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BRAUER GROUPS FOR COMMUTATIVE S-ALGEBRAS

ANDREW BAKER AND BIRGIT RICHTER

Abstract. We investigate a notion of Azumaya algebras in the context of structured ring spectra and give a definition of Brauer groups. We investigate their Galois theoretic properties and discuss examples of Azumaya algebras arising from Galois descent. We construct examples that are related to topological Hochschild cohomology of group ring spectra and we present a $K(n)$-local variant of the notion of Brauer groups.

Introduction

The investigation of Brauer groups of commutative S-algebras is one aspect of the attempt to understand arithmetic properties of structured ring spectra.

In classical algebraic settings, Brauer groups are defined in terms of Azumaya algebras over fields or more generally over commutative rings [3, 2, 32] and are closely involved in Galois theoretic considerations. In this paper we discuss some ideas on Brauer groups for commutative S-algebras and in Section 3 we investigate their behaviour with respect to Galois extensions of commutative S-algebras in the sense of John Rognes [30]. In earlier work, the first named author and Andrey Lazarev discussed notions of Azumaya algebras, but these appear to be technically problematic; see [4], especially sections 2 and 4. Niles Johnson [19] discusses Azumaya objects in the general context of closed autonomous symmetric monoidal bicategories, and his characterization of Azumaya objects in the case of ring spectra resembles ours (see [19, §2]).

We present our definition of topological Azumaya algebras in Section 1 and show that such algebras are always homotopically central (in the sense of Definition 1.2) and separable, and also that the Azumaya property is preserved under base change.

Section 2 we define Brauer groups of commutative S-algebras and in Section 3 we prove a version of Galois descent for topological Azumaya algebras. We use this to construct an example of an Azumaya algebra over real topological K-theory, KO, which can be thought of as a KO-version of the quaternions.

In the case of Eilenberg-Mac Lane spectra we show in Section 4 that an extension $HR \longrightarrow HA$ is topologically Azumaya if and only if the extension of commutative rings $R \longrightarrow A$ is an algebraic Azumaya extension. Furthermore, using recent work of Bertrand Toën [35], we can deduce that the Brauer group $\mathrm{Br}(H\boldsymbol{k})$ is trivial if $\boldsymbol{k}$ is an algebraically closed field.

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Classically, the center of an associative algebra A over a commutative ring R can be described as endomorphisms of A in the category of modules over the enveloping algebra $A^e = A \otimes_R A^o$. For structured ring spectra, the direct analogue of this definition does not yield a homotopy invariant notion. Instead one has to replace A by a cofibrant object in the category of module spectra over the enveloping algebra spectrum, so the center of an associative R-algebra spectrum A is given by the topological Hochschild cohomology spectrum $\mathrm{THH}_R(A, A)$. But this spectrum is not strictly commutative in general, but due to the affirmatively solved Deligne conjecture [27] it is an E_2-spectrum. There are however exceptions and in Section 6 we discuss some examples arising from group ring spectra and their homotopy fixed point spectra.

As usual, when generalizing arguments from ordinary algebra to brave new algebra, technical difficulties are encountered. In the context of Brauer groups one such problem is that we do not know any general argument why an Azumaya algebra spectrum that is trivial with respect to the Brauer relation is itself weakly equivalent to an endomorphism object. In Section 7 we offer a variant of the construction of Brauer groups in the $K(n)$-local context where it appears that the technical difficulties are minimized and we discuss some examples related to EO_2 in Section 8.

1. Azumaya algebras over commutative S-algebras

Throughout, let R be a commutative S-algebra. We work in the categories of R-modules, $\mathscr{M}_R$, and associative R-algebras, $\mathscr{A}_R$. Following [5, 30], we will say that an R-module W is *faithful* if for an R-module X, $W \wedge_R X \sim *$ implies that $X \sim *$.

We recall some ideas from [4]. If A is an R-algebra, we have the topological Hochschild cohomology spectrum

$$\mathrm{THH}_R(A) = \mathrm{THH}_R(A, A) = F_{A \wedge_R A^o}(\widetilde{A}, \widetilde{A}),$$

where $\widetilde{A}$ is a cofibrant replacement for A in the category of left $A \wedge_R A^o$-modules $\mathscr{M}_{A \wedge_R A^o}$. We write $\eta\colon R \longrightarrow \mathrm{THH}_R(A)$ for the canonical map into the R-algebra $\mathrm{THH}_R(A)$; we also write $\mu\colon A \wedge_R A^o \longrightarrow F_R(A, A)$ for the R-algebra map induced by the left and right actions of A and A^o on A.

Definition 1.1. Let A be an R-algebra. Then A is a *weak* (*topological*) *Azumaya algebra over* R if and only if the first two of following conditions hold, while A is a (*topological*) *Azumaya algebra over* R if and only if all three of them hold.

(1) A is a dualizable R-module.
(2) $\mu\colon A \wedge_R A^o \longrightarrow F_R(A, A)$ is a weak equivalence.
(3) A is faithful as an R-module.

Note that this definition of Azumaya algebras over R differs from that in [4] since we demand faithfulness of A over R and not just A-locality of R as an R-module.

If T is an ordinary commutative ring with unit and if B is an associative T-algebra, then the center of B can be identified with the endomorphisms of B as an $B \otimes_T B^o$-module. Therefore $\mathrm{THH}_R(A)$ can be viewed as a homotopy invariant version of the center of A.

Definition 1.2. An R-algebra A is said to be *homotopically central* if the canonical map $\eta\colon R \longrightarrow \mathrm{THH}_R(A)$ is a weak equivalence.

For the following we recall a special case of the Morita theory developed in [4, section 1]. For a topological Azumaya algebra A over R we consider the category of left modules over the endomorphism spectrum $F_R(A,A)$, $\mathscr{M}_{F_R(A,A)}$ and we take a cofibrant replacement $\overline{A}$ of A in this category. The functor

$$F\colon \mathscr{M}_R \longrightarrow \mathscr{M}_{F_R(A,A)}$$

that sends X to $X \wedge_R \overline{A}$ has an adjoint

$$G\colon \mathscr{M}_{F_R(A,A)} \longrightarrow \mathscr{M}_R$$

with $G(Y) = F_{F_R(A,A)}(\overline{A}, Y)$. Then [4, theorem 1.2] implies that this adjoint pair of functors passes to an adjoint pair of equivalences between the corresponding derived categories

$$\mathscr{D}_R \underset{G}{\overset{\widetilde{F}}{\rightleftarrows}} \mathscr{D}_{F_R(A,A)}$$

and as a direct consequence we obtain the following result.

Proposition 1.3 ([4, proposition 2.3])**.** *Every topological Azumaya algebra A over R is homotopically central.*

By proposition 2.3 and definition 2.1 of [4] we also see that any A topological Azumaya algebra over R is dualizable as an $A \wedge_R A^{\mathrm{o}}$-module and $A \wedge_R A^{\mathrm{o}}$ is A-local as a left module over itself.

In classical algebra, Azumaya algebras are in particular separable. Using Morita theory we can deduce the analogous statement for topological Azumaya algebras. Here an R-algebra is *separable* in the sense of [30, definition 9.1.1] if the multiplication $m\colon A \wedge_R A \longrightarrow A$ has a section in the derived category of left $A \wedge_R A^{\mathrm{o}}$-modules, $\mathscr{D}_{A\wedge_R A^{\mathrm{o}}}$.

Proposition 1.4. *Let A be a topological Azumaya R-algebra. Then A is separable.*

Proof. By the remark following [30, definition 9.1.1], it suffices to prove that the induced map

$$m_*\colon \mathrm{THH}_R(A, A \wedge_R A) \longrightarrow \mathrm{THH}_R(A, A)$$

is surjective on $\pi_0(-)$. Denote by $\widetilde{A}$ a cofibrant replacement of A in the category of $A \wedge_R A^{\mathrm{o}}$-modules. Morita equivalence yields the two weak equivalences

$$\widetilde{G} \circ \widetilde{F}(R) \simeq \mathrm{THH}_R(A, A),$$
$$\widetilde{G} \circ \widetilde{F}(A) \simeq \mathrm{THH}_R(A, A \wedge_R A).$$

The functoriality of $\widetilde{G} \circ \widetilde{F}$ ensures that the unit $\eta\colon R \longrightarrow A$ induces a map $\widetilde{G} \circ \widetilde{F}(\eta)$ with

$$R \xrightarrow{\simeq} \widetilde{G} \circ \widetilde{F}(R) \xrightarrow{\widetilde{G}\circ\widetilde{F}(\eta)} \widetilde{G} \circ \widetilde{F}(A) \xleftarrow{\simeq} A.$$

This is given by sending the coefficient module of THH, $\widetilde{A} \simeq R\wedge_R\widetilde{A} \simeq R\wedge_R A$, to $A\wedge_R A \simeq A\wedge_R\widetilde{A}$ using η. Therefore

$$\pi_0(m_*) \circ \pi_0(\widetilde{G} \circ \widetilde{F}(\eta)) = \mathrm{id},$$

and so $\pi_0(m_*)$ is surjective. □

We now describe the behaviour of Azumaya algebras under base change.

