# A new look at infinite matroids

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**Abstract**

It has recently been shown that, contrary to common belief, infinite matroids can be axiomatized in a way very similar to finite matroids. This should make it possible now to extend much of the theory of finite matroids to infinite ones: an aim that had previously been thought to be unattainable, because the popular additional 'finitary' axiom believed to be necessary clearly spoils duality.

We present the five new axiom sets for infinite matroids found in [3]. They come in terms of independent sets, bases, circuits, closure and rank. We then illustrate them by showing what becomes of the usual cycle and bond matroids of a graph when this graph is infinite.

## 1 Introduction

Traditionally, infinite matroids are either ignored entirely or defined like finite ones,[1] with the following additional axiom:

(I4) An infinite set is independent as soon as all its finite subsets are independent.

We shall call such set systems *finitary matroids*.

The additional axiom (I4) reflects the notion of linear independence in vector spaces, and also the absence of (finite) circuits from a set of edges in a graph. More generally, it is a direct consequence of (I4) that circuits, defined as minimal dependent sets, are finite.

An important and regrettable consequence of the additional axiom (I4) is that it spoils duality, one of the key features of finite matroid theory. For example, the cocircuits of an infinite uniform matroid of rank $k$ would be the sets missing exactly $k-1$ points; since these sets are infinite, however, they cannot be the circuits of another finitary matroid. Similarly, every bond of an infinite graph would be a circuit in any dual of its cycle matroid—a set of edges minimal with the property of containing an edge from every spanning tree—but these sets can be infinite and hence will not be the circuits of a finitary matroid.

This situation prompted Rado in 1966 to ask for the development of a theory of non-finitary infinite matroids with duality [10, Problem P531]. In the late

[1] The augmentation axiom is required only for finite sets: given independent sets $I, I'$ with $|I| < |I'| < \infty$, there is an $x \in I' \setminus I$ such that $I + x$ is again independent.

1960s and 70s, a number of such theories were proposed; see [3] for references. One of these, the 'B-matroids' proposed by Higgs [8], were later shown by Oxley [9] to identify the models of any theory of infinite matroids that admitted both duality and minors as we know them. However, Higgs did not present his 'B-matroids' in terms of axioms similar to those for finite matroids. As a consequence, theorems about finite matroids whose proofs rested on these axioms could not be readily extended to infinite matroids, even when this might have been possible in principle.

With the axioms from [3] presented in the next section, this could now change: it should be possible now to extend many more results about finite matroids to infinite matroids, either by

- adapting their proofs based on the finite axioms to the (very similar) new infinite axioms,

or by

- finding a sequence of finite matroids that has the given infinite matroid as a limit, and is chosen in such a way that the instances of the theorem known for those finite matroids imply a corresponding assertion for the limit matroid.

After presenting our new axioms in Section 2, we apply them in Sections 3 and 4 to see what they mean for graphs. We shall see that, for matroids whose circuits are the (usual finite) cycles of a graph, our axioms preserve what would be wrecked by the finitary axiom (I4): that their duals are the matroids whose circuits are the bonds of our graph – even though these can now be infinite.

The converse is also nice. The dual $M^*$ of the (finitary) matroid $M$ whose circuits are the finite bonds of an infinite graph cannot be finitary; indeed, trivial exceptions aside, the duals of finitary matroids are never finitary [14, 1]; see also [4]. So $M^*$ will have to have infinite circuits.[2] Excitingly, these circuits turn out to be familiar objects: when the graph is locally finite, they are the edge sets of the topological circles in its Freudenthal compactification, which already have a fixed place in infinite graph theory quite independently of matroids [6].

We shall see further that, for planar graphs, matroid duality is now fully compatible with graph duality as explored in [2]. Finally, we shall see that Whitney's theorem, that a graph is planar if and only if its cycle matroid has a graphic dual, now has an infinite version too.

# 2 Axioms

We now present our five sets of axioms for finite or infinite matroids, in terms of independent sets, bases, circuits, closure and rank. These axioms were first stated in [3], and proved to be equivalent to each other in the usual sense. For finite or finitary matroids they default to the usual finite matroid axioms.

[2] We are using here that every dependent set contains a minimal such. This is indeed true.

Let $E$ be any non-empty set, finite or infinite; it will be the default ground set for all matroids considered in this paper. We write $2^E$ for its power set. The set of all pairs $(A, B)$ such that $B \subseteq A \subseteq E$ will be denoted by $(2^E \times 2^E)_{\subseteq}$; for its elements we usually write $(A|B)$ instead of $(A, B)$. Unless otherwise mentioned, the terms 'minimal' and 'maximal' refer to set inclusion. Given $\mathcal{E} \subseteq 2^E$, we write $\mathcal{E}^{\max}$ for the set of maximal elements of $\mathcal{E}$, and $\lceil \mathcal{E} \rceil$ for the *down-closure* of $\mathcal{E}$, the set of subsets of elements of $\mathcal{E}$. For $F \subseteq E$ and $x \in E$, we abbreviate $F \smallsetminus \{x\}$ to $F - x$ and $F \cup \{x\}$ to $F + x$. We shall not distinguish between infinite cardinalities and denote all these by $\infty$; in particular, we shall write $|A| = |B|$ for any two infinite sets $A$ and $B$. The set $\mathbb{N}$ contains 0.

