Spanning trees in hyperbolic graphs

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Abstract

In this paper we construct spanning trees in hyperbolic graphs that represent their hyperbolic compactification in a good way: so that the tree has a bounded number of distinct rays to each boundary point. The bound depends only on the (Assouad) dimension of the boundary. As a corollary we sharpen a result of Gromov which says that from every hyperbolic graph with bounded degrees one can construct a tree outside the graph with a continuous surjection from the ends of the tree onto the hyperbolic boundary such that the surjection is finite-to-one. We will construct a tree with these properties inside the hyperbolic graph, which in addition is also a spanning tree of that graph.

1 Introduction

A spanning tree of a graph is called *end-faithful* if the tree contains exactly one ray from each end, starting at the root. Halin [6] proved that every countable graph has an end-faithful spanning tree. So it is a natural question to ask - if we replace the end-compactification of a graph by other compactifications that refine the end-compactification - what we can expect of the spanning tree with respect to the new compactification: Is it possible that the ends of a spanning tree represent the boundary points of another compactification also in a one-to-one correspondence?

In this paper we study such a generalization of end-faithful spanning trees to spanning trees in hyperbolic graphs, replacing the end-compactification by the hyperbolic compactification.

A *hyperbolic graph* G is a locally finite connected graph for which there exists a δ such that for every three vertices every geodesic between two of them is contained in a δ-neighbourhood of the union of any two geodesics between the two other vertices. A *hyperbolic boundary point* is an equivalence class of the following equivalence relation of geodetic rays: Two geodetic rays $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are *equivalent* if $\liminf_{i\to\infty} d(x_i, y_i)$ is bounded. The *hyperbolic boundary* ∂G is the set of all hyperbolic boundary points. This is one of many equivalent definitions of the hyperbolic boundary (see [4, 9] and Section 2 of this paper).

It is not in general possible to have spanning trees which are faithful to hyperbolic boundary points instead of ends. An easy example is the graph G of Figure 1. Its hyperbolic boundary is an arc A. Now consider any spanning tree T of G. Whenever a vertex x separates T into at least two infinite components, any two of these either have a common boundary point or there is a boundary point that separates them on A. In the first case, the common boundary point is the limit of two inequivalent rays, while in the second case the separating boundary point is not a limit of any ray. In either case, therefore, the tree is not faithful to the boundary. We shall return to this example in Section 4.

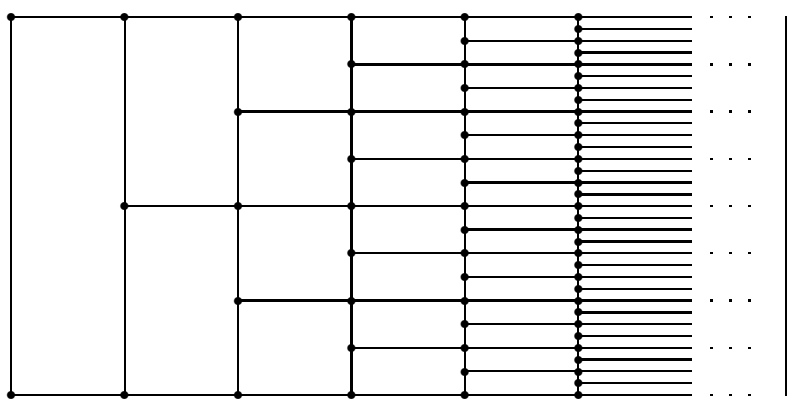

Figure 1: A hyperbolic graph with its boundary

Instead of spanning trees that are faithful to boundary points, we may perhaps hope that we get spanning trees that have only a global bounded number of distinct paths from the root to each boundary point. This is indeed true if the boundary has finite Assouad dimension, which is our main result:

Theorem 1.1. *Let G be a locally finite hyperbolic graph whose boundary ∂G has finite Assouad dimension. Then there exists an $n \in \mathbb{N}$, depending only on the dimension, and a rooted spanning tree T of G, with the following properties:*

(i) Every ray in T converges to a point in the boundary of G;

(ii) for every boundary point η of G there is a ray in T converging to η;

(iii) for every boundary point η of G there are at most n distinct rays in T that start at the root of T and converge to η.

We prove Theorem 1.1 in section 5.

Gromov [5, §7.6] states the following theorem:

Theorem 1.2. *Let X be a locally finite δ-hyperbolic graph with maximum degree $N < \infty$. Then there is a locally finite tree $T(X)$ with maximum degree at*

most $\exp(\exp((\delta+1)N))$ *with a continuous map* $\partial T \to \partial X$ *that is finite-to-one; additionally a boundary point of* X *has at most* $\exp(\exp(\exp((\delta+1)N)))$ *preimages.* □

Gromov constructed the tree $T(X)$ independently of the local structure of the graph X, just depending on the metric of X. Thus a vertex in $T(X)$ may have higher degree than all vertices in X.

