SOME PROPERTIES OF LUBIN-TATE COHOMOLOGY FOR CLASSIFYING SPACES OF FINITE GROUPS

ANDREW BAKER AND BIRGIT RICHTER

Abstract. We consider brave new cochain extensions $F(BG_+, R) \longrightarrow F(EG_+, R)$, where R is either a Lubin-Tate spectrum E_n or the related 2-periodic Morava K-theory K_n, and G is a finite group. When R is an Eilenberg-Mac Lane spectrum, in some good cases such an extension is a G-Galois extension in the sense of John Rognes, but not always faithful. We prove that for E_n and K_n these extensions are always faithful, but not always dualizable. In particular, for G a non-trivial finite p-group, $F(EG_+, K_n) \sim K_n$ is not dualizable over $F(BG_+, K_n)$ and $F(EG_+, E_n) \sim E_n$ is not dualizable over $F(BG_+, E_n)$. Therefore $F(BG_+, K_n) \longrightarrow F(EG_+, K_n)$ and $F(BG_+, E_n) \longrightarrow F(EG_+, E_n)$ are not always Galois extensions.

Introduction

In the algebraic Galois theory of commutative rings [1, 8], faithful flatness is a property implied by separability. However, in the topological analogue, the 'brave new Galois theory' of Rognes [27], this is not true. The simplest counterexample, due to Ben Wieland [28], is provided by the C_2-Galois extension

$$F(BC_{2+}, H\mathbb{F}_2) \longrightarrow F(EC_{2+}, H\mathbb{F}_2) \sim H\mathbb{F}_2$$

which is not faithful. This example relies on the algebraic fact that

$$\pi_*(F(BC_{2+}, H\mathbb{F}_2)) = H^{-*}(BC_2; \mathbb{F}_2)$$

is a polynomial algebra and so has finite global dimension.

In this note we consider this question for a Lubin-Tate spectrum E_n and the related Morava K-theory K_n, and show that for any finite group G, the extension

$$E_n^{BG} = F(BG_+, E_n) \longrightarrow F(EG_+, E_n) \sim E_n \tag{0.1}$$

is faithful as an E_n-module. We also show that the non-commutative extension

$$F(BG_+, K_n) \longrightarrow F(EG_+, K_n) \sim K_n \tag{0.2}$$

is faithful and $F(BG_+, K_n)$ is a faithful E_n-module. A crucial difference from $F(BG_+, H\mathbb{F}_p)$ is that K_n^*BG is always an Artinian algebra over $(K_n)_*$, and so if $K_n^*BG \neq 0$ then it has infinite global dimension by Proposition 1.2. In the special case of a finite non-trivial p-group G we can deduce that K_n is not dualizable as an $F(BG_+, K_n)$-module spectrum (see Theorem 4.2).

Date: 05/08/2010 version 3 arxiv:1005.1662 .

2000 *Mathematics Subject Classification.* Primary 55P43; Secondary 13B05 55N22 55P60.

Key words and phrases. Lubin-Tate spectrum, Morava K-theory, classifying spaces of finite group.

The second author was supported by the Glasgow Mathematical Journal Learning and Research Support Fund; she thanks the Mathematics Department of the University of Glasgow for its hospitality. We would like to thank Ken Brown, Jim Davis, Uli Krähmer and Jesper Møller for helpful comments.

In particular, the extension of (0.2) is not a Galois extension in the associative setting. We demonstrate the failure of (0.1) to be a G-Galois extension in many cases.

Our approach to this involves introducing an analogue of the algebraic socle series for a module over an Artinian ring, and we show that this behaves well enough to prove our result. It is not clear how our work is related to that of Hovey and Lockridge [16], this is something we intend to explore further.

In Section 5 we review some other properties of $K^*(BG_+)$, specifically it is a Frobenius algebra and a zero-dimensional Gorenstein ring.

Notation, etc. In discussing purely algebraic notions we will often use boldface symbols $\boldsymbol{A}, \boldsymbol{M}, \ldots$ to denote rings, modules, etc, while for topological objects such as S-algebras and their modules we will use italic symbols $A, M, \ldots$, thereby reducing the possibility of confusion between the two settings. For an associative S-algebra A, we denote by $\mathscr{D}_A$ the derived category of A-module spectra.

We follow Lam [20] in using the phrase *local ring* to indicate a ring with a unique maximal left ideal (necessarily 2-sided and equal to its Jacobson radical); the quotient of such a ring by its Jacobson radical is a division ring. For non-commutative rings other terminology is often encountered such as *scalar local ring.*

Brave new Galois extensions. The following definition of a Galois extension is due to John Rognes [27]. Examples noted by Ben Wieland [28] after the publication of [27] show that not every Galois extension is faithful.

Let A be a commutative S-algebra and let B be a commutative cofibrant A-algebra. Let G be a finite (discrete) group and suppose that there is an action of G on B by commutative A-algebra morphisms. Then B/A is a *G-Galois extension* if it satisfies the following two conditions:

- The natural map
$$A \longrightarrow B^{hG} = F(EG_+, B)^G$$
is a weak equivalence of A-algebras.
- There is a natural equivalence of B-algebras
$$\Theta \colon B \wedge_A B \xrightarrow{\sim} F(G_+, B)$$
induced from the action of G on the right hand factor of B.

