Axioms for infinite matroids

Henning Bruhn Reinhard Diestel Matthias Kriesell
Paul Wollan

Abstract

We give axiomatic foundations for non-finitary infinite matroids with duality, in terms of independent sets, bases, circuits and closure. This completes the solution to a problem of Rado of 1966.

Introduction

Traditionally, infinite matroids are most often defined like finite ones,[1] with the following additional axiom:

(I4) An infinite set is independent as soon as all its finite subsets are independent.

We shall call such set systems *finitary matroids*.

The additional axiom (I4) reflects the notion of linear independence in vector spaces, and also the absence of (finite) circuits from a set of edges in a graph. More generally, it is a direct consequence of (I4) that circuits, defined as minimal dependent sets, are finite.

An important and regrettable consequence of the additional axiom (I4) is that it spoils duality, one of the key features of finite matroid theory. For example, the cocircuits of an infinite uniform matroid of rank k would be the sets missing exactly $k-1$ points; since these sets are infinite, however, they cannot be the circuits of another finitary matroid. Similarly, every bond of an infinite graph would be a circuit in any dual of its cycle matroid—a set of edges minimal with the property of containing an edge from every spanning tree—but these sets can be infinite and hence will not be the circuits of a finitary matroid.

This situation prompted Rado in 1966 to ask for the development of a theory of non-finitary infinite matroids with duality [19, Problem P531]. Rado's challenge caused a flurry of activity in the late 1960s (see e.g. [14] for references), in which several authors suggested numerous possible notions of infinite matroids. Each of these highlighted one of the aspects of finite matroids (usually closure), some had duality built-in by force, but none came with a set of axioms similar to those known from finite matroids, something that would visibly identify these structures as 'matroids'. As a consequence, none of these notions was widely accepted as a notion of infinite matroids; compare Oxley [17, p. 68].

[1] The augmentation axiom is required only for finite sets: given independent sets I, I' with $|I| < |I'| < \infty$, there is an $x \in I' \setminus I$ such that $I + x$ is again independent.

This situation led to the popular belief, common to this day,[2] that Rado's problem may have no solution, that there may be no theory of infinite matroids with all the usual aspects including duality. And only one of the objects proposed in the 1960s was studied again later: the concept of a 'B-matroid', proposed by Higgs [11] among several similar notions. (These 'B-matroids' were studied further in the late 1970s by Oxley [14, 15, 16], who did a lot to clarify the concept.)

Our aim in this paper is to finally settle Rado's problem, in the affirmative. We propose four equivalent sets of matroid axioms, in terms of independent sets, bases, closure and circuits, that make duality possible. They will allow for infinite circuits, but default to finitary matroids when their circuits happen to be finite. Duality will work as familiar from finite matroids: the cobases are the complements of bases, and there are well-defined and dual operations of contraction and deletion extending the familiar finite operations.

Generic examples of these matroids abound: they include the duals of all finitary matroids, a vast class of structures that can now also be described in matroids terms. (These duals are not normally finitary.) There are also some 'primary' examples that occur naturally. Unlike finitary matroids, for instance, our matroids capture the duality of infinite graphs:[3] between their bonds and finite circuits, and between their finite bonds and the topological circuits of their Freudenthal compactification. Planar infinite graphs too can be characterized by the (non-finitary) duals of their finite-cycle matroids, just as the finite planar graphs are characterized by their matroid duals via Whitney's theorem. There are also some algebraic examples, such as from simplicial homology. Finally, a valuable source of examples will come from the existing literature on Higgs's 'B-matroids', see e.g. [1, 11, 14, 21, 23]. Indeed, building on the work of Oxley [14] we shall prove that Higgs's theory of 'B-matroids' has the same models as our theory. Thus, our matroids have been 'known' for some time—it simply had not been realised that they can be axiomatized in the way known from finite matroid theory.[4]

When developing our axioms we faced two challenges: to avoid the use of cardinalities, and to deal with limits. As concerns the latter, we want every independent set to extend to a basis (so that there can be an equivalent set of basis axioms, in which independent sets are defined as subsets of bases), and we want every dependent set to contain a circuit (so that there can be an equivalent set of circuit axioms, in which independent sets are defined as the sets not containing a circuit). It turns out that we have to require one of these as an additional axiom, but the other will then follow.

---

[2] From the Wikipedia entry at *Matroid* of 15 March 2010: "The theory of infinite matroids is much more complicated than that of finite matroids and forms a subject of its own. One of the difficulties is that there are many reasonable and useful definitions, none of which captures all the important aspects of finite matroid theory. For instance, it seems to be hard to have bases, circuits, and duality together in one notion of infinite matroids."

[3] It was this example that led us to Rado's problem: an observable duality that should, but could not, be described in matroid terms.

[4] Conversely, we were not aware of Higgs's work when we developed our axioms from the concrete dualities that we observed in infinite structures. We are grateful to James Oxley [18] for alerting us to the possibility of such a connection.

Devising axioms without reference to cardinalities is a more serious challenge. Consider two independent sets I_1, I_2 in a finite matroid. How can we translate the property, referred to in the third of the standard independence axioms, that $|I_1| < |I_2|$? If $I_1 \subseteq I_2$, this is equivalent (for finite sets) to $I_1 \subsetneq I_2$, and we can use the latter statement instead. But if $I_1 \not\subseteq I_2$, the only way to designate I_1 as 'smaller' and I_2 as 'larger' is to assume that I_2 is maximal among all the independent sets while I_1 is not—a much stronger statement (for finite matroids) that fails to capture size differences among non-maximal independent sets. Nevertheless, we shall see that this distinction will be enough.

Our paper is organized as follows. In Section 1 we state our axiom systems for infinite matroids, and provisionally define infinite matroids as set systems satisfying the independence axioms. Section 2 is devoted to examples of such infinite matroids. In Section 3 we establish a minimum of basic properties of our infinite matroids (including duality and the existence of minors): those that will enable us in Section 4 to prove that the independence axioms are in fact equivalent to the other axiomatic systems proposed in Section 1. Section 5 provides some alternative axiom systems, which are more technical to state but may be easier to verify, and are hence worth knowing. We also include the mixed set of independence and basis axioms developed for Higgs's 'B-matroids' by Oxley [14, 15]. In Section 6, finally, we illustrate our axioms by examples of set systems that narrowly fail to satisfy them, by missing just one axiom each. In particular, our axioms are shown to be independent.

Any matroid terminology not explained below is taken from Oxley [17]. Let E be any non-empty set, finite or infinite. This set E will be the default ground set for all matroids considered in this paper. We write $\overline{X} := E \setminus X$ for complements of sets $X \subseteq E$, and 2^E for the power set of E. Unless otherwise mentioned, the terms 'minimal' and 'maximal' refer to set inclusion. Given $\mathcal{E} \subseteq 2^E$, we write $\mathcal{E}^{\max}$ for the set of maximal elements of $\mathcal{E}$, and $\lceil \mathcal{E} \rceil$ for the *down-closure* of $\mathcal{E}$, the set of subsets of elements of $\mathcal{E}$. For $F \subseteq E$ and $x \in E$, we abbreviate $F \setminus \{x\}$ to $F - x$ and $F \cup \{x\}$ to $F + x$.

1 Axiom systems for infinite matroids

In this section we present our four systems of axioms for infinite matroids. They are stated, respectively, in terms of independent sets, bases, closures and circuits.

One central axiom that features in all these systems is that every independent set extends to a maximal one, inside any restriction $X \subseteq E$.[5] The notion of what constitutes an independent set, however, will depend on the type of axioms under consideration. We therefore state this extension axiom in more general form right away, without reference to independence, so as to be able to refer to it later from within different contexts.

Let $\mathcal{I} \subseteq 2^E$. The following statement describes a possible property of $\mathcal{I}$.

[5] Interestingly, we shall not need to require that every dependent set contains a minimal one. We need that too, but will be able to prove it; see Section 3.

(M) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{ I' \in \mathcal{I} \mid I \subseteq I' \subseteq X \}$ has a maximal element.

Note that the maximal superset of I in $\mathcal{I} \cap 2^X$ whose existence is asserted in (M) need not lie in $\mathcal{I}^{\max}$.

1.1 Independence axioms

The following statements about a set $\mathcal{I} \subseteq 2^E$ are our *independence axioms*:

(I1) $\emptyset \in \mathcal{I}$.

(I2) $\lceil \mathcal{I} \rceil = \mathcal{I}$, i.e., $\mathcal{I}$ is closed under taking subsets.

(I3) For all $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$ there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.

(IM) $\mathcal{I}$ satisfies (M).

We remark that although (IM) formally depends on our choice of E as well as that of $\mathcal{I}$, this dependence on E is not crucial: if $\mathcal{I}$ satisfies (IM) for some set E large enough that $E \supseteq \bigcup \mathcal{I}$, it does so for every such set E'.

When a set $\mathcal{I} \subseteq 2^E$ satisfies the independence axioms, we call the pair $(E, \mathcal{I})$ a *matroid* on E. We then call every element of $\mathcal{I}$ an *independent* set, every element of $2^E \setminus \mathcal{I}$ a *dependent* set, the maximal independent sets *bases*, and the minimal dependent sets *circuits*. The $2^E \to 2^E$ function mapping a set $X \subseteq E$ to the set

$$\mathrm{cl}(X) := X \cup \{ x \mid \exists I \subseteq X : I \in \mathcal{I} \text{ but } I + x \notin \mathcal{I} \}$$

will be called the *closure operator* on 2^E *associated with* $\mathcal{I}$.

1.2 Basis axioms

The following statements about a set $\mathcal{B} \subseteq 2^E$ are our *basis axioms*:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) Whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there is an element y of $B_2 \setminus B_1$ such that $(B_1 - x) + y \in \mathcal{B}$.

