HAMBURGER BEITRÄGE ZUR MATHEMATIK

Heft 353

Extremal results for random discrete structures

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Version Oktober 2009

EXTREMAL RESULTS FOR RANDOM DISCRETE STRUCTURES

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Abstract. We study thresholds for extremal properties of random discrete structures. We determine the threshold for Szemerédi's theorem on arithmetic progressions in random subsets of the integers and its multidimensional extensions and we determine the threshold for Turán-type problems for random graphs and hypergraphs. In particular, we verify a conjecture of Kohayakawa, Łuczak, and Rödl for Turán-type problems in random graphs. Similar results were obtained by Conlon and Gowers.

1. Introduction

Extremal problems are widely studied in discrete mathematics. Given a finite set Γ and a family $\mathcal{F}$ of subsets of Γ an *extremal result* asserts that any sufficiently large (or dense) subset $G \subseteq \Gamma$ must contain an element from $\mathcal{F}$. Often all elements of $\mathcal{F}$ have the same size, i.e., $\mathcal{F} \subseteq \binom{\Gamma}{k}$ for some integer k, where $\binom{\Gamma}{k}$ denotes the family of all k-element subsets of Γ.

For example, if $\Gamma_n = [n] = \{1, \dots, n\}$ and $\mathcal{F}_n$ consists of all k-element subsets of $[n]$ which form an arithmetic progression, then Szemerédi's celebrated theorem [40] asserts that every subset $Y \subseteq [n]$ with $|Y| = \Omega(n)$ contains an arithmetic progression of length k.

A well known result from graph theory, which fits this framework, is Turán's theorem [41] and its generalization due to Erdős and Stone [12] (see also [10]). Here $\Gamma_n = E(K_n)$ is the edge set of the complete graph with n vertices and $\mathcal{F}_n$ consists of the edge sets of copies of some fixed graph F (say with k edges) in K_n. Here the Erdős-Stone theorem implies that every subgraph $H \subseteq K_n$ which contains at least

$$\left(1 + \frac{1}{\chi(F) - 1} + o(1)\right) \binom{n}{2}$$

edges must contain a copy of F, where $\chi(F)$ denotes the chromatic number of F (see, e.g., [2, 3, 5, 7]).

We are interested in "random versions" of such extremal results. We study the *binomial model* of random substructures. For a finite set Γ_n and a probability $p \in [0, 1]$ we denote by $\Gamma_{n,p}$ the random subset where every $x \in \Gamma_n$ is included in $\Gamma_{n,p}$ independently with probability p. In other words, $\Gamma_{n,p}$ is the finite probability space on the power set of Γ_n in which every elementary event $\{G\}$ for $G \subseteq \Gamma_n$

Date: October 23, 2009.

2000 *Mathematics Subject Classification.* 05C80 (11B25, 05C35).

Key words and phrases. Szemerédi's theorem, Turán's theorem, Erdős-Stone theorem, random sets, thresholds.

occurs with probability

$$\mathbb{P}\left(G = \Gamma_{n,p}\right) = p^{|G|}(1-p)^{|\Gamma_n|-|G|}\,.$$

For example, if Γ_n is the edge set of the complete graph on n vertices, then $\Gamma_{n,p}$ denotes the usual binomial random graph $G(n,p)$ (see, e.g., [4, 25]).

The deterministic extremal results mentioned earlier can be viewed as statements which hold with probability 1 for $p = 1$ and it is natural to investigate the asymptotic of the smallest probabilities for which those results hold. In the context of Szemerédi's theorem for every $k \geq 3$ and $\varepsilon > 0$ we are interested in the smallest sequence $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ of probabilities such that the binomial random subset $[n]_{p_n}$ has asymptotically almost surely (a.a.s., i.e. with probability tending to 1 as $n \to \infty$) the following property: Every subset $Y \subseteq [n]_{p_n}$ with $|Y| \geq \varepsilon|[n]_{p_n}|$ contains an arithmetic progression of length k. Similarly, in the context of the Erdős-Stone theorem, for every graph F and $\varepsilon > 0$ we are interested in the asymptotic of the smallest sequence $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ such that the random graph $G(n,p_n)$ a.a.s. satisfies: every $H \subseteq G(n,p)$ with

$$e(H) \geq \left(1 - \frac{1}{\chi(F)-1} + \varepsilon\right) e(G(n,p_n))\,,$$

contains a copy of F.

We determine the asymptotic growth of the smallest such sequence $\boldsymbol{p}$ of probabilities for those and some related extremal properties including multidimensional versions of Szemerédi's theorem (Theorem 2.3), solutions of density regular systems of equations (Theorem 2.4), an extremal version for solutions of the Schur equation (Theorem 2.5), and extremal problems for hypergraphs (Theorem 2.7). In other words, we determine the *threshold* for those properties. Similar results were obtained by Conlon and Gowers [6].

The new results will follow from a general result (see Theorem 3.3), which allows us to transfer certain extremal results from the classical deterministic setting to the probabilistic setting. In Section 4 we deduce the results stated in the next section from Theorem 3.3.

2. New results

2.1. **Szemerédi's theorem and its multidimensional extension.** We study extremal properties of random subsets of the first n positive integers. One of the best known extremal-type results for the integers is Szemerédi's theorem. In 1975 Szemerédi solved a longstanding conjecture of Erdős and Turán [13] by showing that every subset of the integers of upper positive density contains a arithmetic progression of any finite length. For a set $X \subseteq [n]$ we write

$$X \to_\varepsilon [k] \tag{1}$$

for the statement that every subsets $Y \subseteq X$ with $|Y| \geq \varepsilon|X|$ contains an arithmetic progression of length k. With this notation at hand, we can state (the finite version of) Szemerédi's theorem as follows: for every integer $k \geq 3$ and $\varepsilon > 0$ there exists n_0 such that for every $n \geq n_0$ we have $[n] \to_\varepsilon [k]$.

For fixed $k \geq 3$ and $\varepsilon > 0$ we are interested in the asymptotic behavior of the threshold sequence of probabilities $\boldsymbol{p} = (p_n)$ such that there exist constants

$0 < c < C$ for which

$$\lim_{n\to\infty} \mathbb{P}\left([n]_{q_n} \to_\varepsilon [k]\right) = \begin{cases} 0, & \text{if } q_n \le cp_n \text{ for all } n \in \mathbb{N}, \\ 1, & \text{if } q_n \ge Cp_n \text{ for all } n \in \mathbb{N}. \end{cases} \tag{2}$$

Remark 2.1. We note that the family $\{X \subseteq [n]\colon X \to_\varepsilon [k]\}$ is not closed under supersets. In other words, the property is "$X \to_\varepsilon [k]$" is not a monotone property. However, similar arguments as presented in [25, Proposition 8.6] show that the property "$X \to_\varepsilon [k]$" and the other properties considered in this section have a threshold as displayed in 2.

It is easy to see that if the expected number of arithmetic progressions of length k in $[n]_{q_n}$ is asymptotically smaller than the expected number of elements in $[n]_{q_n}$, then there exists a subset of size $(1 - o(1))|[n]_{q_n}|$, which contains no arithmetic progressions of length k at all. In other word, if

$$q_n^k n^2 \ll q_n n \qquad \Leftrightarrow \qquad q_n \ll n^{-1/(k-1)} \tag{3}$$

then $\mathbb{P}\left([n]_{q_n} \to_\varepsilon [k]\right) \to 0$ for every $\varepsilon < 1$. Consequently, $n^{-1/(k-1)}$ is a lower bound on the threshold for Szemerédi's theorem for arithmetic progressions of length k. For $k = 3$ Kohayakawa, Łuczak, and Rödl [28] established a matching upper bound. Our first result generalizes this for arbitrary $k \ge 3$.

Theorem 2.2. *For every integer $k \ge 3$ and every $\varepsilon \in (0, 1)$ there exit constants $C > c > 0$ such that for any sequence of probabilities $\boldsymbol{q} = (q_n)_{n\in\mathbb{N}}$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left([n]_{q_n} \to_\varepsilon [k]\right) = \begin{cases} 0, & \text{if } q_n \le cn^{-1/(k-1)} \text{ for all } n \in \mathbb{N}, \\ 1, & \text{if } q_n \ge Cn^{-1/(k-1)} \text{ for all } n \in \mathbb{N}. \end{cases}$$

We remark that the 0-statement in Theorem 2.2 (and, similarly, the 0-statements of the other results of this section) follows from standard probabilistic arguments. The 1-statement of Theorem 2.2 follows from our main result, Theorem 3.3.

A multidimensional version of Szeméredi's theorem was obtained by Furstenberg and Katznelson [18]. Those authors showed that for every integer ℓ, every finite subset $F \subset \mathbb{N}^\ell$ and every $\varepsilon > 0$ there exists some integer n_0 such that for $n \ge n_0$ every $Y \subseteq [n]^\ell$ with $|Y| \ge \varepsilon n^\ell$ contains a homothetic copy of F, i.e., there exist some $y_0 \in \mathbb{N}^\ell$ and $\lambda \ne 0$ such that $y_0 + \lambda F = \{y_0 + \lambda f\colon f \in F\} \subseteq Y$. Clearly, the case $\ell = 1$ and $F = [k]$ resembles Szemerédi's theorem. Generalizing the notation introduced in (1), for sets X, $F \subseteq \mathbb{N}^\ell$ and $\varepsilon > 0$ we write $X \to_\varepsilon F$, if every subset $Y \subseteq X$ with $|Y| \ge \varepsilon|X|$ contains a homothetic copy of F.

A simple heuristic, similar to the one in the context of Szeméredi's theorem, suggests that $n^{-1/(|F|-1)}$ is a lower bound on the threshold for the Furstenberg-Katznelson theorem for a configuration $F \subseteq \mathbb{N}^\ell$ in the binomial random subset $[n]^\ell_p$ where elements of $[n]^\ell$ are included with probability p. Our next result shows that, in fact, this gives the correct asymptotic for the threshold.

Theorem 2.3. *For every integer $\ell \ge 1$, every configuration $F \subseteq \mathbb{N}^\ell$ with $|F| \ge 3$, and every $\varepsilon \in (0, 1)$ there exit constants $C > c > 0$ such that for any sequence of probabilities $\boldsymbol{q} = (q_n)_{n\in\mathbb{N}}$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left([n]^\ell_{q_n} \to_\varepsilon F\right) = \begin{cases} 0, & \text{if } q_n \le cn^{-1/(|F|-1)} \text{ for all } n \in \mathbb{N}, \\ 1, & \text{if } q_n \ge Cn^{-1/(|F|-1)} \text{ for all } n \in \mathbb{N}. \end{cases}$$

2.2. **Density regular matrices.** Another extension of Szemerédi's theorem leads to the notion of *density regular* matrices. Arithmetic progressions of length k can be viewed as the set of distinct-valued solutions of the following homogeneous system of $k-2$ linear equations

$$\begin{array}{ccccccc} x_1 & - & 2x_2 & + & x_3 & = & 0\,, \\ x_2 & - & 2x_3 & + & x_4 & = & 0\,, \\ \vdots & & \vdots & & \vdots & & \vdots \\ x_{k-2} & - & 2x_{k-1} & + & x_k & = & 0\,. \end{array}$$

More generally, for an $\ell \times k$ integer matrix A let $\mathcal{S}(A) \subseteq \mathbb{R}^k$ be the set of solutions of the homogeneous system of linear equations given by A. Let $\mathcal{S}_0(A) \subseteq \mathcal{S}(A)$ be those solutions $(x_1, \dots, x_k)$ with all x_i being distinct. We say A is *irredundant* if $\mathcal{S}_0(A) \neq \emptyset$. Moreover, an irredundant $\ell \times k$ integer matrix A is *density regular*, if for every $\varepsilon > 0$ there exists an n_0 such that for all $n \geq n_0$ and every $Y \subseteq [n]$ with $|Y| \geq \varepsilon n$ we have $Y^k \cap \mathcal{S}_0(A) \neq \emptyset$. Szemerédi's theorem, for example, implies that the following $(k-2) \times k$ matrix

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{pmatrix} \tag{4}$$

is density regular for any $k \geq 3$.

