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MODULAR CATEGORIES FROM FINITE CROSSED MODULES

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Abstract

It is known that finite crossed modules provide premodular tensor categories. These categories are in fact modularizable. We construct the modularization and show that it is equivalent to the module category of a finite Drinfeld double.

1 Introduction

Modular tensor categories and, more generally, premodular tensor categories arise as representation categories of certain (weak) Hopf algebras, certain nets of von Neumann algebras and suitable classes of vertex algebras. They have found numerous applications, including the construction of invariants of three-manifolds and links, the construction of low-dimensional quantum field theories and the construction of gates in topological quantum computing. The simplest algebraic object whose representation category is a premodular tensor category is a finite crossed module:

Definition 1.1
A finite crossed module *consists of two finite groups $\mathcal{X}_1$ and $\mathcal{X}_2$, together with a (right)action μ of $\mathcal{X}_1$ on $\mathcal{X}_2$ by group automorphisms, written as $\mu(m,g) = m^g$ and a group homomorphism, called the* boundary map, *$\partial : \mathcal{X}_2 \to \mathcal{X}_1$ that satisfies*

$$\partial(m^g) = g^{-1}(\partial m)g \;\; \text{and} \;\; m^{\partial n} = n^{-1}mn \qquad \text{for all } \; m,n \in \mathcal{X}_2 \; \text{and} \; g \in \mathcal{X}_1.$$

It follows immediately from the definition that the kernel of ∂ is a central subgroup of $\mathcal{X}_2$ and that the image of ∂ is a normal subgroup of $\mathcal{X}_1$. This definition reduces to the usual

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definition of the Drinfeld double $\mathcal{D}(G)$ of a finite group G if the boundary map ∂ is the identity and the action μ is given by conjugation.

Our results hold over any algebraically closed field k of characteristic zero. Any finite crossed module $\mathcal{X}$ gives rise to a category of representations over k which we denote by $\mathcal{M}(\mathcal{X})$. The objects (V,P,Q) of the category $\mathcal{M}(\mathcal{X})$ are finite-dimensional $\mathcal{X}_2$-graded k-vector spaces $V = \oplus_{m\in\mathcal{X}_2} V_m$ with an action $Q:\mathcal{X}_1\to \mathrm{Aut}(V)$ of $\mathcal{X}_1$ such that

$$P(m)Q(g) = Q(g)P(m^g) \text{ for all } m\in\mathcal{X}_2 \text{ and } g\in\mathcal{X}_1 \ .$$

Here $P(m)$ is the projection to the m-th graded component. Morphisms are required to preserve the $\mathcal{X}_2$-grading and the $\mathcal{X}_1$-action. In other words, we consider the category of $\mathcal{X}_1$-equivariant vector bundles on $\mathcal{X}_2$ of finite rank.

We endow the category $\mathcal{M}(\mathcal{X})$ with the structure of a tensor category: the tensor product is the usual tensor product of vector spaces, where the grading on $V\otimes W$ is given by $(V\otimes W)_m = \bigoplus_{nl=m} V_n\otimes W_l$ and the action of $\mathcal{X}_1$ on $V\otimes W$ is the diagonal action. The boundary map ∂ gives the additional structure of a braided tensor category on $\mathcal{M}(\mathcal{X})$: braiding isomorphisms are given by

$$\begin{aligned} R_{VW}: V\otimes W &\to W\otimes V \\ v\otimes w &\mapsto \sum_{m\in\mathcal{X}_2} Q_W(\partial m)w\otimes P_V(m)v \ . \end{aligned} \tag{1}$$

Bantay has shown [Ba] that together with the dualities inherited from the category of finite-dimensional k-vector spaces, where on the dual space the grading is defined by $(V^*)_m = (V_{m^{-1}})^*$ and the action is given by $Q^*(g) = Q(g^{-1})^*$, the category $\mathcal{M}(\mathcal{X})$ has the structure of a premodular tensor category:

Definition 1.2

(i) Let k be an algebraically closed field of characteristic zero. A premodular tensor category *over k is an abelian, k-linear, semi-simple ribbon category $\mathcal{C}$ such that*

(a) The tensor product is linear in each variable and the tensor unit is absolutely simple, $\mathrm{End}(\mathbf{1}) = k$.

(b) There are only finitely many isomorphism classes of simple objects, indexed by a set $\Lambda_\mathcal{C}$.

(ii) The braiding on $\mathcal{C}$ allows to define the S-matrix *with entries in the field k*

$$s_{XY} := tr(R_{YX}\circ R_{XY}) \ , \tag{2}$$

where $X,Y\in\Lambda_\mathcal{C}$. A premodular category is called modular, *if the S-matrix is invertible.*

We refer to [BK, Ka] for the notion of a ribbon category. For a detailed discussion of the premodular tensor category $\mathcal{M}(\mathcal{X})$, including a character theory, we refer to [Ba].

Modular tensor categories are of particular interest, since they allow the construction of a topological field theory [RT, Tu] and thus of invariants of three-manifolds and of knots and links. The category $\mathcal{M}(\mathcal{X})$ associated to a crossed module is known to be modular, iff the boundary map ∂ is an isomorphism [N, Proposition 5.6]. In this case, it is equivalent to the representation category of the Drinfeld double of a finite group. Equivalent categories appear as representation categories of holomorphic orbifold theories, see e.g. [DVVV].

Bruguières [Br] (see also [M1]) has introduced the notion of modularization that associates to any premodular tensor category (obeying certain conditions) a modular tensor category. This tensor category is unique up to equivalence of braided tensor categories. The categories associated to crossed modules obey these conditions [Ba]; hence the question arises whether crossed modules provide a source of new modular tensor categories. A first main result of this note is a negative answer to this question in Theorem 4.1: the modularization yields a modular tensor category equivalent to the category for the Drinfeld double of $\mathcal{X}_2/\ker\partial \cong \operatorname{Im}\partial$.

Bruguières has also given an explicit modularization procedure which is based on a Tannakian subcategory of the premodular tensor category. As a second main result of this note, we determine in Proposition 2.12 the group corresponding to the Tannakian subcategory to be a semi-direct product

$$G(\mathcal{X}) := (\ker\partial)^* \rtimes_{\hat{\mu}} (\operatorname{coker}\partial) \ .$$

Here $(\ker\partial)^*$ is the group of characters of the finite abelian group $\ker\partial$; the semidirect product is explained in equation (7). The regular representation of G then provides a commutative special symmetric Frobenius algebra in the premodular tensor category $\mathcal{M}(\mathcal{X})$. By general arguments, the category of left modules over this algebra is a modular tensor category, see Proposition 3.7.

This note is organized as follows: in section 2 we recollect a few more aspects of crossed modules and their representation category from [Ba] and describe explicitly the full Tannakian subcategory of transparent objects. The transparent object corresponding to the regular representation of $G(\mathcal{X})$ is shown in Section 3 to be a commutative special symmetric Frobenius algebra $\underline{\mathbf{0}}$. We describe the modularization functor as the induction functor with respect to the algebra $\underline{\mathbf{0}}$. In section 4, this description is used to construct an explicit equivalence of categories from the modularization to the representation category of a Drinfeld double.

2 Premodular categories from finite crossed modules

We start by summarizing some more aspects of the premodular category $\mathcal{M}(\mathcal{X})$ defined in the previous section. For any object $V \in \mathcal{M}(\mathcal{X})$, the character is defined as the function

$$\begin{array}{rrcl} \psi: & \mathcal{X}_2 \times \mathcal{X}_1 & \to & k \\ & (m,g) & \mapsto & \operatorname{tr}_V(P(m)\, Q(g)) \ . \end{array}$$

The character theory for finite crossed modules largely parallels (and in fact generalizes) the character theory of finite groups [Ba]. In particular, Maschke's theorem and orthgonality relations hold: for a general field k, the irreducible characters are orthogonal for the non-degenerate symmetric bilinear form

$$< \psi_1, \psi_2 > := \frac{1}{|\mathcal{X}_1|} \sum_{g\in\mathcal{X}_1, m\in\mathcal{X}_2} \psi_1(m, g^{-1}) \psi_2(m,g) \ .$$

For $k = \mathbb{C}$, the irreducible characters are orthonormal with respect to the hermitian scalar product

$$(\psi_1, \psi_2) := \frac{1}{|\mathcal{X}_1|} \sum_{g\in\mathcal{X}_1, m\in\mathcal{X}_2} \overline{\psi_1(m, g)} \psi_2(m,g) \ .$$

We introduce two particularly important objects in $\mathcal{M}(\mathcal{X})$. To this end, we introduce the notation $K := \ker\,\partial, C := \operatorname{coker}\partial$ and $I := \operatorname{Im}\partial$.