One central axiom that features in all our axiom systems is that every independent set extends to a maximal one, even inside any restriction $X \subseteq E$.[3] The notion of what constitutes an independent set, however, will depend on the type of axioms under consideration. We therefore state this extension axiom in more general form first, without reference to independence, so as to be able to refer to it later from within different contexts.

Let $\mathcal{I} \subseteq 2^E$. The following statement describes a possible property of $\mathcal{I}$.

(M) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{ I' \in \mathcal{I} \mid I \subseteq I' \subseteq X \}$ has a maximal element.

Note that the maximal superset of $I$ in $\mathcal{I} \cap 2^X$ whose existence is asserted in (M) need not lie in $\mathcal{I}^{\max}$.

## 2.1 Independence axioms

The following statements about a set $\mathcal{I} \subseteq 2^E$ are our *independence axioms*:

(I1) $\emptyset \in \mathcal{I}$.

(I2) $\lceil \mathcal{I} \rceil = \mathcal{I}$, i.e., $\mathcal{I}$ is closed under taking subsets.

(I3) For all $I \in \mathcal{I} \smallsetminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$ there is an $x \in I' \smallsetminus I$ such that $I + x \in \mathcal{I}$.

(IM) $\mathcal{I}$ satisfies (M).

When a set $\mathcal{I} \subseteq 2^E$ satisfies the independence axioms, we call the pair $(E, \mathcal{I})$ a *matroid* on $E$. We call every element of $\mathcal{I}$ an *independent* set, every element of $2^E \smallsetminus \mathcal{I}$ a *dependent* set, the maximal independent sets *bases*, and the minimal dependent sets *circuits*. This matroid is *finitary* if it also satisfies (I4) from the Introduction, which is equivalent to requiring that every circuit be finite [3].

The $2^E \to 2^E$ function mapping a set $X \subseteq E$ to the set

$$\mathrm{cl}(X) := X \cup \{ x \mid \exists I \subseteq X \colon I \in \mathcal{I} \text{ but } I + x \notin \mathcal{I} \}$$

will be called the *closure operator* on $2^E$ *associated with* $\mathcal{I}$.

[3] Interestingly, we shall not need to require that every dependent set contains a minimal one. We need that too, but it follows [3].

The $(2^E \times 2^E)_{\subseteq} \to \mathbb{N} \cup \{\infty\}$ function $r$ that maps a pair $A \supseteq B$ of subsets of $E$ to

$$r(A|B) := \max\{\, |I \smallsetminus J| : I \supseteq J,\ I \in \mathcal{I} \cap 2^A,\ J \text{ maximal in } \mathcal{I} \cap 2^B \}$$

will be called the *relative rank function* on the subsets of $E$ *associated with* $\mathcal{I}$. This maximum is always attained, and independent of the choice of $J$ [3].

## 2.2 Basis axioms

The following statements about a set $\mathcal{B} \subseteq 2^E$ are our *basis axioms*:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) Whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \smallsetminus B_2$, there is an element $y$ of $B_2 \smallsetminus B_1$ such that $(B_1 - x) + y \in \mathcal{B}$.

(BM) The set $\mathcal{I} := \lceil \mathcal{B} \rceil$ of all $\mathcal{B}$-*independent* sets satisfies (M).

## 2.3 Closure axioms

The following statements about a function $\mathrm{cl}\colon 2^E \to 2^E$ are our *closure axioms*:

(CL1) For all $X \subseteq E$ we have $X \subseteq \mathrm{cl}(X)$.

(CL2) For all $X \subseteq Y \subseteq E$ we have $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.

(CL3) For all $X \subseteq E$ we have $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.

(CL4) For all $Z \subseteq E$ and $x, y \in E$, if $y \in \mathrm{cl}(Z + x) \smallsetminus \mathrm{cl}(Z)$ then $x \in \mathrm{cl}(Z + y)$.

(CLM) The set $\mathcal{I}$ of all cl-*independent* sets satisfies (M). These are the sets $I \subseteq E$ such that $x \notin \mathrm{cl}(I - x)$ for all $x \in I$.

## 2.4 Circuit axioms

The following statements about a set $\mathcal{C} \subseteq 2^E$ are our *circuit axioms*:

(C1) $\emptyset \notin \mathcal{C}$.

(C2) No element of $\mathcal{C}$ is a subset of another.

(C3) Whenever $X \subseteq C \in \mathcal{C}$ and $(C_x \mid x \in X)$ is a family of elements of $\mathcal{C}$ such that $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in C \smallsetminus \left(\bigcup_{x \in X} C_x\right)$ there exists an element $C' \in \mathcal{C}$ such that $z \in C' \subseteq \left(C \cup \bigcup_{x \in X} C_x\right) \smallsetminus X$.

(CM) The set $\mathcal{I}$ of all $\mathcal{C}$-*independent* sets satisfies (M). These are the sets $I \subseteq E$ such that $C \nsubseteq I$ for all $C \in \mathcal{C}$.