Density regular matrices are a subclass of so-called partition regular matrices. This class was studied and characterized by Rado [34] and, for example, it follows from this characterization that $k \geq \ell + 2$ (see [22] for details). In [14] Frankl, Graham, and Rödl characterized irredundant, density regular matrices, being those partition regular matrices A for which $(1, 1, \dots, 1) \in \mathcal{S}(A)$.

Similar as in the context of Theorem 2.2 and Theorem 2.3 the following notation will be useful. For an irredundant, density regular, $\ell \times k$ integer matrix A, $\varepsilon > 0$, and $X \subseteq [n]$ we write $X \to_\varepsilon A$ if for every $Y \subseteq X$ with $|Y| \geq \varepsilon |X|$ we have $Y^k \cap \mathcal{S}_0(A) \neq \emptyset$. The following parameter in connection with Ramsey properties of random subsets of the integers with respect to irredundant, partition regular matrices was introduced by Rödl and Ruciński [36].

Let A be an $\ell \times k$ integer matrix and let the columns be indexed by $[k]$. For a partition $W \dot\cup \overline{W} \subseteq [k]$ of the columns of A, we denote by $A_{\overline{W}}$ the matrix obtained from A by restricting to the columns indexed by $\overline{W}$. Let $\operatorname{rank}(A_{\overline{W}})$ be the rank of $A_{\overline{W}}$, where $\operatorname{rank}(A_{\overline{W}}) = 0$ for $\overline{W} = \emptyset$. We set

$$m(A) = \max_{\substack{W \dot\cup \overline{W} = [k] \\ |W| \geq 2}} \frac{|W| - 1}{|W| - 1 + \operatorname{rank}(A_{\overline{W}}) - \operatorname{rank}(A)}\,. \tag{5}$$

It was shown in [36, Proposition 2.2 (ii)] that for irredundant, partition regular matrices A the denominator of (5) is always at least 1. For example, for A given in (4) we have $m(A) = k - 1$.

It follows from the 0-statement of Theorem 1.1 in [36] that for any irredundant, density regular, $\ell \times k$ integer matrix A of rank ℓ and every $1/2 > \varepsilon > 0$ there exist a $c > 0$ such that for every sequence of probabilities $\boldsymbol{q} = (q_n)$ with $q_n \leq cn^{-1/m(A)}$

we have

$$\lim_{n\to\infty} \mathbb{P}\left([n]_{q_n} \to_\varepsilon A\right) = 0\,. \tag{6}$$

We deduce a corresponding upper bound from Theorem 3.3 and obtain the following result.

Theorem 2.4. *For every irredundant, density regular, $\ell \times k$ integer matrix A with rank ℓ, and every $\varepsilon \in (0, 1/2)$ there exit constants $C > c > 0$ such that for any sequence of probabilities $\boldsymbol{q} = (q_n)_{n\in\mathbb{N}}$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left([n]_{q_n} \to_\varepsilon A\right) = \begin{cases} 0, & \text{if } q_n \le cn^{-1/m(A)} \text{ for all } n \in \mathbb{N}, \\ 1, & \text{if } q_n \ge Cn^{-1/m(A)} \text{ for all } n \in \mathbb{N}. \end{cases}$$

Note that we restrict $\varepsilon < 1/2$ here. With this restriction the 0-statement will follow from a result of Rödl and Ruciński from [36]. The proof of the 1-statement presented in Section 4.2 actually works for all $\varepsilon \in (0, 1)$.

2.3. **An extremal problem related to Schur's equation.** In 1916 Schur [38] showed that every partition of the positive integers into finitely many classes contains a class which contains a solution of the single, homogeneous equation $x_1 + x_2 - x_3 = 0$. Clearly, the corresponding matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ is not density regular, since the set of all odd integers contains no solution. However, it is not hard to show that that every subset $Y \subseteq [n]$ with $|Y| \ge (1/2 + o(1))n$ contains such a solution. Similarly, as above for $\varepsilon > 0$ and $X \subseteq [n]$ we write

$$X \to_{1/2+\varepsilon} \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$$

if every subset $Y \subseteq X$ with $|Y| \ge (1/2 + \varepsilon)|X|$ contains a distinct-valued solution, i.e., $Y^k \cap \mathcal{S}_0\left(\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}\right) \neq \emptyset$.

We are interested in the threshold for the extremal problem of Schur's equation, i.e., for the property $X \to_{1/2+\varepsilon} \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$. In this context the simple heuristic based on the expected number of solutions of the Schur equation in random subsets of the integers suggests that $n^{-1/2}$ is the threshold for this property. Moreover, for Schur's theorem in random subsets of the integers the threshold turned out to be $n^{-1/2}$ as shown in [21, 16]. We show that the threshold of the extremal version of Schur's equation is the same.

Theorem 2.5. *For every $\varepsilon \in (0, 1/2)$ there exit constants $C > c > 0$ such that for any sequence of probabilities $\boldsymbol{q} = (q_n)_{n\in\mathbb{N}}$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left([n]_{q_n} \to_{1/2+\varepsilon} \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}\right) = \begin{cases} 0, & \text{if } q_n \le cn^{-1/2} \text{ for all } n \in \mathbb{N}, \\ 1, & \text{if } q_n \ge Cn^{-1/2} \text{ for all } n \in \mathbb{N}. \end{cases}$$

2.4. **Extremal problems for hypergraphs.** The last result deals with extremal problems for hypergraphs. An ℓ-uniform hypergraph H is a pair (V, E), where the vertex set V is some finite set and the edge set $E \subseteq \binom{V}{\ell}$ is a subfamily of the ℓ-element subsets of V. As usual we call 2-uniform hypergraphs simply graphs. For some hypergraph H we denote by $V(H)$ and $E(H)$ its vertex set and its edge set and we denote by $v(H)$ and $e(H)$ the cardinalities of those sets. For an integer n we denote by $K_n^{(\ell)}$ the complete ℓ-uniform hypergraph on n vertices, i.e., $v(K_n^{(\ell)}) = n$ and $e(K_n^{(\ell)}) = \binom{n}{\ell}$. An ℓ-uniform hypergraph H' is a sub-hypergraph of H, if $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$ and we write $H' \subseteq H$ to denote that. For a subset $U \subseteq V(H)$ we denote by $E(U)$ the edges of H contained in U and we set

$e(U) = |E(U)|$. Moreover, we write $H[U]$ for the sub-hypergraph induced on U, i.e., $H[U] = (U, E(U))$.

For two ℓ-uniform hypergraphs F and H we say H contains a copy of F, if there exists an injective map $\varphi\colon V(F) \to V(H)$ such that $\varphi(e) \in E(H)$ for every $e \in E(F)$. If H contains no copy of F, then we say H is *F-free*. We denote by $\mathrm{ex}(H, F)$ the maximum number of edges of an F-free sub-hypergraph of H, i.e.,

$$\mathrm{ex}(H, F) = \max\{e(H')\colon H' \subseteq H \text{ and } H' \text{ is } F\text{-free}\}\,.$$

Mantel [33], Erdős [8], and Turán [41] were the first to study this function for graphs. In particular, Turán determined $\mathrm{ex}(K_n, K_k)$ for all integers n and k. This line of research was continued by Erdős and Stone [12] and Erdős and Simonovits [10] and those authors showed that for every graph F with chromatic number $\chi(F) \geq 3$ we have

$$\mathrm{ex}(K_n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right)\binom{n}{2}, \tag{7}$$

where $\chi(F)$ is minimum number r such that there exists a partition $V_1 \dot\cup \ldots \dot\cup V_r = V(F)$ such that $E(V_i) = \emptyset$ for every $i \in [r]$. Moreover, it follows from the result of Kövari, Sós, and Turán [31] that

$$\mathrm{ex}(K_n, F) = o(n^2) \tag{8}$$

for graphs F with $\chi(F) \leq 2$.

For an ℓ-uniform hypergraph F we define the *Turán density*

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(K_n^{(\ell)}, F}{\binom{n}{\ell}}\,.$$

For a graph F the Turán density $\pi(F)$ is determined due to (7) and (8). For hypergraphs (8) was extended by Erdős [9] to ℓ-partite, ℓ-uniform hypergraphs. Here an ℓ-uniform hypergraph F is ℓ-partite if its vertex set can be partitioned into ℓ classes, such that every edge intersects every partition class in precisely one vertex. Erdős showed that $\pi(F) = 0$ for every ℓ-partite, ℓ-uniform hypergraph F. For other ℓ-uniform hypergraphs only a few results are known and, for example, determining $\pi(K_4^{(3)})$ is one of the best known open problems in the area. However, one can show that $\pi(F)$ indeed exists for every hypergraph F (see, e.g. [26]).

We study the random variable $\mathrm{ex}(G^{(\ell)}(n,q), F)$ for ℓ-uniform hypergraphs F, where $G^{(\ell)}(n,q)$ denotes the binomial random ℓ-uniform sub-hypergraph of $K_n^{(\ell)}$ with edges of $K_n^{(\ell)}$ included independently with probability q. It is easy to show that

$$\mathrm{ex}(H, F) \geq \pi(F)e(H)$$

for all ℓ-uniform hypergraphs H and F (see, e.g. [25, Proposition 8.4] for a proof for graphs). We are interested in the threshold for the property that a.a.s.

$$\mathrm{ex}(G^{(\ell)}(n,q), F) \leq (\pi(F) + o(1))e(G^{(\ell)}(n,q))\,. \tag{9}$$

Results of that sort appeared in the work of Babai, Simonovits, and Spencer [1] who showed that (9) holds for $F = K_3$ and $q = 1/2$. In fact, it follows from an earlier result of Frankl and Rödl [15] that the same holds as long as $q \gg n^{-1/2}$. The systematic study for graphs was initiated by Kohayakawa and his coauthors. In particular, Kohayakawa, Łuczak, and Rödl formulated a conjecture for the threshold of Turán properties for random graphs (see Conjecture 2.6 below).

For an ℓ-uniform hypergraph F with $e(F) \geq 1$ we set

$$m(F) = \max_{\substack{F' \subseteq F \\ e(F') \geq 1}} d(F') \quad \text{with} \quad d(F') = \begin{cases} \frac{e(F')-1}{v(F')-\ell}\,, & \text{if } v(F') > \ell \\ 1/\ell\,, & \text{if } v(F') = \ell\,. \end{cases} \tag{10}$$

If follows from the definition of $m(F)$, that if $q = \Omega(n^{-1/m(F)})$ then a.a.s. the number of copies of every sub-hypergraph $F' \subseteq F$ in the random hypergraph $G^{(\ell)}(n,q)$ has at least the same order of magnitude, as the number of edges. Recall that a similar heuristic gave rise to the thresholds in the theorem above.

Conjecture 2.6 ([29, Conjecture 1 (i)])**.** *For every graph F with at least one edge and every $\varepsilon > 0$ there exists $C > 0$ such that for every sequence of probabilities $\boldsymbol{q} = (q_n)_{n\in\mathbb{N}}$ with $q_n \geq Cn^{-1/m(F)}$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathrm{ex}(G(n,q_n),F) \leq (\pi(F)+\varepsilon)e(G(n,q_n))\right) = 1\,.$$

Conjecture 2.6 was verified for a few special cases. As already mentioned for $F = K_3$ the conjecture follow from a result in [15]. For F being a clique with 4, 5, or 6 vertices the conjecture was verified by Kohayakawa, Łuczak, and Rödl [29], Gerke, Schickinger, and Steger [20] and Gerke [19]. Moreover, the conjecture is known to be true when F is a cycle due to the work of Füredi [17] (for the cycle of length four) and Haxell, Kohayakawa, and Łuczak [23, 24] (see also [27, 32]) and the conjecture is known to be true for trees. The best current bounds on q for which (9) holds for F being a clique and for arbitrary F were obtained by Szabó and Vu [39] and Kohayakawa, Rödl, and Schacht [30].

We verify this conjecture for all graphs F and the natural analogue of this conjecture for hypergraphs. (For ℓ-partite, ℓ-uniform hypergraphs such a conjecture was made in [37, Conjecture 15].)