Proposition 1.5. *Let A, B, C be R-algebras.*

(1) *If A is an Azumaya algebra over R and if C is a commutative R-algebra, then $A \wedge_R C$ is an Azumaya algebra over C.*
(2) *Conversely, let C be a commutative R-algebra such that C is dualizable and faithful as an R-module. If $A \wedge_R C$ is an Azumaya algebra over C, then A is an Azumaya algebra over R.*
(3) *If A and B are Azumaya algebras over R, then $A \wedge_R B$ is also Azumaya over R.*

Proof. If A is an Azumaya algebra over R, then it is formal to verify that $A \wedge_R C$ is dualizable and faithful over C (compare [30, 4.3.3,6.2.3]). It remains to show that

$$\mu_{A\wedge_R C} \colon (A \wedge_R C) \wedge_C (A \wedge_R C)^{\mathrm{o}} \longrightarrow F_C(A \wedge_R C, A \wedge_R C)$$

is a weak equivalence. Note that since the multiplication in $A \wedge_R C$ is defined componentwise,

$$(A \wedge_R C)^{\mathrm{o}} = A^{\mathrm{o}} \wedge_R C^{\mathrm{o}}.$$

The diagram

$$(1.1)\qquad \begin{tikzcd} (A \wedge_R C) \wedge_C (A \wedge_R C)^{\mathrm{o}} \arrow[rr, "\mu_{A\wedge_R C}"] \arrow[d, "\simeq"'] & & F_C(A \wedge_R C, A \wedge_R C) \arrow[d, "\simeq"] \\ A \wedge_R A^{\mathrm{o}} \wedge_R C \arrow[dr, "\mu_A \wedge_R C"] & & F_R(A, A \wedge_R C) \\ & F_R(A, A) \wedge_R C \arrow[ur, "\nu"] & \end{tikzcd}$$

commutes. Here $\nu \colon F_R(A, A) \wedge_R C \longrightarrow F_R(A, A \wedge_R C)$ denotes the duality map. As A is Azumaya over R we know that ν and μ_A are equivalences, and thus we obtain that the top map is an equivalence as well.

For the converse we assume that $A \wedge_R C$ is Azumaya over C and C is faithful and dualizable as an R-module. If M is an R-module, then $A \wedge_R M \simeq *$ implies that

$$(A \wedge_R C) \wedge_R M \simeq (A \wedge_R C) \wedge_C (C \wedge_R M) \simeq *.$$

Also, the faithfulness of $A \wedge_R C$ over C ensures that $C \wedge_R M \simeq *$. But as we assumed that C is faithful over R, we can conclude that M was trivial.

The fact that A is dualizable over R follows from [30, lemma 6.2.4]. Making use of diagram (1.1) we see that μ_A is also a weak equivalence.

The proof of the third claim is straightforward. □

Later we will consider Azumaya algebras in a Bousfield local setting. Let L be a cofibrant R-module.

Definition 1.6. An L-local R-algebra A is an (L*-local*) *Azumaya algebra* if

(1) A is a dualizable L-local R-module,
(2) the natural morphism of R-algebras $A \wedge_R A^{\mathrm{o}} \longrightarrow F_R(A, A)$ is an L-local equivalence.
(3) A is faithful as an L-local R-module.

Now suppose that M is a dualizable R-module as discussed in [30, 5]; a more detailed discussion of dualizability can be found in [13]. Let $\mathcal{E}_R(M) = F_R(M, M)$ be its endomorphism R-algebra. Then there is a weak equivalence

$$\mathcal{E}_R(M) \simeq F_R(M, R) \wedge_R M. \tag{2.1}$$

In order to identify endomorphism spectra of faithful and dualizable R-modules as trivial Azumaya algebras we need the following auxiliary result.

Lemma 2.1. *Let M be a dualizable R-module.*

(1) *If M is a faithful R-module, then the dual $F_R(M, R)$ is also faithful.*
(2) *If M is L-local with respect to a cofibrant R-module L, then $F_R(M, R)$ is L-local.*

Proof. (1) Dualizability of M implies that the composition

$$M \simeq R \wedge_R M \xrightarrow{\delta\wedge\mathrm{id}} M \wedge_R F_R(M, R) \wedge_R M \xrightarrow{\mathrm{id}\wedge\varepsilon} M \wedge_R R \simeq M$$

is the identity on M. Here $\delta\colon R \longrightarrow M\wedge_R F_R(M, R)$ is the counit, and $\varepsilon\colon F_R(M, R)\wedge_R M \longrightarrow R$ is the evaluation map. Now if N is an R-module for which $F_R(M, R)\wedge_R N \simeq *$, then the identity of $M \wedge_R N$ factors through the trivial map, hence $N \simeq *$ by faithfulness of M.
(2) A similar argument with the functor $F_R(W, -)$ shows that if $L \wedge_R W \sim *$, then the identity map on $F_R(W, F_R(M, R))$ factors through

$$F_R(W, F_R(M, R) \wedge_R M \wedge_R F_R(M, R)) \sim F_R(W \wedge_R M \wedge_R M, M) \sim *. \qquad \square$$

It was shown in [4, proposition 2.11] that if M is a dualizable, cofibrant R-module, then $\mathcal{E}_R(M)$ is a weak topological Azumaya algebra in the sense of [4, definition 2.1].

Proposition 2.2. *If M is a faithful, dualizable, cofibrant R-module, then $\mathcal{E}_R(M)$ is an Azumaya R-algebra.*

Proof. As $\mathcal{E}_R(M)$ is a weak Azumaya algebra, it suffices to show that $\mathcal{E}_R(M)$ is a faithful R-module. Dualizability of M ensures that

$$\mathcal{E}_R(M) \simeq F_R(M, R) \wedge_R M,$$

and this is a smash product of two faithful R-modules which is also faithful. $\square$

This result shows that we can take the R-algebras of the form $\mathcal{E}_R(M)$ with M faithful, dualizable and cofibrant, to be trivial Azumaya algebras when defining a topological version of a Brauer group which we now do.

First we note that every Azumaya algebra is weakly equivalent to a retract of a cell R-module, so the following construction yield a *set* of equivalence classes. Define $\mathrm{Az}(R)$ to be the collection of all Azumaya algebras. Now we introduce our version of the Brauer equivalence relation $\approx$ on $\mathrm{Az}(R)$.

Definition 2.3. If $A_1, A_2 \in \mathrm{Az}(R)$, then $A_1 \approx A_2$ if and only if there are faithful, dualizable, cofibrant R-modules M_1, M_2 for which

$$A_1 \wedge_R F_R(M_1, M_1) \sim A_2 \wedge_R F_R(M_2, M_2)$$

as R-algebras. We denote the sets of equivalence classes of these by $\mathrm{Br}(R)$.

Theorem 2.4. *The set* $\mathrm{Br}(R)$ *is an abelian group with multiplication induced by the smash product* $\wedge_R$*. Furthermore,* Br *is a functor from the category of commutative* S*-algebras to abelian groups.*

Proof. The details involve routine modifications of the approach used in the case of Brauer groups of commutative rings in [2, theorem 5.2]. Note in particular that we need faithfulness in order to ensure the existence of inverses.

Functoriality for morphisms of commutative S-algebras $R \longrightarrow R'$ is achieved by sending an R-algebra A to the R'-algebra $R' \wedge_R A$. □

For a cofibrant R-module L, we can similarly define the sets of L-local Azumaya algebras $\mathrm{Az}_L(R)$ and the associated L-local Brauer group $\mathrm{Br}_L(R)$.

