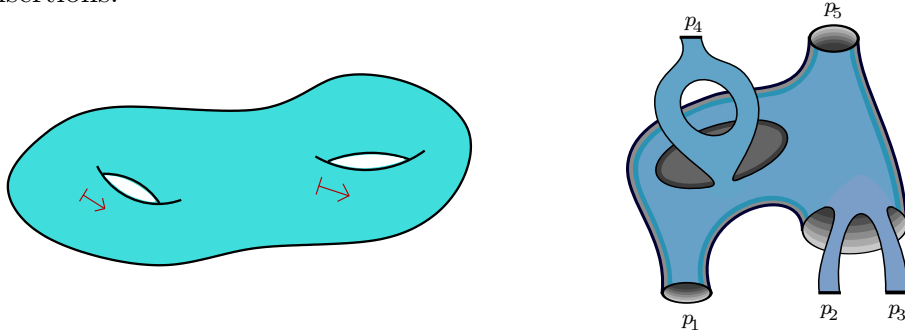
2.2 Consistency conditions

The correlators of a CFT are subject to a system of consistency conditions. There are two basically different types of such conditions. The first are the *Ward identities*, which implement compatibility with the symmetries of the theory that are encoded in $\mathcal{V}$. They select, for any point on $\mathfrak{M}_Y$, a subspace $\mathcal{B}_Y$ in the space of linear forms

$$\mathrm{Hom}_\mathbb{C}(\mathcal{H}_1 \otimes_\mathbb{C} \mathcal{H}_2 \otimes_\mathbb{C} \cdots \otimes_\mathbb{C} \mathcal{H}_m, \mathbb{C}) \qquad (2.3)$$

on the tensor product of state spaces. The relevant subspace is obtained as the space of invariants with respect to a globally defined action of the symmetries. This action can be constructed from the algebra $\mathcal{V}$, which encodes the action of the symmetries locally in the vicinity of an insertion point, together with the geometrical data of the world sheet. The so obtained subspaces $\mathcal{B}_Y$, which for all rational CFTs are finite-dimensional, are called the vector spaces of *conformal blocks*, or also of *chiral blocks*.

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Further, in the rational case the spaces of conformal blocks for all world sheets of a given topology and with given number and decorations of bulk and boundary insertions fit together into the total space of a finite-rank vector bundle over the moduli space $\mathfrak{M}_{\widehat{Y}}$ of geometric data for the *complex double* $\widehat{Y}$ of the world sheet Y. The surface $\widehat{Y}$ is, by definition, the orientation bundle $\mathrm{Or}(Y)$ over Y, modulo an identification of the points over ∂Y:

$$\widehat{Y} = \mathrm{Or}(Y) \Big/_{p \times \{1\} \simeq p \times \{-1\}, \ p \in \partial Y} . \tag{2.4}$$

For instance, the double of a world sheet Y that is closed and orientable is the disconnected sum of two copies of Y with opposite orientation, the double of the disk D is a two-sphere, $\widehat{D} = S^2$, and the double of the Klein bottle is a two-torus, $\widehat{K} = T$.

The symmetries encoded in the conformal vertex algebra $\mathcal{V}$ include in particular the conformal symmetry; accordingly, any $\mathcal{V}$-module carries a representation of the Virasoro algebra. (Also, all of them have the same eigenvalue of the canonical central element of the Virasoro algebra; this value c is called the central charge of $\mathcal{V}$.) This symmetry can be used to endow each bundle of conformal blocks with a projectively flat connection, the Knizhnik-Zamolodchikov connection. The monodromy of this connection furnishes a projective representation $\rho_{\widehat{Y}}$ of the mapping class group $\mathrm{Map}(\widehat{Y})$ of $\widehat{Y}$ (the fundamental group of the moduli space $\mathfrak{M}_{\widehat{Y}}$) on the fibers, i.e. on the spaces of conformal blocks (for details see e.g. [FS,FB,Lo]). Another property of the system of conformal blocks of a rational CFT is the existence of *gluing isomorphisms* between (direct sums of) conformal blocks for surfaces that are related by 'cutting and gluing' (and thus generically have different topology); see e.g. [Lo, Sect. 4] or [FFFS, Sect. 2.5.3].

The bundles of conformal blocks, or also their (local) sections, are sometimes referred to as the correlators of the *chiral CFT* that is associated to the chiral symmetry algebra $\mathcal{V}$. However, generically the bundles do not have global sections, so a chiral CFT behaves quite differently from a conventional quantum field theory, which has single-valued correlators. (Still, chiral CFT has direct applications in physics, for systems in which a chirality (handedness) is distinguished, e.g. in the description of universality classes of quantum Hall fluids.)

The second type of consistency conditions obeyed by the correlators of the full CFT are twofold: first, the *sewing constraints*, or factorization constraints, and second, the *modular invariance* constraints. These are algebraic equations which assert that the image of a correlator under a gluing isomorphism is again a correlator (and thus incorporate the compatibility of correlators on world sheets that are related by cutting and gluing), respectively that the correlator for a world sheet Y is invariant under the action of the group $\mathrm{Map}_{\mathrm{or}}(Y)$ of oriented mapping classes of the world sheet, which is naturally embedded in $\mathrm{Map}(\widehat{Y})$. In rational CFT the compatibility with cutting and gluing allows one to reconstruct all correlators by gluing from a small collection of special correlators. Invariance under $\mathrm{Map}_{\mathrm{or}}(Y)$ is referred to as *modular invariance*, a term reflecting the special case that Y is a torus without insertions. In that case one has $\widehat{T} = T \sqcup -T$ (with $-T$ denoting the torus with opposite orientation), and the relevant representation of $\mathrm{Map}(\widehat{T})$ has the form $\rho_{\widehat{T}} = \rho_T \otimes_{\mathbb{C}} \rho_T^*$ with ρ_T a representation of the modular group $\mathrm{SL}(2, \mathbb{Z}) = \mathrm{Map}_{\mathrm{or}}(T)$.

CFT defined on world sheets Y, with correlators satisfying, in addition to the Ward identities, also all sewing constraints, is referred to as *full CFT*, as opposed to chiral CFT, which is defined on closed oriented surfaces and deals with conformal blocks, which are solutions to the Ward identities only. The problem to be addressed in the rest of the present section and in the

next section is how to obtain a full CFT on world sheets Y from a corresponding chiral CFT on the complex doubles $\widehat{Y}$, and in particular, how to select the correlator on any world sheet Y as an element of the space of conformal blocks on $\widehat{Y}$.

2.3 Solution of the sewing and modular invariance constraints

From now until the end of section 2 we restrict our attention to *rational* CFTs. Basically this means that the relevant representation theory is semisimple and satisfies certain finiteness conditions. Technically, it can be formulated as a set of conditions on the vertex algebra $\mathcal{V}$ [Hua4] (or on the corresponding net of von Neumann algebras [KLM]), and in terms of the decoration category $\mathcal{C}$ it is characterized by $\mathcal{C}$ being a *modular tensor category.*

In short, a modular tensor category $\mathcal{C}$ is a semisimple monoidal category which comes with a braiding, a duality and a twist, which has finitely many simple objects up to isomorphism, for which the tensor unit is simple, and for which the braiding is maximally non-degenerate; for the precise definition, see appendix A.1. We denote the finite set of isomorphism classes of simple objects of $\mathcal{C}$ by $\mathcal{I}$ and select a representative S_i for each class $i \in \mathcal{I}$. As representative for the class of the tensor unit $\mathbf{1}$ we take $\mathbf{1}$ itself, and denote its class by 0, i.e. write $\mathbf{1} = S_0$. As a $\mathcal{V}$-module, $\mathbf{1} \in \mathcal{C} = \mathcal{R}ep(\mathcal{V})$ is just $\mathcal{V}$ itself.

A crucial feature of rational CFTs is that for solving the sewing constraints, and thus for obtaining the correlators for all world sheets, of arbitrary topology and with any number and type of field insertions (as well as for other purposes), *combinatorial* information about the theory is sufficient. Specifically, to account for the chiral symmetries it is enough to treat the category $\mathcal{C}$ as an abstract category (with suitable additional structure), rather than as given concretely as the representation category $\mathcal{R}ep(\mathcal{V})$ of the chiral symmetry algebra. For instance, we can, and will, take $\mathcal{C}$ to be *strict* monoidal, equivalent as a monoidal category (with additional structure) to the non-strict category $\mathcal{R}ep(\mathcal{V})$. Furthermore, one can replace the complex-analytic modular functor that is furnished by the system of conformal blocks by the equivalent [BK, Ch. 4] topological modular functor that is provided [2] by the TFT functor $\mathtt{tft}_{\mathcal{C}}$ that is associated to the modular tensor category $\mathcal{C}$. Hereby the fibers of the bundles of conformal blocks for Y are identified, as representation spaces of mapping class groups, with the state spaces $\mathtt{tft}_{\mathcal{C}}(\widehat{Y})$ of the $\mathcal{C}$-decorated three-dimensional topological field theory $\mathtt{tft}_{\mathcal{C}}$. In this description, the correlator $Cor(Y)$ is just regarded as an element

$$Cor(Y) \in \mathcal{B}_Y \tag{2.5}$$

of such a finite-dimensional vector space. As for the world sheet Y, it can be treated just as a topological manifold, endowed with some additional structure to rigidify the situation. The information that is suppressed in this simplified setting can be restored completely when combining the results obtained in this setting with the explicit expressions for the conformal blocks as vector bundles over $\mathfrak{M}_{\widehat{Y}}$. (The latter are *not* fully known for all surfaces $\widehat{Y}$ and all classes of RCFTs, but determining them is by all means an issue independent of the problems considered here.)

Despite all these simplifications, to actually determine the solution of all sewing and modular invariance constraints is not an easy matter. After all, these constraints constitute infinitely

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[2] See the discussion around formula (2.11) below.

many nonlinear equations, in particular one for each cutting or sewing, in infinitely many variables, namely the correlators for all world sheets. A traditional approach to these constraints has been to establish the general solution to some specific small subset of constraints, most prominently, to the requirement of modular invariance of the torus partition function (i.e., the correlator for a torus without insertions) $Z = Cor(T, \tau, \emptyset)$. The torus partition function is a vector in the space $\mathcal{B}_T$ of conformal blocks for the torus T. The terminology 'partition function' signifies the property of Z that $\mathcal{B}_T$ has a basis [3] $\{\chi_i \otimes \chi_j^* \mid i, j \in \mathcal{I}\}$ in which the expansion coefficients are non-negative integers and the coefficient with $i = j = 0$ is unity (in physical terms, this amounts to uniqueness of the vacuum state). Thus with

$$Z = \sum_{i,j \in \mathcal{I}} Z_{i,j}\, \chi_i \otimes \chi_j^* \qquad (2.6)$$

one has

$$Z_{i,j} \in \mathbb{Z}_{\geq 0} \qquad \text{and} \qquad Z_{0,0} = 1. \qquad (2.7)$$

Modular invariance of Z amounts in this basis to the requirement

$$[Z, \rho_T(\gamma)] = 0 \quad \text{for all} \quad \gamma \in \mathrm{SL}(2,\mathbb{Z}). \qquad (2.8)$$

As conformal blocks depending on the modular parameter τ of the torus, the basis elements χ_i are given by the *characters*

$$\chi_i(\tau) := \mathrm{tr}_{S_i}(\mathrm{e}^{2\pi\mathrm{i}\tau(L_0 - c/24)}) \qquad (2.9)$$

of the irreducible modules of the chiral symmetry algebra $\mathcal{V}$. (In (2.9), c is the central charge of the conformal vertex algebra $\mathcal{V}$ and L_0 is the zero mode of the Virasoro algebra.)

The classification, for a given chiral algebra, of matrices $Z = (Z_{i,j})$ obeying these conditions has produced remarkable results, starting with the discovery of an A-D-E-structure of the solution set for classes of models with simple objects of sufficiently small categorical dimension. It has not led to any deep insight in the structure of the solution set for general rational CFTs, though. And indeed this classification program had to be put into a different perspective by the observation that there exist spurious solutions, i.e. matrices satisfying (2.7) and (2.8) which are unphysical in the sense that they do not fit into a consistent collection of correlators that also satisfies all other sewing and modular invariance constraints. (For some details and references see section 3.4 of [FRS8].)

Additional insight has been gained by the study of boundary conditions that preserve all chiral symmetries. In particular it has been realized that for consistency of such boundary conditions, representations of the fusion rules by matrices with non-negative integral entries, so called NIM-reps, must exist [BPPZ]. This requirement, and similar consistency conditions for unorientable surfaces, turn out to be rather restrictive, see e.g. [Ga, HSS]. These developments have triggered another approach to the solution of the sewing constraints, which we call the *TFT construction* of the correlators of rational CFTs. [4] This construction associates to a world

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[3] The complex conjugation of χ_j reflects the opposite orientation of the second copy of T in the double $\widehat{T} = T \sqcup -T$.

[4] The correlators on world sheets of genus 0 and 1 have also been obtained by direct use of the theory of vertex algebras [HK, Ko]. The relation between these results and the TFT construction is discussed in [KR1].

sheet Y a cobordism $\mathcal{M}_Y$ from $\emptyset$ to $\partial\mathcal{M}_Y$ such that $\partial\mathcal{M}_Y$ is homeomorphic to the complex double $\widehat{Y}$ (see (2.4)) of the world sheet. The correlator for Y is then defined to be the element

$$Cor(Y) = \mathtt{tft}_{\mathcal{C}}(\mathcal{M}_Y)\,1 \tag{2.10}$$

of the vector space $\mathcal{B}_Y = \mathtt{tft}_{\mathcal{C}}(\widehat{Y})$ of conformal blocks. It has been proven that the correlators produced by this construction satisfy *all* sewing and modular invariance constraints.