Let G be a locally finite hyperbolic graph and let T be a subtree of G such that every ray in T converges to some boundary point in G. Let $\iota : T \cup \partial T \to G \cup \partial G$ be the continuous extension of the identity on T. We call the restriction of ι to the boundary of T the *canonical map* $\partial T \to \partial G$.

As a corollary of Theorem 1.1 we obtain the following strengthening of Theorem 1.2:

Theorem 1.3. *Let* G *be a locally finite hyperbolic graph with maximum degree* $N < \infty$*. Then there exists a rooted spanning tree* T *of* G *with a canonical map* $\partial T \to \partial G$ *that has at most* M *preimages of each boundary point of* G*, where* M *is a constant depending only on* N*.*

As the hyperbolic boundary is defined as equivalence classes of geodetic rays, a natural class of spanning trees is the class of *geodetic spanning trees.* These are spanning trees that preserve the distance to the root from the distance-metric of the graph. Thus any ray in any geodetic spanning tree of a locally finite hyperbolic graph converges to a hyperbolic boundary point. But these spanning trees does not fulfill the conclusion (iii) of Theorem 1.1: there is no bound on the maximum number of ends of the tree mapping canonically to a given hyperbolic boundary point; indeed, there can be infinitely many (Example 4.2). However, we shall obtain a lower bound on the maximum number of tree ends mapping to a common hyperbolic boundary point. This bound depends only on the topological dimension of the hyperbolic boundary.

Theorem 1.4. *Let* G *be a locally finite hyperbolic graph whose boundary has topological dimension* $n \in \mathbb{N}$*. Then every rooted geodetic spanning tree* T *has the following property:*

(∗) *There is a boundary point* $\eta \in \partial G$ *with at least* $\frac{n+1}{2}$ *distinct rays starting in the root and converging to* η*.*

In Section 3 we will give explicit definitions of the two dimensional concepts we use, the Assouad dimension and the topological dimension, and state some of their properties. For a more detailled introduction to the Assouad dimension we refer to Luukkainen [8, Appendix A].

2 Hyperbolic graphs

Let $G = (V, E)$ be a graph. A *geodesic* is a path between two vertices x and y with length $d(x, y)$ and denoted by $[x, y]$. A *triangle* is a set of three vertices

(not necessarily distinct) - called *corners* of the triangle - together with paths between each two of these vertices. These paths are called *sides* of the triangle. The triangle is *geodetic* if all sides of the triangle are geodesics. We write $[x, y, z]$ for a geodetic triangle with corners x, y and z.

We are investigating G from a topological point of view, so that every edge of G can be understood as a homeomorphic image of the real interval $[0, 1]$.

The graph G is called *(δ-)hyperbolic* if there exists a δ such that for every geodetic triangle $[x, y, z]$ each of its sides lies in a δ-neighbourhood of the other two sides.

Let o be a vertex in G. The *Gromov-product* (with respect to o) for two vertices x and y is $(x, y)_o := \frac{1}{2}(d(x, o) + d(y, o) - d(x, y))$. If it is obvious by the context that we use o as the base-point for the product, we simply write (x, y). An easy proposition is due to Gromov.

Proposition 2.1. [5, 1.1B] *Let G be a graph and $o \in VG$. If*

$$(x, y)_o \geq \min\{(x, z)_o, (y, z)_o\} - \delta$$

for all $x, y, z \in VG$, then there is

$$(x, y)_w \geq \min\{(x, z)_w, (y, z)_w\} - 2\delta$$

for every $w \in VG$. □

Another definition of hyperbolicity uses the Gromov-product. So one might expect that this definition depends on the vertex o, but Proposition 2.1 has shown to us that this is not the case. See for example [1, Proposition 2.1] for a proof of the following Proposition.

Proposition 2.2. *A locally finite graph G is hyperbolic if and only if there is a vertex o and some $\delta \in \mathbb{R}_{\geq 0}$ with $(x, y)_o \geq \min\{(x, z)_o, (y, z)_o\} - \delta$ for all $x, y, z \in VG$.* □

We are now introducing the ends of infinite graphs and the hyperbolic boundary of infinite hyperbolic graphs. A *ray* is a one-way infinite path, a *double ray* is a two-way infinite path. Two rays are *equivalent* if no finite set of vertices separates them. An *end* is an equivalence class of rays. A *geodetic ray* is a ray $\pi = x_0 x_1 \ldots$ with $d(x_i, x_j) = |i - j|$ for all $i, j \geq 0$, and a double ray $\ldots x_{-1} x_0 x_1 \ldots$ is a *geodetic double ray* if $d(x_i, x_j) = |i - j|$ for all $i, j \in \mathbb{Z}$. A well-known fact is the following:

Proposition 2.3. [9, (22.12)] *The equivalence of geodetic rays in hyperbolic graphs is an equivalence relation.* □

Hence we are able to define the *hyperbolic boundary* of a hyperbolic graph: A *hyperbolic boundary point* is an equivalence class of geodetic rays. Let ∂G be the set of hyperbolic boundary points, and let $\widehat{G}$ be $G \cup \partial G$.

