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THE CLASSIFYING ALGEBRA FOR DEFECTS

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Abstract

We demonstrate that topological defects in a rational conformal field theory can be described by a classifying algebra for defects – a finite-dimensional semisimple unital commutative associative algebra whose irreducible representations give the defect transmission coefficients. We show in particular that the structure constants of the classifying algebra are traces of operators on spaces of conformal blocks and that the defect transmission coefficients determine the defect partition functions.

1 Introduction

In two-dimensional quantum field theory, in particular in conformal quantum field theory, the one-dimensional structures of boundaries and defect lines have attracted much interest. Boundaries require the notion of boundary conditions. The study of these has provided much structural insight into conformal field theory. Boundary conditions are also of considerable interest in applications, ranging from percolation problems in statistical mechanics and impurity problems in condensed matter systems to D-branes in string theory.

A defect line separates two regions of the world sheet on which the theory is defined. In fact, on the two sides of a defect line there can be two different conformal field theories. Hence defects relate different conformal field theories; this forms the ground for their structural importance. Indeed, defects allow one to determine non-chiral symmetries of a conformal field theory and dualities between different theories. For further applications see e.g. [1, 2, 3, 4].

Among the conformal field theories those with 'large' symmetry algebras are particularly accessible for an explicit treatment. Technically, these are *rational* conformal field theories, for which the representation category of the chiral symmetry algebra is a modular tensor category. Similarly, those boundary conditions and defect lines are particularly tractable which preserve rational chiral symmetries. In fact, rational CFTs are amenable to a precise mathematical treatment [5], in which boundary conditions and defects can be analyzed with the help of certain algebras internal to the representation category of the chiral symmetry algebra.

An earlier approach [6] to boundary conditions can be understood with the help of a classifying algebra [7], the presence of which was established rigorously in [8] for all rational CFTs. The classifying algebra is an algebra over the complex numbers spanned by certain bulk fields. It is semisimple, associative and commutative, and its irreducible representations are in bijection with the elementary boundary conditions. (Any boundary condition of a rational CFT is a superposition, with suitable Chan-Paton multiplicities, of elementary boundary conditions.)

In fact, the classifying algebra allows one to obtain also the corresponding reflection coefficients for bulk fields: the homomorphisms given by its irreducible representations give the bulk field reflection coefficients in the presence of elementary boundary conditions. The latter coefficients determine not only boundary states and boundary entropies [9], but also annulus partition functions. The classifying algebra approach is also of interest in the study of boundary conditions and defects in non-rational theories, like e.g. in Liouville theory, for which a description in terms of algebras in the representation category of the chiral symmetry algebra is not yet available [10, 11, 12, 13].

Obviously, it is desirable to have similar tools at one's disposal for the study of defects. The basic result of this paper is that such tools indeed exist. In the case of defects the role of reflection coefficients for boundary conditions is taken over by *defect transmission coefficients*, which we carefully discuss. We show in particular that these coefficients determine the defect partition functions. As discussed in [14], such quantities are also expected to be relevant for the structure of the moduli space of conformal field theories. Again there is a classifying algebra for defects, spanned by certain pairs of bulk fields, which determines these coefficients, namely through the homomorphisms given by its irreducible representations.

One way to capture aspects of defect lines is to think of them as boundary conditions for a doubled theory, the so-called folding trick. From this point of view it is reasonable to expect that a classifying algebra for defects indeed exists. In fact, this algebra has already been used successfully on a heuristic basis in the study of Liouville theory [12]. As we will see, the folding trick does not render the problem trivial, though, since when resorting to this mechanism one loses essential information about defects, like e.g. the fusion of defect lines, a structure that is crucial for the relation between defects and non-chiral symmetries and dualities. Indeed, general defect transmission coefficients cannot be understood as boundary reflection coefficients of the doubled theory, see the discussion after (2.7) below.

This paper is organized as follows. In section 2 we review basic properties of defects and explain the fundamental role played by the defect transmission coefficients. Section 3 summarizes our strategy for the derivation of the classifying algebra for defects. It involves the comparison of two different ways of factorizing correlators with defect fields, which we carry out in Sections 4 and 5, respectively (two technical steps are relegated to appendices). In section 6 we establish the main properties of the classifying algebra: it is a semisimple unital commutative associative algebra over the complex numbers, whose irreducible representations furnish the defect transmission coefficients. Section 7 finally shows how defect partition functions can be expressed in terms of the defect transmission coefficients.

2 Defects and defect transmission coefficients

Let us start by discussing a few general aspects of codimension-one defects that separate different quantum field theories. Our treatment is adapted to the case of two-dimensional conformal quantum field theories, but part of the discussion is relevant to non-conformal and to higher-dimensional theories as well.

In the case of *rational* conformal field theories, our statements have in fact the status of mathematical theorems. The understanding of rational conformal field theory is based on the use of 'large' chiral symmetry structures. Technically, this means that the representation category of the chiral symmetry structure, say of a conformal vertex algebra, is a modular tensor category. To keep this technically important tool available also in the discussion of defects that separate different quantum field theories, we assume that all theories in question share the same chiral symmetry structure and that the defects preserve all those symmetries. But chiral symmetries do not determine the local quantum field theory completely – a fact that sometimes is, slightly inappropriately, expressed by saying that for a given (chiral) CFT there can be different modular invariant partition functions. As a consequence we can still realize non-trivial situations in which two different rational CFTs are separated by a defect. In addition, as we will see, even defects adjacent to one and the same CFT carry important structural information.

We will say that different local conformal field theories based on the same chiral symmetry realize different *phases* of the theory. An instructive example one may wish to keep in mind is given by the conformal sigma models based on the group manifolds SU(2) and SO(3), respectively. It makes sense to consider world sheets that are composed of several regions in which different phases of the theory live and which are separated by one-dimensional phase boundaries. We refer to these phase boundaries as *defect lines*; more specifically, the interface between regions in phases A and B constitutes an *A-B-defect*. For instance, in the sigma model example, in each region one deals with maps to either SU(2) or SO(3), depending on the phase of the region. At the defect line a transition condition for these maps must be specified; it determines the type of defect assigned to the defect line.

Defects can be characterized by the behavior of the energy-momentum tensor in the vicinity of the defect line. In particular, for a *conformal* defect the difference $T-\bar{T}$ between holomorphic and antiholomorphic components is continuous across the defect line (see e.g. [15]). Among the conformal defects there are the totally reflective defects, for which $T=\bar{T}$ on either side of the defect line, and the totally transmissive ones, for which T and $\bar{T}$ individually are continuous across the defect line. Totally transmissive defects are tensionless, i.e. they can be deformed on the world sheet without affecting the value of a correlator, as long as they are not taken through any field insertion or another defect line. To allow us to attain a maximal mathematical control of the situation, a defect should preserve even more chiral symmetries than the conformal symmetries, namely the ones of a rational chiral algebra. We refer to totally transmissive defects with this stronger property as *topological defects*. Such topological defects have been studied in e.g. [16, 17, 18, 19, 20, 21, 22]. All the defects considered in the present paper are topological; accordingly we will usually suppress the qualification 'topological' in the sequel.

Let us explain how properties of defects are reflected in the mathematical description of rational CFTs. As has been shown in [23], for a rational CFT the phases A are in bijection with certain Frobenius algebras internal to the representation category $\mathcal{C}$ of the chiral symmetry algebra, and for given phases A and B the (topological) A-B-defects are the objects of the category $\mathcal{C}_{A|B}$ of A-B-bimodules in $\mathcal{C}$. Morita equivalent Frobenius algebras give equivalent local conformal field theories, so that with regard to all observable aspects a phase can be identified with a Morita class of Frobenius algebras. Similarly, different defects can be physically equivalent. Equivalence classes of A-B-defects correspond to isomorphism classes of objects of the bimodule category $\mathcal{C}_{A|B}$; we refer to such classes of defects as *defect types*.

The category of bimodules over Frobenius algebras in the category of chiral symmetries can be analyzed explicitly. One derives the following finiteness statements, which are specific to rational theories:

- The number of inequivalent phases is finite.
- The number of elementary defect types is finite.

Indeed, for any two phases A and B there is a finite number of inequivalent elementary, or *simple*, A-B-defects. Every defect is a finite superposition (finite direct sum) of such simple defects.

Fusion of defects

A central feature of all defects that are transparent to both chiralities of the stress-energy tensor, and thus in particular of all topological defects, is that there is an operation of *fusion* of defects [16, 17, 23, 18]:

- Defects with matching adjacent phases can be *fused*.

Specifically, when an A-B-defect $X \equiv X_{A|B}$ and a B-C-defect $Y \equiv Y_{B|C}$ are running parallel to each other, then due to the transparency property, we can take a smooth limit of vanishing distance in which the two defects constitute an A-C-defect $X \otimes_B Y$. This fusion operation is associative up to equivalence of defects. The fused defect $X \otimes_B Y$ is, in general, not simple, even if both X and Y are simple.

- For every defect X there is a *dual defect* $X^\vee$.

Namely, another operation that we can perform on a defect line is to change its orientation, and this results in a new defect type, the dual defect. More precisely, for an A-B-defect X the dual defect $X^\vee$ corresponding to the defect line with opposite orientation is a B-A-defect.

These two operations have again a clear-cut representation theoretic meaning: Fusion is the tensor product over B of the bimodules X and Y, and the categories of bimodules have a duality which implements orientation reversal.

- For each phase there is an *invisible defect*, the fusion with which does not change any defect type.

The presence of this distinguished defect signifies that the tensor category $\mathcal{C}_{A|A}$ of A-A-bimodules has a tensor unit. The tensor unit of $\mathcal{C}_{A|A}$, and thus the invisible A-A-defect, is in fact just the algebra A, seen as a bimodule over itself. For any A-B-defect X the fused defects $X \otimes_B X^\vee$ and $X^\vee \otimes_A X$ contain the invisible defects A and B, respectively, as sub-bimodules (with multiplicity one if X is simple).

- For each phase A there is a *fusion ring* $\mathcal{F}_A$ of A-A defect types.
  The A-B-defect types span a left module over $\mathcal{F}_A$ and a right module over $\mathcal{F}_B$.

Indeed, the fusion of A-A defects in a rational CFT is sufficiently well-behaved such that it induces the structure of a fusion ring on the set of A-A defect types. This defect fusion ring has a distinguished basis consisting of simple defect types. (But there does not exist a braiding of defect lines, hence the fusion ring of defects is not, in general, commutative).

The defect fusion ring contains a surprising amount of information. To elucidate this, consider the subset of those A-A-defects X which when fused with their dual defect just give the invisible defect or, in other words, for which $X^\vee \otimes_A X$ is isomorphic to A as an A-A-bimodule. Such defects, which we call *invertible* (or group-like) defects, turn out to be particularly interesting:

- Non-chiral *internal symmetries* of a full CFT in phase A are in bijection with the equivalence classes of invertible A-A-defects.

This insight [20, Sect. 3.1] has important consequences in applications; we will return to this effect and to its generalization to Kramers-Wannier dualities in a moment. The types of invertible defects form a group, the *Picard group* of the phase A. The group ring of the Picard group is a subring of the fusion ring $\mathcal{F}_A$.

Defect fields

In the language of bimodules, another aspect of defects becomes obvious: Defects can also be joined. At such a junction, a coupling needs to be chosen. The possible couplings of an A-B-defect X and a B-C-defect Y to an A-C-defect Z are given by the space $\mathrm{Hom}_{A|C}(X \otimes_B Y, Z)$ of bimodule morphisms. Moreover, defect lines can start and end at field insertion points – at insertions of so-called *disorder fields*. A field insertion on a defect line – a *defect field* – can also change the type of a defect.

- Disorder fields are special instances of defect fields, namely those which turn the invisible defect into a non-trivial defect or inversely. Bulk fields are special instances of disorder fields, turning the invisible defect to the invisible defect.

Just like bulk fields, defect fields carry two chiral labels U, V which correspond to representations of the chiral symmetry algebra. In representation theoretic terms, the space of defect fields changing an A-B-defect X to another A-B-defect Y is given by the space $\mathrm{Hom}_{A|B}(U \otimes^+ X \otimes^- V, Y)$ of bimodule morphisms, where $U \otimes^+ X \otimes^- V$ carries a specific structure of A-B-bimodule that is determined by the (left, respectively right) actions of the algebras A and B on X and by the braiding of $\mathcal{C}$.

For *rational* conformal field theories one can classify defect fields. The corresponding partition functions obey remarkable consistency relations:

- The expansion coefficients, in the basis of characters, of the partition functions of a torus with two parallel defect lines inserted furnish a NIM-rep of the double fusion algebra [16, 23].

Acting with defects

We next turn to another important consequence of the fusion structure on defects: their action on various quantities of a rational CFT.

- Defects can be fused to boundary conditions.

Specifically, when an A-B-defect $X \equiv X_{A|B}$ and a boundary condition $M \equiv M_B$ adjacent to phase B are running parallel to each other, then in the case of a topological defect one can take the limit of vanishing distance between defect line and boundary, after which defect and boundary condition together constitute a boundary condition $X \otimes_B M$ adjacent to phase A. Representation theoretically one deals again with the tensor product over B, now between a bimodule and a left module. Especially, the space of boundary conditions in phase A carries an action of the A-A-type defects and in particular of their Picard group.

- Defects act on bulk fields.

Namely, consider a bulk field insertion in phase B that is encircled by an A-B-defect. Invoking again the transparency property we can shrink the circular defect line to zero size, whereby we obtain another bulk field, now in phase A. This way the defect gives rise to a linear map on the space of bulk fields. The action of defects on bulk fields is sufficient to distinguish inequivalent defects (i.e. non-isomorphic bimodules) [20, Prop. 2.8]. As a consequence, any two different defect-induced internal symmetries can already be distinguished by their action on bulk fields [20, Sect. 3.1].

Combining the various actions of defects on other structures on the world sheet, one arrives at a notion of *inflating an A-B-defect in phase A*, by which one can relate correlators on arbitrary world sheets in phase A to correlators in phase B [20, Sect. 2.3].

Defects and dualities

An A-B-defect X is called a *duality defect* iff there exists a B-A-defect X' such that the fused defect $X \otimes_B X'$ is a superposition of only invertible A-A-defects [20, Thm. 3.9]. These generalize the invertible defects:

- *Order-disorder dualities* of Kramers-Wannier type can be deduced from the existence of duality defects.

In short, the relationship between defects and internal symmetries generalizes to order-disorder dualities.

- *T-duality* between free boson CFTs is a variant of order-disorder duality.

More precisely, T-duality is generated by a special kind of duality defect, namely one in the *twisted* sector of the $\mathbb{Z}_2$-orbifold of the free boson theory, corresponding to a $\mathbb{Z}_2$-twisted representation of the U(1) current algebra ([24, Sect. 5.4], compare also [25, 26]). While ordinary order-disorder dualities in general relate correlators of bulk fields to correlators of genuine disorder fields, in the case of T-duality the resulting disorder fields are in fact again local bulk fields.

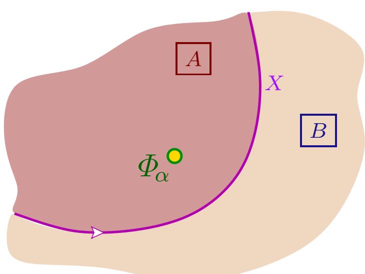
If two phases A and B can be separated by a duality defect X, then the torus partition function for phase A can be written in terms of partition functions with defect lines of phase B, in a manner reminiscent of the way the partition function of an orbifold is expressed as a sum over twisted sectors. [20, Prop. 3.13]. (For $A = B$, one thus deals with an ‘auto-orbifold property’, as first observed in [27].) The relevant orbifold group consists of the types of invertible defects that appear in the fusion of X with $X^\vee$.

This orbifold construction can be generalized to arbitrary pairs of phases A and B: correlators in phase B can be obtained from those in phase A as a generalized orbifold corresponding to an A-A-defect which (as an object of $\mathcal{C}$) is given by $A \otimes B \otimes A$ [22].

Transmission through a defect

To further investigate defects, and in particular to derive a classifying algebra for them, we need to consider yet another phenomenon arising from the presence of defects, namely the transmission of bulk fields through a defect. Let us describe this phenomenon in some detail. When a bulk field passes through a defect, it actually turns into a superposition of *disorder* fields. This process is similar to the excitation of boundary fields that results when a bulk field approaches the boundary of the world sheet.

Consider a bulk field Φ_α in phase A close to an A-B-defect X. Since the defect is topological, we can deform the defect line around Φ_α. Fusing the resulting two parallel pieces of defect line (with opposite orientation) then amounts to the creation of disorder fields in phase B with the same chiral labels as Φ_α, connected by defects Y to the original defect X. In the case of rational CFTs these can be expanded in a basis of elementary disorder fields $\Theta_{Y,B;\gamma}$. Pictorially, this process is described schematically as follows (compare [20, (2.28)]):



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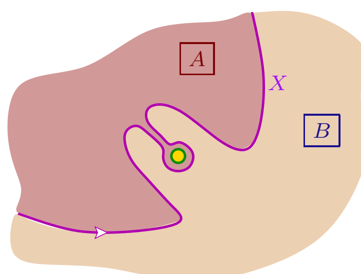
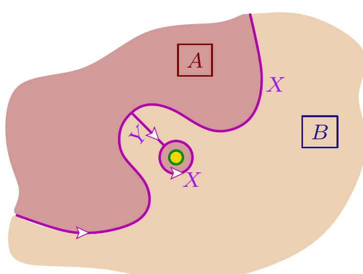
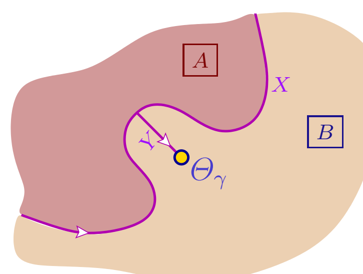


$= \sum_Y \sum_\tau$



$= \sum_{Y,\tau} \sum_\gamma d^{\alpha\gamma}_{A,X,B;Y,\tau}$



(2.1)

Here the Y-summation is over isomorphism classes of simple B-B-defects, while the τ-summation is over a basis of the space of B-bimodule morphisms from $X^\vee \otimes_A X$ to Y. (The notations for defect fields will be explicated after (2.6) below.)

Now take the bulk field $\Phi_\alpha \equiv \Phi_\alpha^{\imath\jmath}$ to have chiral labels $\imath, \jmath$ (corresponding to simple objects $U_\imath$ and $U_\jmath$ of the representation category of the chiral symmetry algebra, so that the label α takes values in the bimodule morphism space $\mathrm{Hom}_{A|A}(U_\imath \otimes^+ A \otimes^- U_\jmath, A)$). Then the expansion (2.1) of the bulk field Φ_α in phase A into disorder fields $\Theta_{Y,B;\gamma}$ in phase B reads more explicitly

$$\Phi_\alpha^{\imath\jmath}(z) = \sum_Y \sum_\tau \sum_\gamma d^{\imath\jmath,\alpha\gamma}_{A,X,B;Y,\tau}\, \Theta^{\imath\jmath}_{Y,B;\gamma}(z) \tag{2.2}$$

(as an equality valid inside correlators with suitable bulk fields). Note that the coefficients of this expansion do not depend on the position of the insertion point of the bulk, respectively disorder, field, simply because we can freely move a topological defect line and keep the position of the insertion point fixed while deforming the defect.

Next consider the situation that we perform the manipulations in (2.1) on a sphere on which besides the bulk field Φ_α in phase A there is one further field insertion, another bulk field Φ_β in phase B, so that we are dealing with the correlator C_X of two bulk fields Φ_α and Φ_β separated by the defect X on a sphere. Then in the sum over Y in (2.2) only a single summand gives a non-zero contribution to the correlator, namely the one with $Y = B$, and in this case the label τ takes only a single value, which we denote by '$\circ$'. The corresponding disorder fields $\Theta_{B,B;\gamma}$ are just ordinary bulk fields in phase B, and thus the resulting contribution to C_X involves the coefficient $(c^{\rm bulk}_{B;\imath,\jmath})_{\gamma\beta}$ (in a standard basis, compare [28, App. C.2]) of an ordinary two-point function $C(\Phi^{\imath\jmath\,(B)}_\alpha, \Phi^{\bar\imath\bar\jmath\,(B)}_\gamma)$ of bulk fields on the sphere. Furthermore, if $Y = B$, then the circular defect line carrying the defect X, which no longer encloses any field insertion in phase A, can be shrunk to zero size and thus be completely removed from the world sheet at the expense of multiplying the correlator with $\dim(X)\,/\dim(B)$. The contribution of $Y = B$ to the expansion (2.2), and thus to the correlator C_X, is therefore given by

$$\frac{\dim(X)}{\dim(B)} \sum_\gamma d^{\imath\jmath,\alpha,\gamma}_{A,X,B;B,\circ}\, (c^{\rm bulk}_{B;\imath,\jmath})_{\gamma\beta} =: \frac{1}{S_{0,0}} \dim(X)\, d^{\imath\jmath,\alpha\beta}_X \tag{2.3}$$

(keeping the factor $\dim(X)/S_{0,0}$ on the right hand side will prove to be convenient).

The so defined numbers $d^{\imath\jmath,\alpha\beta}_X$ thus encode the contribution of a bulk field (in the guise of a particular disorder field) to the expansion (2.2). The presence of a *bulk* field in that expansion may be described as the effect of *transmitting* the original bulk field Φ_α from phase A to phase B through the defect X; accordingly we will refer to the numbers $d^{\imath\jmath,\alpha\beta}_X$ as *defect transmission coefficients*. Equivalently one may also think of $d^{\imath\jmath,\alpha\beta}_X$ as a particular operator product coefficient, namely for the operator product of the two bulk fields Φ_α and Φ_β in different phases separated by the defect X. Another way of interpreting the defect transmission coefficients is as the matrix elements of the action of the defect X on bulk fields [24] (i.e. for the operators that implement the shrinking of a defect line around a bulk field), and yet another way to view the situation is as a scattering of bulk fields in the background of the defect, as studied in [29]. The defect transmission coefficients also appear naturally in the expansion of the partition function on a torus with defect lines into characters, see formula (7.11) below.

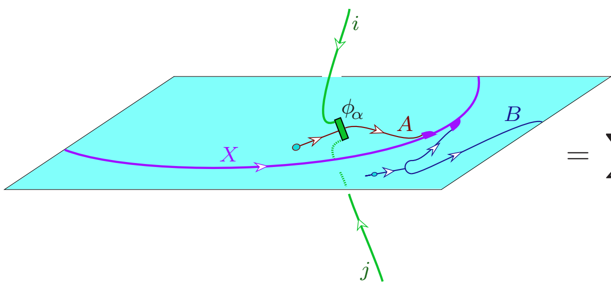
Since, as noted above, defect types are completely characterized by their action on bulk fields, the set of two-point correlators of bulk fields separated by a simple defect on a sphere, and thus the collection of defect transmission coefficients, carries essential information about simple defect types.

