On the excluded minor structure theorem for graphs of large treewidth

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September 30, 2009

Abstract

At the core of the Robertson-Seymour theory of graph minors lies a powerful structure theorem which captures, for any fixed graph H, the common structural features of all the graphs not containing H as a minor. Robertson and Seymour prove several versions of this theorem, each stressing some particular aspects needed at a corresponding stage of the proof of the main result of their theory, the graph minor theorem.

We prove a new version of this structure theorem: one that seeks to combine maximum applicability with a minimum of technical ado, and which might serve as a canonical version for future applications in the broader field of graph minor theory. Our proof departs from a simpler version proved explicitly by Robertson and Seymour. It then uses a combination of traditional methods and new techniques to derive some of the more subtle features of other versions as well as further useful properties, with substantially simplified proofs.

1 Introduction

Graphs in this paper are finite and may have loops and multiple edges. Otherwise we use the terminology of [6]. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges.

The theory of graph minors was developed by Robertson and Seymour, in a series of 23 papers published over more than twenty years, with the aim of proving a single result: the *graph minor theorem*, which says that in any infinite collection of finite graphs there is one that is a minor of another. As with other deep results in mathematics, the body of theory developed for the proof of the graph minor theorem has also found applications elsewhere, both within graph theory and computer science. Yet many of these applications rely not only on the general techniques developed by Robertson and Seymour to handle graph minors, but also on one particular auxiliary result that is also central to the proof of the graph minor theorem: a result describing the structure of all graphs G not containing some fixed other graph H as a minor.

This structure theorem has many facets. It roughly says that every graph G as above can be decomposed into parts that can each be 'almost' embedded in a surface of bounded genus (the bound depending on H only), and which fit together in a tree structure [6, Thm. 12.4.11]. Although later dubbed a 'red herring' (in the search for the proof of the graph minor theorm) by Robertson and Seymor themselves [14], this simplest version of the structure theorem is the one that appears now to be best known, and which has also found the most algorithmic applications [2, 3, 4, 10].

A particularly simple form of this structure theorem applies when the excluded minor H is planar: in that case, the said parts of G—the parts that fit together in a tree-structure and together make up all of G—have bounded size, i.e., G has bounded *tree-width*. If H is not planar, the graphs G not containing H as a minor have unbounded tree-width, and therefore contain arbitrarily large grids as minors and arbitrarily large walls as topological minors [6]. Such a large grid or wall identifies, for every low-order separation of G, one side in which most of that grid or wall lies. This is formalized by the notion of a *tangle*: the larger the tree-width of G, the larger the grid or wall, the order of the separations for which this works, and (thus) the *order* of the tangle. Since adjacent parts in our tree-decomposition of G meet in only a bounded number of vertices and thus define low-order separations, our large-order tangle 'points to' one of the parts, the part G' that contains most of its defining grid or wall.

The more subtle versions of the structure theorem, such as Theorem (13.4) from Graph Minors XVII [15], now focus just on this part G' of G. Like every part in our decomposition, it intersects every other part in a controlled way. Every such intersection consists of a bounded number of vertices, of which some lie in a fixed *apex set* $A \subseteq V(G')$ of bounded size, while the others are either at most 3 vertices lying on a face boundary of the portion G_0 of G' embedded in the surface, or else lie in (a common bag of) a so-called *vortex*, a ring-like subgraph of G' that is not embedded in the surface and meets G_0 only in (possibly many) vertices of a face boundary of G_0. The precise structure of these vortices, of which G' has only boundedly many, will be the focus of our attention for much of the paper. Our theorem describes in detail both the inner structure of the vortices and the way in which they are linked to each other and to the large wall, by disjoint paths in the surface. These are the properties that have been used in applications of the structure theorems such as [1, 7], and which will doubtless be important also in future applications. An important part of the proofs is a new technique for analyzing vortices. We note that these techniques have also been independently developed by Geelen and Hyuhn [8].

This paper is organized as follows. In Section 2 we introduce the terminology we need to state our results, as well as the theorem from Graph Minors XVI [14] on which we shall base our proof. Section 3 explains how we can find the tree-decomposition indicated earlier, with some additional information on how the parts of the tree-decomposition overlap. Section 4 collects some lemmas about graphs embedded in a surface, partly from the literature and partly new. In Section 5 we show how a given near-embedding of a graph can be simplified in various ways if we allow ourselves to remove a bounded number of vertices (which, in applications of these tools, will be added to the apex set). Section 6 contains lemmas showing how to obtain path systems with nice properties. Section 7 contains the proof of our structure theorem. In the last section, we give an alternative definition of vortex decompositions and show that our result works with these 'circular' decompositions as well.

2 Structure Theorems

A *vortex* is a pair $V = (G, \Omega)$, where G is a graph and $\Omega =: \Omega(V)$ is a linearly ordered set $(w_1, \dots, w_n)$ of vertices in G. These vertices are the *society vertices* of the vortex; their number n is its *length*. We do not always distinguish notationally between a vortex and its underlying graph or vertex set; for example, a *subgraph of* V is just a subgraph of G, a subset of V is a subset of $V(G)$, and so on. Also, we will often use Ω to refer both to the linear order of the vertices $w_1, \dots, w_n$ as well as the set of vertices $\{w_1, \dots, w_n\}$.

A path-decomposition $\mathcal{D} = (X_1, \dots, X_m)$ of G is a *decomposition of* V if $m = n$ and $w_i \in X_i$ for all i. The *depth* of the vortex V is the minimum width of a path-decomposition of G that is a decomposition of V.

The *adhesion* of our decomposition $\mathcal{D}$ of V is the maximum value of $|X_{i-1} \cap X_i|$, taken over all $1 < i \leq n$. We define the *adhesion* of a vortex V as the minimum adhesion of a decomposition of that vortex.

When $\mathcal{D}$ is a decomposition of a vortex V as above, we write $Z_i := (X_i \cap X_{i+1}) \setminus \Omega$, for all $1 \leq i < n$. These Z_i are the *adhesion sets* of $\mathcal{D}$. We call $\mathcal{D}$ is *linked* if

- all these Z_i have the same size;
- there are $|Z_i|$ disjoint Z_{i-1}–Z_i paths in $G[X_i] - \Omega$, for all $1 < i < n$;
- $X_i \cap \Omega = \{w_{i-1}, w_i\}$ for all $1 \leq i \leq n$, where $w_0 := w_1$.

Note that $X_i \cap X_{i+1} = Z_i \cup \{w_i\}$, for all $1 \leq i < n$ (Fig. 1).

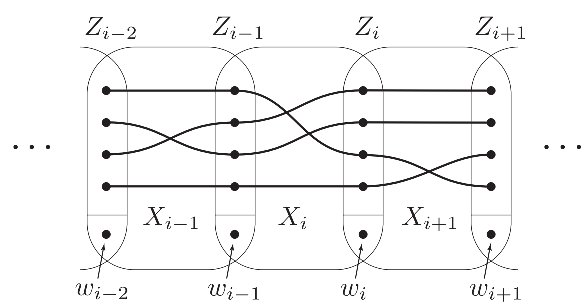

Figure 1: A linked vortex decomposition

The union of the Z_{i-1}–Z_i paths in a linked decomposition of V is a disjoint union of X_1–X_n paths in G; we call the set of these paths a *linkage* of V with respect to $(X_1, \dots, X_m)$.

Clearly, if V has a linked decomposition as above, then G has no edges between non-consecutive society vertices, since none of the X_i could contain both ends of such an edge. Conversely, if G has no such edges then V does have a linked decomposition: just let X_i consist of all the vertices of $G - \Omega$ plus w_{i-1}

and w_i. We shall be interested in linked vortex decompositions whose adhesion is small, unlike in this example.

Let $(G, \Omega) =: V$ be a vortex and v a vertex of some supergraph of V. Clearly, $(G - v, \Omega \setminus \{v\})$ is a vortex, too, which we denote by $V - v$. If the length of V is larger than 2, this operation cannot increase the adhesion q of V: This is clear for $v \notin \Omega$, so suppose $\Omega = (w_1, \ldots, w_n)$ with $v = w_k$ for some $1 \leq k \leq n$. We may assume without loss of generality that $k \neq n$. Take a decomposition $(X_1, \ldots, X_n)$ of V of adhesion q. Then, it is easy to see that

$$(X_1, \ldots, X_{k-1}, (X_k \cup X_{k+1}) \setminus \{w_k\}, X_{k+1}, \ldots, X_n)$$

is a decomposition of $V - v$ of adhesion at most q. We shall not be interested in the adhesion of vortices of length at most 2. For a vertex set $A \subseteq V$ we denote by $V - A$ the vortex we obtain by deleting the vertices in A subsequently from V. For a set of vortices $\mathcal{V}$ we define $\mathcal{V} - A := \{V - A : V \in \mathcal{V}, V - A \neq \emptyset\}$.

A (directed) *separation* of a graph G is an ordered pair (A, B) of non-empty subsets of $V(G)$ such that $G[A] \cup G[B] = G$. The number $|A \cap B|$ is the *order* of (A, B). Whenever we speak of separations in this paper, we shall mean such directed separations.

A set $\mathcal{T}$ of separations of G, all of order less than some integer θ, is a *tangle of order* θ if the following holds:

(1) For every separation (A, B) of G of order less than θ, either (A, B) or (B, A) lies in $\mathcal{T}$.

(2) If $(A_i, B_i) \in \mathcal{T}$ for $i = 1, 2, 3$, then $G[A_1] \cup G[A_2] \cup G[A_3] \neq G$.

Note that if $(A, B) \in \mathcal{T}$ then $(B, A) \notin \mathcal{T}$; we think of A as the 'small side' of the separation (A, B), with respect to this tangle.

For a tangle $\mathcal{T}$ of order θ in a graph G, let $Z \subseteq V(G)$ be a vertex set with $|Z| < \theta$. Let $\mathcal{T} - Z$ denote the set of all separations (A', B') of $G - Z$ of order less than $\theta - |Z|$ such that there exists a separation $(A, B) \in \mathcal{T}$ with $Z \subseteq A \cap B$, $A - Z = A'$ and $B - Z = B'$. It is shown in [13, Theorem (6.2)] that $\mathcal{T} - Z$ is a tangle of order $\theta - |Z|$ in $G - Z$.

For a positive integer α, a graph G is *α-nearly embeddable* in a surface Σ if there is a subset $A \subseteq V(G)$ with $|A| \leq \alpha$ such that there are integers $\alpha' \leq \alpha$ and $n \geq \alpha'$ for which $G - A$ can be written as the union of $n + 1$ edge-disjoint graphs $G_0, \ldots, G_n$ with the following properties:

(i) For all $1 \leq i \leq j \leq n$ and $\Omega_i := V(G_i \cap G_0)$, the pairs $(G_i, \Omega_i) =: V_i$ are vortices and $G_i \cap G_j \subseteq G_0$ when $i \neq j$.

(ii) The vortices $V_1, \ldots, V_{\alpha'}$ are disjoint and have adhesion at most α; we denote the set of these vortices by $\mathcal{V}$. We will sometimes refer to these vortices as *large* vortices.

(iii) The vortices $V_{\alpha'+1}, \ldots, V_n$ have length at most 3; we denote the set of these vortices by $\mathcal{W}$. These are the *small* vortices of the near-embedding.

(iv) There are closed discs in Σ with disjoint interiors $D_1, \ldots, D_n$ and an embedding $\sigma : G_0 \hookrightarrow \Sigma - \bigcup_{i=1}^{n} D_i$ such that $\sigma(G_0) \cap \partial D_i = \sigma(\Omega_i)$ for all i and the generic linear ordering of Ω_i is compatible with the natural cyclic ordering of its image (i.e., coincides with the linear ordering of $\sigma(\Omega_i)$ induced by $[0,1)$ when ∂D_i is viewed as a suitable homeomorphic copy of $[0,1]/\{0,1\}$). For $i = 1, \ldots, n$ we think of the disc D_i as *accommodating* the (unembedded) vortex V_i, and denote D_i as $D(V_i)$.

We call $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ an *α-near embedding* of G in Σ or just *near-embedding* if the bound is clear from the context. It *captures* a tangle $\mathcal{T}$ if the 'large side' B of an element $(A,B) \in \mathcal{T} - Z$ is never contained in a vortex.