(BM) The set $\mathcal{I} := \lceil \mathcal{B} \rceil$ of all $\mathcal{B}$-*independent* sets satisfies (M).

1.3 Closure axioms

The following statements about a function $\mathrm{cl}\colon 2^E \to 2^E$ are our *closure axioms*:

(CL1) For all $X \subseteq E$ we have $X \subseteq \mathrm{cl}(X)$.

(CL2) For all $X \subseteq Y \subseteq E$ we have $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.

(CL3) For all $X \subseteq E$ we have $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.

(CL4) For all $Z \subseteq E$ and $x, y \in E$, if $y \in \mathrm{cl}(Z + x) \setminus \mathrm{cl}(Z)$ then $x \in \mathrm{cl}(Z + y)$.

(CLM) The set $\mathcal{I}$ of all cl-*independent* sets satisfies (M). These are the sets $I \subseteq E$ such that $x \notin \mathrm{cl}(I - x)$ for all $x \in I$.

1.4 Circuit axioms

The following statements about a set $\mathcal{C} \subseteq 2^E$ are our *circuit axioms*:

(C1) $\emptyset \notin \mathcal{C}$.

(C2) No element of $\mathcal{C}$ is a subset of another.

(C3) Whenever $X \subseteq C \in \mathcal{C}$ and $(C_x \mid x \in X)$ is a family of elements of $\mathcal{C}$ such that $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in C \smallsetminus \left(\bigcup_{x \in X} C_x\right)$ there exists an element $C' \in \mathcal{C}$ such that $z \in C' \subseteq \left(C \cup \bigcup_{x \in X} C_x\right) \smallsetminus X$.

(CM) The set $\mathcal{I}$ of all $\mathcal{C}$-*independent* sets satisfies (M). These are the sets $I \subseteq E$ such that $C \nsubseteq I$ for all $C \in \mathcal{C}$.

Axiom (C3) defaults for $|X| = 1$ to the usual ('strong') circuit elimination axiom for finite matroids. In particular, it implies that adding an element to a basis creates at most one circuit; the existence of such a (*fundamental*) circuit will follow from Lemma 3.8. For $|X| > 1$, the inclusion of a specified element z in C' is not just a convenience but essential: without it, the statement would in general be false even for finite matroids. (Take $X = C$ to be the rim of a wheel in its cycle matroid.) We shall see in Section 6 that the usual finite circuit elimination axiom is too weak to guarantee a matroid (Example 6.4).

We did not try to devise a set of rank axioms.

2 Examples

The purpose of this section is to show that the infinite matroids just defined do occur in nature: we give a small collection of natural examples from contexts in which, working on other problems, we encountered these matroids, and which made us look for a general definition.

Before we start, let us note that, for finite set systems, our definition of a matroid coincides with the usual definition. Indeed, finite matroids defined as usual are matroids in our sense: this is most easily seen in terms of our basis or closure axioms, which for finite E coincide with the usual basis or closure axioms. More generally, all finitary matroids are matroids in our sense: axiom (BM) follows by Zorn's Lemma applied to the independent subsets of X containing I.

Conversely, if a matroid in our sense happens to be finite or finitary (i.e., satisfies (I4) in addition to our axioms), it also satisfies the usual axioms for finite or finitary matroids: the finite augmentation axiom (see the introduction) is easy to deduce from Lemma 3.7 below, applied in our matroid's restriction to $I \cup I'$. Since every dependent set contains a circuit (Lemma 3.8), a matroid in our sense is finitary if and only if it has only finite circuits.

In what follows we shall concentrate on non-finitary matroids.

2.1 Generic non-finitary matroids

Since classical finitary matroids are matroids in our sense, and our matroids have duals, we at once have a large class of new matroids: duals of finitary matroids that are not themselves finitary. We already saw an example in the introduction: the duals of uniform matroids of finite rank. We remark that having a non-finitary dual is the rule rather than the exception for a finitary matroid: Las Vergnas [21] and Bean [1] showed that the only finitary matroids with finitary duals are the direct sums of finite matroids.

2.2 Cycle and bond matroids in graphs

There are two standard matroids associated with a graph G, both finitary: the *finite-cycle matroid* $M_{\mathrm{FC}}(G)$ whose circuits are the edge sets of the (finite) cycles of G, and the *finite-bond matroid* $M_{\mathrm{FB}}(G)$ whose circuits are the finite bonds of G. (A *bond* is a minimal non-empty cut.) In a finite graph these two matroids are dual.

When G is infinite, the dual of $M_{\mathrm{FC}}(G)$ is not $M_{\mathrm{FB}}(G)$ but the full *bond matroid* $M_{\mathrm{B}}(G)$. This is the matroid whose circuits are all the bonds of G, finite or infinite: these, as is easy to show, are the minimal edge sets meeting all the spanning trees of G (connected), the bases of $M_{\mathrm{FC}}(G)$. Similarly, the dual of $M_{\mathrm{FB}}(G)$ is no longer $M_{\mathrm{FC}}(G)$ but a matroid $M_{\mathrm{C}}(G)$ whose circuits can be infinite.

This matroid $M_{\mathrm{C}}(G)$ has a natural topological characterization [3]. When G is connected and locally finite, it is particularly natural: its circuits are the edge sets of the topological circles in $|G|$, the compact topological space obtained from G by adding its ends.[6] Its bases are the edge sets of the topological spanning trees of G, the arc-connected standard subspaces of $|G|$ that contain every vertex (and every end) but lose their connectedness if any edge is deleted.

Theorem 2.1. [3] *For a locally finite connected graph G, the dual of its finite-bond matroid $M_{\mathrm{FB}}(G)$ is the matroid $M_{\mathrm{C}}(G)$ whose circuits are the edge sets of the topological circles in $|G|$ and whose bases are the edge sets of the topological spanning trees of G.*

2.3 Matroids describing the duality of planar graphs

Whitney's theorem [7] says that a finite graph G is planar if and only if the dual of its cycle matroid is *graphic*, i.e., is the cycle matroid of some other graph. Our matroids allow us to extend this to infinite graphs, as follows.

Thomassen [20] showed that any reasonable notion of duality for infinite graphs requires that these are *finitely separable*: that any two vertices can be separated by a finite set of edges. The class of finitely separable graphs is slightly larger than that of locally finite graphs, and just right for duality: while locally finite graphs can have duals that are not locally finite (with respect to any

[6] This space, also known as the Freudenthal compactification of G, is the natural setting for most problems about locally finite graphs that involve paths and cycles. It has been extensively studied; see [7, 6] for an introduction and overview.

reasonable notion of duality, e.g. geometrically in the plane), duals of finitely separable graphs, as defined formally below, are again finitely separable.

Since bonds can be infinite, we need a notion of infinite circuits. The notion that works for finitely separable graphs extends that defined for locally finite graphs in Section 2.2: a *circuit* will be the edge sets of a topological circle in the quotient space $\tilde{G}$ of $|G|$ obtained by identifying every vertex with all the ends from which it cannot be separated by finitely many edges.[7] (Note that, since G is finitely separable, no two vertices are identified with the same end.) These edge sets do form the circuits of a matroid, which we call the *cycle matroid* of G. This is in fact the matroid $M_{\mathrm{C}}(G)$ defined in Section 2.2, the dual of the finite-bond matroid $M_{\mathrm{FB}}(G)$:

Theorem 2.2. [3] *The cycle matroid of a finitely separable graph is the dual of its finite-bond matroid.*

A finitely separable graph G^* is a *dual* of a finitely separable graph G with the same edge set if the bonds of G^* are precisely the circuits of G, the edge sets of the topological circles in $\tilde{G}$. It has been shown in [4] that, if G is 3-connected, this graph G^* is unique, 3-connected, and has G as its unique dual, so $G^{**} = G$.

By Theorem 2.2, graph duality commutes with matroid duality:

Corollary 2.3. *If G and G^* are dual finitely separable graphs, then*

$$M^*_{\mathrm{FC}}(G) = M_{\mathrm{C}}(G^*) \quad \textit{and} \quad M^*_{\mathrm{FB}}(G) = M_{\mathrm{B}}(G^*).$$

□

Call a (finite or infinite) matroid *graphic* if it is the cycle matroid of a finitely separable graph. The infinite version of Whitney's theorem [4] can now be described in matroid terms too:

Theorem 2.4. [3] *A finitely separable graph is planar if and only if its finite-cycle matroid has a graphic dual.*

2.4 The algebraic cycle matroid of a graph

Another natural matroid in a locally finite graph G is its *algebraic cycle matroid*: the matroid whose circuits are the *elementary algebraic cycles* of G, the edge sets that induce degree either 0 or 2 at every vertex. When G is infinite, these are the edge sets of its (finite) cycles and those of its *double rays*, its 2-way infinite paths.

The elementary algebraic cycles do not form a matroid in every infinite graph: we shall see in Section 6 that they do not satisfy our circuit axioms when G is the *Bean graph* shown in Figure 1. However, Higgs proved (for his 'B-matroids'; but cf. Theorem 5.1) that this is essentially the only counterexample:

[7]Equivalently: by finitely many vertices. Another way of obtaining $\tilde{G}$ is to start not from $|G|$ but directly from G: we simply add only those ends that are not dominated by a vertex in this way, while making rays of the other ends converge to the vertex dominating that end. See [6, 8] for details.

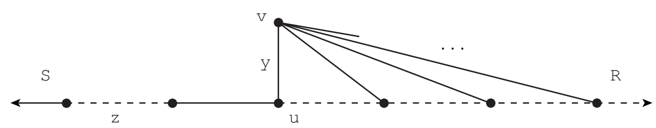

Figure 1: The Bean graph

Theorem 2.5 (Higgs [10])**.** *The elementary algebraic cycles of an infinite graph G are the circuits of a matroid on its edge set $E(G)$ if and only if G contains no subdivision of the Bean graph.*

Corollary 2.6. *The elementary algebraic cycles of any locally finite graph are the circuits of a matroid.* □

The dual of the algebraic cycle matroid of a graph G can also be described: it is the matroid whose circuits are the bonds separating a finite part of G from the infinite rest [3].