We are also giving a second topological definition of the hyperbolic boundary: A sequence $(x_i)_{i \geq 0}$ *converges* to a vertex x if $\lim_{i \to \infty} (x_i, x) = 0$. A sequence $(x_i)_{i \geq 0}$

converges to ∞ if $\lim_{i,j\to\infty}(x_i, x_j) \to \infty$. Like above, it is independent of the choice of o, so we just wrote (x_i, x_j) instead of $(x_i, x_j)_o$. Two sequences $(x_i)_{i\geq 0}$, $(y_j)_{j\geq 0}$ are *equivalent* if $\lim_{i,j\to\infty}(x_i, y_j) = \infty$. In hyperbolic graphs this equivalence is indeed an equivalent relation. The hyperbolic boundary can also be defined as equivalence classes of this equivalence relation. A sequence $(x_i)_{i\geq 0}$ *tends* to a boundary point if it is in its equivalence class. In [4] the equivalence of these definitions is shown.

A third way to define the boundary is by defining a metric d_ε on G and then defining $\widehat{G}$ as the completion of G induced by d_ε. Let $\varepsilon > 0$ with $\varepsilon' := \exp(\varepsilon\delta) - 1 \leq \sqrt{2} - 1$. Let

$$\varrho_\varepsilon(x, y) := \exp(-\varepsilon(x, y)),$$

$$\varrho_\varepsilon(x_0, \ldots, x_n) := \sum_{i=1}^{n} \varrho_\varepsilon(x_{i-1}, x_i)$$

and

$$d_h(x, y) := \inf\{\varrho_\varepsilon(c) \mid c \text{ chain between } x \text{ and } y\}.$$

It is easy to check that d_ε is a metric on G.

An important theorem about the hyperbolic boundary is the following.

Theorem 2.4. [4, Proposition 7.2.9] *If G is a locally finite hyperbolic graph, then $(\widehat{G}, d_\varepsilon)$ is a compact metric space.* □

We will now define a topology on G, which is compatible with the topology of $\widehat{G}$ which is induced by d_ε.

For two vertices and/or hyperbolic boundary points a and b we define the *Gromov-product* (once more):

$$(a, b) := \sup \liminf_{i,j\to\infty}(x_i, y_j)$$

where the supremum is taken over all sequences $(x_i)_{i\geq 0} \to a$ and $(y_i)_{i\geq 0} \to b$. Obviously it is just the same as the previous definition for vertices, so we were allowed to use the same symbol. Let $N_k(x) := \{y \in \widehat{G} | (x, y) > k\}$ for every $x \in \partial G$ and every $k \in \mathbb{R}_{\geq 0}$ and let $B_r(x) = \{y \in VG | d(x, y) < r\}$ for every $x \in VG$ and $r \in \mathbb{R}_{\geq 0}$.

Proposition 2.5. [1, Proposition 4.8] *Let G be a locally finite hyperbolic graph. The union of the sets $B_r(x)$ for all $x \in VG$ and all $r \in \mathbb{R}_{\geq 0}$ and $N_k(x)$ for all $x \in \partial G$ and all $k \in \mathbb{R}_{\geq 0}$ form a basis for a topology on $\widehat{G}$.* □

This topology is compatible with the metrics d_ε, which makes the boundary to a compact metric space by Proposition 2.6.

Proposition 2.6. [4, Proposition 7.3.10] *Let G be a locally finite hyperbolic graph. There exists a metric d_ε on $\widehat{G}$ such that $(\widehat{G}, d_\varepsilon)$ is a compact metric*

space and such that the metric is compatible with the just defined topology in the sense that

$$\varepsilon' \cdot \exp(-\varepsilon \cdot (\eta, \nu)) \leq d_\varepsilon(\eta, \nu) \leq \exp(-\varepsilon \cdot (\eta, \nu))$$

for all $\eta, \nu \in \partial G$ *and for* $\varepsilon' = \exp(\varepsilon\delta) - 1$.

In addition every ε *with* $\varepsilon' \leq \sqrt{2} - 1$ *has this property.* □

Proposition 2.6 is the reason why it possible that we will use the metric in some place, the topology in some other place, and sometimes use them both together.

Proposition 2.7. [4, Proposition 7.5.17] *Let* G *be a locally finite hyperbolic graph. There exists a continuous surjection from the hyperbolic boundary of* G *to its set of ends whose fibres are the connected components of* ∂G. □

We will state some propositions that we will need later.