Furthermore, B/A is a *faithful G-Galois extension* if it also satisfies

- B is faithful as an A-module, *i.e.*, for an A-module M,
$$B \wedge_A M \sim * \quad \Longrightarrow \quad M \sim *.$$

1. Recollections on modules over Artinian algebras

Let $\boldsymbol{D}$ be a division ring. A ring $\boldsymbol{A}$ equipped with homomorphisms of rings $\eta \colon \boldsymbol{D} \longrightarrow \boldsymbol{A}$ and $\varepsilon \colon \boldsymbol{A} \longrightarrow \boldsymbol{D}$ is an *augmented $\boldsymbol{D}$-algebra* if the following diagram commutes.

$$\begin{tikzcd} \boldsymbol{D} \arrow[rr, "="] \arrow[dr, "\eta"'] & & \boldsymbol{D} \\ & \boldsymbol{A} \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

The augmentation ε splits the unit η. We also say that $\boldsymbol{A}$ is an *Artinian local* $\boldsymbol{D}$-algebra if it is Artinian and local.

If $\boldsymbol{A}$ is an Artinian local augmented $\boldsymbol{D}$-algebra, then the Jacobson radical of $\boldsymbol{A}$ is

$$\boldsymbol{J} = \operatorname{rad}(\boldsymbol{A}) = \ker \varepsilon.$$

By [19, theorem 4.12], $\boldsymbol{J}$ is nilpotent, say $\boldsymbol{J}^e = 0$ and $\boldsymbol{J}^{e-1} \neq 0$.

Lemma 1.1. *Let $\boldsymbol{M}$ be a left $\boldsymbol{A}$-module. If $\boldsymbol{D} \otimes_{\boldsymbol{A}} \boldsymbol{M} = 0$ then $\boldsymbol{M} = 0$.*

Proof. Comparing the two horizontal exact sequences

$$\begin{tikzcd}
 & \boldsymbol{J} \otimes_{\boldsymbol{A}} \boldsymbol{M} \arrow[r] \arrow[d] & \boldsymbol{A} \otimes_{\boldsymbol{A}} \boldsymbol{M} \arrow[r] \arrow[d, "\cong"] & \boldsymbol{D} \otimes_{\boldsymbol{A}} \boldsymbol{M} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \boldsymbol{J}\boldsymbol{M} \arrow[r] & \boldsymbol{M} \arrow[r] & \boldsymbol{M}/\boldsymbol{J}\boldsymbol{M} \arrow[r] & 0
\end{tikzcd}$$

we see that if $\boldsymbol{D} \otimes_{\boldsymbol{A}} \boldsymbol{M} = 0$ then

$$\boldsymbol{M} = \boldsymbol{J}\boldsymbol{M} = \ldots = \boldsymbol{J}^e\boldsymbol{M} = 0.$$

□

Let $\boldsymbol{M}$ be a left $\boldsymbol{A}$-module. The *socle* of $\boldsymbol{M}$ is the submodule

$$\operatorname{soc}^1 \boldsymbol{M} = \operatorname{soc} \boldsymbol{M} = \{x \in \boldsymbol{M} : \boldsymbol{J}x = 0\},$$

which can also be characterised as the sum of all the simple $\boldsymbol{A}$-submodules of $\boldsymbol{M}$. The *socle series* of $\boldsymbol{M}$ is the increasing sequence of submodules

$$0 = \operatorname{soc}^0 \boldsymbol{M} \subseteq \operatorname{soc}^1 \boldsymbol{M} \subseteq \ldots \subseteq \operatorname{soc}^k \boldsymbol{M} \subseteq \operatorname{soc}^{k+1} \boldsymbol{M} \subseteq \ldots \subseteq \boldsymbol{M},$$

where for each k the following is a pullback square

$$\begin{tikzcd}
\operatorname{soc}^{k+1} \boldsymbol{M} \arrow[r] \arrow[d] & \operatorname{soc}(\boldsymbol{M}/\operatorname{soc}^k \boldsymbol{M}) \arrow[d] \\
\boldsymbol{M} \arrow[r] & \boldsymbol{M}/\operatorname{soc}^k \boldsymbol{M}
\end{tikzcd}$$

so we have

$$\operatorname{soc}^k \boldsymbol{M} = \{x \in \boldsymbol{M} : \boldsymbol{J}^k x = 0\},$$

and

$$\operatorname{soc}^e \boldsymbol{M} = \boldsymbol{M}.$$

In fact, for small k

$$\operatorname{soc}^k \boldsymbol{M} \subset \operatorname{soc}^{k+1} \boldsymbol{M},$$

until we reach a value $k = k_0$ for which $\operatorname{soc}^{k_0} \boldsymbol{M} = \boldsymbol{M}$.

It is also clear that given a homomorphism $\varphi\colon \boldsymbol{M} \longrightarrow \boldsymbol{N}$ of $\boldsymbol{A}$-modules there are compatible homomorphisms

$$\operatorname{soc}^k \boldsymbol{M} \longrightarrow \operatorname{soc}^k \boldsymbol{N}.$$

For details on the socle series see [19], especially Ex. 4.18, and [5].

We end this section with a result that supplies an algebraic backdrop for some of our later work. We give a proof suggested by K. Brown.

Proposition 1.2. *Let $\boldsymbol{A}$ be a local left-Artinian ring which is not a division ring. Then*

$$\operatorname{proj}\dim(\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})) = \operatorname{gl}\dim \boldsymbol{A} = \infty,$$

where $\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})$ is the unique simple left $\boldsymbol{A}$-module.

Proof. Since $\boldsymbol{A}$ is local, it has only one simple module and therefore

$$\operatorname{proj}\dim(\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})) = \operatorname{gl}\dim \boldsymbol{A}.$$

Also, since $\boldsymbol{A}$ is Artinian it has a left ideal $\boldsymbol{I}$ isomorphic to $\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})$. The corresponding exact sequence

$$0 \to \boldsymbol{I} \longrightarrow \boldsymbol{A} \longrightarrow \boldsymbol{A}/\boldsymbol{I} \to 0 \tag{1.1}$$

cannot split since $\boldsymbol{A}$ is local and therefore it has no non-trivial idempotents.