It is instructive to compare the defect transmission coefficients introduced above with the analogous quantities in the case of boundaries, which are the *reflection coefficients* $b^{\imath,\alpha}_M \equiv b^{\imath,\alpha;0\circ}_M$. These are the numbers which appear as the coefficient of the boundary identity field $\Psi^{MM;0}$ in the short-distance expansion

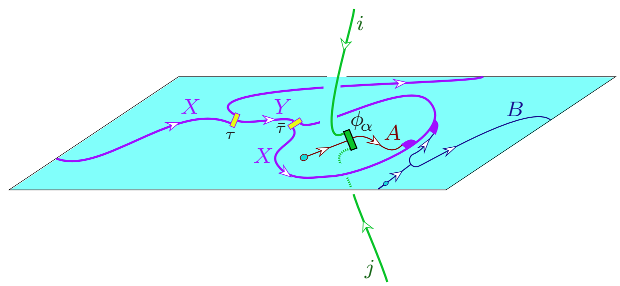
$$\Phi^{\imath\jmath}_\alpha(r\mathrm{e}^{\mathrm{i}\sigma}) \sim \sum_{j\in\mathcal{I}} \sum_\beta (r^2{-}1)^{-2\Delta_\imath+\Delta_j}\, b^{\imath,\alpha;j\beta}_M\, \Psi^{MM;j}_\beta(\mathrm{e}^{\mathrm{i}\sigma}) \qquad \text{for} \ \ r\to 1 \tag{2.4}$$

of the bulk field $\Phi^{\imath\bar\imath}_\alpha$ into boundary fields when it approaches the boundary of the (unit) disk with boundary condition M (see [30, 31, 32] or [33, Sect. 2.7]). In contrast to the expansion (2.2), the coefficients in (2.4) involve a non-trivial dependence $(r^2{-}1)^{-2\Delta_\imath+\Delta_j}$ on the position of the bulk field, in agreement with the fact that the boundary of the world sheet, unlike a topological defect, in general cannot be deformed without changing the value of a correlator.

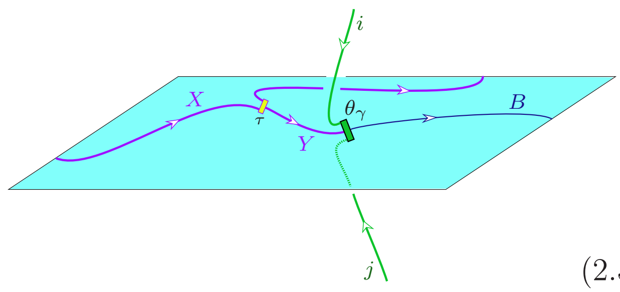
When formulating rational CFT with the help of the TFT construction, correlators are described as invariants of three-manifolds [5]. The third dimension allows for a geometric separation of left- and right-movers, whereby the two chiral labels of a bulk field Φ_α^{ij} label ribbons which run through the regions above and below the world sheet, respectively. When applied to the transmission of a bulk field through a defect as shown in (2.1), this amounts to the equalities



$$= \sum_Y \sum_\tau$$



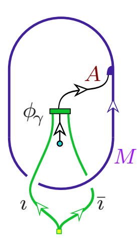
$$= \sum_{Y,\tau} \sum_\gamma \; d_{A,X,B;Y,\tau}^{ij,\alpha\gamma}$$



(2.5)

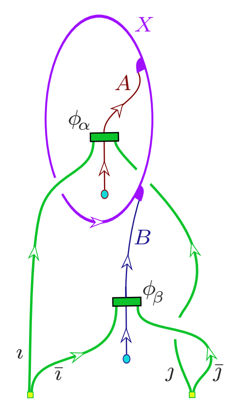
which translate into mathematical identities for corresponding correlators. It follows in particular that the defect transmission coefficients, and likewise the reflection coefficients, can be expressed as invariants of ribbon graphs in the three-sphere S^3, and thus as morphisms in $\mathrm{End}(\mathbf{1}) \cong \mathbb{C}$. One finds

$$\dim(M)\; b_M^{\imath,\gamma} \;=$$



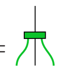
and

$$\dim(X)\; d_X^{\imath\jmath,\alpha\beta} \;=$$

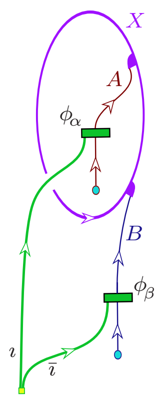


(2.6)

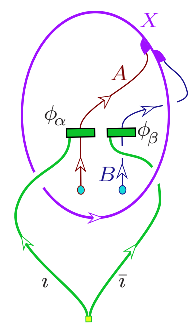
As for the notations used here and in the sequel, we follow the conventions listed in the appendix of [8]. In particular, in a rational CFT the set $\mathcal{I}$ of chiral sectors, i.e. isomorphism classes of simple objects of the representation category $\mathcal{C}$ of the chiral symmetry algebra, is finite. We denote representatives of these isomorphism classes by U_i with $i \in \mathcal{I}$, reserving the label $i = 0$ for the vacuum sector, i.e. for the tensor unit of $\mathcal{C}$, $U_0 = \mathbf{1}$. Boundary fields adjacent to phase A which change the boundary condition from M to N (both of which are left A-modules) are denoted by $\Psi_\alpha \equiv \Psi_\alpha^{MN;j}$ with $\alpha \equiv \psi_\alpha \in \mathrm{Hom}_A(M \otimes U_i, N)$ a morphism of A-modules, bulk fields in phase A by $\Phi_\alpha \equiv \Phi_\alpha^{ij}$ with $\alpha \equiv \phi_\alpha \in \mathrm{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$ a morphism of A-A-bimodules, and general defect fields changing an A-B-defect X to Y by $\Theta_\alpha \equiv \Theta_{X,Y;\alpha}^{ij}$ with $\alpha \equiv \theta_\alpha \in \mathrm{Hom}_{A|B}(U_i \otimes^+ X \otimes^- U_j, Y)$; see

appendix A.5 of [8] for more details. The object $U^\vee$ of $\mathcal{C}$ is the one dual to U, and $\bar\imath\in\mathcal{I}$ is the label such that the simple object $U_{\bar\imath}$ is isomorphic to $U_\imath^\vee$. For any $i\in\mathcal{I}$ the intertwiner space $\mathrm{Hom}(\mathbf{1},U_i\otimes U_i^\vee)$ is one-dimensional and we choose a basis Υ^i of that space. In pictures like in (2.6) the tensor unit $\mathbf{1}$ is invisible, and the basis Υ^i is depicted by a piece of ribbon graph that looks as .
The morphisms corresponding to bulk fields are depicted as $\phi_\alpha =$  . When displaying ribbon graphs in three-manifolds, we use blackboard framing, whereby ribbons are depicted as lines (with an arrow indicating the orientation of the ribbon core). For other aspects of the graphical calculus see e.g. the appendix of [34] and the references given there.

The graphical calculus will be a crucial tool in our analysis of the classifying algebra for defects. Let us give two simple applications of this calculus which are easy consequences of (2.6). First, the equality



$=$



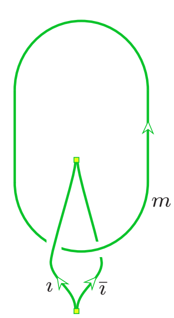
(2.7)

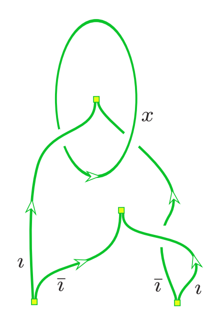
demonstrates that those defect transmission coefficients for which one of the chiral labels is 0 coincide with specific reflection coefficients, namely

$$d^{\imath 0,\alpha\beta}_{X_{A|B}} = b^{\imath,\alpha\otimes\widehat\beta}_{X_{A\otimes B^-}} \,, \tag{2.8}$$

where B^- denotes the algebra opposite to B (i.e., the one obtained from B by replacing the product m_B of B with the composition of m_B and an inverse self-braiding of B), the boundary condition $X_{A\otimes B^-}$ is the A-B-bimodule $X_{A|B}$ regarded as a left $A\otimes B^-$-module, and the morphism $\phi_{\widehat\beta}$ is obtained from ϕ_β by composition with the inverse braiding of $U_{\bar\imath}$ and B. (In contrast, for generic defect transmission coefficients a similar relation with reflection coefficients does not exist; this illustrates the limitations of the folding trick.)

Second, in the so-called Cardy case, which is the phase for which the Frobenius algebra A is (Morita equivalent to) the tensor unit $\mathbf{1}$, both the elementary boundary conditions and the simple defects are labeled by the same set $\mathcal{I}$ as the chiral sectors, whereby the reflection coefficients and defect transmission coefficients reduce to

$\dim(U_m)\, b^{\imath,\circ}_m \;=$ 

and $\dim(U_x)\, d^{\imath\bar\imath,\circ\circ}_x \;=$ 

(2.9)

With the help of the identity $\dim(U_j) = S_{j,0}/S_{0,0}$ and elementary braiding and fusing moves (see e.g. the formulas (2.45) – (2.48) of [23]), one sees that these are nothing but simple multiples of entries of the modular matrix S:

$$b_m^{\imath,\circ} = \frac{1}{\dim(U_\imath)}\,\frac{S_{\bar\imath,m}}{S_{0,m}} \qquad \text{and} \qquad d_x^{\imath\bar\imath,\circ\circ} = \frac{\theta_\imath}{\dim(U_\imath)^2}\,\frac{S_{\bar\imath,x}}{S_{0,x}} \tag{2.10}$$

($\theta_j = \exp(-2\pi\mathrm{i}\Delta_j)$ is the twist of U_j). This illustrates the multifaceted role played in rational CFT by the matrix S: Besides representing the modular transformation $\tau \mapsto -1/\tau$ on the characters of the *chiral* CFT, in the Cardy case it also gives the reflection coefficients as well as the defect transmission coefficients of the *full* CFT. [1]

3 The classifying algebra for defects – synopsis

Let us briefly present the basic ingredients of our construction. The derivation of the classifying algebra for boundary conditions employs the fact that there are two types of factorization operations by which one can relate the correlator on any world sheet without defect lines to more fundamental correlators: on the one hand, boundary factorization, which involves a cutting of the world sheet along an interval that connects two points on its boundary, and on the other hand, bulk factorization, for which one cuts along a circle in the interior of the world sheet. Comparing the two types of factorization for the correlator of two bulk fields on the disk, one arrives at a quadratic identity for the reflection coefficients $b_M^{\imath,\alpha}$. To obtain the classifying algebra for defects in a rational CFT we proceed along similar lines. We select an appropriate correlation function C_X involving a circular defect line separating phases A and B. By comparing two different factorizations of C_X we derive a quadratic identity for the defect transmission coefficients $d_X^{\imath\jmath,\alpha\beta}$.

The algebra structure on $\mathcal{D}_{A|B}$

Suppressing multiplicity labels, the quadratic identity for the defect transmission coefficients that we are going to derive in this paper reads

$$d_X^{ij}\, d_X^{kl} = \sum_{p,q} C_{pq}^{ij;kl}\, d_X^{pq} \tag{3.1}$$

with complex numbers $C_{pq}^{ij;kl}$ that do not depend on the simple defect X. It is natural to interpret the latter coefficients as the structure constants of a multiplication on the vector space

$$\mathcal{D}_{A|B} := \bigoplus_{\imath,\jmath\in\mathcal{I}} \mathrm{Hom}_{A|A}(U_\imath \otimes^+ A \otimes^- U_\jmath, A) \otimes_{\mathbb{C}} \mathrm{Hom}_{B|B}(U_{\bar\imath} \otimes^+ B \otimes^- U_{\bar\jmath}, B)\,, \tag{3.2}$$

and doing so indeed endows the space $\mathcal{D}_{A|B}$ with the structure of an associative algebra over $\mathbb{C}$. Each summand in the expression (3.2) is the space of a *pair of bulk fields*, namely a bulk field with chiral labels $\imath, \jmath$ in phase A and one with conjugate chiral labels $\bar\imath, \bar\jmath$ in phase B, and the structure constants $C_{pq}^{ij;kl}$ describe the product on $\mathcal{D}_{A|B}$ in a natural basis of such pairs of bulk fields. Further, by making use of the TFT approach to RCFT correlators, the derivation of this

[1] The equality between the topological defect operators of a full CFT and the so-called Verlinde loop operators of the corresponding chiral CFT, which holds [3] for the Cardy case, is another consequence of this versatility of S. There is thus in particular no reason to expect that this equality will survive beyond the Cardy case.

result at the same time supplies a description of the structure constants $C^{ij;kl}_{pq}$ as invariants of ribbon graphs in the three-manifold $S^2\times S^1$ and thus as traces on spaces of conformal blocks. This description allows us to establish that the product on $\mathcal{D}_{A|B}$ obtained this way is indeed associative, as well as commutative, and moreover that there is a unit, that the algebra $\mathcal{D}_{A|B}$ with this product is semisimple, and that the irreducible $\mathcal{D}_{A|B}$-representations are in bijection with the types of simple defects separating the phases A and B.

The strategy for obtaining the boundary classifying algebra (revisited)

Our procedure is in fact largely parallel to the derivation of the classifying algebra $\mathcal{B}_A$ for boundary conditions in [8]. It is therefore instructive to recapitulate the main steps of that derivation. The relevant correlation function $C=C_M$ is in this case the correlator of two bulk fields on a disk in phase A with elementary boundary condition M. Correlators of a full rational CFT are elements of an appropriate space of conformal blocks; in the TFT approach, the coefficients of a correlator in an expansion in a basis of the space of conformal blocks are expressed as invariants of ribbon graphs in a closed three-manifold; in the case of C_M, this three-manifold is the three-sphere S^3.

By boundary factorization of C_M one obtains an expression in which two additional boundary fields are inserted, and which can be written as a linear combination of products of two factors, with each factor corresponding to the correlator of one bulk field and one boundary field Ψ on a disk with boundary condition M. Moreover, as one deals with a space of conformal blocks at genus zero, this space has a distinguished subspace, corresponding to the *vacuum channel*. By restriction to this subspace one can extract the term in the linear combination for which the boundary fields Ψ are identity fields, so that one deals just with correlators $C(\Phi^{\imath\bar\imath}_\alpha;M)$ for one left-right symmetric bulk field $\Phi^{\imath\bar\imath}_\alpha$ on a disk with boundary condition M. Now a correlator of bulk fields on a disk is naturally separated in a product of a normalized correlator and of the boundary vacuum two-point function $c^{\rm bnd}_{M,0}$, which by formula (C.3) of [28] equals the (quantum) dimension of M. When doing so for the correlators $C(\Phi^{\imath\bar\imath}_\alpha;M)$, the resulting expansion coefficients $c(\Phi^{\imath\bar\imath}_\alpha;M)$ of these correlators in a standard basis of conformal two-point blocks on the sphere become precisely the reflection coefficients, i.e.

$$c(\Phi^{\imath\bar\imath}_\alpha;M)=\dim(M)\,b^{\imath,\alpha}_M. \tag{3.3}$$

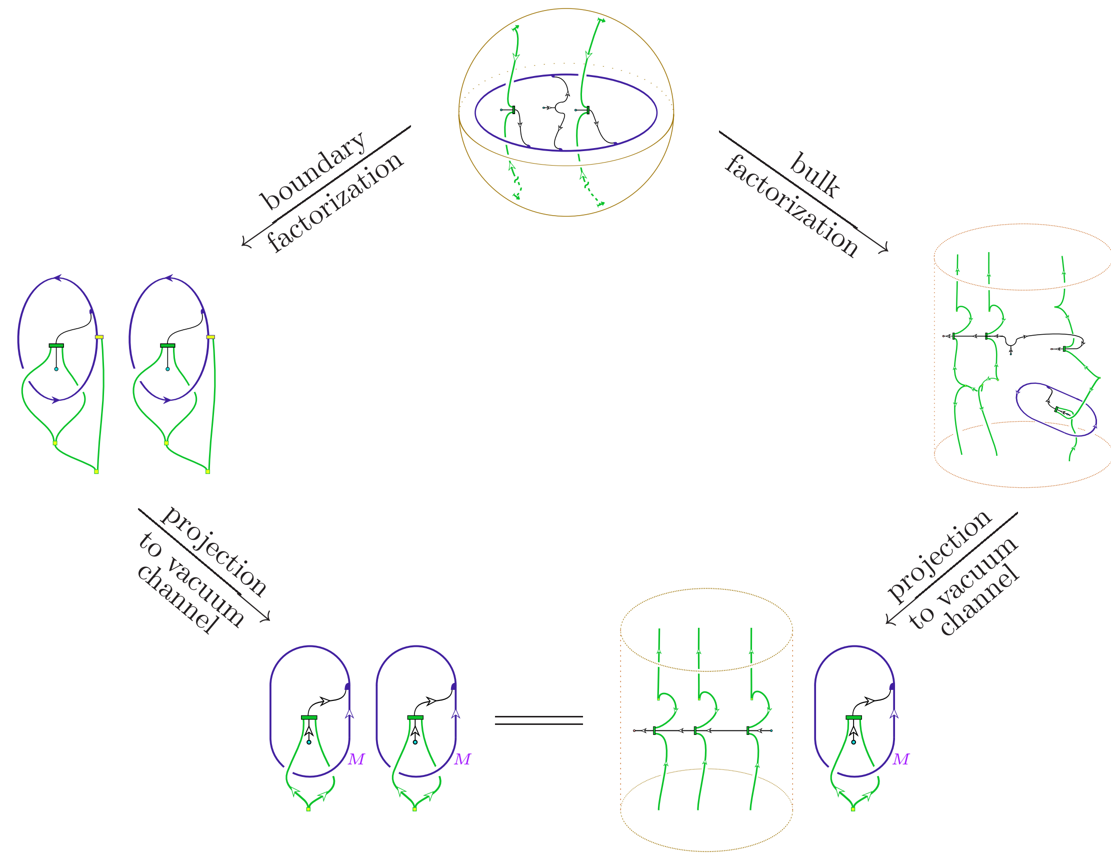
Bulk factorization of C_M yields a correlator with two additional bulk field insertions. Relating this new factorized correlator to the unfactorized one involves a surgery along a solid torus embedded in the relevant three-manifold. This amounts to a modular S-transformation (see e.g. [28, Sects. 5.2,5.3]), and accordingly now the coefficients are described by ribbon graphs in $S^2\times S^1$ rather than in the three-sphere; this implies that each such coefficient can be interpreted as the trace of an endomorphism of the space of three-point conformal blocks on S^2. After again specializing to the vacuum channel (as well as simplifying the result with the help of the *dominance* property of the category of chiral sectors of a rational CFT, i.e. the fact that any morphism between objects V and W of the category can be written as a sum of morphisms between simple subquotients of V and of W) one can separate the expression for the coefficients into the product of a reflection coefficient with the trace of an endomorphism that involves the insertion of three left-right symmetric bulk fields on S^2.

Comparing the results of boundary and bulk factorization of C_M in the vacuum channel one thus finds that the reflection coefficients $b^{\imath,\alpha}_M$ satisfy

$$b^{\imath,\alpha}_M\,b^{k,\beta}_M=\sum_{q\in\mathcal{I}}\sum_{\gamma=1}^{\mathrm{Z}_{q\bar q}}C_{\imath\alpha,k\beta}^{\;\;q\gamma}\,b^{q,\gamma}_M \tag{3.4}$$

with complex numbers $C_{\imath\alpha,k\beta}^{\;\;q\gamma}$ that do not depend on the elementary boundary condition M. The latter numbers, which are obtained as traces on spaces of conformal blocks [35], can be used to define an associative multiplication on the space $\mathcal{B}_A := \bigoplus_{\imath\in\mathcal{I}} \mathrm{Hom}_{A|A}(U_\imath \otimes^+ A \otimes^- U_{\bar\imath}, A)$ of left-right symmetric bulk fields, i.e. they play the role of the structure constants of the algebra $\mathcal{B}_A$.

The derivation of (3.4) is summarized schematically in the following diagram, where some simple prefactors and all summations are suppressed:



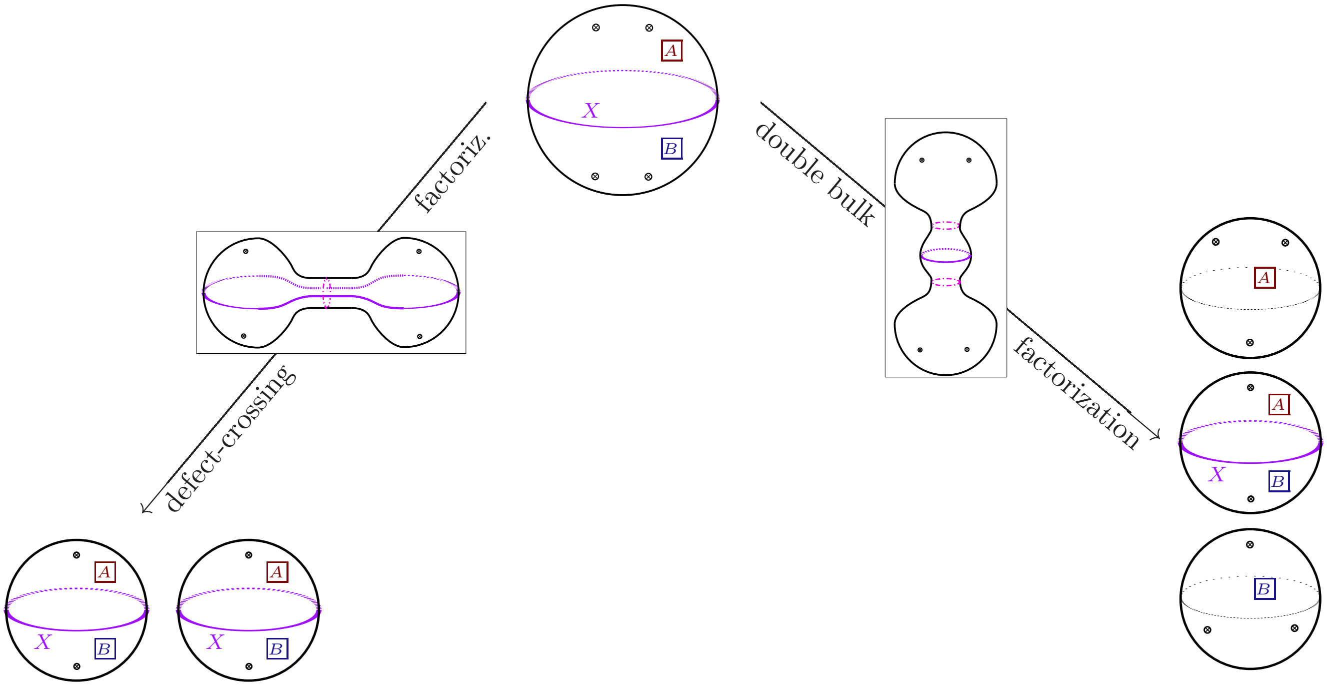
(3.5)

The strategy for obtaining the defect classifying algebra $\mathcal{D}_{A|B}$

As we will demonstrate, each of the steps in the description above has a counterpart in the derivation of the classifying algebra for defects. More specifically, we have already pointed out that just like the collection of one-point functions of bulk fields on the disk characterizes an elementary boundary condition, for characterizing a simple defect one needs the two-point functions of bulk fields on a world sheet that is a two-sphere containing a circular defect line, with the two bulk insertions on opposite sides of the defect line. Accordingly, the relevant correlator to start from is now the one for four bulk fields on a two-sphere, with a circular simple A-B-defect along the equator that separates the bulk fields into two pairs, one pair on the Northern hemisphere in phase A and the other pair on the Southern hemisphere in phase B. We label the defect line by X and denote this correlator by C_X.