Theorem 2.7. *For every ℓ-uniform hypergraph F with at least one vertex contained in at least two edges and every $\varepsilon \in (0, 1-\pi(F))$ there exist constants $C > c > 0$ such that for any sequence of probabilities $\boldsymbol{q} = (q_n)_{n\in\mathbb{N}}$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathrm{ex}\left(G^{(\ell)}(n,q_n)\right) \leq (\pi(F)+\varepsilon)e\left(G^{(\ell)}(n,q_n)\right)\right) = \begin{cases} 0, & \text{if } q_n \leq cn^{-1/m(F)} \text{ for all } n \in \mathbb{N}, \\ 1, & \text{if } q_n \geq Cn^{-1/m(F)} \text{ for all } n \in \mathbb{N}. \end{cases}$$

In Section 4 we will deduce the 1-statements of Theorems 2.3, 2.4, 2.5, and 2.7 from the main result, Theorem 3.3, which we present in the next section. The proofs of the 0-statements will be more elementary and will be also given in Section 4.

3. Main technical result

The main result will be phrased in the language of hypergraphs. We will study sequences of hypergraphs $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$. In the context of Theorem 2.2 one may think of $V_n = [n]$ and E_n being the arithmetic progressions of length k. In the context of Theorems 2.3, 2.4, and 2.5 the corresponding hypergraphs the reader should have in mind are defined in a very similar way. Moreover, for Theorem 2.7 one should think of $V_n = E(K_n^{(\ell)})$ being the edge set of the complete hypergraph $K_n^{(\ell)}$ and edges of E_n correspond to copies of F in $K_n^{(\ell)}$.

In order to transfer an extremal result from the classical, deterministic setting to the probabilistic setting we will require that a stronger quantitative version of the extremal result holds (see Definition 3.1 below). Roughly speaking, we will require that a sufficiently dense sub-structure not only contains one copy of the special configuration (not only one arithmetic progression or not only one copy of F), but instead the number of those configurations should be of the same order as the total number of those configurations in the given underlying ground set.

Definition 3.1. *Let $\boldsymbol{H} = (H_n)_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs and $\alpha \geq 0$. We say $\boldsymbol{H}$ is α-**dense** if the following is true.*

For every $\varepsilon > 0$ there exist $\zeta > 0$ and n_0 such that for every $n \geq n_0$ and every $U \subseteq V(H_n)$ with $|U| \geq (\alpha+\varepsilon)|V(H_n)|$ we have

$$|E(H_n[U])| \geq \zeta |E(H_n)|.$$

The second condition in Theorem 3.3 imposes a lower bound on the smallest probability for which we can transfer the extremal result to the probabilistic setting (see Definition 3.2). For a k-uniform hypergraph $H = (V, E)$, $i \in [k-1]$, $v \in V$, and $U \subseteq V$ we denote by $\deg_i(v, U)$ the number of edges of H containing v and having at least i vertices in $U \setminus \{v\}$. More precisely,

$$\deg_i(v, U) = \big|\{e \in E\colon\ |e \cap (U \setminus \{v\})| \geq i \text{ and } v \in e\}\big| \,. \tag{11}$$

For $q \in (0,1)$ we let $\mu_i(H, q)$ denote the expected value of the sum over all such degrees squared with $U = V_q$ being the binomial random subset of V

$$\mu_i(H, q) = \mathbb{E}\left[\sum_{v\in V} \deg_i^2(v, V_q)\right] .$$

Definition 3.2. *Let $\boldsymbol{H} = (H_n)_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ be a sequence of probabilities, and let $K \geq 1$. We say $\boldsymbol{H}$ is $(K, \boldsymbol{p})$-**bounded** if the following is true.*

For every $i \in [k-1]$ there exists n_0 such that for every $n \geq n_0$ and $q \geq p_n$ we have

$$\mu_i(H_n, q) \leq K q^{2i} \frac{|E(H_n)|^2}{|V(H_n)|} \,. \tag{12}$$

With those definitions at hand, we can state the main result.

Theorem 3.3. *Let $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ be a sequence of probabilities satisfying $p_n^k |E_n| \to \infty$ as $n \to \infty$, and let $\alpha \geq 0$ and $K \geq 1$. If $\boldsymbol{H}$ is α-dense and $(K, \boldsymbol{p})$-bounded, then the following holds.*

For every $\delta > 0$ and $(\omega_n)_{n\in\mathbb{N}}$ with $\omega_n \to \infty$ as $n \to \infty$ there exists $C \geq 1$ such that for every $q_n \geq C p_n$ the following holds a.a.s. for V_{n,q_n}: If $W \subseteq V_{n,q_n}$ with $|W| \geq (\alpha+\delta)|V_{n,q_n}|$, then $E(H_n[W]) \neq \emptyset$.

The proof of Theorem 3.3 is based on induction on k and for the induction we will strengthen the statement (see Lemma 3.4 below).

For a k-uniform hypergraph $H = (V, E)$ subsets $W \subseteq U \subseteq V$, and any integer $i \in \{0, 1, \dots, k\}$ we consider those edges of $H[U]$ which have at least i vertices in W and we denote this family by

$$E_U^i(W) = \{e \in E(H[U])\colon\ |e \cap W| \geq i\} \,.$$

Note that

$$E_U^0(W) = E(H[U]) \quad \text{and} \quad E_U^k(W) = E(H[W]) \tag{13}$$

for every $W \subseteq U$.

Lemma 3.4. *Let $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ be a sequence of probabilities satisfying $p_n^k|E_n| \to \infty$ as $n \to \infty$, and let $\alpha \geq 0$ and $K \geq 1$. If $\boldsymbol{H}$ is α-dense and $(K, \boldsymbol{p})$-bounded, then the following holds.*

For every $i \in [k]$, $\delta > 0$, and $(\omega_n)_{n\in\mathbb{N}}$ with $\omega_n \to \infty$ as $n \to \infty$ there exist $\xi > 0$, $b > 0$, $C \geq 1$, and n_0 such that for all β, $\gamma \in (0,1]$ with $\beta\gamma \geq \alpha + \delta$, every $n \geq n_0$, every q with $1/\omega_n \geq q \geq Cp_n$ the following holds.

If $U \subseteq V_n$ with $|U| \geq \beta|V_n|$, then the binomial random subset U_q satisfies with probability at least

$$1 - 2^{-bq|V_n|}$$

the following property: For every subset $W \subseteq U_q$ with $|W| \geq \gamma|U_q|$ we have

$$\left|E_U^i(W)\right| \geq \xi q^i |E_n|\,.$$

Theorem 3.3 follows directly from Lemma 3.4 applied with $i = k$, $\beta = 1$, $\gamma = \alpha + \delta$, and $U = V_n$.

3.1. **Probabilistic tools.** We will use Chernoff's inequality in the following form (see, e.g., [25, Corollary 2.3]).

Theorem 3.5 (Chernoff's inequality)**.** *Let $X \subseteq Y$ be finite sets and $p \in (0,1]$. For every $0 < \varrho \leq 3/2$ we have*

$$\mathbb{P}\left(\left||X \cap Y_p| - p|X|\right| \geq \varrho p|X|\right) \leq 2\exp(-\varrho^2 p|X|/3)\,. \qquad \square$$

We also use an approximate concentration result for $(K, \boldsymbol{p})$-bounded hypergraphs. The $(K, \boldsymbol{p})$-boundedness only bounds the expected value of the quantity $\sum_v \deg_i^2(v, V_p)$. In the proof of Lemma 3.4 we need an exponential upper tail bound and, unfortunately, it is known that such bounds usually not exist. However, it was shown by Rödl and Ruciński in [35] that on the prize of deleting a few elements such bound can be obtained. We will again apply this idea in the proof of Lemma 3.4.

Proposition 3.6 (Upper tail [35, Lemma 4])**.** *Let $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ be a sequence of probabilities, and let $K \geq 1$. If $\boldsymbol{H}$ is $(K, \boldsymbol{p})$-bounded, then the following holds.*

For every $i \in [k-1]$ and every $\eta > 0$ there exist $b > 0$ and n_0 such that for every $n \geq n_0$ and every $q \geq p_n$ the binomial random subset $V_{n,q}$ has the following property with probability at least $1 - 2^{-bq|V_n|+1+\log_2 k}$. There exists a set $X \subseteq V_{n,q}$ with $|X| \leq \eta q|V_n|$ such that

$$\sum_{v\in V_n} \deg_i^2(v, V_{n,q} \setminus X) \leq 4^k k^2 K q^{2i} \frac{|E_n|^2}{|V_n|}\,.$$

The proof follows the lines of [35, Lemma 4] and we include it for completeness.

Proof. Suppose $\boldsymbol{H}$ is $(K, \boldsymbol{p})$-bounded and $i \in [k-1]$ and $\eta > 0$ are given. We set

$$b = \frac{\eta}{4(k-1)^2}$$

and n_0 be sufficiently large, so that (12) holds for every $n \geq n_0$ and $q \geq p_n$.

For every $j = i, \dots, 2(k-1)$ we consider the family $\mathscr{S}_j$ defined as follows

$$\mathscr{S}_j = \Big\{(S, v, e, e')\colon\ S \subseteq V_n,\ v \in V_n,\ e, e' \in E_n \text{ such that } |S| = j,\\ v \in e \cap e',\ S \subseteq (e \cup e') \setminus \{v\},\ |e \cap S| \geq i \text{ and } |e' \cap S| \geq i\Big\}.$$

Let $\mathcal{S}_j$ be the random variable denoting the number of elements (S, v, e, e') from $\mathscr{S}_j$ with $S \in \binom{V_{n,q}}{j}$. By definition we have $\sum_{j=i}^{2k-2} \mathbb{E}\left[\mathcal{S}_j\right] \leq 4^{k-1}\mu_i(H_n, q)$ and due to the $(K, \boldsymbol{p})$-boundedness of $\boldsymbol{H}$ we have

$$\max_{j=i,\dots,2(k-1)} \mathbb{E}\left[\mathcal{S}_j\right] \leq \sum_{j=i}^{2k-2} \mathbb{E}\left[\mathcal{S}_j\right] \leq 4^{k-1}\mu_i(H_n, q) \leq 4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|}.$$

Let Z_j be the random variable denoting the number of sequences

$$((S_r, v_r, e_r, e'_r))_{r\in[z]} \in \mathscr{S}_j^z$$

of length

$$z = \left\lceil \frac{\eta q |V_n|}{4(k-1)^2} \right\rceil \leq \left\lceil \frac{\eta q |V_n|}{2(k-1)j} \right\rceil$$

which satisfy

(i) the sets S_r are contained in $V_{n,q}$ and
(ii) the sets S_r are mutually disjoint, i.e., $S_{r_1} \cap S_{r_2} = \emptyset$ for all $1 \leq r_1 < r_2 \leq z$.

Clearly, we have

$$\mathbb{E}\left[Z_j\right] \leq |\mathscr{S}_j|^z q^{jz} = \left(\mathbb{E}\left[\mathcal{S}_j\right]\right)^z \leq \left(4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|}\right)^z.$$

On the other hand, if

$$\sum_{v\in V_n} \deg_i^2(v, V_{n,q} \setminus X) \geq 4^k k^2 K q^{2i} \frac{|E_n|^2}{|V_n|} \geq \sum_{j=i}^{2k-2} j \cdot 2 \cdot 4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|}$$

for any $X \subseteq V_{n,q}$ with $|X| \leq \eta q |V_n|$, then there exists some $j_0 \in \{i, \dots, 2k-2\}$ such that

$$Z_{j_0} \geq \left(2 \cdot 4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|}\right)^z.$$

Markov's inequality bounds the probability of this event by

$$\mathbb{P}\left(\exists j_0 \in \{i, \dots, 2k-2\}\colon\ Z_{j_0} \geq 2^z \left(4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|}\right)^z\right)\\ \leq \sum_{j=i}^{2k-2} \mathbb{P}\left(Z_j \geq 2^z \left(4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|}\right)^z\right) \leq 2k \cdot 2^{-z} \leq 2^{-bq|V_n| + 1 + \log_2 k},$$

which concludes the proof of Proposition 3.6. □

3.2. **Proof of Lemma 3.4.** Let $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ be a sequence of probabilities, and let $\alpha \geq 0$ and $K \geq 1$ such that $\boldsymbol{H}$ is α-dense and $(K, \boldsymbol{p})$-bounded. We prove Lemma 3.4 by induction on i.