Proposition 2.8. [9, (22.11) and (22.15)] *Let* G *be a locally finite hyperbolic graph with two distinct boundary points* η *and* ν. *Let* o *be a vertex in* G, $(x_i)_{i\in\mathbb{N}}$ *a geodetic ray converging to* η, *and* $(y_j)_{j\in\mathbb{N}}$ *a geodetic ray converging to* ν. *Then the following two properties holds:*

(i) There is a geodetic ray in G *starting in* o *and having only finitely many vertices different from* $(x_i)_{i\in\mathbb{N}}$.

(ii) There is a geodetic double ray having only finitely many vertices different from $(x_i)_{i\in\mathbb{N}}$ *and* $(y_j)_{j\in\mathbb{N}}$. *One side of the geodetic double ray converges to* η, *the other to* ν. □

Proposition 2.9. [1, Lemma 4.6 (4)] *Let* G *be a* δ*-hyperbolic graph. Then the inequalities*

$$(x, y) \leq d(z, [x, y]) \leq (x, y) + 2\delta$$

holds for all $x, y, z \in VG$. □

Proposition 2.10. [4, Remark 7.2.7] *Let* G *be a locally finite* δ*-hyperbolic graph, and let* η *and* ν *be hyperbolic boundary points. There is*

$$(\eta, \nu) - 2\delta \leq \liminf(x_i, y_j) \leq (\eta, \nu)$$

for all sequences $(x_i)_{i\in\mathbb{N}} \to \eta$ *and* $(y_i)_{i\in\mathbb{N}} \to \nu$. □

A direct consequence of the Propositions 2.9 and 2.10 is Proposition 2.11.

Proposition 2.11. *Let* G *be a locally finite* δ*-hyperbolic graph, let* η *and* ν *be hyperbolic boundary points of* G, *and let* o *be the base-point of the Gromov-product.*

$$(\eta, \nu) - 2\delta \leq d(o, \pi) \leq (\eta, \nu) + 2\delta$$

holds for all geodetic double rays π *from* η *to* ν. □

The proof of the following Lemma can be found for example in [3, Lemma 8.1.2].

Lemma 2.12. *Let $V_0, V_1, \ldots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph with $VG = \bigcup_{i\in\mathbb{N}} V_i$. Assume that every vertex v in a set V_n with $n \geq 1$ has a neighbour $f(v)$ in V_{n-1}. Then G contains a ray $v_0v_1\ldots$ with $v_n \in V_n$ for all n.* □

A *spanning tree* of a graph G is a tree with vertex set VG whose edge set is a subset of EG. A *geodetic spanning tree with root r* is a spanning tree with *root* $r \in VG$ such that for every $v \in VG$ there is $d_G(r, v) = d_T(r, v)$.

3 Dimensions of topological spaces

Let us introduce the first dimension, just depending on the topology of a space: Let X be a topological space. A *refinement* $\mathcal{U}$ of an open cover $\mathcal{V}$ of X is an open cover of X such that for every $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ with $U \subseteq V$. X has *topological dimension at most* n if every open cover has a refinement such that each $x \in X$ lies in at most $n + 1$ elements of the refinement, and X has *topological dimension* n (notation; $\dim(X) = n$) if it has topological dimension at most n but not topological dimension at most $n - 1$. If there exists no $n \in \mathbb{N}$ such that X has topological dimension at most n then X has *infinite topological dimension*. Let X be an n-dimensional topological space, and let $\mathcal{U}$ be an open cover of X; $\mathcal{U}$ is *critical* if there exists no refinement $\mathcal{V}$ such that each $x \in X$ lies in at most n sets $V \in \mathcal{V}$.

Let us now introduce the second dimension, depending on the metric of a space: Let X be a metric space. For $\alpha, \beta > 0$ let $S(\alpha, \beta)$ be the maximal cardinality of a subset V of X with $\alpha \leq d_X(x, y) \leq \beta$ for all $x \neq y \in V$. Let n be the infimum of all $s \geq 0$ such that there is a $C \geq 0$ with $S(\alpha, \beta) \leq C(\frac{\beta}{\alpha})^s$ for all $0 < \alpha \leq \beta$. Then n is called the *Assouad dimension* of the metric space X (notation: $\dim_A(X) = n$).

Furthermore we introduce a property of metric spaces. Let X be a metric space. X is *doubling* if there is $\kappa \geq 1$ such that every ball of radius r can be covered by at most 2^κ balls of radius at most $\frac{r}{2}$. Let $\dim_2(X)$ be the infimum of all κ such that X is doubling with this κ. A subset Y of X has *diameter* $\sup\{d(x, y) \mid x, y \in Y\}$ (notation: $\operatorname{diam}(Y)$), and a set $\mathcal{Y}$ of subsets of X has *diameter* $\operatorname{diam}(\mathcal{Y}) = \sup\{\operatorname{diam}(Y) \mid Y \in \mathcal{Y}\}$. For every $r \geq 0$, a family $\mathcal{B} = (B_i)_{i\in I}$ of subsets of X has *r-multiplicity* at most n if every subset of X with diameter at most r intersects with at most n members of the family. A point $x \in X$ has *r-multiplicity* at most n in $\mathcal{B}$ if $\overline{B_r(x)}$ intersects with at most n members of the family $\mathcal{B}$ non-trivially.