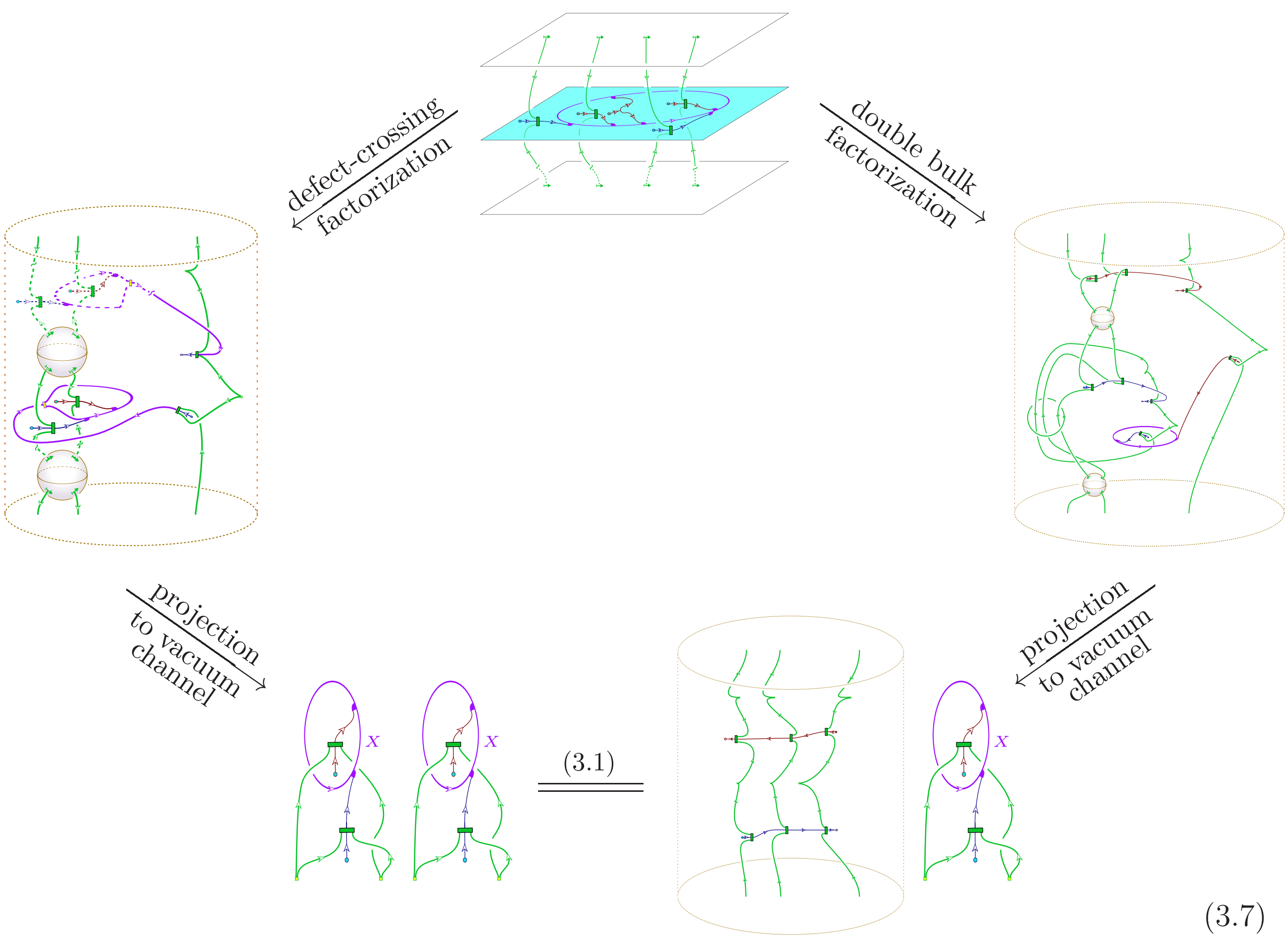
Again we consider two different factorizations of C_X; the role of bulk factorization is taken over by a *double bulk factorization*, while instead of boundary factorization we now must consider *bulk factorization across the defect line*, to which for brevity we refer to as *defect-crossing factorization*.

This is indicated schematically in the following picture:



(3.6)

More explicitly, together with the projection to the vacuum channel we proceed according to the following picture, which is the analogue of (3.5) (suppressing simple prefactors and summations):



(3.7)

When comparing this description with the corresponding picture (3.5) for the boundary case, one must bear in mind that what is depicted in the middle row of (3.5) (i.e. after implementing factorization, but before projecting to the vacuum channel) are the coefficients of the correlator in a chosen basis of four-point blocks on the sphere, whereas the middle row of (3.7) gives results for the entire correlator. Accordingly the ribbon graphs shown in the middle row of (3.7) are embedded in a manifold with boundary (namely $S^2 \times S^1$ with two four-punctured three-balls removed, see (4.7) and (5.8) for details) rather than in a closed three-manifold. We have chosen this alternative description because, unlike in the boundary case, the complexity of the ribbon graphs does not get significantly reduced when one expands the correlator in a basis (and subsequently invokes dominance).

Our task in sections 4 and 5 of this paper will be to explain the various ingredients of the picture (3.7) in appropriate detail. In section 4 we introduce the correlator C_X and perform the defect-crossing factorization, followed by projection to the vacuum channel. Section 5 is devoted to the double bulk factorization and ensuing projection to the vacuum channel, with some of the details deferred to appendices A and B. Afterwards we are in a position to compare the two factorizations, whereby we arrive, in section 6, at the precise form (6.1) of the equality in the bottom line of the picture (3.7); this allows us to define the classifying algebra $\mathcal{D}_{A|B}$ and establish its various properties.

Let us finally note that, in accordance with (2.10), in the Cardy case the classifying algebra $\mathcal{D}_{A|B}$, as well as the classifying algebra for boundary conditions, just coincides with the chiral fusion rules. In a suitable basis (corresponding to re-normalizing the defect transmission, respectively reflection, coefficients by the fractions of dimensions and twist eigenvalues that appear on the right hand side of (2.10)) the structure constants are nothing but the fusion rule multiplicities. For $\mathcal{D}_{A|B}$ this is demonstrated explicitly in (6.7).

4 Defect-crossing factorization

As already pointed out in section 2, in order to characterize a simple defect $X \equiv X_{A|B}$ separating phases A and B, one needs the correlators $C(\Phi_\alpha; X; \Phi_\beta)$ of two bulk fields $\Phi_\alpha \equiv \Phi_\alpha^{\imath\jmath\,(A)}$ and $\Phi_\beta \equiv \Phi_\beta^{kl\,(B)}$ on the Northern and Southern hemisphere which live in phases A and B, respectively, and are separated by a circular defect line labeled by X and running along the equator. Such a correlator can be non-zero only if $k = \bar\imath$ and $l = \bar\jmath$, and in that case it can be written as

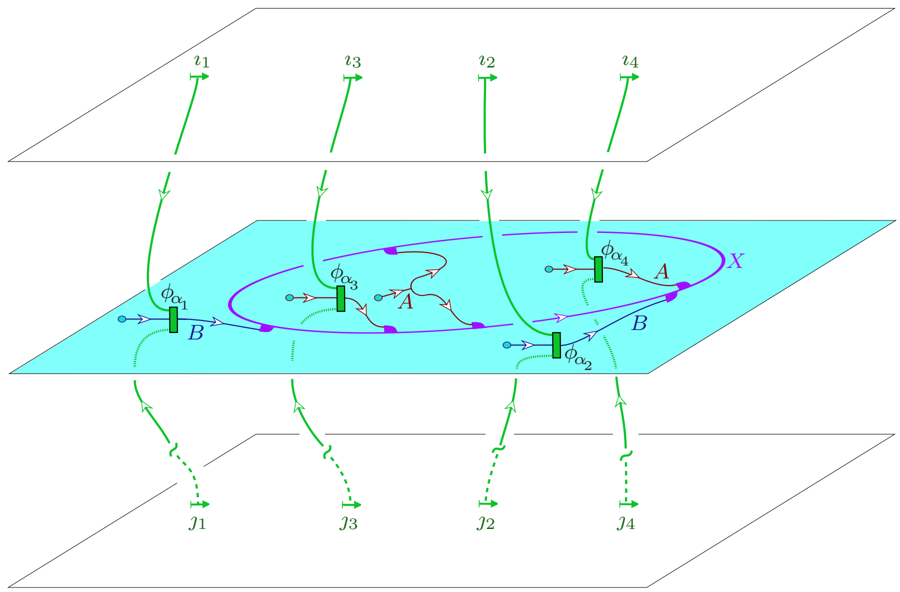
$$C(\Phi_\alpha^{\imath\jmath\,(A)}; X; \Phi_\beta^{\bar\imath\bar\jmath\,(B)}) = \frac{1}{S_{0,0}} \dim(X)\, d_X^{\imath\jmath,\alpha\beta}\, \mathcal{B}(\imath\bar\imath) \otimes \mathcal{B}^-(\jmath\bar\jmath) \tag{4.1}$$

with $\mathcal{B}(\imath\bar\imath)$ a standard basis for the (one-dimensional) space of two-point conformal blocks on a sphere with standard (outward) orientation and $\mathcal{B}^-(\jmath\bar\jmath)$ a basis for the corresponding blocks on a sphere with opposite orientation. Via the TFT construction, the numbers $d_X^{\imath\jmath,\alpha\beta}$ appearing in (4.1) are easily seen to be precisely those defined in (2.2), i.e. they are defect transmission coefficients. (This is again in analogy with the situation for boundary conditions: For the one-point functions of bulk fields on a disk with elementary boundary condition M one has an expression analogous to (4.1) with a single factor of $\mathcal{B}(\imath\bar\imath)$, and the corresponding coefficients $b_M^{\imath,\alpha}$ coincide with the expansion coefficients $b_M^{\imath,\alpha;0\circ}$ in (2.4), i.e. with the reflection coefficients.)

To gain information about properties of defect transmission coefficients with the help of factorization, the obvious starting point is the correlator C_X for four bulk fields $\Phi_{\alpha_r} \equiv \Phi_{\alpha_r}^{\imath_r\jmath_r}$ on a two-sphere, separated by the defect X into two pairs, as already described above. We label the bulk fields Φ_{α_r} such that the ones with $r=3,4$ are in phase A and those with $r=1,2$ are in phase B. The TFT construction provides an expression

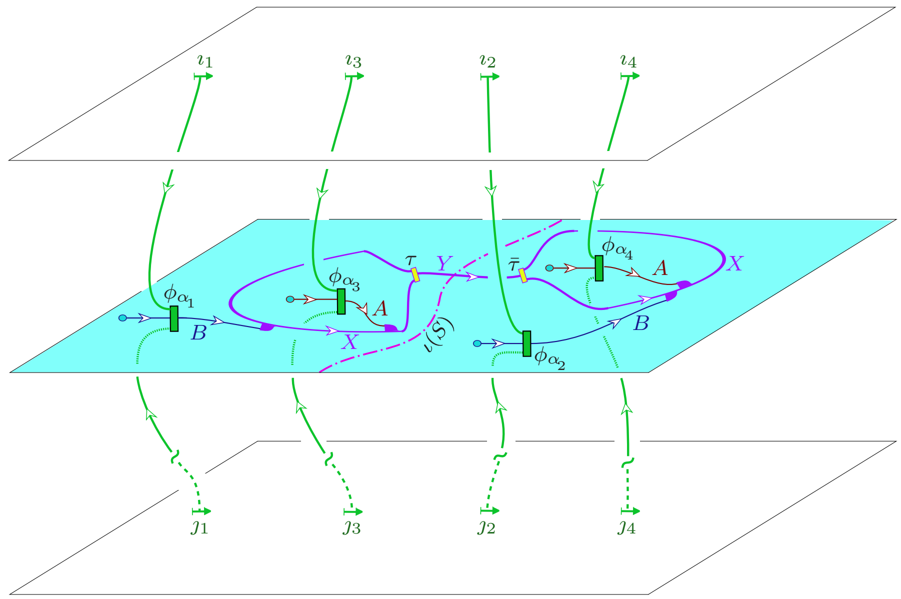
$$C_X \equiv C(\Phi_{\alpha_3}^{\imath_3\jmath_3\,(A)},\Phi_{\alpha_4}^{\imath_4\jmath_4\,(A)};X;\Phi_{\alpha_1}^{\imath_1\jmath_1\,(B)},\Phi_{\alpha_2}^{\imath_2\jmath_2\,(B)}) = Z(\mathcal{M}_{\mathrm{X}}) \tag{4.2}$$

for C_X as the invariant of the *connecting manifold* $\mathcal{M}_{\mathrm{X}}$, a three-manifold with embedded ribbon graph that is associated to X. In the case at hand, the connecting manifold $\mathcal{M}_{\mathrm{X}}$ is $S^2\times[-1,1]$ (a two-sphere times an interval) as a three-manifold, and including the ribbon graph it looks as follows [23]:

$\mathcal{M}_{\mathrm{X}}$ $=$  (4.3)

(the three accented horizontal pieces of $\mathcal{M}_{\mathrm{X}}$ are parts of the spheres $S^2\times\{-1\}$, $S^2\times\{0\}$, and $S^2\times\{1\}$, respectively).

We now want to perform a factorization of this correlator along a circle in the bulk that separates the bulk fields with labels $r=1,3$ from those with $r=2,4$. When applying this procedure directly to the correlator as displayed in (4.3), the cutting circle would intersect the defect line twice, with the defect line oriented in opposite directions and thus corresponding once to the defect line labeled by $X=X_{A|B}$ and once to a defect line labeled by the dual $X^\vee=X_{B|A}^\vee$. To circumvent the resulting complications, we take advantage of the fact that defects have a fusion structure. We deform the defect line in such a way that the relevant pieces labeled by X and $X^\vee$ are close to each other and thus form the fused defect $X^\vee\otimes_A X$, and then use dominance in the tensor category of defects to expand $X^\vee\otimes_A X$ into a direct sum of simple B-B-defects Y. Hereby (4.3) is rewritten as

$\mathcal{M}_{\rm X} \;=\; \sum_Y \sum_\tau$  (4.4)

where the Y-summation is over isomorphism classes of simple B-B-defects, while the τ-summation is over a basis of the space $\mathrm{Hom}_{B|B}(X^\vee \otimes_A X, Y)$ of B-bimodule morphisms and $\bar\tau$ is an element of the dual basis of $\mathrm{Hom}_{B|B}(Y, X^\vee \otimes_A X)$.

In (4.4) also the embedded cutting circle $\iota(S)$ is indicated, as a dashed-dotted line. Note that, in agreement with the expectation from the folding trick, the situation displayed in (4.4) bears some resemblance with the one for a boundary factorization for a doubled theory; still we deal with a genuine *bulk* factorization, albeit one for which the cutting circle crosses a defect line. Such a factorization works just like in the case without defect lines, as discussed in Section 5 of [28], up to the following modifications (details will be given elsewhere): First, some ribbons labeled by B – i.e. by the tensor unit of the category of B-B-bimodules – namely those appearing in picture (5.4) below, are exchanged with ribbons labeled by the defect Y – i.e. by another B-B-bimodule (compare (5.4) to (4.6)). Second, the bulk two-point function on the sphere gets replaced by the correlator $c^{\rm def}_{Y,pq}$ for the sphere with an Y-defect line running between two disorder fields.

More explicitly, the factorization results in the expression

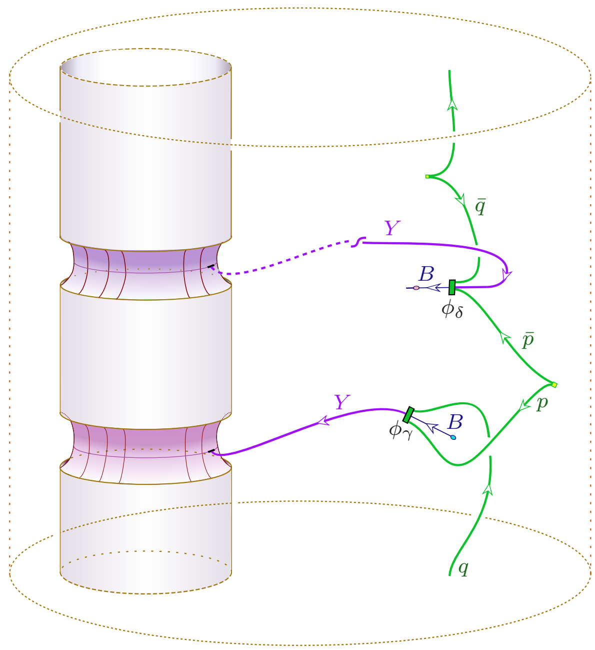
$$C_X = \sum_Y \sum_\tau \sum_{p,q\in\mathcal{I}} \sum_{\gamma,\delta} \dim(U_p)\, \dim(U_q)\, (c^{\rm def\,-1}_{Y,pq})_{\delta\gamma}\, Z(\mathcal{M}^{Y,\tau}_{pq\gamma\delta}) \tag{4.5}$$

for the correlator, with appropriate cobordisms $\mathcal{M}^{Y,\tau}_{pq\gamma\delta}$. To obtain these cobordisms one has to cut the three-manifold in (4.4) along the connecting intervals over the cutting circle $\iota(S)$, which yields the disconnected sum of two three-balls, and glue the so obtained manifold to a specific manifold $\mathcal{T}^{B,Y}_{pq\gamma\delta}$.[2] The following picture describes $\mathcal{T}^{B,Y}_{pq\gamma\delta}$ as the exterior of a solid torus embedded in the closed three-manifold $S^2\times S^1$ (with the S^1-factor running vertically, i.e. top and bottom are identified, and with the boundary of the excised solid torus oriented inwards), to which we will refer as the *factorization torus*; the sticky[2] annular parts which are to be identified with corresponding sticky

[2] Both the three-balls and $\mathcal{T}^{B,Y}_{pq\gamma\delta}$ are actually manifolds with corners. Their corners separate the annular parts along which the gluing is performed from the rest of the manifold; for brevity we refer to such annuli as the *sticky* parts of a manifold with corners. Also note that $\mathcal{T}^{B,Y}_{pq\gamma\delta}$ is the analogue of the solid torus with corners that appears in picture (3.4) of [8].

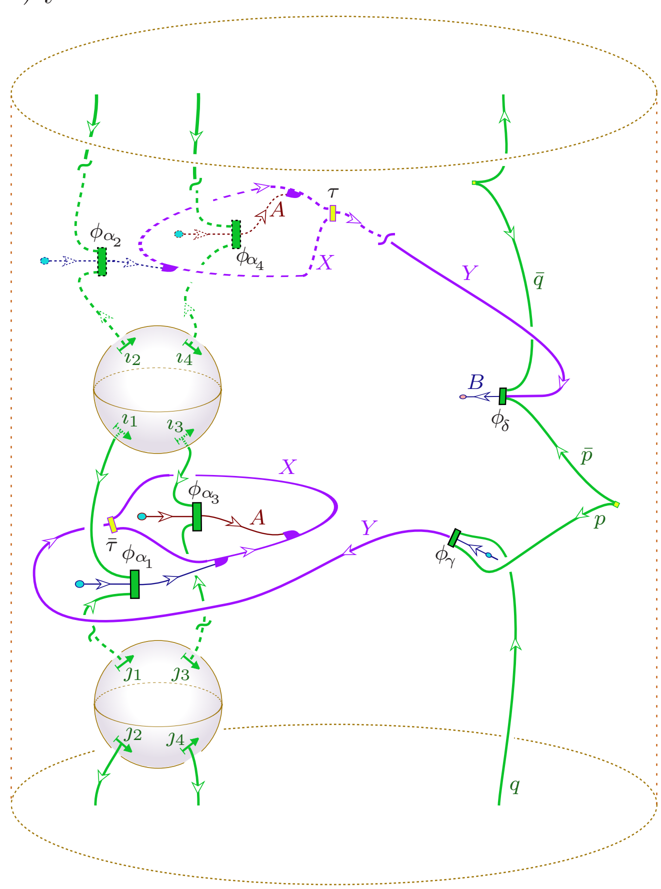
parts of the two three-balls are indicated by a darker shading:

$$\mathcal{T}^{B,Y}_{pq\gamma\delta} \;= \qquad (4.6)$$



Gluing the two three-balls to (4.6) yields

$$\mathcal{M}^{Y,\tau}_{pq\gamma\delta} \;= \qquad (4.7)$$



Note that the boundary of $\mathcal{M}^{Y,\tau}_{pq\gamma\delta}$ has two components, each of which is a a two-sphere on which four ribbons start.