In order to relate Azumaya algebras to Galois theory, we require the following notions modelled on algebraic analogues.

Definition 2.5. Let $R \longrightarrow R'$ be an extension of commutative S-algebras. Then the Azumaya algebra $R \longrightarrow A$ is *split* by $R \longrightarrow R'$ (or just by R') if $R' \wedge_R A \approx R'$, or equivalently if $A \in \mathrm{Ker}(\mathrm{Br}(R) \longrightarrow \mathrm{Br}(R'))$. We define the *relative Brauer group*

$$\mathrm{Br}(R'/R) = \mathrm{Ker}(\mathrm{Br}(R) \longrightarrow \mathrm{Br}(R')).$$

Similarly we can define a *relative* L*-local Brauer group*

$$\mathrm{Br}_L(R'/R) = \mathrm{Ker}(\mathrm{Br}_L(R) \longrightarrow \mathrm{Br}_L(R')).$$

In practise, we will use this when $R \longrightarrow R'$ is a faithful G-Galois extension for some finite group G.

3. Galois extensions and Azumaya algebras

Consider a map of commutative S-algebras $A \longrightarrow B$, which we often denote by B/A. If A is cofibrant as a commutative S-algebra, B is cofibrant as a commutative A-algebra, and if G is a finite group which acts on B by morphisms of commutative A-algebras, then following John Rognes [30], then we call B/A a G*-Galois extension* if the canonical maps $i\colon A \longrightarrow B^{hG}$ and $h\colon B \wedge_A B \longrightarrow F(G_+, B)$ are weak equivalences.

In addition to these conditions, we will assume that B is faithful as an A-module spectrum. This is a further restriction as there are examples of Galois extensions which are not faithful. The following example is due to Ben Wieland (see [31]).

Remark 3.1. Let p be a prime. Then the $\mathbb{Z}/p$-Galois extension

$$F(B\mathbb{Z}/p_+, H\mathbb{F}_p) \longrightarrow F(E\mathbb{Z}/p_+, H\mathbb{F}_p) \sim H\mathbb{F}_p$$

is not faithful. To its eyes the $\mathbb{Z}/p$-Tate spectrum of $H\mathbb{F}_p$ appears trivial, but it is not.

Let $B\langle G\rangle$ be the twisted group algebra over B, *i.e.*, the A-algebra whose underlying A-module is $B \wedge G_+$ whose multiplication is the composition $\widetilde{\mu}$

$$\begin{tikzcd}
B \wedge G_+ \wedge B \wedge G_+ \arrow[r, "\mathrm{id}\wedge\Delta\wedge\mathrm{id}"] \arrow[rrdd, bend right, "\widetilde{\mu}"'] & B \wedge G_+ \wedge G_+ \wedge B \wedge G_+ \arrow[r, "\mathrm{id}\wedge\nu\wedge\mathrm{id}"] & B \wedge G_+ \wedge B \wedge G_+ \arrow[d, "(23)"] \\
 & & B \wedge B \wedge G_+ \wedge G_+ \arrow[d, "\mu_B\wedge\mu_G"] \\
 & & B \wedge G_+
\end{tikzcd}$$

where Δ is the diagonal, ν denotes the G-action on B, μ_B is the multiplication of B and μ_G the multiplication in G. Then $\widetilde{\mu}$ factors through $(B \wedge G_+) \wedge_A (B \wedge G_+)$ and turns $B\langle G\rangle$ into an A-algebra. Note that $B\langle G\rangle$ is an associative algebra but in general it lacks commutativity. More precisely, we know that the morphism $j\colon B\langle G\rangle \longrightarrow F_A(B,B)$ is a weak equivalence of A-algebras for every G-Galois extension $A \longrightarrow B$. In particular, $B\langle G\rangle$ gives rise to a trivial element in the Brauer group of A.

Lemma 3.2. *Let B/A be a faithful G-Galois extension and let M be a $B\langle G\rangle$-module which is of the form $B \wedge_A N$ for some A-module N, where the $B\langle G\rangle$-module structure is given by the B-factor of $B \wedge_A N$. Then there is a weak equivalence of A-modules $N \simeq M^{hG}$.*

Proof. Consider $B \wedge_A M = B \wedge_A B \wedge_A N$. As B is G-Galois over A, the latter term is equivalent to $F(G_+, B) \wedge_A N$ and this in turn is equivalent to $F(G_+, B \wedge_A N)$ because G_+ is finite. As B is dualizable over A, the homotopy fixed point spectrum $(B \wedge_A M)^{hG}$ is equivalent to $B \wedge_A M^{hG}$.

There is a chain of equivalences of B-modules

$$B \wedge_A N \xrightarrow{\simeq} F(G_+, B \wedge_A N)^{hG} \xrightarrow{\simeq} (B \wedge_A B \wedge_A N)^{hG} = (B \wedge_A M)^{hG} \xleftarrow{\simeq} B \wedge_A M^{hG},$$

and the result follows by faithfulness of B over A. $\square$

The following two results give analogues of Galois descent of algebraic Azumaya algebras as in [32, proposition 6.11].

Proposition 3.3. *Suppose that C is an Azumaya algebra over B for which the natural morphism $B \wedge_A C^{hG} \longrightarrow C$ is a weak equivalence of $B\langle G\rangle$-modules. Then C^{hG} is also an Azumaya algebra over A.*

Proof. We know from [30, lemma 6.2.4] that the A-algebra C^{hG} is dualizable as an A-module.

As C is Azumaya over B, we know that $C \wedge_B C^{\mathrm{o}} \simeq F_B(C,C)$. Also, dualizability of C^{hG} over A guarantees that

$$\begin{aligned}
B \wedge_A F_A(C^{hG}, C^{hG}) &\simeq F_A(C^{hG}, B \wedge_A C^{hG}) \\
&\cong F_B(B \wedge_A C^{hG}, B \wedge_A C^{hG}) \\
&\simeq F_B(C,C) \simeq C \wedge_B C^{\mathrm{o}},
\end{aligned}$$

and so

$$\begin{aligned}
C \wedge_B C^{\mathrm{o}} &\simeq (B \wedge_A C^{hG}) \wedge_B (B \wedge_A (C^{hG})^{\mathrm{o}}) \\
&\simeq B \wedge_A (C^{hG} \wedge_A (C^{hG})^{\mathrm{o}}).
\end{aligned}$$

As B is faithful over A, this shows that

$$C^{hG} \wedge_A (C^{hG})^{\mathrm{o}} \simeq F_A(C^{hG}, C^{hG}).$$

Since C is faithful as a B-module and B is faithful as an A-module, we know that C is faithful as an A-module. Assume that for an A-module M we have $C^{hG} \wedge_A M \simeq *$. This is the case if and only if

$$B \wedge_A C^{hG} \wedge_A M \simeq C \wedge_A M \simeq *$$

because B is a faithful A-module. Now faithfulness of C over A implies that C^{hG} is also faithful over A. □

Suppose that B/A is a faithful G-Galois extension in the sense of Rognes [30], where G is a finite group. Now let $H \lhd K \leqslant G$ so that B/B^{hH} is a faithful H-Galois extension, K acts on B^{hH} by B^{hK}-algebra maps and $B^{hK} \longrightarrow B^{hH}$ is a faithful K/H-Galois extension, in particular,

$$B^{hK} \sim (B^{hH})^{h(K/H)}. \tag{3.1}$$

By [30, lemma 6.1.2(b)], the twisted group ring $B\langle H\rangle \sim F_{B^{hH}}(B, B)$ is an Azumaya algebra over B^{hH}, and K acts on $B\langle H\rangle$ by extending the action on B by conjugation on H, so we will write $B\langle H_c\rangle$ to emphasize this.

If $K = Q \ltimes H$ is a semi-direct product or H is abelian, the quotient $Q = K/H$ acts by conjugation on H.

Note that as in algebra, there is an isomorphism of $A[K]$-modules

$$A[K] \cong \prod_K A.$$

The algebraic version of this isomorphism is given by

$$\sum_{k\in K} a_k k \leftrightarrow (a_{k^{-1}})_{k\in K}$$

and we will use the topological analogue of this.

Our next result is based on [32, proposition 6.11(b)].