If $\operatorname{proj}\dim(\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})) = \operatorname{gl}\dim \boldsymbol{A} < \infty$, then (1.1) would give

$$\operatorname{proj}\dim(\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})) + 1 = \operatorname{proj}\dim(\boldsymbol{A}/\boldsymbol{I}) \leqslant \operatorname{gl}\dim \boldsymbol{A} = \operatorname{proj}\dim(\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})),$$

which is impossible. □

Remark 1.3. We end this section by noting that the above discussion works as well if we assume that $\boldsymbol{A}$ is graded, provided this is suitably interpreted. In our work below we are interested in $\mathbb{Z}$-gradings which are also 2-periodic, *i.e.*, for all $n \in \mathbb{Z}$, $(-)_{n+2} = (-)_n$. This can be interpreted as a $\mathbb{Z}/2$-grading, which allows us to use the framework of [9].

2. Socle series in topology

Let D be an S-algebra for which π_*D is a 2-periodic graded division ring, *i.e.*, $\pi_0 D = \boldsymbol{D}$ is a non-trivial division ring and $\pi_1 D = 0$. Suppose that A is an S-algebra both under and over D, giving the following diagram of morphisms of S-algebras.

$$\begin{tikzcd} D \arrow[rr, "="] \arrow[dr, "\eta"'] & & D \\ & A \arrow[ur, "\varepsilon"'] & \end{tikzcd} \tag{2.1}$$

We assume that $\boldsymbol{A} = \pi_*A$ is an Artinian local augmented $\boldsymbol{D}$-algebra, so that the augmentation ideal $\ker\varepsilon$ is the Jacobson radical of $\boldsymbol{A}$, $\operatorname{rad}(\boldsymbol{A})$, and also $\operatorname{rad}(\boldsymbol{A})^e = 0$ and $\operatorname{rad}(\boldsymbol{A})^{e-1} \neq 0$.

Remark 2.1. Let M be a left A-module. Then $\boldsymbol{M} = \pi_*M$ is a left $\boldsymbol{A}$-module and its socle $\operatorname{soc}\boldsymbol{M}$ is a $\boldsymbol{D}$-module through both the unit η and the augmentation ε, and these module structures agree since $\operatorname{rad}(\boldsymbol{A}) = \ker\varepsilon$.

Theorem 2.2. *There are functors $\operatorname{soc}^k\colon \mathscr{D}_A \longrightarrow \mathscr{D}_A$ for $0 \leqslant k \leqslant e$ such that*
(a) *for each k, $\pi_*(\operatorname{soc}^k M) = \operatorname{soc}^k \boldsymbol{M}$;*
(b) *there are natural transformations $\operatorname{soc}^k M \longrightarrow \operatorname{soc}^{k+1} M$ giving a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \pi_*\operatorname{soc}^1 M \arrow[r] \arrow[d, "\cong"] & \pi_*\operatorname{soc}^2 M \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \pi_*\operatorname{soc}^e M \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \operatorname{soc}^1 \boldsymbol{M} \arrow[r] & \operatorname{soc}^2 \boldsymbol{M} \arrow[r] & \cdots \arrow[r] & \operatorname{soc}^e \boldsymbol{M} \arrow[r] & 0 \end{tikzcd}$$

which is natural with respect to morphisms of A-modules.

Proof. As $\boldsymbol{D}$ is a division ring, $\operatorname{soc}\boldsymbol{M}$ is a finite dimensional D_*-vector space. Since M is a D-module via the unit we can find a morphism of D-modules

$$\bigvee_j \Sigma^{s(j)} D \longrightarrow M \tag{2.2}$$

to realize an algebraic isomorphism

$$\bigoplus_j D_{*-s(j)} \xrightarrow{\cong} \operatorname{soc}\boldsymbol{M}.$$

Now Remark 2.1 implies that the morphism of (2.2) is actually one of A-modules. We set $\operatorname{soc}M = \bigvee_j \Sigma^{s(j)} D$.

Now we can repeat this on the cofibre $M/\operatorname{soc}M$ of the map $\operatorname{soc}M \longrightarrow M$, obtaining $\operatorname{soc}(M/\operatorname{soc}M) \longrightarrow M/\operatorname{soc}M$. We then define $\operatorname{soc}^2 M$ using the right hand pullback square in the diagram

$$\begin{tikzcd}
\operatorname{soc}M \arrow[r] \arrow[d, "="'] & \operatorname{soc}^2 M \arrow[r] \arrow[d] & \operatorname{soc}(M/\operatorname{soc}M) \arrow[d] \\
\operatorname{soc}M \arrow[r] & M \arrow[r] & M/\operatorname{soc}M
\end{tikzcd}$$

from which we see that $\pi_0(\operatorname{soc}^2 M) \cong \operatorname{soc}^2 \boldsymbol{M}$. Continuing in this way we inductively build the socle tower

$$* \to \operatorname{soc}^1 M \longrightarrow \operatorname{soc}^2 M \longrightarrow \ldots \longrightarrow \operatorname{soc}^{e-1} M \longrightarrow \operatorname{soc}^e M = M,$$

using pullback squares

$$\begin{tikzcd}
\operatorname{soc}^{k+1} M \arrow[r] \arrow[d] & \operatorname{soc}(M/\operatorname{soc}^k M) \arrow[d] \\
M \arrow[r] & M/\operatorname{soc}^k M
\end{tikzcd}$$

for each k. These satisfy

$$\pi_*(\operatorname{soc}^k M) = \operatorname{soc}^k \boldsymbol{M}.$$

□

An important consequence of this construction is that there is a minimal k_0 for which $\operatorname{soc}^{k_0} M = M$, so since $\operatorname{soc}^{k_0-1} \boldsymbol{M} \neq \boldsymbol{M}$, using the fibre sequence

$$\operatorname{soc}^{k_0-1} M \longrightarrow M \longrightarrow M/\operatorname{soc}^{k_0-1} M, \tag{2.3}$$

we obtain $\pi_*(M/\operatorname{soc}^{k_0-1} M) \neq 0$.