To obtain an expression involving the defect transmission coefficients, we do not need the whole correlator C_X, but only its component $c_{X;0}$ in the *vacuum channel* for both left- and right-movers. C_X is an element of the vector space $B(U_{\imath_1},U_{\imath_3},U_{\imath_2},U_{\imath_4}) \otimes_{\mathbb{C}} B^-(U_{\jmath_1},U_{\jmath_3},U_{\jmath_2},U_{\jmath_4})$, with B and B^- spaces of conformal four-point blocks on the sphere with outward and inward orientation, respectively. For the component $c_{X;0}$ to be non-zero it is necessary that

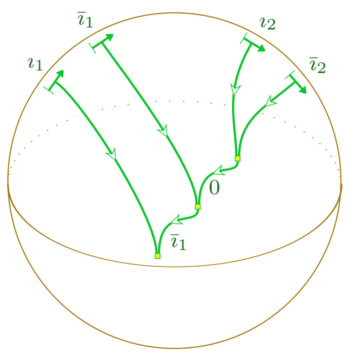
$$\imath_3 = \bar\imath_1\,, \quad \imath_4 = \bar\imath_2\,, \quad \jmath_3 = \bar\jmath_1 \quad \text{and} \quad \jmath_4 = \bar\jmath_2\,. \tag{4.8}$$

In this case the component of C_X in the vacuum channel is the projection of C_X to the one-dimensional subspace that is spanned by

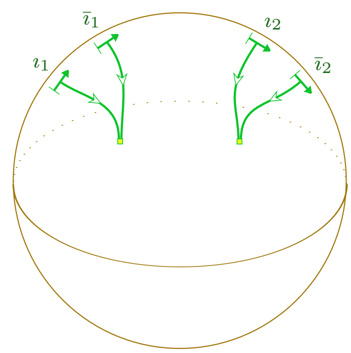
$$\mathcal{B}(\imath_1\bar\imath_1\imath_2\bar\imath_2)_0 \otimes \mathcal{B}^-(\jmath_1\bar\jmath_1\jmath_2\bar\jmath_2)_0\,, \tag{4.9}$$

where $\mathcal{B}(\imath_1\bar\imath_1\imath_2\bar\imath_2)_0$ is the basis element of $B(U_{\imath_1},U_{\bar\imath_1},U_{\imath_3},U_{\bar\imath_3})$ that corresponds to the propagation of the subobject $\mathbf{1}$ of $U_{\imath_1}\otimes U_{\bar\imath_1}$ in the intermediate channel in a chiral factorization of four-point blocks into tensor products of three-point blocks, and analogously for $\mathcal{B}^-(\jmath_1\bar\jmath_1\jmath_2\bar\jmath_2)_0$. The vector $\mathcal{B}(\imath_1\bar\imath_1\imath_2\bar\imath_2)_0$ is the value of the modular functor on the cobordism

$\widehat{\mathcal{B}}(\imath_1\bar\imath_1\imath_2\bar\imath_2)_0 \ :=$



$\equiv$



(4.10)

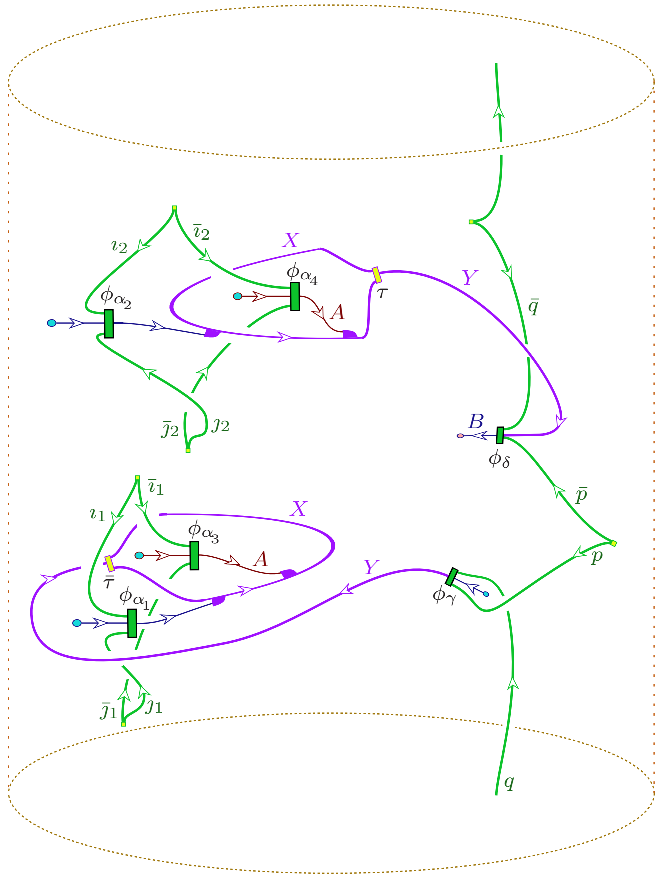
(compare the picture (2.14) of [8]). Also recall from [8] that there is a canonical projection from the space of four-point blocks to its vacuum channel subspace.

After restriction to the situation (4.8), we obtain the component $c_{X;0}$ of the correlator C_X in the vacuum channel by gluing the basis element dual to (4.9) to the cobordism (4.7). This results in the following expression for the projection to the vacuum channel:

$$\begin{aligned} c_{X;0} &\equiv c\big(\Phi_{\alpha_3}^{\bar\imath_1\bar\jmath_1\,(A)},\Phi_{\alpha_4}^{\bar\imath_2\bar\jmath_2\,(A)};X;\Phi_{\alpha_1}^{\imath_1\jmath_1\,(B)},\Phi_{\alpha_2}^{\imath_2\jmath_2\,(B)}\big)_0 \\ &= \frac{1}{S_{0,0}^2}\sum_Y\sum_\tau\sum_{p,q,\gamma,\delta}\dim(U_p)\,\dim(U_q)\,(c^{\mathrm{def}\,-1}_{Y,pq})_{\delta\gamma}\,Z(\breve{\mathcal{M}}^{Y,\tau}_{pq\gamma\delta}) \end{aligned} \tag{4.11}$$

with $\breve{\mathcal{M}}^{Y,\tau}_{pq\gamma\delta}$ the ribbon graph

$$\breve{\mathcal{M}}^{Y,\tau}_{pq\gamma\delta} \;=\; \tag{4.12}$$



in the closed three-manifold $S^2\times S^1$.

Next we note that the invariant $Z(\breve{\mathcal{M}}^{Y,\tau}_{pq\gamma\delta})$ is the trace of an endomorphism of the space of conformal one-point blocks on S^2. This space is zero unless the chiral field insertion is the identity, and hence $Z(\breve{\mathcal{M}}^{Y,\tau}_{pq\gamma\delta})$ can be non-zero only if $p=q=0$. Moreover, since Y is a simple defect and B is a simple Frobenius algebra and thereby simple as a B-B-defect as well, the spaces $\mathrm{Hom}_{B|B}(U_0\otimes^+B\otimes^-U_0,Y)\cong\mathrm{Hom}_{B|B}(B,Y)$ and $\mathrm{Hom}_{B|B}(Y,U_0\otimes^+B\otimes^-U_0)\cong\mathrm{Hom}_{B|B}(Y,B)$ of bimodule morphisms are zero unless $Y=B$, in which case they are one-dimensional with natural basis given by id_B. Furthermore, for $Y=B$, the morphism τ in (4.12) is a basis of the one-dimensional space $\mathrm{Hom}_{B|B}(X^\vee\otimes_A X,B)$, and thus τ and the dual basis morphism $\bar\tau\in\mathrm{Hom}_{B|B}(B,X^\vee\otimes_A X)$ can be written as

$$\tau=\lambda_X\,\delta_X \quad\text{and}\quad \bar\tau=\tilde\lambda_X\tilde\beta_X \qquad\text{with}\qquad \lambda_X\,\tilde\lambda_X=\dim(B)\,/\dim(X)\,, \tag{4.13}$$

where δ_X is the evaluation morphism for the right duality and $\tilde\beta_X$ the coevaluation morphism for the left duality in the category of B-bimodules. (Their precise definition and the expression for the product $\lambda_X\,\tilde\lambda_X$ are given in formulas (2.20) and (3.48), respectively, of [20]. Note that $\dim(\cdot)$ is the dimension as an object of $\mathcal{C}$, rather than as an object of the category of B-bimodules.)

This way the sum in the expression (4.11) for $c_{X;0}$ reduces to a single summand, and up to the explicit prefactors already obtained, this term is just the product of two defect transmission coefficients as given in (2.6). We thus arrive at

$$c_{X;0}=\frac{1}{S_{0,0}^2}\,\dim(B)\,\dim(X)\,(c^{\mathrm{def}\,-1}_{B,00})_{\circ\circ}\,d_X^{\bar\imath_1\bar\jmath_1,\alpha_3\alpha_1}\,d_X^{\bar\imath_2\bar\jmath_2,\alpha_4\alpha_2}. \tag{4.14}$$

Finally we note that the defect fields that are relevant to the defect field two-point function $c_{B,00}^{\rm def}$ appearing here are actually just ordinary bulk fields in phase B, so that we have $c_{B,00}^{\rm def} = S_{0,0}/\dim(B)$ by formula (C.14) of [28]. As a consequence we can rewrite (4.14) as

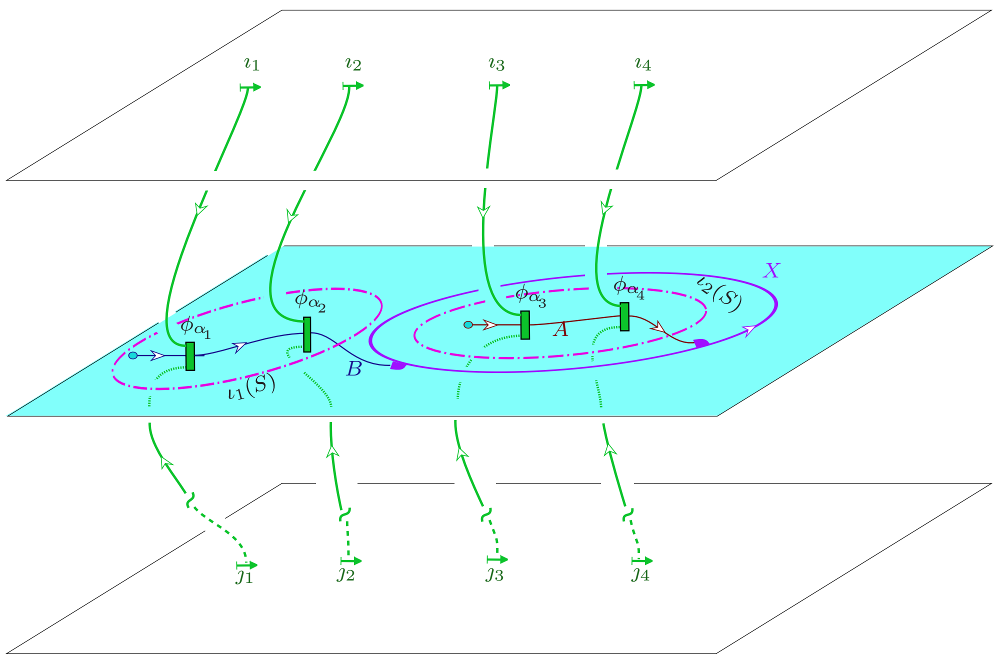
$$c(\Phi_{\alpha_3}^{\bar\imath_1\bar\jmath_1\,(A)},\Phi_{\alpha_4}^{\bar\imath_2\bar\jmath_2\,(A)};X;\Phi_{\alpha_1}^{\imath_1\jmath_1\,(B)},\Phi_{\alpha_2}^{\imath_2\jmath_2\,(B)})_0 = \frac{1}{S_{0,0}}\,\dim(X)\,d_X^{\bar\imath_1\bar\jmath_1,\alpha_3\alpha_1}\,d_X^{\bar\imath_2\bar\jmath_2,\alpha_4\alpha_2}. \tag{4.15}$$

5 Double bulk factorization

The double bulk factorization of the correlator C_X is a factorization along two circles, each of which encircles the two bulk field insertions on one of the two hemispheres. In the context of the folding trick, this corresponds to a single bulk factorization in the folded theory. We perform both factorizations simultaneously.

We indicate the two cutting circles $\iota_1(S)$ and $\iota_2(S)$ as dashed-dotted lines in the following redrawing (with a slightly different choice of dual triangulation) of the three-manifold (4.3):

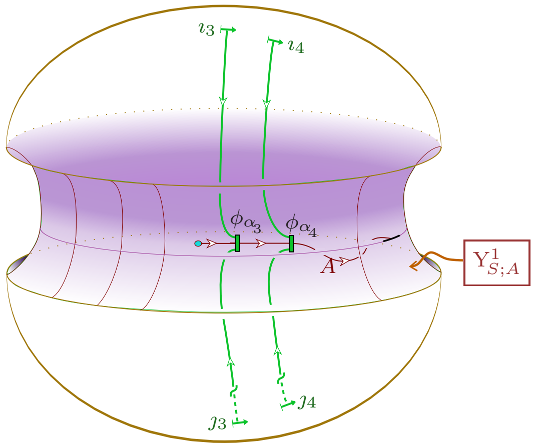
$\mathcal{M}_X \;=$



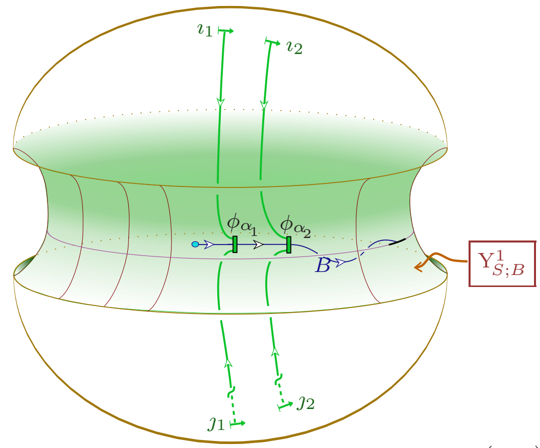
(5.1)

Cutting $\mathcal{M}_X$ along the connecting intervals over the two cutting circles $\iota_1(S)$ and $\iota_2(S)$ yields the disconnected sum of three three-manifolds with corners. Two of these, to be denoted by $\mathcal{M}^A$ and $\mathcal{M}^B$, are *nibbled apples*, i.e. three-balls with one sticky annulus (analogously as in formula (3.6) of [8]). We denote these sticky annuli by $\mathrm{Y}_{S,A}^1$ and $\mathrm{Y}_{S,B}^1$, respectively, and again indicate them in the pictures below by an accentuated shading. The three-balls $\mathcal{M}^A$ and $\mathcal{M}^B$ contain the two bulk insertions on the Northern and Southern hemisphere, respectively; we depict them as follows:

$$\mathcal{M}^{A}_{\imath_3\jmath_3,\imath_4\jmath_4} \;=$$



$$\mathcal{M}^{B}_{\imath_1\jmath_1,\imath_2\jmath_2} \;=$$

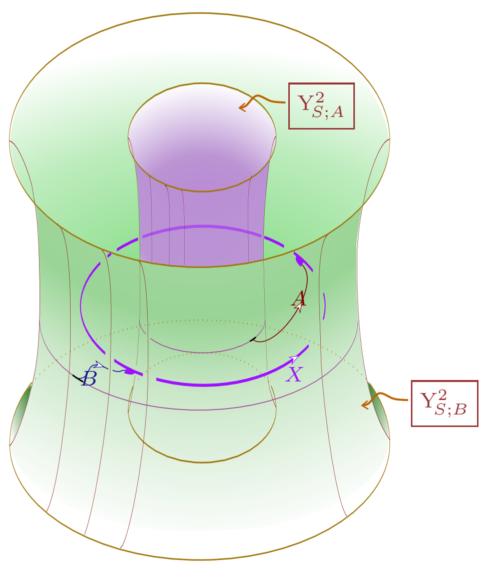


(5.2)

For each of the nibbled apples, a single phase – either A or B – is relevant.

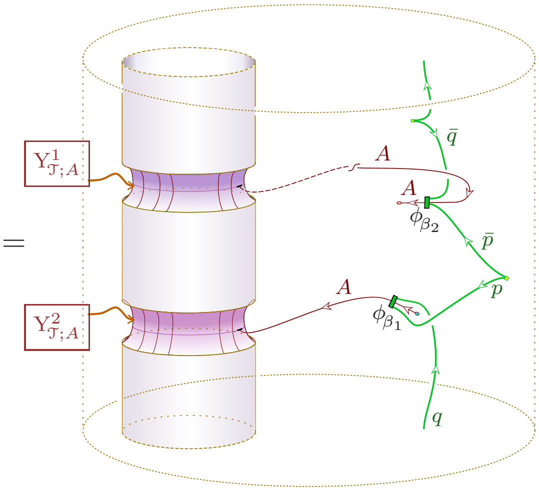
The third connected component $\mathcal{M}^{AB}$, which contains the equatorial region of the world sheet and thereby in particular the defect line, may be described as a *millstone* with two sticky annuli $\mathrm{Y}^2_{S,A}$ and $\mathrm{Y}^2_{S,B}$, one for each phase A and B:

$$\mathcal{M}^{AB}_{X} \;= \qquad (5.3)$$

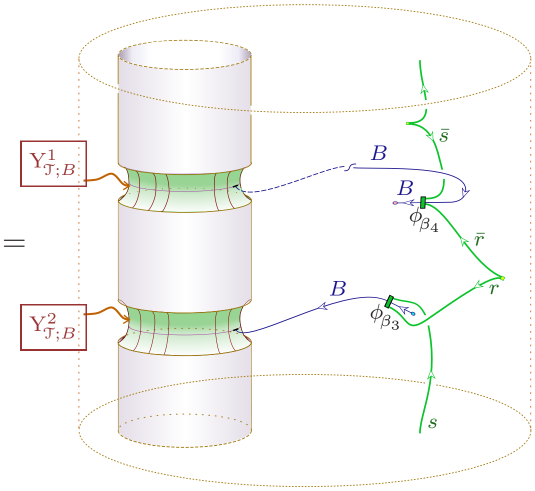


As we are performing *two* bulk factorizations, the sticky parts of the manifolds (5.2) and (5.3) are to be identified pairwise with corresponding sticky annuli $\mathrm{Y}_{\mathcal{T}}$ on *two* factorization tori $\mathcal{T}^A$ and $\mathcal{T}^B$. The latter are of the same type as the one in picture (4.6), except that the Y-ribbons are replaced by A- and B-ribbons, respectively:

$\mathcal{T}^A_{pq\beta_1\beta_2}$ =



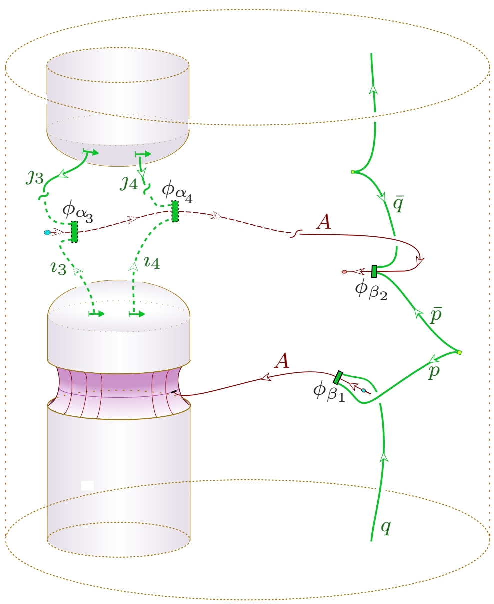
$\mathcal{T}^B_{rs\beta_3\beta_4}$ =



(5.4)

Thus altogether we must perform four identifications of pairs of sticky annuli. Three of them are straightforward. The manifold (with corners) $\mathcal{N}^{A;\,\imath_3\jmath_3,\imath_4\jmath_4}_{pq\beta_1\beta_2}$ obtained by gluing the nibbled apple $\mathcal{M}^A_{\imath_3\jmath_3,\imath_4\jmath_4}$ to the factorization torus $\mathcal{T}^A_{pq\beta_1\beta_2}$ (by identifying $\mathrm{Y}^1_{S,A}$ with $\mathrm{Y}^1_{\mathcal{T},A}$) is

$\mathcal{N}^{A;\,\imath_3\jmath_3,\imath_4\jmath_4}_{pq\beta_1\beta_2}$ =



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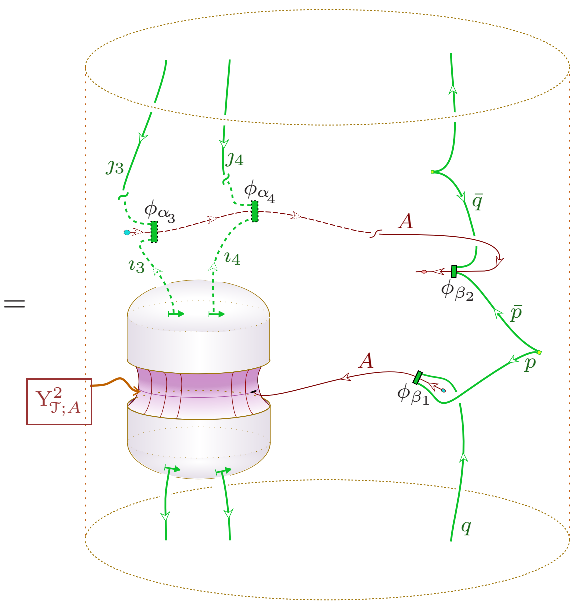
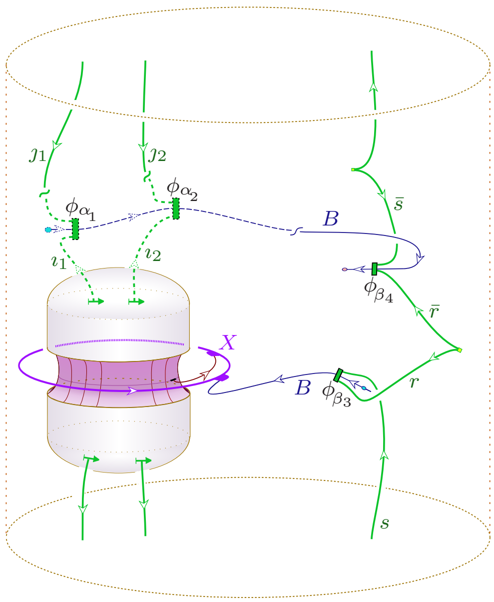


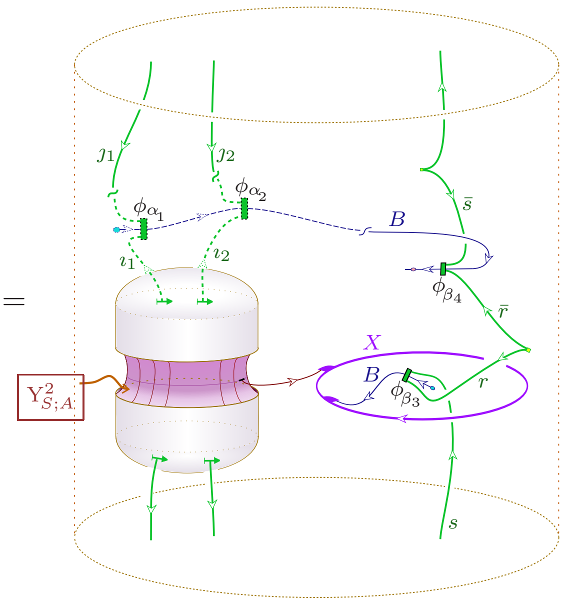
(5.5)

where the second equality is just a redrawing corresponding to a shift along the S^1-direction of $S^2 \times S^1$.

