HOPF ALGEBRAS AND FROBENIUS ALGEBRAS
IN FINITE TENSOR CATEGORIES

Christoph Schweigert [a]   and   Jürgen Fuchs [b]

[a] *Organisationseinheit Mathematik, Universität Hamburg*
Bereich Algebra und Zahlentheorie
Bundesstraße 55, D – 20 146 Hamburg

[b] *Teoretisk fysik, Karlstads Universitet*
Universitetsgatan 21, S – 651 88 Karlstad

Abstract

We discuss algebraic and representation theoretic structures in braided tensor categories $\mathcal{C}$ which obey certain finiteness conditions. Much interesting structure of such a category is encoded in a Hopf algebra $\mathcal{H}$ in $\mathcal{C}$. In particular, the Hopf algebra $\mathcal{H}$ gives rise to representations of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on various morphism spaces. We also explain how every symmetric special Frobenius algebra in a semisimple modular category provides additional structure related to these representations.

1 Braided finite tensor categories

Algebra and representation theory in semisimple ribbon categories has been an active field over the last decade, having applications to quantum groups, low-dimensional topology and quantum field theory. More recently, partly in connection with progress in the understanding of logarithmic conformal field theories, there has been increased interest in tensor categories that are not semisimple any longer, but still obey certain finiteness conditions [EO].

Owing to the work of various groups (for some recent results see e.g. [GT, NT]), examples of such categories are by now rather explicitly understood, at least as abelian categories. In this section we describe a class of categories that has received particular attention. This will allow us to define the structure of a semisimple modular tensor category. To extend the notion of modular tensor category to the non-semisimple case requires further categorical constructions involving Hopf algebras and coends; these will be introduced in section 2. These constructions also provide representations of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on certain morphism spaces. In section 3 we show that symmetric special Frobenius algebras in semisimple modular tensor categories give rise to structures related to such $\mathrm{SL}(2,\mathbb{Z})$-representations.

Let $\Bbbk$ be an algebraically closed field of characteristic zero and $\mathcal{V}ect_{\text{fin}}(\Bbbk)$ the category of finite-dimensional $\Bbbk$-vector spaces.

Definition 1.1.
A finite category $\mathcal{C}$ is an abelian category enriched over $\mathcal{V}ect_{\text{fin}}(\Bbbk)$ with the following additional properties:

1. Every object has finite length.

2. Every object $X \in \mathcal{C}$ has a projective cover $P(X) \in \mathcal{C}$.

3. The set I of isomorphism classes of simple objects is finite.

It can be shown that an abelian category is a finite category if and only if it is equivalent to the category of (left, say) modules over a finite-dimensional $\Bbbk$-algebra.

We will be concerned with finite categories that have additional structure. First, they are tensor categories, i.e., for our purposes, sovereign monoidal categories:

Definition 1.2.
A tensor category over a field $\Bbbk$ is a $\Bbbk$-linear abelian monoidal category $\mathcal{C}$ with simple tensor unit $\mathbf{1}$ and with a left and a right duality in the sense of [Ka, Def. XIV.2.1], such that the category is sovereign, i.e. the two functors

$$?^{\vee}, \, {}^{\vee}? : \quad \mathcal{C} \to \mathcal{C}^{\text{opp}}$$

that are induced by the left and right dualities coincide.
Thus for any object $V \in \mathcal{C}$ there exists an object $V^{\vee} = {}^{\vee}V \in \mathcal{C}$ together with morphisms

$$b_V : \quad \mathbf{1} \to V \otimes V^{\vee} \quad and \quad d_V : \quad V^{\vee} \otimes V \to \mathbf{1}$$

(right duality) and

$$\tilde{b}_V : \quad \mathbf{1} \to V^{\vee} \otimes V \quad and \quad \tilde{d}_V : \quad V \otimes V^{\vee} \to \mathbf{1}$$

(left duality), obeying the relations

$$(\text{id}_V \otimes d_V) \circ (b_V \otimes \text{id}_V) = \text{id}_V \quad and \quad (d_V \otimes \text{id}_{V^{\vee}}) \circ (\text{id}_{V^{\vee}} \otimes b_V) = \text{id}_{V^{\vee}}$$

and analogous relations for the right duality, and the duality functors not only coincide on objects, but also on morphisms, i.e.

$$(d_V \otimes \text{id}_{U^{\vee}}) \circ (\text{id}_{V^{\vee}} \otimes f \otimes \text{id}_{U^{\vee}}) \circ (\text{id}_{V^{\vee}} \otimes b_U) = (\text{id}_{U^{\vee}} \otimes \tilde{d}_V) \circ (\text{id}_{U^{\vee}} \otimes f \otimes \text{id}_{V^{\vee}}) \circ (\tilde{b}_U \otimes \text{id}_{V^{\vee}})$$

for all morphisms $f : U \to V$.