Proposition 3.4. *Suppose that $K = Q \ltimes H$ is a semi-direct product, or that H is abelian. Then the B^{hK}-algebra $B\langle H_c\rangle^{hQ}$ is Azumaya, and*

$$B^{hH} \wedge_{B^{hK}} B\langle H_c\rangle^{hQ} \sim B\langle H_c\rangle.$$

Hence the Azumaya algebra $B\langle H_c\rangle^{hQ}$ over B^{hK} is split by B^{hH}.

Proof. Note that we can assume that $G = K$ and $B^{hK} = A$. Making use of a faithful base change, it suffices to assume that B is the trivial K-Galois extension, $B = \prod_K A$.

There are isomorphisms of $A[K]$-modules

$$\begin{aligned} B\langle H_c\rangle &\cong {}_{\mathrm{diag}}(\prod_K A \wedge_A A[H_c]) \\ &\cong {}_{\mathrm{left}}(\prod_K A \wedge_A A[H]) \\ &\cong {}_{\mathrm{left}}(A[K] \wedge_A A[H]) \\ &\cong {}_{\mathrm{left}}(A[K \times H]), \end{aligned} \tag{3.2}$$

where ${}_{\mathrm{diag}}(-)$ and ${}_{\mathrm{left}}(-)$ indicate the diagonal and left K-actions respectively, the second isomorphism is the standard equivariant shear map similar to the map sh of [30, section 3.5], and $K \times H$ is viewed as a K-set through the action on the left hand factor. As a Q-set, K decomposes into free orbits indexed on H. On taking Q-homotopy fixed points we obtain an equivalence of A-modules

$$B \langle H_c \rangle^{hQ} \cong A[H \times H]. \tag{3.3}$$

There is a map of A-modules

$$B^{hH} \xrightarrow{\text{unit}} B^{hH} \wedge_A B \langle H_c \rangle^{hQ} \longrightarrow B \langle H_c \rangle$$

which is also a map of $B^{hH} \langle Q \rangle$-modules. Applying $\pi_*(-)$ and working algebraically with $\pi_*(A)$-modules, using [32, proposition 6.11(b)] it follows that we have an isomorphism

$$\pi_*(B^{hH} \wedge_A B \langle H_c \rangle^{hQ}) \cong \pi_*(B \langle H_c \rangle),$$

and therefore a weak equivalence

$$B^{hH} \wedge_A B \langle H_c \rangle^{hQ} \xrightarrow{\sim} B \langle H_c \rangle$$

of $B^{hH} \langle Q \rangle$-modules. Now Proposition 3.3 shows that $B \langle H_c \rangle^{hQ}$ is Azumaya over B^{hK}. $\square$

Here is an example which is analogous to the quaternions viewed as a real Azumaya algebra which splits over the complex numbers. Recall that the quaternions can be generated as a real algebra by the two complex matrices

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Example 3.5. Let $\mathbf{4} = \{1, 2, 3, 4\}$, and let $C = F(\mathbf{4}_+, KU)$, which is equivalent to four copies of the complex K-theory spectrum KU. We view this as a KU-algebra by imposing 2×2-matrix multiplication on C. Then

$$C \sim F_{KU}(KU \vee KU, KU \vee KU),$$

so C is a trivial Azumaya algebra over KU. Consider the group homomorphism

$$\kappa \colon \mathbb{Z}/2 \longrightarrow \Sigma_4; \quad \kappa(\tau) = (14)(23),$$

and let $\mathbb{Z}/2$ act on C by

$$\tau_\bullet(f)(i) = \tau f(\kappa(\tau)(i)),$$

where Σ_4 acts on $\mathbf{4}$ through its defining action and $\mathbb{Z}/2$ acts on KU via maps of commutative KO-algebras. The homotopy fixed point spectrum $C^{h\mathbb{Z}/2}$ is a KO-algebra spectrum, and furthermore we claim that

$$KU \wedge_{KO} C^{h\mathbb{Z}/2} \simeq C.$$

As KU is dualizable over KO and $\mathbf{4}$ is finite, we have

$$\begin{aligned} KU \wedge_{KO} F(\mathbf{4}_+, KU)^{h\mathbb{Z}/2} &\simeq (KU \wedge_{KO} F(\mathbf{4}_+, KU))^{h\mathbb{Z}/2} \\ &\simeq F(\mathbf{4}_+, KU \wedge_{KO} KU)^{h\mathbb{Z}/2}. \end{aligned}$$

Since KU is $\mathbb{Z}/2$-Galois over KO,

$$KU \wedge_{KO} KU \simeq F(\mathbb{Z}/2\mathbb{Z}_+, KU)$$

and thus the above term is weakly equivalent to

$$F(\mathbf{4}_+, F(\mathbb{Z}/2_+, KU))^{h\mathbb{Z}/2} \simeq F((\mathbf{4}\times\mathbb{Z}/2))_+, KU)^{h\mathbb{Z}/2}.$$

Here the $\mathbb{Z}/2$-action on $\mathbf{4}\times\mathbb{Z}/2$ is given by

$$\tau(i,x) = (\kappa(\tau)(i), \tau(x))$$

for $i \in \mathbf{4}$, $x \in \mathbb{Z}/2$. Thus the homotopy fixed point spectrum is equivalent to $C = F(\mathbf{4}_+, KU)$.

Using the homotopy fixed point spectral sequence we can calculate the algebra structure on $F(\mathbf{4}_+, KU)^{h\mathbb{Z}/2}$. This spectral sequence has the form

$$\mathrm{E}_2^{s,t} = H^{-s}(\mathbb{Z}/2, \pi_t(F(\mathbf{4}_+, KU))) \Longrightarrow \pi_{t+s}(F(\mathbf{4}_+, KU)^{h\mathbb{Z}/2}). \tag{3.4}$$

The homotopy groups of $F(\mathbf{4}_+, KU)$ give KU_*^4 with the multiplicative structure of the 2×2-matrices over KU_*. Here the action of τ on $(\lambda_1 u^r, \lambda_2 u^r, \lambda_3 u^r, \lambda_4 u^r)$ with $\lambda_i \in \mathbb{Z}$ is given by

$$\tau(\lambda_1 u^r, \lambda_2 u^r, \lambda_3 u^r, \lambda_4 u^r) = ((-1)^r\lambda_4 u^r, (-1)^r\lambda_3 u^r, (-1)^r\lambda_2 u^r, (-1)^r\lambda_1 u^r).$$

We consider the standard resolution for calculating the cohomology of $\mathbb{Z}/2$. The cocycles with respect to the coboundary that is induced by $(\mathrm{id}-\tau)$ are given by elements of the form $(\lambda u^r, \mu u^r, (-1)^r\mu u^r, (-1)^r\lambda u^r)$ for integers λ and μ and such 4-tuples are equal to the image of $(\lambda u^r, \mu u^r, 0, 0)$ under the coboundary that is induced by $(\mathrm{id}+\tau)$. Similarly, the cocycles with respect to the norm $(\mathrm{id}+\tau)$ can be seen to be coboundaries.

Thus the E_2-term is trivial in positive cohomological degrees, so we only have to determine the invariants in KU_*^4 under the $\mathbb{Z}/2$-action. Here, additively two copies of KU_* remain, but the multiplication arises from matrix multiplication:

$$\begin{aligned}\begin{pmatrix} au^n & bu^n \\ (-1)^n bu^n & (-1)^n au^n \end{pmatrix} \cdot \begin{pmatrix} cu^m & du^m \\ (-1)^m du^m & (-1)^m cu^m \end{pmatrix} \\ = \begin{pmatrix} (ac+(-1)^m bd)u^{n+m} & (ad+(-1)^m bc)u^{n+m} \\ ((-1)^n bc + (-1)^{n+m} ad)u^{n+m} & ((-1)^n bd + (-1)^{n+m} ac)u^{n+m} \end{pmatrix}.\end{aligned}$$

4. Azumaya algebras over Eilenberg-Mac Lane spectra

In this section we consider the case of Azumaya algebras over the Eilenberg-Mac Lane spectrum of a commutative ring. In Toën [35], the algebraic notion of a *derived Azumaya algebra* over a commutative ring is introduced as a special case of the more general notion for simplicial rings. First we explain how the topological and algebraic notions are related.