Let G_0' be the graph resulting from G_0 by joining any two nonadjacent vertices $u, v \in G_0$ that lie in a common vortex $V \in \mathcal{W}$; the new edge uv of G_0' will be called a *virtual edge*. By embedding these virtual edges disjointly in the discs $D(V)$ accommodating their vortex V, we extend our embedding $\sigma \colon G_0 \hookrightarrow \Sigma$ to an embedding $\sigma' \colon G_0' \hookrightarrow \Sigma$. We shall not normally distinguish G_0' from its image in Σ under σ'.

A vortex (G_i, Ω_i) is *properly attached* to G_0 if it satisfies the following two requirements. First, for every pair of distinct vertices $u, v \in \Omega_i$ the graph G_i must contain an Ω_i-path (one with no inner vertices in Ω_i) from u to v. Second, whenever $u, v, w \in \Omega_i$ are distinct vertices (not necessarily in this order), there are two internally disjoint Ω_i-paths in G_i linking u to v and v to w, respectively.

It is easy to see that for a vortex (G_i, Ω_i) properly attached to G_0, the vortex $(G_i - v, \Omega_i \setminus \{v\})$ is properly attached to G_0 for any vertex $v \in \Omega_i$.

The *distance in* Σ of two vertices $x, y \in \Sigma$ is the minimal value of $|G_0' \cap C|$ taken over all curves C in the surface that link x to y and meet the graph only in vertices. The *distance in* Σ of two vortices V and W is the minimum distance in Σ of a vertex in $\Omega(V)$ from a vertex in $\Omega(W)$. Similar, the *distance in* Σ of to subgraphs H and H' of G_0' is the minimum distance in Σ of a vertex in H from a vertex in H'.

A cycle C in Σ is *flat* if C bounds an open disc $D(C)$ in Σ. Disjoint cycles $C_1, \ldots, C_n$ in Σ are *concentric* if they bound discs $D(C_1) \supseteq \ldots \supseteq D(C_n)$ in Σ. A set $\mathcal{P}$ of paths intersects $C_1, \ldots, C_n$ *orthogonally*, and is *orthogonal to* $C_1, \ldots, C_n$, if every path P in $\mathcal{P}$ intersects each of the cycles in a (possibly trivial but non-empty) subpath of P.

Let G be a graph embedded in a surface Σ and Ω a subset of its vertices. Let $C_1, \ldots, C_n$ be cycles in G that are concentric in Σ. The cycles $C_1, \ldots, C_n$ *enclose* Ω if $D(C_n) \setminus \partial D(C_n)$ contains Ω. They *tightly enclose* Ω if the following holds:

> *For every vertex $v \in V(C_k)$, for all $1 \le k \le n$, there is a vertex $w \in \Omega$ such that the distance of v and w in Σ is at most $n-k+2$.*

For a near-embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of some graph G in a surface Σ and concentric cycles $C_1, \ldots, C_n$ in G_0', a vortex $V \in \mathcal{V}$ is *(tightly) enclosed* by these cycles if $C_1, \ldots, C_n$ cycles (tightly) enclose $\Omega(V)$.

A flat triangle is a *boundary triangle* if it bounds a disc that is a face of G_0' in Σ.

For positive integers r, define a graph H_r as follows. Let $P_1, \dots, P_r$ be r vertex disjoint ('horizontal') paths of length $r-1$, say $P_i = v_1^i \dots v_r^i$. Let $V(H_r) = \bigcup_{i=1}^r V(P_i)$, and let

$$E(H_r) = \bigcup_{i=1}^r E(P_i) \cup \left\{ v_j^i v_j^{i+1} \mid \; i, j \text{ odd; } 1 \le i < r; \; 1 \le j \le r \right\} \\ \cup \left\{ v_j^i v_j^{i+1} \mid \; i, j \text{ even; } 1 \le i < r; \; 1 \le j \le r \right\}.$$

We call the paths P_i the *rows* of H_r; the paths induced by the vertices $\{v_j^i, v_{j+1}^i : 1 \le i \le r\}$ for an odd index i are its *columns*. The 6-cycles in H_r are its *bricks*. In the natural plane embedding of H_r, these bound its 'finite' faces. The outer cycle of the unique maximal 2-connected subgraph of H_r is the *boundary cycle* of H_r.

Any subdivision $H = TH_r$ of H_r will be called an *r-wall* or a *wall of size r*. The *bricks* and the *boundary cycle* of H are its subgraphs that form subdivisions of the bricks and the boundary cycle of H_r, respectively. An embedding of H in a surface Σ is a *flat* embedding, and H is *flat* in Σ, if the boundary cycle C of H bounds a disc $D(H)$ that contains a vertex of degree 3 of $H - C$.

For topological concepts used but not defined in this paper we refer to [6, Appendix B]. When we speak of the *genus* of a surface Σ we always mean its Euler-genus, the number $2 - \chi(\Sigma)$.

A closed curve C in Σ is *genus-reducing* if the (one or two) surfaces obtained by capping the holes of the components of $\Sigma \backslash C$ have smaller genus than Σ. Note that if C separates Σ and one of the two resulting surfaces is homeomorphic to S^2, the other is homeomorphic to Σ. Hence in this case C was not genus-reducing.

The *representativity* of an embedding $G \hookrightarrow \Sigma \not\simeq S^2$ is the smallest integer k such that every genus-reducing curve C in Σ that meets G only in vertices meets it in at least k vertices.

An α-near embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of a graph G in some surface Σ is *β-rich* for some integer β if the following statements hold:

(i) G_0' contains a flat r-wall H for an integer $r \ge \beta$.

(ii) The representativity of G_0' in Σ is at least β.

(iii) For every vortex $V \in \mathcal{V}$ there are β concentric cycles $C_1(V), \dots, C_\beta(V)$ in G_0' tightly enclosing V and bounding open discs $D_1(V) \supseteq \dots \supseteq D_\beta(V)$, such that $D_\beta(V)$ contains $\Omega(V)$ and $D(H)$ does not intersect $D_1(V) \cup C_1(V)$. For distinct vortices $V, W \in \mathcal{V}$, the discs $\overline{D_1(V)}$ and $\overline{D_1(W)}$ are disjoint.

(iv) Every two vortices in $\mathcal{V}$ have distance at least β in Σ.

(v) Let $V \in \mathcal{V}$ with $\Omega(V) = (w_1, \dots, w_n)$. Then there is a linked decomposition of V of adhesion at most α and a path P in $V \cup \bigcup \mathcal{W}$ with $V(P \cap G_0) = \Omega(V)$, avoiding all the paths of the linkage of V, and traversing $w_1, \dots, w_n$ in their order.

(vi) For every vortex $V \in \mathcal{V}$, its set of society vertices $\Omega(V)$ is linked in G_0' to branch vertices of H by a set $\mathcal{P}(V)$ of β disjoint paths having no inner vertices in H.

(vii) For every vortex $V \in \mathcal{V}$, the paths in $\mathcal{P}(V)$ intersect the cycles $C_1(V), \dots, C_\beta(V)$ orthogonally.

(viii) All vortices in $\mathcal{W}$ are properly attached to G_0.

With these concepts, we can state the theorem that is the main result of this paper:

Theorem 1. *For every graph R there is an integer α such that for every integer β there is an integer $w = w(R, \beta)$ such that the following holds. Every graph G with $tw(G) \geq w$ that does not contain R as a minor has an α-near, β-rich embedding in some surface Σ in which R cannot be embedded.*

A direct implication of Theorem (3.1) from [14], stated with this terminology, reads as follows:

Theorem 2. *For every graph R there exist integers $\theta, \alpha \geq 0$ such that the following holds: Let G be a graph that does not contain R as a minor and $\mathcal{T}$ be a tangle in G of order at least θ. Then G has an α-near embedding with apex set A into a surface Σ in which R cannot be drawn and this embedding captures $\mathcal{T} - A$.*

3 Finding a tree-decomposition

The following lemma shows that we can slightly modify a given α-near embedding by embedding some more vertices of the graph in the surface, so that all the small vortices are properly attached to G_0.

Lemma 3. *Given an integer α and an α-near embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of some graph G in a surface Σ, there exists an α-near embedding $(\hat{\sigma}, \hat{G}_0, A, \mathcal{V}, \hat{\mathcal{W}})$ of G in Σ such that $G_0 \subseteq \hat{G}_0$ and $\hat{\sigma}|_{G_0} = \sigma$ and each vortex in $\hat{\mathcal{W}}$ is properly attached to $\hat{G}_0$.*

Proof. Let us consider two modifications of our near-embedding, each resulting in another α-near embedding.

(1) Let $V := (G_i, \Omega_i) \in \mathcal{W}$ be a vortex of length 3. If there is a vertex $v \in V(G_i) \setminus \Omega_i$, that separates one society vertex $w \in \Omega_i$ from the other society vertices $\{w', w''\} := \Omega_i \setminus \{w\}$, we can write (G_i, Ω_i) as the union of two disjoint vortices $V^1 := (G_i^1, \{w\})$ and $V^2 := (G_i^2, \{w', w''\})$ whose intersection is $\{v\}$. Let G_0^+ denote the graph we obtain from G_0 if we add v to its vertex set and let σ^+ be an embedding of G_0^+ in Σ that maps v onto a point in $D(V)$ and coincides with σ everywhere else. It is easy to see that $(\sigma, G_0^+, A, \mathcal{V}, (\mathcal{W} \setminus \{V\}) \cup \{V^1, V^2\})$ is an α-near embedding of G in Σ.

(2) Let $V \in \hat{\mathcal{W}}$ be a vortex of length at least 2. If there are two society vertices $w, w' \in \Omega(V)$ that cannot be linked by a path having only its endvertices in $\Omega(V)$, we can write V as the union of two vortices V^1 and V^2 with $w \in \Omega(V^1)$ and $w' \in \Omega(V^2)$ whose intersection is contained in $\Omega(V)$ and whose societies are smaller than $|\Omega(V)|$. Similar to the case before, it is easy to see that $(\sigma, G_0, \hat{A}, \hat{\mathcal{V}}, (\hat{\mathcal{W}} \setminus \{V\}) \cup \{V^1, V^2\})$ is an α-near embedding of G in Σ, respecting $\mathcal{T}$.

We can iterate these two modifications only finitely often: Every application of (1) increases the number of embedded vertices of the graph G; the application of (2) lexicographically reduces (n_3, n_2) when n_2 and n_3 denote the number of small vortices of length 2 and 3, respectively.

Let $(\hat{\sigma}, \hat{G}_0, A, \mathcal{V}, \hat{\mathcal{W}})$ be the α-near embedding obtained by applying the two modifications as often as possible. Now every vortex $W \in \hat{\mathcal{W}}$ is properly attached to $\hat{G}_0$, as otherwise one could perform a further modification step. Clearly, $G_0 \subseteq \hat{G}_0$ and $\hat{\sigma}|_{G_0} = \sigma$. □

Given two graphs G and H, we say that H is *properly attached* to G if the vortex $(H, V(H) \cap V(G))$ is properly attached to G.

We will use Theorem 2 to prove the following result, which strengthens Theorem (1.3) of [14].

Theorem 4. *For every graph R there exist integers α and θ such that for every graph G that does not contain R as a minor and every $Z \subseteq V(G)$ with $|Z| \leq 3\theta - 2$ there is a rooted tree-decomposition $(V_t)_{t \in T}$ of G with root r such that for every $t \in T$, there is a surface Σ_t in which R cannot be embedded, and the torso G_t of V_t has an α-near embedding $(\sigma_t, G_{t,0}, A_t, \mathcal{V}_t, \emptyset)$ into Σ_t with the following properties:*

(i) *All vortices have depth at most α.*

(ii) *For every $t' \in T$ with $tt' \in E(T)$ and $t \in rTt'$ there is a vertex set X which is either*

   (a) *one part of a vortex decomposition or*

   (b) *a subset of $V(G_{t,0})$ and induces in $G_{t,0}$ a K_1, a K_2 or a boundary triangle*

   such that $V_t \cap V_{t'} \subseteq X \cup A_t$. If (b) holds, $G_{t'} - A_t$ is properly attached to $G_{t,0}$.