(i) The tensor unit $\underline{\mathbf{1}}$ is defined on a one-dimensional vector space in the graded component for $e \in \mathcal{X}_2$ and with trivial action of $\mathcal{X}_1$.

(ii) The vacuum object is the triple $\underline{\mathbf{0}} = (V_{\underline{\mathbf{0}}}, P_{\underline{\mathbf{0}}}, Q_{\underline{\mathbf{0}}})$ with the vector space

$$V_{\underline{\mathbf{0}}} = k[\ker \partial] \otimes k(\operatorname{coker} \partial) \equiv k[K] \otimes k(C) \ .$$

On the distinguished basis $(x \otimes \delta_{Iy})_{x \in K, Iy \in C}$ we set for $m \in \mathcal{X}_2,\ g \in \mathcal{X}_1$:

$$P_{\underline{\mathbf{0}}}(m)(x \otimes \delta_{Iy}) = \delta(m^y, x)(x \otimes \delta_{Iy}) \qquad Q_{\underline{\mathbf{0}}}(g)(x \otimes \delta_{Iy}) = (x \otimes \delta_{Igy})$$

A direct calculation using the explicit form (1) of the braiding gives the S-matrix defined in (2) in terms of the characters: the entry corresponding to the irreducible representations $p, q \in \Lambda_{\mathcal{M}(\mathcal{X})}$ is

$$s_{pq} = \sum_{m,n \in \mathcal{X}_2} \psi_p(m, \partial n)\psi_q(n, \partial m) \ .$$

It is convenient to introduce a normalization factor to obtain a non-degenerate symmetric matrix:

$$S_{pq} := \frac{s_{pq}}{|\mathcal{X}|} = \frac{1}{|\mathcal{X}|} \sum_{m,n \in \mathcal{X}_2} \psi_p(m, \partial n)\psi_q(n, \partial m)$$

with $|\mathcal{X}| := |\mathcal{X}_1| \cdot |\ker \partial| = |\mathcal{X}_2| \cdot |\operatorname{coker} \partial|$. For later reference, we associate to each $p \in \Lambda_{\mathcal{M}(\mathcal{X})}$ the number

$$\omega_p := \frac{1}{d_p} \sum_{m \in \mathcal{X}_2} \psi_p(m, \partial m) \ ,$$

where d_p is the categorical dimension of p (which coincides with the dimension of the underlying vector space). It gives the eigenvalue of the twist θ_p on the simple object p and it can be shown to satisfy the equality

$$\psi_p(m, g\partial m) = \omega_p \cdot \psi_p(m, g) \ . \tag{3}$$

Remark 2.1.

(i) Given any simple object $p \in \Lambda_{\mathcal{M}(\mathcal{X})}$, we have $S_{\underline{\mathbf{1}}p} = \frac{d_p}{|\mathcal{X}|}$, where $\underline{\mathbf{1}}$ is the tensor unit.

(ii) The multiplicity $\mu_p = \dim_k \operatorname{Hom}(p, \underline{\mathbf{0}})$ of the irreducible representation p in $\underline{\mathbf{0}}$ equals

$$\mu_p = D[S^2]_{\underline{\mathbf{1}}p} \ , \tag{4}$$

where $D := |\operatorname{coker} \partial| \cdot |\ker \partial|$. This follows in a straightforward calculation by expressing the multiplicity in terms of characters as $\mu_p = (\psi_p, \psi_{\underline{\mathbf{0}}})$ and then using orthogonality relations to compare with the matrix element of S^2.

We recall the following definition from [Br]:

Definition 2.2

An object X of a braided tensor category $\mathcal{C}$ is called transparent, *if the equation $R_{Y,X} = R^{-1}_{X,Y}$ holds for all $Y \in \mathcal{C}$. We denote the set of isomorphism classes of simple transparent objects by $T_{\mathcal{C}}$.*

The following observations are straightforward:

Remark 2.3.

(i) Direct summands of transparent objects and direct sums of transparent objects are transparent.

(ii) The vacuum object $\underline{\mathbf{0}}$ of $\mathcal{M}(\mathcal{X})$ is transparent. This follows from a straightforward calculation using the explicit form (1) of the braiding.

Lemma 2.4.
An irreducible representation $p \in \Lambda_{\mathcal{M}(\mathcal{X})}$ is a direct summand of the vacuum object $\underline{\mathbf{0}}$, if and only if the p-th row of the S-matrix is collinear to the row $(S_{\underline{\mathbf{1}}q})_{q\in\Lambda_{\mathcal{M}(\mathcal{X})}}$.

In this case, the multiplicity μ_p equals $\alpha = \mu_p = d_p$, where $S_{pq} = \alpha S_{\underline{\mathbf{1}}q}$. Moreover, the twist on any such simple object with $\mu_p > 0$ is the identity.

Proof:
Suppose, there is an $\alpha \in k$ such that $S_{pq} = \alpha S_{\underline{\mathbf{1}}q}$ for all $q \in \Lambda_{\mathcal{M}(\mathcal{X})}$. Specializing to $q = \underline{\mathbf{1}}$ yields $\frac{d_p}{|\mathcal{X}|} = \frac{\alpha}{|\mathcal{X}|}$ and hence the first identity $\alpha = d_p > 0$. We compute the multiplicity μ_p:

$$\mu_p \overset{(4)}{=} D[S^2]_{\underline{\mathbf{1}}p} = D \sum_{q\in\Lambda_{\mathcal{M}(\mathcal{X})}} S_{\underline{\mathbf{1}}q} S_{pq} = \alpha D \sum_{q\in\Lambda_{\mathcal{M}(\mathcal{X})}} S_{\underline{\mathbf{1}}q}^2 = \alpha \frac{D}{|\mathcal{X}|^2} \sum_{q\in\Lambda_{\mathcal{M}(\mathcal{X})}} d_p^2 = \alpha \ ,$$

where in the last step we have used a generalization of Burnside's Theorem [Ba] for the characters.

Conversely, suppose $\mu_p > 0$ so that p is a direct summand of the transparent object $\underline{\mathbf{0}}$ and hence by Remark 2.3 transparent itself. For any $q \in \Lambda_{\mathcal{M}(\mathcal{X})}$ we conclude $R_{qp} \circ R_{pq} = \mathrm{id}_{p\otimes q}$ and thus

$$S_{pq} = \frac{1}{|\mathcal{X}|}\mathrm{tr}(\mathrm{id}_{p\otimes q}) = \frac{d_p d_q}{|\mathcal{X}|} = d_p S_{\underline{\mathbf{1}}q} \ .$$

The equalities

$$\begin{aligned}\mu_p &= \frac{1}{|\mathcal{X}_2|} \sum_{n\in\mathcal{X}_2, m\in K} \psi_p(m, \partial n) \overset{(3)}{=} \frac{1}{|\mathcal{X}_2|} \sum_{n\in\mathcal{X}_2, m\in K} \psi_p(m, \partial n m^{-1})\, \omega_p \\ &= \frac{1}{|\mathcal{X}_2|} \sum_{\tilde{n}\in\mathcal{X}_2, m\in K} \psi_p(m, \partial \tilde{n})\, \omega_p = \mu_p \omega_p\end{aligned}$$

show that $\mu_p > 0$ implies $\omega_p = 1$. □

For a premodular category $\mathcal{C}$, consider the set of isomorphism classes of those simple objects $X \in \mathcal{C}$ for which the row $(s_{XY})_{Y\in\Lambda_\mathcal{C}}$ is collinear with the row $(s_{\underline{\mathbf{1}}Y})_{Y\in\Lambda_\mathcal{C}}$ of the tensor unit:

$$M_\mathcal{C} = \{X \in \Lambda_\mathcal{C} \mid \forall Y \in \Lambda_\mathcal{C} : s_{XY} = \dim X \dim Y\} \ .$$

Corollary 2.5.
We have the following identities for the category $\mathcal{M}(\mathcal{X})$:

(i) $M_{\mathcal{M}(\mathcal{X})} = T_{\mathcal{M}(\mathcal{X})}$.