Analogously one glues the nibbled apple $\mathcal{M}^B_{\imath_1\jmath_1,\imath_3\jmath_3}$ to the factorization torus $\mathcal{T}^B_{rs\beta_3\beta_4}$ by identifying $\mathrm{Y}^1_{S,B}$ with $\mathrm{Y}^1_{\mathcal{T},B}$. And next $\mathcal{M}^{AB}_X$ is glued to $\mathcal{T}^B_{rs\beta_3\beta_4}$ by identifying the two sticky annuli $\mathrm{Y}^2_{S,B}$

with $\mathrm{Y}^2_{\mathcal{J},B}$ as well. Together these last two gluings yield

$\mathcal{N}^{B;X,\imath_1\jmath_1,\imath_2\jmath_2}_{rs\beta_3\beta_4} \;=$ 

$=$ 

(5.6)

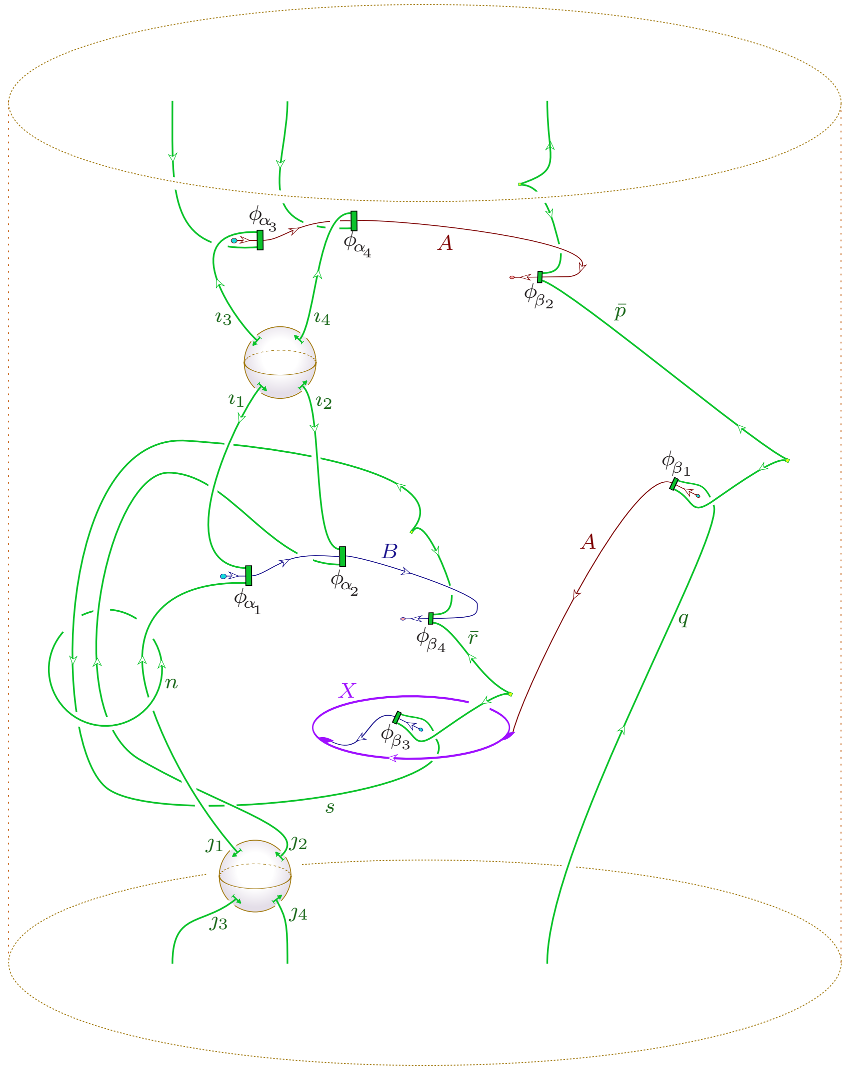
Here the first picture is obtained after performing the same deformation that led to the second picture in (5.5), while the second equality follows by deforming the X-ribbon around the horizontal S^2.

Note that as a three-manifold (with corners), both $\mathcal{N}^{A;\imath_3\jmath_3,\imath_4\jmath_4}_{pq\beta_1\beta_2}$ and $\mathcal{N}^{B;X,\imath_1\jmath_1,\imath_2\jmath_2}_{rs\beta_3\beta_4}$ is $S^2 \times S^1$ with a three-ball cut out, and with a sticky annulus on its spherical boundary. It remains to identify also the latter two sticky annuli $\mathrm{Y}^2_{S,A}$ and $\mathrm{Y}^2_{\mathcal{J},A}$ with each other. This gluing is a bit harder to describe than the previous ones. We relegate the details to appendix A and present only the result for the correlator $C_X = C(\Phi^{\imath_3\jmath_3\,(A)}_{\alpha_3}, \Phi^{\imath_4\jmath_4\,(A)}_{\alpha_4}; X; \Phi^{\imath_1\jmath_1\,(B)}_{\alpha_1}, \Phi^{\imath_2\jmath_2\,(B)}_{\alpha_2})$:

$$C_X = \sum_{p,q,r,s\in\mathcal{I}} \dim(U_p)\dim(U_q)\dim(U_r)\dim(U_s) \sum_{\beta_1,\beta_2,\beta_3,\beta_4} ({c^{\mathrm{bulk}}_{A;p,q}}^{-1})_{\beta_2\beta_1}\, ({c^{\mathrm{bulk}}_{B;r,s}}^{-1})_{\beta_4\beta_3} \sum_{n\in\mathcal{I}} S_{0,n}\, Z(\mathcal{M}^{\beta_1\beta_2\beta_3\beta_4}_{n;pqrs})\, 1\,, \tag{5.7}$$

with $c^{\mathrm{bulk}}_{A;uv}$ the matrix of coefficients of the bulk field two-point function on a sphere in phase A, and with $\mathcal{M}_{n;pqrs}^{\beta_1\beta_2\beta_3\beta_4}$ the ribbon graph

$$\mathcal{M}_{n;pqrs}^{\beta_1\beta_2\beta_3\beta_4} \quad = \tag{5.8}$$



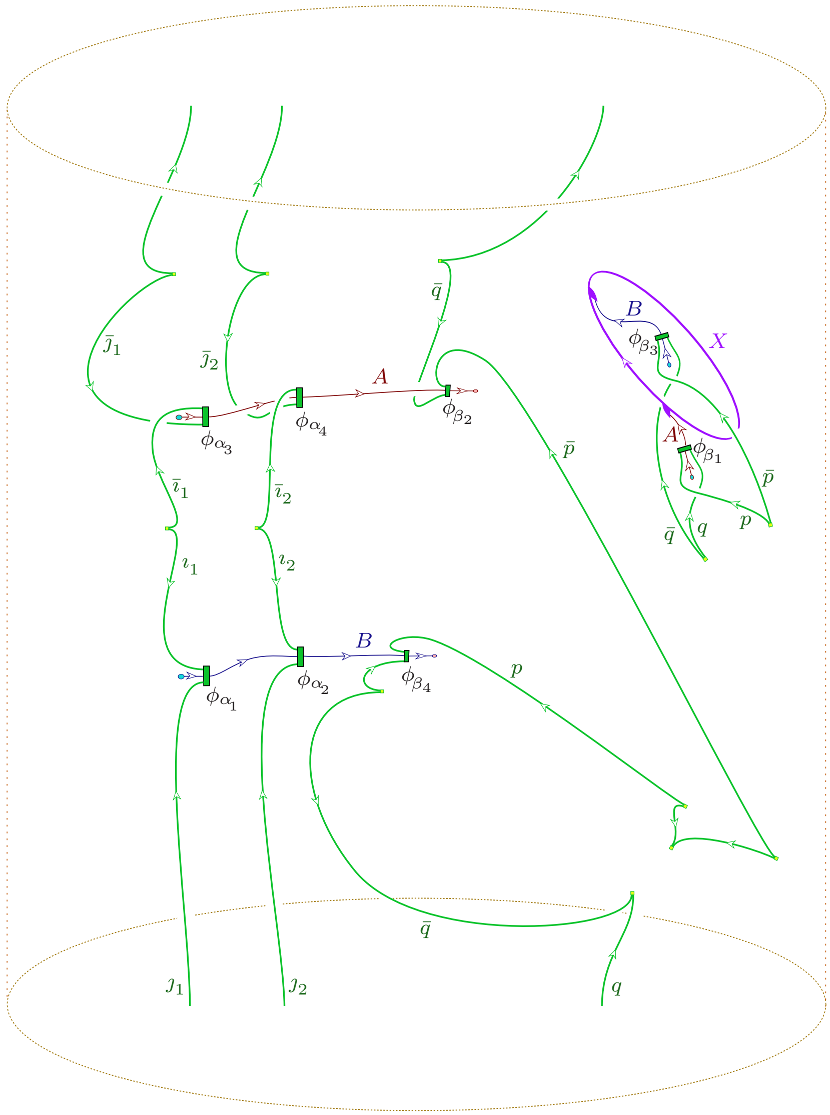
in the three-manifold that is obtained by removing two four-punctured three-balls from the closed manifold $S^2 \times S^1$.

Our next task is to project the correlator (5.7) to the vacuum channel spanned by the vector (4.9) in the space space $B(U_{\imath_1}, U_{\imath_3}, U_{\imath_2}, U_{\imath_4}) \otimes_{\mathbb{C}} B^-(U_{\jmath_1}, U_{\jmath_3}, U_{\jmath_2}, U_{\jmath_4})$ of conformal blocks. To this end we compose the cobordism (5.8) with the basis element dual to (4.9). After some straightforward manipulations of the resulting ribbon graph in $S^2 \times S^1$, the details of which are given in appendix B, this yields the coefficient $c_{X;0} = c(\Phi_{\alpha_3}^{\bar\imath_1\bar\jmath_1\,(A)}, \Phi_{\alpha_4}^{\bar\imath_2\bar\jmath_2\,(A)}; X; \Phi_{\alpha_1}^{\imath_1\jmath_1\,(B)}, \Phi_{\alpha_2}^{\imath_2\jmath_2\,(B)})_0$ of the vacuum channel as

$$c_{X;0} = \frac{1}{S_{0,0}^3} \sum_{p,q\in\mathcal{I}} \dim(U_p)^2 \dim(U_q)^2 \sum_{\beta_1,\beta_2,\beta_3,\beta_4} ({c^{\mathrm{bulk}}_{A;p,q}}^{-1})_{\beta_2\beta_1} \, ({c^{\mathrm{bulk}}_{B;\bar p,\bar q}}^{-1})_{\beta_4\beta_3} \, Z(\mathcal{M}_{0;pq\bar p\bar q}^{\beta_1\beta_2\beta_3\beta_4}) \tag{5.9}$$

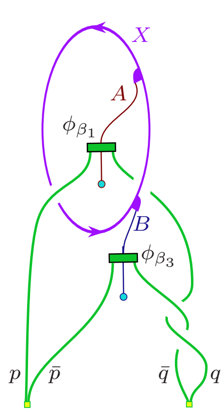
with the ribbon graph

$$\mathcal{M}^{\beta_1\beta_2\beta_3\beta_4}_{0;pq\bar{p}\bar{q}} \;=\; \tag{5.10}$$



in $S^2 \times S^1$.

When evaluating the invariant of (5.10), the connected component of the ribbon graph that contains the ribbon with the defect X just gives a scalar factor. By comparison with (2.6) this number can be written as



$$= \;\; \dim(X)\, \mathsf{R}^{-\,(\bar q\, q)0}_{\circ\,\circ}\; d_X^{pq,\beta_1\beta_3} \tag{5.11}$$

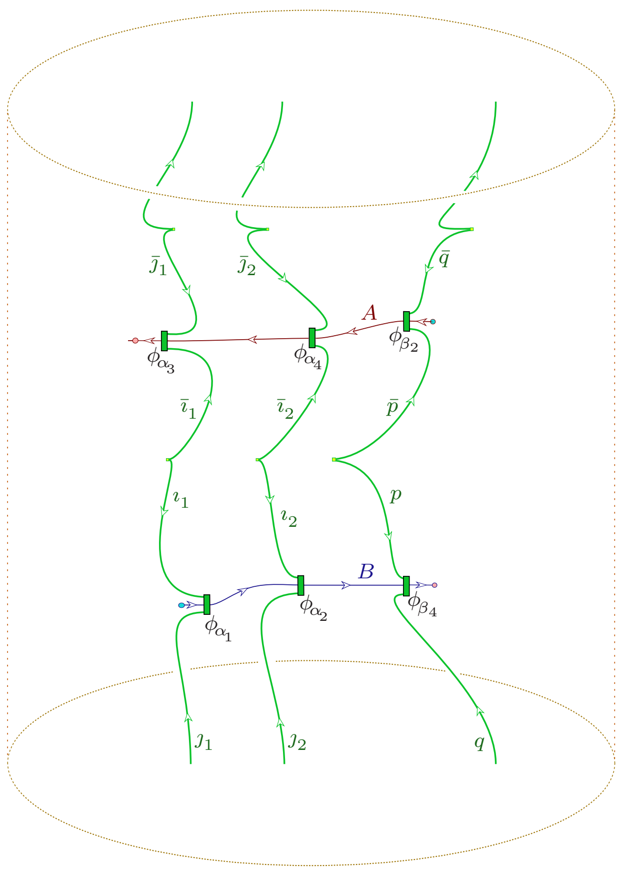
where R^- is an inverse braiding matrix (as defined e.g. in (2.42) of [23]).

The other component of the ribbon graph in (5.10) can be slightly simplified further by first rotating the pieces involving bulk fields in phase A by 180 degrees (and using the bimodule morphism properties of the bulk fields to rearrange the resulting form of the A-ribbons) and then deforming the p- and q-ribbons in such a way that in particular all the braidings involving an A-ribbon are removed. This gives rise to a twist of the $\jmath_1$- and of the $\jmath_2$-ribbon, to an inverse twist of the p-ribbon, and to double a twist as well as a self-braiding of the q-ribbon. Note that the latter braiding precisely cancels the inverse braiding from (5.11), and that the twist values θ_p and θ_q are equal, because p and q are the chiral labels of a non-zero bulk field. In addition we can use the relation between basis three-point couplings and (co)evaluation morphisms (see formulas (2.34) and (2.35) of of [23]), which amounts to factors of $\dim(U_p)^{-1}$ and $\dim(U_q)^{-1}$. We then end up with

$$c_{X;0} = \frac{1}{S_{0,0}^3} \dim(X)\, \theta_{\jmath_1}\, \theta_{\jmath_2} \sum_{p,q\in\mathcal{I}} \dim(U_p) \dim(U_q)\, \theta_q \sum_{\beta_1,\beta_2,\beta_3,\beta_4} ({c^{\rm bulk}_{A;p,q}}^{-1})_{\beta_2\beta_1}\, ({c^{\rm bulk}_{B;\bar p,\bar q}}^{-1})_{\beta_4\beta_3}\, Z(\mathcal{K}^{\alpha_3\alpha_4\beta_2;\alpha_1\alpha_2\beta_4}_{\imath_1\imath_2p;\jmath_1\jmath_2q})\, d_X^{pq,\beta_1\beta_3} \tag{5.12}$$

with the ribbon graph

$$\mathcal{K}^{\alpha_3\alpha_4\beta_2;\alpha_1\alpha_2\beta_4}_{\imath_1\imath_2p;\jmath_1\jmath_2q} \;= \tag{5.13}$$



in $S^2\times S^1$. Note that the invariants $Z(\mathcal{K}^{\alpha_3\alpha_4\beta_2;\alpha_1\alpha_2\beta_4}_{\imath_1\imath_2p;\jmath_1\jmath_2q})$ are traces on spaces of three-point conformal blocks on the sphere.

6 The classifying algebra

What we have achieved so far is to express the vacuum channel coefficient $c_{X;0}$ of the correlator (4.2) in two different ways: defect-crossing factorization leads according to (4.15) to a product of two defect transmission coefficients, while double bulk factorization yields the linear combination (5.12) of defect transmission coefficients. We now compare these two expressions for $c_{X;0}$ and thereby find that

$$d_X^{ij,\alpha\beta}\, d_X^{kl,\gamma\delta} = \sum_{p,q\in\mathcal{I}} \sum_{\mu,\nu} C_{pq,\mu\nu}^{ij,\alpha\beta;kl,\gamma\delta}\, d_X^{pq,\mu\nu} \tag{6.1}$$

for all $i,j,k,l \in \mathcal{I}$ and all $\alpha,\beta,\gamma,\delta$ in the relevant spaces of bimodule morphisms, with coefficients that do not depend on the simple defect X:

$$C_{pq,\mu\nu}^{ij,\alpha\beta;kl,\gamma\delta} = \frac{1}{S_{0,0}^2} \dim(U_p)\dim(U_q)\,\theta_j\,\theta_l\,\theta_q \sum_{\kappa,\lambda} ({c_{A;pq}^{\mathrm{bulk}}}^{-1})_{\kappa\mu}\, ({c_{B;\bar p\bar q}^{\mathrm{bulk}}}^{-1})_{\lambda\nu}\, Z(\mathcal{K}_{\bar\imath\bar kp;\bar\jmath\bar lq}^{\alpha\gamma\kappa;\beta\delta\lambda}), \tag{6.2}$$

where the cobordism $\mathcal{K}_{\bar\imath\bar kp;\bar\jmath\bar lq}^{\alpha\gamma\kappa;\beta\delta\lambda}$ is given by (5.13).

Now consider the finite-dimensional complex vector space

$$\mathcal{D}_{A|B} = \bigoplus_{p,q\in\mathcal{I}} \mathrm{Hom}_{A|A}(U_p \otimes^+ A \otimes^- U_q, A) \otimes_{\mathbb{C}} \mathrm{Hom}_{B|B}(U_{\bar p} \otimes^+ B \otimes^- U_{\bar q}, B) \tag{6.3}$$

that we introduced in (3.2). Recall that this is the space of pairs $(\phi_A^{pq}, \phi_B^{\bar p\bar q})$ of bulk fields in phase A with chiral labels p,q and in phase B with labels $\bar p, \bar q$, respectively. The dimension of the space of bulk fields ϕ_A^{ij} is given by the entry $\mathrm{Z}_{ij}(A)$ of the matrix $\mathrm{Z}(A)$ that describes the torus partition function (in phase A) in the standard basis of characters. Thus the dimension of the vector space $\mathcal{D}_{A|B}$ is

$$\dim_{\mathbb{C}}(\mathcal{D}_{A|B}) = \mathrm{tr}\,(\mathrm{Z}(A)\,\mathrm{Z}(B)^{\mathrm{t}}), \tag{6.4}$$

which according to remark 5.19(ii) of [23] equals the number of isomorphism classes of simple A-B-bimodules, i.e. the number of types of simple A-B-defects.

Choosing a basis $\{\phi_A^{ij,\alpha} \,|\, \alpha = 1,2,...,\mathrm{Z}_{ij}(A)\}$ for each space $\mathrm{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$ of bulk fields in phase A and analogously for those in phase B, a basis of $\mathcal{D}_{A|B}$ is given by

$$\{\phi^{pq,\alpha\beta}\} = \{\phi_A^{pq,\alpha} \otimes \phi_B^{\bar p\bar q,\beta} \,|\, p,q \in \mathcal{I},\ \alpha = 1,2,...,\mathrm{Z}_{pq}(A),\ \beta = 1,2,...,\mathrm{Z}_{pq}(B)\}. \tag{6.5}$$

We can define a multiplication on $\mathcal{D}_{A|B}$ by using the coefficients (6.2) as structure constants in the basis (6.5), i.e. by setting

$$\phi^{ij,\alpha\beta} \cdot \phi^{kl,\gamma\delta} := \sum_{p,q\in\mathcal{I}} \sum_{\mu,\nu} C_{pq,\mu\nu}^{ij,\alpha\beta;kl,\gamma\delta}\, \phi^{pq,\mu\nu}. \tag{6.6}$$

This product on $\mathcal{D}_{A|B}$ turns out to behave neatly: we have the following

Theorem:

(1) The complex vector space $\mathcal{D}_{A|B}$ endowed with the product (6.6) is a semisimple commutative unital associative algebra.

(2) The (one-dimensional) irreducible representations of the algebra $\mathcal{D}_{A|B}$ are in bijection with the types of simple topological defects separating the phases A and B, i.e. with the isomorphism classes of simple A-B-bimodules. Their representation matrices are furnished by the defect transmission coefficients.

Let us first have a look at the specialization of the classifying algebra $\mathcal{D}_{A|B}$ to the Cardy case. The rest of this section will then be devoted to the proof of the theorem in the general case.

The Cardy case

In the Cardy case, i.e. for $A = B$ being (Morita equivalent to) the tensor unit $\mathbf{1}$, the invariant of $\mathcal{K}_{\imath_1\imath_2 p;\jmath_1\jmath_2 q}^{\alpha_3\alpha_4\beta_2;\alpha_1\alpha_2\beta_4}$ in (5.13) is non-zero only if $\jmath_1 = \bar\imath_1$, $\jmath_2 = \bar\imath_2$ and $q = \bar p$, with all labels $\alpha_1, \dots, \beta_4$ taking only a single value $\circ$. And in this case the invariant reduces, after straightening out the ribbons, to the trace over the identity morphism of the object $U_{\imath_1} \otimes U_{\imath_2} \otimes U_p$ which, in turn, is given by the fusion rule coefficient $N_{\imath_1\imath_2}^{\ \ \ \bar p}$. The structure constants (6.2) then take the form

$$C_{p\bar p,\circ\circ}^{\imath\bar\imath,\circ\circ;\jmath\bar\jmath,\circ\circ} = \frac{\theta_\imath\,\theta_\jmath}{\theta_p}\,\frac{\dim(U_p)^2}{\dim(U_\imath)^2\dim(U_\jmath)^2}\,N_{\imath\jmath}^{\ \ p}\,. \tag{6.7}$$

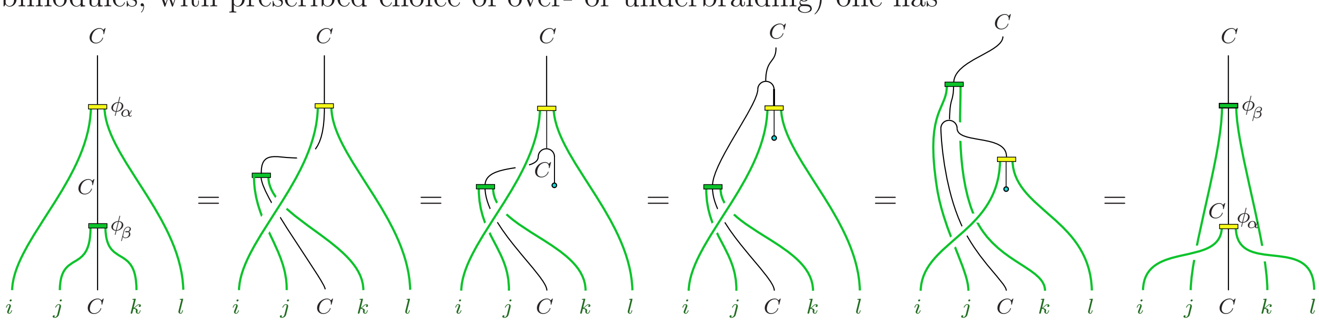
Thus we recognize $\mathcal{D}_{A|B}$ as the *fusion algebra* that describes the fusion product of the chiral sectors of the theory (or in other words, as the complexified Grothendieck ring of the category $\mathcal{C}$ of chiral sectors), albeit expressed in a non-standard basis related to the standard one by a rescaling with $\theta_\imath/\dim(U_\imath)^2$.