Lemma 2.3. *The A-module D satisfies $\pi_*(D \wedge_A D) \neq 0$.*

Proof. There is a diagram of left D-modules induced from (2.1)

$$\begin{tikzcd}
D \wedge_D D \arrow[rr, "="] \arrow[dr] & & D \wedge_D D \\
& D \wedge_A D \arrow[ur] &
\end{tikzcd}$$

in which $D \wedge_D D \cong D$. Applying $\pi_*(-)$ we see that $\pi_*(D \wedge_A D) \neq 0$. □

Theorem 2.4. *Let M be an A-module for which $\pi_* M \neq 0$. Then $\pi_*(D \wedge_A M) \neq 0$,* i.e., *$D$ is a faithful A-module.*

Proof. Using the socle series we can find a fibration sequence as in (2.3),

$$M' \longrightarrow M \longrightarrow M'', \tag{2.4}$$

where $\boldsymbol{M}'' = \pi_* M'' \neq 0$, $J\boldsymbol{M}'' = 0$ and there is a short exact sequence

$$0 \to \pi_*(M') \longrightarrow \pi_*(M) \longrightarrow \pi_*(M'') \to 0. \tag{2.5}$$

As remarked in the proof of Theorem 2.2, M'' is weakly equivalent to a wedge of copies of suspensions of the A-module D. So $\pi_*(M'')$ is a direct sum of copies of suspensions of $\pi_*(D)$, hence by Lemma 2.3, $\pi_*(M'') \neq 0$. The fibre sequence (2.4) induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_*(D \wedge_D M'') \arrow[r] \arrow[d] & \pi_*(D \wedge_D M) \arrow[r, two heads] \arrow[d] & \pi_*(D \wedge_D M'') \arrow[d] \arrow[dd, bend left=60, "="] \\
 & \pi_*(D \wedge_A M'') \arrow[r] & \pi_*(D \wedge_A M) \arrow[r, two heads] & \pi_*(D \wedge_A M'') \arrow[d] \\
 & & & \pi_*(D \wedge_D M'')
\end{tikzcd}$$

in which an non-zero element $x \in \pi_*(D \wedge_D M'')$ lifts to $\pi_*(D \wedge_D M)$ and so is in the image of composition passing through $\pi_*(D \wedge_A M)$. Therefore $\pi_*(D \wedge_A M) \neq 0$. $\square$

3. Lubin-Tate cohomology of classifying spaces

We will denote by E any Lubin-Tate spectrum such as E_n or E_n^{nr}, and then K will denote the corresponding version of Morava K-theory see [3] for details. The spectrum E is a commutative S-algebra, while K is an E-algebra in the sense of [11]. The homotopy groups $\pi_* E$ and $\pi_* K$ are 2-periodic and $\pi_0 E$ is Noetherian; $\pi_0 K$ is a field, although K is only homotopy commutative if p is and odd prime, while when $p = 2$ it is not even that. Nevertheless, we will view K as a kind of 'topological division ring'.

The following lemma will allows us in certain circumstances to relate modules over $E^{BG} = F(BG_+, E)$ to modules over $K^{BG} = F(BG_+, K)$.

Lemma 3.1. *For any E^{BG}-module M, there is isomorphism of K-modules*

$$K \wedge_{E^{BG}} M \cong (K \wedge_E E) \wedge_{K \wedge_E E^{BG}} (K \wedge_E M).$$

In particular, there is an isomorphism of K-modules

$$K \wedge_{E^{BG}} E \cong K \wedge_{K^{BG}} K.$$

Proof. This follows from an obvious generalization of [11, proposition III.3.10]. Since there are isomorphisms of E-algebras $K \cong K \wedge_E E$ and $K^{BG} \cong K \wedge_E E^{BG}$, for any E^{BG}-module M,

$$\begin{aligned}
K \wedge_{E^{BG}} M &\cong K \wedge_E (E \wedge_{E^{BG}} M) \\
&\cong (K \wedge_K K) \wedge_E (E \wedge_{E^{BG}} M) \\
&\cong (K \wedge_E E) \wedge_{K \wedge_E E^{BG}} (K \wedge_E M).
\end{aligned}$$

$\square$

Theorem 3.2. *Let G be a finite group.*
(a) *The K-cohomology $K^*(BG_+)$ is a finite dimensional K^*-vector space and the E-cohomology $E^*(BG_+)$ is a finitely generated E^*-module.*