(iii) *For every $t' \in T$ with $tt' \in E(T)$ and $t' \in rTt$ the overlap $V_t \cap V_{t'}$ is contained in $A_{t'}$.*

Further $Z \subseteq A_r$. We say that the part V_r accomodates *Z.*

Proof. Applying Theorem 2 with the given graph R yields two constants $\hat{\alpha}$ and $\hat{\theta}$. Let $\theta := \max(\hat{\theta}, 3\hat{\alpha} + 1)$ and $\alpha := 4\theta - 2$.

The proof proceeds by induction on $|G|$. We may assume that $|Z| = 3\theta - 2$, since if it is smaller we add arbitrary vertices to Z. Note, we may assume that such vertices exist, as the theorem is trivial for $|G| < \alpha$.

We may assume that

$$\textit{There is no separation } (A,B) \textit{ of order at most } \theta \textit{ such that both } |Z \setminus A| \textit{ and } |Z \setminus B| \textit{ are of size at least } |A \cap B|. \tag{1}$$

Otherwise, let $Z_A := (A \cap Z) \cup (A \cap B)$. By assumption, $|A \cap B| \leq |Z \setminus A|$ and therefore, $|Z_A| \leq |Z|$. We apply our theorem inductively to $G[A]$ and Z_A, which yields a tree-decomposition of $G[A]$ with one part G_A such that the apex set of the embedding of its torso contains Z_A. Similarly, we apply the theorem to $G[B]$ and $Z_B := (B \cap Z) \cup (A \cap B)$. We combine these two tree-decompositions by joining a new part $Z \cup (A \cap B)$ to both G_A and G_B and obtain a tree-decomposition of G with the desired properties of the theorem: The new part contains at most $|Z| + |A \cap B| \leq 4\theta - 2$ vertices, so all these can be put into the apex set of an α-near embedding. Further, the new part contains Z. This proves (1).

Let $\mathcal{T}$ be the set of separations (A, B) of G of order less than θ such that $|Z \cap B| > |Z \cap A|$. With this definition,

$$\mathcal{T} \textit{ is a tangle of } G \textit{ of order } \theta. \tag{2}$$

For every separation (A, B) of G of order less than θ, one of the sets $Z \setminus B$ and $Z \setminus A$ contains at least θ vertices, as $|Z| = 3\theta - 2$, but not both by (1). Therefore, property (i) of the definition of a tangle holds. We deduce further, that for every $(A, B) \in \mathcal{T}$, the small side A contains less than θ vertices from Z. Hence, the union of three small sides cannot be $V(G)$ as it contains at most $3\theta - 3$ vertices from Z, which shows property (ii) and proves (2).

From (1) and the definition of $\mathcal{T}$ we conclude

$$|(A \setminus B) \cap Z| < |A \cap B| \textit{ for every } (A, B) \in \mathcal{T}. \tag{3}$$

Theorem 2 gives us an $\hat{\alpha}$-near embedding $(\sigma, G_0, \hat{A}, \hat{\mathcal{V}}, \hat{\mathcal{W}})$ of G in some surface Σ that captures $\mathcal{T}$. At a high level, our plan is now to split up G at separators consisting of apex vertices, society vertices $\Omega(V)$ for $V \in \hat{\mathcal{W}}$ and vertices of single parts of vortex decompositions of vortices in $\hat{\mathcal{V}}$. We obtain a part that contains G_0 and which we know how to embed α-nearly; this part is going to be one part of a new tree-decomposition. We find tree-decompositions for all subgraphs of G that we split off inductively and eventually combine these tree-decompositions to a new one that satisfies our theorem.

By Lemma 3, we may assume that all small vortices are properly attached to G_0. Let us consider such a vortex $(G_i, \Omega_i) \in \hat{\mathcal{W}}$. Our embedding captures $\mathcal{T}$, therefore the separation $(V(G_i) \cup \hat{A}, V(G \setminus G_i) \cup \hat{A})$, whose order is smaller than $3 + |\hat{A}| \leq \theta$, lies in $\mathcal{T}$. By (3), G_i contains less than θ vertices of Z. Thus, $Z' := \Omega_i \cup \hat{A} \cup (Z \cap G_i)$ contains at most $3 + \hat{\alpha} + \theta \leq 3\theta - 1$ vertices. We apply our theorem inductively to the smaller graph $G[V(G_i) \cup \hat{A}]$ with Z'. Let H^i be a part of the resulting tree-decomposition $(T^i, \mathcal{H}^i)$ that accomodates Z'. Note, that $H^i - \hat{A}$ is properly attached to G_0.

For every vortex $(G_i, \Omega_i) \in \hat{\mathcal{V}}$ with $\Omega_i = \{w_1^i, \ldots, w_{n(i)}^i\}$ let us choose a fixed decomposition $(\hat{X}^i_1, \ldots, \hat{X}^i_{n(i)})$ of depth at most $\hat{\alpha}$. We define

$$X^i_j := \begin{cases} \big(\hat{X}^i_1 \cap \hat{X}^i_2\big) \cup \{w^i_1\} & \text{for } j = 1 \\ \big(\hat{X}^i_j \cap (\hat{X}^i_{j-1} \cup \hat{X}^i_{j+1})\big) \cup \{w^i_j\} & \text{for } 1 < j < n(i) \\ \big(\hat{X}^i_{n(i)} \cap \hat{X}^i_{n(i)-1}\big) \cup \{w^i_{n(i)}\} & \text{for } j = n(i) \end{cases}$$

By G^-_i we denote the graph on $X^i_1 \cup \ldots \cup X^i_{n(i)}$ where every X^i_j induces a complete graph but no further edges are present. Now, as the adhesion of (G_i, Ω_i) is at most $\hat{\alpha}$, every X^i_j contains at most $2\hat{\alpha} + 1$ vertices and thus, $(X^i_1, \ldots, X^i_{n(i)})$ is a decomposition of the vortex $V^-_i := (G^-_i, \Omega_i)$ of depth at most $2\hat{\alpha} + 1 \leq \alpha$. Let $\mathcal{V}$ denote the set of these new vortices.

For every $j = 1, \ldots, n(i)$, the pair

$$\big(\hat{X}^i_j \cup \hat{A}, (V(G) \setminus (\hat{X}^i_j \setminus X^i_j)) \cup \hat{A}\big)$$

is a separation of order at most $|X^i_j \cup \hat{A}| \leq 2\hat{\alpha} + 1 + \hat{\alpha} \leq \theta$. As before, our embedding captures $\mathcal{T}$ and thus, the separation lies in $\mathcal{T}$. By (3), at most $\theta - 1$ vertices from Z lie in $\hat{X}^i_j$. Let $Z' := X^i_j \cup \hat{A} \cup (Z \cap \hat{X}^i_j)$. This set contains at most $3\theta - 1$ vertices and, similar to before, we can apply our theorem inductively to the smaller graph $G[\hat{X}^i_j \cup \hat{A}]$ with Z'. We obtain a tree-decomposition $(T^i_j, \mathcal{H}^i_j)$ of this graph, with one part H^i_j accomodating Z'.

Now, with $V_0 := V(G_0) \cup \hat{A}$, we can write

$$G = G[V_0] \cup \big(\bigcup \mathcal{W}\big) \cup \big(\bigcup \{G[\hat{X}^i_j] : V_i \in \mathcal{V}, 1 \leq j \leq n(i)\}\big).$$

By induction, we obtained tree-decompositions for all vortices in $\mathcal{W}$ and all the graphs $G[\hat{X}^i_j]$ with the required properties. We can now construct a tree-decomposition of G: We just add a new vertex v_0 representing V_0 to the union of all the trees T^i and T^i_j and add edges from v_0 to every vertex representing an H^i or an H^i_j we found in our proof.

We still have to check that the torso of the new part V_0 can be α-nearly embedded as desired. But this is easy: Let G'_0 be the graph resulting from G_0 if we add an edge xy for every two nonadjacent vertices x and y that lie in a common vortex $V \in \mathcal{W}$. We can extend the embedding $\sigma : G_0 \hookrightarrow \Sigma$ to an embedding $\sigma' : G'_0 \hookrightarrow \Sigma$ by mapping the new edges disjointly to the discs $D(V)$. Then, $G' := G'_0 \cup \bigcup G^-_i$ is the torso of V_0 in our new tree-decomposition and with $(\sigma', G'_0, \hat{A} \cup Z, \mathcal{V}, \emptyset)$ we have an α-near embedding of G' in Σ whose apex set contains Z. $\square$

4 Graphs on Surfaces

Our first tool is the so-called *grid-theorem* from [12]; see [6] for a short proof.

Theorem 5. *For every integer k there exists an integer $f(k)$ such that every graph of tree-width at least $f(k)$ contains a wall of size at least k.*

Every large enough wall embedded in a surface contains a large flat subwall:

Lemma 6. *For all integers k, g there is an integer $\ell = \ell(k, g)$ such that any wall of size ℓ embedded in a surface of genus at most g contains a flat wall of size k.*

Proof. Let ℓ be chosen so that every ℓ-wall contains $g + 1$ disjoint k-walls. By [6, Lemma B.6], any ℓ-wall Γ in a surface Σ of genus g contains a k-wall Γ' each of whose bricks bounds an open disc in Σ. If none of these open discs contains a point of Γ, the wall Γ' is flat. Otherwise, the disc containing a point of Γ contains all the other k-walls we considered, and thus, all these are flat. $\square$

Lemma 7. *Let G be a graph of tree-width at least w. Then in every tree-decomposition of G the torso of at least one part also has tree-width at least w.*

Proof. If every torso has a tree-decomposition of width at most $w - 1$, we can combine these into a tree-decomposition of G of width at most $w - 1$. $\square$

Let Σ be a (closed) surface. For $D \subseteq \Sigma$ the boundary of D is denoted by ∂D and $\overline{D} := D \cup \partial D$.

Let G be a graph embedded in Σ. For a face F of G, let S be the set of vertices that lie on ∂F. If we delete S and add a new vertex v to G with neighbours $N(S)$, we obtain a graph G'. It is easy to see that we can extend the induced embedding of $G - S$ to an embedding of G'. We say that the graph G' embedded in Σ was obtained from G by *contracting F to v*.

The following lemma is from Demaine and Hajiaghayi [5].

Lemma 8. *For every two integers t and g there exists an integer $s = s(t, g) > 0$ such that the following holds. Let G be a graph of tree-width at least s embedded in some surface Σ of genus g. If G' is obtained from G by contracting a face to a vertex, then G' has tree-width at least t.*

Our next lemma is due to Mohar and Thomassen [11]:

Lemma 9. *Let $G \hookrightarrow \Sigma \neq S^2$ be an embedding of representativity at least $2k+2$ for some $k \in \mathbb{N}$. Then, for every face F of G in Σ there are k concentric cycles $(C_1, \ldots, C_k)$ in G such that $F \subseteq D(C_k) \setminus \partial D(C_k)$.*

For an oriented curve C and points $x, y \in C$ we denote by xCy the subcurve of C with endpoints x, y that is oriented from x to y. For an embedded graph G in a surface Σ, a face f of G and a closed curve C in Σ, let $\mathcal{C}(C, f)$ denote the number of components of $C \cap f$.

Lemma 10. *Let G be a graph embedded in a surface Σ and F be the set of faces of G. For an integer $r > 0$, consider all genus-reducing curves C in Σ that hit G in vertices only and satisfy $|C \cap G| < r$. Let C be chosen so that $\sum_{f \in F} \mathcal{C}(C, f)$ is minimal. Then, $\mathcal{C}(C, f) \leq 1$ for all $f \in F$.*

Proof. Suppose there is an $f \in F$ with $\mathcal{C}(C, f) > 1$. Then there is a component D of $f - C$ such that $\overline{D}$ contains two distinct components X, Y of $f \cap C$. Let us choose a fixed orientation of C. Let $C \cap \partial D = \{x, y, z, w\}$ such that $xCy, zCw \subseteq D$. wCz contains xCy as C is connected, so x, y, z, w appear in this (cyclic) order on C. Now we distinct two cases:

(a) y, w lie in the same component of $\partial D - \{x, z\}$. Let C' be a curve in D linking x, w, oriented from x to w. Then, $wCx \cup xC'w =: C''$ is a closed curve in Σ that crosses V one time fewer than C and does not hit G in more vertices than C does. C'' does not separate Σ as y and z are connected by yCz, and by [6, Lemma B.4], it follows that C'' is a genus-reducing curve. This is a contradiction to the choice of C.