The first part of the theorem is now a well-known statement about fusion algebras of rational conformal field theories, while the second part reduces to the fact that the inequivalent one-dimensional irreducible representations R_x, $x \in \mathcal{I}$, of the fusion algebra are given by the generalized quantum dimensions $(S_{i,x}/S_{0,x})_{i\in\mathcal{I}}$. And indeed according to (2.10) in the Cardy case the defect transmission coefficients are nothing but rescaled generalized quantum dimensions.

Commutativity

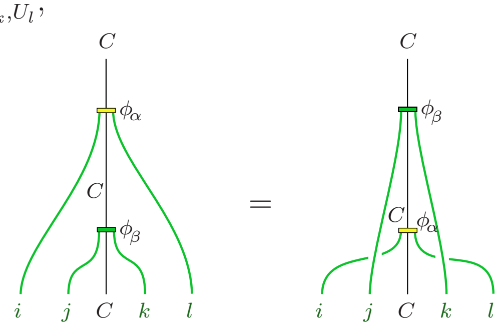
Note that the fusion algebra of defects is, in general, *not* commutative, as the tensor category of defects is not braided. Nevertheless the classifying algebra $\mathcal{D}_{A|B}$ *is* commutative.

That $\mathcal{D}_{A|B}$ is commutative and has a unit is easy to see. For commutativity, the basic observation is that by simple properties of the braiding and of bimodule morphisms (of braided-induced bimodules, with prescribed choice of over- or underbraiding) one has



(6.8)

for any $\phi_\alpha \in \mathrm{Hom}_{C|C}(U_i \otimes^+ C \otimes^- U_l, C)$ and $\phi_\beta \in \mathrm{Hom}_{C|C}(U_j \otimes^+ C \otimes^- U_k, C)$, as well as a similar identity with the over-braiding c_{U_i,U_j} replaced by the under-braiding c_{U_j,U_i}^{-1} and the under-braiding c_{U_l,U_k}^{-1} by the over-braiding c_{U_k,U_l},



(6.9)

Applying the identity (6.8) to the bulk fields Φ_{α_3} and Φ_{α_4} in the ribbon graph (5.13), and the identity (6.9) to the bulk fields Φ_{α_1} and Φ_{α_2}, one sees that the invariant of that ribbon graph is symmetric under simultaneous exchange of the corresponding quadruples of labels,

$$Z(\mathcal{K}^{\alpha\gamma\kappa;\beta\delta\lambda}_{ikp;jlq}) = Z(\mathcal{K}^{\gamma\alpha\kappa;\delta\beta\lambda}_{kip;ljq})\,. \tag{6.10}$$

Applying (6.8) and (6.9) also to other pairs of fields one shows likewise that the invariant is even *totally* symmetric in the three quadruples $(ij\alpha\beta)$, $(kl\gamma\delta)$ and $(pq\kappa\lambda)$. The symmetry (6.10) induces a symmetry of the structure constants (6.2):

$$C^{ij,\alpha\beta;kl,\gamma\delta}_{pq,\mu\nu} = C^{kl,\gamma\delta;ij,\alpha\beta}_{pq,\mu\nu}\,. \tag{6.11}$$

Thus the product (6.6) is commutative,

$$\phi^{ij,\alpha\beta}\cdot\phi^{kl,\gamma\delta} = \phi^{kl,\gamma\delta}\cdot\phi^{ij,\alpha\beta}. \tag{6.12}$$

Unitality

Using dominance one easily shows that if one of the bulk fields in each phase is an identity field, then the invariant of (5.13) is essentially a product of two two-point functions on the sphere, i.e.

$$Z(\mathcal{K}^{\beta\delta\circ;\alpha\gamma\circ}_{\imath k0;\jmath l0}) = \frac{S_{0,0}^2}{\dim(U_\imath)\dim(U_\jmath)\,\theta_\imath\,\theta_\jmath}\,\delta_{\bar\imath,k}\;\delta_{\bar\jmath,l}\,(c^{\text{bulk}}_{B;\imath,\jmath})_{\gamma\alpha}\,(c^{\text{bulk}}_{A;\bar\imath,\bar\jmath})_{\beta\delta} \tag{6.13}$$

and analogously for $\imath=\jmath=0$ and for $k=l=0$ (recall that the invariant is totally symmetric). This implies that

$$C^{00,\circ\circ;ij,\alpha\beta}_{pq,\mu\nu} = \delta_{i,p}\,\delta_{j,q}\,\delta_{\alpha,\mu}\,\delta_{\beta,\nu} = C^{ij,\alpha\beta;00,\circ\circ}_{pq,\mu\nu}\,, \tag{6.14}$$

and thus the basis element $\phi^{00,\circ\circ}$ is a unit for the product (6.6).

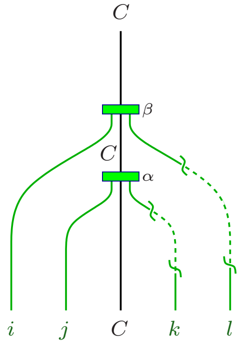
Likewise it follows that the map from $\mathcal{D}_{A|B}\otimes\mathcal{D}_{A|B}$ to $\mathbb{C}$ defined by

$$\phi^{ij,\alpha\beta}\otimes\phi^{kl,\gamma\delta}\mapsto C^{ij,\alpha\beta;kl,\gamma\delta}_{00,\circ\circ} \tag{6.15}$$

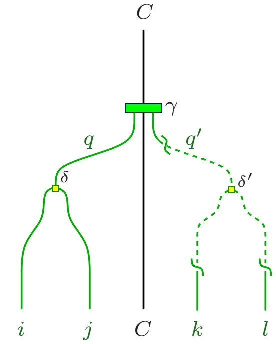
is a non-degenerate bilinear form on $\mathcal{D}_{A|B}$.

Associativity

For establishing associativity it proves to be convenient to express the structure constants (6.2) in terms of the coefficients that appear in the identity (compare section A.2 of [8])



$$= \sum_{q,q'\in\mathcal{I}}\sum_{\gamma=1}^{\mathrm{Z}_{qq'}}\sum_{\delta=1}^{N_{ij}^{\ q}}\sum_{\delta'=1}^{N_{ij}^{\ q'}} \mathsf{F}[C]^{(ijkl)}_{\alpha\beta\gamma,qq'\delta\delta'} \tag{6.16}$$

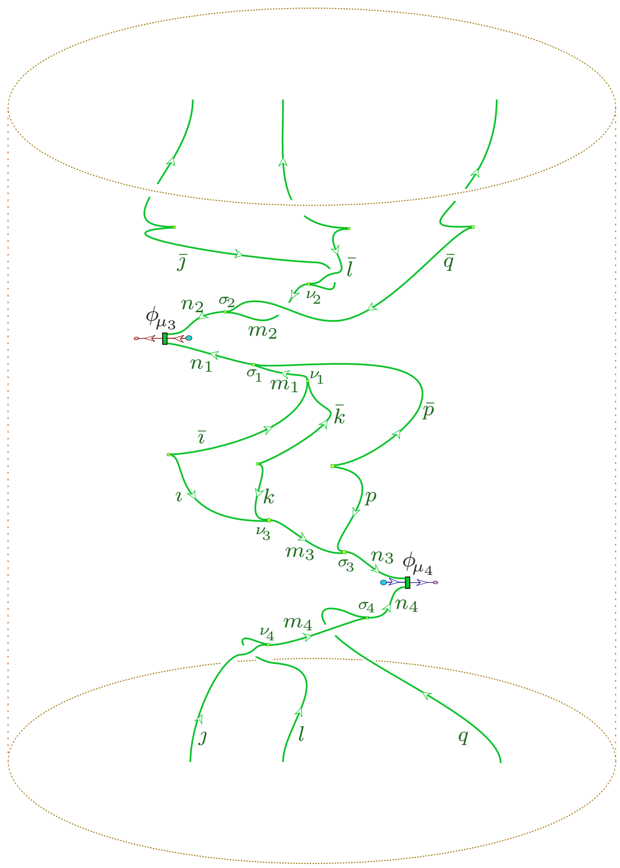


between bimodule morphisms. By a four-fold application of this identity we obtain

$$
\begin{aligned}
Z(\mathcal{K}^{\alpha\gamma\kappa;\beta\delta\lambda}_{\imath kp;\jmath lq}) = & \sum_{m_1,m_2,m_3,m_4} \sum_{\mu_1,\mu_2,\nu_1,\nu_2,\nu_3,\nu_4} \mathsf{F}[A]^{(\bar\imath\bar k\bar l\bar\jmath)}_{\gamma\alpha\mu_1,m_1m_2\nu_1\nu_2}\, \mathsf{F}[B]^{(k\imath\jmath l)}_{\beta\delta\mu_2,m_3m_4\nu_3\nu_4} \\
& \sum_{n_1,n_2,n_3,n_4} \sum_{\mu_3,\mu_4,\sigma_1,\sigma_2,\sigma_3,\sigma_4} \mathsf{F}[A]^{(m_1\bar p\bar q m_2)}_{\kappa\mu_1\mu_3,n_1n_2\sigma_1\sigma_2}\, \mathsf{F}[B]^{(pm_3m_4q)}_{\mu_2\lambda\mu_4,n_3n_4\sigma_3\sigma_4}\, Z(\mathcal{L}^{\alpha\gamma\kappa;\beta\delta\lambda}_{\imath kp;\jmath lq})
\end{aligned}
\tag{6.17}
$$

with

$$\mathcal{L}^{\alpha\gamma\kappa;\beta\delta\lambda}_{\imath kp;\jmath lq} \quad = \tag{6.18}$$



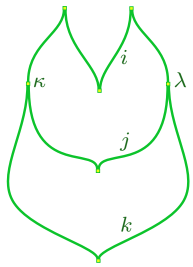
Since the space of one-point blocks with insertion U_i on the sphere vanishes unless $i=0$, only $n_1=n_2=n_3=n_4=0$ gives a non-zero contribution in the summations in (6.17). In the resulting expression we use the identities

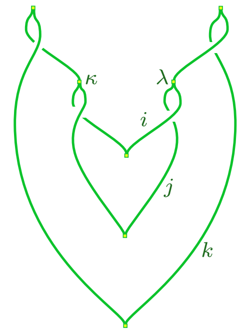
$$S_{0,0}\,(c^{\text{bulk}}_{C;pq})_{\beta\gamma} = \dim(C)\,\mathsf{F}[C]^{(p\bar p\bar q q)}_{\gamma\beta\circ,00\circ\circ} = S_{0,0}\,\mathsf{R}^{(p\,\bar p)0}_{\circ\,\circ}\,\mathsf{R}^{-\,(q\,\bar q)0}_{\circ\,\circ}\,(c^{\text{bulk}}_{C;\bar p\bar q})_{\gamma\beta} \tag{6.19}$$

which follow by performing some elementary fusing and braiding moves on the result (C.14) of [28] for the bulk two-point function; they allow us to cancel the factors ${c^{\text{bulk}}_{A;pq}}^{-1}$ and ${c^{\text{bulk}}_{B;\bar p\bar q}}^{-1}$ in (6.2). Doing so, further summations over morphism spaces become trivial. We then end up with the expression

$$C^{n\jmath,\alpha\beta;kl,\gamma\delta}_{pq,\mu\nu} = \dim(U_p)\dim(U_q)\,\theta_\jmath\,\theta_l\,\theta_q\,\mathsf{R}^{(p\,\bar p)0}_{\circ\,\circ}\,\mathsf{R}^{-\,(q\,\bar q)0}_{\circ\,\circ} \sum_{\kappa,\lambda,\rho,\sigma} \mathsf{F}[A]^{(\imath kl\jmath)}_{\gamma\alpha\mu,pq\kappa\lambda}\,\mathsf{F}[B]^{(\bar k\bar\imath\bar\jmath\bar l)}_{\beta\delta\nu,\bar p\bar q\rho\sigma}\,\omega^{p\bar k\bar\imath}_{\kappa\rho}\,\varpi^{\bar\jmath\bar l q}_{\lambda\sigma} \tag{6.20}$$

for the structure constants, where we introduced the morphisms

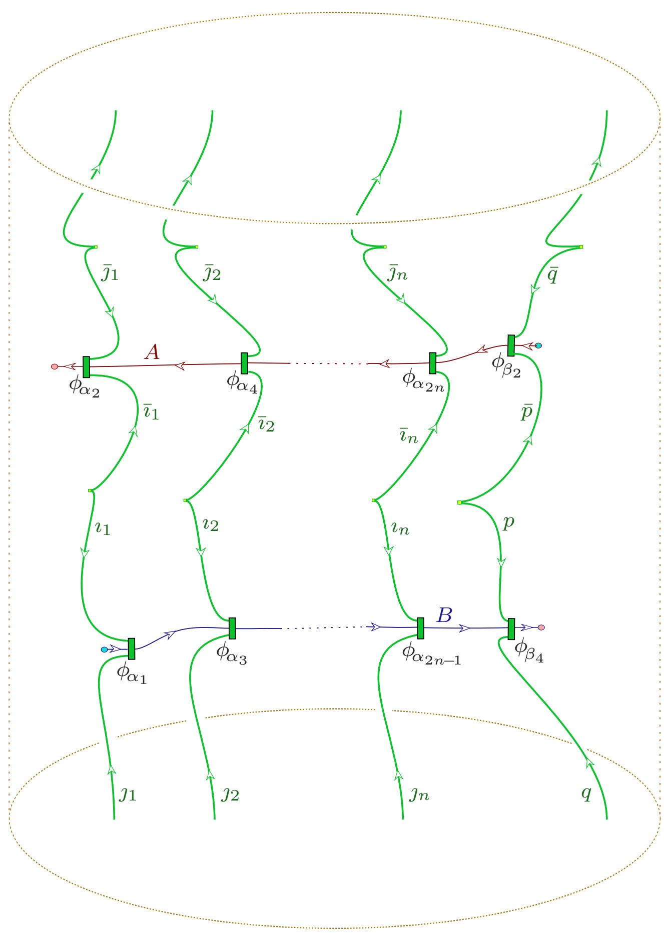
$$\omega^{ijk}_{\kappa\lambda} \;:=$$ 

and $$\varpi^{ijk}_{\kappa\lambda} \;:=$$ 

(6.21)

in $\mathrm{End}(\mathbf{1}) \cong \mathbb{C}$.

Next we observe that in the same way as we obtained an ordinary binary product on $\mathcal{D}_{A|B}$, we can endow $\mathcal{D}_{A|B}$ also with an n-ary product for any integer $n \geq 2$. To define the structure constants of these products in the basis (6.5), we use the same expression as in (6.2), with the only modification that the ribbon graph (5.13) is replaced by another ribbon graph $\mathcal{K}^{\alpha_2\alpha_4\ldots\alpha_{2n}\beta_2;\alpha_1\alpha_3\ldots\alpha_{2n-1}\beta_4}_{\imath_1\imath_2\ldots\imath_n p;\jmath_1\jmath_2\ldots\jmath_n q}$ in $S^2 \times S^1$ in which for each additional basis element in the argument of the product there is an extra ribbon along the S^1-direction and two extra bimodule morphisms:

$$\mathcal{K}^{\alpha_2\ldots\alpha_{2n}\beta_2;\alpha_1\ldots\alpha_{2n-1}\beta_4}_{\imath_1\ldots\imath_n p;\jmath_1\ldots\jmath_n q} \;=$$ 

(6.22)

By analogous arguments as for the binary product one checks that all the n-ary products are totally commutative, i.e. invariant under any permutation of their arguments, and one obtains expressions for the structure constants that generalize those for the binary product in an obvious manner. In particular, the structure constants of the ternary product can be written as

$$\begin{aligned} C_{pq,\mu\nu}^{\imath\jmath,\alpha\beta;kl,\gamma\delta;mn\kappa\lambda} = & \frac{1}{S_{0,0}} \dim(U_p) \dim(U_q)\, \theta_\jmath\, \theta_l\, \theta_n\, \theta_q\, \mathsf{R}^{(p\,\bar p)0}_{\circ\circ}\, \mathsf{R}^{-\,(q\,\bar q)0}_{\circ\circ} \\ & \sum_{r,s\in\mathcal{I}} \sum_{\rho_1,\rho_2,\sigma_1,\sigma_2,\sigma_3,\sigma_4,\tau_1,\tau_2,\tau_3,\tau_4} \dim(U_r) \dim(U_s)\, (\theta_s)^2\, \mathsf{R}^{-\,(\bar r\, r)0}_{\circ\circ}\, \mathsf{R}^{(\bar s\, s)0}_{\circ\circ} \\ & \mathsf{F}[A]^{(\imath k l \jmath)}_{\gamma\alpha\rho_1,\bar r\bar s\sigma_1\sigma_2}\, \mathsf{F}[B]^{(\bar k \bar\imath \bar\jmath \bar l)}_{\beta\delta\rho_2,rs\sigma_3\sigma_4}\, \mathsf{F}[A]^{(\bar r m n \bar s)}_{\lambda\rho_1\mu,pq\tau_1\tau_2}\, \mathsf{F}[B]^{(\bar m r s \bar n)}_{\rho_2\lambda\nu,\bar p\bar q\tau_3\tau_4}\, \omega^{\bar r\bar k\bar\imath}_{\sigma_1\sigma_3}\, \varpi^{\bar\jmath\bar l\bar s}_{\sigma_2\sigma_4}\, \omega^{p\bar m r}_{\tau_1\tau_3}\, \varpi^{s\bar n q}_{\tau_2\tau_4} \end{aligned} \tag{6.23}$$

with the factors of ω and ϖ as defined in (6.21).

By comparison with the expression (6.20) for the structure constants of the binary product it is the easy to see (using also the identity $\mathsf{R}^{-\,(u\,\bar u)0}_{\circ\circ} = \theta_u^2\, \mathsf{R}^{(u\,\bar u)0}_{\circ\circ}$) that the structure constants of the ternary product can be expressed through those of the binary one as

$$C_{pq,\mu\nu}^{\imath\jmath,\alpha\beta;kl,\gamma\delta;mn\kappa\lambda} = \sum_{r,s\in\mathcal{I}} \sum_{\rho,\sigma} C_{rs,\rho\sigma}^{\imath\jmath,\alpha\beta;kl,\gamma\delta}\, C_{pq,\mu\nu}^{rs,\rho\sigma;mn,\kappa\lambda}. \tag{6.24}$$

This proves associativity: it is an elementary fact that a commutative algebra endowed with a totally commutative ternary product is associative if at least one bracketing of a twofold binary product equals the ternary product.

Semisimplicity

Obviously, the formula (6.1) expresses the fact that the defect transmission coefficients for a given simple defect X constitute the (1×1) representation matrices for a one-dimensional irreducible $\mathcal{D}_{A|B}$-representation.

We now invoke theorem 4.2 of [20], according to which the $\dim_{\mathbb{C}}(\mathcal{D}_{A|B}) \times \dim_{\mathbb{C}}(\mathcal{D}_{A|B})$-matrix furnished by the defect transmission coefficients (with rows and columns labeled by simple A-B-bimodules and by pairs of bulk fields, respectively) is non-degenerate. This means that non-isomorphic simple bimodules give rise to inequivalent one-dimensional representations of $\mathcal{D}_{A|B}$ and thus implies that the number $n_{\mathrm{simp}}(\mathcal{D}_{A|B})$ of inequivalent irreducible representations of $\mathcal{D}_{A|B}$ is at least as large as its dimension,

$$n_{\mathrm{simp}}(\mathcal{D}_{A|B}) \geq \dim_{\mathbb{C}}(\mathcal{D}_{A|B})\,. \tag{6.25}$$

On the other hand, as any finite-dimensional associative algebra, $\mathcal{D}_{A|B}$ is isomorphic, as a module over itself, to the direct sum over all inequivalent indecomposable projective $\mathcal{D}_{A|B}$-modules, each one occurring with a multiplicity given by the dimension of the corresponding irreducible module (see e.g. Satz G.10 of [36]), so that in particular

$$\dim_{\mathbb{C}}(\mathcal{D}_{A|B}) \geq n_{\mathrm{simp}}(\mathcal{D}_{A|B})\,. \tag{6.26}$$

Thus in fact the number $n_{\mathrm{simp}}(\mathcal{D}_{A|B})$ of inequivalent irreducible representations equals the dimension $\dim_{\mathbb{C}}(\mathcal{D}_{A|B})$. This is only possible if every irreducible representation is one-dimensional and projective, which in turn implies that $\mathcal{D}_{A|B}$ is semisimple.

Representation matrices

Besides semisimplicity the previous arguments show at the same time that the $\dim_{\mathbb{C}}(\mathcal{D}_{A|B})$ irreducible modules obtained this way already exhaust all irreducible modules of $\mathcal{D}_{A|B}$. Together with the fact that $\dim_{\mathbb{C}}(\mathcal{D}_{A|B})$ equals the number of isomorphism classes of simple A-B-bimodules (see (6.4)), this establishes part (2) of the theorem.