Besides the modular tensor category $\mathcal{C}$, the TFT construction requires one additional datum, namely a certain *Frobenius algebra* internal to $\mathcal{C}$. For each Morita class of such algebras the construction gives one independent solution. The relevant algebras will be described in section 2.6. The term TFT construction alludes to a tool that is crucial in the construction, namely the $\mathcal{C}$-decorated three-dimensional *topological field theory* (TFT) $\mathtt{tft}_{\mathcal{C}}$; this is [Tu] a projective symmetric monoidal functor

$$\mathtt{tft}_{\mathcal{C}}: \quad \mathcal{C}ob^{\mathcal{C}}_{3,2} \to \mathcal{V}ect_{\mathbb{C}} \tag{2.11}$$

from a category $\mathcal{C}ob^{\mathcal{C}}_{3,2}$ of three-dimensional cobordisms to finite-dimensional complex vector spaces. Such a TFT associates to an oriented surface a vector space, called the TFT state space of the surface, and to a cobordism between surfaces a linear map between the corresponding state spaces. Both the surfaces and the cobordisms come with additional data. In a $\mathcal{C}$-*decorated* TFT these data include embedded arcs on the surfaces and embedded ribbon graphs in the cobordisms (in such a way that for any world sheet Y the double $\widehat{Y}$ is an object of $\mathcal{C}ob^{\mathcal{C}}_{3,2}$ and the cobordism $\mathcal{M}_Y$ is a morphism of $\mathcal{C}ob^{\mathcal{C}}_{3,2}$); the arcs are labeled by objects of $\mathcal{C}$, and the ribbon graphs by objects and morphisms of $\mathcal{C}$. It is an important result that to any (semisimple) modular tensor category there can indeed be associated a TFT functor $\mathtt{tft}_{\mathcal{C}}$; for details we refer to [Tu] and [BK, Ch. 4]; a brief summary can be found in [FFFS, Sect. 2.5]. The linear maps obtained by the functor $\mathtt{tft}_{\mathcal{C}}$ furnish an invariant of oriented three-manifolds and of framed links in these manifolds [RT].

In connection with the use of $\mathtt{tft}_{\mathcal{C}}$ in the prescription (2.10) a word of caution is in order. Namely, from a rational conformal vertex algebra $\mathcal{V}$ and the modular tensor category $\mathcal{C} = \mathcal{R}ep(\mathcal{V})$ one obtains representations of mapping class groups in two distinct ways. The first is via the KnizhnikZamolodchikov connection on the bundles of conformal blocks; in this case the mapping class groups arise as the fundamental groups of the moduli spaces $\mathfrak{M}_{\widehat{Y}}$. The second way is via three-dimensional TFT and the embedding of mapping class groups in cobordisms; this construction uses directly the modular tensor category $\mathcal{C}$, while the former is based on the vertex algebra $\mathcal{V}$ and its conformal blocks. That the two constructions give equivalent representations is known for genus 0 and 1, but to establish equivalence model-independently and for any genus will require a considerably improved understanding of conformal blocks and in particular of their factorization properties for general rational vertex algebras.

2.4 The TFT construction of RCFT correlators

The formula (2.10) gives the correlator $Cor(Y)$ as the invariant of a three-manifold $\mathcal{M}_Y$. This manifold, to which we will refer as the *connecting manifold*, comes with a prescription for an embedded ribbon graph Γ_Y; by abuse of notation we use the symbol $\mathcal{M}_Y$ also to refer to the manifold including the graph Γ_Y. In (2.10), the three-manifold $\mathcal{M}_Y$ is regarded as a cobordism from $\emptyset$ to the boundary $\partial\mathcal{M}_Y$. Thus taking the invariant, i.e. applying the TFT functor $\mathtt{tft}_{\mathcal{C}}$,

yields a linear map from $\mathtt{tft}_\mathcal{C}(\emptyset) = \mathbb{C}$ to $\mathtt{tft}_\mathcal{C}(\partial \mathcal{M}_Y) = \mathtt{tft}_\mathcal{C}(\widehat{Y})$, which when applied to $1 \in \mathbb{C}$ then gives the correlator – a vector in $\mathtt{tft}_\mathcal{C}(\widehat{Y})$, i.e. in the space of conformal blocks for the world sheet Y.

The TFT construction is a prescription on how to obtain the connecting manifold $\mathcal{M}_Y$, including its embedded ribbon graph Γ_Y, for any arbitrary world sheet Y. The details of this prescription are a bit lengthy. While many of the details will not concern us here, the following features are pertinent (for more information see e.g. [FjFRS1, App. B] or [FSSt, App. A.4,A.5]):

▪ The finite-dimensional vector space of which $Cor(Y)$ is an element is the space $\mathcal{B}_Y$ of conformal blocks (recall that $\mathcal{B}_Y$ is defined with the help of the complex double $\widehat{Y}$ of Y).

▪ The connecting manifold $\mathcal{M}_Y$ is obtained by suitably 'filling up' the double $\widehat{Y}$ so as to get a three-manifold having $\widehat{Y}$ as its boundary. A guideline for this filling-up procedure is that it should not introduce any new topological information; indeed $\mathcal{M}_Y$ has Y as a deformation retract, i.e. can be viewed as a fattening of the world sheet. (Moreover, in the extension from two to three dimensions there should not, in physics terminology, be any additional dynamical information should be involved; this is accounted for by taking the field theory relevant to $\mathcal{M}_Y$ to be a *topological* theory.) Concretely, as a manifold $\mathcal{M}_Y$ is the total space of an interval bundle over Y modulo an identification over ∂Y. For instance, when Y is closed and orientable, then $\mathcal{M}_Y = X \times [-1, 1]$, the connecting manifold $\mathcal{M}_D$ for the disk is a three-ball, and the connecting manifold for the Klein bottle is a full torus.

▪ By construction, the world sheet Y is naturally embedded in the 'equatorial plane' $Y \times \{0\}$ of $\mathcal{M}_Y$. A crucial ingredient of the ribbon graph Γ_Y in $\mathcal{M}_Y$ is a dual triangulation Γ_Y° of the embedded world sheet. As an additional datum, beyond the modular tensor category $\mathcal{C}$ and the combinatorial data for Y, the decoration of the ribbon graph Γ_Y thereby involves the selection of an object A of $\mathcal{C}$, whose role is to label each of the ribbons in the dual triangulation Γ_Y°. This object A cannot be chosen arbitrarily; indeed, in order for the correlator (2.10) to be well-defined, A is required to carry the structure of a *simple symmetric special Frobenius algebra* in $\mathcal{C}$. (When allowing for unoriented world sheets, A must come with an additional structure, a braided analogue of an involution.) Each of the coupons of Γ_Y° is labeled by either the product or coproduct morphism of A; the various properties of A then ensure that the invariant $\mathtt{tft}_\mathcal{C}(\mathcal{M}_Y)$ is independent of the particular choice of dual triangulation of Y.

2.5   Main results of the TFT construction

The basic result of the TFT construction is the statement that the pair $(\mathcal{C}, A)$ is necessary and sufficient for obtaining a solution to the sewing constraints of rational conformal field theory. In more detail, one has the following

THEOREM:

(i) The pair $(\mathcal{C}, A)$, with $\mathcal{C}$ a modular tensor category and A a simple symmetric special Frobenius algebra in $\mathcal{C}$, supplies all required decoration data, and the TFT construction yields a family $\{ Cor_{(\mathcal{C},A)}(Y) \}$ of correlators (on oriented world sheets arbitrary topology and with arbitrary field insertions) that satisfy all sewing and modular invariance constraints of rational CFT.

(ii) Every solution to the sewing constraints of a non-degenerate rational CFT based on the chiral data $\mathcal{C}$ can be obtained via the TFT construction with a simple symmetric special Fro-

benius algebra A in $\mathcal{C}$. Once a boundary condition of the CFT is selected, this algebra A is determined uniquely up to isomorphism.

(iii) Morita equivalent algebras give rise to equivalent families of correlators.

We add a few remarks; for more details the reader should consult the original literature.

■ The assertion (i), first stated in [FRS1] and [FRS2, Sect. 5], was proved in [FjFRS1]. A crucial input of the proof is that every sewing operation can be obtained by a finite sequence of finitely many distinct local moves. This allows one to reduce the infinitely many sewing constraints to three types of manipulations: action of (a generator of) the relevant mapping class group, boundary factorization, and bulk factorization. These three types of constraints are treated in theorems 2.2, 2.9 and 2.13 of [FjFRS1], respectively.

■ Part (ii) was established in [FjFRS2]. In the proof the boundary condition is taken to be the same on all segments of ∂Y; different choices of boundary condition give rise to Morita equivalent algebras. The qualification 'non-degenerate' refers to properties of the vacuum state and of the two-point correlators of boundary fields on the disk and of bulk fields on the sphere, all of which are expected to hold in a rational CFT.

■ Part (iii) was claimed in [FRS1]. The precise meaning of equivalence – namely equality of correlators up to a factor $\gamma^{-\chi(Y)/2}$ with $\chi(Y)$ the Euler characteristic of Y and γ the quotient of the dimensions of the two Morita equivalent algebras involved – has been formulated in section 3.3 of [FrFRS2], where also a proof of the statement is presented. Note that in a (semisimple) modular tensor category a Frobenius structure can be transported along Morita contexts.

■ As they stand, parts (i) and (iii) apply to oriented world sheets. There is a version for unoriented (in particular, unorientable) world sheets as well (for part (i), see theorems 2.4, 2.10 and 2.14 of [FjFRS1]), but it requires a few modifications. In particular, as already mentioned, the algebra A must possess an additional structure $\sigma \in \text{End}(A)$, which is a braided version of an involution [FRS3] ((A, σ) is then called a *Jandl algebra*), and the notion of Morita equivalence gets generalized accordingly [FRS6, def. 13].

Instrumental in all these findings is another fundamental result: All relevant concepts from rational CFT have a precise mathematical formalization that is entirely expressible in terms of the data $\mathcal{C}$ and A. These may be collected in the form of a dictionary between physical concepts and mathematical structures. Some entries of this dictionary are the following:

physical concept	mathematical structure
phase of a CFT	symmetric special Frobenius algebra A in $\mathcal{C}$
boundary condition	A-module $M \in \mathcal{C}_A$
defect line separating phases A and B	A-B-bimodule $X \in \mathcal{C}_{A|B}$
boundary field changing the boundary condition from M to M'	module morphism $\Psi \in \text{Hom}_A(M \otimes U, M')$
bulk fields in region with phase A	bimodule morphism $\Phi \in \text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A)$
defect field changing the defect type from X to X'	bimodule morphism $\Theta \in \text{Hom}_{A|B}(U \otimes^+ X \otimes^- V, X')$
CFT on unoriented world sheets	A a Jandl algebra
non-chiral internal symmetry	element of Picard group $\text{Pic}(\mathcal{C}_{A|A})$ [FrFRS2]

In the expressions for bulk and defect fields, the notation $\otimes^{\pm}$ refers to bimodule structures of the corresponding objects which are obtained by combining the product of the algebra(s) and suitable braidings, e.g.

$$U \otimes^+ A := (\, U \otimes A, \, (\mathrm{id}_U \otimes m) \circ (c_{U,A}^{-1} \otimes \mathrm{id}_A), \, \mathrm{id}_U \otimes m \,)$$

$$\text{and} \quad U \otimes^- A := (\, U \otimes A, \, (\mathrm{id}_U \otimes m) \circ (c_{A,U} \otimes \mathrm{id}_A), \, \mathrm{id}_U \otimes m \,), \tag{2.12}$$

and analogously in the other cases (see e.g. [FRS5, eqs. (2.17,18)]).

One may note that the structures appearing in the list above naturally fit into a bi-category, with the phases, defect lines and defect fields being, respectively, the objects, 1-morphisms and 2-morphisms.

2.6 Frobenius algebras in $\mathcal{C}$

As the algebra A plays a central role in the TFT construction, let us pause to collect some information about the relevant class of algebras. We find it convenient to represent the relevant structural morphisms graphically. (For an introduction to the graphical calculus for strict monoidal categories see e.g. [JS, Maj, Kas] or section 2.1 of [FRS2]. Note in particular that the positions of the various pieces in such pictures are relevant only up to suitable isotopy.) We draw the pictures such that the domain of a morphism is at the bottom and the codomain is at the top.

The various properties of A are defined as follows.

■ A (unital, associative) *algebra*, or monoid, in a monoidal category $\mathcal{C}$ consists of an object A together with morphisms $m = \vcenter{\hbox{$\curlywedge$}} \in \mathrm{Hom}(A \otimes A, A)$ (the product, or multiplication, morphism) and $\eta = \vcenter{\hbox{$\mathord{\downarrow}$}} \in \mathrm{Hom}(\mathbf{1}, A)$ (the unit morphism) such that the associativity and unit properties

$$\tag{2.13}$$

are satisfied.

■ Likewise, a *coalgebra* in $\mathcal{C}$ is an object C together with coproduct and counit morphisms $\Delta = \vcenter{\hbox{$\curlyvee$}} \in \mathrm{Hom}(A, A \otimes A)$ and $\varepsilon = \vcenter{\hbox{$\mathord{\uparrow}$}} \in \mathrm{Hom}(A, \mathbf{1})$ such that

$$\tag{2.14}$$

■ A *Frobenius algebra* in $\mathcal{C}$ is a quintuple $(A, m, \eta, \Delta, \varepsilon)$ such that (A, m, η) is an algebra in $\mathcal{C}$, (A, Δ, ε) is a coalgebra, and the two structures are connected by the requirement that the coproduct is a bimodule morphism, i.e. that

$$\tag{2.15}$$

■ A *symmetric* algebra in a sovereign monoidal category $\mathcal{C}$ (that is, a monoidal category endowed with a left and a right duality which coincide as functors) is an algebra for which the equality

$$\raisebox{-2em}{\includegraphics{eq216}}\qquad\qquad(2.16)$$

of morphisms in $\mathrm{Hom}(A, A^\vee)$ holds. If the algebra is also Frobenius, then both of these are *iso*morphisms. Note that the equality only makes sense if the left and right dual objects $A^\vee$ and $^\vee A$ are equal, as is ensured if $\mathcal{C}$ is sovereign.