To give an example, the category of finite-dimensional left modules over any finite-dimensional complex Hopf algebra H is a finite tensor category. We also note the following properties of tensor categories which are direct consequences of the definition:

1. The tensor product functor $\otimes$ is exact in both arguments.

2. For any object $U \in \mathcal{C}$ there is a cyclic trace $\text{tr}: \text{End}(U) \to \text{End}(\mathbf{1})$, and thus a dimension $\dim(U) = \text{tr}(\text{id}_U)$.

The categories of our interest have in addition a braiding:

Definition 1.3.
A braiding *on a tensor category $\mathcal{C}$ is a natural isomorphism*

$$c: \quad \otimes \to \otimes^{\mathrm{opp}}$$

that is compatible with the tensor product, i.e. satisfies

$$c_{U \otimes V, W} = (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}) \quad and \quad c_{U, V \otimes W} = (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W).$$

We choose a set $\{U_i\}_{i \in I}$ of representatives for the isomorphism classes of simple objects and take the tensor unit to be the representative of its isomorphism class, writing $\mathbf{1} = U_0$.

We are now ready to formulate the notion of a modular tensor category. Our definition will, however, still be preliminary, as it has the disadvantage of being sensible only for semisimple categories.

Definition 1.4.
A semisimple *modular tensor category is a semisimple finite braided tensor category such that the matrix $(S_{ij})_{i,j \in I}$ with entries*

$$S_{ij} := \mathrm{tr}(c_{U_j, U_i} \circ c_{U_i, U_j})$$

is non-degenerate.

Two remarks are in order:

Remarks 1.5.

1. *The representation categories of several algebraic structures give examples of semisimple modular tensor categories:*

   (a) *Left modules over connected factorizable ribbon weak Hopf algebras with Haar integral over an algebraically closed field* [NTV].

   (b) *Local sectors of a finite μ-index net of von Neumann algebras on $\mathbb{R}$, if the net is strongly additive and split* [KLM].

   (c) *Representations of selfdual C_2-cofinite vertex algebras with an additional finiteness condition on the homogeneous components and which have semisimple representation categories* [Hu].

2. *By the results of Reshetikhin and Turaev* [RT, T], *every $\mathbb{C}$-linear semisimple modular tensor category $\mathcal{C}$ provides a three-dimensional topological field theory, i.e. a tensor functor*

$$\mathrm{tft}_{\mathcal{C}}: \quad \mathrm{cobord}_{3,2}^{\mathcal{C}} \to \mathcal{V}ect_{\mathrm{fin}}(\mathbb{C}).$$

Here $\mathrm{cobord}_{3,2}^{\mathcal{C}}$ is a category of three-dimensional cobordisms with embedded ribbon graphs that are decorated by objects and morphisms of $\mathcal{C}$.
There are also various results for the case of non-semisimple modular categories. We refer to [He, L1, V] *for the construction of three-manifold invariants, to* [L1] *for the construction of representations of mapping class groups, and to* [KL] *for an attempt to unify these constructions in terms of a topological quantum field theory defined on a double category of manifolds with corners.*

2 Hopf algebras, coends and modular tensor categories

Our goal is to study some algebraic and representation theoretic structures in tensor categories of the type introduced above. To simplify the exposition, we suppose that we have replaced the tensor category $\mathcal{C}$ by an equivalent strict tensor category.

Definition 2.1.
A (unital, associative) algebra *in a (strict) tensor category $\mathcal{C}$ is a triple consisting of an object $A \in \mathcal{C}$, a multiplication morphism $m \in \mathrm{Hom}(A \otimes A, A)$ and a unit morphism $\eta \in \mathrm{Hom}(\mathbf{1}, A)$, subject to the relations*

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m) \qquad and \qquad m \circ (\eta \otimes \mathrm{id}_A) = \mathrm{id}_A = m \circ (\mathrm{id}_A \otimes \eta)\,.$$

which express associativity and unitality.
Analogously, a coalgebra *in $\mathcal{C}$ is a triple consisting of an object C, a comultiplication $\Delta\colon C \to C \otimes C$ and a counit $\varepsilon\colon C \to \mathbf{1}$ obeying coassociativity and counit conditions.*

Similarly one generalizes other basic notions of algebra to the categorical setting and introduces modules, bimodules, comodules, etc. (For a more complete exposition we refer to [FRS1].)

To proceed we observe that the multiplication of an algebra A endows both A itself and $A \otimes A$ with the structure of an A-bimodule. Further, if the category $\mathcal{C}$ is braided, then the object $A \otimes A$ can be endowed with the structure of a unital associative algebra by taking the morphisms $(m \otimes m) \circ (\mathrm{id}_A \otimes c_{A,A} \otimes \mathrm{id}_A)$ as the product and $\eta \otimes \eta$ as the counit.