Axiom (C3) defaults for $|X| = 1$ to the usual ('strong') circuit elimination axiom for finite matroids. In particular, it implies that adding an element to a basis creates at most one circuit; the fact that it does create such a (*fundamental*) circuit is trivial when bases are defined from these circuit axioms (as maximal sets not containing a circuit), while if we start from the independence axioms it follows from the fact, mentioned before, that every dependent set contains a minimal one [3]. We remark that the usual finite circuit elimination axiom is too weak to guarantee a matroid [3].

### 2.5 Rank axioms

The following statements about a function $r\colon (2^E \times 2^E)_{\subseteq} \to \mathbb{N} \cup \{\infty\}$ are our (relative) *rank axioms*:

(R1) For all $B \subseteq A \subseteq E$ we have $r(A|B) \leq |A \setminus B|$.

(R2) For all $A, B \subseteq E$ we have $r(A|A \cap B) \geq r(A \cup B|B)$.

(R3) For all $C \subseteq B \subseteq A \subseteq E$ we have $r(A|C) = r(A|B) + r(B|C)$.

(R4) For all families $(A_\gamma)$ and $B$ such that $B \subseteq A_\gamma \subseteq E$ and $r(A_\gamma|B) = 0$ for all $\gamma$, we have $r(A|B) = 0$ for $A := \bigcup_\gamma A_\gamma$.

(RM) The set $\mathcal{I}$ of all *$r$-independent* sets satisfies (M). These are the sets $I \subseteq E$ such that $r(I|I - x) > 0$ for all $x \in I$.

For finite matroids, these axioms (with (R4) and (RM) becoming redundant) are easily seen to be tantamount to the usual axioms for an absolute rank function $R$ derived as $R(A) := r(A|\emptyset)$, or conversely with $r(A|B) := R(A) - R(B)$ for $B \subseteq A$.

## 3 Bond and cycle matroids

In this section we develop the theory of our axioms to see what it yields for the usual matroids for graphs when these are infinite. See [3] for applications to other structures than graphs. All our graphs may have parallel edges and loops.

A well-known matroid associated with a finite graph $G$ is its cycle matroid: the matroid whose circuits are the edge sets of the cycles in $G$. The bases of this matroid are the edge sets of the *spanning forests* of $G$, the sets that form a spanning tree in every component of $G$. This construction works in infinite graphs too: the edge sets of the finite cycles in $G$ form the circuits of a finitary matroid $M_{\mathrm{FC}}(G)$, whose bases are the edge sets of the spanning forests of $G$. We shall call $M_{\mathrm{FC}}(G)$ the *finite-cycle matroid of* $G$. Similarly, we let the *finite-bond matroid* $M_{\mathrm{FB}}(G)$ of $G$ be the matroid whose circuits are the finite bonds of $G$. (A *bond* is a minimal non-empty cut.) This, too, is a finitary matroid.

If $G$ is finite, then $M_{\mathrm{FC}}(G)$ and $M_{\mathrm{FB}}(G)$ are dual to each other. For infinite $G$, however, things are different. As remarked earlier, the duals of finitary matroids are not normally finitary [4], so the duals of $M_{\mathrm{FC}}(G)$ and $M_{\mathrm{FB}}(G)$ will in general have infinite circuits. In the case of $M_{\mathrm{FC}}(G)$, its cocircuits are the expected ones, the (finite or infinite) bonds:

**Theorem 1.** *Let $G$ be any graph.*

(i) *The bonds of $G$, finite or infinite, are the circuits of a matroid $M_{\mathrm{B}}(G)$, the* bond matroid *of $G$.*

(ii) *The bond matroid of $G$ is the dual of its finite-cycle matroid $M_{\mathrm{FC}}(G)$.*

We defer the proof of Theorem 1 to Section 4; it is essentially the same as for finite graphs, although now the bonds can be infinite.

Similarly, the dual of $M_{\mathrm{FB}}(G)$ will in general have infinite circuits. Ideally, these would form some sort of 'infinite cycles' in $G$. 'Infinite cycles' have indeed been considered before for graphs, though in a purely graph-theoretic context: there is a topological such notion that makes it possible to extend classical results about cycles in finite graphs (such as Hamilton cycles) to infinite graphs, see [6] and [12] in this issue. Rather strikingly, it turns out that these 'infinite cycles' are the solution also to our problem: their edge sets are precisely the (possibly infinite) cocircuits of $M_{\mathrm{FB}}(G)$.

In order to define those 'infinite cycles', we need to endow our given graph $G$ with a topology. A *ray* is a one-way infinite path. Two rays are *edge-equivalent* if for any finite set $F$ of edges there is a component of $G - F$ that contains subrays of both rays. The equivalence classes of this relation are the *edge-ends of $G$*, whose set we denote by $\mathcal{E}(G)$.