2.5 A matroid without finite circuits or cocircuits

All the concrete examples of infinite matroids we have seen so far are either finitary or cofinitary. The algebraic cycle matroids discussed in the last section, however, can have both infinite circuits and infinite cocircuits. The following example is an extreme case, in that its circuits and cocircuits are all infinite:

Example 2.7. *The matroid of the elementary algebraic cycles in the $\aleph_0$-regular tree T_∞ has no finite circuit and no finite cocircuit.*

Proof. Clearly, the elementary algebraic cycles of T_∞ are just the edge sets of its double rays. Since T_∞ does not contain the Bean graph as a subdivision, they are the circuits of a matroid MT_∞ on the edge set of T_∞, by Theorem 2.5.[8]

To show that every cocircuit is infinite we borrow Lemma 3.11 from Section 3, which says that a circuit and a cocircuit never meet in exactly one element. Since for any finite edge set F in T_∞ it is easy to find a double ray meeting F in exactly one edge, we deduce that F cannot be a cocircuit. □

2.6 Representability and thin independence

An important class of finite matroids are the representable matroids [22]. However as matroids defined by linear independence are finitary, the dual of an infinite representable matroid will not, except in trivial cases, be representable. Representability thus seems to be a concept too narrow for infinite matroids.

Can we find a notion of independence, different from linear independence, that includes some cofinitary matroids? We do not have a complete answer to this question, but think that the following notion of *thin independence* may be worth considering.

[8]This can also be seen directly. Checking (C1–3) is easy; see [2] for a direct proof of (CM).

Let F be a field, and let A be some set. We say that a family $\Phi = (\varphi_i)_{i \in I}$ of functions $\varphi_i \colon A \to F$ is *thin* if for every $a \in A$ there are only finitely many $i \in I$ with $\varphi_i(a) \neq 0$. Given such a thin family Φ of functions, their pointwise sum $\sum_{i \in I} \varphi_i$ is another $A \to F$ function. We say that a family Ψ of $A \to F$ functions, not necessarily thin, is *thinly independent* if for every thin subfamily $\Phi = (\varphi_i)_{i \in I}$ of Ψ and every corresponding family $(\lambda_i)_{i \in I}$ of coefficients $\lambda_i \in F$ we have $\sum_{i \in I} \lambda_i \varphi_i = 0 \in F^A$ only when $\lambda_i = 0$ for all $i \in I$.

Unlike linear independence, thin independence does not always define a matroid.[9] The following theorem from [3] gives a sufficient condition for when it does:

Theorem 2.8. *If a family E of $A \to F$ functions is thin, then its thinly independent subfamilies form the independent sets of a matroid on E.*

An application of Theorem 2.8 will be given in Section 2.7.

Let us refer to this matroid as the *thin-sums matroid* of the functions $A \to F$ considered, and say that a given matroid can be *thinly represented over F* if it is isomorphic (in the obvious sense) to such a matroid. Note that, for finite matroids, thin representability coincides with ordinary representability.

2.7 The algebraic cycle matroid of a complex

Let us show that the algebraic cycle matroid of a graph $G = (V, E)$ can be thinly represented over $\mathbb{F}_2$, for any G for which it is defined (cf. Theorem 2.5). We represent an edge $e = uv$ by the map $V \to \mathbb{F}_2$ assigning 1 to both u and v, and 0 to every other vertex. Then a set $F \subseteq E$ of edges becomes a family $(\varphi_f)_{f \in F}$ of $V \to \mathbb{F}_2$ functions, not necessarily thin, which is thinly independent if and only if F contains no elementary algebraic cycle.

The above example generalizes to higher dimensions. Let K be a locally finite simplicial complex. Let us show that, for each $n \in \mathbb{N}$, the n-dimensional cycles in K define a matroid $M_n(K)$ on the set $\Delta_n(K)$ of its n-simplices, which is thinly representable over $\mathbb{F}_2$.

Formally, we define this matroid as a thin-sums matroid over $\mathbb{F}_2$, representing each simplex $\sigma \in \Delta_n(K)$ by its boundary $\partial\sigma$: this is an $(n-1)$-chain with coefficients in $\mathbb{F}_2$, which we think of as a function $\varphi_\sigma \colon \Delta_{n-1}(K) \to \mathbb{F}_2$.[10] Thus, formally, our ground set E is not $\Delta_n(K)$ itself (the intended reading) but the family $(\varphi_\sigma)_{\sigma \in \Delta_n(K)}$. Since K is locally finite, the sets $F \subseteq E$ are thin families $F = (\varphi_\sigma)_{\sigma \in \Sigma}$ of such functions. Such a family F is thinly independent if and only if it contains no non-trivial n-cycle, that is, has no non-empty subfamily $F' = (\varphi_\sigma)_{\sigma \in \Sigma'}$ such that $\partial\psi = 0$ for the corresponding n-chain $\psi := \sum_{\sigma \in \Sigma'} \sigma$.

Theorem 2.8 thus has the following application:

[9] View the elements of $E = \mathbb{F}_2^{\mathbb{N}}$ as subsets of $\mathbb{N}$, and define sets $I := \{\{1, n\} : n \in \mathbb{N}\}$ and $I' := \{\{n\} : n \in \mathbb{N}\}$. Both I and I' are thinly independent. Moreover, I' is maximally thinly independent but I is not: $I + \mathbb{N}$, for instance, is still thinly independent. Yet, the only $x \in I'$ for which $I + x$ is thinly independent is $x = \{1\}$, which is already contained in I. Thus, (I3) is violated.

[10] In the notation of Section 2.6, we have $A = \Delta_{n-1}(K)$ and index sets $I \subseteq \Delta_n(K)$.

Theorem 2.9. *The minimal n-dimensional cycles of a locally finite simplicial complex form the circuits of a matroid.*

Let us call this matroid the *n-dimensional cycle matroid* of the complex K and denote it by $M_n(K)$. In general, this is a non-finitary matroid; we do not know whether it is always cofinitary.

We remark that even for $n = 1$ it was not entirely trivial to verify (CM) for this matroid. For $n > 1$ we know of no direct proof. The other essential axioms, such as (C3), (I3) or (B2), also appear to be hard to verify directly when the complex is infinite.

3 Basic properties

In this section we prove just enough about infinite matroids $(E,\mathcal{I})$ to enable us in Section 4 to deduce that the various axiom systems given in Section 1 are indeed equivalent. On the way we define duality, deletions and contractions, and show that they behave as for finite matroids. More properties of infinite matroids, especially regarding connectivity, are proved in [5].

Let $(E,\mathcal{I})$ be a fixed matroid, that is, assume throughout this section that $\mathcal{I}$ satisfies the independence axioms. Write $\mathcal{B} := \mathcal{I}^{\max}$ for its set of bases. We start with an observation that can be directly read off the axioms:

(I3′) For all $I \in \mathcal{I}$ and $I' \in \mathcal{B}$ there is a $B \in \mathcal{B}$ such that $I \subseteq B \subseteq I \cup I'$.

Indeed, by (IM) there is a maximal independent subset B of $I \cup I'$ such that $I \subseteq B$. Then $B \in \mathcal{B}$, as otherwise we could use (I3) to extend B further into I' (keeping it independent), contrary to its definition.

Let us establish duality. Define

$$\mathcal{B}^* := \{\, B^* \subseteq E \mid E \setminus B^* \in \mathcal{B} \,\} \quad \big(= \{\, \overline{B} \mid B \in \mathcal{B} \,\}\big)$$

and $\mathcal{I}^* := \lceil \mathcal{B}^* \rceil$.

Theorem 3.1. *If $\mathcal{I}$ satisfies the independence axioms, then so does $\mathcal{I}^*$, with $\mathcal{B}^*$ as its set of bases.*

Proof. Since $\mathcal{I}$ satisfies (I1) and (IM), we have $\mathcal{B}^* \neq \emptyset$ and hence (I2) and (I1) for $\mathcal{I}^*$. Since $\mathcal{B}$ and hence also $\mathcal{B}^*$ is an antichain, we have $\mathcal{I}^{*\max} = \mathcal{B}^*$. To prove (I3) for $\mathcal{I}^*$, let $I^* \in \mathcal{I}^* \setminus \mathcal{B}^*$ be given, with $I^* \cap B = \emptyset$ for $B \in \mathcal{B}$ say, and let also $B' \in \mathcal{B}$ be given; our aim is to extend I^* non-trivially into $\overline{B'}$ while keeping it in $\mathcal{I}^*$.

We first use (I3′) to extend the independent set $B' \setminus I^*$ into B, to a subset $B'' \in \mathcal{B}$ of $(B' \setminus I^*) \cup B$. Then $I^* \subseteq \overline{B''} \in \mathcal{B}^*$, and the inclusion is proper since $I^* \notin \mathcal{B}^*$ by assumption. But

$$\overline{B''} \setminus I^* = \overline{B'' \cup I^*} \subseteq \overline{B' \cup I^*} = \overline{B'} \setminus I^*$$

since $B' \cup I^* \subseteq B'' \cup I^*$. So the extension $\overline{B''}$ of I^* is as desired, completing our proof of (I3)—indeed of its strengthening (I3′)—for $\mathcal{I}^*$.

It remains to prove that $\mathcal{I}^*$ satisfies (M). Let $X \subseteq E$ and $I^* \in \mathcal{I}^* \cap 2^X$ be given. By definition of $\mathcal{I}^*$, there exists a set $B \in \mathcal{B}$ such that $I^* \cap B = \emptyset$. By (IM), $\overline{X}$ has a maximal independent subset I. By (I3′), we can extend I to a subset $B' \in \mathcal{B}$ of $I \cup B \subseteq \overline{I^*}$.