■ A *special* Frobenius algebra in a rigid monoidal category $\mathcal{C}$ is a Frobenius algebra for which the relations

$$\raisebox{-2em}{\includegraphics{eq217}}\qquad\qquad(2.17)$$

are satisfied, i.e. the product is left-inverse to the coproduct and, up to a factor $\dim(A)\neq 0$, the counit is left-inverse to the unit. (Sometimes this property is instead called normalized special, with specialness just meaning that the two equalities hold up to non-zero constants; if A is symmetric Frobenius, the product of these constants is unity.)

■ A *simple* Frobenius algebra is one that is simple as an A-bimodule.

The following is a (non-exhaustive) list of examples of simple symmetric special Frobenius algebras:

(a) In any (strict) monoidal category, the tensor unit, with all structural morphisms being unit morphisms, $A = (\mathbf{1}, \mathrm{id}_\mathbf{1}, \mathrm{id}_\mathbf{1}, \mathrm{id}_\mathbf{1}, \mathrm{id}_\mathbf{1}, \mathrm{id}_\mathbf{1})$.

(b) In any rigid monoidal category $\mathcal{C}$, and for any object U of $\mathcal{C}$, the following algebra: $A = (U^\vee \otimes U, \mathrm{id}_{U^\vee} \otimes \tilde{d}_U \otimes \mathrm{id}_U, \tilde{b}_U, \mathrm{id}_{U^\vee} \otimes b_U \otimes \mathrm{id}_U, d_U)$. Any such algebra is Morita equivalent to the one in (a).

(c) For $\mathcal{C}$ the modular tensor category of integrable modules over the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$ at positive integral value k of the level, the following objects, coming in an A-D-E pattern, are symmetric special Frobenius algebras (we label the simple objects by their highest $\mathfrak{sl}(2)$-weight, i.e. as S_0, S_1, ... , S_k):

$$A(\mathrm{A}) = S_0 \equiv \mathbf{1} \quad (k \in \mathbb{Z}_{>0})\,, \qquad A(\mathrm{D}) = S_0 \oplus S_k \quad (k \in 2\mathbb{Z}_{>0})\,,$$

$$A(\mathrm{E}_6) = S_0 \oplus S_6 \quad (k = 10)\,, \qquad A(\mathrm{E}_7) = S_0 \oplus S_8 \oplus S_{16} \quad (k = 16)\,,$$

$$A(\mathrm{E}_8) = S_0 \oplus S_{10} \oplus S_{18} \oplus S_{28} \quad (k = 28)\,. \qquad\qquad (2.18)$$

These exhaust the set of Morita classes of simple symmetric special Frobenius algebras in this category [KO].

(d) For $K \leq \mathrm{Pic}(\mathcal{C})$ a subgroup of the Picard group (i.e., the group of isomorphism classes of invertible objects) of $\mathcal{C}$, subject to the requirement that the three-cocycle on $\mathrm{Pic}(\mathcal{C})$ that is furnished by the associativity constraint of $\mathcal{C}$, when restricted to the subgroup K is the coboundary of a two-cocycle ω on K, there is an algebra $A(K, \omega) \cong \bigoplus_{g \in K} S_g$, with cohomologous two-cocycles ω giving isomorphic algebras [FRS4]. Algebras of this type are called *Schellekens algebras*; in the physics literature the corresponding CFT models are known as *simple current constructions* [ScY, FSS].
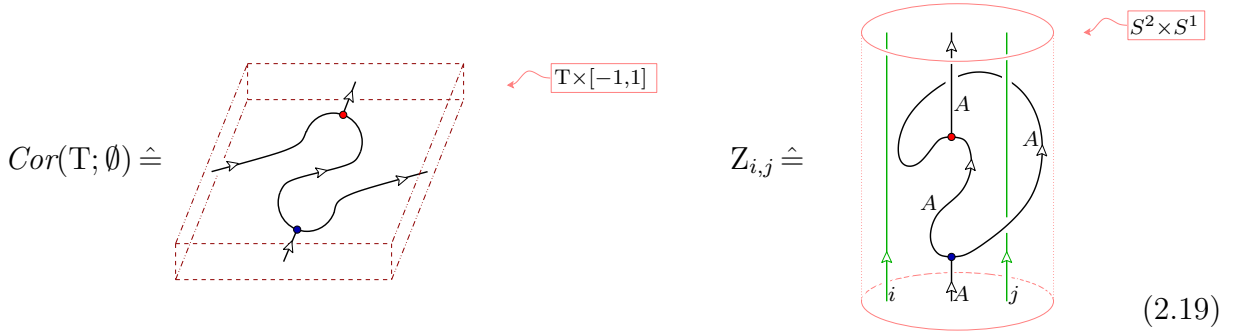
(e) In the m-fold Deligne tensor product $\mathcal{C}^{\boxtimes m}$ of $\mathcal{C}$, there is an algebra structure on the object $A = \bigoplus_{i_1, i_2, \ldots, i_m \in \mathcal{I}} (S_{i_1} \times S_{i_2} \times \ldots \times S_{i_m})^{\oplus N_{i_1 i_2 \ldots i_m}} \in \mathcal{C}^{\boxtimes m}$, with multiplicities given by $N_{i_1 i_2 \ldots i_m} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(S_{i_1} \otimes S_{i_2} \otimes \ldots \otimes S_{i_m}, \mathbf{1})$. Algebras of this type correspond to so-called permutation modular invariants [BFRS2].

2.7 A digest of further results

The TFT construction supplies universal formulas for coefficients of correlation functions, valid for all RCFTs. The coefficients of $Cor(X)$ with respect to a basis in the appropriate space $\mathcal{B}_Y$ of conformal blocks are expressible through the structural data for the modular tensor category $\mathcal{C}$ and the symmetric special Frobenius algebra A.

For some world sheets of particular interest, the construction gives the following results.

■ A basic correlator is the torus partition function $Z = Cor(\mathrm{T}, \tau, \emptyset)$ (2.6). The TFT construction gives Z as the invariant of a ribbon graph in the three-manifold $\mathrm{T} \times [-1, 1]$, and expressing it as in (2.6) in the standard basis of characters yields the coefficients $Z_{i,j}$ as invariants of ribbon graphs in the closed three-manifold $S^2 \times S^1$. Choosing the simplest dual triangulation of the world sheet T, these graphs in the connecting manifold $\mathrm{T} \times [-1, 1]$ look as follows (see (5.24) and (5.30) of [FRS2]):

$$ \tag{2.19} $$

In plain formulas, the resulting expression for the numbers $Z_{i,j}$ looks still a little complicated; in the special case that both $\dim_{\mathbb{C}} \mathrm{Hom}(S_i, A)$ and $\dim_{\mathbb{C}} \mathrm{Hom}(S_i \otimes S_j, S_k)$ for all $i, j, k \in \mathcal{I}$ are either 0 or 1, it reduces to [FRS2, eq. (5.85)]

$$ Z_{i,j} = \mathrm{tft}_{\mathcal{C}} \left(\vcenter{\hbox{\includegraphics{ribbon}}} \right)_{(S^2 \times S^1)} = \sum_{a,b,c \in \mathcal{J}} m_{bc}{}^a \, \Delta_a^{cb} \sum_{k \in \mathcal{I}} \mathsf{G}_{a\,k}^{(c\,b\,j)\,\bar{\imath}} \, \mathsf{R}^{(c\,b)\,a} \, \frac{\theta_k}{\theta_j} \, \mathsf{F}_{k\,a}^{(c\,b\,j)\,\bar{\imath}}, \tag{2.20} $$

with $\mathcal{J} = \{i \in \mathcal{I} \,|\, \dim_{\mathbb{C}} \mathrm{Hom}(S_i, A) = 1\}$, m_{ij}^k and Δ_i^{jk} the structure constants of the product and the coproduct, respectively, θ_i the twist eigenvalues, and F and R coefficients of the associativity constraint and of the braiding with respect to standard bases of morphism spaces (and $\mathsf{G} = \mathsf{F}^{-1}$; for detailed definitions see section 2.2 of [FRS2]).

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- Inserting the data for the algebras of the $\widehat{\mathfrak{sl}}(2)$ A-D-E classification (2.18), one recovers the well-known expressions [CIZ] for the torus partition functions of the $\widehat{\mathfrak{sl}}(2)$ theories. One also shows (Thm 5.1 of [FRS2]) that the partition function is modular invariant, which in terms of the matrix Z means $[\gamma, Z] = 0$ for $\gamma \in \mathrm{SL}(2, \mathbb{Z})$, that $Z_{0,0} = 1$ iff A is simple, and that the coefficients $Z_{i,j}$ are nonnegative integers. In fact,

$$Z_{i,j} = \dim_{\mathbb{C}} \mathrm{Hom}_{A|A}(S_i \otimes^+ A \otimes^- S_j, A). \tag{2.21}$$

One also shows that the torus partition functions obtained for the opposite algebra and the direct sum and tensor product of algebras are $Z^{A^{\mathrm{opp}}} = (Z^A)^{\mathrm{t}}$, $Z^{A \oplus B} = Z^A + Z^B$, and $\widetilde{Z}^{A \otimes B} = \widetilde{Z}^A \widetilde{Z}^B$ with $\widetilde{Z}_{i,j} = Z_{i,\bar{j}}$. Also recall that any correlator on an oriented world sheet, and hence in particular the matrix $Z = Z(A)$, depends only on the Morita class of A.

- As an example of a correlator for a non-orientable world sheet we give the partition function on the Klein bottle K. Recall that for non-orientable world sheets one needs a Jandl algebra (A, σ) rather than just a Frobenius algebra A as an input. One finds [FRS3, Sect. 3.4] that the correlator and the expansion coefficients K_j in the basis $\{\chi_j \mid j \in \mathcal{I}\}$ of conformal blocks of the torus $\mathrm{T} = \widehat{\mathrm{K}}$ are given by

$$\tag{2.22}$$

The coefficients K_j obtained this way satisfy [FRS3, thm. 3.7] $K_j = K_{\bar{j}}$, $K_j + Z_{jj} \in 2\mathbb{Z}$ and $|K_j| \leq \frac{1}{2} Z_{jj}$.

- There is only one Jandl structure on the Frobenius algebra $A = \mathbf{1}$, namely $\sigma = \mathrm{id}_{\mathbf{1}}$. In this particular case the formulas (2.19) for $Z = Z(A)$ and (2.22) for $K = K(A, \sigma)$ reduce to

$$Z_{i,j}(\mathbf{1}) = \delta_{i,\bar{j}} \qquad \text{and} \qquad K_j(\mathbf{1}, \mathrm{id}_{\mathbf{1}}) = \begin{cases} \mathrm{FS}(S_j) & \text{if } j = \bar{j} \\ 0 & \text{else} \end{cases} \tag{2.23}$$

(as first obtained in [FFFS]), with FS denoting the Frobenius-Schur indicator (see e.g. [FFFS, Na, NS]).

- Finally, as an example for a world sheet with non-empty boundary, consider the disk D with one bulk field insertion $\Phi = \Phi_{i,j}$ and with boundary condition M. One finds [FRS5, eq. (4.20)] that the correlator is non-zero only if $j = \bar{i}$, in which case the coefficient of $Cor(\mathrm{D}; \Phi; M)$ in an expansion in a standard (one-dimensional) basis of the space of two-point blocks on the

sphere it is given by

(2.24)

The study of these and other correlators has also motivated the investigation of properties of Frobenius algebras and their representation theory. Here is a list of selected results that arose in this manner:

- For A a special Frobenius algebra in a rigid monoidal category $\mathcal{C}$, every A-module is a submodule of an induced (free) module $(A \otimes U, m \otimes \mathrm{id}_U)$ for some object U of $\mathcal{C}$ [FrFRS1, Lemma 4.8(ii)].

- For A a simple symmetric special Frobenius algebra in a modular tensor category $\mathcal{C}$, every simple A-bimodule is a sub-bimodule of a braided-induced bimodule $U \otimes^+ A \otimes^- V$ (see (2.12)) for some simple objects U, V of $\mathcal{C}$ [FrFRS2, Prop. 4.7].

- For A a symmetric special Frobenius algebra in a modular tensor category the complexified Grothendieck ring $G_0(\mathcal{C}_{A|A}) \otimes_{\mathbb{Z}} \mathbb{C}$ of the category of A-bimodule is isomorphic as s $\mathbb{C}$-algebra to $\bigoplus_{i,j \in I} \mathrm{End}_{\mathbb{C}}\big(\mathrm{Hom}_{A|A}(A \otimes^+ U_i, A \otimes^- U_j)$ [FRS7].

- For A a symmetric special Frobenius algebra in a modular tensor category $\mathcal{C}$ there is an exact sequence
$$1 \to \mathrm{Inn}(A) \to \mathrm{Aut}(A) \to \mathrm{Pic}(\mathcal{C}_{A|A}) \qquad (2.25)$$
of groups [FRS6, Prop. 7]. One can also describe conditions under which there exists a Morita equivalent Frobenius algebra A' for which the corresponding homomorphism from $\mathrm{Aut}(A)$ to $\mathrm{Pic}(\mathcal{C}_{A|A})$ is surjective [BFRS1].

- For A a simple symmetric special Frobenius algebra in a modular tensor category $\mathcal{C}$, the $\mathbb{C}$-vector space $\bigoplus_{i \in \mathcal{I}} \mathrm{Hom}_{A|A}(S_i \otimes^+ A \otimes^- S_{\bar{i}}, A)$ can be endowed with a natural structure of a semisimple unital commutative associative algebra $\mathcal{A}$. The simple modules of this algebra are in bijection with the simple A-modules, i.e. with the elementary boundary conditions of the full CFT defined by $\mathcal{C}$ and A. If A is Morita equivalent to $\mathbf{1}$, then $\mathcal{A}$ is naturally isomorphic to the complexified Grothendieck ring of $\mathcal{C}$ [FSSt].