Let us view the edges of $G$ as disjoint topological copies of $[0, 1]$, and let $X_G$ be the quotient space obtained by identifying these copies in their common vertices. We now define a topological space $\|G\|$ on the point set of $X_G \cup \mathcal{E}(G)$ by taking as our open sets the unions of sets $\widetilde{C}$, where $C$ is a connected component of $X_G - Z$ for some finite set $Z \subset X_G$ of inner points of edges, and $\widetilde{C}$ is obtained from $C$ by adding all the edge-ends represented by a ray in $C$.

When $G$ is connected then $\|G\|$ is a compact topological space [11], although in general it need not be Hausdorff: the common starting vertex of infinitely many otherwise disjoint equivalent rays, for example, cannot be distinguished topologically from the edge-end which those rays represent. However if $G$ is locally finite, then $\|G\|$ coincides with the (Hausdorff) Freudenthal compactification of $G$. See Section 4 for more properties of $\|G\|$.

For any set $X \subseteq \|G\|$ we call

$$E(X) := \{e \in E(G) : \mathring{e} \subseteq X\}$$

the *edge set of $X$*. A subspace $C$ of $\|G\|$ that is homeomorphic to $S^1$ is a *circle* in $\|G\|$. One can show that $\bigcup E(C)$ is dense in $C$, so $C$ lies in the closure of the subgraph formed by its edges [11]. In particular, there are no circles consisting only of edge-ends.

A subspace $X \subseteq \|G\|$ is a *standard subspace* if it is the closure in $\|G\|$ of a subgraph of $G$. A *topological spanning tree* of $G$ is a standard subspace $T$ of $\|G\|$ that is path-connected and contains $V(G)$ but contains no circle. Note that, since standard subspaces are closed, $T$ will also contain $\mathcal{E}(G)$.

**Theorem 2.** *Let $G$ be any connected*[4] *graph.*

(i) *The edge sets of the circles in $\|G\|$ are the circuits of a matroid $M_C(G)$, the* cycle matroid *of $G$.*

(ii) *The bases of $M_C(G)$ are the edge sets of the topological spanning trees of $G$.*

(iii) *The cycle matroid $M_C(G)$ is the dual of the finite-bond matroid $M_{FB}(G)$.*

We shall prove Theorem 2 in Section 4.

In the finite world, matroid duality is compatible with graph duality in that the dual of the cycle matroid of a finite planar graph $G$ is the cycle matroid of its (geometric or algebraic) dual $G^*$. Duality for infinite graphs has come to be properly understood only recently [2]. But now that we have matroid duality as well, it turns out that the two are again compatible. In the remainder of this section we briefly explain how infinite graph duality is defined, and then show its compatiblity with matroid duality.

When one tries to define abstract graph duality so that it satisfies the minimum requirement of capturing the geometric duality of locally finite graphs in the plane (where one has a dual vertex for every face and a dual edge between vertices representing two faces for every edge that lies on the boundary of both these faces), the first thing one realizes is that by taking duals one will leave the class of locally finite graphs: the dual of a ray, for example, is a vertex with infinitely many loops. On the other hand, Thomassen [13] showed that any class of graphs for which duality can be reasonably defined cannot be much larger: these graphs have to be *finitely separable* in that every two vertices can be separated by finitely many edges.[5]

It was finally shown in [2] that the class of finitely separable graphs is indeed the right setting for infinite graph duality, defined as follows. Let $G$ be a finitely separable graph. A graph $G^*$ is called a *dual of $G$* if there is a bijection

$$^* : E(G) \to E(G^*)$$

such that a set $F \subseteq E(G)$ is the edge set of a circle in $\|G\|$ if and only if $F^* := \{e^* \mid e \in F\}$ is a bond of $G^*$.[6] Duals defined in this way behave just as for finite graphs:

**Theorem 3. [2]** *Let $G$ be a countable finitely separable graph.*

(i) *$G$ has a dual if and only if $G$ is planar.*

(ii) *If $G^*$ is a dual of $G$, then $G^*$ is finitely separable, $G$ is a dual of $G^*$, and this is witnessed by the inverse bijection of $^*$.*

(iii) *Duals of 3-connected graphs are unique, up to isomorphism.*

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[4] The theorem extends to disconnected graphs in the obvious way.

[5] Christian, Richter and Rooney [5] define certain dual objects for arbitrary planar graphs; however these objects are "graph-like spaces", not graphs.

[6] We are cheating a bit here, but only slightly. In [2], these circles are taken not in $\|G\|$ but in a slightly different space $\tilde{G}$. However, while the circles in $\tilde{G}$ may differ slightly from those in $\|G\|$, their edge sets are the same. This is not hard to see directly; it also follows from Theorems 6.3 and 6.5 in [7] in conjunction with Satz 4.3 and 4.5 in [11].