(b) *If $K^*(BG_+)$ is concentrated in even degrees, then $E^*(BG_+)$ is a free E^*-module of finite rank and*

$$K^*(BG_+) = K^* \otimes_{E^*} E^*(BG_+) = E^*(BG_+)/\mathfrak{m}E^*(BG_+).$$

(c) *$K^*(BG_+)$ is an augmented Artinian local K^*-algebra whose maximal ideal is nilpotent. Hence $E^*(BG_+)$ is an augmented pro-Artinian local E^*-algebra,*

$$E^*(BG_+) = \lim_r E^*(BG_+)/\mathfrak{m}^r E^*(BG_+).$$

Proof. (a) See [14, 15] for example.
(b) See [17, proposition 2.5].
(c) We can reduce to the case where G is a p-group using the transfer $\Sigma^\infty BG_+ \longrightarrow \Sigma^\infty BG'_+$ associated with a p-Sylow subgroup $G' \leqslant G$. The case of a cyclic p-group C_{p^r} is well known and

$$K^*(BC_{p^r+}) = K^*[y]/(y^{p^r}).$$

The case of a general p-group G of order p^m follows by induction on m since there is always a normal subgroup $N \triangleleft G$ of index p and this permits an argument with the Serre spectral sequence associated with the fibration

$$BN \longrightarrow BG \longrightarrow BC_p$$

as used in [24] to calculate $K^*(BG_+)$ from knowledge of $K^*(BN_+)$ as input. □

It is known that $K^*(BG_+)$ need not be concentrated in even degrees [18].

We are interested in the E-algebras $E^{BG} = F(BG_+, E)$ and $K^{BG} = F(BG_+, K)$, each of which is K-local. Of course the diagonal $BG \longrightarrow BG \times BG$ induces the product on each of these, but only E^{BG} is strictly commutative, while K^{BG} is homotopy commutative when $p \neq 2$ and merely associative when $p = 2$. At the level of homotopy groups, $E^*(BG_+) = \pi_*(E^{BG})$ and $K^*(BG_+) = \pi_*(K^{BG})$ are both graded commutative.

Now we can apply our earlier results to give

Theorem 3.3. *For any finite group G, E and K are faithful K-local E^{BG}-modules.*

Proof. It suffices to show that K is faithful. By Lemma 3.1, for any E^{BG}-module there is an isomorphism

$$K \wedge_{E^{BG}} M \cong (K \wedge_E E) \wedge_{K \wedge_E E^{BG}} (K \wedge_E M).$$

The natural morphism of E-algebras

$$K \wedge_E F(BG_+, E) \longrightarrow F(BG_+, K \wedge_E E)$$

is a weak equivalence since K is a finite cell E-module, so by [11, theorem III.4.2] it is enough to know that

$$(K \wedge_E E) \wedge_{K^{BG}} (K \wedge_E M) \cong K \wedge_{K^{BG}} (K \wedge_E M) \nsim *.$$

If M is K-local and non-trivial, then $K \wedge_{K^{BG}} (K \wedge_E M) \nsim *$, because we know from Theorem 2.4 that K is faithful as a K^{BG}-module. □

4. Dualizability of E^{BG}-modules

In [27, proposition 5.6.3], it was shown that the extension

$$F(BG_+, H\mathbb{F}_p) \longrightarrow F(EG_+, H\mathbb{F}_p) \sim H\mathbb{F}_p$$

is a G-Galois extension for any prime p and any finite group G acting nilpotently on $\mathbb{F}_p[G]$ by conjugation. In this section we discuss the analogous situation for the extensions of E-algebras

$$F(BG_+, E) \longrightarrow F(EG_+, E) \sim E, \quad F(BG_+, K) \longrightarrow F(EG_+, K) \sim K.$$

First we note that by a standard argument making use of the transfer [4], after p-localization, $\Sigma^\infty BG_+$ is a retract of $\Sigma^\infty BG'_+$ where G' is any p-Sylow subgroup of G. When $p \nmid |G|$, we have

$$F(BG_+, E) \sim E, \quad F(BG_+, K) \sim K.$$

If $p \mid |G|$ we might still have that one or both of the restriction maps

$$E_*^{BG} \longrightarrow E_*^{BG'}, \quad K_*^{BG} \longrightarrow K_*^{BG'}$$

has a trivial image. We are interested in the case where these images are not trivial. We start by considering the case of a p-group.

Proposition 4.1. *Suppose that $G = C_{p^r}$ is cyclic of order p^r where $r \geqslant 1$. Then the $E^{BC_{p^r}}$-algebra $F(EC_{p^r}{}_+, E) \sim E$ is not dualizable as an $E^{BC_{p^r}}$-module.*

Proof. We recall ([15, lemma 5.1]) that

$$(E^{BC_{p^r}})_* = E^*[[y]]/([p^r]y),$$

where the p^r-series has the form

$$[p^r]y \equiv y^{p^{rn}} \mod \mathfrak{m}.$$

By the Weierstrass preparation theorem, there is a polynomial

$$\langle p^r \rangle\, y = p^r + \cdots + y^{p^{rn}-1} \equiv y^{p^{rn}-1} \mod \mathfrak{m}$$

for which

$$(E^{BC_{p^r}})_* = E^*[[y]]/(y\, \langle p^r \rangle\, y).$$

Now the $(E^{BC_{p^r}})_*$-module E_* admits the periodic minimal free resolution

(4.1)

$$0 \leftarrow E_* \longleftarrow (E^{BC_{p^r}})_* \xleftarrow{y} (E^{BC_{p^r}})_* \xleftarrow{\langle p^r \rangle y} (E^{BC_{p^r}})_* \xleftarrow{y} (E^{BC_{p^r}})_* \xleftarrow{\langle p^r \rangle y} (E^{BC_{p^r}})_* \longleftarrow \cdots,$$

so for $s \geqslant 0$ we have

$$\mathrm{Tor}_{s,*}^{(E^{BC_{p^r}})_*}(K_*, E_*) = K_*. \tag{4.2}$$

The Künneth spectral sequence

$$\mathrm{E}^2_{s,t}(C_{p^r}) = \mathrm{Tor}_{s,*}^{(E^{BC_{p^r}})_*}(K_*, E_*) \Longrightarrow \pi_{s+t}(K \wedge_{E^{BC_{p^r}}} E) \tag{4.3}$$

is multiplicative. It is standard (compare for instance [25, lemma 6.6]) that for odd primes p, $\mathrm{Tor}_{*,*}^{(E^{BC_{p^r}})_*}(K_*, E_*)$ is the tensor product of a divided power algebra on the generator in bidegree $(2,0)$ and an exterior algebra on the generator in bidegree $(1,0)$, while for $p = 2$ it is an exterior algebra on the generators in bidegrees $(s,0)$ with $s \geqslant 1$. From this we see that the spectral sequence collapses from the E^2-term, and so $\pi_*(K \wedge_{E^{BC_{p^r}}} E)$ is an infinite dimensional K_*-vector space.