- Every simple algebra in a modular tensor category is Morita equivalent to one for which $\dim_{\mathbb{C}} \mathrm{Hom}(\mathbf{1}, A) = 1$ [Os]. This can be used to show [FRS4, Prop. 3.6] that the number of Morita classes of simple symmetric special Frobenius algebras in a modular tensor category is finite. As an immediate consequence, associated to any chiral RCFT there are only finitely many different full RCFTs, in particular only finitely many different physical modular invariant torus partition functions.

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3 Verlinde-like relations in non-rational CFT

3.1 Beyond rational CFT

The results for RCFT correlators reported in section 2 may be summarized by the statement that for *rational* CFT, the transition from chiral to full CFT is fully under control and can be treated in an entirely model-independent manner. In particular all correlators can be constructed combinatorially [5] as elements of the appropriate spaces of conformal blocks of the chiral theory. This does not mean that a complete understanding of rational CFT has been achieved, but the issues that still need to be resolved are confined to the realm of chiral CFT.

In contrast, attempts to understand more general CFTs, like Liouville theory or logarithmic CFTs, at a similar level are still under development. In the present and the next section we discuss two features that are expected to be relevant for making progress in this direction: relations between fusion rules and modular transformation properties of characters, and a generalization of the notion of modular tensor category to chiral symmetry structures with non-semisimple representation categories which is based on the presence of a Hopf algebra in $\mathcal{C}$ that is obtained as a coend.

The first of these observations has been made in a specific class of models which are collectively termed *logarithmic CFTs* (or sometimes rational logarithmic CFTs), and which have been intensively studied in the physics literature. The proper categorical setting remains, however, unclear even for these models. For instance, in spite of much recent progress (see e.g. [HLZ, Hua3, Mi2]), still some mysteries need to be unraveled for the tensor structure of logarithmic CFTs. From the physics perspective, one is tempted to argue that certain features valid in RCFT are still present in the non-rational case, and that they imply specific properties of the vertex algebra and its relevant representation category $\mathcal{C}$. Concretely, the existence of a sensible notion of operator product expansion is expected to imply that $\mathcal{C}$ is monoidal, for obtaining reasonable monodromy properties of conformal blocks $\mathcal{C}$ should be braided, the non-degeneracy of two-point blocks on the disk and on the sphere can be taken as an indication that $\mathcal{C}$ is rigid, while the scaling symmetry points to the possible presence of a twist (balancing). But as any such reasoning is basically heuristic, none of these properties can be taken for granted. For instance, it has been argued, in the context of vertex algebras [Mi2], that rigidity must be relaxed.

For the purposes of the present section we adopt the following setting: we assume that the category $\mathcal{C}$ of chiral data is a braided *finite tensor category*, that is [EO], a $\Bbbk$-linear abelian braided rigid monoidal category for which every object U is of finite length and has a projective cover $P(U)$, which has finitely many simple objects up to isomorphism, and for which the tensor unit $\mathbf{1}$ is simple. Here $\Bbbk$ is an algebraically closed field; in the CFT context, $\Bbbk = \mathbb{C}$. For finite tensor categories the tensor product functor $\otimes$ is exact in both arguments. We continue to denote the set of isomorphism classes of simple objects by $\mathcal{I}$.

While it is not known whether the framework of finite tensor categories will ultimately

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[5] To go beyond the combinatorial framework, one must promote the geometric category of topological world sheets to a category of world sheets with metric, and similarly for the relevant algebraic category of vector spaces. Confidence that this can be achieved comes from the result [BK] that the notions of a ($\mathcal{C}$-decorated) topological topological modular functor and of a ($\mathcal{C}$-decorated) complex-analytic modular functor are equivalent. An intermediate step might consist of the construction of cohomology classes on the moduli spaces of curves using combinatorial models for the moduli space like in [Co].

suffice to understand all aspects of logarithmic CFTs, in any case the relevant categories $\mathcal{C}$ will no longer be semisimple, so that in particular various arguments used in the TFT construction no longer apply. Still one may hope that three-dimensional topological field theory, or at least invariants of links in three-manifolds, will continue to provide tools for studying such CFTs. It is thus a significant result that such invariants exist [Ly1, KR, He, MN], and that even representations of mapping class groups can be constructed [Ly2, Ke2]. However, these invariants and mapping class group representations do not really fit together, and actually it is reasonable to expect that for a full treatment one must work with *extended* TFTs related to higher categorical structures (see e.g. [Lu]) and allow the TFT functor to take values in complexes of vector spaces. But such generalizations have yet to be established.

3.2   The fusion rules of a semisimple modular tensor category

By the *fusion rules*, or fusion algebra, of a chiral CFT, one means the complexified Grothendieck ring

$$\mathcal{F} := G_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}. \tag{3.1}$$

Recall that the Grothendieck group of an abelian category $\mathcal{C}$ is the abelian group $G_0(\mathcal{C})$ generated by isomorphism classes $[U]$ of objects of $\mathcal{C}$ modulo the relations $[W] = [U] + [V]$ for any short exact sequence $0 \to U \to W \to V \to 0$ in $\mathcal{C}$.

To set the observations about fusion rules for logarithmic CFTs into context, it is worth to describe first in some detail the analogous results for (semisimple) rational CFTs. The Grothendieck group of a modular tensor category $\mathcal{C}$ inherits various properties from $\mathcal{C}$. It is a commutative unital ring, with product

$$[U] * [V] := [U \otimes V] \tag{3.2}$$

and unit element $[\mathbf{1}]$, and evaluation at $[\mathbf{1}]$ defines an involution $[U] \mapsto [U^\vee]$. $G_0(\mathcal{C})$ has a distinguished basis $\{[S_i]\}_{i \in \mathcal{I}}$ in which the structure constants $N_{ij}{}^k$, obeying

$$[S_i] * [S_j] = \sum_{k \in \mathcal{I}} N_{ij}{}^k [S_k], \tag{3.3}$$

are non-negative integers, and as a consequence of semisimplicity these structure constants are given by $N_{ij}{}^k = \dim_{\mathbb{C}} \mathrm{Hom}(S_i \otimes S_j, S_k)$.

It follows that the fusion algebra (3.1) of a modular tensor category is a commutative, associative, unital $\mathbb{C}$-algebra for which evaluation at the unit is an involutive automorphism. Together, these properties imply that $\mathcal{F}$ is a *semisimple* algebra. As a consequence, $\mathcal{F}$ has another distinguished basis $\{e_l\}$, consisting of primitive orthogonal idempotents, such that $e_l * e_{l'} = \delta_{l,l'} e_l$. The transformation between the two distinguished bases $\{[S_i]\}$ and $\{e_l\}$ furnishes a unitary matrix $S^\otimes$ such that $[S_i] = \sum_l (S^{\otimes -1})_{0l} S^\otimes_{il} e_l$ with $(S^{\otimes -1})_{0l} \neq 0$. The structure constants $N_{ij}{}^k$ can then be written as

$$N_{ij}{}^k = \sum_l \frac{S^\otimes_{il} S^\otimes_{jl} S^{\otimes^*}_{lk}}{S^\otimes_{0l}}, \tag{3.4}$$

a relation known as the *diagonalization of the fusion rules*.

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One of the distinctive features of modular tensor categories is the fact that the so defined matrix $S^\otimes$ coincides with a multiple of the matrix $s^\circledcirc$ (A.3) whose invertibility enters the definition of a modular tensor category:[6]

$$S^\otimes = S^\circledcirc \equiv S_{00}^\otimes \, s^\circledcirc \,. \tag{3.5}$$

This equality has been established, from three related points of view, in 1989 in [Wi], [MS] and [Ca]. The proof can be formulated entirely in categorical terms, as a series of identities between elements of the morphism space $\mathrm{End}(\mathbf{1}) = \mathbb{C}\,\mathrm{id}_{\mathbf{1}}$ (which we identify with $\mathbb{C}$):

$$\frac{s_{i,k}^\circledcirc}{s_{0,k}^\circledcirc}\, s_{j,k}^\circledcirc \;=\; \frac{s_{i,k}^\circledcirc}{s_{0,k}^\circledcirc}$$

$$=\; \sum_p \sum_\alpha \qquad =\; \sum_p \sum_\alpha \qquad =\; \sum_p N_{ij}{}^p s_{p,k}^\circledcirc \tag{3.6}$$

Here the first two equalities follow directly from the definition of s (together with the fact that simple objects are absolutely simple), the third uses semisimplicity to express $\mathrm{id}_{S_j} \otimes \mathrm{id}_{S_i}$ through a choice of bases in the spaces $\mathrm{Hom}(S_i \otimes S_j, S_p)$ and their dual bases in $\mathrm{Hom}(S_p, S_i \otimes S_j)$, the forth holds by the defining properties of duality morphisms and braiding, and the last is a property of dual bases.

An independent, much stronger statement, can be established when one not just works with $\mathcal{C}$ as a category, but includes information from its concrete realization as the representation category of a rational conformal vertex algebra $\mathcal{V}$. Namely, one can then show that the matrix $S^\circledcirc$ also coincides with the matrix S^χ that implements the modular transformation $\tau \mapsto -\frac{1}{\tau}$ on the space spanned by the characters χ_{S_i} (2.9) of the simple $\mathcal{V}$-modules S_i. Thus the diagonalization of the fusion rules can be rephrased as the *Verlinde formula*

$$N_{ij}{}^k = \sum_l \frac{S_{il}^\chi S_{jl}^\chi S_{lk}^{\chi\,*}}{S_{0l}^\chi} \,. \tag{3.7}$$

The Verlinde formula was conjectured in 1988 [Ve]. It was established, around 1995, for the particular case of WZW models, a class of models for which the vertex algebra $\mathcal{V}$ can be constructed from an affine Lie algebra. For these models the proof can be obtained with methods from algebraic geometry (see e.g. [TUY, Fa, Be, So]). A proof for all rational conformal vertex algebras was achieved in 2004 [Hua1, Hua2].

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[6] To be precise, the equality holds for a specific ordering of the basis $\{e_l\}$ of idempotents. Note that for any semisimple $\mathbb{C}$-algebra the sets of primitive orthogonal idempotents and of irreducible representations are in bijection, albeit not in natural bijection. Implicit in (3.5) is the statement that for $\mathcal{F}$ one specific such bijection is distinguished; in this bijection the braiding enters as an additional structure.

3.3 Verlinde-like formulas for logarithmic minimal models

One reason why the findings above, which all apply to *rational* CFT, are of interest to us here is that Verlinde-like relations have also been found in non-rational CFT, namely for the specific class of so-called *logarithmic minimal models* $\mathcal{L}_{1,p}$, with $p \in \mathbb{Z}_{\geq 2}$. For these models, there is a corresponding vertex algebra $\mathcal{V}_{(1,p)} = \mathcal{V}(\mathcal{L}_{1,p})$ that is quite well understood [Kau, CF, AM, NT]. In particular, $\mathcal{V}_{(1,p)}$ obeys the so-called C_2-cofiniteness condition, and as a consequence the category $\mathcal{C}_{(1,p)} = \mathcal{R}ep(\mathcal{V}_{(1,p)})$ of grading-restricted generalized $\mathcal{V}_{(1,p)}$-modules is a braided finite tensor category (see [NT] and section 5.2 of [Hua4]). Specifically, the number of isomorphism classes of simple objects of $\mathcal{C}_{(1,p)}$ is $2p$; there are two isomorphism classes of simple objects which are projective, while the isomorphism classes of non-projective simple objects come in $p-1$ pairs, with non-trivial extensions existing within each pair, but not among members of different pairs [FGST2, NT]. We write the label set $\mathcal{I}$ as a corresponding disjoint union

$$\mathcal{I} = \bigsqcup_{a=1}^{p+1} \mathcal{I}_a \quad \text{with} \quad \mathcal{I}_a = \{i_a^+, i_a^-\} \quad \text{for} \quad a = 1, 2, \dots, p-1 \tag{3.8}$$

$$\text{and} \quad \mathcal{I}_p = \{i_p\}, \quad \mathcal{I}_{p+1} = \{i_{p+1}\}$$

of $p-1$ two-element subsets and two one-element subsets, with S_i projective for $i \in \mathcal{I}_p \cup \mathcal{I}_{p+1}$ and S_i non-projective for $i \in \mathcal{I} \setminus (\mathcal{I}_p \cup \mathcal{I}_{p+1})$.

Under the modular transformation $\tau \mapsto -1/\tau$ the characters $\chi(\tau)$ of $\mathcal{V}_{(1,p)}$-modules acquire explicit factors of the modular parameter τ or, in other words, logarithms of the parameter $q = e^{2\pi i \tau}$ in which the characters are power series. There is thus no longer an $\mathrm{SL}(2,\mathbb{Z})$-representation on the ($2p$-dimensional) span of $\mathcal{V}_{(1,p)}$-characters. However, there is a $3p-1$-dimensional $\mathrm{SL}(2,\mathbb{Z})$-representation $\widehat{\rho}_{(3p-1)}$ on the space spanned by the $2p$ simple characters $\chi_i(\tau)$ together with $p-1$ *pseudo-characters* $\psi_a(\tau)$. The quantities ψ_a are known explicitly as functions of τ [FGST1, Sect. 2.2]: they are of the form of a product of $i\tau$ times a power series in q. It is expected [FGST1] that this $3p-1$-dimensional space coincides with the space of zero-point conformal blocks on the torus. For any C_2-cofinite vertex algebra this space of conformal blocks can be constructed with the help of certain symmetric linear functions on the endomorphism rings of projective modules [Mi1, Ar2] and can be shown [Mi1, FG] to carry a finite-dimensional $\mathrm{SL}(2,\mathbb{Z})$-representation.