(ii) $\theta_X = \mathrm{id}_X$ *for all transparent objects* X.

(iii) $\sum_{p\in T_{\mathcal{M}(\mathcal{X})}} (d_p)^2 = |\ker \partial||\mathrm{coker}\,\partial|$ *with* $d_p = \dim p$.

Proof:

(i) According to Lemma 2.4, any simple object $p \in M_{\mathcal{M}(\mathcal{X})}$ is contained in the transparent object $\underline{\mathbf{0}}$ and thus transparent itself. The other inclusion is obvious.

(ii) Lemma 2.4 and the first assertion of this corollary imply $\theta_p = \mathrm{id}_p$ for all $p \in T_{\mathcal{M}(\mathcal{X})}$. The assertion follows since a transparent object is a direct sum of simple transparent objects.

(iii) The definition of μ_p and Lemma 2.4 imply

$$|\ker \partial| \cdot |\mathrm{coker}\,\partial| = \dim \underline{\mathbf{0}} = \sum_{p\in\Lambda_{\mathcal{M}(\mathcal{X})}} \mu_p d_p = \sum_{p\in T_{\mathcal{M}(\mathcal{X})}} (d_p)^2 \ .$$

□

Bruguières' modularity criterion [Br, Proposition 1.1] asserts that a premodular category $\mathcal{C}$ is modular if and only if $M_{\mathcal{C}} = \{\mathbf{1}\}$. As an application we obtain:

Proposition 2.6.
The category $\mathcal{M}(\mathcal{X})$ *is modular, if and only if the boundary map* ∂ *is a bijection. In this case,* $\mathcal{M}(\mathcal{X})$ *is equivalent to the representation category of a Drinfeld double.*

Proof:
Lemma 2.4 implies that the row $(S_{pq})_{q\in\Lambda_{\mathcal{M}(\mathcal{X})}}$ is collinear with $(S_{\underline{\mathbf{1}}q})_{q\in\Lambda_{\mathcal{M}(\mathcal{X})}}$, if and only if p has non-vanishing multiplicity in $\underline{\mathbf{0}}$. For the tensor unit, we have multiplicity $\mu_{\underline{\mathbf{1}}} = d_{\underline{\mathbf{1}}} = 1$.

If the boundary map ∂ is a bijection, we have $\underline{\mathbf{0}} \cong \underline{\mathbf{1}}$ and, according to Bruguières' criterion, the category $\mathcal{M}(\mathcal{X})$ is modular. If ∂ is not bijective, we have $\dim V_{\underline{\mathbf{0}}} > 1$ and $\underline{\mathbf{0}}$ contains at least one simple object that is not isomorphic to the tensor unit $\underline{\mathbf{1}}$. Bruguières' criterion now implies that the category is not modular. □

For a proof of this assertion that does not directly use Bruigières' criterion, we refer to [N, Proposition 5.6].

We will now explain why the premodular category $\mathcal{M}(\mathcal{X})$ is modularizable [Ba]. To this end, we repeat some definitions of [Br]:

Definition 2.7

(i) An object X *of a category* $\mathcal{C}$ *is called* retract *of an object* $Y \in \mathcal{C}$, *if there are morphisms* $\iota : X \to Y$ *and* $\pi : Y \to X$ *such that* $\pi \circ \iota = \mathrm{id}_X$.

(ii) A functor $F : \mathcal{C} \to \mathcal{C}'$ *is called* dominant, *if for every object* $X \in \mathcal{C}'$ *there exists an object* $Y \in \mathcal{C}$ *such that* X *is a retract of* $F(Y)$.

(iii) A modularization *of a premodular category* $\mathcal{C}$ *is a dominant ribbon functor* $F : \mathcal{C} \to \mathcal{C}'$ *with* $\mathcal{C}'$ *a modular tensor category. A premodular category is called* modularizable, *if it admits a modularization.*

If a modularization exists, it is unique up to equivalence of braided tensor categories. It is known [Br, Corollary 3.5] that a premodular category over an algebraically closed field of characteristic zero is modularizable, if and only if for all objects $X \in M_{\mathcal{C}}$ one has $X \in T_{\mathcal{C}}$, $\theta_X = \mathrm{id}_X$ and $\dim X \in \mathbb{N}$. We thus obtain for the category $\mathcal{M}(\mathcal{X})$:

Proposition 2.8.
The premodular category $\mathcal{M}(\mathcal{X})$ is modularizable.

Proof:
Corollary 2.5 implies for $p \in M_{\mathcal{M}(\mathcal{X})}$ that $p \in T_{\mathcal{M}(\mathcal{X})}$ and $\theta_p = \mathrm{id}_p$. The assertion follows by [Br, Corollary 3.5]. □

Proposition 2.3 of [Br] allows to detect modularizations among dominant ribbon functors $F : \mathcal{C} \to \mathcal{C}'$ between premodular categories: it is sufficient to check that for any transparent object $X \in M_{\mathcal{C}}$ the image $F(X)$ is trivial in the sense that it is a finite direct sum of the tensor unit of $\mathcal{C}'$.

Let us investigate further the tensor subcategory of transparent objects:

Definition 2.9
A premodular category $\mathcal{C}$ enriched over an algebraically closed field k is called Tannakian, *if there exists a modularization of $\mathcal{C}$ that is equivalent to the category $\mathrm{vect}_f(k)$ of finite-dimensional k-vector spaces.*

We need the following facts proven in [De, Theorem 7.1] and [DM, Theorem 2.11]:

Proposition 2.10.
Let $\mathcal{C}$ be a premodular category over an algebraically closed field k of characteristic zero.

(i) The category $\mathcal{C}$ is Tannakian, if and only if for all simple objects $X \in \Lambda_{\mathcal{C}}$ the twist equals the identity, $\theta_X = \mathrm{id}_X$, and $\dim X \in \mathbb{N}$.

(ii) If $\mathcal{C}$ is Tannakian, it is equivalent as a tensor category to the category of representations of a finite group G on k-vector spaces.

Corollary 2.11.
The full tensor subcategory $\mathcal{M}(\mathcal{X})^T$ of transparent objects of a premodular tensor category is Tannakian.

Proof:
This follows immediately from Proposition 2.10 (i) and Corollary 2.5 (ii). □

We next determine explicitly the finite group G describing the Tannakian subcategory $\mathcal{M}(\mathcal{X})^T$. The action $\mu : \mathcal{X}_2 \times \mathcal{X}_1 \to \mathcal{X}_2$ that is part of the crossed module $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mu, \partial)$ factorizes to an action of $\operatorname{coker} \partial$ on $\mathcal{X}_2$ which can be restricted to an action of $\operatorname{coker} \partial$ on $\ker \partial$:

$$\begin{array}{ccc} \ker \partial \times \operatorname{coker} \partial & \to & \ker \partial \\ (k, Ig) & \mapsto & k^{Ig} := k^g \;. \end{array} \tag{5}$$

Since the subgroup $\ker\,\partial$ is abelian, its irreducible characters form a group $(\ker\,\partial)^*$. We introduce the dual action

$$\begin{array}{rcl}\hat{\mu}:(\ker\,\partial)^*\times\operatorname{coker}\partial & \to & (\ker\partial)^* \\ (\chi,c) & \mapsto & \chi^c(k):=\chi(k^{c^{-1}})\;,\end{array} \tag{6}$$

where we tacitly use the canonical identification $(\ker\partial)^{**}\cong\ker\partial$. We denote by $G(\mathcal{X})$ the semi-direct product

$$G(\mathcal{X}):=(\ker\,\partial)^*\rtimes_{\hat{\mu}}(\operatorname{coker}\partial)\;. \tag{7}$$

Proposition 2.12.
The category $G(\mathcal{X})$-Rep is equivalent, as a tensor category, to the category $\mathcal{M}(\mathcal{X})^T$ of transparent $\mathcal{X}$-representations.