Definition 2.2.
Let $\mathcal{C}$ be a tensor category and $A \in \mathcal{C}$ an object which is endowed with both the structure (A, m, η) of a unital associative algebra and the structure (A, Δ, ε) of a counital coassociative coalgebra.

1. *$(A, m, \eta, \Delta, \varepsilon)$ is called a* Frobenius algebra *iff $\Delta\colon A \to A \otimes A$ is a morphism of bimodules.*

2. *$(A, m, \eta, \Delta, \varepsilon)$ is called a* bialgebra *iff $\Delta\colon A \to A \otimes A$ is a morphism of unital algebras.*

3. *A bialgebra with an antipode $\mathrm{S}\colon A \to A$ (with properties analogous to the classical case) is called a* Hopf algebra.

To construct concrete examples of such structures, we recall a few notions from category theory.

Definition 2.3.
Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F\colon \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \to \mathcal{D}$ be a functor.

1. *For B an object of $\mathcal{D}$, a* dinatural transformation *$\varphi\colon F \Rightarrow B$ is a family of morphisms $\varphi_X\colon F(X,X) \to B$ for every object $X \in \mathcal{C}$ such that the diagram*

$$
\begin{array}{ccc}
F(Y,X) & \xrightarrow{\;F(\mathrm{id}_Y, f)\;} & F(Y,Y) \\
{\scriptstyle F(f,\mathrm{id}_X)}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_Y} \\
F(X,X) & \xrightarrow{\;\;\varphi_X\;\;} & B
\end{array}
$$

commutes for all morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$.

2. A coend *for the functor F is a dinatural transformation $\iota\colon F \Rightarrow A$ with the universal property that any dinatural transformation $\varphi\colon F \Rightarrow B$ uniquely factorizes:*

$$
\begin{array}{ccc}
F(Y,X) & \xrightarrow{\;F(\mathrm{id}_Y,f)\;} & F(Y,Y) \\[2pt]
{\scriptstyle F(f,\mathrm{id}_X)}\big\downarrow & & \big\downarrow{\scriptstyle \iota_Y} \\[2pt]
F(X,X) & \xrightarrow{\quad\iota_X\quad} & A
\end{array}
$$

with φ_Y, φ_X mapping to B.

If the coend exists, it is unique up to unique isomorphism. It is denoted by $\int^X F(X,X)$. The universal property implies that a morphism with domain $\int^X F(X,X)$ can be specified by a dinatural family of morphisms $X^\vee \otimes X \to B$ for each object $X \in \mathcal{C}$.

We are now ready to formulate the following result.

Theorem 2.4. [L2]
In a finite braided tensor category $\mathcal{C}$, the coend

$$
\mathcal{H} := \int^X X^\vee \otimes X
$$

of the functor

$$
\begin{aligned}
F\colon \quad \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} &\to \mathcal{C} \\
(U,V) &\mapsto U^\vee \otimes V
\end{aligned}
$$

exists, and it has a natural structure of a Hopf algebra in $\mathcal{C}$.

Proof:
For a proof we refer e.g. to [V]. Here we only indicate how the structural morphisms of the Hopf algebra are constructed. Owing to the universal property, the counit $\varepsilon_{\mathcal{H}}\colon \mathcal{H} \to \mathbf{1}$ can be specified by the dinatural family

$$
\varepsilon_{\mathcal{H}} \circ \iota_X = d_X\colon \quad X^\vee \otimes X \to \mathbf{1}
$$

of morphisms. Similarly, the coproduct is given by the dinatural family

$$
\Delta_{\mathcal{H}} \circ \iota_{\mathcal{H}} = (\iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^\vee} \otimes b_X \otimes \mathrm{id}_X)\colon \quad X^\vee \otimes X \to \mathcal{H} \otimes \mathcal{H}\,.
$$

It should be appreciated that the braiding does not enter in the coalgebra structure of $\mathcal{H}$. It does enter in the product, though. We refrain from writing out the product as a formula. Instead, we use the graphical formalism [JS, FRS1] to display all structural morphisms $(m_{\mathcal{H}}, \Delta_{\mathcal{H}}, \eta_{\mathcal{H}}, \epsilon_{\mathcal{H}}, \mathrm{S}_{\mathcal{H}})$ of the Hopf algebra $\mathcal{H}$. More precisely, we display dinatural families of