We claim that $X \setminus B'$ witnesses (M) for I^* and $\mathcal{I}^*$, i.e. that $X \setminus B'$ is maximal among the subsets of X that contain I^* and avoid an element of $\mathcal{B}$. Suppose not. Then there is a set $B'' \in \mathcal{B}$ such that $B'' \cap X \subsetneq B' \cap X$. Then

$$I' := (B'' \cap X) \cup (B' \setminus X) \subsetneq B',$$

so $I' \in \mathcal{I} \setminus \mathcal{B}$. We can thus use (I3) to extend I' properly into B'' to a larger independent set I''. But then $I \subseteq I' \setminus X \subsetneq I'' \setminus X$, contradicting the choice of I. □

Given a matroid $M = (E, \mathcal{I})$, we call the matroid $M^* := (E, \mathcal{I}^*)$ specified by Theorem 3.1 the *dual* of M. As usual, we call the bases, circuits, dependent and independent sets of M^* the *cobases, cocircuits, codependent* and *coindependent* sets of M.

Next, let us show that our matroids have restrictions defined in the usual way: that, given a set $X \subseteq E$, the pair $(X, \mathcal{I} \cap 2^X)$ is again a matroid. It will be convenient to use the following duality argument in our proof of this fact:

Lemma 3.2. *If $X \subseteq E$ and $B \in \mathcal{B}$, then $B \cap X$ is maximal in $\mathcal{I} \cap 2^X$ if and only if $\overline{B} \cap \overline{X}$ is maximal in $\mathcal{I}^* \cap 2^{\overline{X}}$.*

Proof. Suppose first that $B \cap X$ is maximal in $\mathcal{I} \cap 2^X$. If $\overline{B} \cap \overline{X}$ is not maximal in $\mathcal{I}^* \cap 2^{\overline{X}}$, there exists some $B' \in \mathcal{B}$ such that $B' \setminus X \subsetneq B \setminus X$. Use (I3′) to extend $I := B \cap X$ to a subset $I' \in \mathcal{B}$ of $I \cup B'$. Then $I' \cap X \supsetneq B \cap X$, since I' is not a proper subset of B. This contradicts our initial assumption about B.

The converse implication follows by taking complements. □

Lemma 3.3. *For every set $X \subseteq E$, the set $\mathcal{I} \cap 2^X$ satisfies* (I3′).

Proof. Let an independent subset I of X and a maximal independent subset I' of X be given. Using (IM) in E, extend I' to a set $B' \in \mathcal{B}$. Note that $I' = B' \cap X$, by the maximality of I'. By Lemma 3.2,

$$\overline{B'} \cap \overline{X} \text{ is maximal in } \mathcal{I}^* \cap 2^{\overline{X}}. \qquad (*)$$

Use (I3′) to extend I into B', to a subset $B \in \mathcal{B}$ of $I \cup B'$. Then $B \cap \overline{X} \subseteq B' \cap \overline{X}$ and hence $\overline{B} \cap \overline{X} \supseteq \overline{B'} \cap \overline{X}$. Thus by $(*)$, the set $\overline{B} \cap \overline{X}$ is maximal in $\mathcal{I}^* \cap 2^{\overline{X}}$. Applying Lemma 3.2 backwards, we deduce that $B \cap X$ is maximal in $\mathcal{I} \cap 2^X$. Since

$$I \subseteq B \cap X \subseteq (I \cup B') \cap X = (I \cap X) \cup (B' \cap X) = I \cup I'$$

(recall that $I' = B' \cap X$), this completes the proof. □

Theorem 3.4. *For every set $X \subseteq E$, the pair $(X, \mathcal{I} \cap 2^X)$ is a matroid.*

Proof. Axioms (I1), (I2) and (IM) hold for the sets in $\mathcal{I} \cap 2^X$ because they hold for $\mathcal{I}$. Axiom (I3) for $\mathcal{I} \cap 2^X$ follows from Lemma 3.3. □

Given a matroid $M = (E, \mathcal{I})$ and $X \subseteq E$, we denote the matroid $(X, \mathcal{I} \cap 2^X)$ as $M|X$ or as $M - \overline{X}$, and call it the *restriction* of M *to* X, or the *minor* of M obtained by *deleting* $\overline{X}$. Following Oxley [17], we call

$$M.X := M/\overline{X} := (M^*|X)^*$$

the *contraction* of M *to* X, or the *minor* of M obtained by *contracting* $\overline{X}$.

Lemma 3.5. *The following statements are equivalent or all sets $I \subseteq X \subseteq E$:*

(i) *I is a basis of $M.X$.*

(ii) *There exists a basis I' of $M - X$ such that $I \cup I' \in \mathcal{B}$.*

(iii) *$I \cup I'' \in \mathcal{B}$ for every basis I'' of $M - X$.*

Proof. (i) means that $X \setminus I$ is a basis of $M^*|X$, a maximal subset of X extending to a basis of M^*. (Equivalently, I is minimal with the property that we can extend it to a basis of M by adding points of $\overline{X}$ only.) By Lemma 3.2, this is equivalent to (ii).

Since $M - X$ is a matroid (Theorem 3.4) it has a basis, so (iii) implies (ii). To prove the converse implication, assume (ii) and let I'' be a basis of $M - X$. Use (3c′) to extend I'' into $B' := I \cup I'$, i.e. to find a set $B'' \in \mathcal{B}$ such that $I'' \subseteq B'' \subseteq I'' \cup B'$. By the minimality of I mentioned in the proof of (i)↔(ii), we have $B'' \cap X \supseteq I$, and by the maximality of I'' as a basis of $M - X$ we have $B'' \setminus X \subseteq I''$. In both cases we trivially also have the converse inclusion, so $B'' = I \cup I''$ as desired. □

Corollary 3.6. *A set $I \subseteq X$ is independent in $M.X$ if and only if $I \cup I' \in \mathcal{I}$ for every independent set I' of $M - X$.*

Proof. The forward implication follows easily from Lemma 3.5 (i)→(iii).

For the backward implication, choose I' as a basis of $M - X$. Use (IM) to extend $I \cup I' \in \mathcal{I}$ to a basis $B \in \mathcal{B}$. Then $B \setminus X = I'$ by the maximality of I', so $B \cap X \supseteq I$ is a basis of $M.X$ by (ii)→(i) of Lemma 3.5. □

Our next aim is to show the counterpart of (IM) for dependent sets: that inside every dependent set we can find a minimal one, a circuit. For the proof we need another lemma:

Lemma 3.7. *If bases B, B' satisfy $|B \setminus B'| < \infty$, then $|B \setminus B'| = |B' \setminus B|$.*

Proof. Suppose not, and choose a counterexample (B, B') with $|B \setminus B'|$ minimum. Then $|B \setminus B'| < |B' \setminus B|$. Pick $x \in B' \setminus B$, and use (I3′) to extend $B' - x$ to a subset $B'' \in \mathcal{B}$ of $(B' - x) \cup B$. As B'' is not a proper subset of B' it meets $B \setminus B'$, so $|B \setminus B''| < |B \setminus B'|$. Hence (B, B'') is not a counterexample, so

$$|B'' \setminus B| = |B \setminus B''| < |B \setminus B'| < |B' \setminus B|\,.$$

But $B'' \setminus B$ differs from $B' \setminus B$ only by x, a contradiction. □

Lemma 3.8. *Every dependent set contains a circuit.*

Proof. By Theorem 3.4, it suffices to assume that $E \notin \mathcal{I}$ and find a circuit in E. Pick a basis $B \in \mathcal{B}$; this exists by (I1) and (IM). Then $B \subsetneq E$; pick $z \in E \setminus B$. We shall prove that

$$C := \{\, x \in B + z \mid B + z - x \in \mathcal{I} \,\} .$$

is a circuit. Note that $z \in C$.

We first show that C is dependent. Suppose not, and use (I3′) to extend C to a subset $B' \in \mathcal{B}$ of $C \cup B = B + z$. Since $B' \setminus B = \{z\}$, we have $|B \setminus B'| = 1$ by Lemma 3.7, say $B \setminus B' = \{y\}$. But then $B + z - y = B' \in \mathcal{B}$, so $y \in C \subseteq B'$ by definition of C. This contradicts the definition of y.

C is minimally dependent, since for every $x \in C$ we have $C - x \subseteq B + z - x \in \mathcal{I}$ by definition of C. □

Recall that a matroid is called *finitary* if any set whose finite subsets are independent is also independent.

Corollary 3.9. *A matroid is finitary if and only if every circuit is finite.*

Proof. A finitary matroid clearly has no infinite circuits. Conversely, a set whose finite subsets are independent cannot contain a finite circuit. Hence if all circuits are finite it contains no circuit, and is therefore independent by Lemma 3.8. □

Let $\mathrm{cl}\colon 2^E \to 2^E$ be the closure operator associated with $\mathcal{I}$.

Lemma 3.10. *If B is a maximal independent subset of X, then* $\mathrm{cl}(X) = \mathrm{cl}(B)$.