We denote by $\widehat{S}^\chi$ the representation matrix for the modular transformation $\tau \mapsto -\frac{1}{\tau}$ in the representation $\widehat{\rho}_{(3p-1)}$. Under such a transformation a pseudo-character is mapped to a linear combination of ordinary characters [FGST1, GT], i.e. $\widehat{S}^\chi$ has the block-diagonal form

$$\widehat{\rho}_{(3p-1)}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \equiv \widehat{S}^\chi = \begin{pmatrix} S^{\chi\chi} & S^{\chi\psi} \\ S^{\psi\chi} & 0 \end{pmatrix}. \tag{3.9}$$

Further, introducing label sets $A_a := \{+, -, \circ\}$ for $a = 1, 2, \dots, p-1$ and $A_p = A_{p+1} := \{\circ\}$ and labeling the characters and pseudo-characters in such a way that

$$\chi_{(a\alpha)} = \begin{cases} \chi_{i_a^\alpha} & \text{for } a = 1, 2, \dots, p-1, \ \alpha = +, -, \\ \psi_a & \text{for } a = 1, 2, \dots, p-1, \ \alpha = \circ, \\ \chi_{i_a} & \text{for } a = p, p+1, \ \alpha = \circ, \end{cases} \tag{3.10}$$

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as well as $\mathbf{1} = S_{(1+)} \equiv S_{i_1^+}$, one shows [GT, Prop. 1.1.2] by direct calculation that the complex numbers

$$N_{(a\alpha),(b\beta)}^{\quad\quad(c\gamma)} = \sum_{d=1}^{p+1} \sum_{\delta \in A_d} \frac{\widehat{S}_{(c\gamma),(d\delta)}^{\chi}}{\left(\widehat{S}_{(1\circ),(d\circ)}^{\chi}\right)^2} \left(\widehat{S}_{(1\circ),(d\circ)}^{\chi} \widehat{S}_{(a\alpha),(d\circ)}^{\chi} \widehat{S}_{(b\beta),(d\delta)}^{\chi} + \widehat{S}_{(1\circ),(d\circ)}^{\chi} \widehat{S}_{(a\alpha),(d\delta)}^{\chi} \widehat{S}_{(b\beta),(d\circ)}^{\chi} \right.$$
$$\left. - \widehat{S}_{(1\circ),(d\delta)}^{\chi} \widehat{S}_{(a\alpha),(d\circ)}^{\chi} \widehat{S}_{(b\beta),(d\circ)}^{\chi} \right) \tag{3.11}$$

are integers, and that those for which all three labels $(a\alpha), (b\beta), (c\gamma)$ correspond to ordinary characters rather than pseudo-characters are nonnegative and coincide with the Jordan-Hölder multiplicities for the corresponding tensor products of simple objects of $\mathcal{C}$. (For labels outside this range, the integers $N_{(a\alpha),(b\beta)}^{\quad\quad(c\gamma)}$ can be negative, though.) The close similarity of (3.11) with the Verlinde formula (3.7) suggests to call it a *Verlinde-like formula*.

The Jordan-Hölder multiplicities obtained this way constitute the structure constants $N_{ij}{}^k$ of the fusion algebra $\mathcal{F} = G_0(\mathcal{C}_{(1,p)}) \otimes_{\mathbb{Z}} \mathbb{C}$ of $\mathcal{C}_{(1,p)}$. This algebra is non-semisimple, so that the fusion rules can no longer be diagonalized. However, there is another type of Verlinde-like relation: one can show [GR,PRR2] that the matrices N_i with $(N_i)_j{}^k = N_{ij}{}^k$) can *simultaneously* be brought to Jordan form N_i^{J} according to

$$N_i = Q\, N_i^{\mathrm{J}}\, Q^{-1}\,, \tag{3.12}$$

with the entries of the similarity matrix Q as well as of all matrices N_i^{J} being expressible through the entries of $\widehat{S}^{\chi}$. While the concrete expressions are somewhat unwieldy and up to now a conceptual understanding of this similarity transformation is lacking, the very existence of the transformation is a quite non-trivial observation.

Besides the fusion algebra $\mathcal{F}$, of dimension $2p$, there is also an algebra of dimension $4p-2$ that arises from the tensor product of $\mathcal{C}_{(1,p)}$. Namely, the (conjectured) tensor product of any two simple objects of $\mathcal{C}_{(1,p)}$ decomposes as a direct sum of simples and of projective covers of simples. As a consequence the tensor product closes among the union of (direct sums of) all simple objects S_i and their projective covers $P_i = P(S_i)$, and one can thus extract a $4p-2$-dimensional *extended fusion algebra* $\widetilde{\mathcal{F}}$ spanned by the set $\{[S_i], [P_i] \,|\, i \in \mathcal{I}\}$. $\widetilde{\mathcal{F}}$ is a commutative unital associative algebra [PRR1]. Moreover, similarly as for $\mathcal{F}$, the matrices $\widetilde{N}_\ell$ made out of the structure constants of $\widetilde{\mathcal{F}}$ can simultaneously be brought to Jordan form by a similarity transformation,

$$\widetilde{N}_\ell = \widetilde{Q}\, \widetilde{N}_\ell^{\mathrm{J}}\, \widetilde{Q}^{-1}\,. \tag{3.13}$$

And again the entries of both $\widetilde{Q}$ and $\widetilde{N}_\ell^{\mathrm{J}}$ can be expressed through the entries of $\widehat{S}^{\chi}$ [Ra], although, again, a deeper understanding of the resulting formulas is lacking. Thus there is again a Verlinde-like relationship.

Finally, even though $\mathrm{SL}(2,\mathbb{Z})$-transformations do not close on the linear span of characters of $\mathcal{V}_{(1,p)}$-modules, it is nevertheless possible to extract an interesting $2p$-dimensional $\mathrm{SL}(2,\mathbb{Z})$-representation $\rho_{(2p)}$ from the modular transformations of characters. This is achieved by either by separating a so-called automorphy factor from the transformation formulas [FuHST], or by decomposing $\widehat{\rho}_{(3p-1)}$ as the 'pointwise' product (i.e. such that $\widehat{\rho}_{(3p-1)}(\gamma) = \rho'_{(3p-1)}(\gamma)\, \rho''_{(3p-1)}(\gamma)$ for any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$) of two commuting $(3p-1)$-dimensional $\mathrm{SL}(2,\mathbb{Z})$-representations $\rho'_{(3p-1)}$ and $\rho''_{(3p-1)}$, one of which restricts to the span of characters [FGST1]. Moreover, this description gives rise to yet another Verlinde-like formula: by a similarity transformation $\mathcal{Q}$ the fusion matrices

N_i (formed, as described above, from the subset $\{N_{ij}{}^k\}$ of the Jordan-Hölder multiplicities (3.11)) are simultaneously brought to *block-diagonal* form $N_i^{\text{b.d.}}$, i.e.

$$N_i = \mathcal{Q} \, N_i^{\text{b.d.}} \mathcal{Q}^{-1} \,, \tag{3.14}$$

and both $\mathcal{Q}$ and the $N_i^{\text{b.d.}}$ have simple expressions in terms of the entries of $\mathcal{S}$. But again the explicit expressions obtained this way are not too illuminating. On the other hand, the block structure of the matrices $N_i^{\text{b.d.}}$ is given by two 1×1-blocks and $p-1$ 2×2-blocks, which precisely matches the structure of nontrivial extensions among simple objects of $\mathcal{C}$.

3.4   Hopf algebras for the $\mathcal{L}_{1,p}$ models

By inspection, the Perron-Frobenius dimensions of the simple objects (i.e. the Perron-Frobenius eigenvalues of the fusion matrices N_i) of $\mathcal{C}_{(1,p)}$ with p prime coincide with the dimensions of the simple modules over the p-restricted enveloping algebra of $\mathfrak{sl}(2,\mathbb{F}_p)$ (see e.g. [Hum]), and the same holds for their respective projective covers. In view of the relationship between modular representations and quantum groups at roots of unity, this may be taken as an indication of the existence of a suitable quantum group with a representation category that is equivalent to $\mathcal{C}_{(1,p)}$ at least as an abelian category.

Such a Hopf algebra indeed exists for any $p \in \mathbb{Z}_{\geq2}$, namely the $2p^3$-dimensional restricted quantum group $\overline{\text{U}_q}(\mathfrak{sl}_2)$ at the value $q = \text{e}^{\pi\text{i}/p}$ of the deformation parameter: there is an equivalence

$$\mathcal{R}ep(\mathcal{V}_{(1,p)}) \simeq \overline{\text{U}_q}(\mathfrak{sl}_2)\text{-mod} \tag{3.15}$$

of abelian categories [FGST1, FGST2, NT].

The two $\text{SL}(2,\mathbb{Z})$-representation $\widehat{\rho}_{(3p-1)}$ and $\rho_{(2p)}$ described in the previous subsection also arise naturally in the study of $\overline{\text{U}_q}(\mathfrak{sl}_2)$. First, it is known [LM, Ke1] that the center of $\overline{\text{U}_q}(\mathfrak{sl}_2)$, which has dimension $3p-1$, carries a representation of $\text{SL}(2,\mathbb{Z})$, obtainable by composing the Frobenius and Drinfeld maps (see (3.33) and (3.30) below) between $\overline{\text{U}_q}(\mathfrak{sl}_2)$ and its dual, and this representation can be shown to be isomorphic with $\widehat{\rho}_{(3p-1)}$ [FGST1]. Second, $\rho_{(2p)}$ is obtained through the multiplicative Jordan decomposition of the ribbon element of $\overline{\text{U}_q}(\mathfrak{sl}_2)$ [FGST1].

In view of these results there is little doubt that the quantum group $\overline{\text{U}_q}(\mathfrak{sl}_2)$ is intimately related with the logarithmic minimal models $\mathcal{L}_{1,p}$. Let us therefore briefly summarize a few main features of $\overline{\text{U}_q}(\mathfrak{sl}_2)$. As an associative algebra, $\overline{\text{U}_q}(\mathfrak{sl}_2)$ is freely generated by three elements $\{E, F, H\}$ modulo the relations

$$E^p = 0 = F^p \,, \qquad K^{2p} = \mathbf{1} \,, \qquad K\,E\,K^{-1} = q^2\,E \,, \qquad K\,F\,K^{-1} = q^{-2}\,F$$

$$\text{and} \qquad E\,F - F\,E = \frac{K - K^{-1}}{q - q^{-1}} \,. \tag{3.16}$$

The coproduct Δ and counit ε act on the generators as

$$\Delta(E) = \mathbf{1} \otimes E + E \otimes K \,, \qquad \Delta(F) = K^{-1} \otimes F + F \,, \qquad \Delta(K) = K \otimes K \qquad \text{and}$$

$$\varepsilon(E) = 0 = \varepsilon(F) \,, \qquad \varepsilon(K) = 1 \,, \tag{3.17}$$

and this extends uniquely to all of $\overline{\text{U}_q}(\mathfrak{sl}_2)$ by requiring that it endows $\overline{\text{U}_q}(\mathfrak{sl}_2)$ with the structure of a bialgebra. This bialgebra structure becomes a Hopf algebra by defining the antipode S to

act as

$$S(E) = -EK^{-1}, \qquad S(F) = -KF \qquad \text{and} \qquad S(K) = K^{-1}. \tag{3.18}$$

The structure of $\overline{U}_q(\mathfrak{sl}_2)$-mod as abelian category, which appears in the equivalence (3.15), has been established with the help of the reduced form (basic algebra) of $\overline{U}_q(\mathfrak{sl}_2)$ as an algebra. This reduced form has the structure

$$\mathrm{Red}(\overline{U}_q(\mathfrak{sl}_2)) \,=\, \mathbb{C} \,\oplus\, \mathbb{C} \,\oplus\, \Big[\mathcal{P}(\bullet \mathrel{\substack{\longrightarrow \\[-0.6ex] \longleftarrow}} \bullet)/_\sim \Big]^{\oplus(p-1)}, \tag{3.19}$$

where the algebra in the square brackets, which is 8-dimensional, is the path algebra of the indicated quiver modulo certain relations [Su, Xi, Ar1]. This direct sum decomposition shows in particular that the $2p$ isomorphism classes of simple $\overline{U}_q(\mathfrak{sl}_2)$-modules can be partitioned into $p-1$ pairs $\mathcal{I}_a = \{i_a^+, i_a^-\}$ precisely as in (3.8) (together with two isolated classes) such that non-trivial extensions only exist between the two members of each pair $\mathcal{I}_a$.

There is also an analogue of the pseudo-characters ψ of $\mathcal{V}_{(1,p)}$: for each of the $p-1$ decomposable projective modules $\mathrm{P}_a := \bigoplus_{j \in \mathcal{I}_a} P_j$ one independent pseudo-character is obtained [FGST3, Se, GT] by inserting a specific linear endomorphism of P_a into the trace. Such linear maps can be non-zero because of the isomorphisms

$$\mathrm{Soc}(P_i) \,\cong\, S_i \,\cong\, \mathrm{Top}(P_i) \tag{3.20}$$

for all $i \in \mathcal{I}$. Note, however, that no non-zero map of this form is a morphism of $\overline{U}_q(\mathfrak{sl}_2)$-modules, and it is in fact not yet known how to obtain such linear maps in a systematic manner from Hopf algebra theory and the R-matrix.