In [15, section IV.2], an equivalence of categories

$$\Psi\colon \mathscr{D}_{HR} \longrightarrow \mathscr{D}_R \tag{4.1}$$

is constructed, where Ψ is defined on a CW HR-module M to be the cellular chain complex $C_*(M)$. By [15, proposition IV.2.5], for CW HR-modules M, N there are isomorphisms of chain complexes of R-modules

$$C_*(M\wedge_{HR} N) \cong C_*(M)\otimes_R C_*(N),$$
$$C_*(F_{HR}(M,N)) \cong \mathrm{Hom}_R(C_*(M), C_*(N)).$$

The inverse functor $\Phi = \Psi^{-1}$ also preserves the monoidal structure, this is an equivalence of symmetric monoidal categories.

Following Toën [35], see remark 1.2, we find that an Azumaya algebra A over HR, corresponds to a derived Azumaya algebra over R. Note that as we are working with associative (but not commutative) HR-algebras, a cofibrant HR-algebra is a retract of a cell HR-module relative to HR by [15, theorem VII.6.2].

We get the following correspondence.

Proposition 4.1. *Let R be a commutative ring such that for any finitely presented R-module M with $\mathrm{Tor}_k^R(M,M)=0$ for $k>0$ we can deduce that M is flat over R.*

Let T be an R-algebra. Then the HR-algebra HT is a topological Azumaya algebra if and only if T is an algebraic Azumaya R-algebra.

Proof. One direction is easy to see: if $R \longrightarrow T$ is an algebraic Azumaya extension, then $HR \longrightarrow HT$ is topologically Azumaya without any additional assumptions on R.

For the converse, from [15, theorem IV.2.1] we have

$$\pi_n(HT \wedge_{HR} HT^{\mathrm{o}}) = \mathrm{Tor}_n^R(T, T^{\mathrm{o}}), \tag{4.2}$$

$$\pi_n(F_{HR}(HT, HT)) = \mathrm{Ext}_R^{-n}(T,T). \tag{4.3}$$

Because $\mathrm{Tor}_s^R = 0 = \mathrm{Ext}_R^s$ when $s<0$, the Azumaya condition

$$\mu\colon HT \wedge_{HR} HT^{\mathrm{o}} \xrightarrow{\simeq} F_{HR}(HT, HT)$$

implies that for $n \neq 0$,

$$\pi_n(HT \wedge_{HR} HT^{\mathrm{o}}) = \mathrm{Tor}_n^R(T, T^{\mathrm{o}}) = 0 = \mathrm{Ext}_R^n(T,T) = \pi_n(F_{HR}(HT, HT)). \tag{4.4}$$

In particular,

$$T \otimes_R T^{\mathrm{o}} = \pi_0(HT \wedge_{HR} HT^{\mathrm{o}}) \cong \pi_0(F_{HR}(HT, HT)) = \mathrm{Hom}_R(T,T). \tag{4.5}$$

According to [35, remark 1.2], the R-module T is finitely presented and flat by assumption, therefore it is finitely generated and projective by the corollary to [25, theorem 7.12].

For faithfulness, suppose that M is a non-trivial R-module. Since HT is a faithful HR-module, $HT \wedge_{HR} HM \nsim *$. Flatness of T over R together with [15, theorem IV.2.1] yields the isomorphisms

$$\pi_*(HT \wedge_{HR} HM) \cong \pi_0(HT \wedge_{HR} HM) \cong T \otimes_R M,$$

and therefore $T \otimes_R M$ is not trivial. □

Proposition 4.2. *For any commutative ring with unit R there is a functor*

$$H\colon \mathrm{Br}(R) \longrightarrow \mathrm{Br}(HR)$$

induced by the functor which sends a ring to its Eilenberg-Mac Lane spectrum.

Proof. Let $[A]$ be an element of $\mathrm{Br}(R)$, then Proposition 4.1 identifies HA as an HR-Azumaya algebra. If $[A]=0$, *i.e.*, if there is a finitely generated faithful projective R-module M with $A \cong \mathrm{Hom}_R(M,M)$, then

$$HA \sim H\,\mathrm{Hom}_R(M,M) \simeq F_{HR}(HM, HM)$$

and therefore HA is trivial in $\mathrm{Br}(HR)$. □

For instance, the assumptions of Proposition 4.1 are satisfied if R is a principal ideal domain.

The situation is drastically different if we consider arbitrary HR-algebra spectra A. For instance, for every R, every R-module spectrum $\Sigma^n HR$ is faithful and dualizable, and therefore $F_{HR}(HR \vee \Sigma^n HR, HR \vee \Sigma^n HR)$ is a trivial topological Azumaya HR-algebra whose homotopy groups spread over positive and negative degrees. This indicates that the Eilenberg-Mac Lane functor of Proposition 4.2 will not induce an isomorphism in general.

We will discuss this for the case of a field $\boldsymbol{k}$. If A is Azumaya over $H\boldsymbol{k}$, then as A is dualizable over $H\boldsymbol{k}$ we know that the homotopy groups of A are concentrated in finitely many degrees, say $\pi_r(A) \neq 0$ only when $-m < r < n$ for some $m, n \geqslant 0$. As $\boldsymbol{k}$ is a field, we have

$$\pi_*(A \wedge_{H\boldsymbol{k}} A^{\mathrm{o}}) \cong \pi_*(A) \otimes_{\boldsymbol{k}} \pi_*(A)^{\mathrm{o}}.$$

Using the fact that μ induces an isomorphism, we can deduce that $n = m$ because otherwise the kernel of $\pi_*(\mu)$ would be nontrivial.

A derived Azumaya algebra over the field $\boldsymbol{k}$ is a differential graded $\boldsymbol{k}$-algebra B_* whose underlying chain complex is a compact generator of the derived category of chain complexes of $\boldsymbol{k}$-vector spaces $\mathscr{D}_{\boldsymbol{k}}$ and the natural map

$$\mu_{B_*} \colon B_* \otimes_{\boldsymbol{k}} B_*^{\mathrm{o}} \longrightarrow \mathrm{Hom}_{\boldsymbol{k}}(B_*, B_*)$$

is an isomorphism in $\mathscr{D}_{\boldsymbol{k}}$. Here $B_* \otimes_{\boldsymbol{k}} B_*^{\mathrm{o}}$ agrees with the derived tensor product because we are working over a field, and similarly, $\mathrm{Hom}_{\boldsymbol{k}}(B_*, B_*)$ is the graded $\boldsymbol{k}$-vector space of derived endomorphisms of B_*. Now we can relate topological $H\boldsymbol{k}$-Azumaya algebras to derived Azumaya algebras over $\boldsymbol{k}$.

Proposition 4.3. *If A is a topological Azumaya algebra over $H\boldsymbol{k}$, then $\pi_*(A)$ is a derived Azumaya algebra over $\boldsymbol{k}$.*

Proof. As A is dualizable over $H\boldsymbol{k}$, its homotopy groups build a finite dimensional graded $\boldsymbol{k}$-vector space and hence $\pi_*(A)$ is a compact generator of $\mathscr{D}_{\boldsymbol{k}}$. The weak equivalence

$$\mu \colon A \wedge_{H\boldsymbol{k}} A^{\mathrm{o}} \longrightarrow F_{H\boldsymbol{k}}(A, A)$$

yields isomorphisms

$$\mu_{\pi_*(A)} \colon \pi_*(A) \otimes_{\boldsymbol{k}} \pi_*(A)^{\mathrm{o}} \cong \pi_*(A \wedge_{H\boldsymbol{k}} A^{\mathrm{o}}) \cong \pi_* F_{H\boldsymbol{k}}(A, A) \cong \mathrm{Hom}_{\boldsymbol{k}}(A_*, A_*)$$

and so $\pi_*(A)$ is a derived Azumaya algebra over $\boldsymbol{k}$. □

Using Proposition 4.3 together with Toën's results of [35, section 1] we obtain the following.

Theorem 4.4. *For any algebraically closed field $\boldsymbol{k}$, the Brauer group of $H\boldsymbol{k}$ is trivial.*

Proof. Let A be a derived Azumaya algebra over $\boldsymbol{k}$. We know from [35, corollary 1.11] that every derived Azumaya algebra over an algebraically closed field $\boldsymbol{k}$, in particular $\pi_*(A)$, is quasi-isomorphic to a graded $\boldsymbol{k}$-vector space $\mathrm{Hom}_{\boldsymbol{k}}(V, V)$ for some finite dimensional graded $\boldsymbol{k}$-vector space V.