Induction start ($i = 1$). For $\delta > 0$ and $(\omega_n)_{n\in\mathbb{N}}$ (which plays no role for the induction start) we appeal to the α-denseness of $\boldsymbol{H}$ and let ζ and n_1 be the constants given by this property for $\varepsilon = \delta/8$. We set

$$\xi = \frac{\delta\zeta}{4k}\,, \quad b = \frac{\delta^3}{193}\,, \quad C = 1\,, \quad \text{and} \quad n_0 = n_1\,.$$

Let $\beta, \gamma \in (0,1]$ satisfy $\beta\gamma \geq \alpha + \delta$, let $n \geq n_0$ be sufficiently large, $q \geq p_n$, and let $U \subseteq V_n$ with $|U| \geq \beta|V_n|$ be given. We consider the set $Y \subseteq U$ defined by

$$Y = \left\{ u \in U\colon \ |\{e \in E(H_n[U])\colon\ u \in e\}| \leq \frac{\zeta|E_n|}{2|V_n|} \right\}.$$

In other words, Y is the set of vertices in U with low degree in $H_n[U]$. Due to the α-denseness of $\boldsymbol{H}$ we have

$$|Y| \leq \left(\alpha + \frac{\delta}{8}\right)|V_n|\,.$$

It follows from Chernoff's inequality that with probability at least

$$1 - 2\exp(-\delta^2 q|U|/48) - 2\exp(-\delta^2 q|V_n|/192) \geq 1 - 2^{-bq|V_n|}$$

we have

$$|U_q| \geq \left(1 - \frac{\delta}{4}\right) q|U| \quad \text{and} \quad |U_q \cap Y| \leq \left(\alpha + \frac{\delta}{4}\right) q|V_n|\,.$$

Consequently, for every $W \subseteq U_q$ satisfying $|W| \geq \gamma|U_q|$ we have

$$\begin{aligned} |W| \geq \gamma|U_q| &\geq \left(1 - \frac{\delta}{4}\right)\gamma q|U| \geq \left(1 - \frac{\delta}{4}\right)\beta\gamma q|V_n| \\ &\geq \left(1 - \frac{\delta}{4}\right)(\alpha + \delta)\, q|V_n| \geq \left(\alpha + \frac{\delta}{2}\right) q|V_n| \geq |U_q \cap Y| + \frac{\delta}{2} q|V_n| \end{aligned}$$

and the definition of Y yields

$$\left|E^1_U(W)\right| \geq |W \setminus Y| \cdot \frac{1}{k}\frac{\zeta|E_n|}{2|V_n|} \geq \frac{\delta}{2} q|V_n| \cdot \frac{1}{k}\frac{\zeta|E_n|}{2|V_n|} = \xi q|E_n|\,.$$

This concludes the proof of the induction start.

Induction step ($i \longrightarrow i+1$). Let $i \geq 1$, $\delta > 0$, and $(\omega_n)_{n\in\mathbb{N}}$ with $\omega_n \to \infty$ as $n \to \infty$ be given. We will expose the random set U_q in several rounds. The number of "main" rounds R will depend on the constant $\xi(i, \delta/8)$, which is given by the induction assumption. More precisely, let

$$\xi' = \xi(i, \delta/8)\,, \quad b' = b(i, \delta/8)\,, \quad C' = C(i, \delta/8)\,, \quad \text{and} \quad n' = n_0(i, \delta/8)$$

be given by the induction assumption applied with $\delta' = \delta/8$. We set

$$R = \left\lceil \frac{4^{k+2}k^2K}{\delta(\xi')^2} + 1 \right\rceil. \tag{14}$$

Overview. Roughly, speaking our argument is as follows. We will expose U_q in R main rounds of the same weight, i.e., we will chose q_R in such a way that $(1-q) = (1-q_R)^R$ and we let $U_q = U^1_{q_R} \cup \cdots \cup U^R_{q_R}$. Since, every subset W,

which we have to consider, contains at least $\gamma \geq \alpha + \delta$ proportion of the elements of U_q there must be at least $\delta R/4$ rounds such that $|U^s_{q_R} \cap W| \geq (\alpha + \delta/2)|U_{q_R}|$. For those rounds we will appeal to the induction assumption, which combined with Proposition 3.6, implies that U contains at least $\Omega((\xi')^2|V_n|)$ elements $u \in U$ with the property that every such u completes "many" elements in $E^i_U(W \cap U^s_{q_R})$ to elements in $E^{i+1}_U(W \cap U^s_{q_R})$. Moreover, in each of these "substantial" rounds $(\xi')^2|V_n|/(4^{k+1}k^2K)$ new "rich" elements u will be created. Consequently, after at most $\delta R/4-1$ of these substantial rounds all but, say, at most $(\alpha+\delta/8)|V_n| < \gamma|V_n|$ elements of U are rich and in the final substantial round $W \cap U_{q_R}$ must contain many rich $u \in U$and therefore create many elements from $E^{i+1}_U(W)$.

However, the error probabilities in the later rounds will have to beat the number of choices for the elements of W in the earlier rounds. For that we will split the earlier main rounds into several subrounds. This does not affect the argument indicated above, since our bound on the number of "rich" elements will be independent of q_R. We now continue with the details of this proof.

Constants. Set

$$\eta = \frac{\delta^2}{9} \tag{15}$$

and let $\hat{b}$ and $\hat{n}$ be given by Proposition 3.6 applied with i and η. We set

$$b^* = \min\left\{\frac{\delta^4}{10^6}, \frac{b'}{3}, \frac{\hat{b}}{3}\right\} \quad \text{and} \quad B = \left\lceil 1 + \frac{1.01^2}{b^*} \right\rceil . \tag{16}$$

Finally, let

$$\xi = \frac{\xi'\delta^2}{18k(RB^{R-1})^{i+1}}, \tag{17}$$

$$b = \min\left\{\frac{\delta^3}{60001RB^{R-1}}, \frac{b^*}{2RB^{R-1}}\right\}, \tag{18}$$

$$C = RB^{R-1}C', \tag{19}$$

and let $n_0 \geq \max\{n', \hat{n}\}$ be sufficiently large.

Suppose β and $\gamma \in (0, 1]$ satisfy

$$\beta\gamma \geq \alpha + \delta .$$

Let $n \geq n_0$ and let q satisfy $1/\omega_n \geq q \geq Cp_n$. Moreover, let $U \subseteq V_n$ be such that $|U| \geq \beta|V_n|$. Note that

$$\min\{\beta, \gamma\} \geq \alpha + \delta \geq \delta > 0 \quad \text{and} \quad |U| \geq (\alpha + \delta)|V_n| .$$

For a simpler notation from now on we suppress the subscript n in p_n, H_n, V_n and E_n.

Details of the induction step. As discussed above we generate the random set U_q in several rounds. We will have R main rounds and for that we choose q_R such that

$$1 - q = (1 - q_R)^R .$$

For $s \in [R]$ we will further split the sth main round into B^{R-s} subrounds. For $s \in [R]$ we set

$$r_s = B^{R-s}$$

and let q_s satisfy

$$(1-q_R) = (1-q_s)^{r_s}\,.$$

Note that for sufficiently large n, due to $q_n \leq 1/\omega_n$ and $\omega_n \to \infty$ we have

$$\left(1+\frac{\delta}{100}\right)\frac{q}{R} \geq q_R \geq \frac{q}{R} \quad \text{and} \quad \left(1+\frac{\delta}{100}\right)\frac{q_R}{B^{R-s}} \geq q_s \geq \frac{q_R}{B^{R-s}}\,, \tag{20}$$

and due to the choice of B we have

$$\sum_{t=1}^{s-1} q_t \leq 1.01\frac{q_R}{B^R}\sum_{t=1}^{s-1} B^t \overset{(16)}{\leq} \frac{b^*}{1.01}\frac{q_R}{B^R}B^s \leq \frac{b^*}{1.01}q_s\,. \tag{21}$$

We proceed as follows we first consider r_1 rounds with probability q_1, which all together establish the first main round and we denote the random subsets obtained by

$$U^1_{q_R} = U^{1,1}_{q_1} \cup \cdots \cup U^{1,r_1}_{q_1}\,.$$

This is followed by r_2 rounds with probability q_2 establishing the second main round. This way we have

$$U_q = U^1_{q_R} \cup \cdots \cup U^R_{q_R}$$

and for all $s \in [R]$

$$U^s_{q_R} = U^{s,1}_{q_s} \cup \cdots \cup U^{1,r_s}_{q_s}\,.$$

Furthermore, let $W \subseteq G_q$ with $|W| \geq \gamma|U_q|$ and let

$$W^s = W \cap U^s_{q_R} \quad \text{and} \quad W^{s,j} = W \cap U^{s,j}_{q_s}$$

for all $s \in [R]$ and $j \in [r_s]$.

In our analysis we focus on "substantial" rounds. For that let $S \subseteq [R]$ be the set defined by $s \in S$ if and only if

$$|W^s| \geq \left(\gamma - \frac{\delta}{2}\right)|U^s_{q_R}|\,.$$

By definition of S, for every $s \in S$ exists some $j_s \in [r_s]$ such that

$$|W^{s,j_s}| \geq \left(\gamma - \frac{\delta}{2}\right)|U^{s,j_s}_{q_s}|$$

and for the rest of the proof we fix such an j_s for every $s \in S$. The following claim is a direct consequence of Chernoff's inequality.

Claim 1. *Let $\mathcal{A}$ denote the event that $|S| \geq \delta R/4$. Then $\mathbb{P}(\mathcal{A}) \geq 1 - 2^{-2bq|V|}$.*

Proof. Due to Chernoff's inequality we have

$$|U^{s,j}_{q_s}| = (1 \pm 0.01\delta)q_s|U|\,. \tag{22}$$

for all $s \in [R]$ and every $j \in [r_s]$ with probability at least

$$1 - 2\sum_{s=1}^{R} r_s \exp(-\delta^2 q_s|U|/30000) \geq 1 - 2^{-2bq|V|}\,,$$

where we used $q_1 \le q_s$, (20), the choice of b in (18) and the fact that n is sufficiently large for the last inequality. Since $|W| \ge \gamma |U_q|$ we have

$$|S| \ge \frac{|W| - R \cdot (1+\delta/100)(\gamma - \delta/2) q_R |U|}{(1+\delta/100) q_R |U|} \\ \ge \frac{(1-\delta/100)\gamma q}{(1+\delta/100) q_R} - \left(\gamma - \frac{\delta}{2}\right) R \overset{(20)}{\ge} \frac{\delta}{4} R$$

with probability at least $1 - 2^{-2bq|V|}$. □

For the rest of the proof we analyze the rounds indexed by (s, j_s) for $s \in S$. For $s \in S$ we set

$$W(s) = \bigcup_{\substack{t \in S \\ t \le s}} W^{t,j_t} \quad \text{and} \quad U(s) = \bigcup_{\substack{t \in S \\ t \le s}} U_{q_t}^{t,j_t} .$$

Note that $W(t) = U(t) = \emptyset$ for all $t < \min_{s \in S} s$. Roughly, speaking we will show for every $s \in S$ that either $E_U^{i+1}(W(s))$ is sufficiently large or $\Omega(|V|)$ new "rich" elements in U will be created. More precisely, for $s \in S$ we consider the following subset $Z^s \subseteq U$ of rich elements

$$Z^s := \left\{ u \in U \colon \ \deg_i(u, W^{s,j_s}, U) \ge \frac{\xi'}{2} q_s^i \frac{|E|}{|V|} \right\} ,$$

where

$$\deg_i(u, W^{s,j_s}, U) \ := \ \big|\big\{ e \in E \colon |e \cap (W^{s,j_s} \setminus \{u\})| \ge i \text{ and } e \subseteq U \big\}\big| . \quad (23)$$

Note that $\deg_i(u, W^{s,j_s}, V) = \deg_i(u, W^{s,j_s})$ and, hence, for every set $U \subseteq V$ and every $u \in V$ we have

$$\deg_i(u, W^{s,j_s}, U) \le \deg_i(u, W^{s,j_s}) . \quad (24)$$

Similarly, as above we set

$$Z(s) = \bigcup_{\substack{t \in S \\ t \le s}} Z^s$$

Claim 2. *For every $s \in S$ and any choice of $W(s-1) \subseteq U(s-1)$ let $\mathcal{B}_{W(s-1)}$ denote the event that $U_{q_s}^{s,j_s}$ satisfies the following properties:*

(i) $|U_{q_s}^{s,j_s}| \le 1.01 q_s |U|$ *and*

(ii) *for every W^{s,j_s} with $|W^{s,j_s}| \ge (\gamma - \delta/2)|U_{q_s}^{s,j_s}|$ either*

$$|E_U^{i+1}(W(s))| \ge \xi q^{i+1} |E| \quad (25)$$

or

$$|Z(s) \setminus Z(s-1)| \ge \frac{(\xi')^2}{4^{k+1} k^2 K} |V| . \quad (26)$$

Then

$$\mathbb{P}\left(\mathcal{B}_{W(s-1)} \mid U(s-1)\right) \ge 1 - 2^{-2b^* q_s |V|} ,$$

where $\mathbb{P}\left(\mathcal{B}_{W(s_0-1)} \mid U(s_0-1)\right) = \mathbb{P}\left(\mathcal{B}_{W(s_0-1)}\right)$ for $s_0 = \min_{s \in S} s$.