For a metric space X it is equivalent that $\dim_2(X)$ is finite and that $\dim_A(X)$ is finite by the following theorem of Luukkainen [8, Theorem A.3].

Theorem 3.1. *Let X be a metric space. X is doubling if and only if X has finite Assouad dimension.* □

It is easy to adapt the proof of [7, Lemma 2.3] for Lemma 3.2.

Lemma 3.2. *Let X be a metric space with $\dim_2(X) = \kappa$ and let $r > 0$. Then X has a covering $\mathcal{B}$ of closed balls of radius r such that $\mathcal{B} = \bigcup_{k=0}^{2^\kappa} \mathcal{B}_k$ and each family $\mathcal{B}_k$ has r-multiplicity at most 1; so $\mathcal{B}$ has r-multiplicity at most 2^κ.*

Furthermore it is possible, to choose a given subset Y of X with $d(x, y) > r$ for all $x, y \in Y$ so that Y is a subset of the set of centers of the balls B_k. □

Remark 3.3. Let X be a metric space. Then there is $\dim(X) \leq \dim_A(X)$. This follows directly from the definitions of both dimensions.

4 Two examples

In this section we will give two examples: The first example is a locally finite hyperbolic graph that has no rooted spanning tree with exactly one ray from the root to each boundary point of G. The second example is a locally finite hyperbolic graph that has only one boundary point but every rooted geodetic spanning tree (T, r) has infinitely many distinct rays starting in r and converging to the only boundary point.

Example 4.1. Let us construct the following graph G: Let the *layer* k be a set of $2^{k-1} + 1$ vertices x_1^k to $x_{2^{k-1}+1}^k$. Let VG be the disjoint union of the layers k for all $k \in \mathbb{N}$. Let two vertices in the same layer k be adjacent if and only if they are x_i^k and x_{i+1}^k for some $i \leq 2^{k-1} + 1$. Two vertices of different layers are adjacent if and only if the vertices are x_i^k and x_j^{k+1} with $2(i-1) + 1 = j$. The resulting graph is a planar graph with one end but whose hyperbolic boundary is homeomorphic to the unit interval $[0, 1]$.

Let us suppose that there exists a rooted spanning tree (T, r) such that every ray in T converges to some boundary point and such that there is exactly one r-η-path for every boundary point η. Then there is a vertex x such that $T - x$ has at least two infinite components. Let C_1 and C_2 be two components of $T - x$ and let π_i be a ray in C_i $(i = 1, 2)$ such that the following properties holds for C_1, C_2, π_1 and π_2.

(i) The graph $G[C_1 \cup C_2]$ has only one end.

(ii) The graph $G[C_1 \cup C_2] - \pi_i$ has precisely two ends for every $i \in \{1, 2\}$.

(iii) All components of $G[C_i] - \pi_i$ that are adjacent to $G[C_j]$ with $i \neq j$ are finite.

Let φ be a homeomorphism from the boundary of G to $[0, 1]$. Let η_i be the boundary point of G to which π_i converges. We may assume that $\varphi(\eta_1) \leq \varphi(\eta_2)$. If $\varphi(\eta_1) \neq \varphi(\eta_2)$, then there is some $\eta \in \partial G$ with $\varphi(\eta_1) < \varphi(\eta) < \varphi(\eta_2)$. This contradicts the choice of π_1 and π_2. Thus $\varphi(\eta_1) = \varphi(\eta_2)$ and $\eta_1 = \eta_2$ since φ is a homeomorphism. This completes Example 4.1.

Example 4.2. Let V_k be a set of 2^k elements such that the V_k are pairwise disjoint. Let G be a graph with vertex set $\bigcup_{k\in\mathbb{N}} V_k$. Any two vertices of the same V_k are adjacent. Furthermore any $x \in V_k$ with $k \neq 0$ has precisely one neighbour in V_{k-1}, two neighbours in V_{k+1}, and no other neighbours.

This graph is obviously a hyperbolic graph with one end and one boundary point. Let T be a geodetic spanning tree in G with root r. For every vertex x in G there is a subgraph H of G such that H is isometric to G and x is mapped to $o \in V_0$. If the graph with $r = o$ has infinitely many distinct r-η-paths for the only boundary point η, then this is the case for any T with arbitrary r. For every vertex y there is a unique geodesic from o to y. Since the only boundary point has infinitely many distinct geodetic rays converging to it, T has to contain them, too. This proves our claim on G.