Proof:

- We construct the equivalence explicitly and define a functor on objects as

$$F:G(\mathcal{X})\text{-Rep}\to\mathcal{M}(\mathcal{X}) \tag{8}$$

which maps the $G(\mathcal{X})$-representation (V,ρ) to the triple (V,P^ρ,Q^ρ) with

$$\begin{aligned}P^\rho(m)&:=\begin{cases}\frac{1}{|K|}\sum_{\chi\in K^*}\chi(m)\rho(\chi,I) & \text{if } m\in K\equiv\ker\,\partial\\ 0 & \text{else}\end{cases}\\ Q^\rho(g)&:=\rho(1,Ig).\end{aligned}$$

Since a linear map commuting with the $G(\mathcal{X})$-action commutes with the action of $\mathcal{X}$ defined by P^ρ and Q^ρ, we can define F on morphisms as the identity so that the functor F is fully faithful. To show that the $\mathcal{X}$-representation (V,P^ρ,Q^ρ) is transparent, consider any $\mathcal{X}$-representation (W,P_W,Q_W) and compute the braiding:

$$\begin{aligned}R_{V,W}&=\sum_{m\in\mathcal{X}_2}Q_W(\partial m)\otimes P^\rho(m)\circ\tau_{V,W}=\sum_{m\in K}Q_W(\partial m)\otimes\frac{1}{|K|}\sum_{\chi\in K^*}\chi(m)\rho(\chi,I)\circ\tau_{V,W}\\ &=Q_W(1)\otimes\sum_{\chi\in K^*}\delta(\chi,1)\rho(\chi,I)\circ\tau_{V,W}\qquad(\text{since } m\in\ker\,\partial)\\ &=(\mathrm{id}_W\otimes\mathrm{id}_V)\circ\tau_{V,W}=\tau_{V,W}\end{aligned}$$

where $\tau_{V,W}:V\otimes W\to W\otimes V$ is the transposition map. Similarly, we find

$$\begin{aligned}R_{W,V}&=\sum_{m\in\mathcal{X}_2}Q^\rho(\partial m)\otimes P_W(m)\circ\tau_{W,V}=\rho(1,I)\otimes\sum_{m\in\mathcal{X}_2}P_W(m)\circ\tau_{W,V}\\ &=(\mathrm{id}_V\otimes\mathrm{id}_W)\circ\tau_{W,V}=\tau_{W,V}\;.\end{aligned}$$

- We next show that F is a strict tensor functor. To check that the tensor unit of $G(\mathcal{X})$-Rep, the trivial representation, is mapped to the tensor unit in $\mathcal{M}(\mathcal{X})$, we remark that for

$m \in \mathcal{X}_2$ and $g \in \mathcal{X}_1$ we have

$$P^{\rho_{\mathbf{1}}}(m) = \frac{1}{|K|} \sum_{\chi \in K^*} \chi(m) \rho_{\mathbf{1}}(\chi, I) = \frac{1}{|K|} \sum_{\chi \in K^*} \chi(m) \mathrm{id}_{\mathbb{C}} = \frac{1}{|K|} \sum_{\chi \in K^*} m(\chi) \mathrm{id}_{\mathbb{C}} = \delta(m, 1) \mathrm{id}_{\mathbb{C}},$$

$$Q^{\rho_{\mathbf{1}}}(g) = \rho_{\mathbf{1}}(1, Ig) = \mathrm{id}_{\mathbb{C}}.$$

Consider two objects (V_1, ρ_1) and (V_2, ρ_2) of $G(\mathcal{X})$-Rep. To show that the tensor product of the images under F equals the image of the tensor product $(V_1 \otimes V_2, P^{\rho_1 \otimes \rho_2}, Q^{\rho_1 \otimes \rho_2})$, we remark

$$\begin{aligned}
P_{V_1 \otimes V_2}(m) &= \sum_{n \in \mathcal{X}_2} P^{\rho_1}(n) \otimes P^{\rho_2}(n^{-1} m) \\
&= \sum_{n \in K} \frac{1}{|K|^2} \sum_{\chi, \tilde{\chi} \in K^*} \chi(n) \tilde{\chi}(n^{-1} m) \rho_1(n, I) \otimes \rho_2(n^{-1} m, I) \\
&= \frac{1}{|K|} \sum_{\chi, \tilde{\chi} \in K^*} \delta(\chi, \tilde{\chi}) \chi(m) \rho_1(m, I) \otimes \rho_2(m, I) \\
&= \frac{1}{|K|} \sum_{\chi \in K^*} \chi(m) \rho_1(m, I) \otimes \rho_2(m, I) = P^{\rho_1 \otimes \rho_2}(m) \ ,
\end{aligned}$$

where the third equality is the generalized orthogonality relation for group characters [Is, Theorem 2.13]. The analogous identity for the action of $\mathcal{X}_1$ is straightforward.

- The functor F being fully faithful, it suffices to show that F is essentially surjective to prove that it is an equivalence of tensor categories.

  Any transparent object is a direct sum of simple transparent objects; hence we can restrict ourselves to simple transparent objects. They are all direct summands of the vacuum object $\underline{\mathbf{0}} = (V_{\underline{\mathbf{0}}}, P_{\underline{\mathbf{0}}}, Q_{\underline{\mathbf{0}}})$ (Lemma 2.4). From this, we conclude that the linear map $P_{\underline{\mathbf{0}}}(m)$ is zero for $m \notin K$ and that the automorphism $Q_{\underline{\mathbf{0}}}(g)$ is constant on the equivalence classes of the cokernel $\operatorname{coker} \partial = \mathcal{X}_1 / I$. Consider thus for a simple transparent object (V, P, Q)

$$\begin{array}{rcl}
\rho : G(\mathcal{X}) & \to & \mathrm{Aut}(V) \\
(\chi, Ig) & \mapsto & \rho(\chi, Ig) := \sum_{k \in K} \chi^{-1}(k) Q(g) P(k).
\end{array} \tag{9}$$

  Direct computations show that this defines an action of the group $G(\mathcal{X})$.

  The image of the $G(\mathcal{X})$-representation (V, ρ) under F is the $\mathcal{X}$-representation

$$\begin{aligned}
P^\rho(m) &= \frac{1}{|K|} \sum_{\chi \in K^*} \chi(m) \rho(\chi, I) = \frac{1}{|K|} \sum_{k \in K, \chi \in K^*} \chi(m) \chi^{-1}(k) Q(1) P(k) \\
&= \frac{1}{|K|} \sum_{k \in K^{**}, \chi \in K^*} m(\chi) k(\chi^{-1}) P(k) = \sum_{k \in K^{**}} \delta(k, m) P(k) = P(m) \qquad \text{if } m \in K \\
P^\rho(m) &= 0 = P(m) \qquad \text{if } m \notin K
\end{aligned}$$

  and

$$Q^\rho(g) = \rho(1, Ig) = \sum_{k \in K} 1(k) Q(g) P(k) = Q(g) \ .$$

  We conclude that F is essentially surjective and thus an equivalence of tensor categories.

$\square$

3 The modularization of $\mathcal{M}(\mathcal{X})$

The vacuum object $\underline{\mathbf{0}}$ carries additional algebraic structure which crucially enters in the modularization of the premodular category $\mathcal{M}(\mathcal{X})$.

Definition 3.1

(i) *An* algebra *in a (strict) tensor category $\mathcal{C}$ is a triple consisting of an object $A \in \mathcal{C}$, a multiplication morphism $m \in \mathrm{Hom}(A \otimes A, A)$ and a unit $\eta \in \mathrm{Hom}(\underline{\mathbf{1}}, A)$ obeying the equations*

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id} \otimes m) \qquad \textit{and} \qquad m \circ (\eta \otimes \mathrm{id}_A) = \mathrm{id}_A = m \circ (\mathrm{id}_A \otimes \eta) \ .$$

A coalgebra *in $\mathcal{C}$ is defined analogously as a triple consisting of an object C, a comultiplication morphism $\Delta : C \to C \otimes C$ and a counit $\epsilon : C \to \underline{\mathbf{1}}$ obeying coassociativity and counit equalities.*

(ii) *An algebra (A, m, η) in a braided tensor category $\mathcal{C}$ is called (braided-)*commutative*, if $m \circ R_{AA} = m$.*

(iii) *An algebra in a tensor category is called* haploid*, if it is simple as a left module over itself, i.e. if $\dim_k \mathrm{Hom}(\underline{\mathbf{1}}, A) = 1$.*

In the sequel we will see, that $\underline{\mathbf{0}}$ even carries the structure of a special symmetric Frobenius algebra:

Definition 3.2
Let $\mathcal{C}$ be a (strict) tensor category.