If $F(EC_{p^r}{}_+, E) \sim E$ were dualizable as an E-module, then it would be a retract of a finite cell E-module, and so $\pi_*(K \wedge_{E^{BC_{p^r}}} E)$ would be a finite dimensional K_*-vector space. The above calculation shows that this is false. $\square$

Theorem 4.2. *Let G be a non-trivial p-group. Then*

$$\operatorname{proj\,dim}_{K^*BG} K_* = \infty.$$

In particular, for all $s \geqslant 0$,

$$\operatorname{Tor}_{2s,*}^{K^*BG}(K_*, K_*) \neq 0$$

and $\pi_*(K \wedge_{K^{BG}} K)$ is an infinite dimensional K_*-vector space. Therefore K is not dualizable as a K^{BG}-module.

Proof. The first statement is a consequence of Proposition 1.2, while the second requires the collapsing Künneth spectral sequence (4.3). To see this, we note that there is a non-trivial epimorphism $G \longrightarrow C_p$, and for some $k \geqslant 1$ a factorisation

$$\begin{tikzcd} C_{p^k} \arrow[r, tail] \arrow[rr, bend left, two heads] & G \arrow[r, two heads] & C_p \end{tikzcd}$$

inducing a homomorphism of K^*-algebras $K^*BC_p \longrightarrow K^*BC_{p^k}$. There are associated morphisms between the associated Künneth spectral sequences:

$$\mathrm{E}^r_{**}(C_p) \longrightarrow \mathrm{E}^r_{**}(G) \longrightarrow \mathrm{E}^r_{**}(C_{p^k}). \tag{4.4}$$

To understand this, we will produce an explicit map between resolutions for K^*BC_p and $K^*BC_{p^k}$.

To ease notation we set $q = p^n$. The canonical surjection $C_{p^k} \longrightarrow C_p$ induces the ring monomorphism

$$K_*[[y]]/(y^q) \longrightarrow K_*[[y]]/(y^{q^k}); \quad y \mapsto y^{q^{k-1}}.$$

Then $K^*BC_{p^k}$ is a free K^*BC_p-module of rank q^{k-1}, with basis $1, y, \ldots, y^{q^{r-1}-1}$. There are resolutions similar to (4.1), and a chain map between them:

(4.5)

$$\begin{tikzcd} 0 \leftarrow K_* & (K^{BC_p})_* a_0 \arrow[l] \arrow[d, "\rho_0"] & (K^{BC_p})_* a_1 \arrow[l, "y"'] \arrow[d, "\rho_1"] & (K^{BC_p})_* a_2 \arrow[l, "y^{q-1}"'] \arrow[d, "\rho_2"] & (K^{BC_p})_* a_3 \arrow[l, "y"'] \arrow[d, "\rho_3"] & \cdots \arrow[l, "y^{q-1}"'] \\ 0 \leftarrow K_* & (K^{BC_{p^k}})_* b_0 \arrow[l] & (K^{BC_{p^k}})_* b_1 \arrow[l, "y"'] & (K^{BC_{p^k}})_* b_2 \arrow[l, "y^{q^k-1}"'] & (K^{BC_{p^k}})_* b_3 \arrow[l, "y"'] & \cdots, \arrow[l, "y^{q^k-1}"'] \end{tikzcd}$$

where for each $s \geqslant 1$,

$$\rho_{2s}(f(y)a_{2s}) = f(y^{q^{k-1}})b_{2s}, \quad \rho_{2s-1}(g(y)a_{2s-1}) = g(y^{q^{k-1}})y^{q^{k-1}-1}b_{2s-1}.$$

On tensoring these with K_* we obtain the map of chain complexes

$$\begin{tikzcd} 0 & K_* a_0 \arrow[l] \arrow[d, "\rho'_0"] & K_* a_1 \arrow[l, "0"'] \arrow[d, "\rho'_1"] & K_* a_2 \arrow[l, "0"'] \arrow[d, "\rho'_2"] & K_* a_3 \arrow[l, "0"'] \arrow[d, "\rho'_3"] & \cdots \arrow[l, "y^{q-1}"'] \\ 0 & K_* b_0 \arrow[l] & K_* b_1 \arrow[l, "0"'] & K_* b_2 \arrow[l, "0"'] & K_* b_3 \arrow[l, "0"'] & \cdots, \arrow[l, "0"'] \end{tikzcd} \tag{4.6}$$

where

$$\rho'_{2s}(a_{2s}) = b_{2s}, \quad \rho'_{2s-1}(a_{2s-1}) = 0.$$

From this it follows that the chain map ρ induces isomorphisms

$$\rho_*\colon \operatorname{Tor}_{2s,*}^{K^*BC_p}(K_*,K_*) \longrightarrow \operatorname{Tor}_{2s,*}^{K^*BC_{p^k}}(K_*,K_*)$$

for $s \geqslant 0$, and trivial maps in odd degrees. So

$$\mathrm{E}^2_{2s,*}(C_p) \longrightarrow \mathrm{E}^2_{2s,*}(C_{p^k})$$

is an isomorphism for each $s \geqslant 0$, and therefore $\mathrm{E}^2_{2s,*}(G))$ contains a summand which survives to $\mathrm{E}^\infty_{2s,*}(G))$. Since every dualizable K^{BG}-module is a retract of a finite cell module, this shows that K cannot be a dualizable K^{BG}-module. $\square$