morphisms so that the identities apply to all $X, Y \in \mathcal{C}$:
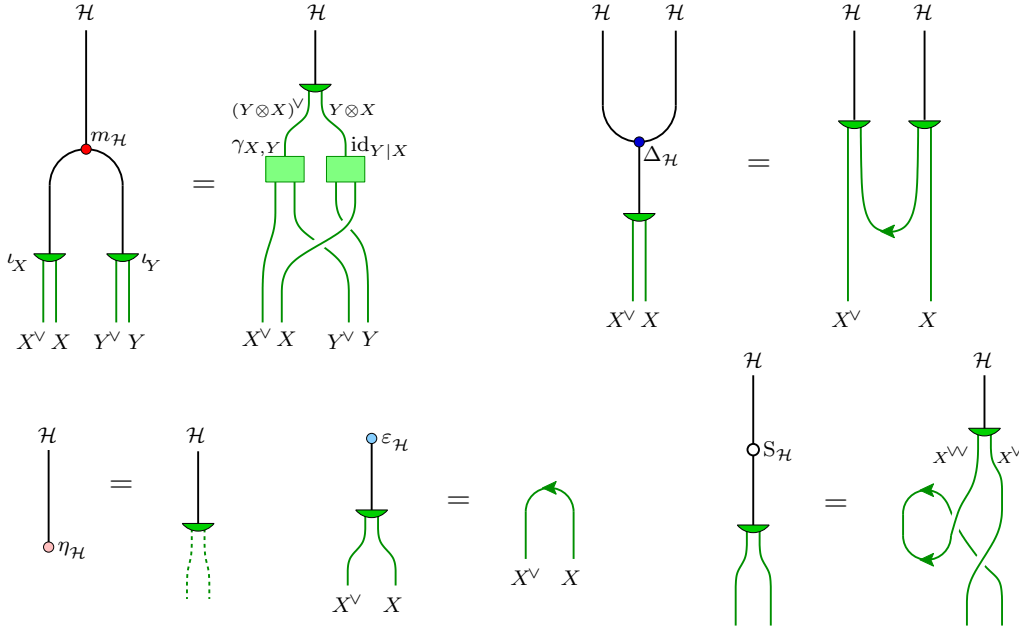
(Here $\gamma_{X,Y}$ is the canonical identification of $X^\vee \otimes Y^\vee$ with $(Y \otimes X)^\vee$, and $\mathrm{id}^{X|Y}$ is the one of $\mathrm{id}_X \otimes \mathrm{id}_Y$ with $\mathrm{id}_{X \otimes Y}$.) □

An explicit description of the Hopf algebra $\mathcal{H} \in \mathcal{C}$ is available in the following specific situations:

Examples 2.5.

1. *For $\mathcal{C} = H$-mod the category of left modules over a finite-dimensional Hopf algebra H, the coend $\mathcal{H} = \int^X X^\vee \otimes X$ is the dual space $H^* = \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$ endowed with the coadjoint representation. The structure morphism for the coend for a module $M \in H$-mod is*

$$\iota_M : \quad M^\vee \otimes M \ \rightarrow \ H^*$$
$$\tilde{m} \otimes m \ \mapsto \ (h \mapsto \langle \tilde{m}, h.m \rangle) \,.$$

   For more details see [V, Sect. 4.5].

2. *If the finite tensor category $\mathcal{C}$ is semisimple, then the Hopf algebra decomposes as an object as $\mathcal{H} = \bigoplus_{i \in I} U_i^\vee \otimes U_i$, see [V, Sect. 3.2].*

The Hopf algebra in question has additional structure: it comes with an integral and with a Hopf pairing.

Definition 2.6.
A left integral *of a bialgebra $(H, m, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ is a non-zero morphism $\mu_l \in \mathrm{Hom}(\mathbf{1}, H)$ satisfying*

$$m \circ (\mathrm{id}_H \otimes \mu_l) = \mu_l \circ \varepsilon \,.$$

A right cointegral *of H is a non-zero morphism $\lambda_r \in \mathrm{Hom}(H, \mathbf{1})$ satisfying*

$$(\lambda \otimes \mathrm{id}_H) \circ \Delta = \eta \circ \lambda \,.$$

Right integrals μ_r and left cointegrals λ_l are defined analogously.

The Hopf algebra $\mathcal{H}$ in any finite braided tensor category has left and right integrals, as can be shown [L2] by a generalization of the classical argument of Sweedler that an integral exists for any finite-dimensional Hopf algebra. If $\mathcal{C}$ is semisimple, then the integral of $\mathcal{H}$ can be given explicitly [Ke, Sect. 2.5]:

$$\mu_l = \mu_r = \bigoplus_{i \in I} \dim(U_i)\, b_{U_i}\,.$$

Remarks 2.7.