Proof. The inclusion $\mathrm{cl}(B) \subseteq \mathrm{cl}(X)$ is trivial since $B \subseteq X$; we show the converse. Let $y \in \mathrm{cl}(X)$ be given, witnessed by an independent set $I \subseteq X$ such that $I + y \notin \mathcal{I}$. By (IM), we can extend I to a maximal independent subset B' of $X + y$. Clearly $y \notin B'$, so $B' \subseteq X$. If $y \in \mathrm{cl}(B)$ we are done. If not then $B + y \in \mathcal{I}$, so B is an independent but not a maximal independent subset of $X + y$. By Lemma 3.3, we may use (I3) in $X + y$ to extend B into B' to an independent subset of X that contains B properly, contradicting its maximality. □

We conclude with a lemma already used in the proof of Example 2.7:

Lemma 3.11. *A circuit and a cocircuit of a matroid never meet in exactly one element.*

Proof. Let C be a circuit, and D a cocircuit, such that $C \cap D = \{x\}$. As $D - x$ is coindependent, it misses a basis B. Apply (I3′) to extend the independent set $C - x$ to a basis $B' \subseteq (C - x) \cup B$. Since C is dependent and $C - x \subseteq B'$, we have $x \notin B'$. Hence $D \cap B' = \emptyset$, contradicting our assumption that D is codependent. □

4 Equivalence of the axiom systems

In this section we prove that our axiom systems are equivalent. In our use of the terms 'dependent', 'independent', 'basis', 'circuit' and 'closure' we stick to their definitions as given in Section 1.1, referring to a set system $\mathcal{I}$ known or assumed to satisfy the independence axioms. When we do not assume this, as will often be the case in this section, we shall use unambiguous other terms defined in the context of the axioms assumed, such as 'maximal $\mathcal{C}$-independent set'.

Theorem 4.1.

(i) *If a set $\mathcal{I} \subseteq 2^E$ satisfies the independence axioms, then the set $\mathcal{B}$ of bases satisfies the basis axioms with $\mathcal{I}$ as the set of $\mathcal{B}$-independent sets.*

(ii) *If a set $\mathcal{B} \subseteq 2^E$ satisfies the basis axioms, then the set $\mathcal{I}$ of $\mathcal{B}$-independent sets satisfies the independence axioms with $\mathcal{B}$ as the set of bases.*

Proof. (i) Let $\mathcal{I}$ satisfy the independence axioms. Applying (IM) with $X := E$, we see that every set in $\mathcal{I}$ extends to a set in $\mathcal{I}^{\max}$. Hence (I1) implies (B1), and $\mathcal{I} = \lceil \mathcal{I}^{\max} \rceil$; in particular, (IM) implies (BM).

To prove (B2), let $B_1, B_2 \in \mathcal{B} := \mathcal{I}^{\max}$ and $x \in B_1 \setminus B_2$ be given. Applying (I3) with $I := B_1 - x$ and $I' := B_2$, we find an element $y \in B_2 \setminus B_1$ such that $B := (B_1 - x) + y \in \mathcal{I}$. We have us show that $B \in \mathcal{I}^{\max}$. If $B \notin \mathcal{I}^{\max}$, we can apply (I3) with $I := B$ and $I' := B_1$ to extend B into B_1 to a set $B' \in \mathcal{I}$. But $B_1 \setminus B = \{x\}$, so this means that $B_1 \subsetneq B' \in \mathcal{I}$, as $y \in B' \setminus B_1$. This contradicts our assumption that $B_1 \in \mathcal{I}^{\max}$.

(ii) Let $\mathcal{B}$ satisfy the basis axioms, and let $\mathcal{I} := \lceil \mathcal{B} \rceil$. Then (B1) implies (I1), (I2) is trivial, and (BM) trivially implies (IM). Since by (B2) no set in $\mathcal{B}$ contains another, we also have $\mathcal{B} = \mathcal{I}^{\max}$.

To prove (I3), let $I \in \mathcal{I} \setminus \mathcal{B}$ and $I' \in \mathcal{B}$ be given. Use (IM) with $X := E$ to extend I to a set $B \in \mathcal{B} = \mathcal{I}^{\max}$, and pick $x \in B \setminus I$. If $x \in I'$, then $I + x \in \mathcal{I}$ is as desired. If not, we can use (B2) with $B_1 := B$ and $B_2 := I'$ to find $y \in I' \setminus B$ such that $(B - x) + y \in \mathcal{B}$. As $I \subseteq B - x$ this yields $I + y \in \mathcal{I}$, as required for (I3). □

Theorem 4.2.

(i) *If a set $\mathcal{I} \subseteq 2^E$ satisfies the independence axioms, then the associated closure operator* cl *satisfies the closure axioms with $\mathcal{I}$ as the set of* cl*-independent sets.*

(ii) *If a function* $\mathrm{cl}\colon 2^E \to 2^E$ *satisfies the closure axioms, then the set $\mathcal{I}$ of* cl*-independent sets satisfies the independence axioms with* cl *as the associated closure operator.*

Proof. (i) Let $\mathcal{I}$ satisfy the independence axioms, and let cl be the associated closure operator. Then (CL1) and (CL2) hold trivially. By (I2), every set in $\mathcal{I}$ is cl-independent. Conversely, a cl-independent set X lies in $\mathcal{I}$: if not, then by (IM) it has a maximal independent subset $I \subsetneq X$, and every $x \in X \setminus I$ satisfies $x \in \mathrm{cl}(I)$, contradicting the cl-independence of X. Hence the cl-independent sets are precisely those in $\mathcal{I}$, and (IM) implies (CLM).

To prove (CL3), let $X \subseteq E$ and $y \in \mathrm{cl}(\mathrm{cl}(X))$ be given. If $y \notin \mathrm{cl}(X)$, there exists a set $I \subseteq \mathrm{cl}(X)$ such that $I \in \mathcal{I}$ but $I + y \notin \mathcal{I}$. By (IM) applied in $\mathrm{cl}(X) + y$, we can extend I to a maximal independent subset B of $\mathrm{cl}(X) + y$. Clearly $y \notin B$, so $B \subseteq \mathrm{cl}(X)$.

Also by (IM), X has a maximal independent subset B', which satisfies $\mathrm{cl}(B') = \mathrm{cl}(X)$ by Lemma 3.10. Hence B' is maximal even among the independent subsets of $\mathrm{cl}(X)$: for every $x \in \mathrm{cl}(X) \setminus X$ we have $x \in \mathrm{cl}(B') \setminus B'$, so $B' + x \notin \mathcal{I}$. If $B' + y \notin \mathcal{I}$, then B' witnesses that $y \in \mathrm{cl}(X)$ and we are done. But if $B' + y \in \mathcal{I}$, then B' is an independent subset of $\mathrm{cl}(X) + y$ though not a maximal one. By Lemma 3.3, we may use (I3) in $\mathrm{cl}(X) + y$ to extend B' into B to yield a larger independent subset of $\mathrm{cl}(X)$, contradicting the maximality of B' in $\mathcal{I} \cap 2^{\mathrm{cl}(X)}$.

Let finally Z, x and y be given for the proof of (CL4). As $y \in \mathrm{cl}(Z + x) \setminus \mathrm{cl}(Z)$, there is an independent set $I \subseteq Z + x$ such that $I + y$ is dependent and $(I - x) + y \in \mathcal{I}$. As $I - x \subseteq Z$, this also witnesses that $x \in \mathrm{cl}(Z + y)$.

(ii) Let $\mathrm{cl}\colon 2^E \to 2^E$ satisfy the closure axioms, and let $\mathcal{I}$ be the set of cl-independent sets. Then $\mathcal{I}$ satisfies (I1) and (I2) trivially, and (IM) is just a restatement of (CLM).

For the remainder of our proof we shall need show the following fact:

Whenever a set $Z \subseteq E$ is cl-independent but $Z + x$ is not (for some $x \in E$), we have $x \in \mathrm{cl}(Z)$. $\quad (*)$

Indeed, if not then some $y \in Z$ lies in the closure of the other elements of $Z + x$. Since $y \notin \mathrm{cl}(Z - y)$ by assumption, this implies $x \in \mathrm{cl}(Z)$ by (CL4).

To prove (I3), let $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$ be given. Use (CLM) to extend I to a maximal element B of $\mathcal{I} \cap 2^{I \cup I'}$. We shall prove that B is maximal in all of $\mathcal{I}$; then $B \setminus I \neq \emptyset$, and any $x \in B \setminus I$ proves (I3).

To show that $B \in \mathcal{I}^{\max}$, consider any $z \in E \setminus B$. Then $z \in \mathrm{cl}(I')$: trivially if $z \in I'$, or by $(*)$ and $I' \in \mathcal{I}^{\max}$ if $z \notin I'$. Similarly, the maximality of B in $\mathcal{I} \cap 2^{I \cup I'}$ implies by $(*)$ that $I' \subseteq \mathrm{cl}(B)$. Hence $z \in \mathrm{cl}(I') \subseteq \mathrm{cl}(\mathrm{cl}(B)) = \mathrm{cl}(B)$ by (CL2) and (CL3). As $z \notin B$, this means that $B + z \notin I$ as desired.

It remains to show that cl coincides with the closure operator cl' associated with $\mathcal{I}$, i.e. that $\mathrm{cl}(X) = \mathrm{cl}'(X)$ for every $X \subseteq E$. To show that $\mathrm{cl}(X) \subseteq \mathrm{cl}'(X)$, consider any $x \in \mathrm{cl}(X)$. If $x \in X$ then $x \in \mathrm{cl}'(X)$, so assume that $x \notin X$. Our assumption of $x \in \mathrm{cl}(X)$ now means that $X + x$ is cl-dependent, that $X + x \notin \mathcal{I}$. By (CLM), X has a maximal cl-independent subset I. Then $X \subseteq \mathrm{cl}(I)$ by $(*)$, so $x \in \mathrm{cl}(X) \subseteq \mathrm{cl}(\mathrm{cl}(I)) = \mathrm{cl}(I)$ by (CL2) and (CL3), showing that $x \in \mathrm{cl}'(X)$.

The converse inclusion, $\mathrm{cl}'(X) \subseteq \mathrm{cl}(X)$, follows easily form $(*)$. □

Theorem 4.3.