Theorem 4.3. *Suppose that G is a non-trivial p-group. Then E is not dualizable as an E^{BG}-module.*

Proof. By Lemma 3.1,

$$K \wedge_{E^{BG}} E \sim K \wedge_{K^{BG}} K,$$

so

$$\pi_*(K \wedge_{E^{BG}} E) \cong \pi_*(K \wedge_{K^{BG}} K)$$

is an infinite dimensional K_*-vector space. $\square$

Using this we can also see that $E^{BG} \longrightarrow F(EG_+, E) \sim E$ is not a Galois extension since if it were then the unramified condition would give

$$E \wedge_{E^{BG}} E \sim \prod_G E$$

and so

$$K \wedge_{E^{BG}} E \sim \prod_G K,$$

making $\pi_*(K \wedge_{E^{BG}} E)$ a finite dimensional K_*-vector space.

Of course, an obvious conjecture is that these results also hold for every finite group G with $K^*(BG_+)$ non-trivial. The topological content of Theorem 4.2 that $K \wedge_{E^{BG}} K$ is infinite dimensional seems hard to verify, even though Proposition 1.2 applies to $K^*(BG_+)$.

A class of groups for which this conjecture does hold is that of *p-nilpotent groups*, which are finite groups G for which a p-Sylow subgroup $P \leqslant G$ has a normal p-complement, *i.e.*, there is a normal subgroup $N \lhd G$ with $p \nmid |N|$ and $G = PN = P \ltimes N$. A convenient summary of the properties of such groups can be found in [22, section 7], see also [26]. By a result of Tate [30], G being p-nilpotent is equivalent to the restriction homomorphism being an isomorphism

$$\operatorname{res}^G_P\colon H^*(BG;\mathbb{F}_p) \xrightarrow{\cong} H^*(BP;\mathbb{F}_p),$$

and in fact it is sufficient that this holds in degree 1. Comparison of the Serre spectral sequences for $K^*_n(BG_+)$ and $K^*_n(BP_+)$ shows that

$$K^*_n(BG_+) \xrightarrow{\cong} K^*_n(BP_+).$$

5. Frobenius algebra and Gorenstein properties of $K^*(BG_+)$

In this section we will discuss some other properties enjoyed by K^{BG} for a finite group G. These are consequences of work by Greenlees, May and Sadofsky on generalized Tate cohomology spectra and Gorenstein conditions.

According to [12, corollary 1.2], there is a weak equivalence of K-modules given by the composite of the adjoint of the transfer with the inclusion into the homotopy fixed points

$$\delta\colon K \wedge BG_+ \longrightarrow F(BG_+, K) = K^{BG}. \tag{5.1}$$

That this is an equivalence follows from the triviality of the corresponding Tate spectrum $t_G K$. In the following, a ring spectrum is a spectrum with a ring structure in the homotopy category and a module spectrum over such a ring spectrum is a module in the homotopy category.

Lemma 5.1. *The map $\delta\colon K \wedge BG_+ \longrightarrow K^{BG}$ is a morphism of K^{BG}-module spectra over the ring spectrum K^{BG}.*

Proof. The module structure is defined using the composition

$$\begin{tikzcd} F(BG_+,K) \wedge K \wedge BG_+ \arrow[r, "\mathrm{id}\wedge\mathrm{id}\wedge\mathrm{diag}"] & F(BG_+,K) \wedge K \wedge BG_+ \wedge BG_+ \arrow[d] & \\ & K \wedge K \wedge BG_+ \arrow[r, "\mathrm{mult}\wedge\mathrm{id}"] & K \wedge BG_+ \end{tikzcd}$$

in which the vertical arrow is the evaluation map. The fact that δ is a module map follows from the fact that the transfer is also one. □

Note that the last statement amounts to working up to homotopy; it is unclear whether the module structure can be rigidified to one over K^{BG} as an S-algebra.

This result has the algebraic consequence that $(K^{BG})_*$ is a Frobenius algebra, hence it is self-injective, see [20, section 1.3B].

Corollary 5.2. *The map $\delta\colon K \wedge BG_+ \longrightarrow K^{BG}$ induces an isomorphism of $(K^{BG})_*$-modules*

$$\mathrm{Hom}_{(K^{BG})_*}((K^{BG})_*, K_*) \cong (K^{BG})_*, \tag{5.2}$$

so $(K^{BG})_*$ is a Frobenius K_*-algebra. Hence $(K^{BG})_*$ is self-injective, and satisfies the Gorenstein condition

$$\mathrm{Ext}^{s,*}_{(K^{BG})_*}(K_*, (K^{BG})_*) = \begin{cases} (K^{BG})_{-*} & \text{if } s = 0, \\ 0 & \text{if } s \neq 0. \end{cases}$$

Proof. Since $(K^{BG})_*$ and $K_*(BG_+)$ are finite dimensional K_*-modules, there are duality isomorphisms

$$(K^{BG})_* \cong \mathrm{Hom}^{-*}_{K_*}(K_*(BG_+), K_*), \quad K_*(BG_+) \cong \mathrm{Hom}^{-*}_{K_*}((K^{BG})_*, K_*).$$

The map induced by δ gives the isomorphism of (5.2). □

When G is abelian, $(K^{BG})_*$ is a bicommutative finite dimensional Hopf algebra over K_*, therefore by the Larson-Sweedler theorem [23, theorem 2.1.3] it is a Frobenius algebra. The last result show that this holds in full generality. More generally, for any $r \geqslant 1$, there is an isomorphism of $(E/\mathfrak{m}^r)^*(BG_+)$-modules

$$(E/\mathfrak{m}^r)_*(BG_+) \cong (E/\mathfrak{m}^r)^*(BG_+),$$

and an isomorphism of $E^*(BG_+)$-modules

$$E_*^\vee(BG_+) = \pi_*(L_{K(n)}(E \wedge BG_+)) \cong E^*(BG_+).$$

For more on these ideas, see [13, 29].