Let

$$M = HV = \bigvee_{i=1}^{n} \Sigma^{m_i} H\boldsymbol{k}$$

be the $H\boldsymbol{k}$-module spectrum such that $\pi_*M \cong V$ as graded $\boldsymbol{k}$-vector spaces. Then A is weakly equivalent to $F_{H\boldsymbol{k}}(A, A)$ since there are isomorphisms

$$\pi_*(A) \cong \mathrm{Hom}_{\boldsymbol{k}}(V, V) \cong \pi_*(F_{H\boldsymbol{k}}(M, M)).$$

Therefore $[A]$ is trivial in the Brauer group $\mathrm{Br}(H\boldsymbol{k})$. $\square$

5. Realizability of algebraic Azumaya extensions

Using Vigleik Angeltveit's obstruction theory [1, theorem 3.5], we can import algebraic Azumaya algebra extensions into topology. Let R be a commutative S-algebra and let $\pi_0 R \longrightarrow A_0$ be an algebraic Azumaya extension. Then

$$A_* := \pi_* R \otimes_{\pi_0 R} A_0$$

is a projective module over $R_* = \pi_* R$ and there is an R-module spectrum A' with $\pi_*(A') \cong A_*$ which can be built as a mapping telescope of an idempotent corresponding to viewing A_* as a direct summand of a free R_*-module. The methods of [5] carry over to give a homotopy associative R-ring spectrum A that realises A_* as the homotopy ring $\pi_* A$.

Angeltveit's obstruction theory [1] then yields the following.

Theorem 5.1. *There is an A_∞ R-algebra structure on A,* i.e., *there is a unique rigidification $r(A)$ of A to an associative R-algebra. The resulting extension $R \longrightarrow r(A)$ is an Azumaya algebra.*

Proof. The existence of the A_∞ structure on A is given by [1, theorem 3.5], because $\pi_*(A \wedge_R A^{\mathrm{o}})$ is separable over A_* and hence the possible obstructions to an A_∞-structure on A (which live in Hochschild cohomology groups of $\pi_*(A \wedge_R A^{\mathrm{o}})$ over A_*) are trivial. The possibility of rigidification follows from [15, II.4]. Uniqueness also follows from the vanishing of all higher Hochschild cohomology groups.

As A_0 is finitely generated projective and faithful over $\pi_0 R$, $r(A)$ is dualizable and faithful as an R-module spectrum. The Azumaya condition

$$\mu\colon A_0 \otimes_{\pi_0 R} A_0^{\mathrm{o}} \cong \mathrm{Hom}_{\pi_0 R}(A_0, A_0)$$

for A_0 guarantees that the μ-map

$$\mu\colon r(A) \wedge_R r(A)^{\mathrm{o}} \longrightarrow F_R(r(A), r(A))$$

is a weak equivalence. $\square$

Corollary 5.2. *There is a natural group homomorphism*

$$r\colon \mathrm{Br}(R_0) \longrightarrow \mathrm{Br}(R); \quad [A] \mapsto [r(A)].$$

For instance in the presence of enough roots of one, we can build generalized quaternionic extensions of ring spectra or consider cyclic extensions.

6. Topological Hochschild cohomology of group rings

We will consider Azumaya algebra extensions that arise as follows. For a finite discrete group G and a commutative S-algebra A, we consider the group A-algebra spectrum $A[G] = A \wedge G_+$. Note, that if G is not abelian, then $A[G]$ is not commutative. We want to identify the extension $\mathrm{THH}_A(A[G]) \longrightarrow A[G]$ as an Azumaya extension in good cases. To that end we document a well-known identification of topological Hochschild cohomology of group rings, see for instance [24, 6.3]. This can we viewed as a topological version of Mac Lane's isomorphisms [23, 7.4.2].

Lemma 6.1. *For A and G as above we have*

$$\mathrm{THH}_A(A[G], A[G]) \simeq (A[G]^c)^{hG} = F_G(EG_+, A[G]^c).$$

Here $A[G]^c$ denotes the naive G-spectrum $A[G]$, where G acts by conjugation on G.

Proof. Topological Hochschild cohomology of $A[G]$ can be described as the totalization of the cosimplicial spectrum that has

$$F_A(A[G]^q, A[G]) \cong F(G_+^q, A[G])$$

as q-cosimplices [27]. First, we mimic the identification that is used in the Mac Lane isomorphism for usual Hochschild cohomology in order to identify this cosimplicial spectrum with the one that has $F(G^q, A[G]^c)$ as q-cosimplices. In algebra this identification is given by $f \mapsto f'$ where

$$f'(g_1, \ldots, g_q) = f(g_1, \ldots, g_q)g_q^{-1} \cdots g_1^{-1}.$$

An analogous identification works on spectrum level. The coface maps in the cosimplicial structure in $F(G^\bullet, A[G]^c)$ are given by

$$\begin{aligned} d_0(f)(g_1, \ldots, g_q) &= g_1 f(g_2, \ldots, g_q)g_1^{-1}, \\ d_i(f)(g_1, \ldots, g_q) &= f(g_1, \ldots, g_i g_{i+1}, \ldots, g_q), \quad (0 < i < q) \\ d_q(f)(g_1, \ldots, g_q &= f(g_1, \ldots, g_{q-1}). \end{aligned}$$

Consider the simplicial model of EG with q-simplices G^{q+1}, with diagonal G-action, and where the i-th face map in EG is given by omitting the i-th group element. We can write the homotopy fixed point spectrum $F_G(EG_+, A[G]^c)$ as

$$F_G(EG_+, A[G]^c) \cong \mathrm{Tot}([q] \mapsto F_G(G^{q+1}, A[G]^c)).$$

Let $\varphi\colon F(G^\bullet, A[G]^c) \longrightarrow F_G(EG_+, A[G]^c)$ be the map that we can describe symbolically as

$$(\varphi f)(g_0, \ldots, g_q) = g_0 f(g_0^{-1}g_1, \ldots, g_{q-1}^{-1}gq)g_0^{-1}.$$

It is then straightforward to check that φ in fact respects the cosimplicial structure. $\square$

Now fix a prime p. Let $\boldsymbol{k}$ be an algebraically closed field of characteristic p and let $H\boldsymbol{k}$ be the corresponding Eilenberg-Mac Lane spectrum realised as a commutative S-algebra. We also adopt the notation of [7]. Thus E_n is the Lubin-Tate spectrum associated with the prime p and the Honda formal group of height n and E_n^{nr} is its maximal unramified Galois extension. These commutative S-algebras have 'residue fields' in the sense of [5, 6], namely K_n and K_n^{nr} respectively, and these are algebras over E_n and E_n^{nr} respectively, but only homotopy commutative when $p \neq 2$ and not even that when $p = 2$.

Theorem 6.2. *Let G be a non-trivial finite discrete group whose order is not divisible by p. Suppose that A is either $H\boldsymbol{k}$ or E_n^{nr}.*

(1) *If G is abelian, then $(A[G]^c)^{hG} \longrightarrow A[G]$ and the trivial extension* $\mathrm{id}\colon A[G] \longrightarrow A[G]$ *are equivalent.*
(2) *If G is non-abelian, then $A[G]$ is a non-trivial $(A[G])^{hG}$-Azumaya algebra.*

Proof. In all cases, we will consider the homotopy fixed point spectral sequence

$$\mathrm{E}_2^{s,t} = H^{-s}(G; A_t[G]^c) \Longrightarrow \pi_{s+t}((A[G]^c)^{hG}).$$

If p does not divide the order of the group G, then this spectral sequence collapses and the only surviving non-trivial terms are the G-invariants

$$\mathrm{E}_2^{0,t} = (A_t[G]^c)^G$$

which can be identified with the center of the group ring $Z(A_*[G])$. In particular, $\pi_*((A[G]^c)^{hG})$ is a graded commutative A_*-algebra.

If G is abelian, then the conjugation action is trivial and as p does not divide $|G|$ we obtain

$$(A[G]^c)^{hG} = F(BG_+, A[G]) \sim A[G],$$

so we have the trivial Azumaya extension. If G is not abelian, then the center of the group ring $A_*[G]$ is a proper subring of $A_*[G]$.

