The self-minor conjecture for infinite trees

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Abstract

We prove Seymour's self-minor conjecture for infinite trees.

1. Introduction

P. D. Seymour conjectured that every infinite graph is a proper minor of itself. This was disproven for uncountable graphs by B. Oporowski [1] but is still open for countable graphs. We prove the conjecture for trees of any cardinality. Our proof is based only on Kruskal's theorem that the finite rooted trees are well-quasi-ordered [2].

2. Terms and Definitions

We use the terms and notation from [3]. In particular, H is a *minor* of G, denoted by $H \preccurlyeq G$, if there is a subset $X \subseteq V(G)$ and a surjective map $\gamma : X \to V(H)$ such that for every vertex $v \in V(H)$ its *branch set* $\gamma^{-1}(v)$ is connected in G and for every edge $vw \in E(H)$ there is a $\gamma^{-1}(v)$–$\gamma^{-1}(w)$ edge in G. Such a map γ is called a *subcontraction from G to H*. A subcontraction from G to H is *proper* if it is not an isomorphism between G and H. For graphs G and H the graph H is a *proper minor of G* if there is a proper subcontraction from G to H. Let (T, r) and (T', r') be rooted trees. A map is a *subcontraction* from (T, r) to (T', r') if it is a subcontraction from T to T' which maps r to r'. If there is such a subcontraction, we call (T', r') a *minor* of (T, r).

Let (T, r) be a rooted tree. The *tree-order* of (T, r) is an order $\leq_r$ on the vertex set of T such that $v_1 \leq_r v_2$ if and only if v_1 lies on the (unique) r–v_2 path in T. Let x be a vertex of a rooted tree (T, r) the vertex set $\lfloor x \rfloor := \{v \in V(T) \mid x \leq_r v\}$ is the *up-closure* of x in (T, r).

A *labelling* of A is a map λ from A to another set B. Then A is *labelled by B* and for $a \in A$ its image $\lambda(a)$ is the *label of* a.

3. Countable trees

In this section we give a proof of the self-minor conjecture for countable trees. The argument starts with trees that consist essentially of a ray. This will clear the way for treating arbitrary countable trees afterwards.

As finite rooted trees are well-quasi ordered, by Kruskals's Theorem [2], the next lemma immediately implies the self-minor conjecture for a ray that has only finite trees attached to its vertices. Let $(N, 0)$ be the *natural ray* on $\mathbb{N}$ with vertex set $\mathbb{N}$, root 0, and edges $\{i, i+1\}$ for $i \in \mathbb{N}$. The natural order of $\mathbb{N}$ and the tree-ordering agree on $\mathbb{N} = V(N)$.

Lemma 1. *Let Q be well-quasi-ordered by $\leq_Q$ and $(N, 0)$ the natural ray. For every labelling $\lambda : V(N) \to Q$, there is a proper subcontraction γ from $(N, 0)$ to $(N, 0)$ such that for every vertex $n \in V(N)$ there is an $m \in \gamma^{-1}(n)$ such that $\lambda(n) \leq_Q \lambda(m)$.*

Proof. For every $i \in \mathbb{N}$ let $F_i := \{j \in \mathbb{N} | i \leq j, \lambda(i) \leq_Q \lambda(j)\}$ be the *successor set* of i. Thus a successor set of i contains the vertices that are at least i such that their label is at least the label of i. In a later subcontraction these are precisely the vertex-sets that need to have a representative in the ith branch set. Coined this term we have two different cases:

1. The successor set is finite only for a finite number of vertices.
2. There is an infinite sequence of finite successor sets.

For both cases there is a short proof of the lemma. In fact, the first case yields to a construction of a proper subcontraction, while the latter one is contradictory.

Case 1. There is only a finite number of vertices $i \in V(N)$ such that F_i is finite, thus there is a largest one k say. Thus for any $j > k$ the set F_j is infinite.

We will define a subcontraction form $(N, 0)$ to $(N, 0)$. On the first k vertices this will be the identity. Then we map $k+1$ to k and this will force the map to be proper. The images of later vertices are defined recursively. We make use of the fact the the F_n are infinite for $n > k$ so that we can map the vertices to larger images again and again.

We recursively define the function μ as follows: Let $\mu(0) = 0$ and let $\mu : V(N) \to V(N)$ be defined for $i < n$. Then there are four cases with different rules for defining $\mu(n)$,

$$\mu(n) := \begin{cases} n & \text{if } n \leq k \\ k & \text{if } n = k+1 \\ \mu(n-1) & \text{if } n > k+1 \text{ and } n \notin F_{\mu(n-1)+1} \\ \mu(n-1)+1 & \text{if } n > k+1 \text{ and } n \in F_{\mu(n-1)+1} \end{cases}$$

This definition makes sure that the map is proper, the inverse images are connected and the ith interval which is $\mu^{-1}(i)$ contains a vertex from $F_{\mu(i)}$. It is surjective as $F_{\mu(n-1)+1}$ is infinite for every $n > k$.