Before we verify Claim 2 we deduce Lemma 3.4 from it. Let $\mathcal{C}$ denote the event that the conclusion of Lemma 3.4 holds. If event $\mathcal{A}$ holds and $\mathcal{B}_{W(s-1)}$ holds for every $s \in S$, then $\mathcal{C}$ must hold, since (26) in Claim 2 can occur at most

$$\frac{4^{k+1}k^2K}{(\xi')^2} \overset{(14)}{<} \frac{\delta}{4}R \leq |S|$$

times and, therefore, (25) in Claim 2 must occur. Below we will verify that this happens with a sufficiently large probability. Setting $\mathbb{P}\left(U(s_0-1)\right) = 1$ for $s_0 = \min_{s\in S} s$, we have

$$\mathbb{P}\left(\neg\mathcal{C}\right) \leq \mathbb{P}\left(\neg\mathcal{A}\right) + \sum_{S\subseteq[R]} \sum_{s\in S} \sum_{U(s-1)} \sum_{W(s-1)} \mathbb{P}\left(\neg\mathcal{B}_{W(s-1)} \mid U(s-1)\right) \mathbb{P}\left(U(s-1)\right),$$

where the first sum runs over all subsets $S \subseteq [R]$ with $|S| \geq \delta R/4$, the third sum runs over all choices of $U(s-1) = \bigcup_{t\in S, t<s} U_{q_t}^{t,j_t}$ with $|U_{q_t}^{t,j_t}| \leq 1.01 q_t|G|$, and the inner sum runs over all $2^{1.01|V|\sum_{t\in S,t<s} q_t}$ choices of $W(s-1) \subseteq U(s-1)$. Therefore, Claims 1 and 2 yield

$$\begin{aligned}\mathbb{P}\left(\neg\mathcal{C}\right) &\leq 2^{-2bq|V|} + 2^R \sum_{s=1}^{R} 2^{1.01|V|\sum_{t=1}^{s-1} q_t} \cdot 2^{-2b^* q_s|V|} \\ &\overset{(21)}{\leq} 2^{-2bq|V|} + 2^R R 2^{-b^* q_1|V|} \overset{(20)}{\leq} 2^{-2bq|V|} + 2^R R 2^{-b^* q|V|/(RB^{R-1})} \overset{(18)}{\leq} 2^{-bq|V|},\end{aligned}$$

where the last inequality holds for sufficiently large n. This concludes the proof of Lemma 3.4 and it is left to verify Claim 2. □

Proof of Claim 2. Let $s \in S$, $W(s-1) \subseteq U(s-1)$ be given. Note that this also defines $Z(s-1)$. We first observe that property (i) of Claim 2 holds with high probability In fact, due to Chernoff's inequality, with probability at least

$$1 - 2\exp(-\delta^2 q_s|U|/30000) \overset{(16)}{\geq} 1 - 2^{-3b^* q_s|V|}$$

we even have

$$|U_{q_s}^{s,j_s}| = (1 \pm 0.01\delta) q_s|U| \tag{27}$$

and below we assume that (27) holds. We distinguish two cases for property (ii).

Case 1 ($|U \setminus Z(s-1)| < (\gamma - 3\delta/4)|U|$)**.** Due to Chernoff's inequality with probability at least

$$1 - 2\exp(-\delta^2(\alpha+\delta/4)q_s|U|/192) \overset{(16)}{\geq} 1 - 2^{-3b^* q_s|V|}$$

we have

$$|U_{q_s}^{s,j_s} \setminus Z(s-1)| \leq \left(\gamma - \tfrac{5}{8}\delta\right)|U_{q_s}^{s,j_s}|.$$

Since $s \in S$ it follows that

$$|W_{q_s}^{s,j_s} \cap Z(s-1)| \geq \frac{\delta}{8}|U_{q_s}^{s,j_s}| \overset{(27)}{\geq} 0.99\frac{\delta}{8} q_s|U| \geq \frac{\delta\beta}{9} q_s|V| \geq \frac{\delta^2}{9} q_s|V|.$$

Hence the definition of $Z(s-1) \subseteq \bigcup_{t\in S,t<s} Z^s$ and $q_1 \leq q_t$ for all $t \in S$ yields

$$\begin{aligned}|E_U^{i+1}(W(s))| &\geq \frac{\delta^2}{9} q_s|V| \cdot \frac{1}{k}\frac{\xi'}{2} q_1^i \frac{|E|}{|V|} \\ &\geq \frac{\xi'\delta^2}{18k} q_1^{i+1}|E| \overset{(20)}{\geq} \frac{\xi'\delta^2}{18k(RB^{R-1})^{i+1}} q^{i+1}|E| \overset{(17)}{\geq} \xi q^{i+1}|E|.\end{aligned}$$

In other words, for this case we showed that alternative (25) happens with probability at least $1-2\cdot 2^{-3b^*q_s|V|}\geq 1-2^{-2b^*q_s|V|}$.

Case 2 ($|U\setminus Z(s-1)|\geq(\gamma-3\delta/4)|U|$)**.** In this case we consider

$$U'=U\setminus Z(s-1)\,.$$

We set

$$\beta'=\frac{|U'|}{|V|}\quad\text{and}\quad\gamma'=\left(\gamma-\frac{7\delta}{8}\right)\frac{|U|}{|U'|}\,.$$

Clearly, $\beta'\in(0,1]$,

$$0<\gamma'\leq\frac{\gamma-7\delta/8}{\gamma-3\delta/4}\leq 1\,,$$

and

$$\beta'\gamma'=\left(\gamma-\frac{7\delta}{8}\right)\frac{|U|}{|V|}\geq\left(\gamma-\frac{7\delta}{8}\right)\beta\geq\gamma\beta-\frac{7\delta}{8}\geq\alpha+\frac{\delta}{8}\,.$$

Hence, we can apply the induction assumption to U'. More precisely, the induction assumption asserts that with probability at least

$$1-2^{b'q_s|V|}$$

every subset $\hat{W}'\subseteq U'_{q_s}$ with $\hat{W}'\geq\gamma'|U'_{q_s}|$ satisfies

$$\left|E^i_{U'}(\hat{W}')\right|\geq\xi' q_s^i|E|\,. \tag{28}$$

Note that, in fact,

$$q_s\overset{(20)}{\geq}\frac{q}{RB^{R-1}}\geq\frac{Cp}{RB^{R-1}}\overset{(19)}{\geq}C'p\,.$$

We split the random subset $U^{s,j_s}_{q_s}=U'_{q_s}\dot\cup U''_{q_s}$, where

$$U'_{q_s}=U^{s,j_s}_{q_s}\setminus Z(s-1)\quad\text{and}\quad U''_{q_s}=U^{s,j_s}_{q_s}\setminus U'_{q_s}\,.$$

Similarly, we split $W^{s,j_s}=W'\dot\cup W''$ where $W'=W^{s,j_s}\cap U'_{q_s}$ and $W''=W^{s,j_s}\cap U''_{q_s}$.

It follows again from Chernoff's inequality that

$$|U'_{q_s}|=\left(1\pm\frac{\delta}{16}\right)q_s|U'| \tag{29}$$

holds with probability at least

$$1-2\exp(-\delta^2q_s|U'|/768)\overset{(16)}{\geq}1-2^{-3b^*q_s|V|}\,.$$

We distinguish two sub-cases depending on the size of W''.

Case 2.1 ($|W''|>\delta|U^{s,j_s}_{q_s}|/8$)**.** In this case, it follows from the $W''\subseteq Z(s-1)$

$$\begin{aligned}|E^{i+1}_U(W(s))|&\geq|W''|\cdot\frac{1}{k}\frac{\xi'}{2}q_1^i\frac{|E|}{|V|}\geq\frac{\delta}{8}|U^{s,j_s}_{q_s}|\cdot\frac{\xi'}{2k}q_1^i\frac{|E|}{|V|}\\&\overset{(27)}{\geq}\frac{\delta}{9}q_s|U|\cdot\frac{\xi'}{2k}q_1^i\frac{|E|}{|V|}\geq\frac{\delta\beta}{9}q_s\cdot\frac{\xi'}{2k}q_1^i|E|\geq\frac{\delta^2\xi'}{18k}q_1^{i+1}|E|\overset{(17)}{\geq}\xi q^{i+1}|E|\,.\end{aligned}$$

In other words, for this case we showed that alternative (25) happens with probability at least $1-2\cdot 2^{-3b^*q_s|V|}\geq 1-2^{-2b^*q_s|V|}$.

Case 2.2 ($|W''|\leq\delta|U^{s,j_s}_{q_s}|/8$)**.**

In this case we appeal to the $(K,\boldsymbol{p})$-boundedness of $\boldsymbol{H}$. It follows from Proposition 3.6 and the choice of η in (15) that with probability at least

$$1-2^{-\hat{b}q_s|V|+1+\log_2 k}$$

there exists a set $X\subseteq U'_{q_s}$ such that

$$\begin{aligned}|X|\le \eta q_s|V| &\overset{(15)}{\le} \frac{\delta^2}{16}q_s|V| \le \frac{\delta}{16}(\alpha+\delta)q_s|V|\\ &\le \frac{\delta}{16}\gamma q_s|V| \le \frac{\delta}{16}q_s|U| \overset{(27)}{\le} \frac{\delta}{8}q_s|U^{s,j_s}_{q_s}| \end{aligned} \tag{30}$$

and

$$\begin{aligned}\sum_{u\in U'}\deg_i^2(u,W'\setminus X,U') &\overset{(24)}{\le} \sum_{u\in U'}\deg_i^2(u,W'\setminus X)\\ &\le \sum_{u\in U'}\deg_i^2(u,U'_{q_s}\setminus X) \le 4^k k^2 K q_s^{2i}\frac{|E|^2}{|V|}\,. \end{aligned} \tag{31}$$

Consider the set

$$\hat{W}'=W'\setminus X\,.$$

Since $s\in S$, it follows from (30) and the assumption of this case that

$$|\hat{W}'|\ge |W^{s,j_s}|-|W''|-|X|\ge \left(\gamma-\frac{\delta}{2}\right)|U^{s,j_s}_{q_s}|-2\frac{\delta}{8}|U^{s,j_s}_{q_s}|\ge\left(\gamma-\frac{3\delta}{4}\right)|U^{s,j_s}_{q_s}|\,.$$

Furthermore assertions (27) and (29) yield

$$\frac{|\hat{W}'|}{|U'_{q_s}|}\ge\left(\gamma-\frac{3\delta}{4}\right)\frac{|U^{s,j_s}_{q_s}|}{|U'_{q_s}|}\ge\frac{(\gamma-3\delta/4)(1-\delta/100)}{1+\delta/16}\frac{|U|}{|U'|}\ge\left(\gamma-\frac{7\delta}{8}\right)\frac{|U|}{|U'|}=\gamma'$$

In other words, $\hat{W}'$ satisfies $|\hat{W}'|\ge\gamma'|U'_{q_s}|$ and from the induction assumption we infer that (28) holds with probability at least $1-2^{-b'q_s|V|}$ and then

$$\sum_{u\in U'}\deg_i(u,\hat{W}',U')\ge|E^i_{U'}(\hat{W}')|\ge\xi' q_s^i|E|\,. \tag{32}$$

For

$$\hat{Z}=\left\{u\in U'\colon\ \deg_i(u,\hat{W}',U')\ge\frac{\xi'}{2}q_s^i\frac{|E|}{|V|}\right\}$$

it follows from the Cauchy-Schwarz inequality

$$\begin{aligned}4^k k^2 K q_s^{2i}\frac{|E|^2}{|V|} &\overset{(31)}{\ge} \sum_{u\in U'}\deg_i^2(u,\hat{W}',U')\ge\sum_{u\in\hat{Z}}\deg_i^2(u,\hat{W}',U')\\ &\ge\frac{1}{|\hat{Z}|}\left(\sum_{u\in\hat{Z}}\deg_i(u,\hat{W}',U')\right)^2\overset{(32)}{\ge}\frac{1}{|\hat{Z}|}\left(\frac{\xi' q_s^i|E|}{2}\right)^2\,.\end{aligned}$$

Consequently,

$$|\hat{Z}|\ge\frac{(\xi')^2}{4^{k+1}k^2K}|V|\,.$$

Since $\hat{Z} \subseteq U' = U \setminus Z(s-1)$ we have $\hat{Z}$ is disjoint from $Z(s-1)$. Furthermore, by definition of $\hat{Z}$ we have $\hat{Z} \subseteq Z^s$. Therefore, (26) of Claim 2 holds with probability at least

$$1 - 2 \cdot 2^{-3b^* q_s |V|} - 2^{-\hat{b} q_s |V| + 1 + \log_2 k} - 2^{-b' q_s |V|} \overset{(16)}{\geq} 1 - 2^{-2b^* q_s |V|},$$

which concludes the proof of Claim 2.