5 Spanning trees in hyperbolic graphs

In this section we will prove our main result Theorem 1.1 and deduce some corollaries from that theorem.

Proof of Theorem 1.1. Let $d_h = d_\varepsilon$ be one of the metrics of Theorem 2.4 such that $(\widehat{G}, d_h)$ is a compact metric space with finite Assouad dimension. In particular, $\varepsilon \geq 0$ and $\exp(\varepsilon\delta) - 1 \leq \sqrt{2} - 1$. Let $r \geq 0$, and let $m = \inf\{(\eta,\mu) \mid \eta,\mu \in \partial G, d_h(\eta,\mu) \leq r\}$. Then by Proposition 2.6 there is a δ' such that any two boundary points η', μ' with $(\eta',\mu') \geq m - 5\delta$ have distance at most $\varepsilon\delta'$, i.e. $d_h(\eta',\mu') \leq \varepsilon\delta'$. For this δ' the following inequality holds.

$$\delta' = \exp(5\varepsilon\delta) \leq (\sqrt{2})^5 < 8$$

Let us first construct the spanning tree T and thereafter we will show that T fulfills the properties (i) to (iii) of the theorem. We will construct the spanning tree inductively. In each step of the induction there is an ε_{j-1} (with $\varepsilon_{j-1} > \varepsilon_j$) such that ∂G is covered by the open balls of radius ε_{j-1} and with those boundary points as centers to which we have already constructed a ray to.

Since ∂G has finite Assouad dimension we may assume by Theorem 3.1 that ∂G is doubling. Let $r \in VG$, and let $N = 2^{\dim_2(\partial G)}$. For the first step choose a boundary point $\eta \in \partial G$. Let $S_0 = \{\eta\} = Y_0$, and let T_0 be the graph consisting of a geodetic ray from r to η.

For the step j of the construction let T_{j-1} be the tree, constructed in the previous step, let S_{j-1} be the set of boundary points for that T_{j-1} contains a ray converging to that boundary point, and let $\mathcal{U}_{j-1}$ be the set of all open ε_{j-1}-balls with centers in S_{j-1}. Furthermore we may assume that $\mathcal{U}_{j-1}$ is an open cover of ∂G and that the tree T_{j-1} has the following properties.

($*$) Every edge in T_{j-1} lies on such a double ray between two boundary points in S_{j-1} that is a geodetic double ray in G or the edge lies in the tree T_0.

($**$) Every ray in T_{j-1} is eventually geodetic.

By Lemma 3.2 there is a closed covering $\mathcal{B}_j$ of ∂G with balls of radius $\frac{\varepsilon_{j-1}}{16}$, with $\frac{\varepsilon_{j-1}}{16}$-multiplicity at most N and such that the set Y_j of centers of these balls is a superset of S_{j-1}. Let $\varepsilon_j = \frac{a}{8N}$ with $a = \frac{\varepsilon_{j-1}}{16}$, and let S_j be a subset of ∂G with $S_{j-1}, Y_j \subseteq S_j$, $d_h(\eta, \mu) \geq \varepsilon_j$ for all $\eta, \mu \in S_j$, $\frac{\varepsilon_{j-1}}{16}$-multiplicity at most $N^{\log_2(8N)}$ and such that $\left\{B_{\varepsilon_j}(s) | s \in S_j\right\}$ is an open cover of ∂G. We obtain this set by applying the definition of doubling $\log_N(N^{\log_2(8N)})$ times to the sets in $\mathcal{B}_j$. For every $\eta \in S_j \setminus S_{j-1}$ we will add a new ray to T_{j-1} and get T_j.

Let $T_j^0 = T_{j-1}$. Let $S_j \setminus S_{j-1} = \left\{\mu_1, \ldots, \mu_{|S_j \setminus S_{j-1}|}\right\}$ with the property that all μ_i with $8\varepsilon_{j-1}$-multiplicity 1 in $\mathcal{B}_{j-1}$ have a smaller index than those that have $2 \cdot 8\varepsilon_{j-1}$-multiplicity 2 in $\mathcal{B}_{j-1}$ and so on until we have those that have $N \cdot 8\varepsilon_{j-1}$-multiplicity N in $\mathcal{B}_{j-1}$. Since the $\mathcal{B}_{j-1}$ have $\frac{\varepsilon_{j-2}}{16}$-multiplicity at most N and $8\varepsilon_{j-1}$ is less than $\frac{\varepsilon_{j-2}}{16}$, the radius of all $B \in \mathcal{B}_{j-1}$, any point in ∂G has $8\varepsilon_{j-1}$-multiplicity at most N. Thus we have enumerated the whole $S_{j-1} \setminus S_{j-1}$.