(i) *If a set $\mathcal{I} \subseteq 2^E$ satisfies the independence axioms, then the set $\mathcal{C}$ of circuits satisfies the circuit axioms with $\mathcal{I}$ as the set of $\mathcal{C}$-independent sets.*

(ii) *If a set $\mathcal{C} \subseteq 2^E$ satisfies the circuit axioms, then the set $\mathcal{I}$ of $\mathcal{C}$-independent sets satisfies the independence axioms with $\mathcal{C}$ as the set of circuits.*

Proof. (i) Let $\mathcal{I}$ satisfy the independence axioms, let $\mathcal{C}$ be the corresponding set of circuits, and let cl be the closure operator associated with $\mathcal{I}$. (I1) implies (C1), and (C2) holds by definition of $\mathcal{C}$. By (I2) and Lemma 3.8, the $\mathcal{C}$-independent sets are precisely those in $\mathcal{I}$, so (IM) implies (CM).

To prove (C3), let $X \subseteq C \in \mathcal{C}$ and $(C_x \mid x \in X)$ and z be given as stated. Let

$$Y := \Big(C \cup \bigcup_{x \in X} C_x \Big) \smallsetminus (X + z)\,.$$

For every $x \in X$ we have $x \in \mathrm{cl}(C_x - x)$ and $(C_x - x) \cap (X + z) = \emptyset$, so

$$X \;\subseteq\; \mathrm{cl}\Big(\bigcup_{x \in X} (C_x - x) \smallsetminus (X + z) \Big) \;\subseteq\; \mathrm{cl}(Y).$$

Hence

$$C - z = (C \smallsetminus (X + z)) \cup X \subseteq Y \cup \mathrm{cl}(Y) = \mathrm{cl}(Y)$$

and therefore

$$z \in \mathrm{cl}(C - z) \subseteq \mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(Y)$$

by Theorem 4.2 (i). So Y has an independent subset I such that $I + z$ is dependent. By Lemma 3.8, $I + z$ contains a circuit, which clearly contains z.

(ii) Let $\mathcal{C}$ satisfy the circuit axioms, and let $\mathcal{I}$ be the set of $\mathcal{C}$-independent sets. Then (I1) and (I2) hold trivially, and (IM) is just a restatement of (CLM). By (C2), no element of $\mathcal{C}$ contains another, so $\mathcal{C}$ is the set of circuits.

To prove (I3), let $I \in \mathcal{I} \smallsetminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$ be given. Use (IM) with $X := E$ to extend I to a set $B \in \mathcal{I}^{\max}$, and pick $z \in B \smallsetminus I$. We show that $B \subseteq I \cup I'$; then $x := z$ is as required for (I3). Suppose $z \in B \smallsetminus (I \cup I')$. Then $I' + z$ contains a set $C \in \mathcal{C}$. We wish to apply (C3) with $X := C \smallsetminus B$. Note that $X \subseteq I' \smallsetminus I$, since $I + z \subseteq B$. For each $x \in X$ we may assume that $I + x$ contains a set $C_x \in \mathcal{C}$, since otherwise $I + x$ witnesses (I3). Then $z \notin I + x \supseteq C_x$ for all $x \in X$, so by (C3) there is a set $C' \in \mathcal{C}$ such that $C' \subseteq \big(C \cup \bigcup_{x \in X} C_x \big) \smallsetminus X$. As $C_x \smallsetminus X \subseteq I \subseteq B$ for every x, and $C \smallsetminus X \subseteq B$ by definition of X, this implies that $C' \subseteq B \in \mathcal{I}$, a contradiction. □

5 Alternative axiom systems and historical links

In the late 1960s and early '70s, a number of researchers—including Bean, Higgs, Klee, Minty and Las Vergnas—responded to Rado's [19] challenge to develop a theory of non-finitary infinite matroids, one that would allow for the kind of duality that was known from finite matroids but incompatible with the finitary infinite matroids that had so far been studied. This resulted in a flurry of related but not easily compatible proposals of how such structures might be defined, of which Higgs's *B-matroids* were but one among many.[11]

It was only several years later that Oxley clarified the situation in two ways: he found a simple set of axioms that characterized Higgs's B-matroids [14], and he showed that the notion of a B-matroid was, in a natural sense, best possible among the various notions proposed (Oxley [16]). In particular, Oxley

[11] Even Higgs himself studied various notions in parallel, including 'C-matroids', 'transitive spaces', 'finitely transitive spaces', 'dually transitive spaces', 'exchange spaces' and 'dually exchange spaces'—as well as two more general structures with duality that he calls 'spaces' and 'matroids'.

showed that it encompasses an approach suggested by Klee [12] that was built on properties of closure operators.

Oxley's axioms for B-matroids are of 'mixed type': they can be stated either in terms of independent sets or in terms of bases, but each version contains elements of the other. Our first result in this section is that our axiom systems from Section 1 are equivalent to Oxley's axioms for B-matroids. Thus, our matroids *are* B-matroids in the sense of Higgs [11]. Readers familiar with B-matroids may view our paper as simply furnishing the theory of B-matroids with sets of axioms each highlighting one (and only one) aspect, in the spirit of finite matroid theory. This might be seen as a completion of the solution to Rado's problem in the line of Higgs and Oxley.

Readers less familiar with B-matroids may view our axiom sets of Section 1, together with the results of Section 3, as our solution offered for Rado's problem. The existing work on B-matroids such as [1, 9, 11, 13, 14, 21] may then still serve as a valuable repository of properties which, with hindsight, our matroids have already been found to have.[12]

The four statements in the theorem below are Oxley's axioms for the independent sets of B-matroids (with (IM) rephrased to fit our terminology):

Theorem 5.1. *A set $\mathcal{I} \subseteq 2^E$ satisfies the independence axioms if and only if it satisfies the following four statements:*

(I1) $\emptyset \in \mathcal{I}$.

(I2) *$\mathcal{I}$ is closed under taking subsets.*

(IB) *Whenever $X \subseteq E$, the sets $I_1, I_2 \subseteq X$ are maximal elements of $\mathcal{I} \cap 2^X$, and $x \in I_1 \setminus I_2$, there exists an element $y \in I_2 \setminus I_1$ such that $(I_1 - x) + y$ is a maximal element of $\mathcal{I} \cap 2^X$.*

(IM) *$\mathcal{I}$ satisfies* (M).

Proof. If $\mathcal{I}$ satisfies the independence axioms, then in particular it satisfies (I1), (I2) and (IM). Statement (IB) is the basis exchange axiom for restrictions, so it holds by Theorems 3.4 and 4.1.

Conversely, if $\mathcal{I}$ satisfies the above four statements, then $\mathcal{I}^{\max}$ satisfies the basis axioms: (B1) follows from (I1) and (IM); (B2) is the case $X = E$ of (IB); and (BM) follows from (IM) and (I2), since these imply that $\mathcal{I} = \lceil \mathcal{I}^{\max} \rceil$. □

Given the 'exchange' nature of axiom (IB), it may seem that the four statements above are better rephrased in terms of bases. And indeed, Oxley [14] notes such a translation: a set $\mathcal{B} \subseteq 2^E$ is the set of bases of a B-matroid if and only if it satisfies (B1), (BM), and (IB) with $\mathcal{I} := \lceil \mathcal{B} \rceil$. These, then, differ from our basis axioms only in that they require our exchange axiom (B2) explicitly for all restrictions to subsets X of E. This strengthening makes it necessary to invoke a notion of independent sets, since the 'bases' of $M|X$ for which (IB) says that (B2) should hold are defined as the maximal subsets of X in $\lceil \mathcal{B} \rceil$. Thus, whichever way we choose to present these axioms, the presentation will involve

[12]It might still be a good idea to look for independent proofs, if only for reasons of accessibility.

both elements of basis exchange and of independence. Divorcing these into separate sets of independence and basis axioms, as we have done in Section 1, made it necessary to prove that requiring (B2) for all restrictions is in fact redundant in the presence of (BM)—which we did in our Theorems 3.4 and 4.1.

As a common feature, all our axiom systems so far have included the explicit requirement (M) that every independent set extends to a maximal one—not only in the whole matroid but inside any given $X \subseteq E$. This is a strong statement, which might not be easy to verify in practice. We therefore tried to replace it with weaker axioms, such as one requiring merely that every set $X \subseteq E$ must have *some* maximal independent subset.

We succeeded in doing this for the independence and the basis axioms, at the expense of having to strengthen the third axioms a little (see below). For the circuit and the closure axioms we found no natural strengthening that would allow a similar substantial weakening of the (M) axiom.

Let us rephrase the independence axioms first:

Theorem 5.2. *A set $\mathcal{I} \subseteq 2^E$ satisfies the independence axioms if and only if it satisfies the following three statements:*

(I1′) *Every set $X \subseteq E$ has a subset that is maximal in $\mathcal{I} \cap 2^X$.*

(I2) *$\mathcal{I}$ is closed under taking subsets.*

(I3′) *For all $I \in \mathcal{I}$ and $I' \in \mathcal{I}^{\max}$ there is a $B \in \mathcal{I}^{\max}$ such that $I \subseteq B \subseteq I \cup I'$.*

Proof. Suppose first that $\mathcal{I}$ satisfies the independence axioms: statements (I1), (I2), (I3) and (IM) from Section 1. At the start of Section 3 we proved that these imply (I3′), and (I1′) follows from (IM) with $I := \emptyset$.

Conversely, assume that $\mathcal{I}$ satisfies (I1′), (I2) and (I3′). Axiom (I1) follows from (I1′) and (I2), and (I3) is a weakening of (I3′). To prove (IM), we begin by re-proving the statement of Lemma 3.3 in Section 3, replacing the use of (IM) in that proof with suitable applications of (I1′) and (I3′). By (I1′), the assertion of Lemma 3.3 will then imply (IM).