For $A = H\boldsymbol{k}$ we can use Artin-Wedderburn theory to obtain a splitting of the semisimple ring $\boldsymbol{k}[G]$ into a product of matrix algebras over the algebraically closed field $\boldsymbol{k}$,

$$\boldsymbol{k}[G] \cong \prod_{i=1}^{r} \mathrm{M}_{m_i}(\boldsymbol{k}),$$

where r agrees with the number of conjugacy classes in G. Thus the center of $\boldsymbol{k}[G]$ is a product of copies of $\boldsymbol{k}$ and is therefore an étale $\boldsymbol{k}$-algebra. By the obstruction theory of Robinson or Goerss-Hopkins [29, 16], there is a unique E_∞ $H\boldsymbol{k}$-algebra spectrum that is weakly equivalent to $(A[G]^c)^{hG}$. By abuse of notation we denote the corresponding commutative $H\boldsymbol{k}$-algebra by $(A[G]^c)^{hG}$.

We have to describe $A[G]$ as an associative $(A[G]^c)^{hG}$-algebra. For this we use [1, theorem 3.5] again. Starting with our commutative model of $(A[G]^c)^{hG}$ we can build a homotopy associative ring spectrum B with $\pi_*(B) \cong A_*[G]$, and as G is finite and discrete this extension is of the form

$$\pi_*(B) \cong \pi_*(A[G]^c)^{hG} \otimes_{\pi_0(A[G]^c)^{hG}} B_0,$$

with $\pi_0(A[G]^c)^{hG} \longrightarrow B_0$ being algebraically Azumaya. Thus we can apply Theorem 5.1 to see that there is an associative $(A[G]^c)^{hG}$-algebra B which models $A[G]$ and such that B is Azumaya over $(A[G]^c)^{hG}$.

For E_n^{nr} we pass to the residue field K_n^{nr}. The homotopy fixed point spectral sequence gives

$$\begin{aligned}\pi_*((E_n^{\mathrm{nr}}[G]^c)^{hG}) &\cong Z((E_n^{\mathrm{nr}})_*[G]) \\ &\cong Z(W\overline{\mathbb{F}}_p[[u_1, \ldots, u_{n-1}]][G])[u^{\pm 1}].\end{aligned}$$

Reducing modulo the maximal ideal $\mathfrak{m} = (p, u_1, \ldots, u_{n-1})$ gives the homotopy groups of the G-homotopy fixed points of $K_n^{\mathrm{nr}}[G]$ with respect to the conjugation action, $Z(\overline{\mathbb{F}}_p[G])[u^{\pm 1}]$ and again we can identify this term as $\prod_{i=1}^{r} \overline{\mathbb{F}}_p$ where r denotes the number of conjugacy classes in G. The idempotents that give rise to these splittings can be lifted to idempotents for

$Z(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G])$ and $W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G]$ and therefore these two algebras also split into products with r factors:

$$W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G] \cong \prod_{i=1}^{r} B_i,$$

$$Z(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G]) \cong \prod_{i=1}^{r} C_i,$$

where

$$B_i/\mathfrak{m} \cong \mathrm{M}_{m_i}(\overline{\mathbb{F}}_p),$$

while for $1 \leqslant i \leqslant r$, the C_i are commutative and satisfy

$$C_i/\mathfrak{m}C_i \cong \overline{\mathbb{F}}_p.$$

Additively we know that $Z(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G])$ is the free module on the conjugacy classes and so we can conclude that $(E_n^{\mathrm{nr}}[G]^c)^{hG}$ is weakly equivalent to $\prod_{i=1}^{r} E_n^{\mathrm{nr}}$ and the latter spectrum can be modelled by a commutative E_n^{nr}-algebra spectrum and $E_n^{\mathrm{nr}}[G]$ is dualizable over $\prod_{i=1}^{r} E_n^{\mathrm{nr}}$.

Artin-Wedderburn theory gives a semisimple decomposition

$$\overline{\mathbb{F}}_p[G] \cong \prod_{i=1}^{r} \mathrm{M}_{d_i}(\overline{\mathbb{F}}_p),$$

and the centre $Z(\overline{\mathbb{F}}_p[G])$ can be identified with the product of the centres of the matrix ring factors. There are associated central idempotents of $\overline{\mathbb{F}}_p[G]$ accomplishing this splitting. By the theory of idempotent lifting described in [21, section 21] for example, these idempotents lift to give an associated splitting

$$W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G] \cong \prod_{i=1}^{r} \mathrm{M}_{d_i}(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]]),$$

and again the centre of $W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G]$ can be identified with the product of the centres of the matrix factors. Notice that $\mathrm{M}_{d_i}(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]])$ is Azumaya over $W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]]$. The rest of the proof involves realising the central idempotents as morphisms of S-algebras, but this is well known to be possible since the projections are Bousfield localisations, see [33]. □

Remark 6.3. For $A = H\mathbb{F}_p$ and $G = C_p$ the extension $F(BC_{p+}, H\mathbb{F}_p) \longrightarrow H\mathbb{F}_p$ is not always Azumaya: Wieland's example of Remark 3.1 shows that $H\mathbb{F}_p$ is not faithful over $F(BC_{p+}, H\mathbb{F}_p)$.

7. Azumaya algebras over Lubin-Tate spectra

From now on will use E to denote E_n, E_n^{nr} or any commutative Galois extension of E_n obtained as a homotopy fixed point algebra $E = (E_n^{\mathrm{nr}})^{h\Gamma}$ for some closed normal subgroup $\Gamma \lhd \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})$. Similarly, K will denote the corresponding residue field of E, so when $E = E_n$ or E_n^{nr} we have $K = K_n$ or K_n^{nr}.

We will work with dualizable K-local E-modules. By [7, section 7] we know that such modules are retracts of finite cell E-modules. If $W \in \mathscr{M}_{E,K}$, then since $\pi_*(K \wedge_E W)$ is a graded vector space over the graded field $K_* = \pi_*(K)$, it follows that

$$K \wedge_E W \sim L_K \bigvee_i \Sigma^{d(i)} K,$$

where the right hand wedge is non-trivial if and only if W is non-trivial in $\mathscr{D}_{E,K}$. In particular, if W is dualizable this wedge is finite and

$$K \wedge_E W \sim \bigvee_i \Sigma^{d(i)} K$$

since W is K-local. For any $X \in \mathscr{M}_{E,K}$,

$$K \wedge_E (W \wedge_E X) \sim L_K \bigvee_i \Sigma^{d(i)} K \wedge_E X,$$

so $W \wedge_E X$ is trivial in $\mathscr{D}_{E,K}$ if and only if both of W and X are trivial in $\mathscr{D}_{E,K}$. Thus every E-module W which is non-trivial as an element of $\mathscr{D}_{E,K}$ is faithful and cofibrant as a K-local E-module; furthermore, every $X \in \mathscr{M}_{E,K}$ is W-local.

By [1], there are many examples of K-local Azumaya algebras over E which have K as their underlying E ring spectrum. These examples have no analogue in the algebraic context since they are not projective E-modules, nor do they split over suitable Galois extensions. Instead we focus on split examples. A good source of these can be found in the situation of [30, section 5.4.3], based on work of Devinatz and Hopkins [12] and we will discuss these in Section 8.

For background ideas on Azumaya algebras graded on a finite abelian group, we follow [9]. We will only consider the case where the grading group is $\mathbb{Z}/2$ with the non-trivial symmetric bilinear map $\mathbb{Z}/2 \longrightarrow \{\pm 1\}$ determining the relevant signs.

Over a field $\boldsymbol{k}$, an (ungraded) Azumaya algebra A is a central simple algebra, so by Wedderburn's theorem, there is an isomorphism of $\boldsymbol{k}$-algebras

$$A \cong \mathrm{M}_r(D),$$

where D is a central division algebra over $\boldsymbol{k}$. If $d = \dim_{\boldsymbol{k}} D$, then

$$\dim_{\boldsymbol{k}} A = (rd)^2,$$

so $\dim_{\boldsymbol{k}} A$ is a square. In the graded case, such restrictions do not always apply, and this has consequences for the topological situation.