(i) *A* Frobenius algebra *in $\mathcal{C}$ is an object with an algebra structure (A, m, η) and a coalgebra structure (A, Δ, ϵ) such that $\Delta : A \to A \otimes A$ is a morphism of A-bimodules:*

$$(\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A) = \Delta \circ m = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta). \tag{10}$$

(ii) *Suppose that the tensor category $\mathcal{C}$ is enriched over the category of k-vector spaces where k is a field. A* special algebra *in $\mathcal{C}$ is an object that is endowed with an algebra and a coalgebra structure such that*

$$\epsilon \circ \eta = \beta_1 \mathrm{id}_{\underline{\mathbf{1}}} \qquad \textit{and} \qquad m \circ \Delta = \beta_A \mathrm{id}_A$$

with invertible elements $\beta_1, \beta_A \in k^\times$.

(iii) *Let $\mathcal{C}$ be a sovereign tensor category, i.e. a category with left and right dualities that coincide as functors from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{opp}}$. A* symmetric algebra *in $\mathcal{C}$ is an algebra (A, m, η) together with a morphism $\epsilon \in \mathrm{Hom}(A, \underline{\mathbf{1}})$ such that the two morphisms*

$\Phi_1, \Phi_2 : A \to A^\vee$

$$\Phi_1 := [(\epsilon \circ m) \otimes \mathrm{id}_{A^\vee}] \circ (\mathrm{id}_A \otimes b_A) \in \mathrm{Hom}(A, A^\vee) \tag{11}$$

$$\Phi_2 := [\mathrm{id}_A^\vee \otimes (\epsilon \circ m)] \circ (\tilde{b}_A \otimes \mathrm{id}_A) \in \mathrm{Hom}(A, A^\vee) \tag{12}$$

are identical.

Here $b_A : \underline{\mathbf{1}} \to A \otimes A^\vee$ and $\tilde{b}_A : \underline{\mathbf{1}} \to A^\vee \otimes A$ are the coevaluations of the two dualities.

Let G be a finite group and k a field. An important example of a symmetric special Frobenius algebra in the symmetric tensor category of $k[G]$-modules is the algebra of functions $k(G)$ on G, the regular representation.

Lemma 3.3.
The essential image of the regular representation of $G(\mathcal{X})$ under the functor F is the the vacuum object $\underline{\mathbf{0}}$.

Corollary 3.4.
Since $k(G(\mathcal{X}))$ is a commutative symmetric Frobenius algebra, the vacuum object $\underline{\mathbf{0}}$ carries a natural structure of a symmetric special Frobenius algebra in $\mathcal{M}(\mathcal{X})^T$ and thus in $\mathcal{M}(\mathcal{X})$.

Proof of Lemma 3.3:
Consider the natural basis $\{(\chi, c)\}_{(\chi,c)\in K^* \rtimes_{\hat{\mu}} C}$ of $k(G(\mathcal{X}))$ of idempotents

$$(\chi, c) \cdot (\tilde{\chi}, \tilde{c}) := \delta(\chi, \tilde{\chi})\delta(c, \tilde{c})(\tilde{\chi}, \tilde{c}) \tag{13}$$

and in which the regular representation $\rho_R : G(\mathcal{X}) \to \mathrm{Aut}(G(\mathcal{X}))$ is given by

$$\rho_R(\chi, c)(\tilde{\chi}, \tilde{c}) := (\chi^{\tilde{c}}\tilde{\chi}, c\tilde{c}) \ . \tag{14}$$

It is convenient to perform a partial Fourier transform with respect to K to introduce also the basis

$$(k, c) := \sum_{\chi\in K^*} \chi(k)(\chi, c). \tag{15}$$

of

$k(G(\mathcal{X}))$ in which the multiplication is

$$(k, c) \cdot (\tilde{k}, \tilde{c}) = \delta(c, \tilde{c})(k\tilde{k}, \tilde{c}) \ .$$

The regular algebra $k(G(\mathcal{X}))$ is mapped under the functor F to the triple $(k(K^* \rtimes_{\hat{\mu}} C), P^R, Q^R)$ with

$$P^R(m) = \begin{cases} \frac{1}{|K|} \sum_{\chi\in K^*} \chi(m)\rho_R(\chi, I) & \text{if } m \in K \\ 0 & \text{else} \end{cases} \tag{16}$$

$$Q^R(g) = \rho_R(1, Ig). \tag{17}$$

We compute the action of P^R and Q^R on the basis $(k, c)_{(k,c)\in K\times C}$:

$$
\begin{aligned}
P^R(m)(\tilde{k}, I\tilde{g}) &= \frac{1}{|K|}\sum_{\chi,\tilde{\chi}\in K^*} \chi(m)\tilde{\chi}(\tilde{k})\rho_R(\chi, I)(\tilde{\chi}, I\tilde{g}) \\
&= \frac{1}{|K|}\sum_{\chi,\tilde{\chi}\in K^*} \chi(m^{\tilde{g}})\tilde{\chi}(\tilde{k})(\chi\tilde{\chi}, I\tilde{g}) = \frac{1}{|K|}\sum_{\chi,\chi'\in K^*} m^{\tilde{g}}(\chi)\tilde{k}(\chi^{-1}\chi')(\chi', I\tilde{g}) \\
&= \sum_{\chi'\in K^*} \delta(m^{\tilde{g}}, \tilde{k})\chi'(\tilde{k})(\chi', I\tilde{g}) = \delta(m^{\tilde{g}}, \tilde{k})(\tilde{k}, I\tilde{g}) && \text{if } m \in K \\
P^R(m)(\tilde{k}, I\tilde{g}) &= 0 = \delta(m^{\tilde{g}}, \tilde{k})(\tilde{k}, I\tilde{g}) && \text{if } m \notin K \\
Q^R(g)(\tilde{k}, I\tilde{g}) &= \sum_{\chi\in K^*} \chi(\tilde{k})\rho_R(1, Ig)(\chi, I\tilde{g}) = \sum_{\chi\in K^*} \chi(\tilde{k})(\chi, Ig\tilde{g}) = (\tilde{k}, Ig\tilde{g}) \ .
\end{aligned}
$$

Since this is precisely the action of $\mathcal{X}$ on $\underline{\mathbf{0}}$, we have proven the assertion. $\square$

Modules over the special symmetric Frobenius algebra $\underline{\mathbf{0}}$ crucially enter in the concrete construction [Br, Lemma 3.3] of the modularization:

Definition 3.5
Let A be an algebra in a strict tensor category $\mathcal{C}$.

*(i) A (left) A-*module *is a pair (X, ρ_X) with $A \in \mathcal{C}$ and $\rho_X \in \mathrm{Hom}_\mathcal{C}(A \otimes X, X)$ such that*

$$\rho \circ (m \otimes \mathrm{id}_X) = \rho \circ (\mathrm{id}_A \otimes \rho) \quad \text{and} \quad \rho \circ (\eta \otimes \mathrm{id}_X) = \mathrm{id}_X \ .$$

(ii) A module (X, ρ_X) over A is called local *or* dyslectic *([Pa, KO, FFRS]), if $\rho_X \circ R_{XA} \circ R_{AX} = \rho_X$.*

(iii) A morphism of A-modules (X, ρ_X) and (Y, ρ_Y) is a morphism $f \in \mathrm{Hom}_\mathcal{C}(X, Y)$ such that

$$f \circ \rho_X = \rho_Y \circ (\mathrm{id}_A \otimes f) \ . \tag{18}$$

(iv) We denote by A-$\mathrm{Mod}_\mathcal{C}$ the category of A-modules in $\mathcal{C}$ and by A-$\mathrm{Mod}_\mathcal{C}^{loc}$ the full subcategory of local A-modules.