We begin by copying the first two paragraphs of the proof of Theorem 3.1, to show that $\mathcal{I}^*$, defined as before Theorem 3.1, satisfies (I3′). (The proof there assumes that the given set I^* is not in $\mathcal{B}^*$; but if it is, there is nothing to show since $\mathcal{B}^* = \mathcal{I}^{*\max}$.) Next, we establish the assertion of Lemma 3.2 by copying its proof; this uses (I3′) for both $\mathcal{I}$ and $\mathcal{I}^*$, but it does not use (IM). Finally, we copy the proof of Lemma 3.3 itself. This proof uses (IM) for $X = E$ in the second line. Instead, we use (I1′) with $X := E$ to find some set $\hat{B} \in \mathcal{B}$, and then apply (I3′) to extend the given set I' into $\hat{B}$ to the desired set $B \in \mathcal{B}$ (where $I' \subseteq B \subseteq I' \cup \hat{B}$). □

We remark that (I1′) is weakest possible with the property of completing (I2) and (I3′) to a full set of independence axioms. Indeed, since the set of finite subsets of an infinite set satisfies (I2) and (I3′) but does not define a matroid, we need to require the existence of a maximal set at least in all of E. Moreover, we want restrictions $M|X$ of a matroid M to be matroids, but the existence of maximal independent sets is not hereditary even in the presence of (I2) and (I3′); see Example 6.3 in Section 6. We thus have to require (I1′) as stated.

However, there is an interesting alternative to (I1′), which we mention without proof. Rather than requiring that in every restriction there is a maximal independent set, we may instead prescribe this for every contraction (cf. Corollary 3.6): Theorem 5.2 remains valid if we replace its statement (I1′) with

(I1″) Every set $X \subseteq E$ has a maximal subset I such that $I \cup I' \in \mathcal{I}$ for every $I' \in \mathcal{I} \cap 2^{\overline{X}}$.

Next, an alternative set of basis axioms. Unlike (B2), the alternative exchange axiom (B2′) does not imply that $\mathcal{B}$ is an antichain, so we have to add this as a new requirement (B0):

Theorem 5.3. *A set $\mathcal{B} \subseteq 2^E$ satisfies the basis axioms if and only if $\mathcal{B}$ satisfies the following three statements:*

(B0) *No element of $\mathcal{B}$ is a subset of another.*

(B1′) *For every $X \subseteq E$ there is a $B \in \mathcal{B}$ such that $B \cap X$ is maximal in $\lceil \mathcal{B} \rceil \cap 2^X$.*

(B2′) *Whenever $B_1, B_2 \in \mathcal{B}$ and $F_1 \subseteq B_1 \setminus B_2$, there exists $F_2 \subseteq B_2 \setminus B_1$ such that $(B_1 \setminus F_1) \cup F_2 \in \mathcal{B}$.*

Proof. Suppose first that $\mathcal{B}$ satisfies the basis axioms: statements (B1), (B2) and (BM) from Section 1. (B2) implies (B0), and (BM) implies (B1′). To prove (B2′), let B_1, B_2 and F_1 be given as stated. Use (BM) to extend $I := B_1 \setminus F_1$ to a maximal subset B of $X := I \cup B_2$ in $\lceil \mathcal{B} \rceil$. We show that $B \in \mathcal{B}$; then (B2′) holds with $F_2 := B \setminus I$.

Suppose $B \notin \mathcal{B}$. Since $B \in \lceil \mathcal{B} \rceil$, there exists a set $B' \in \mathcal{B}$ such that $B \subsetneq B'$. Note that $B' \setminus B \subseteq \overline{I \cup B_2}$, by the maximality of B in $\lceil \mathcal{B} \rceil \cap 2^{I \cup B_2}$. Pick $x \in B' \setminus B$. Applying (B2) to B' and B_2, we find an element $y \in B_2 \setminus B' = B_2 \setminus B$ such that $(B' - x) + y \in \mathcal{B}$. This contradicts the maximality of B in $\lceil \mathcal{B} \rceil \cap 2^{I \cup B_2}$.

Conversely, assume that $\mathcal{B}$ satisfies (B0), (B1′) and (B2′). We shall prove that $\mathcal{I} := \lceil \mathcal{B} \rceil$ satisfies statements (I1′), (I2) and (I3′) from Theorem 5.2. Then, by that theorem, $\mathcal{I}$ satisfies the independence axioms. By Theorem 4.1, then, $\mathcal{I}^{\max}$ will satisfy the basis axioms and $\mathcal{I} = \lceil \mathcal{I}^{\max} \rceil$. Hence $\lceil \mathcal{B} \rceil = \mathcal{I} = \lceil \mathcal{I}^{\max} \rceil$, which implies $\mathcal{B} = \mathcal{I}^{\max}$ since $\mathcal{I}^{\max}$ and $\mathcal{B}$ are both antichains (by definition and by (B0), respectively). So $\mathcal{B}$ satisfies the basis axioms, as desired.

So let us prove that $\mathcal{I} := \lceil \mathcal{B} \rceil$ satisfies (I1′), (I2) and (I3′). Statement (I1′) is just (B1′) for $\mathcal{I} = \lceil \mathcal{B} \rceil$. Statement (I2) is immediate from $\mathcal{I} = \lceil \mathcal{B} \rceil$. For the proof of (I3′), let I and I' be given as stated. Since $I \in \mathcal{I} = \lceil \mathcal{B} \rceil$, there exists a set $B_1 \in \mathcal{B}$ such that $I \subseteq B_1$. Applying (B2′) to B_1 and $B_2 := I' \in \mathcal{I}^{\max} \subseteq \mathcal{B}$ with $F_1 := B_1 \setminus (I \cup I')$, we find a set $F_2 \subseteq I' \setminus B_1$ such that $(B_1 \setminus F_1) \cup F_2 \in \mathcal{B}$. For $B := (B_1 \setminus F_1) \cup F_2$ we now have $I \subseteq B \subseteq I \cup I'$, as required for (I3′). □

As mentioned earlier, we have no alternative systems of circuit or closure axioms. Oxley [16] proved that B-matroids are characterized by the closure axioms given in Section 1, except that he replaced the usual (and our) axiom (CL4) by the stronger axiom

(CL4′) For all $Z \subseteq X \subseteq E$ and $y \in \mathrm{cl}(X) \setminus \mathrm{cl}(Z)$ there exists an $x \in X \setminus Z$ such that $x \in \mathrm{cl}((X - x) + y)$.

(For $|X \setminus Z| = 1$, axiom (CL4′) yields axiom (C4).)

The reason Oxley replaced (CL4) with (CL4′) was historical: Klee [12], in his own response to Rado's [19] challenge, had considered the systems of cl-independent sets for functions $\mathrm{cl}\colon 2^E \to 2^E$ satisfying (CL1–3) and (CL4′). Oxley [15] had shown that these axioms are not strong enough to define a B-matroid,[13] and remedied this defect by adding (CLM).

In the presence of (CLM), however, the strengthening of (CL4) to (CL4′) becomes obsolete, because the first implies the second:

Proposition 5.4. *If* $\mathrm{cl}\colon 2^E \to 2^E$ *satisfies the closure axioms, it even satisfies* (CL4′).

Proof. By Theorem 4.2 (ii), the set $\mathcal{I}$ of cl-independent sets satisfies the independence axioms, and hence defines a B-matroid by Theorem 5.1. The closure operator associated with $\mathcal{I}$, which by Theorem 4.2 (ii) is the function cl, satisfies (CL4′) by Oxley [16, Prop. 3.2.8]. □

6 Examples of non-matroids

In this section we illustrate our axioms by examples of set systems that narrowly fail to be matroids, by missing just one axiom of a given set. In particular, the axioms within each system will be seen to be independent.

We start with an example mentioned before:

Example 6.1. *Let E be an infinite set. The set $\mathcal{I}$ of finite subsets of E satisfies* (I1)–(I3)*, even* (I3′)*. But $\mathcal{I}$ has no maximal element, so it violates* (IM) *and* (I1′)*.*

Our next example shows that, although we can now deal with infinite sets, matroids still live in the discrete world:

Example 6.2. *The usual topological closure operator for subsets of $\mathbb{R}$ satisfies the closure axioms* (CL1)–(CL4) *for $E := \mathbb{R}$, but not* (CLM)*.*

Proof. To see that (CLM) fails, notice that independent sets must be discrete. Hence there is no maximal such set in $\mathbb{R}$. □

Our motivation for the alternative axiom systems given in Section 5 was to replace our axiom (M) with something weaker. It led us to replace it by (I1′), while strengthening the extension axiom (I3) to (I3′) or the basis exchange axiom (B2) to (B2′). We now show that (I1′) cannot be weakened further on the basis of (I2) and (I3′), the other alternative independence axioms from Theorem 5.2.

Example 6.1 shows that we cannot replace (I1′) with (I1): it is not enough that some independent set exists, we need a maximal one. But then (I3′) gives

[13] Here is another simple example. Let cl map every finite subset of E to itself, and every infinite subset to E. Then cl satisfies (CL1–3) and (CL4′), but the cl-independent sets are just the finite subsets of E, which fail to satisfy (M) for $X = E$.

us many more. This led Higgs [9] to ask whether our set of axioms from Theorem 5.2, with

(I0) $\mathcal{I}$ has a maximal element

instead of (I1′), would yield a (B-) matroid. If that was the case, then the axiom (M) common to all our systems would also be too strong: we could extract its 'extension' part as (I3′) and limit its 'existence' part to (I0). Of course, we would still want restrictions of matroids to be matroids, so (I1′)—which is (I0) for restrictions—should, somehow, follow.

Our next example shows that it does not: the example satisfies all axioms other than those of type (M) or type 1′, including (I0), but the set $X \subseteq E$ (see below) has no maximal independent subset.