(b) y, w lie in distinct components of $\partial D - \{x, z\}$. Let C', C'' be disjoint oriented curves in D such that C' links x and w, and is oriented from x to w, and C'' links z and y and is oriented from z to y. If $wCx \cup xC'w$ bounds a disc D' in Σ and $yCz \cup zC''y$ bounds a disc D'' in Σ, then $D \cup D' \cup D''$ is a disc whose closure contains C. By [6, Lemmas B.4–5], this is a contradiction to the assumption that C is a genus-reducing curve. Thus, one of $wCx \cup xC'w$, $yCz \cup zC''y$ is a genus-reducing curve that crosses f one time fewer than C and does not hit G in more vertices than C does, which contradicts the choice of C.

This proves Lemma 10. □

Whenever there are cycles enclosing a vortex V, we can find cycles tightly enclosing V:

Lemma 11. *For an integer $\alpha > 0$, let $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an α-near embedding of some graph G in a surface Σ and let $C_1, \ldots, C_n$ be cycles enclosing a vortex $V \in \mathcal{V}$. Then, there are n cycles $C'_1, \ldots, C'_n$ in G_0, tightly enclosing V.*

Proof. Let us write $D_k := D(C_k)$ for $1 \leq k \leq n$ and $D_{n+1} := D(V)$. Suppose there is a cycle $C \subseteq G'_0 \cap \overline{D}_k \setminus D_{k+1}$ for some $1 \leq k \leq n$ such that there is a vertex $v \in V(C_k \setminus C)$. Then, we can replace C_k by C and obtain a new set of cycles in G_0 enclosing V. By this replacement, we reduce $|G'_0 \cap \overline{D}_k|$, and therefore we can repeate this step only finitely often. Let us choose cycles $C'_1, \ldots, C'_n$ enclosing V such that a replacement as described is not possible.

We claim that these cycles enclose V tightly. To see this, for a vertex $v \in V(C'_k)$ consider the set $\mathcal{F}$ of all faces F of G'_0 with $F \subseteq D_1$ and $v \in \partial F$. For every two neighbours x, y of v that lie in the same face $F \in \mathcal{F}$, there is a path in $G'_0 \cap \partial F$ linking x, y and avoiding v. Therefore, there is a face $\hat{F} \in \mathcal{F}$ such that ∂F contains a vertex $v' \in V(C'_{k+1})$: otherwise, $\bigcup\{\partial F : F \in \mathcal{F}\}$ would contain a path linking both neighbours of v in C'_k, avoiding v and thus, a cycle would live in $G'_0 \cap (\overline{D}_k \setminus \overline{D}_{k+1})$, also avoiding v, contradicting the choice of the C'_i. Now, by induction on k, there is a curve C linking $v' \in V(C'_{k+1})$ and $w \in \Omega(V)$ with $|C \cap G'_0| \leq \alpha - k + 1$. We extend this curve by a curve in F linking v and v' which gives us a curve as desired. □

5 Taming a Vortex

In this section we describe how to obtain a new (and simpler) near-embedding from an old one if we are allowed to move a bounded number of vertices from the embedded part of the graph to the apex set. For example, we might reduce the number of large vortices by combining two of them, or reduce the genus of the surface by cutting along a genus-reducing curve. To describe these changes precisely, let us introduce some new notation.

A near-embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of a graph G in a surface Σ is an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of G if $|A| \leq \alpha_0$ and $|\mathcal{V}| \leq \alpha_1$ and the adhesion of all vortices in $\mathcal{V}$ is at most α_2.

Lemma 12. *Let $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of a graph G in a surface Σ. If there are two vortices $V, W \in \mathcal{V}$ of length at least 4 and a curve C in Σ with endpoints in $D(V)$ and $D(W)$ that hits G in at most d vertices, then there is a vertex set $A' \subseteq V(G_0)$ with $|A'| \leq 2\alpha + 2 + d$ and a vortex $V' \subseteq G - A'$ such that*

$$(\sigma|_{G_0 - A'}, G_0, A \cup A', ((\mathcal{V} \setminus \{V, W\}) - A') \cup \{V'\}, \mathcal{W} - A')$$

is an $(\alpha_0 + 2\alpha_2 + 2 + d, \alpha_1 - 1, \alpha_2)$-near embedding of G in Σ.

Proof. Let us choose decompositions $(X_1, \ldots, X_n)$ of V and $(Y_1, \ldots, Y_m)$ of W of adhesion at most α, where $\Omega(V) = (v_1, \ldots, v_n)$ and $\Omega(W) = (w_1, \ldots, w_m)$. By slightly adjusting C we obtain a curve C' with $C' \cap D(V) = \{v_k\}$ and $C' \cap D(W) = \{w_\ell\}$ for some indices $1 \leq k \leq n$ and $1 \leq \ell \leq m$ and $C' \cap G_0 = (C \cap G_0) \cup \{v_k, w_\ell\}$. Let S be the set of vertices in Σ that are hit by C'. By fattening C' we obtain an open disc D' in Σ with $C' \subseteq D'$ and $D' \cap G_0 - S = \emptyset$ such that $D'' := D' \cup D(V) \cup D(W)$ is an open disc whose closure is a closed disc and $D'' \cap (G_0 - S) = (\Omega(V) \cup \Omega(W)) \setminus \{v_k, w_\ell\}$. We may assume without loss of generality that the orientations of $\partial D''$ induced by $\Omega(V) \setminus \{v_k\}$ and $\Omega(W) \setminus \{w_\ell\}$ agree by reindexing if necessary.

Let $X := (X_k \cap X_{k+1}) \cup \{v_k\}$ if $k < n$ and $X := \{v_k\}$ if $k = n$; let $Y := (Y_\ell \cap Y_{\ell-1}) \cup \{w_\ell\}$ if $\ell < m$ and $Y := \{w_\ell\}$ if $\ell = 1$. Note that $|X| \leq \alpha + 1$ and $|Y| \leq \alpha + 1$.

Let $A' := S \cup X \cup Y$ and $G' := (V \cup W) - A'$. With

$$\Omega' := (v_{k+1}, \ldots, v_n, v_1, \ldots, v_{k-1}, w_{\ell+1}, \ldots, w_m, w_1, \ldots, w_{\ell-1})$$

the tuple $V' := (G', \Omega')$ is a vortex with a decomposition

$$\begin{aligned}(X_{k+1}, \ldots, X_n, X_1, \ldots, X_{k-2}, (X_{k-1} \cup X_k) \setminus X,\\ (Y_\ell \cup Y_{\ell+1}) \setminus Y, Y_{\ell+2}, \ldots, Y_m, Y_1, \ldots, Y_{\ell-1})\end{aligned}$$

of adhesion at most α. Now it is easy to see that A' satisfies the conditions as desired. $\square$

We note that the techniques used in the following lemma have also been independently developed by Geelen and Hyuhn [8].

Lemma 13. *Let $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of some graph G in a surface Σ such that the following statements hold:*

(i) For every vortex $V \in \mathcal{V}$ there are $\alpha_2 + 1$ concentric cycles $C_0(V), \ldots, C_{\alpha_2}(V)$ in G_0' tightly enclosing V.

(ii) For distinct vortices $V, W \in \mathcal{V}$, the discs $D(C_0(V))$ and $D(C_0(W))$ are disjoint.

Then, there is a graph $\tilde{G}_0$ with

$$G_0 \setminus \Big(\bigcup_{V \in \mathcal{V}} D(C_0(V)) \Big) \subseteq \tilde{G}_0,$$

a set $\tilde{A} \subseteq V(G)$ with $|\tilde{A}| \leq \tilde{\alpha} := \alpha_0 + \alpha_1(2\alpha_2 + 2)$ avoiding $\tilde{G}_0$, and sets $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ of vortices in G such that with $\tilde{\sigma} := \sigma|_{\tilde{G}_0'}$ the tuple $(\tilde{\sigma}, \tilde{G}_0, A \cup \tilde{A}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}})$ is an $(\tilde{\alpha}, \alpha_1, \alpha_2 + 1)$-near embedding of G in Σ such that every vortex $V \in \tilde{\mathcal{V}}$ satisfies property (v) of β-rich.

Proof. We will convert the vortices in $\mathcal{V}$ into linked vortices one by one, so let us focus on one vortex $V \in \mathcal{V}$. The idea is as follows: we delete one vertex from each of the enclosing cycles, which gives us a set of $\alpha + 1$ disjoint paths. If necessary, we also delete an adhesion set of V which allows us to assume that the paths are 'aligned' to the vortex. Then, we 'push' these paths as far into the vortex as possible. As the adhesion of the vortex is bounded by α, at least one of the paths remains entirely in the surface. The vertices of this path, later denoted by P_0, become the society vertices of our new vortex whereas the path system shows us that this new vortex is linked.

We choose a decomposition $(X_1', \ldots, X_{n'}')$ and some vertex $v \in C_0(V)$. As $C_0(V), \ldots, C_{\alpha_2}(V)$ enclose V tightly, there is a curve C linking v and $\Omega(V)$ that hits at most $\alpha_2 + 2$ vertices of G_0'. Let S denote the set of these vertices. Clearly, S consists of exactly one vertex from each $C_i(V)$, $0 \leq i \leq \alpha_2$ and one society vertex w_j' of V.

If $j = 1$ or $j = n'$, we put $n := n'$, $Z := \emptyset$ and $X_i := X_i'$ for $1 \leq i \leq n$; otherwise, let $n := n' - 1$, $Z := Z_j \cup \{w_j'\}$ and

$$(X_1, \ldots, X_n) := (X_{j+1}' \backslash Z, X_{j+2}' \backslash Z, \ldots, X_{n'}' \backslash Z, X_1' \backslash Z, \ldots, (X_{j-1}' \cup X_j') \backslash Z)$$

Now, $(X_1, \ldots, X_n)$ is a path decomposition of $V - Z$ of adhesion at most α_2. Let $(w_1, \ldots, w_n)$ denote the society of $V - Z$.

Let us choose one fixed orientation for all the cycles $C_i(V)$ enclosing V. For every $0 \leq i \leq \alpha_2$ let x_i denote the successor and y_i the predecessor of the unique vertex in $S \cap V(C_i)$. Let $X := \{x_0, \ldots, x_{\alpha_2}\}$ and $Y := \{y_0, \ldots, y_{\alpha_2}\}$. Now, we delete $S \cup Z$, a set of at most $2\alpha_2 + 1$ vertices. We put

$$G' := \Big(\big(G_0' \cap D(C_0(V))\big) \cup V \Big) - (S \cup Z).$$

Clearly, the graph G' still contains a set of $\alpha_2 + 1$ disjoint X–Y-paths. Let us show that

$$\text{\textit{Every set $\mathcal{P}$ of more than α_2 disjoint X–Y-paths in G' contains a path P that is a path in G_0'.}} \tag{4}$$

Let $P_0 \in \mathcal{P}$ be the path starting in x_0. This path is disjoint to $V - \Omega(V)$: Otherwise, let w_q be the first vertex on P sending an edge of P to $V - \Omega(V)$.

As the subpath $P_0 w_q$ of P_0 lives entirely in the planar graph $G_0' \cap D(C_0(V))$, the set $V(Pw_q) \cup Z_q$ separates X from Y. Thus, all $\alpha_2 + 1$ paths in $\mathcal{P}$ have to pass Z_q, a set containing at most α_2 vertices, a contradiction. This proves (4).

We conclude that, for every such set $\mathcal{P}$ of paths, the path $P_0 \in \mathcal{P}$ starting in x_0 ends in y_0. This path P_0, together with v and the edges $x_0 v$, $v y_0$ form a cycle in G_0'. This cycle bounds a disc $D(\mathcal{P})$ in Σ containing $\Omega(V)$ and we define

$$G(\mathcal{P}) := \Big(\big(G_0' \cap D(\mathcal{P})\big) \cup V \Big) - (S \cup Z).$$

Clearly, $G(\mathcal{P})$ contains $\mathcal{P}$.

Let us fix a set $\mathcal{P}$ of paths in G' that link X to Y and are such that $G(\mathcal{P})$ is minimal. Assume P has length r and let the vertices of P be labeled $p_0, p_1, \ldots, p_r$.