Thus μ is a proper subcontraction from $(N, 0)$ to $(N, 0)$ and every branch set contains a vertex with at least one suitable label.

Case 2. As there are infinitely many finite successor sets there is a strictly increasing sequence $(n_i)_{i \in \mathbb{N}}$ say, in $\mathbb{N}$ such that $(F_{n_i})_{i \in \mathbb{N}}$ is a disjoint family of finite successor sets. As the labels of $(n_i)_{i \in \mathbb{N}}$ are well-quasi-ordered there are positive integers j and j' with $n_j < n_{j'}$ and $\lambda(n_j) \leq_Q \lambda(n_{j'})$. Thus $n_{j'} \in F_{n_j}$ and $F_{n_j} \cap F_{n_{j'}}$ is not empty containing at least $n_{j'}$. A contradiction. □

Let us prove the self-minor conjecture for countable trees using Lemma 1 for trees without T_2-minor[1]. For rayless and such trees containing a T_2-minor we will provide a self contained construction of a suitable subcontraction.

Lemma 2. *Every countable rooted tree has a proper subcontraction onto itself.*

Proof. Let (T, r) be a countable rooted tree, then there are three cases:

1. T does not contain a ray.
2. T contains a ray but no T_2 as a minor.
3. T contains a T_2 as a minor.

Case 1. Let there be no ray in T. Let x have infinite degree and let it be maximal in the tree order $\leq_r$ with that property. Such a vertex exists as we required T to be rayless and connected. In $T - x$ there is one component for every neighbor of x. As x has infinite degree there are infinitely many components. At most one component is not contained in the up-closure of x. Every component in that up-closure is finite, thus there are infinitely many finite components above x. By adding x with its incident edge every such component induces a finite rooted tree with root x. Kruskal proved that finite rooted trees are well-quasi-ordered by the minor relation [2]. Thus there is an increasing sequence $(T_1, x) \preccurlyeq (T_2, x) \preccurlyeq \ldots$ say, of these rooted trees above x. We can delete $T_1 - x$ and map (T_{i+1}, x) to (T_i, x) with subcontractions. These subcontractions are extendable by the identity outside of $\bigcup T_i$. We then have a subcontraction from (T, r) to (T, r) which is proper as $T_1 - x$ contains at least one vertex.

Case 2. Let there be a ray but no T_2 minor in T. Then the following algorithm to find a T_2 minor in T fails at some point.

1. Let (T', r) be the trivial tree on one vertex, in this case the root.
2. Mark the maximal vertices in (T', r).
3. Choose for every marked vertex x two incomparable vertices a, b say, in its up-closure such that there is a ray in the up-closure of a and one in the up-closure of b.
4. Add the x–a path and the x–b path to (T', r) for all the maximal vertices.
5. Repeat step 2. to 5.

The only step the above algorithm may fail is the third one. Thus there is a maximal vertex in (T', r) at some point so that there are no two incomparable vertices with rays in their up-closure above it. Thus there is an $x \in V(T)$ such that $(T[\lfloor x \rfloor], x)$ contains only one ray that starts in x. Let R be this ray. $(T[\lfloor x \rfloor], x)$ essentially consists of one ray, i. e. in every vertex of R only a finite rooted tree—disjoint to R except in its root—is attached.

We may indeed assume these trees to be finite as we can use that infinite rayless rooted trees have a proper subcontraction onto themselves. Let us regard the attached trees as the labels of the vertices they are attached to. We may apply Lemma 1 since finite rooted trees are well-quasi-ordered [2]. Thus there is a proper subcontraction

[1]The T_2 is the binary tree, i. e. the infinite tree with one vertex of degree 2, which is its root in the rooted version, and all other vertices having degree 3.

from (R, x) to (R, x) with respect to this particular labelling and the minor relation on finite rooted trees. Such a proper subcontraction induces one from $(T[\lfloor x \rfloor], x)$ to $(T[\lfloor x \rfloor], x)$. This subcontraction is extendable by the identity to the rest of (T, r).

Case 3. T contains a T_2-minor. A T_2 contains a $T_{\aleph_0}$ as a minor. This can be seen by coloring the edges of the T_2 such that every vertex has one white and one black edge to its above neighbors and contracting all white edges. Thus there is a proper subcontraction γ say, from T to $T_{\aleph_0}$. As T is connected we may demand that r is mapped to a vertex of $T_{\aleph_0}$ by γ. The rooted tree $(T_{\aleph_0}, \gamma(r))$ contains every countable rooted tree even as a proper subgraph. □

4. Arbitrary cardinalities

In this section we prove the self-minor conjecture for trees of arbitrary cardinality. This result does not extend to graphs in general as B. Oporowski gives a counterexample in [1]. This part of the problem looked straight forward first but the singular cardinalities of cofinality ω demanded some effort from us.