Remark 3.6.
Let $\mathcal{C}$ be a braided tensor category and A be a commutative algebra in $\mathcal{C}$. The following elementary facts from commutative algebra are still valid in this setting:

(i) Every left A-module (M, ρ) has a structure of a right A-module with $(M, \rho \circ R_{M,A})$.

(ii) Let M, N be two left A-modules. Then

$$M \otimes_A N := \mathrm{coker}\,(\rho_M \circ R_{M,A} \otimes \mathrm{id}_N - \mathrm{id}_M \otimes \rho_N)$$

endows the category A-$\mathrm{Mod}_\mathcal{C}$ with the structure of a tensor category.

In fact, the modularization functor was constructed in [Br, Proposition 3.2] as an induction functor for a special commutative symmetric Frobenius algebra obtained as the regular algebra in a Tannakian subcategory. We conclude

Proposition 3.7.
The induction functor

$$\begin{aligned} \mathrm{Ind}_{\underline{\mathbf{0}}} : \mathcal{M}(\mathcal{X}) &\to \overline{\mathcal{M}(\mathcal{X})} := \underline{\mathbf{0}}\text{-Mod} \\ X &\mapsto (\underline{\mathbf{0}} \otimes X, m \otimes \mathrm{id}_X) \end{aligned}$$

is a modularization of $\mathcal{M}(\mathcal{X})$.

Proof:
By [FS, Proposition 5.11], the induction functor is a tensor functor and by [FS, Proposition 5.17] it is compatible with duality. Since $\underline{\mathbf{0}}$ is a special Frobenius algebra, the induction functor is dominant by [FS, Lemma 4.15]. From the explicit form of the braiding given in [FFRS, Proposition 3.21] one deduces that the induction functor respects the braiding and is thus a ribbon functor. The category of modules over the regular algebra $k(G(\mathcal{X}))$ in $G(\mathcal{X})$-Rep is equivalent to the category of k-vector spaces. Hence, for any transparent object $X \in \mathcal{C}$, the induced module is isomorphic to a direct sum of $\underline{\mathbf{0}}$. Thus by [Br, Proposition 2.3], the induction functor is a modularization. $\square$

4 Explicit description of the modularization

We now wish to describe the modularization $\overline{\mathcal{M}(\mathcal{X})}$ explicitly by showing that it is equivalent to the category of representations of a crossed module $\overline{\mathcal{X}}$ with bijective boundary map and thus to the representation category of an ordinary Drinfeld double. To this end, we consider

$$\overline{\mathcal{X}} := (I, \mathcal{X}_2/K, \overline{\mu}, \overline{\partial}) \tag{19}$$

with action

$$\begin{aligned} \overline{\mu} : I \times \mathcal{X}_2/K &\to \mathcal{X}_2/K \\ (g, Km) &\mapsto K\mu(g, m) = Km^g \end{aligned}$$

and boundary map

$$\begin{aligned} \overline{\partial} : \mathcal{X}_2/K &\to I \\ Km &\mapsto \partial(m) \ . \end{aligned}$$

All maps are well-defined, since $x \in K$ and $g \in I$ implies $x^g \in K$ and $\partial(x) = 1$. A direct computation shows that this defines a crossed module; the bijectivity of $\overline{\partial}$ is obvious.

Theorem 4.1.
The modularization of the representation category $\mathcal{M}(\mathcal{X})$ *of a crossed module* $\mathcal{X}$ *is equivalent, as a ribbon category, to the category of representations* $\mathcal{M}(\overline{\mathcal{X}})$ *of the crossed module* $\overline{\mathcal{X}}$.

Our proof proceeds in two steps. We first introduce the crossed module

$$\mathcal{X}' = (I, \mathcal{X}_2, \mu' = \mu|_{I\times\mathcal{X}_2}, \partial)$$

where we restrict to the image I of ∂. By abuse of notion, we denote the boundary map of this crossed module again by ∂; this map is surjective. We denote by $\underline{\mathbf{0}}'$ the vacuum object of $\mathcal{M}(\mathcal{X}')$ which is again a commutative special symmetric Frobenius algebra. We then construct a functor

$$\underline{\mathbf{0}}\text{-Mod}_{\mathcal{M}(\mathcal{X})} \xrightarrow{F} \underline{\mathbf{0}}'\text{-Mod}_{\mathcal{M}(\mathcal{X}')} \ .$$

Proposition 4.4 asserts that F is an equivalence of abelian categories.

In a second step, we construct a functor

$$\underline{\mathbf{0}}'\text{-Mod}_{\mathcal{M}(\mathcal{X}')} \xrightarrow{F'} \mathcal{M}(\overline{\mathcal{X}})$$

and show in Proposition 4.5 that it provides an equivalence of abelian categories as well. We finally endow the two functors F and F' with the structure of braided tensor functors and thus show that the categories $\underline{\mathbf{0}}\text{-Mod}_{\mathcal{M}(\mathcal{X})}$ and $\mathcal{M}(\overline{\mathcal{X}})$ are equivalent as braided tensor categories.

This implies also that the categories are equivalent as ribbon categories: any braided equivalence $G : \mathcal{C} \to \mathcal{D}$ of ribbon categories is an equivalence of ribbon categories. To see this, define on the image of $\mathcal{C}$ under G a new duality by $G(X)^* := G(X^\vee)$. The new duality is isomorphic to the duality in $\mathcal{D}$, thus $G(X^\vee) \cong G(X)^\vee$. Since in any ribbon category the twist can be expressed in terms of the dualities and the braiding, the equivalence is also compatible with the twist.

This concludes our argument that the categories $\overline{\mathcal{M}(\mathcal{X})}$ and $\mathcal{M}(\overline{\mathcal{X}})$ are equivalent as ribbon categories.

We first construct a functor $F : \underline{\mathbf{0}}\text{-Mod}_{\mathcal{M}(\mathcal{X})} \to \underline{\mathbf{0}}'\text{-Mod}_{\mathcal{M}(\mathcal{X}')}$ by restricting the group-action of $\mathcal{X}_1$ to the group-action of $I = \mathrm{Im}\partial$.

Construction of F

- To construct the functor F, we spell out the data contained in an object of $\underline{\mathbf{0}}\text{-Mod}_{\mathcal{M}(\mathcal{X})}$. Such an object consists of a $\mathcal{X}$-representation (V, P_V, Q_V) and a k-linear map $\rho : V_{\underline{\mathbf{0}}} \otimes V \to V$ such that

  (i) $\rho_V(x \otimes \delta_{Iy}, \rho_V(\tilde{x} \otimes \delta_{I\tilde{y}}, v)) = \delta(Iy, I\tilde{y})\rho_V(x\tilde{x} \otimes \delta_{I\tilde{y}}, v)$ ($\underline{\mathbf{0}}$-action)

  (ii) $\rho_V(1 \otimes \sum_{Iy\in C} \delta_{Iy}, v) = v$ (unitality of $\underline{\mathbf{0}}$-action)

  (iii) $\rho_V \circ P_{\underline{\mathbf{0}}V} = P_V \circ \rho_V$

  (iv) $\rho_V \circ Q_{\underline{\mathbf{0}}V} = Q_V \circ \rho_V$ (unitality of $\underline{\mathbf{0}}$-action)

  We introduce the simplified notation with $v \in V$:

  $$x \otimes \delta_{Iy}.v := \rho_V(x \otimes \delta_{Iy}, v) \quad \text{and} \quad \delta_{Iy}.v := 1 \otimes \delta_{Iy}.v \ .$$

- With the notation $V_{Iy} := 1 \otimes \delta_{Iy}.V$, (i) and (ii) imply the decomposition of V as a direct sum of vector spaces

  $$V = \bigoplus_{Iy\in C} V_{Iy} \ .$$

Similarly, we conclude that for every $x \in K$ the action $x \otimes \delta_{Iy}.\_$ is an automorphism of vector spaces

$$x \otimes \delta_{Iy}.V_{Iy} = V_{Iy} \ . \tag{20}$$

We next show that for all $m \in \mathcal{X}_2$, $Iy \in C$, we have

$$P_V(m)V_{Iy} \subset V_{Iy} \ . \tag{21}$$

Indeed,

$$\begin{aligned} P_{\underline{\mathbf{0}}V}(m)(x \otimes \delta_{Iy} \otimes v) &= \sum_{n \in \mathcal{X}_2} P_{\underline{\mathbf{0}}}(n)(x \otimes \delta_{Iy}) \otimes P_V(n^{-1}m)v \\ &= \sum_{n \in \mathcal{X}_2} \delta(n^y, x)(x \otimes \delta_{Iy}) \otimes P_V(n^{-1}m)v \\ &= (x \otimes \delta_{Iy}) \otimes P_V((x^{y^{-1}})^{-1}m)v \end{aligned}$$

and from (iii), we conclude

$$P_V(m)(x \otimes \delta_{Iy}.v) = x \otimes \delta_{Iy}.(P_V((x^{y^{-1}})^{-1}m)v) \ .$$