Let us build new rays to the μ_i one by one in the order they are enumerated. For every μ_i there is an $\eta \in S_{j-1}$ with $d_h(\mu_i, \eta) \leq \varepsilon_{j-1}$. Let π be a geodetic double ray from μ_i to η such that the new ray uses an infinite subray of the existing ray in T_j^{i-1} to η. This is possible by Proposition 2.8 and since the rays in T_j^{i-1} are eventually geodetic by construction and by the property $(**)$. It might happen that π intersects with T_j^{i-1} non-trivially apart from the common infinite subray to η. But then a common vertex is part of a geodetic double ray between two other boundary points of $S_{j-1} \cup \{\mu_1, \ldots, \mu_{i-1}\}$ in T_j^{i-1} (by the construction of T_j^{i-1}). To at least one of the endpoints (say η') of that geodetic double ray, μ_i has distance less than $\delta' \varepsilon_{j-1} \leq 8\varepsilon_{j-1}$ as $d(o, [\mu_i, \eta']) \geq d(o, [\mu_i, \eta])$ by the hyperbolicity of G and $(\mu_i, \eta) - 5\delta \leq (\mu_i, \eta')$ by Proposition 2.11. Hence $d_h(\mu_i, \eta') \leq \varepsilon_{j-1}\delta'$. So we are adding an infinite subray to T_j^{i-1} and get a tree T_j^i.

Let us call μ_i *connected* to η if π intersects with T_j^{i-1} only on the common infinite subray to η and *connected* to η' else. If μ_i is connected to η then μ_i is *eventually connected* to η. If μ_i is connected to η' and $\eta' \in S_{j-1}$ then μ_i is *eventually connected* to η', and if μ_i is connected to η' but $\eta' \notin S_{j-1}$ then μ_i is *eventually connected* to the boundary point, η' is eventually connected to.

Let T_j be the union of all T_j^i, in other words

$$T_j = T_j^{|S_j \setminus S_{j-1}|}.$$

By the construction it is clear that $(*)$ and $(**)$ hold for T_j and that T_j is a tree.

Let $T' = \bigcup_{i \in \mathbb{N}} T_i$. Since all T_i are trees and $T_{i-1} \subseteq T_i$, T' is a tree.

There are two remaining things: First we have to add—without creating new rays—every vertex that lies in $G - T'$ with some edge to T' and then we have to check the properties (i) to (iii) of the theorem.

We are adding the vertices of $G - T'$ recursively by their distance to r to T'. First we can easily extend the tree by adding all finite components of $G - T'$ to

T'. Then we add every vertex with distance $d(r, G - T')$ to T' by a path lying outside of $B_{d(r,G-T')}(r)$. There might be vertices for that does not exist such a path. We do not add these. Let T_1' be the new tree. Then the vertices in $G - T_1'$ with distance $d(r, G - T_1')$ lie in finite components of $G - T_1'$. For the following step we keep in mind the largest distance d_1 from r to a vertex lying on $T_1' - T'$. In the step of recursion first we add again every finite component of $G - T_i'$. We take again paths to T_i' that are lying outside $B_{d_1}(r)$. Once more there can be vertices that cannot be connected to T_i' in such a way. These will be treated at the beginning of the next step.

Let $T = \bigcup_{i\in\mathbb{N}} T_i'$. Obviously T is a spanning tree of G and there is not any new ray created on the way from T' to T. Thus to prove the properties (i) to (iii) of the theorem, we only need to prove them for T'.

Let us first prove two claims.

Claim 5.1. *Let μ_{i_1} and μ_{i_2} be elements of $S_j \setminus S_{j-1}$ with $d_h(\mu_{i_1}, \mu_{i_2}) \leq 8\varepsilon_{j-1}$ and such that both do not have $(n-1)8\varepsilon_{j-1}$-multiplicity $n-1$ but $n8\varepsilon_{j-1}$-multiplicity n in $\mathcal{B}_{j-1}$. Then for any $B \in \mathcal{B}_{j-1}$ with $d_h(\mu_{i_1}, B) \leq n8\varepsilon_{j-1}$ there is $d_h(\mu_{i_2}, B) \leq n8\varepsilon_{j-1}$ and vice versa.*

Proof. Since the $(n-1)8\varepsilon_{j-1}$-multiplicity of both μ_{i_1} and μ_{i_2} must be n, every set with distance at most $n8\varepsilon_{j-1}$ to μ_{i_k} has distance at most $(n-1)8\varepsilon_{j-1}$ to μ_{i_k} and thus distance at most $n8\varepsilon_{j-1}$ to μ_{i_l} with $k \neq l$. □