□

4. Proof of the new results

In this section we prove Theorems 2.2, 2.3, 2.4, 2.5, and 2.7. While the involved 0-statements will follow from standard probabilistic arguments, the 1-statement of those results will follow from Theorem 3.3.

4.1. Proof of Theorems 2.2 and 2.3.

Clearly Theorem 2.2 follows from Theorem 2.3 applied with $\ell = 1$ and $F = [k]$ and it suffices to verify Theorem 2.3.

The 0-statement of Theorem 2.3. We start with the 0-statement of the theorem. Let $F \subseteq \mathbb{N}^\ell$ with $|F| \geq 3$ and $\varepsilon > 0$ be given and set

$$c = \left(\frac{1 - 2\varepsilon}{2} \right)^{1/(|F|-1)}.$$

We distinguish different cases depending on the sequence $\boldsymbol{q} = (q_n)$.

Case 1 ($q_n \ll n^{-(\ell+1)/|F|}$)**.** In this case the expected number of homothetic copies of F in $[n]^\ell_{q_n}$ tends to 0. Hence, we infer from Markov's inequality that a.a.s. $[n]^\ell_{q_n}$ contains no homothetic copy of F, which yields the claim in that rage.

Case 2 ($n^{-\ell} \ll q_n \ll n^{-1/(|F|-1)}$)**.** In this range the expected number of homothetic copies of F in $[n]^\ell_{q_n}$ is asymptotically smaller than the expected number of elements in $[n]^\ell_{q_n}$. Moreover, it follows from Chernoff's inequality that a.a.s. $|[n]^\ell_{q_n}|$ is very close to its expectation. Consequently, it follows from Markov's inequality that a.a.s. the number of homothetic copies of F in $[n]^\ell_{q_n}$ is $o(|[n]^\ell_{q_n}|)$. Therefore, by removing one element from every homothetic copy of F in $|[n]^\ell_{q_n}|$ a.a.s. we obtain a subset Y of size $|Y| \geq \varepsilon |[n]^\ell_{q_n}|$, which contains no homothetic copy of F at all, which yields the 0-statement in this case.

Note that due to $|F| \geq 3$ the ranges considered in Cases 1 and 2 overlap. Similarly, the range considered in the case below overlaps with the one from Case 2.

Case 3 ($n^{-(\ell+1)/|F|} \ll q_n \leq cn^{-1/(|F|-1)}$)**.** Again appealing to Chernoff's inequality applied to the size of $[n]^\ell_{q_n}$ we infer that it suffices to show that a.a.s. the number of homothetic copies of F in $[n]^\ell_{q_n}$ is at most $(1-2\varepsilon) q_n n^\ell$.

Let Z_F be the random variable denoting the number of such copies. Clearly, $\mathbb{E}[Z_F] \leq q_n^{|F|} n^{\ell+1}$ and standard calculations show that the variance of Z_F satisfies

$$\operatorname{Var}[Z_F] = O\left(q_n^{2|F|-1} n^{\ell+2} + q_n^{|F|} n^{\ell+1} \right).$$

Consequently, Chebyshev's inequality yields

$$\mathbb{P}\left(Z_F \geq 2 q_n^{|F|} n^{\ell+1} \right) \leq \frac{\operatorname{Var}[Z_F]}{q_n^{2|F|} n^{2\ell+2}} = O\left(\frac{1}{q_n n^\ell} + \frac{1}{q_n^{|F|} n^{\ell+1}} \right) = o(1),$$

due to the range of q_n we consider in this case. Hence, the claim follows from the choice of c, which yields

$$2q_n^{|F|}n^{\ell+1} \le (1-2\varepsilon)q_n n^{\ell}\,. \qquad \square$$

The 1-statement of Theorem 2.3. We now turn to the 1-statement of Theorem 2.3. We first note that if $q_n = \Omega(1)$, then the theorem follows directly from Chernoff's inequality combined with the original result of Furstenberg and Katznelson. Hence we can assume w.l.o.g. $q_n = o(1)$.

Let $F \subseteq \mathbb{N}^\ell$ with $k = |F| \ge 3$ and $\varepsilon \in (0,1)$. We want to apply Theorem 3.3. For that we consider the following sequence of k-uniform hypergraphs $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$. Let $V_n = [n]^\ell$ and let every homothetic copy of F form an edge in E_n. In particular, $|E_n| = n^{\ell+1}$. We set $p_n = n^{-1/(k-1)}$, $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ and $\alpha = 0$. Clearly, for those definitions the conclusion of Theorem 3.3 yields the 1-statement of Theorem 2.3. In order to apply Theorem 3.3 we have to verify the following three conditions

(a) $p_n^k|E_n| \to \infty$ as $n \to \infty$,
(b) $\boldsymbol{H}$ is α-dense, and
(c) $\boldsymbol{H}$ is $(K,\boldsymbol{p})$-bounded.

By definition of p_n and H_n we have

$$p_n^k|E_n| = \Omega\left(n^{-k/(k-1)}n^{\ell+1}\right) = \Omega\left(n^{\ell-1/(k-1)}\right)\,,$$

which yields (a), as $\ell \ge 1$ and $k \ge 3$.

Condition (b) holds, due to work of Furstenberg and Katznelson [18]. In fact, it follows from the result in [18], that for every configuration $F \subseteq \mathbb{N}^\ell$ and every $\varepsilon > 0$ there exist $\zeta > 0$ and n_0 such that for every $n \ge n_0$ every subset $U \subseteq [n]^\ell$ with $|U| \ge \varepsilon n^\ell$ contains at least $\zeta n^{\ell+1}$ homothetic copies of F. In other words, $\boldsymbol{H}$ is 0-dense.

Hence, it is only left to verify condition (c). We have to show that for every $i \in [k-1]$ and $q \ge p_n = n^{-1/(k-1)}$ we have

$$\mu_i(H_n,q) = \mathbb{E}\left[\sum_{v\in V_n}\deg_i^2(v,V_q)\right] = O\left(q^{2i}\frac{|E_n|^2}{|V_n|}\right) = O\left(q^{2i}n^{\ell+2}\right)\,. \tag{33}$$

It follows from the definition of $\deg_i$ in (11) that $\mu_i(H_n,q)$ is the expected number of pairs (F_1,F_2) of homothetic copies of F which share at least one point v and at least i points different from v of each copy are contained in $[n]_q^\ell$. The expected number of such pairs (F_1,F_2) which share exactly one point can be bounded by $O\left(q^{2i}n^{\ell+2}\right)$. Since for every fixed homothetic copy F_1 there exist only constantly many (independent of n) other copies F_2, which share two points with F_1, the expected number of such pairs (F_1,F_2) with $|F_1 \cap F_2| \ge 2$ is bounded by

$$O\left(q^i n^{\ell+1}\right) = O\left(q^{2i}n^{\ell+2}\right)\,,$$

since $q \ge Cp_n \ge Cn^{-1/(k-1)} \ge Cn^{-1/i}$. Consequently, (33) holds, which concludes the proof of Theorem 2.3. $\square$

4.2. **Proof of Theorems 2.4.** The proof of the 0-statement follows directly from the 0-statement of Theorem 1.1 in [36]. Those authors showed that for every irredundant, density regular $\ell \times k$ matrix with rank ℓ there exists a constant $c > 0$ such that for $q_n \le cn^{-m(A)}$ a.a.s. $[n]_{q_n}$ can be partitioned into two classes such that

none of them contains a distinct-valued solution of the homogeneous system given by A. Clearly, this implies the 0-statement of Theorems 2.4 for every $\varepsilon \in (0, 1/2)$.

The 1-statement of Theorem 2.4. First we note that if $q_n = \Omega(1)$, then the statement follows directly from Chernoff's inequality combined with the definition of irredundant, density regular matrix.

Let A be an irredundant, density regular $\ell \times k$ integer matrix and $\varepsilon > 0$ For the application of Theorem 3.3 we consider the following sequence of k-uniform hypergraphs $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$. Let $V_n = [n]$ and for every distinct-valued solution $(x_1, \dots, x_k)$ let $\{x_1, \dots, x_k\}$ be an edge of E_n. Moreover we set $p_n = n^{-1/m(A)}$, $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ and $\alpha = 0$. The 1-statement of Theorem 2.4 then follows from the conclusion of Theorem 3.3 and we have to verify the same three conditions (a)-(c) as in the proof of the 1-statement of Theorem 2.3.

It was shown in [36, Proposition 2.2 (ii)] that $m(A) \geq k - 1$ and due to Rado's characterization of partition regular matrices (which contains the class of all density regular matrices) we have $k - \ell \geq 2$, which yields $|E_n| = \Omega(n^2)$. Therefore, we have

$$p^k |E_n| = \Omega(n^{-k/(k-1)} \cdot n^2) = \Omega\big(n^{\frac{k-2}{k-1}}\big)$$

and, hence, condition (a) is satisfied.

Moreover, based on the Furstenberg-Katznelson theorem from (*FuKa78*) it was shown by Frankl, Graham, and Rödl in [14, Theorem 2], that the sequence of hypergraphs $\boldsymbol{H}$ defined above is 0-dense, i.e., condition (b) is fulfilled.

Consequently, it suffices to verify that $\boldsymbol{H}$ is $(K, \boldsymbol{p})$-bounded for some $K \geq 1$. For $i \in [k-1]$ and $q \geq p_n = n^{-1/m(A)}$ we have to show that

$$\mu_i(H_n, q) = O\left(q^{2i} \frac{|E_n|^2}{n}\right).$$

Recalling the definitions of $\mu_i(H_n, q)$ and $H_n = ([n], E_n)$ we have

$$\mu_i(H_n, q) = \mathbb{E}\left[\sum_{x\in[n]} \deg_i^2(x, V_{n,q})\right] = \sum_{x\in[n]} \mathbb{E}\left[\deg_i^2(x, V_{n,q})\right]. \tag{34}$$

Note that $\mathbb{E}\left[\deg_i^2(x, V_{n,q})\right]$ is the expected number of pairs $(X, Y) \in [n]^k \times [n]^k$ such that

(i) $x \in X \cap Y$,

(ii) $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ are solutions of $\mathcal{L}(A)$, where

$$A\boldsymbol{x} = A\boldsymbol{y} = \boldsymbol{0}$$

for $\boldsymbol{x} = (x_1, \dots, x_k)^t$ and $\boldsymbol{y} = (y_1, \dots, y_k)^t$, and

(iii) $|X \cap ([n]_q \setminus \{x\})| \geq i$ and $|Y \cap ([n]_q \setminus \{x\})| \geq i$.