On the other hand, for $p > 2$ the equivalence (3.15) does not extend to an equivalence between $\mathcal{R}ep(\mathcal{V}_{(1,p)})$ and $\overline{U}_q(\mathfrak{sl}_2)$-mod as *monoidal* categories. Indeed, whereas the monoidal category $\mathcal{R}ep(\mathcal{V}_{(1,p)})$ can be endowed with a braiding, the category $\overline{U}_q(\mathfrak{sl}_2)$-mod cannot, as $\overline{U}_q(\mathfrak{sl}_2)$ is not quasitriangular. $\overline{U}_q(\mathfrak{sl}_2)$ is, however, 'almost' quasitriangular, in the following sense: adjoining a square root of the generator K to $\overline{U}_q(\mathfrak{sl}_2)$ one obtains a Hopf algebra $\widetilde{U}$ which contains $\overline{U}_q(\mathfrak{sl}_2)$ as a Hopf subalgebra and which *is* quasitriangular, with an explicitly known R-matrix $R \in \widetilde{U} \otimes \widetilde{U}$; and the corresponding monodromy matrix $Q = R_{21} \cdot R$ is not only an element of $\widetilde{U} \otimes \widetilde{U}$, but even of $\overline{U}_q(\mathfrak{sl}_2) \otimes \overline{U}_q(\mathfrak{sl}_2)$. This is also reflected in the structure of tensor products of indecomposable $\overline{U}_q(\mathfrak{sl}_2)$-modules: most of them satisfy $V \otimes V' \cong V' \otimes V$, while those which do not obey this relation involve pairs U, U' of indecomposable $\overline{U}_q(\mathfrak{sl}_2)$-modules such that $U \otimes V \cong V \otimes U'$ and $U' \otimes V \cong V \otimes U$ for any module V and such that $U \oplus U'$, but not U or U' individually, lifts to an $\widetilde{U}$-module [Su, KoS].

3.5 Factorizable ribbon Hopf algebras

It is tempting to expect that there are further interesting CFT models whose category $\mathcal{C}$ of chiral data is equivalent, as an abelian category, to H-mod for some non-semisimple finite-dimensional Hopf algebra H. For instance, in [FGST3] candidates for such Hopf algebras are constructed for the case of the logarithmic minimal models $\mathcal{L}_{p,p'}$, with $p, p' \in \mathbb{Z}_{\geq 2}$ coprime and $p < p'$. Just like for the subclass of $\mathcal{L}_{1,p}$ models such an equivalence will usually not extend to an equivalence of monoidal categories. In particular, for any finite-dimensional Hopf (or, more generally, quasi-Hopf) algebra H, the category H-mod is a finite tensor category [EO], whereas

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the results of [GRW] indicate that already for $\mathcal{L}_{2,3}$ the category $\mathcal{C}$ is no longer a finite tensor category, as e.g. the tensor product of $\mathcal{C}$ is not exact.

Still, the structures observed in logarithmic CFT models and in suitable classes of Hopf algebras are sufficiently close to vindicate a deeper study of such Hopf algebras. In this regard it is gratifying that a Verlinde-like formula has been established [CW] for a particular subalgebra, namely the Higman ideal, of any finite-dimensional factorizable ribbon Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic zero.

We will describe this Verlinde-like formula in section 3.6. To appreciate its status, some information about the following aspects of finite-dimensional factorizable ribbon Hopf algebras $H = (H, m, \eta, \Delta, \varepsilon, \mathrm{S})$ will be needed:

- Chains of subalgebras in the center of H and in the space of central forms in $H^\star = \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$;

- the notions of quasitriangular, ribbon, and factorizable Hopf algebras;

- the Drinfeld and Frobenius maps between H and $H^\star$.

We first recall that the *character* χ_M of a H-module $M = (M, \rho)$ is the map that assigns to $x \in H$ the number $\tilde{d}_M \circ (\rho \otimes \mathrm{id}_{M^*}) \circ (x \otimes b_M)$. In graphical description, the character χ_M is given by

$$\chi_M \;=\; \tilde{d}_M \circ (\rho \otimes \mathrm{id}_{M^\vee}) \circ (\mathrm{id}_H \otimes b_M) \;=\; \qquad\qquad\qquad (3.21)$$

(The graphical description used here and below has the advantage that most results translate directly from $\mathcal{V}ect_{\Bbbk}$ to more general ribbon categories, and thus to the setting of section 4.)

The relevant chain of ideals in the center $Z(H)$ is

$$Z_0(H) \;\subseteq\; \mathrm{Hig}(H) \;\subseteq\; \mathrm{Rey}(H) \;\subseteq\; Z(H)\,, \qquad\qquad (3.22)$$

which features the *Reynolds ideal* $\mathrm{Rey}(H) = \mathrm{Soc}(H) \cap Z(H)$ of H, the *Higman ideal* (or projective center), i.e. the image of the trace map $\tau\colon H \to H$ (with respect to some pair of dual bases of H), and the span $Z_0(H)$ of those central primitive idempotents e for which He is a simple left H-module. For the space

$$C(H) := \{x \in H^\star \,|\, x \circ m = x \circ m \circ c_{H,H}\} \qquad\qquad (3.23)$$

of *central forms* (or class functions, or symmetric linear functions) there is a similar chain of subspaces

$$C_0(H) \;\subseteq\; I(H) \;\subseteq\; R(H) \;\subseteq\; C(H) \qquad\qquad (3.24)$$

with $R(H)$ the span of characters of all H-modules, $I(H)$ the span of characters of all projective H-modules, and $C_0(H)$ the span of characters of all simple projective H-modules. (3.24) is obtained from the chain (3.22) of ideals in H via a symmetrizing form, and H is semisimple iff any of the inclusions in these chains is an equality [CW, Prop. 2.1 & Cor. 2.3]. (In the example of $H = \overline{\mathrm{U}}_q(\mathfrak{sl}_2)$, the dimensions of the respective spaces are $2 < p+1 < 2p < 3p-1$.)

A *quasitriangular* Hopf algebra H is a Hopf algebra with an *R-matrix*, i.e. with an invertible element $R \in H \otimes H$ satisfying $\Delta^{\mathrm{op}} = \mathrm{ad}_R \circ \Delta$ and

$$(\Delta \otimes \mathrm{id}_H)(R) = R_{13}\,R_{23} \qquad \text{and} \qquad (\mathrm{id}_H \otimes \Delta)(R) = R_{13}\,R_{12}\,. \tag{3.25}$$

These properties in turn imply

$$R_{12}\,R_{13}\,R_{23} = R_{23}\,R_{13}\,R_{12} \qquad \text{and} \qquad (\varepsilon \otimes \mathrm{id}_H) \circ R = \eta = (\mathrm{id}_H \otimes \varepsilon) \circ R \tag{3.26}$$

as well as, by the invertibility of the antipode,

$$(\mathrm{S} \otimes \mathrm{id}_H) \circ R = R^{-1} = (\mathrm{id}_H \otimes \mathrm{S}^{-1}) \circ R \qquad \text{and} \qquad (\mathrm{S} \otimes \mathrm{S}) \circ R = R\,. \tag{3.27}$$

Here we made the canonical identification $H \equiv \mathrm{Hom}_{\Bbbk}(\Bbbk, H)$ analogously as $H^\star = \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$, so that in particular $R \in \mathrm{Hom}(\Bbbk, H \otimes H)$. We describe R pictorially as

$$R \;\; = \;\; \quad\quad \in H \otimes H\,. \tag{3.28}$$

From R one forms the *monodromy matrix*

$$Q := R_{21} \cdot R \equiv (m \otimes m) \circ (\mathrm{id}_H \otimes c_{H,H} \otimes \mathrm{id}_H) \circ [(c_{H,H} \circ R) \otimes R] \;\in H \otimes H\,. \tag{3.29}$$

A *ribbon* Hopf algebra H is a quasitriangular Hopf algebra with a *ribbon element*, i.e. a central element $v \in Z(H)$ such that $\mathrm{S} \circ v = v$ and $\varepsilon \circ v = 1$ as well as $\Delta \circ v = (v \otimes v) \cdot Q^{-1}$. The defining properties of the ribbon element imply that $v^2 = m \circ (u \otimes (\mathrm{S} \circ u))$ with $u \in H$ the *canonical element* $u := m \circ (\mathrm{S} \otimes \mathrm{id}_H) \circ R_{21}$. The product $b := m \circ (v^{-1} \otimes u)$ is called the *balancing element* of the ribbon Hopf algebra H; b is group-like and satisfies $\mathrm{S}^2 = \mathrm{ad}_b$. A ribbon Hopf algebra is called *factorizable* iff the *Drinfeld map*, i.e. the morphism Φ from $H^\star$ to H defined as

$$\Phi \; := \; (d_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_{H^\vee} \otimes Q) \; \equiv \quad\quad\quad\quad \tag{3.30}$$

is invertible (as a linear map) or, equivalently [FGST1], iff the monodromy matrix (3.29) can be written as $Q = \sum_\ell h_\ell \otimes k_\ell$ with $\{h_\ell\}$ and $\{k_\ell\}$ two bases of H.

A *left integral* of H is a morphism $\Lambda \in \mathrm{Hom}(\mathbf{1}, H)$ such that $m \circ (\mathrm{id}_A \otimes \Lambda) = \Lambda \circ \varepsilon$, and a *right integral* of $H^\star$ (or right cointegral of H) is a morphism $\lambda \in \mathrm{Hom}(H, \mathbf{1})$ such that $(\lambda \otimes \mathrm{id}_H) \circ \Delta = \eta \circ \lambda$. Pictorially,

$$\tag{3.31}$$

respectively. Given a left integral $\Lambda \in H$ and a right integral $\lambda \in H^\star$, the *Frobenius map* Ψ and its inverse $\Psi^- \equiv \Psi^{-1}$ are the morphisms

$$\Psi : \quad H \to H^\star : \quad h \mapsto \lambda \leftharpoonup S(h) \qquad \text{and} \qquad \Psi^- : \quad H^\star \to H : \quad p \mapsto \Lambda \leftharpoonup p \,, \qquad (3.32)$$

i.e.

$$(3.33)$$

(For a graphical proof that these two morphisms are indeed each others' inverses see appendix A.2.) The Frobenius map is a morphism of left H-modules (and of right $H^\star$-modules):

$$(3.34)$$

For any finite-dimensional factorizable ribbon Hopf algebra H over an algebraically closed field $\Bbbk$ of characteristic zero, the Frobenius map and Drinfeld map furnish an algebra isomorphism $Z(H) \cong C(H)$, as well as isomorphisms between the respective other members in the two chains (3.22) and (3.24) [CW].

3.6 A Verlinde-like formula for the Higman ideal

We now describe the result of [CW] announced in section 3.5. Recall first that for H a finite-dimensional ribbon Hopf algebra, one defines the *s-matrix* of H as the square matrix whose entries are obtained by composing characters and the Drinfeld map Φ as follows:

$$s_{i,j} \;=\; \chi_j \circ (\Phi(\chi_i)) \;=\;$$

$$(3.35)$$

The composition of the flip map with the action $(\rho_{S_i} \otimes \rho_{S_j})(R)$ of R provides a non-trivial braiding on H-mod. The s-matrix given by (3.35) is then precisely the matrix $s^{\circledcirc}$ defined in formula (A.3).

The result of [CW] can be regarded as a specific generalization of the formula (3.4) (the diagonalization of the fusion rules). It is a statement about the Higman ideal $\mathrm{Hig}(H)$ which reduces to (3.4) (together with the identification (3.5)) for H semisimple, i.e. for $\mathrm{Hig}(H) = H$. To cover the non-semisimple case we need two generalizations $\check{s} \equiv \check{s}^{\circledcirc}$ and $\hat{s} \equiv \hat{s}^{\circledcirc}$ of the s-matrix. They are defined by

$$\check{s}_{i,j} := \chi_j \circ (\widehat{\Phi}(\chi_i)) \qquad \text{and} \qquad \hat{s}_{i,j} := (\widehat{\Psi}(\mathrm{S} \circ e_j)) \circ (\widehat{\Phi}(\chi_i)) \tag{3.36}$$

for $i, j \in \mathcal{I}$, respectively, and involve, besides the primitive idempotents $e_j \in H$ (satisfying $P_j = H\, e_j$), the *modified Frobenius map* $\widehat{\Psi} \colon H \to H^\star$ and *modified Drinfeld map* $\widehat{\Phi} \colon H^\star \to H$ which are defined by

$$\widehat{\Psi} \colon h \mapsto (\lambda \leftharpoonup b) \leftharpoonup \mathrm{S}^{-1}(h) \qquad \text{and} \qquad \widehat{\Phi} \colon p \mapsto \Phi(p \leftharpoonup b^{-1}), \tag{3.37}$$

where b is the balancing element of H. ($\widehat{\Phi}$ is an algebra isomorphism $C(H) \xrightarrow{\cong} Z(H)$, and the modifications of Drinfeld and Frobenius maps cancel, $\widehat{\Phi} \circ \widehat{\Psi} = \Phi \circ \Psi$.) Graphically,

$$\tag{3.38}$$

We now consider the following two bases for the Higman ideal $\mathrm{Hig}(H)$: the images $\{\widehat{\Phi}(\chi_{P_j})\}$ under the modified Drinfeld map of the characters of the indecomposable projective H-modules $P_j = P(S_j)$, and the images $\{\tau(e_j)\}$ under the trace map τ of the primitive orthogonal idempotents e_j that correspond to the P_j. It can be shown [CW, Prop. 3.11] that the matrix Σ that transforms the latter basis to the former satisfies

$$\Sigma = (\mathrm{M}(C))^{-1}\, \mathrm{M}(C\,\hat{s}), \tag{3.39}$$

where C is the Cartan matrix (with entries $C_{i,j} = \chi_{P_i} \circ e_j$) of H and $\mathrm{M}(A)$ stands for the $\kappa \times \kappa$ major minor of a matrix A, with

$$\kappa := \dim_{\Bbbk}(\mathrm{Hig}(H)). \tag{3.40}$$

The tensor product of a projective H-module with any module is again projective. Hence for $i, j, k \in \{1, 2, \dots, \kappa\}$ one can define nonnegative integers $\widehat{N}_{ij}{}^k$ by

$$\chi_{S_i} \chi_{P_j} = \sum_{k=1}^{m} \widehat{N}_{kj}{}^i \chi_{P_k}. \tag{3.41}$$

Consider now the matrices $\widehat{N}_i$ that are given by $\widehat{N}_i := (\widehat{N}_{jk}{}^i)$. As shown in [CW, Thm. 3.14], the basis transformation matrix Σ simultaneously diagonalizes these matrices $\widehat{N}_i$, and the entries of the resulting diagonal matrices are, up to factors of dimensions $d_k = \dim(S_k)$, given by the entries of the matrix $\check{s}$:

$$\Sigma^{-1} \widehat{N}_i \Sigma = \left(\mathrm{diag}(d_j^{-1} \check{s}_{i,j})\right)_{j=1,\dots,\kappa}. \tag{3.42}$$

This is the Verlinde-like relation announced above. Regarding (3.41) as (the appropriate analogue of) fusion rules for the Higman ideal $\mathrm{Hig}(H)$, it asserts that the matrix Σ given by (3.39) diagonalizes the fusion rules for $\mathrm{Hig}(H)$.