By this choice, the vertices of P_0 have the following property:

For every vertex $p_i \in V(P_0)$, $i \geq 1$, there is a separator of size $\alpha + 1$, containing p_i and separating X from Y in G''. (5)

Suppose the opposite for some $p_i \in V(P_0)$. Then, there is a set $\mathcal{P}'$ of α_2 disjoint paths in $G(\mathcal{P}) - p_i$ that links X to Y. With (4), this is a contradiction to our choice of $G(\mathcal{P})$.

Let us pick for each $i = 1, \ldots, r$ a separation (A_i, B_i) with $p_i \in S_i := A_i \cap B_i$ such that $X \subseteq A_i$, $Y \subseteq B_i$ and $(|S_i|, |B_i|)$ is lexicographically minimal. Clearly, each S_i contains exactly one vertex from each path in $\mathcal{P}$. We also have the following property:

$B_i \supsetneq B_j$ for all $1 \leq i < j \leq r$. (6)

Let us assume that the statement was false for some fixed i, j with $1 \leq i < j \leq r$. We will show that we could replace (A_j, B_j) by the separation $(A_i \cup A_j, B_i \cap B_j)$. This will give us a contradiction as $|B_i \cap B_j| < |B_j|$. Clearly, the new separator $S' := (A_i \cup A_j) \cap (B_i \cap B_j)$ contains at least one vertex from each path of $\mathcal{P}$. We only have to show that S' also contains at most one vertex from each path in $\mathcal{P}$, the other conditions are clear.

Suppose that there is a path $P \in \mathcal{P}$ such that S' contains two vertices x, y from P. Neither S_i nor S_j contain both x and y, so we may assume that $x \in S_i \setminus S_j$ and $y \in S_j \setminus S_i$. We may further assume that $x \in Py$. But now, $x \in S'$ implies $x \in B_j$ and therefore, Px contains a vertex of S_j distinct to x. This means, that S_j contains two vertices of P, a contradiction. This proves (6).

We set $\Omega := V(P_0)$ and with $X_1 := A_1$, $X_i := A_i \cap B_{i-1}$ for $1 < i < n$ and $X_n := B_{n-1}$, it is easy to check that $(X_1, \ldots, X_n)$ is a linked path decomposition of the vortex $(G(\mathcal{P}), \Omega)$. Finally, consider all $W \in \mathcal{W}$ such that $\Omega(W)$ intersects with $G(\mathcal{P})$. If $\Omega(W) \subseteq G(\mathcal{P})$, we add W to $G(\mathcal{P})$; otherwise, we remove all edges of $G_0'[\Omega(W)]$ from $G(\mathcal{P})$. Now, property (v) from β-rich follows for the new vortex $G(\mathcal{P})$. □

Lemma 14. *Let $z > 0$ be an integer and $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of some graph G in a surface Σ such that every two vortices in $\mathcal{V}$ have distance at least z. If the representativity of G'_0 in Σ is smaller than z, then there is a vertex set $A' \subseteq V(G_0)$ with $|A'| < a := 2\alpha_2 + 2 + z$, such that one of the following statements holds:*

a) There exists a set $\mathcal{V}'$ of vortices in G, a surface Σ' with $g(\Sigma') < g(\Sigma)$ and an $(\alpha_0 + a, \alpha_1 + 1, \alpha_2)$-near embedding

$$(\sigma', G_0 - A', A \cup A', \mathcal{V}', \mathcal{W} - A')$$

of G in Σ'.

b) There exists a separation (A_1, A_2) of G with $A_1 \cap A_2 = A'$ such that for $i = 1, 2$ there are surfaces Σ_i and $(\alpha_0 + a, \alpha_1 + 1, \alpha_2)$-near embeddings $(\sigma^i, G^i_0, A^i, \mathcal{V}^i, \mathcal{W}^i)$ of $G[A_i]$ into Σ_i such that $g(\Sigma_i) < g(\Sigma)$.

Proof. Let C be a genus-reducing curve in Σ that hits at most z vertices of G'_0. Let us assume that C hits the society of a large vortex V. It cannot hit another large vortex as the distance in Σ of two large vortices is larger than z. By Lemma 10 we may assume that $\mathcal{C}(C, f) = 1$ for the face f of G'_0 containing $D(V)$ and we can modify C such that there are society vertices $x, y \in \Omega(V)$ with $\partial D(V) \cap C = \{x, y\}$. After this modification, C hits at most $z+2$ vertices. Deleting the two appropriate adhesion sets splits V into two vortices V', V'' and it is easy to find two disjoint discs $D(V'), D(V'')$ in the surface to accomodate them. Let A' be the union of the deleted adhesion sets and the vertices hit by C. This set contains up to a vertices.

Now, A' is a separation of G. We delete C from Σ and cap the holes of the resulting components. If $\Sigma \setminus C$ has one component, statement a) follows, otherwise b) is true. □

Lemma 15. *Let $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of some graph G in a surface Σ. Let $V \in \mathcal{V}$ a vortex and $(C_1, \ldots, C_\ell)$ cycles tightly enclosing V. Then, there is a set $\mathcal{X}$ of internally disjoint, closed discs with $\bigcup \mathcal{X} = D := \overline{D(C_1(V))}$ such that the following holds: For every disc $\Delta \in \mathcal{X}$ there is a separator S of G'_0 of order at most $2\ell + 2$ such that*

- $S = G'_0 \cap \partial\Delta$
- $|S \cap C_i| \leq 2$ *for each* $i = 1, \ldots, \ell$
- $|S \cap \Omega(V)| \leq 2$

Further, for this disc Δ there is a set $S' \subseteq V$ of order at most $2\alpha_2$ such that there is a separation (X_1, X_2) of G with $X_1 \cap X_2 = S \cup S'$ and $X_1 \cap \Delta = G_0 \cap \Delta$.

Proof. Pick a point $v \in \mathcal{D}(V)$. Let $(x_1, \ldots, x_n)$ denote the vertices of $C_1(V)$ ordered linearly in a way compatible with a cyclic orientation of $C_1(V)$. We will inductively define curves linking $x_1, \ldots, x_n$ and v. First, let us choose for all $i = 1, \ldots, n$ a curve L'_i linking x_i and v, oriented towards v, such that $L'_1 \cap G'_0$ consists of exactly one vertex from each of the cycles $C_1, \ldots, C_\ell$ and one society vertex $w_i \in \Omega(V)$. Note that, for $1 \leq i \leq n$ for each $z \in L_i'$, $|L_i'z \cap (C_1 \cup \ldots \cup C_\ell)| = k$ if and only if $z \in \overline{D(C_k)} \setminus \overline{D(C_{k+1})}$ where $D(C_{\ell+1}) := \emptyset$ and also that $\{w_1, \ldots, w_n\}$ need not contain all society vertices of V. Now, we put $L_1 := L_1'$ and for $i = 2, \ldots, n$ we define L_i inductively: If $L_{i-1} \cap L_i = \{v\}$, we define $L_i := L_i'$. Otherwise, let z be the first point on L_i' contained in $L_{i-1} \cap L_i'$. Then, $L_i := L_i'z \cup zL_i$. We further define $S_i := (L_i \cup L_{i+1}) \cap G_0'$ for $1 \leq i \leq n$ where $L_{n+1} := L_1$.

The components of $D \setminus (L_1 \cup L_2)$ are two discs. Let Δ_1 denote the disc with $x_1x_2 \subseteq \partial\Delta_1$ and let Δ_1' refer to the other one. We define discs $\Delta_2, \ldots, \Delta_n$ and $\Delta_2', \ldots, \Delta_{n-1}'$ inductively: For $1 < i < n$ let Δ_i denote the component of $\Delta_{i-1}' \setminus L_i$ containing x_ix_{i+1} and again let Δ_i' refer to the other one. Clearly, these components are discs. We put $\Delta_n := \Delta_{n-1}'$.

If $w_i = w_{i+1}$ for some $1 \leq i \leq n$ we define $S_i' := \emptyset$, otherwise let S_i' consist of two appropriate adhesion contained in bags assigned to w_i and w_{i+1} for some decomposition of V. The statement of the theroem now follows with $\mathcal{X} := \{\Delta_1, \ldots, \Delta_n\}$ □

6 Streamlining Path Systems

In this section we provide tools that allow us to find path systems in near-embeddings that satisfy properties (vi) and (vii).

Lemma 16. *Let G be a graph and A, B, C subsets of $V(G)$ with $|B| = 2k-1$ for some integer k. If there is a set $\mathcal{P}$ of $2k-1$ disjoint paths linking A and B and a set $\mathcal{Q}$ of $2k-1$ disjoint paths linking B and C, there are k disjoint paths in G linking A and C.*

Proof. Let $P_1, \ldots, P_{2k-1}$ be pairwise disjoint paths linking A and B and let $Q_1, \ldots, Q_{2k-1}$ be pairwise disjoint paths linking B and C such that the endvertex of P_i is the first vertex of Q_i for $1 \leq i \leq 2k-1$. For every set $S \subseteq V(G)$ with $|S| < k$, at least k paths in $\mathcal{P}$ and at least k paths in $\mathcal{Q}$ avoid all vertices in S. Two of these paths contain a common vertex of B and therefore, there is a path linking A and C avoiding S. □

Lemma 17. *Let G be a graph embedded in a surface, let Γ be a flat wall of size $6(2k-1)^2 + k$ in G for some natural number k and Ω a subset of $V(G)$ avoiding $D(\Gamma)$ such that there exist $(2k-1)^2$ disjoint paths linking Ω to the branch vertices of Γ. Then, Γ contains a wall Γ_0 of size k such that there are k disjoint paths having no inner vertex in Γ_0 and linking Ω to branch vertices of Γ_0 which lie on the boundary cycle of Γ_0.*

Proof. Let Γ_0 be a k-wall in Γ such that there exist $3(2k-1)^2 + k$ cycles Γ, enclosing Γ_0. Let us choose a set $\mathcal{P}$ of $(2k-1)^2$ disjoint paths linking Ω to the branch vertices of Γ such that $|E(\mathcal{P}) \setminus E(\Gamma)|$ is minimal. Each branch vertex v of Γ is incident with three subdivided edges of Γ. We say that $P \in \mathcal{P}$ *occupies* these three edges if v is the terminal vertex of P. Clearly, each subdivided edge that intersects with $\mathcal{P}$ is occupied by some path in $\mathcal{P}$.

We claim that no path $P \in \mathcal{P}$ can intersect with Γ_0. Otherwise P would intersect $3(2k-1)^2$ disjoint cycles in Γ but did not have its terminal vertex on one of these cycles. This means that more than $3(2k-1)^2$ edges would be occupied by paths in $\mathcal{P}$, a contradiction.

Now, either at least $2k-1$ rows or at least $2k-1$ columns contain terminal vertices of paths in $\mathcal{P}$. We may assume the latter, without loss of generality, and pick from each such column one path $P \in \mathcal{P}$ having its terminal vertex on this column. It is easy to see that this set of vertices can be linked by disjoint paths to branch verties of Γ_0 which lie on the boundary cycle of Γ_0. Lemma 16 finishes this proof. $\square$

Let G be a graph and $X, Y \subseteq V(G)$ with $|X| = |Y| =: k$. An X–Y *linkage* in G is a set of k disjoint paths in G such that each of these paths has one end in X and the other end in Y.

An X–Y linkage $\mathcal{P}$ in G is *singular* if $V(\bigcup \mathcal{P}) = V(G)$ and G does not contain any other X–Y linkage. The next lemma will be used in the proof of Lemma 19.

Lemma 18. *If a graph G contains a singular X–Y linkage $\mathcal{P}$ for vertex sets $X, Y \subseteq V(G)$, then G has path-width at most $|\mathcal{P}|$.*

Proof. Let $\mathcal{P}$ be a singular X–Y linkage in G. Applying induction on $|G|$, we show that G has a path-decomposition $(X_0, \ldots, X_n)$ of width at most $|\mathcal{P}|$ such that $X \subseteq X_0$. Suppose first that every $x \in X$ has a neighbour $y(x)$ in G that is not its neighbour on the path $P(x) \in \mathcal{P}$ containing x. Then $y(x) \notin P(x)$ by the uniqueness of $\mathcal{P}$. The digraph on $\mathcal{P}$ obtained by joining for every $x \in X$ the 'vertex' $P(x)$ to the 'vertex' $P(y(x))$ contains a directed cycle D. Let us replace in $\mathcal{P}$ for each $x \in X$ with $P(x) \in D$ the path $P(x)$ by the X–Y path that starts in x, jumps to $y(x)$, and then continues along $P(y(x))$. Since every 'vertex' of D has in- and outdegree both 1 there, this yields an X–Y linkage with the same endpoints as $\mathcal{P}$ but different from $\mathcal{P}$. This contradicts our assumption that $\mathcal{P}$ is singular. Thus, there exists an $x \in X$ without any neighbours in G other than (possibly) its neighbour on $P(x)$. Consider this x.