For fixed x and (X, Y) let $w \geq 1$ be the largest integer such that there exist indices $i_1, \dots, i_w$ and $j_1, \dots, j_w$ for which

$$x_{i_1} = y_{j_1}, \dots, x_{i_w} = y_{j_w}. \tag{35}$$

Consequently,

$$x \in \{x_{i_1}, \dots, x_{i_w}\} = \{y_{j_1}, \dots, y_{j_w}\} \tag{36}$$

Set $W_1 = \{i_1, \dots, i_w\}$ and $W_2 = \{j_1, \dots, j_w\}$.

For fixed sets W_1, $W_2 \subseteq [k]$ we are going to describe all $(2k - w)$-tuples $X \cup Y$ satisfying (ii) and (35). To this end consider the $2\ell \times (2k-w)$ matrix B, which arises

from two copies A_1 and A_2 of A with permuted columns. We set $A_1 = (A_{\overline{W}_1} \mid A_{W_1})$ and $A_2 = (A_{W_2} \mid A_{\overline{W}_2})$ where for every $\alpha = 1, \dots, w$ the column of A_{W_1} which is indexed by i_α aligns with that column of A_{W_2} which is indexed by j_α. Then let

$$B = \left(\begin{array}{c|c|c} A_{\overline{W}_1} & A_{W_1} & \mathbf{0} \\ \hline \mathbf{0} & A_{W_2} & A_{\overline{W}_2} \end{array}\right).$$

Without loss of generality we may assume that $\operatorname{rank}(A_{\overline{W}_1}) \geq \operatorname{rank}(A_{\overline{W}_2})$ and, therefore,

$$\operatorname{rank}(B) \geq \operatorname{rank}(A) + \operatorname{rank}(A_{\overline{W}_1}).$$

Clearly, the number of $(2k-w)$-tuples $X \cup Y$ satisfying (ii) and (35) equals the number of solutions of the homogeneous system given by B, which is $O(n^{2k-w-\operatorname{rank}(B)})$. Since A is an irredundant, partition regular matrix, it follows from [36, Proposition 2.2 (i)] that $\operatorname{rank}(A') = \operatorname{rank}(A)$ for every matrix A' obtained from A by removing one column. Consequently, any matrix B' obtained from B by removing one of the middle columns (i.e., one of the w columns of B which consist of a column of A_{W_1} and a columns of A_{W_2}) satisfies

$$\operatorname{rank}(B') \geq \operatorname{rank}(A) + \operatorname{rank}(A_{\overline{W}_1}) = \ell + \operatorname{rank}(A_{\overline{W}_1}).$$

Therefore, it follows from (36) that the number of such $(2k-w)$-tuples that also satisfy condition (i) for some fixed $x \in [n]$ is at most

$$O(n^{2k-w-1-\ell-\operatorname{rank}(A_{\overline{W}_1})}). \tag{37}$$

Finally, we estimate the probability that a $(2k-w)$-tuple $X \cup Y$ satisfying (i), (ii), and (35) also satisfies (iii). Since $|X \cap Y \cap ([n]_q \setminus \{x\})| = j \leq w-1$ and $q \leq 1$ this probability is bounded by

$$\sum_{j=0}^{w-1} q^{2i-j} = O(q^{2i-w+1}).$$

In view of (37) we obtain

$$\begin{aligned} &\sum_{x \in [n]} \mathbb{E}\left[\deg_i^2(x, V_{n,q})\right] \\ &\qquad = \sum_{x \in [n]} \sum_{w=1}^{k} \sum_{\substack{W_1, W_2 \subseteq [k] \\ |W_1| = |W_2| = w}} O(n^{2k-w-1-\ell-\operatorname{rank}(A_{\overline{W}_1})} q^{2i-w+1}). \end{aligned} \tag{38}$$

Note that if $w = 1$, then again due to [36, Proposition 2.2 (i)] we have $\operatorname{rank}(A_{\overline{W}_1}) = \ell$ and, therefore, the contribution of those terms satisfies

$$\sum_{x \in [n]} \sum_{\substack{W_1, W_2 \subseteq [k] \\ |W_1| = |W_2| = 1}} O(n^{2k-2\ell-2} q^{2i}) = O(n^{2k-2\ell-1} q^{2i}) = O\left(q^{2i} \frac{|E_n|^2}{n}\right). \tag{39}$$

For $w \geq 2$ and $W_1 \subseteq [k]$ with $|W_1| = w$ we obtain from the definition of $m(A)$ and $q \geq n^{-1/m(A)}$ that

$$q^{w-1} \geq n^{-w+1-\operatorname{rank}(A_{\overline{W}_1})+\ell}.$$

Consequently,

$$\sum_{x\in[n]}\sum_{w=2}^{k}\sum_{\substack{W_1,W_2\subseteq[k]\\|W_1|=|W_2|=w}} O(n^{2k-w-1-\ell-\mathrm{rank}(A_{\overline{W}_1})}q^{2i-w+1})$$

$$= \sum_{x\in[n]}\sum_{w=2}^{k}\sum_{\substack{W_1,W_2\subseteq[k]\\|W_1|=|W_2|=w}} O(n^{2k-2-2\ell}q^{2i})$$

$$= O(n^{2k-2\ell-1}q^{2i}) = O\left(q^{2i}\frac{|E_n|^2}{n}\right). \quad (40)$$

Finally, combining (34), (38), (39), and (40) we obtain

$$\mu_i(H_n,q) = O\left(q^{2i}\frac{|E_n|^2}{n}\right),$$

which concludes the proof of the 1-statement of Theorem 2.4. □

4.3. **Proof of Theorems 2.5.** The proof is similar to the proof of Theorem 2.3 and we only sketch the main ideas.

The 0-statement of Theorem 2.5. We recall that for the validity of the statement $X \to_{1/2+\varepsilon} (1 \quad 1 \quad -1)$ we only consider distinct-valued of the Schur equation and we call such a solutions *Schur-triples*. The expected number of Schur-triples contained in $[n]_{q_n}$ is bounded by $q_n^3n^2$. Consequently, the 0-statement follows from Markov's inequality if $q_n \ll n^{-2/3}$. In the middle range $n^{-}1 \ll q_n \ll n^{-1/2}$ it follows, on the one hand, from Chernoff's inequality that a.a.s. $|[n]_{q_n}| \geq q_nn/2$. On the other hand, due to Markov's inequality a.a.s. the number of Schur-triples in $[n]_{q_n}$ is $o(q_nn)$ and, hence, the statement holds in this range of q_n. Finally, if $n^{-2/3} \ll q_n \leq cn^{-1/2}$ for sufficiently small $c > 0$, then using Chebyshev's inequality one obtains the upper bound of

$$(1-(1/2+\varepsilon))q_nn/2$$

on the number of Schur-triples in $[n]_{q_n}$, which holds a.a.s. Consequently, in view of Chernoff's inequality, a.a.s. the random set $[n]_{q_n}$ contains a subset of size $(1/2+\varepsilon)|[n]_{q_n}|$, which contains no Schur-triple. □

The 1-statement of Theorem 2.5. Here the we consider a sequence of 3-uniform hypergraphs, where $V_n = [n]$ and E_n corresponds to all Schur-triples in $[n]$ and we set $p_n = n^{-1/2}$ and $\alpha = 1/2$. For given $\varepsilon \in (0,1/2)$ we want to appeal to Theorem 3.3 and for that we assume $q_n = o(1)$. Again the 1-statement of Theorem 2.5 follows from Theorem 3.3 and we have to verify the three conditions (a)-(c) as in the proof of the 1-statement of Theorem 2.3.

Condition (a) follows from the definition of p_n and condition (c) follows from similar considerations as in the proof of Theorem 2.3 for $\ell = 1$ and $k = 3$.

In order to verify condition (b) we have to show that for every $\varepsilon > 0$ there exist $\zeta > 0$ and n_0 such that for $n \geq n_0$ every subset $A \subseteq [n]$ with $|A| \geq (1/2+\varepsilon)n$ contains at least ζn^2 Schur-triples.

So let $A \subseteq [n]$ satisfy $(1/2+\varepsilon)n$ and set $A_1 = A \cap \{1,\dots,(1-\varepsilon)n\}$ (ignoring floors and ceilings). It follows that for every $z \in A \setminus A_1$ there are at least

$$\left(\frac{1}{2}+\varepsilon\right)n - \varepsilon n - \frac{(1-\varepsilon)n}{2} = \frac{\varepsilon}{2}n$$

pairs $x \le y$ with $x, y \in A$ such that $x+y=z$. Hence, if $|A \setminus A_1| \ge 3\varepsilon^2 n/2$, then A contains at least $3\varepsilon^3 n^2/4 - n$ Schur-triples and the claim follows.

On the other hand, if $|A \setminus A_1| < 3\varepsilon^2 n/2$, then we have

$$|A_1| \ge \left(\frac{1}{2}+\varepsilon\right)n - \frac{3\varepsilon^2}{2}n = \left(\frac{1}{2}+\frac{3\varepsilon}{2}\right)(1-\varepsilon)n\,.$$

In other words, we obtained a density increment of $\varepsilon/2$ on the interval $(1-\varepsilon)n$ and the conclusion follows from iterating the above argument.

This concludes the proof of condition (b) and, therefore, Theorem 3.3 yields the proof of the 1-statement of Theorem 2.5 for sequences $\boldsymbol{q}$ satisfying $q_n = o(1)$. The remaining case, when $q_n = \Omega(1)$ then follows by similar arguments as given in [25, Proposition 8.6] and we omit the details. □

4.4. **Proof of Theorems 2.7.**

The 0-statement of Theorem 2.7. Let F be an ℓ-uniform hypergraph with at least one vertex of degree 2 and $\varepsilon \in (0, 1-\pi(F))$. We set

$$c = \frac{1-\pi(F)-\varepsilon}{4}\,.$$

For the proof of the 0-statement we consider different ranges of $\boldsymbol{q} = (q_n)_{n\in\mathbb{N}}$ depending on the density of the densest sub-hypergraph of F and depending on $m(F)$. Let F' be the densest sub-hypergraph of F with $e(F') \ge 1$, i.e., F' maximizes $e(F')/v(F')$. Moreover, let F'' be one of those sub-hypergraphs for which

$$d(F'') = m(F)$$

(see (10) for the definition of those parameters). Note that $e(F'') \ge 2$, since F contains a vertex of degree at least two. We consider the following three ranges for $\boldsymbol{q}$.

Case 1 ($q_n \ll n^{-v(F')/e(F')}$)**.** In this range the expected number of copies of F' in $G^{(\ell)}(n,q_n)$ tends to 0 and, therefore, the statement follows from Markov's inequality.

Case 2 ($n^{-\ell} \ll q_n \ll n^{-1/m(F)}$)**.** It follows from the definition of $m(F)$, that in this range the expected number of copies of F'' in $G^{(\ell)}(n,q_n)$ is asymptotically smaller than the expected number of of edges of $G^{(\ell)}(n,q_n)$. Therefore, applying Markov's inequality to the number of copies of F'' and Chernoff's inequality to the number of edges $G^{(\ell)}(n,q_n)$ we obtain that a.a.s. the number of copies of F'' satisfies $o(e(G^{(\ell)}(n,q_n)))$. Hence, a.a.s. we can obtain an F''-free, and consequently, an F''-free sub-hypergraph of $G^{(\ell)}(n,q_n)$ by removing only $o(e(G^{(\ell)}(n,q_n)))$ edges, which yields the statement for this range of q_n.

We note that $n^{-\ell} \ll n^{-v(F')/e(F')}$ since F contains a vertex of degree 2. In other words, the interval considered in Case 2 overlaps with the interval from Case 1. Similarly, the range considered in the case below overlaps with the one from Case 2.

Case 3 ($n^{-v(F')/e(F')} \ll q_n \le cn^{-1/m(F)}$)**.** Applying again Chernoff's inequality to $e(G^{(\ell)}(n,q_n))$ we see that it suffices to show that a.a.s. the number of copies of F'' is at most $(1-(\pi(F)+\varepsilon))q_n n^\ell/2$.