- From (iv) we conclude that for all $Iy, I\tilde{y} \in C$, we have vector space isomorphisms $Q(\tilde{y}y^{-1}) : V_{Iy} \to V_{I\tilde{y}}$ and that we have for all $h \in I$

$$Q_V(h)V_{Iy} = V_{Iy} \ . \tag{22}$$

  Indeed, we find with $g \in \mathcal{X}_2$

$$Q_{\underline{\mathbf{0}}V}(g)(x \otimes \delta_{Iy} \otimes v) = Q_{\underline{\mathbf{0}}}(g)(x \otimes \delta_{Iy}) \otimes Q_V(g)v = (x \otimes \delta_{Igy}) \otimes Q_V(g)v$$

  and thus by (iv)

$$Q_V(g)(x \otimes \delta_{Iy}.v) = x \otimes \delta_{Igy}.(Q_V(g)v) \ .$$

- From equations (20) - (22) we conclude that every subvector space V_{Iy} is invariant under the action of $x \in K$, $m \in \mathcal{X}_2$ and $h \in I$. In particular, every vector space V_{Iy} is a $\mathcal{X}'$-representation. It becomes a $\underline{\mathbf{0}}' = \mathbb{C}[K]$-module by setting $\rho'_V(x, v) := \rho_V(x \otimes \delta_{Iy}, v)$.

  All these $\underline{\mathbf{0}}'$-modules are isomorphic. We select the $\underline{\mathbf{0}}'$-module V_I as the image of the functor F:

$$F(V, P_V, Q_V, \rho_V) := (V_I, P_V, Q_V|_I, \rho_V(\_ \otimes \delta_I, \_)) \ .$$

  On morphisms, we set

$$F(\phi : V \to W) := \phi|_{V_I} : V_I \to W_I \ .$$

  Indeed, the image of the vector space V_I under $\phi : V \to W$ is contained in W_I, since ϕ commutes with the action, $\phi(V_I) = \phi(\rho_V(1 \otimes \delta_I, V)) = \rho_W(1 \otimes \delta_I, \phi(V)) \subset W_I$.

Proposition 4.2.
The functor F presented in the above construction provides an equivalence of abelian categories $\underline{\mathbf{0}}\text{-Mod}_{\mathcal{M}(\mathcal{X})} \simeq \underline{\mathbf{0}}'\text{-Mod}_{\mathcal{M}(\mathcal{X}')}$.

Proof:
We show that the functor is fully faithful and essentially surjective. To show essential surjectivity, consider an object (W, P'_W, Q'_W, ρ'_W) in $\underline{\mathbf{0}}'\text{-Mod}_{\mathcal{M}(\mathcal{X}')}$.

To find the preimage, we use induction from I to x_1: consider the object (V, P_V, Q_V, ρ_V) in $\underline{\mathbf{0}}'\text{-Mod}_{\mathcal{M}(\mathcal{X}')}$ with

$$V = \bigoplus_{Iy \in C} W_{Iy}$$

and action

$$Q_V = \mathrm{Ind}_I^{\mathcal{X}_1} Q'_W : \mathcal{X}_1 \to \mathrm{End}(V) \ .$$

We introduce a $\mathcal{X}_2$-grading by

$$P_V(m) w_{Iy} = (P_V(m) w)_{Iy}$$

and the structure of a $\underline{\mathbf{0}}$-module by

$$\rho_V(x \otimes \delta_{Iy}, w_{I\tilde{y}}) := \delta(Iy, I\tilde{y})(\rho'_W(x, w))_{Iy} \ .$$

A straightforward calculation shows that the image of this object under F is (W, P'_W, Q'_W, ρ'_W).

To show that F is fully faithful, we note that a morphism $\phi : V \to W$ from (V, P_V, Q_V, ρ_V) to (W, P_W, Q_W, ρ_W) is determined by its restriction to V_I, since for any $v \in V$, we have

$$\begin{aligned} \phi(v) &= \sum_{Iy \in C} \phi(1 \otimes \delta_{Iy}.v) = \sum_{Iy \in C} \phi(Q(y)Q(y^{-1})1 \otimes \delta_{Iy}.v) \\ &= \sum_{Iy \in C} Q(y)\phi \underbrace{(1 \otimes \delta_I.Q(y^{-1})v)}_{\in V_I} \ . \end{aligned}$$

□

We next construct an equivalence $F' : \underline{\mathbf{0}}'\text{-Mod}_{\mathcal{M}(\mathcal{X}')} \to \mathcal{M}(\overline{\mathcal{X}})$. The idea is to take coinvariants with respect to the action of the kernel $K := \ker \partial$.

Construction of F'

- To construct an equivalence $F' : \underline{\mathbf{0}}'\text{-Mod}_{\mathcal{M}(\mathcal{X}')} \to \mathcal{M}(\overline{\mathcal{X}})$ we spell out explicitly the data contained in an object (W, P_W, Q_W, ρ_W) of $\underline{\mathbf{0}}'\text{-Mod}_{\mathcal{M}(\mathcal{X}')}$: here (W, P_W, Q_W) is an object of $\mathcal{M}(\mathcal{X}')$ and $\rho_W : \mathbb{C}[K] \otimes W \to W$ is a k-linear map such that
    (i) $\rho_W(x, \rho_W(\tilde{x}, w)) = \rho(x\tilde{x}, w)$
    (ii) $\rho_W(1, w) = w$
    (iii) $\rho_W \circ P_{\underline{\mathbf{0}}'W} = P_W \circ \rho_W$
    (iv) $\rho_W \circ Q_{\underline{\mathbf{0}}'W} = Q_W \circ \rho_W$

We introduce the shorthand notation $x.w$ for $\rho_W(x,w)$.

- From the first two axioms we conclude that for all $x \in K$, the action is an isomorphism. As a $\mathcal{X}'$-module, we decompose W

$$W = \bigoplus_{m \in \mathcal{X}_2} W_m \quad \text{with} \quad W_m := P(m)W \ .$$

From (iii) we conclude as in the construction of F

$$x.P(m)w = P(xn)x.w \ . \tag{23}$$

Thus the action of x implies for $Km = Kn$ in $\mathcal{X}_2/K$ the isomorphy of vector spaces $W_m \cong W_n$. Again as in the construction of F, we conclude

$$x.(Q(g)w) = Q(g)(x.w) \quad \text{for all} \quad g \in \mathcal{X}_1 \ . \tag{24}$$

- We now define F' by taking coinvariants with respect to the action of $K = \ker \partial$. On objects, we have

$$F'(W, P_W, Q_W, \rho_W) := (W_K, P_{W_K}, Q_{W_K})$$

with

$$W_K := W/(x.w - w | x \in K, w \in W)$$
$$P_{W_K}(Km)\overline{w} := \overline{P_W(m)w}$$
$$Q_{W_K}(h)\overline{w} := \overline{Q_W(h)w} \ .$$

From (23) and (24) we deduce that the maps P_{W_K} and Q_{W_K} are well-defined. On morphisms, we consider the restriction

$$F(f : W \to V) := (f_K : W_K \to V_K)$$

and obtain a k-linear functor F'.

Proposition 4.3.
The functor F' presented in the above construction provides an equivalence of abelian categories $\underline{\mathbf{0}}'$-$\mathrm{Mod}_{\mathcal{M}(\mathcal{X}')} \simeq \mathcal{M}(\overline{\mathcal{X}})$.