1. *If the left and right integrals of $\mathcal{H}$ coincide, then the integral can be used as a Kirby element and provides invariants of three-manifolds [V]. If the category $\mathcal{C}$ is the category of representations of a finite-dimensional Hopf algebra, this is the Lyubashenko [L1] invariant.*

2. *The category $\mathcal{C}$ is semisimple if and only if the morphism $\varepsilon \circ \mu \in \mathrm{Hom}(\mathbf{1},\mathbf{1})$ does not vanish, i.e. iff the constant $\mathcal{D}^2$ of proportionality in*

$$\varepsilon \circ \mu = \mathcal{D}^2 \, \mathrm{id}_{\mathbf{1}}$$

   is non-zero. (This generalizes Maschke's theorem.) This constant, in turn, which in the semisimple case (with $\mu_l = \mu_r$ normalized as above) has the value $\mathcal{D}^2 = \sum_{i \in I}(\dim U_i)^2$, crucially enters the normalizations in the Reshetikhin-Turaev construction of topological field theories (see e.g. chapter II of [T]).
   Invariants based on nonsemisimple categories, like the Hennings invariant, vanish on many three-manifolds. This can be traced back to the vanishing of $\varepsilon \circ \mu$ [CKS].

3. *Any Hopf algebra H in $\mathcal{C}$ that has a left integral μ and a right cointegral λ with $\lambda \circ \mu \neq 0$ is naturally also a Frobenius algebra, with the same algebra structure.*

Definition 2.8.
A Hopf pairing of a Hopf algebra H in $\mathcal{C}$ is a morphism

$$\omega_H: \quad H \otimes H \to \mathbf{1}$$

such that

$$\omega_H \circ (m \otimes \mathrm{id}_H) = (\omega_H \otimes \omega_H) \circ (\mathrm{id}_H \otimes c_{H,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \mathrm{id}_H \otimes \Delta)\,,$$
$$\omega_H \circ (\mathrm{id}_H \otimes m) = (\omega_H \otimes \omega_H) \circ (\mathrm{id}_H \otimes c_{H,H}^{-1} \otimes \mathrm{id}_H) \circ (\Delta \otimes \mathrm{id}_H \otimes \mathrm{id}_H)$$
$$and \quad \omega_H \circ (\eta \otimes \mathrm{id}_H) = \varepsilon = \omega_H \circ (\mathrm{id}_H \otimes \eta)\,.$$

As one easily checks, a non-degenerate Hopf pairing gives an isomorphism $H \to H^\vee$ of Hopf algebras.

The dinatural family of morphisms

$$(d_X \otimes d_Y) \circ [\mathrm{id}_{X^\vee} \otimes (c_{Y^\vee,X} \circ c_{X,Y^\vee}) \otimes \mathrm{id}_Y]$$

induces a bilinear pairing $\omega_{\mathcal{H}}: \mathcal{H} \otimes \mathcal{H} \to \mathbf{1}$ on the coend $\mathcal{H} = \int^X X^\vee \otimes X$ of a finite braided tensor category. It endows [L1] the Hopf algebra $\mathcal{H}$ with a symmetric Hopf pairing.

We are now finally in a position to give a conceptual definition of a modular finite tensor category without requiring it to be semisimple:

Definition 2.9.
A modular finite tensor category *is a braided finite tensor category for which the Hopf pairing* $\omega_{\mathcal{H}}$ *is non-degenerate.*

Example 2.10.
The category H-mod of left modules over a finite-dimensional factorizable ribbon Hopf algebra H is a modular finite tensor category [LM, L1].

One can show [L2, Thm. 6.11] that if $\mathcal{C}$ is modular in the sense of definition 2.9, then the left integral and the right integral of $\mathcal{H}$ coincide.

As the terminology suggests, there is a relation with the modular group $SL(2,\mathbb{Z})$. To see this, we will now obtain elements $S, T \in \text{End}(\mathcal{H})$ that satisfy the relations for generators of $SL(2,\mathbb{Z})$.

Recall the notion of the center $Z(\mathcal{C})$ of a category as the algebra of natural endotransformations of the identity endofunctor of $\mathcal{C}$ [Ma]. Given such a natural transformation $(\phi_X)_{X \in \mathcal{C}}$ with $\phi_X \in \text{End}(X)$, one checks that $(\iota_X \circ (\text{id}_{X^\vee} \otimes \phi_X)_{X \in \mathcal{C}}$ is a dinatural family, so that the universal property of the coend gives us a unique endomorphism $\overline{\phi}_{\mathcal{H}}$ of $\mathcal{H}$ such that the diagram

$$
\begin{array}{ccc}
X^\vee \otimes X & \xrightarrow{\ \text{id} \otimes \phi_X\ } & X^\vee \otimes X \\
{\scriptstyle \iota_X} \downarrow & & \downarrow {\scriptstyle \iota_X} \\
\mathcal{H} & \dashrightarrow{\ \overline{\phi}_{\mathcal{H}}\ } & \mathcal{H}
\end{array}
$$

commutes, leading to an injective linear map $Z(\mathcal{C}) \to \text{End}(\mathcal{H})$.