4 Hopf algebras and coends

4.1 Hopf algebras in braided monoidal categories

One idea to comprehend CFT beyond the rational case is to generalize the notion of modular tensor category, as defined in appendix A.1, in such a way that semisimplicity of the category is not required. As it turns out, such a generalization can be achieved with the help of a Hopf algebra that, for a large class of rigid braided monoidal categories, can be defined as the coend of a suitable functor. We will thus need some background information about coends and about Hopf algebras in braided categories.

A *bialgebra* in a braided monoidal category $\mathcal{C}$ is an object H of $\mathcal{C}$ endowed with a product m, unit η, coproduct Δ and counit ε such that $(A, m.\eta)$ is an algebra and (A, Δ, ε) is a coalgebra (compare section 2.6), and such that the coproduct and counit are (unital) algebra morphisms, i.e.

$$\tag{4.1}$$

A *Hopf algebra* $(H, m, \eta, \Delta, \varepsilon, \mathrm{S})$ in $\mathcal{C}$ is a bialgebra together with an antipode, i.e. an endomorphism S of H satisfying

$$\tag{4.2}$$

Just like in the case of Frobenius algebras, the use of graphical calculus is not only convenient for visualizing definitions, but also for giving proofs. For instance, it can be checked graphically that the antipode is an algebra- and coalgebra-antihomomorphism. As another simple illustration, that the left coadjoint action, i.e.

$$\in \operatorname{Hom}(H \otimes H^\vee, H^\vee) \tag{4.3}$$

endows the dual $H^\vee$ with the structure of a left module over H is seen by the following sequence of equalities:

$$\tag{4.4}$$

It is interesting to note that the self-braiding $c_{H,H}$ of a Hopf algebra is already determined by the structural morphisms of H. Indeed [Sc],

$$\tag{4.5}$$

In particular the self-braiding is symmetric, $c_{H,H}^2 = \operatorname{id}_{H\otimes H}$; accordingly, in pictures like those above the distinction between over- and underbraiding is immaterial, even if the braiding on $\mathcal{C}$ is not symmetric.

Below we will also need the notion of a *Hopf pairing* of a Hopf algebra H; this is a morphism

$\omega \colon H \otimes H \to \mathbf{1}$ satisfying

(4.6)

as well as

(4.7)

As is easily checked, a non-degenerate Hopf pairing gives an isomorphism $H \to H^\vee$ of Hopf algebras.

4.2   Dinatural transformations and coends

We now summarize some pertinent information about coends. For $\mathcal{C}$ and $\mathcal{D}$ categories and F a functor $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, a *dinatural transformation* $F \Rightarrow B$ from F to an object $B \in \mathcal{D}$ is a family of morphisms $\varphi = \{\varphi_X \colon F(X,X) \to B\}_{X \in \mathcal{C}}$ such that the diagram

$$
\begin{array}{ccc}
F(Y,X) & \xrightarrow{\ F(\mathrm{id}_Y, f)\ } & F(Y,Y) \\
{\scriptstyle F(f,\mathrm{id}_X)}\downarrow & & \downarrow{\scriptstyle \varphi_Y} \\
F(X,X) & \xrightarrow{\ \varphi_X\ } & B
\end{array}
$$

(4.8)

commutes for all $X, Y \in \mathcal{C}$ and all $f \colon X \to Y$. Pictorially:

(4.9)

for all $X, Y \in \mathcal{C}$ and all $f \colon X \to Y$.

A *Coend* (A, ι) for a functor $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an initial object in the category of dinatural transformations from F to a constant. In other words, it is a dinatural transformation (A, ι) with the universal property that any dinatural transformation $(B, \varphi) \colon F \Rightarrow B$ uniquely

factorizes, i.e. there is a unique morphism $A \to B$ such that the two triangles in

$$
\begin{array}{ccc}
F(Y,X) & \xrightarrow{F(\mathrm{id}_Y,f)} & F(Y,Y) \\
\downarrow{\scriptstyle F(f,\mathrm{id}_X)} & & \downarrow{\scriptstyle \iota_Y} \quad \searrow{\scriptstyle \varphi_Y} \\
F(X,X) \xrightarrow{\iota_X} & A & \\
\searrow{\scriptstyle \varphi_X} & & \searrow \\
& & B
\end{array} \tag{4.10}
$$

commute for all $X, Y \in \mathcal{C}$ and all $f \colon X \to Y$.

If the coend exists, it is unique up to unique isomorphism. One denotes it by

$$
A = \int^X F(X,X), \tag{4.11}
$$

so that in particular $\iota_X \colon F(X,X) \to \int^X F(X,X)$. Assuming that arbitrary coproducts exist, an equivalent description of $\int^X F(X,X)$ (see e.g. section V.1 of [May]) is as the coequalizer of the morphisms

$$
\coprod_{f \colon Y \to Z} F(Y,Z) \underset{t}{\overset{s}{\rightrightarrows}} \coprod_{X \in \mathcal{C}} F(X,X) \tag{4.12}
$$

whose restrictions to the 'fth summand' are $s_f = F(f,\mathrm{id})$ and $t_f = F(\mathrm{id},f)$, respectively. An immediate consequence of the universal property of the coend is that to give a morphism with domain $\int^X F(X,X)$ and codomain Y is equivalent to give a family $\{f_X\}$ of morphisms from $F(X,X)$ to Y such that (Y,f) is a dinatural transformation.

Provided that the coends exist, the assignment of the coend of F to a functor F is functorial, i.e. extends to a functor

$$
\int^X \colon \quad \mathcal{FUN}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{D}) \to \mathcal{D}. \tag{4.13}
$$

This can be seen by considering, for a given a natural transformation $\alpha \colon F \to F'$ between functors $F, F' \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, the diagram

$$
\begin{array}{c}
F(Y,X) \\
\end{array} \tag{4.14}
$$

with morphisms $F(f^\vee,id_X)$, $\alpha_{Y,X}$, $F(id_Y,f)$ to $F(X,X)$, $F'(Y,X)$, $F(Y,Y)$; then $\alpha_{X,X}$, $F'(f^\vee,id_X)$, $F'(id_Y,f)$, $\alpha_{Y,Y}$ to $F'(X,X)$, $F'(Y,Y)$; then i'_X, i'_Y to $\int^X F'(X,X)$

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in which the upper left and upper right squares commute by the defining property of the natural transformation α, while the lower square commutes by the dinaturality of the coend of F'. Hence also the outer square commutes, and thus it defines a dinatural transformation to the object $\int^X F'(X,X)$. By the universal property of $\int^X F(X,X)$ there is then a unique morphism $\int^X \alpha \colon \int^X F(X,X) \to \int^X F'(X,X)$ in $\mathcal{D}$.

4.3 Coends in braided finite tensor categories

The following result has been established in [Ly1, Ke2]: For $\mathcal{C}$ a braided finite tensor category, the coend

$$\mathcal{H} = \int^X X^\vee \otimes X \tag{4.15}$$

of the functor $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \ni (X,Y) \longmapsto X^\vee \otimes Y \in \mathcal{C}$ exists and has a natural structure of a Hopf algebra in $\mathcal{C}$.

The proof of this result can e.g. be found in [Vi]. Here we only present the structure morphisms, which are described (using the universal property of the coend) in terms of families of morphisms from $X^\vee \otimes X$ to $\mathcal{H}$: The algebra structure is given by

$$m_{\mathcal{H}} \circ (\iota_X \otimes \iota_Y) := \iota_{Y \otimes X} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{Y \otimes X}) \circ (\mathrm{id}_{X^\vee} \otimes c_{X,Y^\vee \otimes Y}) \qquad \text{and} \qquad \eta_{\mathcal{H}} := \iota_{\mathbf{1}} \tag{4.16}$$

($\gamma_{X,Y}$ is the canonical identification of $X^\vee \otimes Y^\vee$ with $(Y \otimes X)^\vee$; in the definition of $\eta_{\mathcal{H}}$ it is used that $\mathbf{1}^\vee \otimes \mathbf{1} \cong \mathbf{1}$), the coalgebra structure by

$$\Delta_{\mathcal{H}} \circ \iota_X := (\iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^\vee} \otimes b_X \otimes \mathrm{id}_X) \qquad \text{and} \qquad \varepsilon_{\mathcal{H}} \circ \iota_X := d_X \tag{4.17}$$

(note that the braiding does not enter the coproduct and counit), and the antipode is

$$\mathrm{S}_{\mathcal{H}} \circ \iota_X := (d_X \otimes \iota_{X^\vee}) \circ (\mathrm{id}_{X^\vee} \otimes c_{X^{\vee\vee},X} \otimes \mathrm{id}_{X^\vee}) \circ (b_{X^\vee} \otimes c_{X^\vee,X}). \tag{4.18}$$

In pictures, the structure morphisms look as follows:

$$\tag{4.19}$$

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(In the picture for $m_{\mathcal{H}}$, $\mathrm{id}_{X|Y}$ is the identification of $\mathrm{id}_X \otimes \mathrm{id}_Y$ with $\mathrm{id}_{X \otimes Y}$.)

The situations discussed in sections 2 and 3.5 – modular tensor categories and finite-dimensional Hopf algebras – provide concrete examples for the hopf algebra $\mathcal{H}$. First, if $\mathcal{C}$ is semisimple with finite set $\mathcal{I}$ of isomorphism classes of simple objects S_i, then $\mathcal{H}$ decomposes as an object as [Vi, Sect. 3.2]

$$\mathcal{H} \cong \bigoplus_{i \in \mathcal{I}} S_i^{\vee} \otimes S_i \,, \tag{4.20}$$

and the structural morphisms ι_X are just combinations of the embedding morphisms of the simple subobjects of X and $X^{\vee}$ in $\mathcal{H}$.

Second, if $\mathcal{C}$ is equivalent to the representation category H-mod of a finite-dimensional ribbon Hopf algebra H, then $\mathcal{H}$ is given by the dual space $H^{\star} = \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$ endowed with the left coadjoint H-action (4.3), and with structural morphisms (see [Ke2, Lemma 3] [Vi, Sect. 4.5])

$$\iota_X: \quad X^{\vee} \otimes X \ni \tilde{x} \otimes x \longmapsto \big(h \mapsto \langle \tilde{x}, h.x \rangle \big), \tag{4.21}$$

i.e.

$$\tag{4.22}$$

4.4 Three-manifold invariants from the coend $\mathcal{H}$

For any finite braided tensor category $\mathcal{C}$, the coend $\mathcal{H}$ is not just a Hopf algebra in $\mathcal{C}$, but comes with further structure, in particular with integrals and with a Hopf pairing. This way $\mathcal{H}$ gives rise both to three-manifold invariants and to representations of mapping class groups [Ly1, Ly2, Ke2, KL, Vi].

As can be seen by a similar argument as for finite-dimensional Hopf algebras in $\mathcal{V}ect_{\Bbbk}$ [Ly2], $\mathcal{H}$ has left and right integrals. If the left and right integrals coincide (when properly normalized), then they can be used as a Kirby element and thus provide an invariant of three-manifolds [Vi]. This invariant has been first studied in [Ly1]. If $\mathcal{C} = H$-mod for a finite-dimensional ribbon Hopf algebra H, the invariant reduces to the one constructed in [KR, He].

If $\mathcal{C}$ is a (semisimple) modular tensor category, then the integral of $\mathcal{H}$ is two-sided and can be given explicitly in terms of the duality morphisms $b_i \in \mathrm{Hom}(S_i \otimes S_i^{\vee})$ [Ke2, Sect. 2.5]:

$$\Lambda_{\mathcal{H}} = \bigoplus_{i \in I} \dim(S_i) \, b_{S_i} \,, \tag{4.23}$$

The three-manifold invariant obtained from this integral is, up to normalization [CKS], the same as the one constructed in [RT].

4.5 A Hopf pairing for $\mathcal{H}$ and modular tensor categories

The second structure on $\mathcal{H}$ that is of interest to us is a symmetric Hopf pairing $\omega_{\mathcal{H}}$ of $\mathcal{H}$. It is given by [Ly1]

$$(4.24)$$

i.e. is induced by the dinatural family of morphisms $(d_X \otimes d_Y) \circ [\mathrm{id}_{X^\vee} \otimes (c_{Y^\vee,X} \circ c_{X,Y^\vee}) \otimes \mathrm{id}_Y]$.

Now notice the similarity of the morphism on the right hand side of (4.24) with the one in the expression (A.3) for the matrix $s^{\textcircled{\tiny o}}$. Since invertibility of $s^{\textcircled{\tiny o}}$ is the crucial ingredient of the definition of a (semisimple) modular tensor category, the following definition is a natural generalization to the non-semisimple case: A *modular finite tensor category* is a braided finite tensor category for which the Hopf pairing $\omega_{\mathcal{H}}$ given by (4.24) is non-degenerate.