To define this example, let $X = \{x_0, x_1, \dots\}$ and $Y = \{y_0, y_1, \dots\}$ be disjoint infinite sets, and let G be the graph with vertex set $E := X \cup Y$ and edge set $\{\, x_i y_i \mid i \in \mathbb{N} \,\}$, an infinite matching. Let $\mathcal{B}$ be the set of all transversals of the edges that meet X only finitely, i.e., of all subsets U of $X \cup Y$ satisfying $|U \cap \{x_i, y_i\}| = 1$ for every i and $|U \cap X| < \infty$ (Fig. 2). Let us call the elements of $\mathcal{B}$ the *skew transversals* of G. Put $\mathcal{I} := \lceil \mathcal{B} \rceil$, call its elements *independent* and the elements of $2^E \setminus \mathcal{I}$ *dependent* sets. Let $\mathcal{C}$ be the set of the minimal dependent sets, or *circuits*, and let cl be the closure operator associated with $\mathcal{I}$.

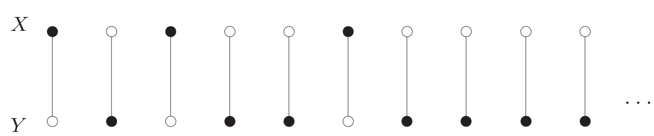

Figure 2: The set of black vertices is an element of $\mathcal{B}$.

Example 6.3. *Let $\mathcal{B}$ be the set of skew transversals of the graph G shown in Figure 2, and let $\mathcal{I}$, $\mathcal{C}$ and* cl *be defined as above.*

(i) *$\mathcal{I}$ satisfies* (I0)–(I3) *and* (I3′), *but not* (I1′) *or* (IM).

(ii) *$\mathcal{B}$ satisfies* (B0)–(B2) *and* (B2′), *but not* (B1′) *or* (BM).

(iii) *Not every dependent set contains a circuit.*

(iv) *$\mathcal{C}$ satisfies* (C1)–(C3), *but not* (CM).

(v) *The operator* cl *satisfies* (CL1)–(CL4), *but not* (CLM).

Proof. There are two properties of a set $U \subseteq E$ that will each make it dependent: meeting X infinitely, and containing an edge $\{x_i, y_i\}$. Every single edge is a circuit, while dependent transversals meeting X infinitely contain no circuit. So the circuits are precisely the single edges. X itself, then, is dependent but contains no circuit, showing (iii). Its independent subsets are its finite subsets. It thus has no maximal independent subset and hence violates (I1′) and (B1′). This proves (i) and (ii). (The other axioms are easy to check.) Since no two distinct circuits meet, (C3) holds vacuously. Since X violates (M) for $\mathcal{I}$, none of (IM), (BM), (CM) or (CLM) holds. Finally, the closure $\mathrm{cl}(U)$ of a set U is obtained by adding to U all those x_i for which $y_i \in U$, and all those y_i for which $x_i \in U$. Statement (v) follows. □

As we saw in Section 2, the elementary algebraic cycles of a graph G, the edge sets of its finite cycles and double rays, are the circuits of a matroid if and only if G contains no subdivision of the Bean graph (Fig. 1). This means that, at least in this class of examples, the Bean graph 'only just' fails to define a matroid. And indeed, we shall see below that its elementary algebraic cycles again violate exactly one of the axioms in each set.

To prove this, we need a formal definition of the Bean graph. Let it be the graph consisting of a ray R and a ray S starting at a common vertex u but otherwise disjoint, plus a vertex v adjacent to all the vertices of R. Write y for the edge uv, and pick an edge $z \in E(S)$. Let $\mathcal{C}$ be the set of the elementary algebraic cycles of this graph, and call them *circuits*. Write $\mathcal{I}$ for the set of edge sets not containing a circuit, put $\mathcal{B} := \mathcal{I}^{\max}$, let cl be the closure operator associated with $\mathcal{I}$, and call the sets in $2^{E(G)} \setminus \mathcal{I}$ *dependent*.

Example 6.4. *Let $\mathcal{C}$ be the set of elementary algebraic cycles of the Bean graph, and let $\mathcal{I}$, $\mathcal{B}$ and* cl *be as defined above.*

(i) *$\mathcal{C}$ satisfies* (C1)*,* (C2)*,* (CM) *and the usual finite circuit elimination axiom, but not the infinite elimination axiom* (C3)*.*

(ii) *Every dependent set contains a circuit.*

(iii) *$\mathcal{I}$ satisfies* (I0), (I1) (*even* (I1′)), (I2) *and* (IM)*, but not* (I3) *or* (I3′)*.*

(iv) *$\mathcal{B}$ satisfies* (B0)*,* (B1) (*even* (B1′)) *and* (BM)*, but not* (B2) *or* (B2′)*.*

(v) *The operator* cl *satisfies* (CL1), (CL2)*,* (CL4) *and* (CLM)*, but not* (CL3)*.*

Proof. (i) Assertions (C1) and (C2) are trivial. For a proof of (CM), let X be any set of edges, and let $I \subseteq X$ be an independent set. Using Zorn's Lemma we find a maximal subset B of X that contains I but contains no finite circuit. (Such a set B consists of a spanning tree in each component of the corresponding subgraph). If B contains no double ray either, it is a maximal independent subset of X containing I. If it does, then this double ray D is unique: since any double ray in the Bean graph has to link its two ends, two distinct double rays would form a finite cycle between them. As I is independent, D contains an edge $x \notin I$. Now $B - x$ is a maximal independent subset of X as required by (CM). The circuit elimination axiom for just two circuits is easily checked by straightforward case analysis for the current graph, or proved in general as for finite graphs by considering vertex degrees.

To see that (C3) fails, consider the circuit $C := E(R) \cup E(S)$, its subset $X := E(R)$, the edge $z \in E(S) \subseteq C$, and for every $x \in X$ as C_x the unique triangle containing x.

(ii) is trivial.

(iii) While (I0) is easy (consider $E(R) \cup E(S) + y - z$), (I1) and (I2) are trivial. We have already proved (IM), which implies (I1′). To see that (I3) and

(I3′) fail, consider as I the set of unbroken edges in Figure 1, and as I' the set $E(R) \cup E(S) + y - z$. Then $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ (since we can add z and remain independent), while $I' \in \mathcal{I}^{\max}$. But, clearly, I does not extend properly to any independent subset of $I \cup I'$.

(iv) $\mathcal{B}$ is the set of spanning trees not containing a double ray. It clearly satisfies (B0) and (B1). We have already proved (BM), and this implies (B1′). To see that (B2) fails (and with it (B2′)), consider as B_1 the set of unbroken edges in Figure 1 plus the edge z, and as B_2 the set $E(R) \cup E(S) + y - z$. Then we cannot delete z from B_1, add an edge of $B_2 \setminus B_1$, and remain independent.

(v) While (CL1), (CL2) and (CL4) are trivial, we have already proved (CLM). To see that (CL3) fails, consider as X the set of unbroken edges in Figure 1. Its closure $\mathrm{cl}(X)$ contains all the broken edges except z, but $\mathrm{cl}(\mathrm{cl}(X))$ contains z as well. □

References

[1] D.W.T. Bean, *A connected finitary co-finitary matroid is finite*, Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium, vol. 17, 1976, pp. 115–19.

[2] H. Bruhn, *Graphs and their circuits: from finite to infinite*, Habilitationsschrift Universität Hamburg, 2009.

[3] H. Bruhn and R. Diestel, *Infinite matroids in graphs*, in preparation.

[4] ______, *Duality in infinite graphs*, Comb., Probab. Comput. **15** (2006), 75–90.

[5] H. Bruhn and P. Wollan, *Connectivity in infinite matroids*, preprint 2010.

[6] R. Diestel, *Locally finite graphs with ends: a topological approach*, `http://arxiv.org/abs/0912.4213`, 2009.

[7] ______, *Graph Theory* (4th edition), Springer-Verlag, 2010.

[8] R. Diestel and D. Kühn, *Topological paths, cycles and spanning trees in infinite graphs*, Europ. J. Combinatorics **25** (2004), 835–862.

[9] D. A. Higgs, *Equicardinality of bases in b-matroids*, Can. Math. Bull. **12** (1969), 861–862.

[10] D.A. Higgs, *Infinite graphs and matroids*, Recent Progress in Combinatorics, Proceedings Third Waterloo Conference on Combinatorics, Academic Press, 1969, pp. 245–53.

[11] ______, *Matroids and duality*, Colloq. Math. **20** (1969), 215–220.

[12] V. Klee, *The greedy algorithm for finitary and confinitary matroids*, Combinatorics, Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc., 1971.

[13] G.J. Minty, *On the axiomatic foundation of the theories of directed linear graphs, electrical networks and network-programming*, J. Math. and Mech. **15** (1966), 485–520.

[14] J.G. Oxley, *Infinite matroids*, Proc. London Math. Soc. **37** (1978), no. 3, 259–72.

[15] ______, *Some problems in combinatorial geometries*, Ph.D. thesis, Oxford University, 1978.

[16] ______, *Infinite matroids*, Matroid applications (N. White, ed.), Encycl. Math. Appl., vol. 40, Cambridge University Press, 1992, pp. 73–90.

[17] ______, *Matroid theory*, Oxford University Press, 1992.

[18] ______, *personal communication*, 2005.

[19] R. Rado, *Abstract linear dependence*, Colloq. Math. **14** (1966), 257–64.

[20] C. Thomassen, *Duality of infinite graphs*, J. Combin. Theory (Series B) **33** (1982), 137–160.

[21] M. Las Vergnas, *Sur la dualité en théorie des matroïdes*, Théorie des Matroïdes, Lecture notes in mathematics, vol. 211, Springer-Verlag, 1971, pp. 67–85.

[22] Geoff Whittle, *Recent work in matroid representation theory*, Discrete Math. **302** (2005), 285–296.

[23] Jerzy Wojciechowski, *Infinite matroidal version of Hall's matching theorem*, J. London Math. Soc. **71** (2005), 563–578.