If $P(x)$ is trivial, then x is isolated in G and $x \in X \cap Y$. By induction, $G - x$ has a path-decomposition $(X_1, \ldots, X_n)$ of width at most $|\mathcal{P}| - 1$ with $X \setminus \{x\} \subseteq X_1$. Add $X_0 := X$ to obtain the desired path-decomposition of G. If $P(x)$ is not trivial, let x' be its second vertex, and replace x in X by x' to obtain X'. By induction, $G - x$ has a path-decomposition $(X_1, \ldots, X_n)$ of width at most $|\mathcal{P}|$ with $X' \subseteq X_1$. Add $X_0 := X \cup \{x'\}$ to obtain the desired path-decomposition of G. $\square$

The next lemma is a weaker version of Theorem 10.1 of [9].

Lemma 19. *Let s, s', and t be positive integers with $s \geq s' + t$. Let G' be a graph embedded in the plane and let $X \subseteq V(G')$ a set of vertices lying on a face boundary of G'. Let $(C_1, \ldots, C_s)$ be concentric cycles in G', tightly enclosing X. Let G'' be another graph, with $V(G') \cap V(G'') \subseteq V(C_1)$. Assume that $G' \cup G''$ contains an X–Y linkage $\mathcal{P}$ with $Y \subseteq C_1$. Then there exists an X–Y linkage $\mathcal{P}'$ in $G' \cup G''$ such that $\mathcal{P}'$ is orthogonal to $C_{t+1}, \ldots, C_s$.*

Proof. Assume the lemma is false, and let G', G'', $\mathcal{P}$, and $(C_1, \ldots, C_s)$ form a counterexample containing a minimal number of edges. To simplify the notation, we let $G = G' \cup G''$. By minimality, it follows that the graph $G = \bigcup_1^s C_i \cup \mathcal{P}$. Also, for all $P \in \mathcal{P}$ and for all $1 \leq i \leq s$, every component of $P \cap C_i$ is a single vertex. If $P \cap C_i$ had a component that was a non-trivial path containing an edge e, then G'/e would form a counterexample with fewer edges. Similarly, we conclude that $V(G) = V(\mathcal{P})$.

We claim:

$\mathcal{P}$ is singular. (7)

As noted above, $E(\mathcal{P})$ is disjoint from $E\left(\bigcup_1^s C_i\right)$. It follows that if there exists an X–Y linkage $\overline{\mathcal{P}}$ distinct from $\mathcal{P}$, then at least one of the edges of $\mathcal{P}$ is not contained in $\overline{\mathcal{P}}$. We conclude that the subgraph $\bigcup_1^s C_i \cup \overline{\mathcal{P}}$ forms a counterexample to the claim with fewer edges, a contradiction. This proves (7).

There is no subpath $Q \subseteq \mathcal{P}$ in $D(C_j)$ with both endpoints in C_j for some j and otherwise disjoint from $\bigcup_i V(C_i)$. (8)

Otherwise, this would violate our choice of the cycles C_i as tightly enclosing X.

A *local peak* of $\mathcal{P}$ is a subpath $Q \subseteq \mathcal{P}$ such that Q has both endpoints on C_j for some $j > 1$ and every internal vertex of $Q \cap \left(\bigcup_{i \neq j} V(C_i)\right) \subseteq V(C_{j-1})$.

We claim the following.

For all $j > 1$, there does not exist a local peak with endpoints in C_j. (9)

Fix Q to be a local peak with endpoints in C_j with Q chosen over all such local peaks so that j is maximal. Assume Q is a subpath of $P \in \mathcal{P}$. Let the endpoints of Q be x and y. Lest we re-route P through C_j and find a counter-example containing fewer edges, there exists a component $P' \in \mathcal{P}$ intersecting the subpath of C_j linking x and y. By planarity, P' either contains a subpath internally disjoint from the union of the C_i with both endpoints in C_s, or P' contains a subpath forming a local peak with endpoints in C_{j+1}. The former is a contradiction to our choice of a minimal counterexample, the latter a contradiction to our choice of Q. This proves (9).

An immediate consequence of (8) and (9) is the following. For every $P \in \mathcal{P}$, let x be the endpoint of P in X and let y be the vertex of $V(C_1) \cap V(P)$ closest to x on P. Define the path $\overline{P}$ be the subpath xPy of P. The path $\overline{P}$ is orthogonal to the cycles $C_1, \ldots, C_s$. In fact, $\overline{P} \cap C_i$ is a single vertex for each $1 \leq i \leq s$. The final claim will complete the proof.

For all $P \in \mathcal{P}$, the path $P - \overline{P}$ does not intersect C_{t+1}. (10)

To see (10) is true, fix $P \in \mathcal{P}$ such that $(P - \overline{P}) \cap C_{t+1} \neq \emptyset$. It follows now from (8) and (9) that $P - \overline{P}$ contains a subpath Q with one endpoint in C_{t+1} and one endpoint in C_1 such that Q is orthogonal to the cycles $C_{t+1}, C_t \ldots, C_1$. By the planarity of G', we see that G contains a subgraph isomorphic to a subdivision of the $(t+1) \times (t+1)$ grid. This contradicts (7) and Lemma 18, proving (10).

We conclude that $\mathcal{P}$ is orthogonal to the s' disjoint cycles $C_s, C_{s-1}, \ldots, C_{t+1}$. This contradicts our choice of G, and the lemma is proven. □

7 Proof of the Main Result

Before proceeding with the proof of Theorem 1, we will need one more lemma. A result similar to the following lemma can be found in [5].

Lemma 20. *For every integer t and every integer $\alpha > 0$ there is an integer $s > 0$ such that the following holds. Let G be a graph of tree-width at least s and $(\sigma, G_0, A, \mathcal{V}, \emptyset)$ be an α-near embedding of G in a surface Σ such that all vortices $V \in \mathcal{V}$ have depth at most α. Then, G_0 has tree-width at least t.*

Proof. For a given surface Σ, let r be an integer such that for every graph H embedded in Σ, the contraction of α disjoint faces of H to vertices leaves a graph of tree-width at least $t + \alpha$. Such an integer exists by Lemma 8.

Let G be a graph as stated with tree-width $s > \alpha r$. Let G_0^+ be the graph we obtain if for every vortex $V_i \in \mathcal{V}$ with $\Omega(V_i) = (w_1, \ldots, w_{n(i)})$ we add all edges $w_j w_{j+1}$ for $1 \leq j \leq n(i)$ where $n + 1 := 1$ to G_0 if not already in G_0. Clearly, σ can be extended to an embedding of G_0^+ by embedding the new edges in the according discs $D(V_i)$.

The tree-width of G_0^+ is at least r. (11)

Otherwise, choose a tree-decomposition $(V_t)_{t \in T}$ of G_0^+ of width less than r. For every vortex $V_i \in \mathcal{V}$ choose a fixed decomposition $(X_1^i, \ldots, X_{n(i)}^i)$ of depth at most α. For all $t \in T$ we define

$$V_t' := V_t \cup \bigcup \{X_j^i : w_j^i \in V_t\}$$

Note that, as all vortices are disjoint and thus, every vertex in V_t can be a society vertex of at most one vortex, we have $|V_t'| \leq \alpha |V_t| < \alpha r$. We claim that $(V_t')_{t \in T}$ is a tree-decomposition of $G \cup G_0^+$. To see this, pick a vertex $v \in V_{t_1} \cap V_{t_3}$ for distinct $t_1, t_3 \in T$. We have to show that $v \in V_{t_2}'$ for all $t_2 \in t_1 T t_3$. Let us assume that $v \notin V(G_0^+)$ as the other cases are easy. By construction, there is a vortex V_i with $\Omega(V_i) = (w_1, \ldots, w_{n(i)})$ such that for some $w_j, w_k \in \Omega(V_i)$, we have $v \in X_j^i \cap X_k^i$ and $w_j \in V_{t_1}$ and $w_k \in V_{t_3}$. We may assume without loss of generality that $j < k$. By construction of G_0^+, there is a path $w_j, w_{j+1}, \ldots, w_k$ in G_0^+. As V_{t_2} separates V_{t_1} from V_{t_3} in $G \cup G_0^+$, there is a vertex $w_\ell \in V_{t_2}$ for some $j < \ell < k$. As $v \in X_\ell^i$, we have $v \in V_{t_2}$ as desired.

Clearly, $(V_t)_{t \in T}$ is a tree-decomposition of G as well, but has width less than αr, a contradiction to our choice of G. This proves (11).

For every vortex $V \in \mathcal{V}$ there is a face $F \subseteq D(V)$ of G_0^+ with $\Omega(V) = \partial F \cap G_0^+$. By the choice of r, contracting all these faces to vertices yields a graph of tree-width at least $t + \alpha$. Removing the new vertices, of which we have at most α, results in the graph $G_0 \setminus \bigcup \mathcal{V}$ with tree-width at least t. Thus, the graph G_0 has tree-width at least t as well, proving the lemma. □

We now proceed with the proof of Theorem 1.

Proof of Theorem 1. Let $\hat{\alpha}$ be the integer from Theorem 4 for R and $\hat{\gamma}$ be an integer such that for every surface Σ with $g(\Sigma) > \hat{\gamma}$, the graph R can be embedded in Σ. By Theorem 5 and the Lemmas 20, 6 and 7, there is an integer w such that if the tree-width of our graph G is larger than w, the following holds: There is a tree-decomposition $(V_t)_{t\in T}$ of G such that the torso $\hat{G}$ of one part V_{t_0} has an $\hat{\alpha}$-near embedding $(\hat{\sigma}, \hat{G}_0, \hat{A}, \hat{\mathcal{V}}', \emptyset)$ into a surface $\hat{\Sigma}$ in which R cannot be embedded and $\hat{G}_0'$ contains a flat wall of size at least

$$6^{\hat{\alpha}+2\hat{\gamma}+1}(\beta + \hat{\alpha} + 1)p$$

where $p := 2\hat{\alpha}\beta + 4\hat{\alpha} + 4$. We will show, that with these constants, which only depend on R and β, we find an α-near embedding of G where

$$\alpha := \hat{\alpha} + p(2g(\hat{\Sigma}) + \hat{\alpha}) + 2\hat{\alpha}^2$$

that is 'almost' β-rich: The near embedding satisfies all the desired properties except for (vi) and (vii). Instead, we only find paths linking the societies of large vortices to arbitrary branch vertices of a large wall. But this can be remedied: We apply the result for $12(2\beta - 1)^2 + \beta$ instead for β and with Lemmas 17 and 19 we obtain a β-rich near-embedding as desired.

First, we will convert the near-embedding of the torso $\hat{G}$ into a near-embedding of the whole graph G by accomodating in to the vortices each part of the graph not yet included in the near-embedding. To accomplish this, we will use property (ii) from Theorem 4 of our tree-decomposition: pick a component T' of $T - t_0$ and let t' be the vertex in this component adjacent to t_0 in T. Let X denote the vertex set for t' as in (ii) and $Y := (\bigcup_{t\in T'} V_t) \setminus \hat{A}$. If (a) holds, we add $G[Y]$ to the vortex $V \in \hat{\mathcal{V}}'$ containing X as a part of its decomposition. This modification neither changes $\hat{G}_0$ nor does it increase the adhesion of V. If (b) holds, $(G[Y], X)$ defines a vortex of length at most 3 which is properly attached to $\hat{G}_0$.