Since $\mathcal{H}$ has in particular the structure of a coalgebra and $\mathbf{1}$ the structure of an algebra, the vector space $\text{Hom}(\mathcal{H}, \mathbf{1})$ has a natural structure of a $\Bbbk$-algebra. Concatenating with the counit $\varepsilon_{\mathcal{H}}$ gives a map

$$
Z(\mathcal{C}) \longrightarrow \text{End}(\mathcal{H}) \xrightarrow{\ (\varepsilon_{\mathcal{H}})_*\ } \text{Hom}(\mathcal{H}, \mathbf{1})\,,
$$

which can be shown [Ke, Lemma 4] to be an isomorphism of $\Bbbk$-algebras. The vector space on the right hand side is dual to the vector space $\text{Hom}(\mathbf{1}, \mathcal{H})$, of which one can think as the appropriate substitute for the space of class functions. Hence $\text{Hom}(\mathbf{1}, \mathcal{H})$ would be a natural starting point for constructing a vector space assigned to the torus T^2 by a topological field theory based on $\mathcal{C}$.
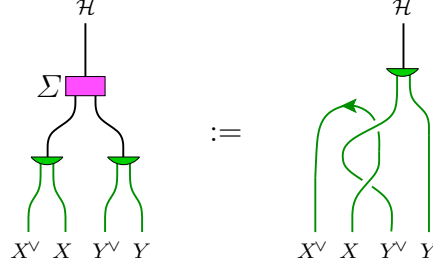
If the category $\mathcal{C}$ is a ribbon category, we have the ribbon element $\nu \in Z(\mathcal{C})$. We set

$$
T_{\mathcal{H}} := \overline{\nu}_{\mathcal{H}} \ \in \ \text{End}(\mathcal{H})\,.
$$

Pictorially,
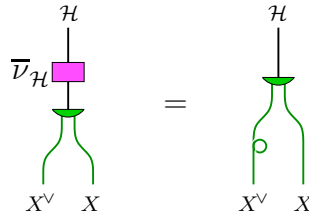
Another morphism $\Sigma\colon \mathcal{H}\otimes\mathcal{H}\to\mathcal{H}$ is obtained from the following family of morphisms which is dinatural both in X and in Y:

Composing this endomorphism of $\mathcal{H}$ with a left or right integral $\mu\colon \mathbf{1}\to\mathcal{H}$ one arrives at an endomorphism

$$S_{\mathcal{H}} := \Sigma\circ(\mathrm{id}_{\mathcal{H}}\otimes\mu) \in \mathrm{End}(\mathcal{H})\,.$$

For $\xi\in\Bbbk^{\times}$, denote by $\Bbbk_\xi\mathrm{SL}(2,\mathbb{Z})$ the twisted group algebra of $\mathrm{SL}(2,\mathbb{Z})$ with relations $S^4=1$ and $(ST)^3=\xi\,S^2$. The previous construction and the following result are due to Lyubashenko.

Theorem 2.11. [L2, Sect. 6]
Let $\mathcal{C}$ be modular. Then the two-sided integral of $\mathcal{H}$ can be normalized in such a way that the endomorphisms $S_{\mathcal{H}}$ and $T_{\mathcal{H}}$ of $\mathcal{H}$ provide a morphism of algebras

$$\Bbbk_\xi\mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{End}(\mathcal{H})$$

for some $\xi\in\Bbbk^{\times}$.

Since for every $U\in\mathcal{C}$ the morphism space $\mathrm{Hom}(U,\mathcal{H})$ is, by push-forward, a left module over the algebra $\mathrm{End}(\mathcal{H})$, we obtain this way projective representations of $\mathrm{SL}(2,\mathbb{Z})$ on all vector spaces $\mathrm{Hom}(U,\mathcal{H})$.

To set the stage for the results in the next section, we consider the map

$$\begin{aligned}\mathrm{Obj}(\mathcal{C}) &\to \mathrm{Hom}(\mathbf{1},\mathcal{H})\\ U &\mapsto \chi_U\end{aligned}$$

with

$$\chi_U\colon \quad 1 \xrightarrow{\;b_U\;} U^{\vee}\otimes U \xrightarrow{\;\iota_U\;} \mathcal{H}\,.$$

It factorizes to a morphism of rings

$$K_0(\mathcal{C}) \to \mathrm{Hom}(1,\mathcal{H}) = \mathrm{tft}_{\mathcal{C}}(T^2)\,.$$

If the category $\mathcal{C}$ is semisimple, then $\mathrm{Hom}(1,\mathcal{H})\cong\bigoplus_{i\in I}\mathrm{Hom}(1,U_i^{\vee}\otimes U_i)$, so that $\{\chi_{U_i}\}_{i\in I}$ constitutes a basis of the vector space $\mathrm{Hom}(\mathbf{1},\mathcal{H})$. If $\mathcal{C}$ is not semisimple, these elements are still linearly independent, but they do not form a basis any more. Pseudo-characters [Mi, GT] have been proposed as a (non-canonical) complement of this linearly independent set.