At the time, the reason for defining graph duality as above was purely graph-theoretic: it appeared (and still appears) to be the unique way to make all three statements of Theorem 3 true for infinite graphs. As matroid duality was developed independently of graph duality, it is thus remarkable—and adds to the justification of both notions—that the two are once more compatible, as far as remains possible in an infinite setup:

**Theorem 4.** *Let $G$ and $G^*$ be a pair of countable dual graphs, each finitely separable, and defined on the same edge set $E$. Then*

$$M_{\mathrm{FB}}(G) = (M_{\mathrm{C}}(G))^* = (M_{\mathrm{B}}(G^*))^* = M_{\mathrm{FC}}(G^*).$$

*Proof.* The first equality is Theorem 2 (iii). The last equality is Theorem 1 (ii) (after dualizing). The middle equality follows from $M_{\mathrm{C}}(G) = M_{\mathrm{B}}(G^*)$, which is a direct consequence of the definition of a dual graph. □

Finally, we obtain an infinite analogue of Whitney's theorem that a finite graph is planar if and only if the dual of its cycle matroid is 'graphic', ie., is the cycle matroid of another finite graph. Let us call a matroid *finitely graphic* if it is isomorphic to the finite-cycle matroid of a graph.

**Theorem 5.** *A countable finitely separable graph is planar if and only if its cycle matroid has a finitely graphic dual.*

*Proof.* Let $G$ be a countable finitely separable graph. If $G$ is planar it has a dual $G^*$, and $(M_{\mathrm{C}}(G))^* = M_{\mathrm{FC}}(G^*)$ by Theorem 4.

For the converse direction, assume that $(M_{\mathrm{C}}(G))^*$ is finitely graphic. Then there exists a graph $H$ with the same edge set as $G$ such that $(M_{\mathrm{C}}(G))^* = M_{\mathrm{FC}}(H)$. As $(M_{\mathrm{FC}}(H))^* = M_{\mathrm{B}}(H)$ by Theorem 1 and matroid duals are unique, we obtain $M_{\mathrm{C}}(G) = M_{\mathrm{B}}(H)$. Hence the edge sets of the circles in $\|G\|$, which by Theorem 2 (i) are the circuits of $M_{\mathrm{C}}(G)$, are precisely the bonds of $H$. So $H$ is a dual of $G$, and $G$ is planar by Theorem 3 (i). □

We remark that Theorem 5 would remain valid if we strengthened the notion of 'finitely graphic' by requiring that the graph referred to in its definition be finitely separable. Indeed, if $G$ is planar then its dual $G^*$, which we used in the forward implication of Theorem 5 as a witness that $(M_{\mathrm{C}}(G))^*$ is graphic, is finitely separable by Theorem 3 (ii).

We believe, but have been unable to prove, that in Theorem 5 one can shift the finiteness assumption from the dual to the primal matroid, as follows. Call a matroid *graphic* if it is isomorphic to the cycle matroid of a graph.

**Conjecture 6.** *A countable finitely separable graph is planar if and only if its finite-cycle matroid has a graphic dual.*

Conjecture 6 can be proved, by similar methods as above, if we strengthen the notion of 'graphic' so as to require that the graph referred to is finitely separable.

## 4 Proof of Theorems 1 and 2

We begin with the easy proof of Theorem 1, which we restate:

**Theorem 1.** *Let $G$ be any graph.*

(i) *The bonds of $G$, finite or infinite, are the circuits of a matroid $M_{\mathrm{B}}(G)$, the* bond matroid *of $G$.*

(ii) *The bond matroid of $G$ is the dual of its finite-cycle matroid $M_{\mathrm{FC}}(G)$.*

*Proof.* For simplicity we assume that $G$ is connected; the general case is very similar. From [3] we know that $M_{\mathrm{FC}}(G)$ has a dual; let us call this dual $M_{\mathrm{B}}(G)$, and show that its circuits are the bonds of $G$. By definition of matroid duality, the circuits of $M_{\mathrm{B}}(G)$ are the minimal edges sets that meet every spanning tree of $G$.

We show first that every bond $B$ of $G$ is a circuit of $M_{\mathrm{B}}(G)$, a minimal set of edges meeting every spanning tree. Since $B$ is a non-empty cut, it is the set of edges across some partition of the vertex set of $G$. Every spanning tree meets both sides of this partition, so it has an edge in $B$. On the other hand, we can extend any edge $e \in B$ to a spanning tree of $G$ that contains no further from $B$, since by the minimality of $B$ as a cut its two sides are connected in $G$. Hence $B$ is minimal with the property of meeting every spanning tree.

Conversely, let $B$ be any set of edges that is minimal with the property of meeting every spanning tree. We show that $B$ contains a bond; by the implication already shown, and its minimality, it will then *be* that bond. Since $G$ has a spanning tree, we have $B \neq \emptyset$; let $e \in B$. If $B$ contains no bond, then every bond has an edge not in $B$. The subgraph $H$ formed by all these edges is connected and spanning in $G$, as otherwise the edges of $G$ from the component $C$ of $H$ containing $e$ to any fixed component of $G - C$ would form a bond of $G$ with no edge in $H$, contradicting its definition. So $H$ contains a spanning tree. This misses $B$, contradicting the choice of $B$. $\square$

We prove Theorem 2 for countable graphs; the proof for arbitrary graphs can be deduced from this by considering a quotient space of $\|G\|$ as explained in [11]. For the remainder of this section, let $G$ be a fixed countable connected graph.