We perform this modification for all components of $T - t_0$. Let us collect in a set $\hat{\mathcal{W}}$ all new vortices defined in case (b), and we let $\hat{\mathcal{V}}$ denote the set of the new, large vortices obtained in case (a). By merging vortices if necessary, we may assume that there exist no two vortices $W, W' \in \hat{\mathcal{W}}$ with $\Omega(W) \subseteq \Omega(W')$. The resulting $\hat{\alpha}$-near embedding $\hat{\varepsilon} := (\hat{\sigma}, \hat{G}_0, \hat{A}, \hat{\mathcal{V}}, \hat{\mathcal{W}})$ is a near-embedding of G.

Now, we are interested in finding near-embeddings $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ into surfaces Σ such that

- *All vortices in $\mathcal{W}$ are properly attached to G_0*
- *All vortices in $\mathcal{V}$ have adhesion at most $\hat{\alpha}$*
- $g(\Sigma) \leq g(\hat{\Sigma})$ $(\star)$
- $|\mathcal{V}| \leq |\hat{\mathcal{V}}| + (g(\hat{\Sigma}) - g(\Sigma))$
- $|A| \leq |\hat{A}| + p(2(g(\hat{\Sigma}) - g(\Sigma)) + |\hat{\mathcal{V}}| - |\mathcal{V}|)$

and further

G_0' *contains a flat wall* Γ_0 *of size at least* $6^{|\mathcal{V}|+2g(\Sigma)+1}\mu.$ $(\star\star)$

where

$$\mu := \mu(\sigma, G_0, A, \mathcal{V}, \mathcal{W}) := (\beta + \hat{\alpha} + 1)p$$

Such near-embeddings exist, as $\hat{\varepsilon}$ satisfies ($\star$) and ($\star\star$).

Our next task is to find, among all such near embeddings, one near-embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ with the following additional properties (P1)–(P4), where

$$\lambda := \lambda(\sigma, G_0, A, \mathcal{V}, \mathcal{W}) := |\mathcal{V}|(\beta + \hat{\alpha} + 1)$$

(P1) Every two vortices have distance at least $2\lambda + 3$ in Σ.

(P2) For every vortex $V \in \mathcal{V}$ there exist λ cycles $(C_1, \ldots, C_\lambda)$ tightly enclosing V. If $\Sigma \not\simeq S^2$, the representativity of G_0' in Σ is at least λ.

(P3) For all distinct vortices $V, W \in \mathcal{V}$, the discs $D(C_1(V))$ and $D(C_1(W))$ are disjoint.

(P4) G_0 contains a flat λ-wall Γ such that $D(\Gamma)$ and $D(C_1(V))$ are disjoint for every $V \in \mathcal{V}$.

From all near-embeddings satisfying ($\star$) and ($\star\star$) let us pick one minimizing $(g(\Sigma), |\mathcal{V}|)$ lexicographically. We will denote this near-embedding by ε. We will show that either ε itself has the properties (P1)–(P4) or we can find a disc in Σ such that, roughly said, the part of our graph nearly-embedded in this disc can be considered as a near-embedding in S^2 with these properties.

For the next steps in the proof, we will repeatedly make use of the following fact: for an integer ℓ, consider a flat wall W of size 6ℓ in G_0' and a vortex $V \in \mathcal{V}$ tightly enclosed by $k < \ell$ cycles $C_1(V), \ldots, C_k(V)$. In W, we can find two subwalls W_1, W_2 of size ℓ and ℓ concentric cycles $C_1(W_1), \ldots, C_\ell(W_1)$ around W_1 and ℓ concentric cycles $C_1(W_2), \ldots, C_\ell(W_2)$ around W_2 such that $D(C_1(W_1))$ and $D(C_1(W_2))$ are disjoint. In particular, W_1 and W_2 have distance at least $2\ell + 2$ in Σ. Further, any two vertices picked from the cycles $C_1(V), \ldots, C_k(V)$ have distance at most 2ℓ in Σ. Now, a comparison of the distances shows that one of the walls W_1, W_2 is disjoint to all the cycles $C_1(V), \ldots, C_k(V)$. Also note, that if we delete a set X of k vertices from a wall Γ of size $\ell > k$, at most k rows and at most k columns of Γ are hit by X and thus, $\Gamma - X$ contains a wall of size at least $\ell - k$.

First, we see that the near-embedding ε has property (P1). Otherwise we apply Lemma 12 with $d := 2\lambda$. This gives a vertex set A' of size at most $2\hat{\alpha} + 2 + d \leq p$ and a near-embedding $\varepsilon' := (\sigma', G_0, A \cup A', \mathcal{V}', \mathcal{W}')$ with $|\mathcal{V}'| \leq |\mathcal{V}| - 1$ of G in Σ. Then, ($\star$) holds for ε' and it is easy to verify ($\star\star$) as well but lexicographically, $(g(\Sigma), |\mathcal{V}'|) < (g(\Sigma), |\mathcal{V}|)$, which contradicts the choice of ε.

To show properties (P2) and (P3) we consider two distinct cases: when $\Sigma \simeq S^2$ and when $\Sigma \not\simeq S^2$.

First, we assume that $\Sigma \simeq S^2$. For a vortex $V \in \mathcal{V}$ we can find a subwall $\Gamma(V)$ of size $6^{|\mathcal{V}|+2g(\Sigma)}\mu$ in Γ_0 together with λ concentric cycles $C_1, \ldots, C_\lambda$ enclosing $V(\Gamma(V))$ such that $D(C_1)$ does not meet $D(V)$. These cycles can be considered as concentric cycles enclosing V. Applying Lemma 11 shows that there also exist λ cycles $C_1(V), \ldots, C_\lambda(V)$ tightly enclosing V, where $D(C_1(V))$ does not meet $D(\Gamma(V))$. This shows (P2).

To prove that (P3) holds, let us assume, to reach a contradiction, that for two distinct vortices $V, W \in \mathcal{V}$ the discs $D(C_1(V))$ and $D(C_1(W))$ intersect. By (P1), all the cycles $C_1(V), \ldots, C_\lambda(V), C_1(W), \ldots, C_\lambda(W)$ are disjoint and we may assume that $D(C_1(V)) \subseteq D(C_1(W))$. By Lemma 15, there is a disc $\Delta \subseteq D(C_1(W))$ containing $D(V)$ and a separation (X_1, X_2) of G of order at most $2\lambda + 2\alpha \leq p$ such that $X_1 \cap \Delta = G_0 \cap \Delta$. Now, let $\tilde{\mathcal{W}}$ be the set of all vortices of $\mathcal{W} - X_1$ with a non-empty society plus the vortex $(G[X_1], \emptyset)$, and similar, let $\tilde{\mathcal{V}}$ be the set of all vortices of $\mathcal{V} - X_1$ with a non-empty society. Clearly, $|\tilde{V}| < |\mathcal{V}|$ as $\Omega(V) \subseteq X_1$. It is easy to see now that

$$(\sigma|_{G_0 - X_1}, G_0 - X_1, A \cup (X_1 \cap X_2), \tilde{\mathcal{V}}, \tilde{\mathcal{W}})$$

is a near-embedding of G into Σ, satisfying $(\star)$, and with $\Gamma(V)$ a sufficient large wall lives in $G_0 - X_1$ such that $(\star\star)$ holds as well. Otherwise, we have that $(g(\Sigma), |\tilde{V}|) < (g(\Sigma), |\mathcal{V}|)$, a contradiction to our choice of ε. This proves (P3).

We now consider the case when $\Sigma \not\approx S^2$. Our plan is to apply Lemma 9. Thus, we must first show that the representativity of G_0' in Σ is at least $2\lambda + 2$. Assume, to reach a contradiction, that the representativity is at most $2\lambda + 2$. We can apply Lemma 14 with $z := 2\lambda + 2$. If a) from Lemma 14 holds, we have a near-embedding ε' of G into a surface Σ' with $g(\Sigma') < g(\Sigma)$ and $|\mathcal{V}'| \leq |\mathcal{V}| + 1$. The properties $(\star)$ and $(\star\star)$ are easy to verify and thus, $(g(\Sigma'), |\mathcal{V}'|) < (g(\Sigma), |\mathcal{V}|)$ is a contradiction to our choice of ε. If b) holds, one of the graphs ${G'}_0^1, {G'}_0^2$ contains a sufficient large wall and therefore, one of the near-embeddings $\varepsilon_1, \varepsilon_2$ satisfies the conditions $(\star)$ and $(\star\star)$. Again, as $g(\Sigma_1), g(\Sigma_2) < g(\Sigma)$, this is a contradiction to our choice of ε. This shows that the representativity of G_0' in Σ is at least $2\lambda + 2$ and we apply Lemma 9 to each of the faces of G_0' that contain the society $\Omega(V)$ of one vortex $V \in \mathcal{V}$. Together with Lemma 11, this implies property (P2).

As before, to show property (P3), we assume that for two distinct vortices $V, W \in \mathcal{V}$, the discs $D(C_1(V))$ and $D(C_1(W))$ intersect. Again we may assume that $D(C_1(V)) \subseteq D(C_1(W))$ and application of Lemma 15 gives us a disc $\Delta \subseteq D(C_1(W))$ containing $D(V)$ and a separation (X_1, X_2) of G of order at most $2\lambda + 2\alpha \leq p$ such that $X_1 \cap \Delta = G_0 \cap \Delta$. As noted earlier, there exists a flat subwall Γ of Γ_0 of size at least $6^{|\mathcal{V}| + 2g(\Sigma)}\mu$ that is disjoint to all the cycles $(C_1(W), \ldots, C_\lambda(W))$. If $\Gamma \subseteq D(C_1(W)) \setminus \Delta$, we can reduce the number of vortices of our near-embedding, leading to the same contradiction as above. Otherwise, $\Gamma \subseteq \Delta$. In this case, let $\tilde{\mathcal{W}}$ be the set of all vortices of $\mathcal{W} - X_2$ with a non-empty society plus the vortex $(G[X_2], \emptyset)$, and similar, let $\tilde{\mathcal{V}}$ be the set of all vortices of $\mathcal{V} - X_2$ with a non-empty society. Now,

$$(\sigma|_{G_0 - X_2}, G_0 - X_2, A \cup (X_1 \cap X_2), \tilde{\mathcal{V}}, \tilde{\mathcal{W}})$$

can be considered a near-embedding of G into S^2. Property $(\star\star)$ is clearly true and one easily checks $(\star)$ as well. This is a contradiction to our choice of ε as $(g(S^2), |\mathcal{V}'|) < (g(\Sigma), |\mathcal{V}|)$. We conclude that property (P3) holds.

Let us enumerate the vortices $\mathcal{V} =: \{V_1, \ldots, V_\ell\}$. We will show (P4) by proving inductively that for $1 \leq k \leq \ell$, there is a flat wall $\Gamma_k \subseteq \Gamma_{k-1}$ of size

$$s(k) := 6^{|\mathcal{V}| - k + 2g(\Sigma) + 1}\mu$$

avoiding $D(C_1(V_1)), \dots, D(C_1(V_k))$. By ($\star\star$), there is a flat $s(0)$-wall in G_0. Let us assume that the statement is true for some k with $1 \leq k < \ell$. With the same arguments as before, we find a subwall $\Gamma_{k+1} \subseteq \Gamma_k$ of size $s(k+1)$ that avoids all cycles $C_1(V_k), \dots, C_\lambda(V_k)$.

If $\Gamma_{k+1} \subseteq \Sigma \setminus D(C_1(V_k))$, the statement is true for $k+1$, completing the induction.

Otherwise, Lemma 15 gives us a disc Δ containing Γ_{k+1} and a separation (X_1, X_2) of G of order at most $2\lambda + 2\alpha \leq p$ such that $X_1 \cap \Delta = G_0 \cap \Delta$. By (P3), this disc Δ does not contain any society $\Omega(W)$ for some large vortex $W \neq V$. We let $\tilde{\mathcal{W}}$ be the set of all vortices of $\mathcal{W} - X_2$ with a non-empty society plus the vortex $(G[X_2], \emptyset)$, and $\tilde{V} := V - X_2$. We can consider

$$(\sigma|_{G_0 - X_2}, G_0 - X_2, A \cup (X_1 \cap X_2), \{\tilde{V}\}, \tilde{\mathcal{W}})$$

as a near-embedding of G into S^2, and with similar arguments as before, we can use Γ_{k+1} to find both a wall of size at least λ together with λ concentric cycles tighly enclosing $\tilde{V}$ in S^2 such that all of ($\star$),($\star\star$) and (P1)–(P4) hold.