Proof:
To show that F' is essentially surjective, we construct for $(V, P_V, Q_V) \in \mathcal{M}(\mathcal{X}')$ an object $(W, P_W, Q_W, \rho_W) \in \underline{\mathbf{0}}'$-$\mathrm{Mod}_{\mathcal{M}(\mathcal{X})}$ with

$$W := \bigoplus_{x \in K} W_x \ ,$$

where $W_x \cong V$ as a $\mathcal{X}_1$-representation for all $x \in K$. On W we define an action of K by

$$x.w_{\tilde{x}} := w_{x\tilde{x}} \ .$$

To define an action of $\mathcal{X}_2$, choose representatives (Km) and set

$$P_W(xm)w_{\tilde{x}} := \delta(x, \tilde{x}) P_{W_x}(Km) w_x \ .$$

The image under F of the object constructed is isomorphic to (V, P_V, Q_V), showing essential surjectivity.

As in the proof of proposition 4.2, we conclude that a morphism $\phi : V \to W$ is uniquely determined by the map ϕ_K induced on coinvariants. □

It remains to endow the functors with more structure.

Proposition 4.4.
Consider the morphism

$$\begin{aligned} \varphi_0 : \mathbb{C}[K] &\to \mathbb{C}[K] \otimes \mathbb{C}(\delta_I) \\ x &\mapsto x \otimes \delta_I \end{aligned}$$

and for all objects V, W of $\underline{\mathbf{0}}'$-$\mathrm{Mod}_{\mathcal{M}(\mathcal{X}')}$ the morphisms:

$$\begin{aligned} \varphi_2(V, W) : F(V) \otimes_{\underline{\mathbf{0}}'} F(W) &\to F(V \otimes_{\underline{\mathbf{0}}} W) \\ v_I \otimes_{\underline{\mathbf{0}}'} w_I &\mapsto v_I \otimes_{\underline{\mathbf{0}}} w_I \ . \end{aligned}$$

These morphisms endow the functor $F : \underline{\mathbf{0}} - \mathrm{Mod}_{\mathcal{M}(\mathcal{X})} \to \underline{\mathbf{0}}'$-$\mathrm{Mod}_{\mathcal{M}(\mathcal{X}')}$ with the structure of a braided tensor functor.

Proof:
Bijectivity of φ_0 is obvious. To check bijectivity of $\varphi_2(V, W)$, we note that the equation $v \otimes_{\underline{\mathbf{0}}} w = \delta_I.(v \otimes_{\underline{\mathbf{0}}} w) = \delta_I.v \otimes_{\underline{\mathbf{0}}} \delta_I.w$ for $v \otimes_{\underline{\mathbf{0}}} w \in F(V \otimes W)$ implies $v \otimes_{\underline{\mathbf{0}}} w = \varphi_2(\delta_I.v \otimes_{\underline{\mathbf{0}}'} \delta_I.w)$ and hence surjectivity. On the other hand, $\varphi_2(V, W)(v_I \otimes_{\underline{\mathbf{0}}'} w_I) = 0$ implies $v_I = 0$ and $w_I = 0$ and thus injectivity of φ_2. The verification that $(F, \varphi_0, \varphi_2)$ is a tensor functor is routine.

To show that the tensor functor $(F, \varphi_0, \varphi_2)$ is braided, we have to check that for any pair of objects $(V, P_V, Q_V), (W, P_W, Q_W)$ the diagram

$$\begin{array}{ccc} F(V) \otimes_{\underline{\mathbf{0}}'} F(W) & \xrightarrow{\varphi_2} & F(V \otimes_{\underline{\mathbf{0}}} W) \\ \downarrow{\scriptstyle R_{F(V),F(W)}} & & \downarrow{\scriptstyle F(R_{V,W})} \\ F(W) \otimes_{\underline{\mathbf{0}}'} F(V) & \xrightarrow{\varphi_2} & F(W \otimes_{\underline{\mathbf{0}}} V) \end{array}$$

commutes; indeed,

$$\begin{aligned} F(R_{V,W}) \circ \varphi_2(V, W)(v \otimes_{\underline{\mathbf{0}}'} w) &= F(R_{V,W})(v \otimes_{\underline{\mathbf{0}}} w) \\ &= \sum_{n \in \mathcal{X}_2} Q_W(\partial' m) w \otimes_{\underline{\mathbf{0}}} P_V(m) v \\ &= \varphi_2 \left(\sum_{n \in \mathcal{X}_2} Q_W(\partial' m) w \otimes_{\underline{\mathbf{0}}'} P_V(m) v \right) \\ &= \varphi_2 \circ R_{F(V),F(W)}(v \otimes_{\underline{\mathbf{0}}'} w) \ . \end{aligned}$$

Hence $(F, \varphi_0, \varphi_2)$ is a braided tensor functor.

□

Proposition 4.5. *Consider the morphisms*

$$\begin{aligned} \varphi_0' : \mathbb{C} &\to (\mathbb{C}[K])_K \\ \lambda &\mapsto \lambda \overline{x} \qquad\qquad x \in K \end{aligned}$$

and for all objects V, W of $\mathcal{M}(\overline{\mathcal{X}})$

$$\begin{aligned} \varphi_2'(V,W) : V_K \otimes W_K &\to (V \otimes_{\underline{\mathbf{0}}'} W)_K \\ \overline{v} \otimes \overline{w} \quad &\mapsto \overline{v \otimes_{\underline{\mathbf{0}}'} w}\ . \end{aligned}$$

These morphisms endow the functor $F' : \underline{\mathbf{0}}'\text{-Mod}_{\mathcal{M}(\mathcal{X}')} \to \mathcal{M}(\overline{\mathcal{X}})$ with the structure of a braided tensor functor.

Proof:
We first remark that φ_0' is well-defined, since for $x, x' \in K$ we have $\overline{x} = \overline{x'}$ in $(\mathbb{C}[K])_K$. The bijectivity of φ_0 is immediate from $\dim_k(\mathbb{C}[K])_K = 1$ and $\ker\, \varphi_0' = 0$.

To check that also $\varphi_2'(V,W)$ is well-defined, we first remark that the action of $x \in K$ on $v \otimes_{\underline{\mathbf{0}}'} w \in V \otimes_{\underline{\mathbf{0}}'} W$ reads

$$x.(v \otimes_{\underline{\mathbf{0}}'} w) = x.v \otimes_{\underline{\mathbf{0}}'} w = v \otimes_{\underline{\mathbf{0}}'} x.w\ .$$

Now take $v, v' \in V$ and $w, w' \in W$ such that

$$\overline{v} \otimes \overline{w} = \overline{v'} \otimes \overline{w'}\ .$$

Then we can find $x, \tilde{x} \in K$ such that $v' = x.v$ and $w' = \tilde{x}.w$ and we have by the proceeding remark

$$\overline{v \otimes_{\underline{\mathbf{0}}'} w} = \overline{x\tilde{x}.(v \otimes_{\underline{\mathbf{0}}'} w)} = \overline{x.v \otimes_{\underline{\mathbf{0}}'} \tilde{x}.w}$$

and thus

$$\varphi_2'(\overline{v} \otimes \overline{w}) = \varphi_2'(\overline{v'} \otimes \overline{w'})\ .$$

An inverse of φ_2' can be given directly by

$$\varphi_2'^{-1}(\overline{v \otimes_{\underline{\mathbf{0}}'} w}) = \overline{v} \otimes \overline{w}\ .$$

One checks by direct computations that $(F', \varphi_0', \varphi_2')$ is a tensor functor. Finally, $(F', \varphi_0', \varphi_2')$ is braided, since we have

$$\begin{aligned} F'(R_{VW}) \circ \varphi_2'(\overline{v} \otimes \overline{w}) &= F'(R_{VW})(\overline{v \otimes_{\underline{\mathbf{0}}'} w}) \\ &= \sum_{Km \in \mathcal{X}_2/K} \overline{Q(\bar{\partial}Km)w \otimes_{\underline{\mathbf{0}}'} P(Km)v} \\ &= \varphi_2' \left(\sum_{Km \in \mathcal{X}_2/K} \overline{Q(K\bar{\partial}m)w} \otimes \overline{P(Km)v} \right) \\ &= \varphi_2' \circ R_{V_K W_K}(\overline{v} \otimes \overline{w})\ . \end{aligned}$$

□

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