We shall call two points in $\|G\|$ *(topologically) indistinguishable* if they have the same open neighbourhoods. Clearly two vertices or edge-ends $x, y \in \|G\|$ are indistinguishable if they cannot be separated by finitely many edges. (If both are edge-ends, then $x = y$.) On the other hand, two such points that can be separated by finitely many edges have disjoint open neighbourhoods. Inner points of edges are always distinguishable from all other points.

We shall need a few lemmas. Some of these are quoted from Schulz [11]; the others are adaptations of results proved in [7] for the special case that $G$ is finitely separable. We remark that it is also possible to reduce Theorem 2 formally to that case by replacing $G$ with a quotient graph as explained in [11].

**Lemma 7.** **[11]** $\|G\|$ *is a compact space.*

**Lemma 8.** *Let $X \subseteq \|G\|$ be a closed subspace. Suppose there are disjoint non-empty open subsets $O_1, O_2$ of $X$ such that $X = O_1 \cup O_2$. Then the set $F$ of edges with one endvertex in $O_1 \cap V(G)$ and the other in $O_2 \cap V(G)$ is finite.*

*Proof.* Suppose that $F$ is infinite. As a closed subspace of $\|G\|$, the set $X \cap O_1$ is compact. It therefore contains an accumulation point $x$ of endvertices of edges in $F$. Then $x$ is also an accumulation point of their neighbours in $X \cap O_2$, and thus lies in $X \cap O_2$ as well. This contradicts our assumption that $O_1 \cap O_2 = \emptyset$. □

In a Hausdorff space, every topological $x$–$y$ path contains an injective such path, an *$x$–$y$ arc*. Since $\|G\|$ is not necessarily Hausdorff we cannot assume this shortcut lemma in general, but it holds in the relevant case:

**Lemma 9.** **[11]** *If two points $x, y \in V(G) \cup \mathcal{E}(G)$ are separated by a finite set of edges, then every topological $x$–$y$ path contains an $x$–$y$ arc.*

**Lemma 10.** **[11]** *Let $x, y \in V(G) \cup \mathcal{E}(G)$, and let $(A_\gamma)_{\gamma<\lambda}$ be a transfinite sequence of $x$–$y$ arcs in $\|G\|$. Then there exists a topological $x$–$y$ path $P$ and a dense subset $P^*$ of $P$ so that for all $p \in P^*$ the arcs $A_\gamma$ containing $p$ form a cofinal subsequence.*

**Lemma 11.** *Every closed connected subspace $X$ of $\|G\|$ is path-connected.*

*Proof.* Suppose $X$ is connected but not path-connected. Then there are $x, y \in V(G) \cup \mathcal{E}(G)$ contained in different path-components. In particular, $x$ and $y$ are topologically distinguishable, so they are separated by finitely many edges. Let $e_1, e_2, \ldots$ be a (possibly finite) enumeration of the edges in $E(G) \setminus E(X)$, let $F_i := \{e_1, \ldots, e_i\}$ for all $i$. If there exists an $i$ such that $x$ and $y$ lie in the closures of different graph-theoretical components of $G - F_i$, then picking an inner point outside $X$ from every edge in $F_i$ we obtain a finite set $Z \subseteq \|G\| \setminus X$ witnessing that $x$ and $y$ lie in distinct open sets of $X$ whose union is all of $X$, contradicting our assumption that $X$ is connected.

Hence for every $i$ the points $x$ and $y$ lie in the closure $\overline{C}_i$ of the same component $C_i$ of $G - F_i$. So for each $i$ there is a path, ray or double ray connecting $x$ to $y$ in $\overline{C}_i$, and with Lemma 9 we then obtain an $x$–$y$ arc $A_i$ in $\overline{C}_i$. . By Lemma 10 this implies that there is a topological $x$–$y$ path $P$ and a dense subset $P^* \subseteq P$ such that for every $p \in P^*$ the arcs $A_i$ containing $p$ form a cofinal subsequence. Suppose there exists a $j$ such that $\mathring{e}_j \subseteq P$. Then there must be a point $p \in \mathring{e}_j \cap P^*$. However, none of the $A_i$ with $i \geq j$ contains $\mathring{e}_j$. Thus, $P$ does not use any edge outside $X$. As $X$ is closed, this implies that $P \subseteq X$. The required $x$–$y$ arc in $X$ can be found inside $P$ by Lemma 9. □

**Lemma 12.** *Let $F \subseteq E(G)$ be a set of edges whose closure in $\|G\|$ contains no circle. Then $G$ has a topological spanning tree whose edge set contains $F$.*

*Proof.* Let $G = (V, E)$, let $e_1, e_2, \ldots$ be an enumeration of the edges in $E \setminus F$, and set $T_0 := E$. Inductively, if the closure of $(V, T_{i-1} - e_i)$ is connected in $\|G\|$ then set $T_i := T_{i-1} - e_i$; otherwise put $T_i := T_{i-1}$. Finally, we set $T := \bigcap_{i=0}^{\infty} T_i$.