Definition 3. Let (T, r) be a rooted tree. Let $\mathcal{A}_i$ be the set of vertices with distance i from the root and $\alpha_i := |\mathcal{A}_i|$ for all $i \in \mathbb{N}$. The family $(\alpha_i)_{i \in \mathbb{N}}$ is called the *cardinal family* of (T, r).

Lemma 4. *For every infinite rooted tree there is a proper subcontraction onto itself.*

Proof. Let (T', r) be a counterexample of smallest possible cardinality κ say. By Lemma 2 we know that $\kappa > \aleph_0$. Let (T, r) be the maximal subtree in (T', r) such that every vertex has a ray in its up-closure. Let X be the set of vertices in $V(T') \setminus V(T)$ that are adjacent to T. Every vertex in X has a finite up-closure, as there is a maximal vertex of infinite degree otherwise and we may construct a proper subcontraction just as in the countable rayless case. Additionally there is no vertex in T that is adjacent to infinitely many vertices of X for the same reason. Thus there are only finitely many vertices that are deleted from T' for every vertex that remains in T, this means that there are κ vertices left in T.

Let $(\alpha_i)_{i \in \mathbb{N}}$ be the cardinal family of (T, r). It holds that $\kappa = \bigcup_{i=0}^{\infty} \alpha_i$ as T is connected. Now there are two cases: In the first case, for every vertex $x \in V(T)$ the cardinal family of $(T[\lfloor x \rfloor], x)$ contains κ as a member. In the second case there is a vertex $z \in V(T)$ such that $(T[\lfloor z \rfloor], z)$ has a cardinal family $(\beta_i)_{i \in \mathbb{N}}$ with $\beta_i < \kappa$ for all $i \in \mathbb{N}$.

Case 1. For every $x \in V(T)$ there is a distance class in $(T[\lfloor x \rfloor], x)$ with κ vertices, thus there is a vertex with degree κ above every vertex. This information enables us to construct a subdivided T_κ: We start at the root r and choose a vertex above the root with degree κ. In the next step we add a path from every maximal vertex in the already chosen subtree to a vertex with degree κ above it. After ω steps we constructed a subdivided T_κ with an attached path to the root. By contracting this path and suppressing all the vertices of degree 2 we obtain a T_κ as a minor of T'.

Case 2. Let $z \in V(T)$ be a vertex without κ in its cardinal family. We may assume $|\lfloor z \rfloor| = \kappa$ by induction as a proper subcontraction of $(T[\lfloor z \rfloor], z)$ would be extendable to the whole tree. There is no largest member β say, of the cardinal family $(\beta_i)_{i \in \mathbb{N}}$ of

$(T[\lfloor z \rfloor], z)$ as this would satisfy $\kappa = \bigcup_{i=0}^{\infty} \beta_i \leq \bigcup_{i=0}^{\infty} \beta = \aleph_0 \beta = \beta$. This holds for every vertex above z, too. Thus for a vertex $y \in \lfloor z \rfloor$ the rooted tree $(T[\lfloor y \rfloor], y)$ with cardinal family $(\alpha_i)_{i \in \mathbb{N}}$ satisfies that for every i there is a $j > i$ such that $\alpha_j > \alpha_i$. In other words we will find ever larger cardinals in every such cardinal family. This implies that there is a vertex of degree at least β_i above x for every $i \in \mathbb{N}$ and $x \in \lfloor z \rfloor$.

We will choose a set of rays in $(T[\lfloor z \rfloor], z)$ which we will contract afterwards and therewith construct a T_κ. The idea is basically the same as in the situation where we constructed a $T_{\aleph_0}$ as a minor in a T_2. The collecting of rays works as follows: Let $\mathcal{R}_0 = \emptyset$ and let $\mathcal{R}_i$ be defined for all $i < j$. Let $X_j \subseteq V(T)$ be the set of vertices that are disjoint to the elements of $\mathcal{R}_i$ with minimal distance to the root. Choose a ray R_x for every $x \in X_j$ such that $V(R_x) \subseteq \lfloor x \rfloor$ such that R_x has κ neighbors. Such a ray exists as every vertex has a vertex of degree β_i for every $i \in \mathbb{N}$ in its up-closure. Let $\mathcal{R}_j := \mathcal{R}_{j-1} \cup \{R_x | x \in X_j\}$. As the up-closures of vertices that have the same distance from the root are disjoint the simultaneous chosen rays are mutually disjoint. Let $\mathcal{R} := \bigcup_{i \in \mathbb{N}} \mathcal{R}_i$. We now contract all rays in $\mathcal{R}$. The resulting graph is a T_κ as every ray has the initial vertices of κ distinct rays that belong to $\mathcal{R}$ in its neigborhood.

In both cases we ended up with a T_κ which contains every tree of cardinality κ as a proper minor. Since the T_κ is regular and we made sure that the root of (T', r) is contained in some branch set there is a proper subcontraction as requested. $\square$

This completes the proof of the self-minor conjecture for trees by choosing an arbitrary root:

Theorem 5. *Every infinite tree is a proper minor of itself.* $\square$

References

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