Let $Z_{F''}$ be the random variable denoting the number of such copies. Clearly, $\mathbb{E}[Z_{F''}] \le q_n^{e(F'')} n^{v(F'')}$ and standard calculations show that the variance of Z_F satisfies

$$\mathrm{Var}[Z_{F''}] = O\left(\frac{q_n^{2e(F'')} n^{2v(F'')}}{\min_{F^*\subseteq F, e(F^*)\ge 1} q_n^{e(F^*)} n^{v(F^*)}}\right) = O\left(\frac{q_n^{2e(F'')} n^{2v(F'')}}{q_n^{e(F')} n^{v(F')}}\right),$$

due to the choice of F' being the densest sub-hypergraph of F. Since $q_n \gg n^{-v(F')/e(F')}$ we have $q_n^{e(F')} n^{v(F')} \to \infty$ and, therefore,

$$\mathrm{Var}[Z_{F''}] = o\left(q_n^{2e(F'')} n^{2v(F'')}\right)$$

Consequently, Chebyshev's inequality yields

$$\mathbb{P}\left(Z_{F''} \ge 2q_n^{e(F'')} n^{v(F'')}\right) \le \frac{\mathrm{Var}[Z_{F''}]}{q_n^{2e(F'')} n^{2v(F'')}} = o(1).$$

Moreover, since $q_n \le cn^{-1/m(F)}$ and $e(F'') \ge 2$ it follows from the choice of c that

$$2q_n^{e(F'')} n^{v(F'')} \le \frac{1-(\pi(F)+\varepsilon)}{2} q_n n^\ell,$$

which yields the 0-statement in this case. □

The 1-statement of Theorem 2.7. Let F be an ℓ-uniform hypergraph with at least Theorem 3.3. For that we consider the sequence of k-uniform hypergraphs $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$ where $V_n = E(K_n^{(\ell)})$ and edges of E_n correspond to copies of F in K_n. Moreover, we set $p_n = n^{-1/m(F)}$ and $\alpha = \pi(F)$. Clearly, for this set up the conclusion of Theorem 3.3 yields the 1-statement of Theorem 2.7 for sequences $\boldsymbol{q}$ with $q_n = o(1)$. In order to apply Theorem 3.3 we have to verify the three conditions (a)-(c) stated in the proof of the 1-statement of Theorem 2.3.

Condition (a) follows from the definitions of p_n and E_n combined. In fact, since F contains a vertex of degree at least 2 we have $m(F) \ge 1/(\ell-1)$ and $p_n|E_n| = \Omega(n)$. Such a result was obtained by Erdős and Simonovits [11, Theorem 1] and, hence, it is left to verify condition (c) only.

To this end observe that H_n is a regular hypergraph with $\binom{n}{\ell}$ vertices and every vertex is contained in $\Theta(n^{v(F)-\ell})$ edges and that $|E_n| = \Theta(n^{v(F)})$. We will show that for $q \ge n^{-1/m(F)}$ and $i \in [k-1]$ we have

$$\mu_i(H_n, q) = \mathbb{E}\left[\sum_{v\in V_n} \deg_i^2(v, V_{n,q})\right] = \sum_{v\in V} \mathbb{E}\left[\deg_i^2(v, V_{n,q})\right] = O\left(q^{2i}\frac{|E_n|^2}{|V_n|}\right).$$

Due to the definition of $\boldsymbol{H}$ every $v \in V_n$ corresponds to an edge $e(v)$ in $K_n^{(\ell)}$. Therefore, the number $\mathbb{E}\left[\deg_i^2(v, V_{n,q})\right]$ is the expected number of pairs (F_1, F_2) of copies F_1 and F_2 of F in $K_n^{(\ell)}$ satisfying $e(v) \in E(F_1) \cap E(F_2)$ and both copies F_1 and F_2 have at least i edges in $E(G^{(\ell)}(n,q)) \setminus \{e(v)\}$. Summing over all such pairs

F_1 and F_2 we obtain

$$\begin{aligned}\mathbb{E}\left[\deg_i^2(v,V_{n,q})\right] &\leq \sum_{F_1,F_2\colon e(v)\in E(F_1)\cap E(F_2)} \sum_{j=0}^{|E(F_1)\cap E(F_2)|-1} q^{2i-j} \\ &= O\left(\sum_{F_1,F_2\colon e(v)\in E(F_1)\cap E(F_2)} q^{2i-(|E(F_1)\cap E(F_2)|-1)}\right)\end{aligned} \tag{41}$$

since $q \leq 1$. Furthermore,

$$\begin{aligned}&\sum_{F_1,F_2\colon e(v)\in E(F_1)\cap E(F_2)} q^{2i-(|E(F_1)\cap E(F_2)|-1)} \\ &\qquad = O\left(\sum_{J\colon e(v)\in E(J} n^{2v(F)-2v(J)} q^{2i-(e(J)-1)}\right),\end{aligned} \tag{42}$$

where the sum on the right-hand side is indexed all hypergraphs $J \subseteq K_n^{(\ell)}$ which contain $e(v)$ and which are isomorphic to a sub-hypergraph of F. It follows from the definition of $m(F)$ and $q \geq n^{-1/m(F)}$ that $n^{v(J)}q^{e(J)} = \Omega(qn^{\ell})$. Combining this with (41) and (42) we obtain

$$\begin{aligned}\mathbb{E}\left[\deg_i^2(v,V_{n,q})\right] &= O\left(\sum_{J\colon e(v)\in E(J)} n^{2v(F)-2v(J)} q^{2i-(e(J)-1)}\right) \\ &= O\left(\sum_{J\colon e(v)\in E(J)} n^{2v(F)-v(J)-\ell} q^{2i}\right).\end{aligned}$$

Moreover, since $v(J) \geq \ell$ we have

$$\mathbb{E}\left[\deg_i^2(v,V_{n,q})\right] = O\left(\sum_{J\colon e(v)\in E(J)} n^{2v(F)-2\ell} q^{2i}\right),$$

and, consequently,

$$\mu_i(H_n,q) = \sum_{v\in V_n} O(n^{2v(F)-2\ell}q^{2i}) = O(n^{2v(F)-\ell}q^{2i}) = O\left(q^{2i}\frac{|E_n|^2}{|V_n|}\right).$$

This concludes the proof of condition (c) and, therefore, Theorem 3.3 yields the proof of the 1-statement of Theorem 2.7 for sequences $\boldsymbol{q}$ satisfying $q_n = o(1)$. The remaining case, when $q_n = \Omega(1)$ then follows by similar arguments as given in [25, Proposition 8.6] and we omit the details. □

References

1. L. Babai, M. Simonovits, and J Spencer, *Extremal subgraphs of random graphs*, J. Graph Theory **14** (1990), no. 5, 599–622. 2.4
2. B. Bollobás, *Extremal graph theory*, London Mathematical Society Monographs, vol. 11, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978. 1
3. ———, *Modern graph theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998. 1
4. ———, *Random graphs*, second ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001. 1

5. J. A. Bondy and U. S. R. Murty, *Graph theory*, Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008. 1
6. D. Conlon, personal communication. 1
7. R. Diestel, *Graph theory*, third ed., Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 2005. 1
8. P. Erdős, *On sequences of integers no one of which divides the product of two others and on some related problems*, Mitt. Forsch.-Inst. Math. und Mech. Univ. Tomsk **2** (1938), 74–82. 2.4
9. ———, *On extremal problems of graphs and generalized graphs*, Israel J. Math. **2** (1964), 183–190. 2.4
10. P. Erdős and M. Simonovits, *A limit theorem in graph theory*, Studia Sci. Math. Hungar **1** (1966), 51–57. 1, 2.4
11. ———, *Supersaturated graphs and hypergraphs*, Combinatorica **3** (1983), no. 2, 181–192. 4.4
12. P. Erdős and A. H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091. 1, 2.4
13. P. Erdős and P. Turán, *On some sequences of integers.*, J. Lond. Math. Soc. **11** (1936), 261–264. 2.1
14. P. Frankl, R. L. Graham, and V. Rödl, *Quantitative theorems for regular systems of equations*, J. Combin. Theory Ser. A **47** (1988), no. 2, 246–261. 2.2, 4.2
15. P. Frankl and V. Rödl, *Large triangle-free subgraphs in graphs without K_4*, Graphs Combin. **2** (1986), no. 2, 135–144. 2.4, 2.4
16. E. Friedgut, V. Rödl, and M. Schacht, *Ramsey properties of random discrete structures*, submitted. 2.3
17. Z. Füredi, *Random Ramsey graphs for the four-cycle*, Discrete Math. **126** (1994), no. 1-3, 407–410. 2.4
18. H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for commuting transformations*, J. Analyse Math. **34** (1978), 275–291 (1979). 2.1, 4.1
19. S. Gerke, *Random graphs with constraints*, Habilitationsschrift, Institut für Informatik, Technische Universität München, 2005. 2.4
20. S. Gerke, T. Schickinger, and A. Steger, *K_5-free subgraphs of random graphs*, Random Structures Algorithms **24** (2004), no. 2, 194–232. 2.4
21. R. Graham, V. Rödl, and A. Ruciński, *On Schur properties of random subsets of integers*, J. Number Theory **61** (1996), no. 2, 388–408. 2.3
22. R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey theory*, second ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., New York, 1990, A Wiley-Interscience Publication. 2.2
23. P. E. Haxell, Y. Kohayakawa, and T. Łuczak, *Turán's extremal problem in random graphs: forbidding even cycles*, J. Combin. Theory Ser. B **64** (1995), no. 2, 273–287. 2.4
24. ———, *Turán's extremal problem in random graphs: forbidding odd cycles*, Combinatorica **16** (1996), no. 1, 107–122. 2.4
25. S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000. 1, 2.1, 2.4, 3.1, 4.3, 4.4
26. G. Katona, T. Nemetz, and M. Simonovits, *On a problem of Turán in the theory of graphs*, Mat. Lapok **15** (1964), 228–238. 2.4
27. Y. Kohayakawa, B. Kreuter, and A. Steger, *An extremal problem for random graphs and the number of graphs with large even-girth*, Combinatorica **18** (1998), no. 1, 101–120. 2.4
28. Y. Kohayakawa, T. Łuczak, and V. Rödl, *Arithmetic progressions of length three in subsets of a random set*, Acta Arith. **75** (1996), no. 2, 133–163. 2.1
29. ———, *On K^4-free subgraphs of random graphs*, Combinatorica **17** (1997), no. 2, 173–213. 2.6, 2.4
30. Y. Kohayakawa, V. Rödl, and M. Schacht, *The Turán theorem for random graphs*, Combin. Probab. Comput. **13** (2004), no. 1, 61–91. 2.4
31. T. Kövari, V. T. Sós, and P. Turán, *On a problem of K. Zarankiewicz*, Colloquium Math. **3** (1954), 50–57. 2.4
32. B. Kreuter, *Probabilistic versions of ramsey's and turán's theorems*, Ph.D. thesis, Humboldt-Universität zu Berlin, 1997. 2.4
33. W. Mantel, *Vraagstuk XXVIII*, Wiskundige Opgaven **10** (1907), 60–61. 2.4
34. R. Rado, *Studien zur Kombinatorik*, Math. Z. **36** (1933), no. 1, 424–470. 2.2

35. V. Rödl and A. Ruciński, *Threshold functions for Ramsey properties*, J. Amer. Math. Soc. **8** (1995), no. 4, 917–942. 3.1, 3.6, 3.1
36. ———, *Rado partition theorem for random subsets of integers*, Proc. London Math. Soc. (3) **74** (1997), no. 3, 481–502. 2.2, 2.2, 2.2, 4.2, 4.2, 4.2, 4.2
37. V. Rödl, A. Ruciński, and M. Schacht, *Ramsey properties of random k-partite, k-uniform hypergraphs*, SIAM J. Discrete Math. **21** (2007), no. 2, 442–460 (electronic). 2.4
38. I. Schur, *Über die Kongruenz* $x^m + y^m \equiv z^m (mod\ p)$, Jahresber. Deutsch. Math.-Verein. **25** (1916), 114–117. 2.3
39. T. Szabó and V. H. Vu, *Turán's theorem in sparse random graphs*, Random Structures Algorithms **23** (2003), no. 3, 225–234. 2.4
40. E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245, Collection of articles in memory of Juriĭ Vladimirovič Linnik. 1
41. P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452. 1, 2.4

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