3 Frobenius algebras and braided induction

In this section we show that symmetric special Frobenius algebras (i.e. Frobenius algebras with two further properties, to be defined below) in a modular tensor category allow one to specify interesting structure related to the $\mathrm{SL}(2,\mathbb{Z})$-representation that we have just explained.

Given an algebra A in a braided (strict) tensor category, we consider the two tensor functors

$$\alpha_A^\pm : \quad \mathcal{C} \;\to\; A\text{-bimod}$$
$$U \;\mapsto\; \alpha_A^\pm(U)$$

which assign to an object $U \in \mathcal{C}$ the bimodule $(A \otimes U, \rho_l, \rho_r)$ for which the left action is given by multiplication and the right action by multiplication composed with a braiding,

$$\rho_l = m \otimes \mathrm{id}_U \in \mathrm{Hom}(A \otimes A \otimes U, A \otimes U)$$
$$\text{and} \qquad \rho_r = (m \otimes \mathrm{id}_U) \circ (\mathrm{id}_A \otimes c_{U,A}^{\pm 1})\,.$$

We call these functors *braided induction* functors. They have been introduced, under the name α-induction, in operator algebra theory [LR, X, BE]. For more details in a category-theoretic framework we refer to [O, Sect. 5.1].

We pause to recall that [VZ] an *Azumaya algebra* A is an algebra for which the two functors $\alpha_A^\pm$ are equivalences of tensor categories. This should be compared to the textbook definition of an Azumaya algebra in the tensor category of modules over a commutative $\Bbbk$-algebra A, requiring in particular the morphism

$$\psi_A : \quad A \otimes A^{\mathrm{opp}} \;\to\; \mathrm{End}(A)$$
$$a \otimes a' \;\mapsto\; (x \mapsto a \cdot x \cdot a')$$

to be an isomorphism of algebras. Indeed, in this situation for an Azumaya algebra A one has the following chain of equivalences:

$$A\text{-bimod} \xrightarrow{\;\sim\;} A \otimes A^{\mathrm{opp}}\text{-mod} \xrightarrow{\;\psi_A\;} \mathrm{End}(A)\text{-mod} \xrightarrow{\;\text{Morita}\;} \mathcal{V}\!ect(\Bbbk)\,.$$

We now introduce the properties of an algebra A to be symmetric and special.

Definition 3.1.
Let $\mathcal{C}$ be a tensor category.

1. *For $\mathcal{C}$ enriched over the category of $\Bbbk$-vector spaces, a* special algebra *in $\mathcal{C}$ is an object A of $\mathcal{C}$ that is endowed with an algebra structure (A, m, η) and a coalgebra structure (A, Δ, ε) such that*
$$\varepsilon \circ \eta = \beta_1 \,\mathrm{id}_{\mathbf{1}} \quad \text{and} \quad m \circ \Delta = \beta_A \,\mathrm{id}_A$$
with invertible elements $\beta_1, \beta_A \in \Bbbk^\times$.

2. *A* symmetric algebra *in $\mathcal{C}$ is an algebra (A, m, η) together with a morphism $\varepsilon \in \mathrm{Hom}(A, \mathbf{1})$ such that the two morphisms*
$$\Phi_1 := [(\varepsilon \circ m) \otimes \mathrm{id}_{A^\vee}] \circ (\mathrm{id}_A \otimes b_A) \in \mathrm{Hom}(A, A^\vee) \qquad \text{and} \tag{1}$$
$$\Phi_2 := [\mathrm{id}_{A^\vee} \otimes (\varepsilon \circ m)] \circ (\tilde{b}_A \otimes \mathrm{id}_A) \in \mathrm{Hom}(A, A^\vee) \tag{2}$$
are identical.

Special algebras are in particular separable, and as a consequence their categories of modules and bimodules are semisimple. A class of examples of special Frobenius algebras is supplied by the Frobenius algebra structure on a Hopf algebra H in $\mathcal{C}$, provided H is semisimple.

We now consider the case of a semisimple modular tensor category $\mathcal{C}$ and introduce for any algebra A in $\mathcal{C}$ the square matrix $(Z_{ij})_{i,j\in I}$ with entries

$$Z_{ij}(A) := \dim_{\Bbbk} \operatorname{Hom}_{A|A}(\alpha_A^-(U_i), \alpha_A^+(U_j^\vee))\,,$$

where $\operatorname{Hom}_{A|A}$ stands for homomorphisms of bimodules. Identifying A-bimod with the tensor category of module endofunctors of A-mod, one sees that the non-negative integers $Z_{ij}(A)$ only depend on the Morita class of A.