Theorem 7.1. *Suppose that p is an odd prime and let A be a K-local Azumaya algebra over E. Then $\pi_*(K \wedge_E A)$ is an Azumaya algebra over K_*.*

Proof. The ring K_* is a 2-periodic graded field which we will view as $\mathbb{Z}/2$-graded, and $\pi_*(K \wedge_E A)$ will also be viewed as a $\mathbb{Z}/2$-graded K_*-algebra.

Now we have isomorphisms of K_*-algebras

$$\begin{aligned}
\pi_*(K \wedge_E A) \otimes_{K_*} \pi_*(K \wedge_E A)^{\mathrm{o}} &\cong \pi_*(K \wedge_E A) \otimes_{K_*} \pi_*(K^{\mathrm{o}} \wedge_E A^{\mathrm{o}}) \\
&\cong \pi_*(K \wedge_E (A \wedge_E A^{\mathrm{o}})) \\
&\cong \pi_*(K \wedge_E F_E(A, A)).
\end{aligned}$$

Since A and K are strongly dualizable, using results of [15] we have

$$K \wedge_E F_E(A, A) \sim F_K(K \wedge_E A, K \wedge_E A),$$

so the universal coefficient spectral sequence over K yields

$$\pi_*(K \wedge_E F_E(A, A)) \cong \mathrm{End}_{K_*}(\pi_*(K \wedge_E A)).$$

Therefore $\pi_*(K \wedge_E A)$ is a K_*-Azumaya algebra. □

Corollary 7.2. *If $\pi_*(K \wedge_E A)$ is concentrated in even degrees then its dimension is a square,* i.e., *for some natural number m,*

$$\dim_{K_*} \pi_*(K \wedge_E A) = m^2.$$

In fact we have

Proposition 7.3. *If $\pi_*(K \wedge_E A)$ is concentrated in even degrees then $\pi_*(A)$ is a $\mathbb{Z}/2$-graded algebra Azumaya algebra over E_0. In particular, as an E-module A is equivalent to a wedge of m^2 copies of E, where*

$$m^2 = \dim_{K_*} \pi_*(K \wedge_E A) = \operatorname{rank}_{E_*} \pi_*(A).$$

Proof. By [7] (see section 7 and the proof of theorem 5.1), the E_*-module $\pi_*(A)$ is finitely generated, free and concentrated in even degrees, hence

$$\pi_*(A) \otimes_{E_*} \pi_*(A)^{\mathrm{o}} \cong \pi_*(A \wedge_E A^{\mathrm{o}}) \cong \pi_*(F_E(A, A)) \cong \operatorname{Hom}_{E_*}(\pi_*(A), \pi_*(A)),$$

where the last isomorphism follows from the collapsing of the universal coefficient spectral sequence. $\square$

We define $\mathrm{Az}_K(E)$ to be the collection of all cofibrant K-local topological Azumaya algebras over E, and introduce the following equivalence relation $\approx$ on $\mathrm{Az}_K(E)$.

- If $A, B \in \mathrm{Az}_K(E)$, then $A \approx B$ if and only if there are faithful, dualizable, cofibrant E-modules U, V for which there is an equivalence in the derived category of K-local E-algebras
$$A \wedge_E F_E(U, U) \sim B \wedge_E F_E(V, V).$$

We will denote the set of equivalence classes of $\approx$ by $\mathrm{Br}_K(E)$; this is indeed a set since every dualizable K-local E-module is a retract of a finite cell E-module. We could equally well require that

$$A \wedge_E F_E(U, U)' \sim B \wedge_E F_E(V, V)',$$

where $F_E(U, U)' \xrightarrow{\sim} F_E(U, U)$ and $F_E(V, V)' \xrightarrow{\sim} F_E(V, V)$ are cofibrant replacements.

The following lemma is a topological analogue of a standard algebraic result, see [2] for example.

Proposition 7.4. *If $A, B \in \mathrm{Az}_K(E)$ and $A \approx B$, then there is a faithful, dualizable cofibrant E-module W for which*

$$A \wedge_E B^{\mathrm{o}} \sim F_E(W, W).$$

In particular, if $A \in \mathrm{Az}_K(E)$ and $A \approx E$, then there is a faithful, dualizable, cofibrant E-module W for which $A \sim F_E(W, W)$.

Proof. It is easy to reduce this the second case, so we will assume that as E-algebras,

$$A \wedge_E F_E(U, U) \sim F_E(V, V).$$

This means that given a cofibrant replacement

$$F_E(U, U)' \xrightarrow{\sim} F_E(U, U)$$

in the category of E-algebras, there is a weak equivalence of E-algebras

$$\alpha \colon A \wedge_E F_E(U, U)' \xrightarrow{\sim} F_E(V, V).$$

Let $\alpha' \colon F_E(U,U)' \longrightarrow F_E(V,V)$ be the induced morphism. Then we can consider V as a module over $A \wedge_E F_E(U,U)'$, denoted $\alpha^* V$ in the notation of [15, theorem III.4.2], where

$$\alpha^* \colon \mathscr{D}_{F_E(V,V),K} \longrightarrow \mathscr{D}_{A\wedge_E F_E(U,U)',K}$$

is the pullback functor.

If $\widetilde{U}$ is a cofibrant replacement of U as an $F_E(U,U)'$-module, we set $W = F_E(\widetilde{U}, V)$. Then by [4, theorem 1.2], there is a Morita equivalence between the derived categories $\mathscr{D}_{E,K}$ and $\mathscr{D}_{F_E(U,U)',K}$, and under this equivalence we have

$$W \wedge_E \widetilde{U} \sim V$$

as $F_E(U,U)'$-modules. Now applying $K^E_*(-)$ we find that

$$K^E_* A \otimes_{K_*} K^E_*(F_E(U,U)) \cong \mathrm{End}_{K_*}\left(K^E_* W \otimes_{K_*} K^E_* \widetilde{U}\right).$$

Since $K^E_*(F_E(U,U)) \cong \mathrm{End}_{K_*} K^E_* \widetilde{U}$ and there is a monomorphism $K^E_* A \longrightarrow \mathrm{End}_{K_*} K^E_* W$, we have

$$K^E_* A \cong \mathrm{End}_{K_*} K^E_* W$$

by [9, lemma 2.9]. Using this we see that the morphism of E-algebras $A \longrightarrow F_E(W,W)$ is a K-equivalence and hence an equivalence since A is K-local. $\square$

8. Some examples of K_n-local Azumaya algebras

We now recall Proposition 3.4. By work of Devinatz and Hopkins [12], and subsequently Davis [10], as explained in [30, theorem 5.4.4], for each pair of closed subgroups

$$H \leqslant G \leqslant \mathbb{G}_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathbb{S}_n$$

of the Morava stabilizer group, there is an associated pair of homotopy fixed point spectra $E^{hG} \longrightarrow E^{hH}$, and if $H \lhd G$ then this is a K-local G/H-Galois extension. In particular, when $H \leqslant \mathbb{G}_n$ is finite, $E^{hH} \longrightarrow E$ is a K-local H-Galois extension.

A particularly interesting source of examples is provided by taking G to be a maximal finite subgroup of $\mathbb{G}_n$. If p is odd and $n = (p-1)k$ with $p \nmid k$, or $p = 2$ and $n = 2k$ with k odd, then such maximal subgroups are unique up to conjugation and then the homotopy fixed point spectrum E^{hG} is denoted EO_n. Here is an example, studied in [30, section 5.4.3].

Example 8.1. At the prime $p = 2$, $\mathbb{G}_2$ has a maximal finite subgroup $G_{48} = C_2 \ltimes \widehat{A}_4$, where $\widehat{A}_4 \cong C_3 \ltimes Q_8$ is the binary tetrahedral group. Therefore E_2/EO_2 is a G_{48}-Galois extension. Applying Proposition 3.4 we see that there are Azumaya algebras over EO_2 of the form

$$(E_2\langle \widehat{A}_4 \rangle)^{hC_2}, \quad (E_2\langle Q_8 \rangle)^{h(C_2\times C_3)}, \quad (E_2\langle C_2 \rangle)^{h(C_2\ltimes A_4)}.$$

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Mathematics Department, University of Glasgow, Glasgow G12 8QW, Scotland.
E-mail address: `a.baker@maths.gla.ac.uk`
URL: `http://www.maths.gla.ac.uk/~ajb`

Fachbereich Mathematik der Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany.
E-mail address: `richter@math.uni-hamburg.de`
URL: `http://www.math.uni-hamburg.de/home/richter/`