It can be shown [Ly2, Thm. 6.11] that if $\mathcal{C}$ is modular in this sense, then the left and right integrals of $\mathcal{H}$ coincide. An example for a (non-semisimple) modular tensor category is provided by the category H-mod of left modules over any (non-semisimple) finite-dimensional factorizable ribbon Hopf algebra H [LM, Ly1].

The terminology suggests that there is a relation with the modular group $\mathrm{SL}(2,\mathbb{Z})$. In the case of semisimple modular tensor categories, $\mathrm{SL}(2,\mathbb{Z})$ arises as the mapping class group of the torus, which as noted in section 2.2 naturally acts on the space of zero-point conformal blocks on the torus. But $\mathrm{SL}(2,\mathbb{Z})$ appears in the non-semisimple case as well: there are endomorphisms $S_{\mathcal{H}}$ and $T_{\mathcal{H}}$ of $\mathcal{H}$ which satisfy the relations for the generators of (a twisted group algebra of) $\mathrm{SL}(2,\mathbb{Z})$ [Ly2]. These endomorphisms will be given below.

Denote by $Z(\mathcal{C})$ the *center* of $\mathcal{C}$, i.e. the algebra of natural endotransformations of the identity functor of $\mathcal{C}$. An element of $Z(\mathcal{C})$ is a family $(\phi_X)_{X \in \mathcal{C}}$ of endomorphisms $\phi_X \in \mathrm{End}(X)$, and one can check that the composition $(\iota_X \circ (\mathrm{id}_{X^\vee} \otimes \phi_X))_{X \in \mathcal{C}}$ gives a dinatural family. By the universal property of the coend there is thus a unique endomorphism $\overline{\phi}_{\mathcal{H}}$ of $\mathcal{H}$ such that the diagram

$$(4.25)$$

commutes for all $X \in \mathcal{C}$; this furnishes an injective linear map $Z(\mathcal{C}) \to \mathrm{End}(\mathcal{H})$. By concatenation with the counit $\varepsilon_{\mathcal{H}}$ this gives a map

$$Z(\mathcal{C}) \longrightarrow \mathrm{End}(\mathcal{H}) \xrightarrow{(\varepsilon_{\mathcal{H}})_*} \mathrm{Hom}(\mathcal{H},\mathbf{1}) . \tag{4.26}$$

Now the vector space $\mathrm{Hom}(\mathcal{H},\mathbf{1})$ has a natural structure of a $\Bbbk$-algebra (since $\mathcal{H}$ is in particular a coalgebra and $\mathbf{1}$ is an algebra); it can be shown [Ke2, Lemma 4] that (4.26) is an isomorphism of $\Bbbk$-algebras.

If the category $\mathcal{C}$ is a ribbon category (e.g. if $\mathcal{C}$ is sovereign, i.e. if the left and right dualities are equal as functors), then the family $(\theta_X)_{X\in\mathcal{C}}$ of twist isomorphisms constitutes an element $\nu \in Z(\mathcal{C})$, the ribbon element. Denote by

$$T_{\mathcal{H}} := \overline{\nu}_{\mathcal{H}} \ \in \ \mathrm{End}(\mathcal{H}) \tag{4.27}$$

the endomorphism of $\mathcal{H}$ obtained by applying the map defined in (4.25) to the ribbon element. Pictorially,

$$\tag{4.28}$$

Next consider the family of morphisms on the right hand side of

$$\tag{4.29}$$

This family is dinatural both in X and in Y, and hence (4.29) defines a morphism $\Sigma\colon \mathcal{H}\otimes\mathcal{H}\to\mathcal{H}$. By composition with a left or right integral Λ of $\mathcal{H}$ one arrives at an endomorphism

$$S_{\mathcal{H}} := \Sigma \circ (\mathrm{id}_{\mathcal{H}}\otimes\Lambda) \ \in \ \mathrm{End}(\mathcal{H}) \,. \tag{4.30}$$

It was established in [Ly2, Sect. 6] that for any (not necessarily semisimple) modular tensor category $\mathcal{C}$, the two-sided integral of $\mathcal{H}$ can be suitably normalized in such a way that the endomorphisms (4.27) and (4.30) furnish a morphism

$$\Bbbk_\xi\mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{End}(\mathcal{H}) \tag{4.31}$$

of $\Bbbk$-algebras, where for $\xi\in\Bbbk^\times$, $\Bbbk_\xi\mathrm{SL}(2,\mathbb{Z})$ denotes the twisted group algebra of $\mathrm{SL}(2,\mathbb{Z})$ with relations $S^4 = 1$ and $(ST)^3 = \xi\, S^2$.

Since for every object U of $\mathcal{C}$ the morphism space $\mathrm{Hom}(U,\mathcal{H})$ is, by push-forward, a left module over the algebra $\mathrm{End}(\mathcal{H})$, one obtains this way projective representations of $\mathrm{SL}(2,\mathbb{Z})$ on all vector spaces $\mathrm{Hom}(U,\mathcal{H})$.

If $\mathcal{C}$ is semisimple, the vector space $\mathrm{Hom}(\mathbf{1},\mathcal{H})$ coincides with the space of conformal blocks of the torus, $\mathrm{Hom}(\mathbf{1},\mathcal{H})\cong\mathtt{tft}_{\mathcal{C}}(\mathrm{T})$, and the $\mathrm{SL}(2,\mathbb{Z})$-representation obtained this way is precisely the representation on the characters (2.9) of a rational CFT as described in section 2.2. Also note that one can think of $\mathrm{Hom}(\mathbf{1},\mathcal{H})$, which is dual to the space $\mathrm{Hom}(\mathbf{1},\mathcal{H})$ on the right hand side of (4.26), as the appropriate substitute for the space of class functions. $\mathrm{Hom}(\mathbf{1},\mathcal{H})$ would therefore be a natural starting point for constructing a vector space assigned to the torus T by a topological field theory based on $\mathcal{C}$.

Let us finally comment on a relationship with the pseudo-characters that we encountered in section 3.3. Consider the map $\mathcal{X} \colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Hom}(\mathbf{1}, \mathcal{H})$ given by

$$U \longmapsto \mathcal{X}_U := \iota_U \circ \tilde{b}_U \; = \qquad\qquad\qquad (4.32)$$

The morphisms $\mathcal{X}_U$ can be regarded as generalizations of characters: The map $\mathcal{X}$ factorizes to a morphism of rings $K_0(\mathcal{C}) \to \mathrm{Hom}(1, \mathcal{H})$, i.e. it is are additive under exact sequences and one has

$$\mathcal{X}_X \cdot \mathcal{X}_Y := m_{\mathcal{H}} \circ (\mathcal{X}_Y \otimes \mathcal{X}_X) = \mathcal{X}_{X \otimes Y} \,. \qquad\qquad (4.33)$$

The latter is seen as follows:

$$m_{\mathcal{H}} \circ (\mathcal{X}_Y \otimes \mathcal{X}_X) = \qquad\qquad = \qquad\qquad = \qquad\qquad (4.34)$$

Furthermore, if the category $\mathcal{C}$ is semisimple, then $\mathrm{Hom}(\mathbf{1}, \mathcal{H}) \cong \bigoplus_{i \in \mathcal{I}} \mathrm{Hom}(\mathbf{1}, S_i^{\vee} \otimes S_i)$, so that $\{\mathcal{X}_{S_i}\}_{i \in \mathcal{I}}$ constitutes a basis of the vector space $\mathrm{Hom}(\mathbf{1}, \mathcal{H})$. If $\mathcal{C}$ is not semisimple, these elements are still linearly independent, but they do not form a basis any more. A (non-canonical) complement of this linearly independent set is in certain cases provided by an analogue of pseudo-characters.

A Appendix

A.1 Semisimple modular tensor categories

The category of chiral data of a rational conformal field theory is a semisimple modular tensor category. It arises as the representation category of a conformal vertex algebra that is rational in the sense that it obeys the C_2-cofiniteness condition and certain conditions on its homogeneous subspaces [Hua1]. But there are also various other algebraic structures which give rise to modular tensor categories, for instance (for some more details and references see e.g. section 3.3 of [FRS8]) nets of von Neumann algebras on the real line which have finite μ-index, are strongly additive and possess the split property [KLM], as well as connected ribbon factorizable weak Hopf algebras over $\mathbb{C}$ with a Haar integral [NTV].

A (semisimple) *modular tensor category* is, by definition, a semisimple abelian $\mathbb{C}$-linear monoidal category with simple tensor unit $\mathbf{1}$ and with the set $\mathcal{I}$ of isomorphism classes of simple objects being finite, which has a ribbon structure for which the braiding is maximally non-degenerate.

We will spell out the last two ingredients of this definition. We denote the tensor product bifunctor by $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and the monoidal unit by $\mathbf{1}$. A *ribbon* category [Kas, Ch. XIV.3] $\mathcal{C}=(\mathcal{C},\otimes,\mathbf{1},c,b,d,\theta)$ is a monoidal category endowed with four specific families of morphisms: a duality (evaluation and coevaluation), a braiding, and a twist. A (right) *duality* assigns to every $U\in\mathcal{C}$ another object $U^\vee$, called the (right-) dual of U, and morphisms $b_U\in\mathrm{Hom}(\mathbf{1},U\otimes U^\vee)$ and $d_U\in\mathrm{Hom}(U^\vee\otimes U,\mathbf{1})$, called coevaluation and evaluation, respectively. A *braiding* is a family of isomorphisms $c_{U,V}\in\mathrm{Hom}(U\otimes V,V\otimes U)$, one for each pair $U,V\in\mathcal{C}$, and a *twist* is a family of isomorphisms $\theta_U\in\mathrm{End}(U)$, one for each $U\in\mathcal{C}$.

Every monoidal category is equivalent to one that is *strict* monoidal, i.e. for which one has equalities $U\otimes(V\otimes W)=(U\otimes V)\otimes W$ and $U\otimes\mathbf{1}=U=\mathbf{1}\otimes U$ for $U,V,W\in\mathcal{C}$ rather than just natural isomorphisms of objects. We replace any non-strict monoidal category by an equivalent strict one; this allows for a simple graphical notation for morphisms, in which the tensor product of morphisms is just juxtaposition, see section 2.6. In this notation the braiding and twist and their inverses and the duality are depicted as follows:

$$c_{U,V} = \qquad c_{U,V}^{-1} = \qquad \theta_U = \qquad \theta_U^{-1} = \qquad\qquad (A.1)$$

$$b_U = \qquad d_U =$$

(Since $\mathbf{1}$ is a strict tensor unit, we have adopted the convention that it is invisible in such pictures.) With these conventions, the relations to be satisfied by the duality, braiding and

twist are

$$(A.2)$$

One may think of the lines in these pictures as ribbons, with the twist θ describing a 2π-rotation of a ribbon around its core. The equalities (A.2) are then precisely the relations satisfied by ribbons embedded in the three-sphere.

Finally, that the braiding is maximally non-symmetric means that the $|\mathcal{I}| \times |\mathcal{I}|$-matrix $s^{\circledcirc}$ with entries

$$s_{i,j}^{\circledcirc} := \quad \text{}$$

$$(A.3)$$

(in $\mathrm{End}(\mathbf{1}) = \mathbb{C}\,\mathrm{id}_{\mathbf{1}}$, which we identify with $\mathbb{C}$) is non-degenerate.

A.2 Invertibility of the Frobenius map

Here we give a graphical proof of the following statement:
For any Hopf algebra H in an additive ribbon category $\mathcal{C}$ that has an invertible antipode and a left integral $\Lambda \in \mathrm{Hom}(\mathbf{1}, H)$ and right cointegral $\lambda \in \mathrm{Hom}(H, \mathbf{1})$ such that $\lambda \circ \Lambda$ is invertible, the morphisms Ψ (the Frobenius map) and Ψ^- defined in (3.33) are mutually inverse.

Without loss of generality, we may normalize the integrals such that $\lambda \circ \Lambda = \mathrm{id}_{\mathbf{1}}$. Then we have the following chain of equalities:

$$(A.4)$$

The first equality combines $\lambda \circ \Lambda = \mathrm{id}_{\mathbf{1}}$ with simple properties of unit, counit and antipode, the second implements the defining properties of λ and Λ, the third is the compatibility of product and coproduct in a bialgebra, the fourth is the antialgebra property of the antipode together with a slight deformation of the graph, the fifth uses associativity and coassociativity, and the last follows by the defining property of the antipode. Using finally the defining properties of the unit and counit and composing from the left with S^{-1} and from the right with S, one arrives at

$$\Psi^- \circ \Psi = \mathrm{id}_H \,. \tag{A.5}$$

To show that Ψ^- is also a right-inverse, we proceed in two steps. First we show that

$$\tag{A.6}$$

Here the first equality uses the defining properties of the unit and counit and of λ and Λ, the fifth follows by a combination of the compatibility of product and coproduct and of properties of the antipode, while all other equalities are just properties of the antipode. In the second step we compose the result just obtained with the (co)integrals and then use again the defining property of Λ, yielding

$$\tag{A.7}$$

and thus

$$\lambda \circ S \circ \Lambda = \mathrm{id}_{\mathbf{1}} . \tag{A.8}$$

Re-inserting this identity into (A.6) finally proves that

$$\Psi \circ \Psi^- = \mathrm{id}_H . \tag{A.9}$$

Similarly as for finite-dimensional Hopf algebras over a field or, more generally, over a over commutative ring (see e.g. [Pa, KaS]), as a corollary of the result above one shows that any Hopf algebra $(H, m, \eta, \Delta, \varepsilon, S)$ of the form assumed above is naturally also a Frobenius algebra $(H, m, \eta, \Delta_{\mathrm{F}}, \varepsilon_{\mathrm{F}})$, with the same algebra structure and with Frobenius counit $\varepsilon_{\mathrm{F}} = \lambda$ and Frobenius coproduct

$$\tag{A.10}$$

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