7 The defect partition function

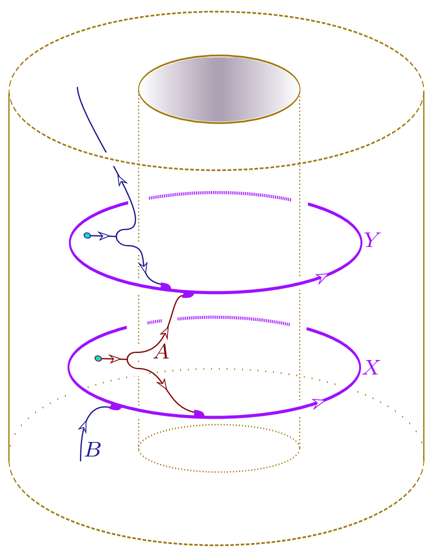
In a rational CFT the classifying algebra $\mathcal{B}_A$ for boundary conditions governs in particular also the annulus partition functions. Denote by $\mathrm{A}_{k,M}^{\ \ N}$ the expansion coefficients, in the basis of characters, of the partition function for an annulus in phase A with elementary boundary conditions M and N on its two boundary circles. One finds that the annulus coefficients $\mathrm{A}_{k,M}^{\ \ N}$ can be naturally expressed in terms of products of reflection coefficients [6,37], i.e. of irreducible $\mathcal{B}_A$-representations. Specifically [38],

$$\mathrm{A}_{k,M}^{\ \ N} = \dim(M)\dim(N) \sum_{q\in\mathcal{I}} S_{k,q}\,\theta_q \sum_{\gamma,\delta=1}^{\mathrm{Z}_{q\bar q}} ({c^{\mathrm{bulk}}_{A;q\bar q}}^{-1})_{\delta\gamma}\, b_N^{q,\gamma}\, b_M^{\bar q,\delta}. \tag{7.1}$$

The purpose of this section is to demonstrate that information about the partition functions on a torus with parallel defect lines is encoded in an analogous manner in the defect transmission coefficients, and hence in the representation theory of the classifying algebra $\mathcal{D}_{A|B}$ of defect lines.

Consider the partition function $Z_{\mathrm{T}}^{X|Y}$ of a torus T with two circular defect lines labeled by an A-B-defect X and a B-A-defect Y and running parallel to a non-contractible cycle that represents a basis element of the first homology of T. Such partition functions were introduced in [16], where they are termed 'generalized twisted partition functions'. In the framework of the TFT construction the partition functions $Z_{\mathrm{T}}^{X|Y}$ were already studied in [23, Sect. 5.10]; in that framework $Z_{\mathrm{T}}^{X|Y}$ is described as the invariant of the ribbon graph

$$\mathcal{M}_{\mathrm{T}}^{X|Y} \;=\; \tag{7.2}$$

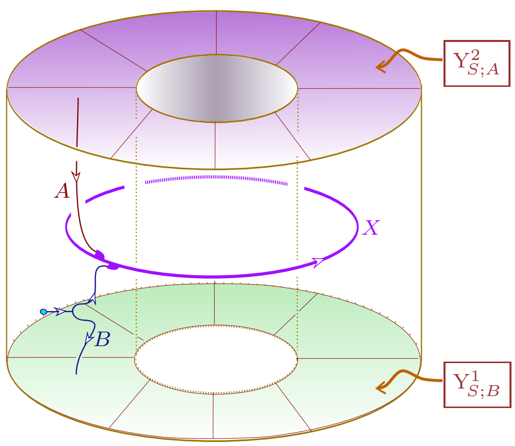


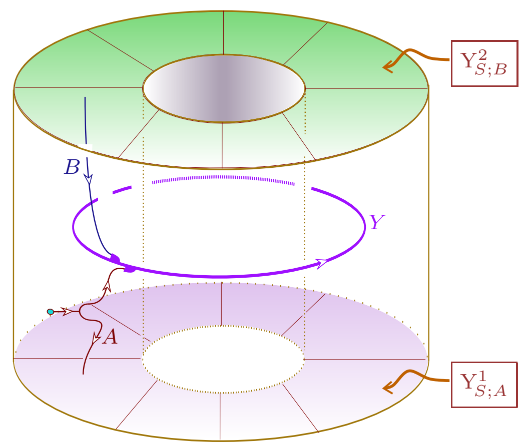
in the connecting three-manifold for the torus, i.e. in $(S^1\times S^1)\times[-1,1]$, which is drawn here as a horizontal annulus $S^1\times[-1,1]$ times a vertically running circle (thus top and bottom of the picture are to be identified).

Our aim is to express $Z_{\mathrm{T}}^{X|Y}$ in terms of the defect transmission coefficients for the two defects X and Y. To this end we perform a double bulk factorization of (7.2), with the two cutting circles running parallel to the defects lines (and thus lying in horizontal planes in the picture (7.2)) in such a way that each of the two full tori (with corners) that result from the cutting contains one of the defect lines. This way we obtain a description of $Z_{\mathrm{T}}^{X|Y}$ as a sum

$$Z_{\mathrm{T}}^{X|Y} = \sum_{p,q,r,s\in\mathcal{I}} \dim(U_p)\dim(U_q)\dim(U_r)\dim(U_s) \\ \sum_{\beta_1,\beta_2,\beta_3,\beta_4} ({c_{A;p,q}^{\mathrm{bulk}}}^{-1})_{\beta_2\beta_1} ({c_{B;r,s}^{\mathrm{bulk}}}^{-1})_{\beta_4\beta_3} Z(\mathcal{M}^{\mathrm{T},XY}_{pq\beta_1\beta_2,rs\beta_3\beta_4})\,1\,, \qquad (7.3)$$

where in each summand $\mathcal{M}^{\mathrm{T},XY}_{pq\beta_1\beta_2,rs\beta_3\beta_4}$ is a three-manifold with embedded ribbon graph that can be described as follows. $\mathcal{M}^{\mathrm{T},XY}_{pq\beta_1\beta_2,rs\beta_3\beta_4}$ is obtained from four pieces $\mathcal{M}_{\mathrm{T}}^X$, $\mathcal{M}_{\mathrm{T}}^Y$, $\mathcal{T}_{\mathrm{T}}^A$ and $\mathcal{T}_{\mathrm{T}}^B$ (which are three-manifolds with corners) by pairwise identification of sticky annuli that are located on their boundaries, in much the same way as the three-manifold (5.1) can be obtained by pairwise identification of sticky annuli on the pieces (5.2), (5.3) and (5.4). Indeed, two of the four pieces – $\mathcal{T}_{\mathrm{T}}^A$ and $\mathcal{T}_{\mathrm{T}}^B$ – are precisely given by full tori with corners of the form displayed in (5.4), while the other two pieces, which are factorization tori (and thus full tori with corners as well), look as

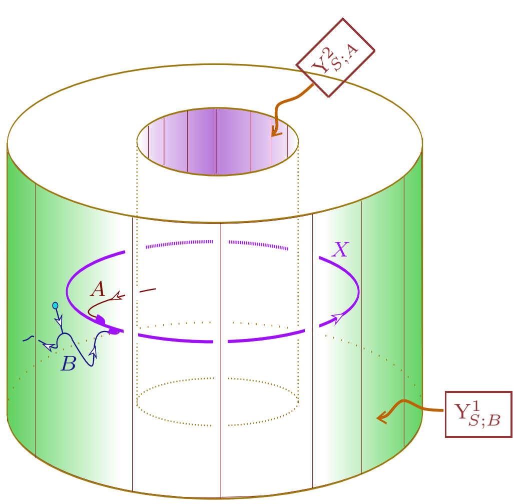
$\mathcal{M}_{\mathrm{T}}^X \;=$ 

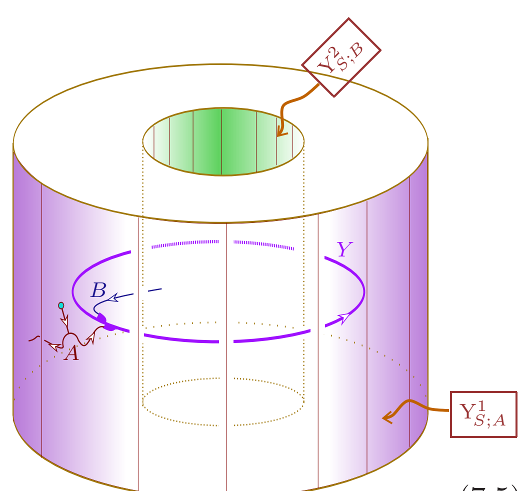
$\mathcal{M}_{\mathrm{T}}^Y \;=$ 

(7.4)

(Here and below, the sticky annuli are denoted by $\mathrm{Y}^\ell_{S;C}$ with $\ell\in\{1,2\}$ and $C\in\{A,B\}$, and are accentuated by their shading, analogously as we already did in pictures like (5.2). The corresponding sticky annuli of the other two pieces $\mathcal{T}_{\mathrm{T}}^A$ and $\mathcal{T}_{\mathrm{T}}^B$ will be denoted by $\mathrm{Y}^\ell_{\mathcal{T};C}$, as in (5.4).)

For dealing with the manifolds $\mathcal{M}_{\mathrm{T}}^X$ and $\mathcal{M}_{\mathrm{T}}^Y$ more easily when performing the various identifications, we redraw them in a slightly deformed manner, such that they look as follows:

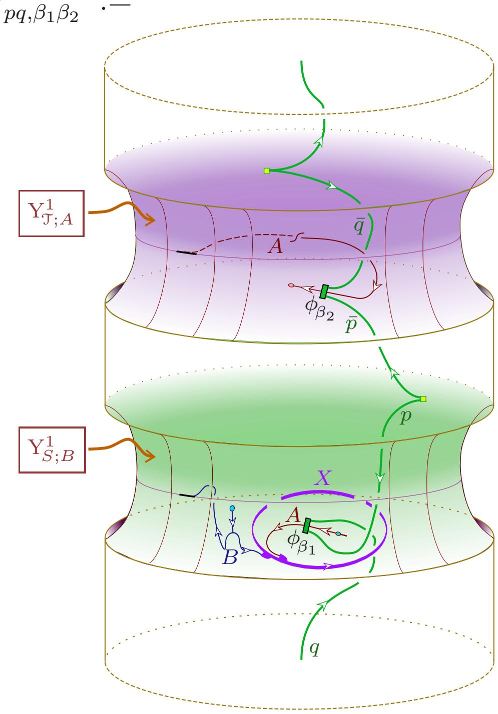
$\mathcal{M}_{\mathrm{T}}^X \;=$ 

$\mathcal{M}_{\mathrm{T}}^Y \;=$ 

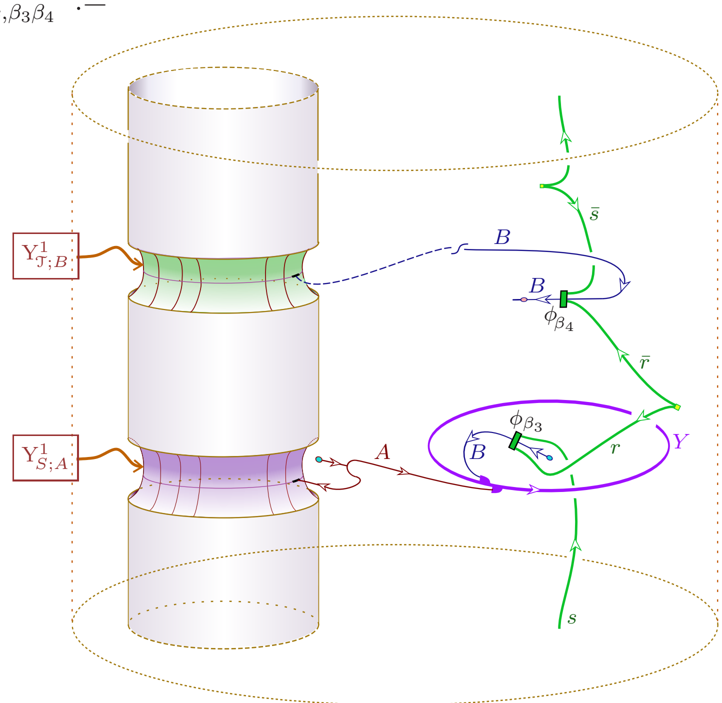
(7.5)

Gluing $\mathcal{M}^{X}_{\mathrm{T}}$ to $\mathcal{T}^{A}_{\mathrm{T}}$, respectively $\mathcal{M}^{Y}_{\mathrm{T}}$ to $\mathcal{T}^{B}_{\mathrm{T}}$, yields the two manifolds

$\mathcal{N}^{A,X}_{pq,\beta_1\beta_2} :=$



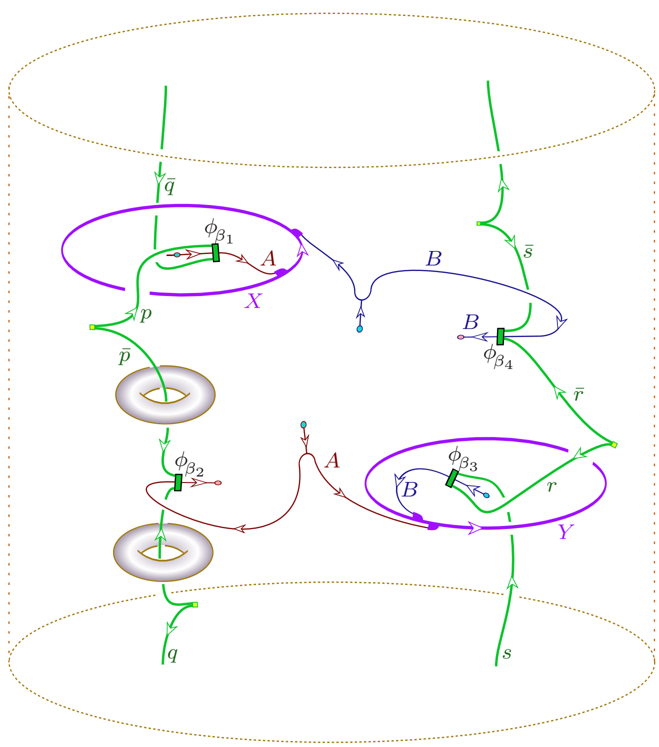
$\mathcal{N}^{B,Y}_{rs,\beta_3\beta_4} :=$



(7.6)

(the second of these is drawn in the same way as we did in (5.4), i.e. with the full torus turned inside out, while the first shows the full torus directly). The remaining two pairwise identifications of sticky annuli are then immediate, leading to

$\mathcal{M}^{\mathrm{T},XY}_{pq\beta_1\beta_2,rs\beta_3\beta_4} =$



(7.7)

which is a ribbon graph in $S^2 \times S^1$ with two full tori cut out.

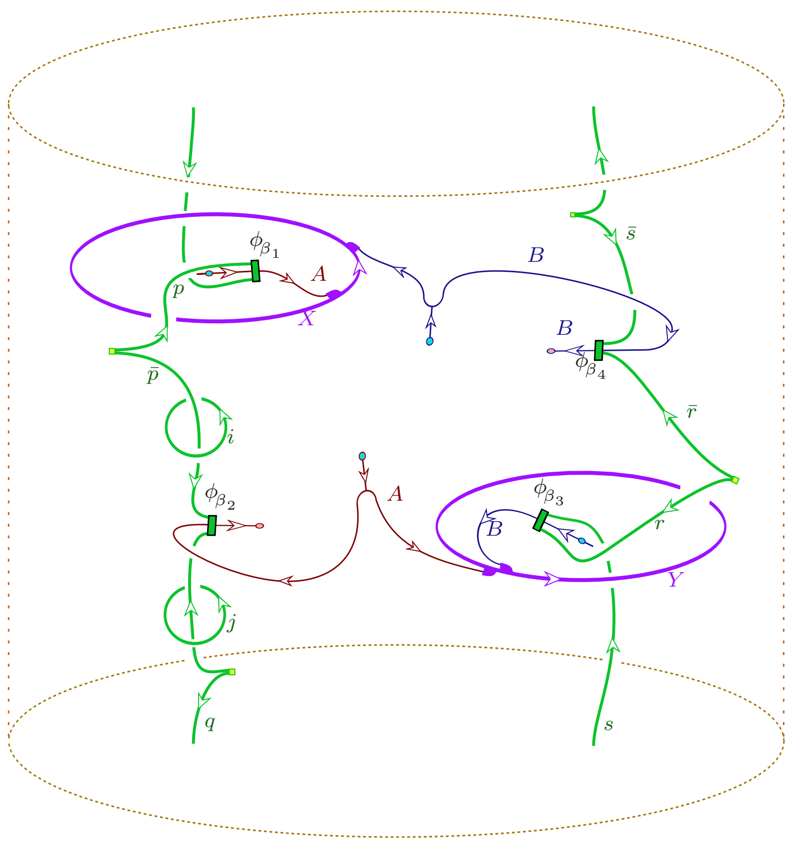
We now extract the coefficients of the partition function $Z_{\mathrm{T}}^{X|Y}$ with respect to a basis of the space of conformal blocks for the boundary of $\mathcal{M}_{\mathrm{T}}^{X|Y}$. This boundary is the complex double $\mathrm{T} \sqcup -\mathrm{T}$ of the torus. Every way to represent T as the boundary of a full torus gives rise to a basis of the space of conformal blocks on T. The basis element $|\chi_i; \mathrm{T}\rangle$ is given as the invariant of a ribbon graph consisting of an annular ribbon that runs along the core of the torus and is labeled by the simple object U_i (see e.g. [23, Sect. 5.2]). To compute the coefficients in the corresponding basis, we compose the cobordism (7.7) with the cobordisms for the elements of the dual basis.

The interpretation as a partition function requires that there is a basis in which the coefficients are non-negative integers. The coefficients $Z_{\mathrm{T},ij}^{X|Y}$ in this basis are obtained by gluing the solid tori to (7.7) precisely in the way suggested by the pictorial description of the boundary $\partial\mathcal{M}_{pq\beta_1\beta_2,rs\beta_3\beta_4}^{\mathrm{T},XY} \cong \mathrm{T} \sqcup -\mathrm{T}$ in (7.7), in particular without applying any large diffeomorphism. That this is the correct gluing is e.g. seen by noticing that in order for $Z_{\mathrm{T}}^{X|Y}$ to have a proper interpretation as a partition function, the 'time' direction of T must be taken along the defect lines, and keeping track of this prescription for the time direction while performing the manipulations that lead from (7.2) to (7.7).[3] This way we find

$$Z_{\mathrm{T},ij}^{X|Y} = \sum_{p,q,r,s\in\mathcal{I}} \dim(U_p)\dim(U_q)\dim(U_r)\dim(U_s) \sum_{\beta_1,\beta_2,\beta_3,\beta_4} ({c_{A;p,q}^{\mathrm{bulk}}}^{-1})_{\beta_2\beta_1} ({c_{B;r,s}^{\mathrm{bulk}}}^{-1})_{\beta_4\beta_3}\, Z(\mathcal{M}_{pq\beta_1\beta_2,rs\beta_3\beta_4}^{\mathrm{T},XY;ij})\, 1 \tag{7.8}$$

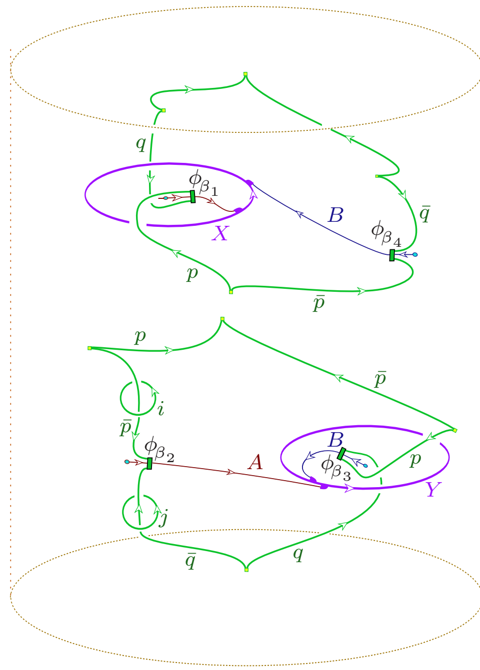
with

$$\mathcal{M}_{pq\beta_1\beta_2,rs\beta_3\beta_4}^{\mathrm{T},XY;ij} \;:= \tag{7.9}$$



[3] This is in agreement with the result (5.151) of [23] for the coefficients of $Z_{\mathrm{T}}^{X|Y}$. Note, however, that in the conventions of [23] the two defect lines have opposite orientation, while here we have chosen the same orientation for both of them.

Again the invariant of this ribbon graph can be evaluated with the help of dominance, which we apply to both $id_{U_p} \otimes id_{U_{\bar r}}$ and $id_{U_q} \otimes id_{U_{\bar s}}$. In both cases only the tensor unit can give a non-zero contribution to the invariant $Z(\mathcal{M}^{\mathrm{T},XY;ij}_{pq\beta_1\beta_2,rs\beta_3\beta_4})$, so that we need $r = p$ and $s = q$, and in this case the invariant is equal to the one of the ribbon graph



$$(7.10)$$

in $S^2 \times S^1$. This invariant, in turn, reduces to the product of the two morphisms in $\mathrm{End}(\mathbf{1}) \cong \mathbb{C}$ that are given by the two components of the ribbon graph. Each of these two morphisms can be simplified with the help of elementary fusing and braiding moves, whereby they are seen to be essentially given by defect transmission coefficients. We suppress the details; the final result is

$$Z^{X|Y}_{\mathrm{T},ij} = \frac{\dim(X)\dim(Y)}{S_{0,0}^2} \sum_{p,q\in\mathcal{I}} S_{i,p}\, S^*_{j,q} \sum_{\beta_1,\beta_2,\beta_3,\beta_4} ({c^{\mathrm{bulk}}_{A;p,q}}^{-1})_{\beta_2\beta_1}\, ({c^{\mathrm{bulk}}_{B;p,q}}^{-1})_{\beta_4\beta_3}\, d_X^{pq,\beta_1\beta_4}\, d_Y^{pq,\beta_3\beta_2}\,. \qquad (7.11)$$

From this result we can deduce various properties of the numbers $Z^{X|Y}_{\mathrm{T},ij}$. We mention two of them. First, for $X = Y = A = B$ we recover the ordinary torus partition function $Z_{\mathrm{T}}(A)$, in agreement with theorem 5.23(ii) of [23]: The identity

$$d_A^{pq,\alpha\beta} = \frac{S_{0,0}}{\dim(A)}\, (c^{\mathrm{bulk}}_{A;p,q})_{\alpha,\beta} \qquad (7.12)$$

together with $\dim_{\mathbb{C}} \mathrm{Hom}_{A|A}(U_{\bar p} \otimes^+ A \otimes^- U_{\bar q}, A) = \mathrm{Z}(A)_{\bar p\bar q}$ implies that

$$Z^{A|A}_{\mathrm{T},ij} = \sum_{p,q\in\mathcal{I}} S_{i,p}\, S^*_{j,q} \sum_{\beta_2=1}^{\mathrm{Z}(A)_{\bar p\bar q}} 1 = \sum_{p,q\in\mathcal{I}} S^{-1}_{i,\bar p}\, \mathrm{Z}(A)_{\bar p\bar q}\, S_{\bar q,j} = \mathrm{Z}(A)_{ij}\,, \qquad (7.13)$$

where the last equality holds by modular invariance of $Z_{\mathrm{T}}(A)$.