In this setting, and in case that the algebra A is symmetric and special, we can make the following statements.

Theorem 3.2. [FRS1, Thm. 5.1(i)]
For $\mathcal{C}$ a semisimple modular tensor category and A a special symmetric Frobenius algebra in $\mathcal{C}$, the morphism

$$\sum_{i,j\in I} Z_{ij}(A)\, \chi_i \otimes \chi_j \ \in \operatorname{Hom}(\mathbf{1}, \mathcal{H}) \otimes_{\Bbbk} \operatorname{Hom}(\mathbf{1}, \mathcal{H}) \tag{3}$$

is invariant under the diagonal action of $\operatorname{SL}(2,\mathbb{Z})$.

Remarks 3.3.

1. *In conformal field theory, the expression (3) has the interpretation of a partition function for bulk fields.*

2. *For semisimple tensor categories based on the* sl(2) *affine Lie algebra, an A-D-E pattern appears* [KO].

We finally summarize a few other results that hold under the assumption that $\mathcal{C}$ is a semisimple modular tensor category and A a symmetric special Frobenius algebra in $\mathcal{C}$. To formulate them, we need the following ingredients: The *fusion algebra*

$$R_{\mathcal{C}} := K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \Bbbk$$

is a separable commutative algebra with a natural basis $\{\, [U_i]\, \}_{i\in I}$ given by the isomorphism classes of simple objects. The matrix S introduced in definition 1.4 provides a natural bijection from the set of isomorphism classes of irreducible representations of $R_{\mathcal{C}}$ to I.

Theorem 3.4. [FRS1, Thm. 5.18]
For any special symmetric Frobenius algebra A the vector space $K_0(A\text{-mod}) \otimes_{\mathbb{Z}} \Bbbk$ is an $R_{\mathcal{C}}$-module. The multiset $\operatorname{Exp}(A\text{-mod})$ that contains the irreducible $R_{\mathcal{C}}$-representations, with their multiplicities in this $R_{\mathcal{C}}$-module, can be expressed in terms of the matrix $Z(A)$:

$$\operatorname{Exp}(A\text{-mod}) \,=\, \operatorname{Exp}(Z(A)) := \{i \in I \text{ with multiplicity } Z_{ii}(A)\}\,.$$

The observation that the vector space $K_0(A\text{-mod}) \otimes_{\mathbb{Z}} \Bbbk$ has a natural basis provided by the classes of simple A-modules gives

Corollary 3.5.
The number of isomorphism classes of simple A-modules equals $\operatorname{tr}(Z(A))$.

The category A-bimod of A-bimodules has the structure of a tensor category. From the fact that A is a symmetric special Frobenius algebra, it follows [FS] that A-bimod inherits left and right dualities from $\mathcal{C}$. Hence the tensor product on A-bimod is exact and thus $K_0(A\text{-bimod})$ is a ring. The corresponding $\Bbbk$-algebra can again be described in terms of the matrix $Z(A)$:

Theorem 3.6. [O, FRS2]
There is an isomorphism

$$K_0(A\text{-}bimod) \otimes_{\mathbb{Z}} \Bbbk \;\cong\; \bigoplus_{i,j \in I} \mathrm{Mat}_{Z_{ij}(A)}(\Bbbk)\,,$$

of $\Bbbk$-algebras, with $\mathrm{Mat}_n(\Bbbk)$ denoting the algebra of $\Bbbk$-valued $n \times n$-matrices.

Corollary 3.7.
The number of isomorphism classes of simple A-bimodules equals $\mathrm{tr}(ZZ^{\mathrm{t}})$.

Theorem 3.8. [FFRS, Prop. 4.7]
Any A-bimodule is a subquotient of a bimodule of the form $\alpha_A^+(U) \otimes_A \alpha_A^-(V)$ for some pair of objects $U, V \in \mathcal{C}$.

4   Outlook

We conclude this brief review with a few comments. First, all the results about algebra and representation theory in braided tensor categories that we have presented above are motivated by a construction of correlation functions of a rational conformal field theory as elements of vector spaces which are assigned by a topological field theory to a two-manifold. For details of this construction we refer to [SFR] and the literature given there.

In the conformal field theory context the matrix Z describes the partition function of bulk fields. The three-dimensional topology involved in the RCFT construction provides in particular a motivation for using the different braidings which lead to the functors α_A^+ and α_A^- as well as in the definition of $Z(A)$.

To extend the results obtained in connection with rational conformal field theory to non-semisimple finite braided tensor categories remains a major challenge. Intriguing first results include, at the level of chiral data, a generalization of the Verlinde formula (see [GT] and references given there), and at the level of partition functions, the bulk partition functions for logarithmic conformal field theories in the $(1, p)$-series found in [GR].

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