From all near-embeddings $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of G into surfaces Σ satisfying ($\star$) and (P1)–(P4) let us choose one minimizing $|\mathcal{V}|$.

There is no separation (X_1, X_2) of G_0' of order less than $\beta + \hat{\alpha} + 1$ such that all the branch vertices of Γ are contained in X_1 and $\Omega(V) \subseteq X_2$ for one vortex $V \in \mathcal{V}$. Assume the opposite. Then, we may assume that for every vortex $V' \in \mathcal{V}$, the society vertices $\Omega(V')$ are contained either in X_1 or in X_2: Otherwise, one of the cycles $C_1(V'), \dots, C_\lambda(V')$ enclosing V' is not hit by $X_1 \cap X_2$ and is therefore contained in either X_1 or X_2. As this (planar) cycle C is a separator of G_0', we can add all vertices embedded in $D(C)$ to one part of the separation. Similar as before, we add $X_1 \cap X_2$ to the apex set A and add a new small vortex $W := (G[X_2], \emptyset)$ to $\mathcal{W}$. This new near-embedding of G still satisfies ($\star$) and (P1)–(P4) but we have reduced $|\mathcal{V}|$ with this operation, a contradiction to our choice of the near-embedding. Therefore, for every large vortex $V \in \mathcal{V}$, the society $\Omega(V)$ is connected to the branch vertices of Γ by $\beta + \hat{\alpha} + 1$ many disjoint paths.

Application of Lemma 13 gives a subgraph $\tilde{G}_0$ of G_0, a vertex set $\tilde{A} \subseteq V(G)$ with $|\tilde{A}| \leq 2\hat{\alpha}^2$ and a near-embedding $\tilde{\varepsilon} := (\tilde{\sigma}, \tilde{G}_0, A_0, \tilde{\mathcal{V}}, \tilde{\mathcal{W}})$ of G such that every vortex $V \in \tilde{\mathcal{V}}$ has a linked decomposition of adhesion at most $\hat{\alpha}$ and there are still at least $|\mathcal{V}|\beta$ cycles enclosing every $\tilde{V} \in \tilde{\mathcal{V}}$. Further, of the $\hat{\alpha} + \beta + 1$ paths linking ΩV to the wall for the vortex $V \in \mathcal{V}$ contained in $\tilde{V}$, at least β are not hit by the set of vertices of size at most $\hat{\alpha}$ that was deleted by the application of the lemma. Thus, $\Omega(\tilde{V})$ is still linked to the wall by at least β many paths. Finally, as described in the beginning, Lemma 17 finishes the proof. $\square$

8 Circular Vortex-Decompositions

In Graph Minors XVII [15], the structure theorem is stated with vortices having a circular instead of a linear structure. For most applications, the linear decompositions as discussed so far in this paper are sufficient, but sometimes the circular structure is necessary. In this section, we introduce circular vortex decompositions and point out how we can derive a new lemma from the proof of Lemma 13 that yields circular linkages for them. It is easy to see that we can apply this new lemma instead of Lemma 13 at the end of the proof of Theorem 1 and therefore, we can choose to have circular linkages for the large vortices when we apply the theorem.

For the remainder of this paper, we call decompositions of vortices as defined in Section 2 *linear decompositions* to distinguish them more clearly from the circular decompositions which we introduce now:

Let $V := (G, \Omega)$ be a vortex with $\Omega = (w_1, \ldots, w_n)$. Let us regard the ordering of Ω as a cyclic ordering. A tuple $\mathcal{D} := (X_1, \ldots, X_n)$ of subsets of $V(G)$ is a *circular decomposition* of V if the following properties are satisfied:

(i) $w_i \in X_i$ for all $1 \leq i \leq n$.

(ii) $X_1 \cup \ldots \cup X_n = V(G)$.

(iii) When $w_i < w_j < w_k < w_\ell$ are society vertices of V ordered with respect to the cyclic ordering Ω, then $X_i \cap X_k \subseteq X_j \cup X_\ell$

(iv) Every edge of G has both ends in X_i for some $1 \leq i \leq n$.

The *adhesion* of our circular decomposition $\mathcal{D}$ of V is the maximum value of $|X_{i-1} \cap X_i|$, taken over all $1 \leq i \leq n$. We define the *circular adhesion* of V as the minimum adhesion of a circular decomposition of that vortex.

When $\mathcal{D}$ is a circular decomposition of a vortex V as above, we write $Z_i := (X_i \cap X_{i+1}) \setminus \Omega$, for all $1 \leq i < n$. These Z_i are the *adhesion sets* of $\mathcal{D}$. We call $\mathcal{D}$ *linked* if

- all these Z_i have the same size;
- there are $|Z_i|$ disjoint Z_{i-1}–Z_i paths in $G[X_i] - \Omega$, for all $1 \leq i \leq n$;
- $X_i \cap \Omega = \{w_{i-1}, w_i\}$ for all $1 \leq i \leq n$.

Note that $X_i \cap X_{i+1} = Z_i \cup \{w_i\}$, for all $1 \leq i \leq n$ (Fig. 1).

The union of the Z_{i-1}–Z_i paths in a circular decomposition of V is a disjoint union of cycles in G each of which traverses the adhesion sets of $\mathcal{D}$ in cyclic order (possibly several times); We call the set of these cycles a *circular linkage* of V with respect to $\mathcal{D}$.

As described in Section 2 for linear decompositions, we see that we can delete a vertex from a circular decomposition of some vortex and obtain a new circular decomposition. This operation does not increase the adhesion but might decrease the number of society vertices.

Clearly, a linear decomposition of some vortex is a circular decomposition as well and it is easy to see that one can obtain a linear from a circular decomposition, if one deletes the overlap of two subsequent bags: Let $V := (G, \Omega)$ a vortex and $(X_1, \ldots, X_n)$ a circular decomposition of V. Delete the set $X_{i-1} \cap X_i$ from V for some index $1 \leq i \leq n$. We obtain a circular decomposition $\mathcal{D} := (X'_1, \ldots, X'_{n'})$ of $V - Z$ with $n' \leq n$. By shifting the indices if necessary we may assume that $X'_{n'} \cap X'_1$ is empty. $\mathcal{D}$ is a linear decomposition of $V - Z$: Pick a vertex $v \in X'_j \cap X'_\ell$ for indices $1 \leq j < \ell \leq n'$. This vertex avoids either X_1 or X_n, let us assume the former. We apply property (iii) from the definition of a circular decomposition to w_1, w_j, w_k, w_ℓ for any k with $j \leq k \leq \ell$ and conclude that $v \in X'_k$.

To distinguish near-embeddings with linear decompositions from near-embeddings with circular decompositions, we will call the latter explicitly *near-embeddings with circular vortices*. Also, for a $(\alpha_0, \alpha_1, \alpha_2)$-near embedding with circular vortices let the third bound α_2 denote an upper bound for the circular adhesion of the large vortices.

We give a modified definition of β-rich to comply with the new concepts. For near-embeddings with circular decompositions we replace property (v) by the following:

(v') For every vortex $V \in \tilde{\mathcal{V}}$ there exists a circular, linked decomposition $\mathcal{D}$ of V of adhesion at most α_2 and there exists a cycle C in $V \cup \bigcup \mathcal{W}$ with $V(C \cap G_0) = \Omega(V)$, avoiding all the paths of a circular linkage of V and traversing $w_1, \ldots, w_n$ in their order.

Lemma 21. *Let $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ be an $(\alpha_0, \alpha_1, \alpha_2)$-near embedding of some graph G in a surface Σ such that the following statements hold:*

(i) For every vortex $V \in \mathcal{V}$ there are α_2+1 concentric cycles $C_0(V), \ldots, C_{\alpha_2}(V)$ in G'_0 tightly enclosing V.

(ii) For distinct vortices $V, W \in \mathcal{V}$, the discs $D(C_0(V))$ and $D(C_0(W))$ are disjoint.

Then, there is a graph $\tilde{G}_0$ with

$$G_0 \setminus \Big(\bigcup_{V \in \mathcal{V}} D(C_0(V)) \Big) \subseteq \tilde{G}_0,$$

a set $\tilde{A} \subseteq V(G)$ with $|\tilde{A}| \leq \tilde{\alpha} := \alpha_0 + \alpha_1(\alpha_2 + 2)$ avoiding $\tilde{G}_0$, and sets $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ of vortices in G such that with $\tilde{\sigma} := \sigma|_{\tilde{G}'_0}$ the tuple $(\tilde{\sigma}, \tilde{G}_0, A \cup \tilde{A}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}})$ is an $(\tilde{\alpha}, \alpha_1, \alpha_2 + 1)$-near embedding with circular vortices of G in Σ such that every vortex $V \in \mathcal{V}$ satisfies property (v') of β-rich.

Proof. This lemma can be proven almost exactly like Lemma 13. To avoid completely rewriting the proof, we just point out the differences.

The curve C in the surface hits the vertex set S which consists of exactly one vertex from each $C_i(V)$ and one society vertex w'_j of V. We split each vertex in $S \setminus \{w'_j\}$: For each $0 \leq i \leq \alpha_2$, we replace $v \in S \cap V(C_i(V))$ by two new vertices x_i, y_i and connect them with edges to the former neighbours of v such that C does not intersect any edges or vertices. The vertices $x_0, \ldots, x_{\alpha_2}$ and $y_0, \ldots, y_{\alpha_2}$ form the sets X and Y, respectively.

In the remainder of the proof we delete the set Z instead of $S \cup Z$. At the end, we identify the vertex pairs (x_i, y_i) for $0 \leq i \leq \alpha_2$ and obtain a linked, circular decomposition as desired. $\square$

References

[1] T. Böhme, K. Kawarabayashi, J. Maharry, B. Mohar, Linear connectivity forces large complete bipartite minors, J. Comb. Theory Ser. B 99 (2009), 557–582.

[2] E. D. Demaine, F. Fomin, M. Hajiaghayi, and D. Thilikos, Subexponential parameterized algorithms on bounded-genus graphs and H-minor-free graphs, *J. ACM* **52** (2005), 1–29.

[3] E. D. Demaine, M. Hajiaghayi, and K. Kawarabayashi, Algorithmic graph minor theory: Decomposition, approximation and coloring, Proc. 46th Ann. IEEE Symp. Found. Comp. Sci., Pittsburgh, PA, 2005, pp. 637–646.

[4] E. D. Demaine, M. Hajiaghayi, and B. Mohar, Approximation algorithms via contraction decomposition, *Proc. 18th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'07)*, 278–287, (2007).

[5] E. Demaine, M. Hajiaghayi, "Linearity of Grid Minors in Treewidth with Applications through Bidimensionality", Combinatorica, 28 (2008), 19–36

[6] R. Diestel, "Graph Theory", 3rd ed. Springer-Verlag, 2005

[7] R. Diestel, K. Kawarabayashi, P. Wollan, The Erdős-Pósa property for clique minors in highly connected graphs, manuscript 2009.

[8] T. Hyuhn, Dissertation, University of Waterloo, 2009.

[9] K. Kawarabayashi, S. Norine, R. Thomas, P. Wollan, K_6 minors in large 6-connected graphs, manuscript 2009.

[10] K. Kawarabayashi and B. Mohar, Approximating the chromatic number and the list-chromatic number of minor-closed family of graphs and odd-minor-closed family of graphs, *Proc. the 38th ACM Symposium on Theory of Computing (STOC'06)*, (2006), 401–416.

[11] B. Mohar, C. Thomassen, "Graphs on Surfaces", Johns Hopkins University Press, 2001

[12] N. Robertson, P. Seymour, "Graph minors. V: Excluding a Planar Graph", J. Comb. Theory Ser. B 41 (1986), 92–114.

[13] N. Robertson, P. Seymour, "Graph minors. X: Obstructions to tree-Decomposition", J. Comb. Theory Ser. B 52 (1991), 153–190.

[14] N. Robertson, P. Seymour, "Graph minors. XVI: Excluding a non-planar graph", J. Comb. Theory Ser. B 89 (2003), 43–76.

[15] N. Robertson, P. Seymour, "Graph minors. XVII: Taming a Vortex", J. Comb. Theory Ser. B 77 (1999), 162–210.

Version 30 Sept 2009

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Research partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research, by C & C Foundation, by Kayamori Foundation and by Inoue Research Award for Young Scientists.

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