In order to show that $T$ is the edge set of a topological spanning tree, let us first check that the closure $X$ of $(V, T)$ is connected. Suppose there are two disjoint non-empty open sets $O_1$ and $O_2$ of $X$ with $X = O_1 \cup O_2$. Then Lemma 8 implies that the cut $S$ consisting of the edges with one endvertex in $O_1$ and the other in $O_2$ is finite. If $j$ is the largest integer with $e_j \in S$ then, however, the closure of $(V, T_j)$ is not connected, a contradiction. Thus, $\overline{T} = X$ is connected and therefore spanning. Moreover, $\overline{T}$ is path-connected, by Lemma 11.

Secondly, we need to show that $\overline{T}$ is acirclic. So, suppose that $\overline{T}$ contains a circle $C$. Since every circle lies in the closure of its edges but the closure of $\bigcup F$ contains no circle, $E(C) \setminus F$ is non-empty. Pick $j$ minimal with $e_j \in E(C) \setminus F$. Since $e_j$ was not deleted from $T_{j-1}$ when $T_j$ was formed, the closure $Y$ of $(V, T_{j-1} - e_j)$ is disconnected. So there are two disjoint non-empty open subsets $O_1, O_2$ of $Y$ such that $Y = O_1 \cup O_2$. The endvertices of $e_j$ do not lie in the same $O_i$, since adding $e_j$ to that $O_i$ would then yield a similar decomposition of the closure of $(V, T_{j-1})$, contradicting its connectedness. But now the connected subset $C \setminus \mathring{e}_j$ of $Y$ meets both $O_1$ and $O_2$, a contradiction. Thus, $\overline{T}$ does not contain any circle and is therefore a topological spanning tree. $\square$

**Lemma 13.** *Let $C_1$ and $C_2$ be two circles in $\|G\|$. Then $E(C_2) \subseteq E(C_1)$ implies that $E(C_1) = E(C_2)$.*

*Proof.* We first prove the following:

*For every point $x \in \overline{C_1} \setminus C_1$ there is a point $y \in C_1$ such that $x$ and $y$ are indistinguishable.* (1)

Indeed, consider a $z \in \|G\|$ that is distinguishable from all points in $C_1$. Thus, we may pick for every $p \in C_1$ two disjoint open neighbourhoods $O_z^p$ and $O_p$ of $z$ and $p$, respectively. Note that $C_1$ is compact, being a continuous image of the compact space $S^1$. Thus, there is a finite subcover $O_{p_1} \cup \ldots \cup O_{p_n}$ of $C_1$. Then, the open set $\cap_{i=1}^n O_z^{p_i}$ is disjoint from $C_1$ and contains $z$. Hence, $z$ does not lie in the closure of $C_1$. This proves (1).

Next, suppose that $E(C_2)$ is a proper subset of $E(C_1)$, and pick $e \in E(C_2)$ and $f \in E(C_1) \setminus E(C_2)$. Since $X := C_1 \setminus (\mathring{e} \cup \mathring{f})$ is disconnected there exist two disjoint non-empty open sets $O_1'$ and $O_2'$ of $X$ with $X = O_1' \cup O_2'$. For $j = 1, 2$, denote by $I_j$ the set of points $x$ in $\|G\|$ for which there is a $y \in O_j'$ such that $x$ and $y$ are indistinguishable. Then $O_1 := O_1' \cup I_1$ and $O_2 := O_2' \cup I_2$ are disjoint and open subsets of $X \cup I_1 \cup I_2$. Moreover, it follows from (1) that $\overline{C_1} \setminus (\mathring{e} \cup \mathring{f}) = X \cup I_1 \cup I_2$. Therefore, $O_1$ and $O_2$ are two disjoint non-empty open sets of $\overline{C_1} \setminus (\mathring{e} \cup \mathring{f})$ with $\overline{C_1} \setminus (\mathring{e} \cup \mathring{f}) = O_1 \cup O_2$.

As $C_2 \setminus \mathring{e}$ is a connected subset of $\overline{C_1} \setminus (\mathring{e} \cup \mathring{f})$ it lies in $O_1$ or in $O_2$, let us say in $O_1$. Then $\tilde{O}_1 := O_1 \cup \mathring{e}$ and $O_2$ are two disjoint non-empty open subsets of $\overline{C_1} \setminus \mathring{f}$ with $\overline{C_1} \setminus \mathring{f} = \tilde{O}_1 \cup O_2$. By (1), this means that also $C_1 \setminus \mathring{f}$ is disconnected. But $C_1 \setminus \mathring{f}$ is a continuous image of a connected space, and hence connected. $\square$

**Lemma 14.** *Let $T$ be a standard subspace of $\|G\|$. Then the following statements are equivalent:*

(i) *$T$ is a topological spanning tree of $\|G\|$.*

(ii) *$T$ is maximally acirclic, that is, it does not contain a circle but adding any edge in $E(G) \setminus E(T)$ creates one.*

(iii) *$E(T)$ meets every finite bond, and is minimal with this property.*

*Proof.* Let us first prove a part of (iii)→(i) before dealing with all the other implications.

$$\text{\textit{If } } E(T) \text{ \textit{meets every finite bond then} } T \text{ \textit{is spanning and path-connected.}} \tag{2}$$

Suppose that the closure $X$ of $(V(G), E(T))$ is not connected. Then there are two disjoint non-empty open sets $O_1$ and $O_2$ of $X$ with $X = O_1 \cup O_2$. From Lemma 8 we get that the cut consisting of the edges with one endvertex in $O_1$ and the other in $O_2$ is finite. Since each of $O_1$ and $O_2$ needs to contain a vertex, this cut is non-empty. Hence, $E(T)$ misses a finite bond, a contradiction. Therefore, $T = X$ is connected and then, by Lemma 11, path-connected.