References

[1] M. Auslander & O. Goldman, The Brauer group of a commutative ring, Trans. Amer. Math. Soc. **97** (1960), 367–409.

[2] A. Baker & B. Richter, Realizability of algebraic Galois extensions by strictly commutative ring spectra, Trans. Amer. Math. Soc. **359** (2007), 827–57.

[3] ———, Galois extensions of Lubin-Tate spectra, Homology, Homotopy Appl. **10** (2008), 27–43.

[4] J. C. Becker & D. H. Gottlieb, The transfer map and fiber bundles, Topology **14** (1975), 1–12.

[5] D. J. Benson, Representations and cohomology, I: Basic representation theory of finite groups and associative algebras, 2nd edition, Cambridge University Press (1998).

[6] ———, Representations and cohomology, II: Cohomology of groups and modules, 2nd edition, Cambridge University Press (1998).

[7] H. Cartan & S. Eilenberg, Homological Algebra, Princeton University Press (1956).

[8] S. U. Chase, D. K. Harrison & A. Rosenberg, Galois theory and Galois cohomology of commutative rings, Mem. Amer. Math. Soc. **52** (1965), 15–33.

[9] L. N. Childs, G. Garfinkel, M. Orzech, The Brauer group of graded Azumaya algebras, Trans. Amer. Math. Soc. **175** (1973), 299–326.

[10] C. W. Curtis & I. Reiner, Representation Theory of Finite Groups and Associative Algebras, reprint of the 1962 original, AMS Chelsea Publishing (2006).

[11] A. Elmendorf, I. Kriz, M. Mandell & J. P. May, Rings, modules, and algebras in stable homotopy theory, Mathematical Surveys and Monographs **47** (1999).

[12] J. P. C. Greenlees & H. Sadofsky, The Tate spectrum of v_n-periodic complex oriented theories, Math. Z. **222** (1996), 391–405.

[13] J. P. C. Greenlees & N. Strickland, Varieties and local cohomology for chromatic group cohomology rings, Topology **38** (1999), 1093–1139.

[14] M. J. Hopkins, N. J. Kuhn & D. C. Ravenel, Morava K-theories of classifying spaces and generalized characters for finite groups, Lect. Notes in Math. **1509** (1992), 186–209.

[15] ———, Generalized group characters and complex oriented cohomology theories, J. Amer. Math. Soc. **13** (2000), 553–594.

[16] M. Hovey & K. Lockridge, Semisimple ring spectra, New York J. Math. **15** (2009), 219–243.

[17] M. Hovey & N. P. Strickland, Morava K-theories and localisation, Mem. Amer. Math. Soc. **139** no. 666 (1999).

[18] I. Kriz & K. P. Lee, Odd-degree elements in the Morava $K(n)$ cohomology of finite groups, Topology Appl. **103** (2000), 229–241.

[19] T. Y. Lam, A First Course in Noncommutative Rings, 2nd edition, Graduate Texts in Mathematics **131**, Springer (2001).

[20] ———, Lectures on Modules and Rings, Graduate Texts in Mathematics **189**, Springer (1999).

[21] L. G. Lewis, J. P. May, M. Steinberger & J. E. McClure, Equivariant Stable Homotopy Theory, Lect. Notes in Math., **1213** (1986).

[22] J. Møller, Frobenius categories for finite groups of Lie type, Carski Trakt, Poland (2010), `http://www.math.ku.dk/~moller/talks/frobenius.pdf`.

[23] S. Montgomery, Hopf algebras and their actions on rings, Regional Conference Series in Mathematics **82**, Amer. Math. Soc. (1993).

[24] D. C. Ravenel, Morava K-theories and finite groups, Contemp. Math. **12** (1982), 289–292.

[25] D. C. Ravenel, W. S. Wilson, Morava K-theories of Eilenberg-MacLane spaces and the Conner-Floyd conjecture, Amer. J. Math. **102** (1980), 691–748.

[26] D. J. S. Robinson, A course in the theory of groups, 2nd edition, Springer-Verlag (1996).

[27] J. Rognes, Galois extensions of structured ring spectra, Mem. Amer. Math. Soc. **192** no. 898 (2008), 1–97.
[28] ———, A Galois extension that is not faithful, http://folk.uio.no/rognes/papers/unfaithful.pdf
[29] N. P. Strickland, $K(N)$-local duality for finite groups and groupoids, Topology **39** (2000), 733–772.
[30] J. Tate, Nilpotent quotient groups, Topology **3** suppl. 1 (1964), 109–111.

Mathematics Department, University of Glasgow, Glasgow G12 8QW, Scotland.
E-mail address: a.baker@maths.gla.ac.uk
URL: http://www.maths.gla.ac.uk/~ajb

Fachbereich Mathematik der Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany.
E-mail address: birgit.richter@uni-hamburg.de
URL: http://www.math.uni-hamburg.de/home/richter/