Second, we observe that the defect transmission coefficients for defect lines with opposite orientation are related by

$$d_{X^\vee}^{\imath\jmath,\alpha\beta} = \mathsf{R}^{(\imath\,\bar\imath)0}_{\circ\,\circ}\, \mathsf{R}^{-\,(\jmath\,\bar\jmath)0}_{\circ\,\circ}\, d_X^{\bar\imath\bar\jmath,\beta\alpha}. \qquad (7.14)$$

When combined with the identity

$$\mathsf{R}^{(\imath\,\bar\imath)0}_{\circ\,\circ}\,\mathsf{R}^{-\,(\jmath\,\bar\jmath)0}_{\circ\,\circ}\,\big(c^{\mathrm{bulk}}_{A;\imath,\jmath}{}^{-1}\big)_{\alpha,\beta} = \big(c^{\mathrm{bulk}}_{A;\bar\imath,\bar\jmath}{}^{-1}\big)_{\beta,\alpha} \tag{7.15}$$

for the bulk field two-point functions on the sphere, it follows that

$$Z^{X^\vee|Y^\vee}_{\mathrm{T}\,,\bar\imath\bar\jmath} = Z^{Y|X}_{\mathrm{T}\,,\imath\jmath}\,. \tag{7.16}$$

This reproduces the result of theorem 5.23(iii) of [23].

A The final gluing in the double bulk factorization

In this appendix we present the derivation of the result (5.7) and (5.8) for the gluing of the two three-manifolds (with corners) (5.5) and (5.6), which is the last of the four pairwise identifications of sticky annuli that result from the double bulk factorization. According to the discussion leading to the expressions (5.5) and (5.6) for the three-manifolds $\mathcal{M}^A \equiv \mathcal{N}^{A;\,\imath_3\jmath_3,\imath_4\jmath_4}_{pq\beta_1\beta_2}$ and $\mathcal{N}^{B;X} \equiv \mathcal{N}^{B;X,\imath_1\jmath_1,\imath_2\jmath_2}_{rs\beta_3\beta_4}$, the correlator $C_X \equiv C(\Phi_{\alpha_3},\Phi_{\alpha_4};X;\Phi_{\alpha_1},\Phi_{\alpha_2})$ can be written as

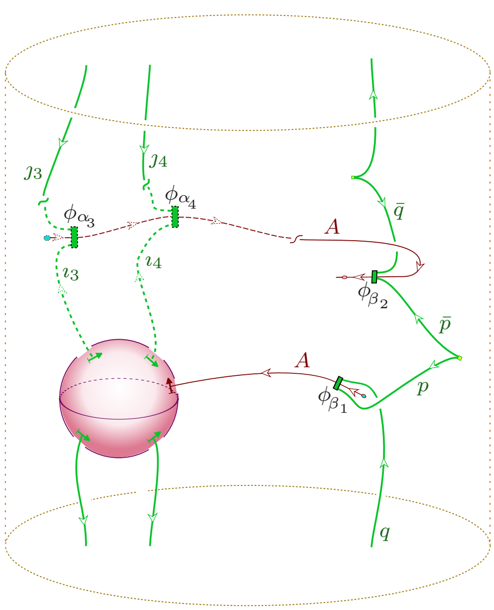
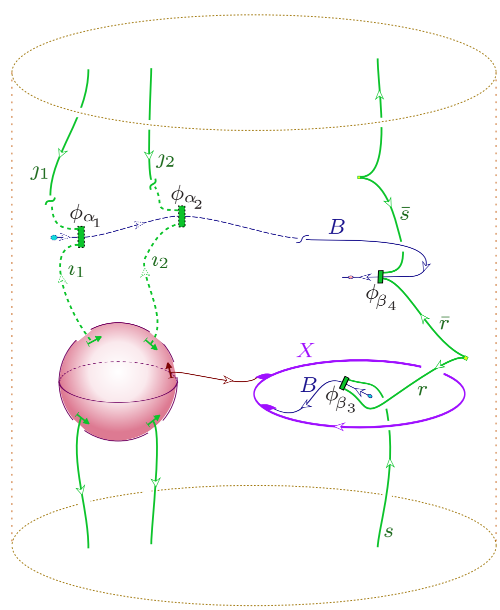
$$C_X = \sum_{p,q,r,s\in\mathcal{I}} \sum_{\beta_1,\beta_2,\beta_3,\beta_4} \dim(U_p)\dim(U_q)\dim(U_r)\dim(U_s) \\ \qquad\qquad (c^{\text{bulk}\,-1}_{A;p,q})_{\beta_2\beta_1}\,(c^{\text{bulk}\,-1}_{B;r,s})_{\beta_4\beta_3}\, Z(\mathcal{M}^{\beta_1\beta_2\beta_3\beta_4}_{pqrs})\,1\,, \tag{A.1}$$

where $\mathcal{M}^{\beta_1\beta_2\beta_3\beta_4}_{pqrs}$ is the three-manifold obtained by identifying the two sticky annuli $\mathrm{Y}^2_{\mathcal{S},A}$ and $\mathrm{Y}^2_{\mathcal{T},A}$ which are subsets of the boundary of $\mathcal{M}^A$ and of $\mathcal{N}^{B;X}$, respectively, and where $c^{\text{bulk}}_{A;uv}$ is the matrix of coefficients of the two-point function of bulk fields (in phase A) on the sphere in a standard basis of blocks. (For more details about the bulk field two-point function see appendix A.6 of [8]. According to theorem 2.13 of [28], each bulk factorization gives rise to a factor of $c^{\text{bulk}\,-1}_{A;uv}$.)

To properly display the manifold $\mathcal{M}^{\beta_1\beta_2\beta_3\beta_4}_{pqrs}$ including its embedded ribbon graph requires some care. We proceed as follows. First we give an alternative description of the three-manifolds $\mathcal{M}^A$ and $\mathcal{N}^{B;X}$, and thereby of $\mathcal{M}^{\beta_1\beta_2\beta_3\beta_4}_{pqrs}$, by cutting each of them along a two-sphere, such that they are written as the compositions

$$\mathcal{N}^A = \widehat{\mathcal{N}}^A \circ \mathcal{S}^A \qquad \text{and} \qquad \mathcal{N}^{B;X} = \widehat{\mathcal{N}}^{B;X} \circ \mathcal{S}^B\,. \tag{A.2}$$

Here each of $\widehat{\mathcal{N}}^A \equiv \widehat{\mathcal{N}}^{A;\,\imath_3\jmath_3,\imath_4\jmath_4}_{pq\beta_1\beta_2}$ and $\widehat{\mathcal{N}}^{B;X} \equiv \widehat{\mathcal{N}}^{B;X;\imath_1\jmath_1,\imath_2\jmath_2}_{rs\beta_3\beta_4}$ is topologically an $S^2\times S^1$ with a three-ball cut out, while each of $\mathcal{S}^A$ and $\mathcal{S}^B$ is a three-manifold with corners, namely a spherical shell $S^2\times[-1,1]$ that has a sticky annulus (namely $\mathrm{Y}^2_{\mathcal{S},A}$ and $\mathrm{Y}^2_{\mathcal{T},A}$, respectively) as part of its 'inner' boundary component. (Even though $\mathcal{S}^A$ and $\mathcal{S}^B$ are thus not cobordisms themselves, the compositions in (A.2) are in the sense of cobordisms, since the relevant identifications do not involve the sticky parts $\mathrm{Y}^2_{\mathcal{S},A}$ and $\mathrm{Y}^2_{\mathcal{T},A}$ of the boundaries of $\mathcal{S}^A$ and $\mathcal{S}^B$.) Including their ribbon graphs, $\widehat{\mathcal{N}}^A$ and $\widehat{\mathcal{N}}^{B;X}$ are given by

$\widehat{\mathcal{N}}^A \;=$  $\qquad \widehat{\mathcal{N}}^{B;X} \;=$ 

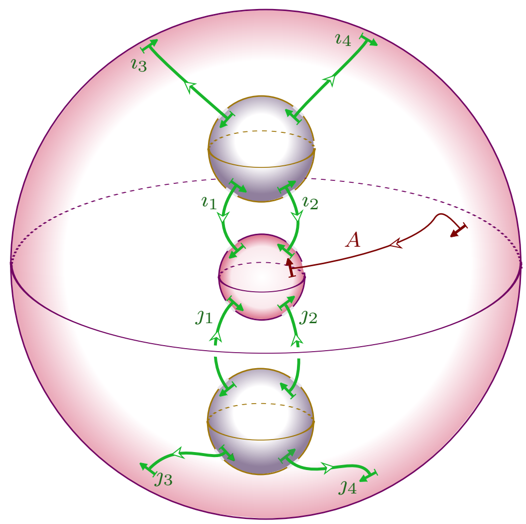
(A.3)

The resulting description of the three-manifold $\mathcal{M}_{pqrs}^{\beta_1\beta_2\beta_3\beta_4}$ is as the composition

$$\mathcal{M}_{pqrs}^{\beta_1\beta_2\beta_3\beta_4} = \mathcal{S} \circ (\widehat{\mathcal{N}}^A \sqcup \widehat{\mathcal{N}}^{B;X}) \tag{A.4}$$

of cobordisms, where $\mathcal{S}$ is the manifold (without corners) that is obtained by gluing the spherical shells $\mathcal{S}^A$ and $\mathcal{S}^B$ along the two sticky annuli on their respective inner boundary components. It is easy to see that as a result of this identification $\mathcal{S}$ is topologically a three-ball with three three-balls cut out. But to be able to visualize this gluing, we need to change the embedding of one of the two spherical shells into $\mathbb{R}^3$ (or rather, S^3) by "turning it inside out", such that the sticky annulus is now located on the outer rather than the inner boundary component. After this manipulation the identification of the sticky annuli on $\mathcal{S}^A$ and $\mathcal{S}^B$ is straightforward, yielding

$\mathcal{S}$ =

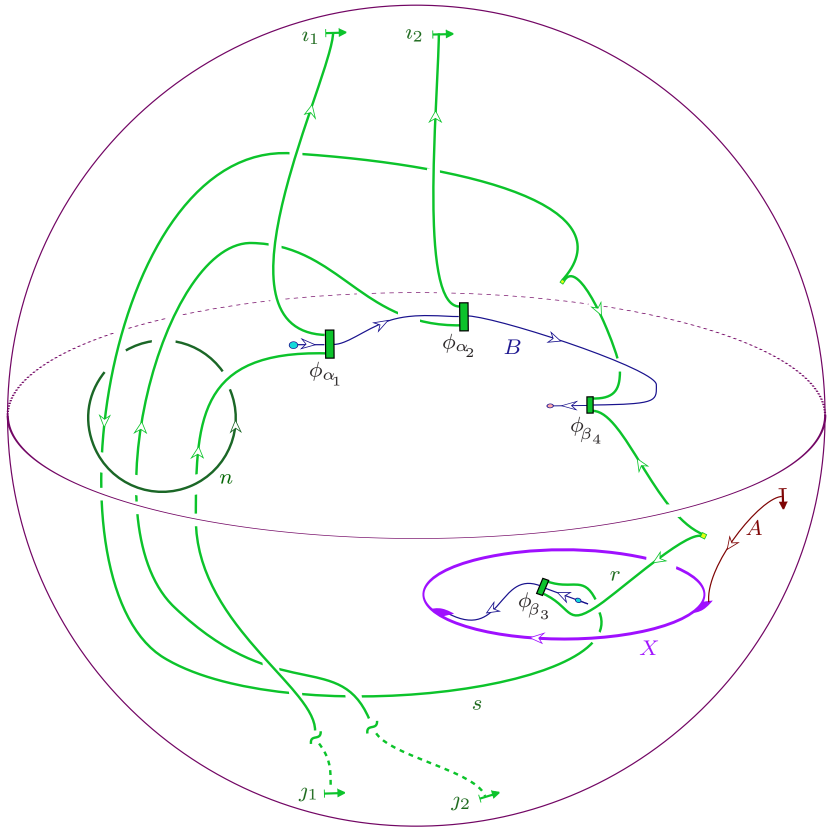


(A.5)

According to the composition in (A.4), the boundary spheres of the three cobordisms $\mathcal{S}$, $\widehat{\mathcal{N}}^A$ and $\widehat{\mathcal{N}}^{B;X}$ are to be identified, in the obvious manner that is already apparent from the labeling of the arcs on their respective boundaries. Again one the these two gluings, say of $\mathcal{S}$ to $\widehat{\mathcal{N}}^A$, is easy. To achieve the other gluing in a convenient way, we first perform surgery on $\widehat{\mathcal{N}}^{B;X}$ along a torus which is a tubular neighbourhood of a surgery link. This yields a sum over ribbon graphs in S^3 with one three-ball cut out, where the summation is over the different labelings by elements of $\mathcal{I}$ of the resulting surgery link. Furthermore, S^3 with a three-ball cut out is topologically again a three-ball, and we may present it as the interior, rather than the exterior, of a two-sphere in S^3.

This way we arrive at

$$\widehat{\mathcal{N}}^{B;X} \;=\; \sum_{n\in\mathcal{I}} S_{0,n} \qquad \text{(A.6)}$$



Here the surgery link is the annular ribbon labeled by $n\in\mathcal{I}$.

With the so obtained description of $\widehat{\mathcal{N}}^{B;X}$ the gluing with $\mathcal{S}$ is readily performed, yielding the resulted quoted in (5.7) and (5.8).

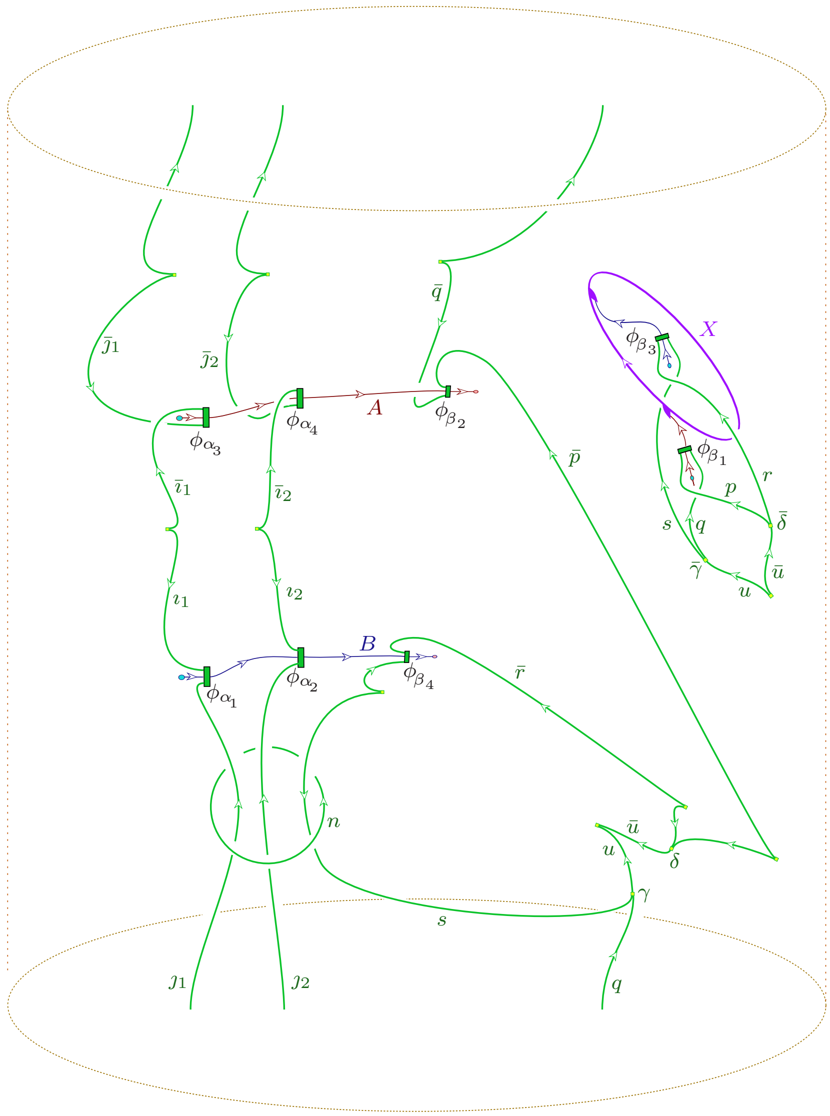
B Projection of (5.7) to the vacuum channel

Here we explain how to obtain the vacuum channel component $c_{X;0} = c(\Phi_{\alpha_3}, \Phi_{\alpha_4}; X; \Phi_{\alpha_1}, \Phi_{\alpha_2})_0$ of the correlator C_X as given by (5.7), by composing the cobordism (5.8) with the basis element dual to (4.9). This composition yields a ribbon graph in $S^2\times S^1$. After slightly deforming this graph so as to reduce the number of braidings, we use dominance for $\mathrm{End}(U_s\otimes U_q\otimes U_p\otimes U_r)$, which results in an additional summation over $u\in\mathcal{I}$ and over three-point couplings $\gamma\in\mathrm{Hom}(U_s\otimes U_q, U_u)$ and $\delta\in\mathrm{Hom}(U_p\otimes U_r, U_{\bar u})$. Hereby we obtain

$$\begin{aligned} c_{X;0} = \frac{1}{S_{0,0}^2} \sum_{p,q,r,s\in\mathcal{I}} & \dim(U_p)\dim(U_q)\dim(U_r)\dim(U_s) \\ & \sum_{\beta_1,\beta_2,\beta_3,\beta_4} ({c^{\mathrm{bulk}}_{A;p,q}}^{-1})_{\beta_2\beta_1} ({c^{\mathrm{bulk}}_{B;r,s}}^{-1})_{\beta_4\beta_3} \sum_{n\in\mathcal{I}} S_{0,n} \sum_{u\in\mathcal{I}} \sum_{\gamma,\delta} Z(\widetilde{\mathcal{M}}^{\gamma\delta,\beta_1\beta_2\beta_3\beta_4}_{n,u;pqrs}) \,, \end{aligned} \qquad \text{(B.1)}$$

where the ribbon graph in $\widetilde{\mathcal{M}}^{\gamma\delta,\beta_1\beta_2\beta_3\beta_4}_{n,p;pqrs}$ consists of two connected components, of which the one containing the X-ribbon is ‘localized’ in the S^1-direction:

$$\widetilde{\mathcal{M}}^{\gamma\delta,\beta_1\beta_2\beta_3\beta_4}_{n,u;pqrs} \;=\; \tag{B.2}$$

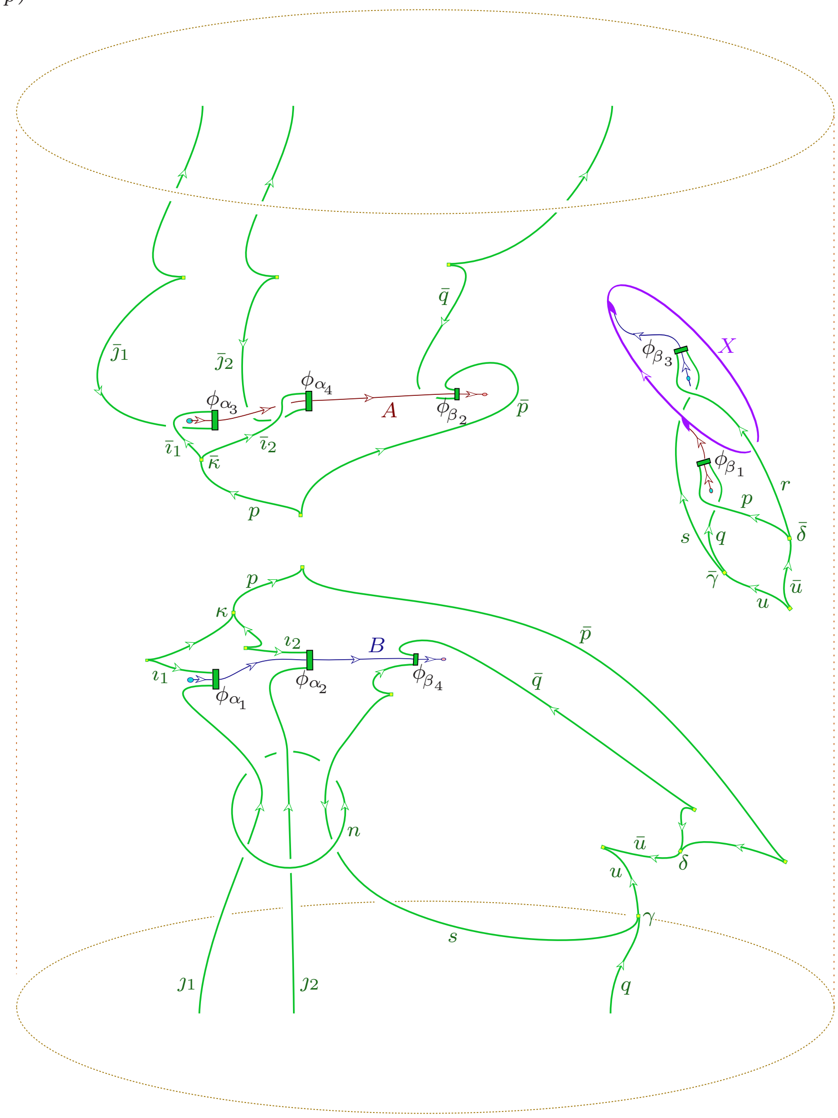


Next we use dominance once more, this time for the space $\mathrm{End}(U_{\bar\imath_1} \otimes U_{\bar\imath_2} \otimes U_{\bar p})$. In the resulting summation over intermediate simple objects only the contribution from the tensor unit gives a non-zero contribution, so that

$$\widetilde{\mathcal{M}}^{\gamma\delta,\beta_1\beta_2\beta_3\beta_4}_{n,u;pqrs} = \sum_\kappa \widetilde{\widetilde{\mathcal{M}}}{}^{\kappa,\gamma\delta,\beta_1\beta_2\beta_3\beta_4}_{n,u;pqrs} \tag{B.3}$$

with $\kappa \in \mathrm{Hom}(U_{\bar\imath_1} \otimes U_{\bar\imath_2}, U_p)$ and

$$\widetilde{\widetilde{\mathcal{M}}}{}^{\kappa,\gamma\delta,\beta_1\beta_2\beta_3\beta_4}_{n,u;pqrs} \quad = \tag{B.4}$$



At this point we can make use of the fact that a horizontal section of the three-manifold is the two-sphere S^2: we can deform the U_n-ribbon around the horizontal two-sphere in such a way that it just encircles the U_u-ribbon. As a consequence, we can trade the U_n-ribbon for a scalar factor $S_{u,n}/S_{u,0}$. Owing to the unitarity of the S-matrix we can then perform the n-summation according to $\sum_{n\in\mathcal{I}} S_{0,n} S_{u,n} = \delta_{0,u}$. As a result only $u=0$ contributes in the summation over u, and accordingly also the γ- and δ-summations consist of only a single term; moreover, for this contribution to be non-zero we need $r=\bar p$ and $s=\bar q$. Inserting this information into the formula (B.1) for $c_{X;0}$ gives the result quoted in (5.9).

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