(i) → (ii) Consider any edge $e \notin E(G) \setminus E(T)$. If the endvertices $u$ and $v$ of $e$ cannot be separated by finitely many edges then $e - u$ (and also $e - v$) is a circle in $\|G\|$. Otherwise, any topological $u$–$v$ path contains an $u$–$v$ arc by Lemma 9. In particular, $T$ contains an $u$–$v$ arc that together with $e$ forms a circle.

(ii) → (iii) Suppose that $E(T)$ misses a finite bond $F$. Pick $e \in F$, and let $C$ be a circle in $T \cup e$ through $\mathring{e}$. Pick an inner point of every edge in $F$ and denote the set of these points by $Z$. Then the two components of $\|G\| \setminus Z$, each of which contains an endvertex of $e$, form two disjoint open sets containing $T$. However, $C \setminus \mathring{e} \subseteq T$ is a connected set that meets both of these disjoint open sets, which is impossible. Thus, $E(T)$ meets every finite bond. In particular, $T$ is spanning and path-connected, by (2).

Let $f$ be any edge in $E(T)$, and let us show that $E(T) - f$ misses some finite bond. Denote the endvertices of $f$ by $r$ and $s$, and observe that $r$ and $s$ can be separated by finitely many edges as $T$ is acirclic. Denote by $K_r$ and $K_s$ the path-components of $T \setminus \mathring{f}$ containing $r$ and $s$, respectively. By Lemma 9 and as $T$ does not contain any circle, $K_r$ and $K_s$ are distinct, and thus disjoint. As $T$ is path-connected, it follows that $T \setminus \mathring{f}$ is the disjoint union of the open sets $K_r$ and $K_s$. Now Lemma 8 yields that there are only finitely many edges with one endvertex in $K_r$ and the other in $K_s$. As $T$ is spanning this means that $E(T) - f$ misses a finite cut.

(iii) → (i) By (2), we only need to check that $T$ does not contain any circle. Suppose there exists a circle $C \subseteq T$, and pick some $e \in E(C)$. By the minimality of $E(T)$ there exists a finite bond $F$ so that $F$ is disjoint from $T \setminus \mathring{e}$. Then, however, picking inner points from the edges in $F$ yields a set $Z$, so that the connected set $C \setminus \mathring{e}$ is contained in $\|G\| \setminus Z$ but meets two components of $\|G\| \setminus Z$, which is impossible. □

We can finally prove our main theorem, which we restate:

**Theorem 2.**

(i) *The edge sets of the circles in* $\|G\|$ *are the circuits of a matroid* $M_{\mathrm{C}}(G)$, *the* cycle matroid *of* $G$.

(ii) *The bases of* $M_{\mathrm{C}}(G)$ *are the edge sets of the topological spanning trees of* $G$.

(iii) *The cycle matroid* $M_{\mathrm{C}}(G)$ *is the dual of the finite-bond matroid* $M_{\mathrm{FB}}(G)$.

*Proof.* To bypass the need to verify any matroid axioms, we *define* $M_{\mathrm{C}}(G)$ as the dual of $M_{\mathrm{FB}}(G)$ (which we know exists [3]), ie., as the matroid whose bases $B$ are the complements of the bases of $M_{\mathrm{FB}}(G)$. These latter are the maximal edge sets not containing a finite bond, so the bases $B$ of $M_{\mathrm{C}}(G)$ are the minimal edge sets meeting every finite bond. By Lemma 14 below, this is equivalent to $B$ being the edge set of a topological spanning tree of $\|G\|$.

We have defined $M_{\mathrm{C}}(G)$ so as to make (iii) true, and shown (ii). It remains to show (i): that the circuits of $M_{\mathrm{C}}(G)$ are the edge sets of the circles in $\|G\|$. Since no circuit of a matroid contains another circuit, and since by Lemma 13 no edge set of a circle contains another such set, it suffices to show that every circuit contains the edge set of a circle, and conversely every edge set of a circle contains a circuit.

For the first of these statements note that, by assertion (ii), a circuit $D$ of $M_{\mathrm{C}}(G)$ does not extend to the edge set of a topological spanning tree. Hence by Lemma 12 its closure $\overline{\bigcup D}$ in $\|G\|$ contains a circle $C$. For the second statement, note that the edge set $D$ of a circle $C$ is not contained in the edge set of a topological spanning tree $T$, because $T$ is closed and would therefore contain $\overline{\bigcup D} \supseteq C$, contradicting its definition. Hence $D$ is dependent in $M_{\mathrm{C}}(G)$, and therefore contains a circuit [3]. $\square$

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