Claim 5.2. *Let μ_{i+1} be connected to $\mu \in S_j$ in T_j^i. Then there is $d_h(\eta, \mu_{i+1}) \leq 8N\varepsilon_{j-1}$. If $\mu \in S_{j-1}$ is eventually connected to η in T_{j-1}, then $d_h(\eta, \mu) \leq 8N\varepsilon_{j-1}\mathrm{diam}(\mathcal{B}_{j-1}) \leq \frac{1}{2}N\varepsilon_{j-1}\varepsilon_{j-2}$.*

Proof. Any boundary point η with $8\varepsilon_{j-1}$-multiplicity 1 in $\mathcal{B}_{j-1}$ can only be connected to a boundary point μ with $d_h(\eta, \mu) \leq 8\varepsilon_{j-1}$ by the construction. Both these boundary points must lie in the same $B \in \mathcal{B}_{j-1}$. By induction we know that η is eventually connected to a boundary point μ' such that η and μ' lie in the same $B \in \mathcal{B}_{j-1}$. Let us assume that η has $k8\varepsilon_{j-1}$-multiplicity k in $\mathcal{B}_{j-1}$. Our aim was to connect η to a boundary point $\mu \in S_{j-1}$ with $d(\mu, \eta) \leq \varepsilon_{j-1}$. But by our construction η is connected to a boundary point ν with $d_h(\eta, \nu) \leq 8\varepsilon_{j-1}$ and $\nu \in S_{j-1} \cup \{\mu_1, \ldots, \mu_{i-1}\}$ if $\mu = \mu_i$. By claim 5.1 and induction, ν is contained in an element $B \in \mathcal{B}_{j-1}$ which is responsible for the $k8\varepsilon_{j-1}$-multiplicity of η in $\mathcal{B}_{j-1}$. This proves the first statement of claim 5.2. For the second statement it follows by induction that ν', the boundary point η is eventually connected to, lies in one of those $B \in \mathcal{B}_{j-1}$ which are responsible for the $k8\varepsilon_{j-1}$-multiplicity of η in $\mathcal{B}_{j-1}$ and since $\mathrm{diam}(\mathcal{B}_{j-1}) \leq \frac{\varepsilon_{j-2}}{16}$ we have $d_h(\eta, \nu') \leq k8\varepsilon_{j-1}\mathrm{diam}(\mathcal{B}_{j-1})$, the second statement of claim 2. □

Let us finally prove (i) to (iii) of the theorem. For a closed ball $B_k \in \mathcal{B}_k$ let B_k' denote B_k together with all other at most N^8 closed balls in $\mathcal{B}_k$ with distance at most $8N\varepsilon_k$ to B_k.

By $(**)$ we constructed the new rays always such that they have an infinite geodetic subray. Thus all we have to prove for (i) is that every ray we created

by the construction of infinitely many rays converges to some boundary point. Let us assume that π is a ray in T with the property that there exists infinitely many finite subpaths of π such that each of these subpaths was used by the construction of another ray. Since $\widehat{G}$ is compact, π has at least one limit point η in ∂G. Thus we have to prove that there exists no second limit point. Let $B_k \in \mathcal{B}_k$ be one of the closed balls in step k which contains η. Any second boundary point must lie - like η does - in $\bigcap_{k\in\mathbb{N}} B'_k$ by claim 5.2. Since $\bigcap_{k\in\mathbb{N}} B'_k$ is a set with at most one element, π has precisely one accumulation point.

For the proof of (ii) let η be a boundary point of G. Then for each k there is at least one closed ball B_k in the step k such that η is contained in B_k. In the construction (since $\eta \in B_k$) we have chosen a boundary point x_k in B_k with $d_h(x_k, \eta) \leq \varepsilon_k$ and constructed a ray π_k to x_k. Since G is locally finite, there is an infinite path π such that each edge of that path is contained in infinitely many of the rays to the x_k. π must have η as an accumulation point by the Gromov-product, by claim 2 and by the choice of the rays π_k. As (i) holds, π has precisely one accumulation point, η.

To any closed ball $B \in \mathcal{B}_k$ in step k there are at most N^8 closed balls in the step $k-1$ sending rays to boundary points in B and additional each ball sends at most $N^{\log_2(8N)}$ many rays to boundary points in B. Thus the number of rays to one boundary point in bounded by $N^{8+\log_2(8N)}$ and hence bounded by a function depending only on the doubling property of ∂G. Since for given ε the doubling property depends only on the Assouad dimension, this proves the only remaining part (iii) of Theorem 1.1. □

Theorem 1.1 tells us that the number of distinct rays to a boundary point is finite and bounded, if the Assouad dimension of a hyperbolic boundary is finite. Since the Assouad dimension depends on the metric, it would be good if the existence on an upper bound does not depend on the metric we used for the completion of G. Bonk and Schramm [2, section 6 and 9] showed that this is indeed the case: If one hyperbolic metric d_ε on G induces a boundary with finite Assouad dimension, then all hyperbolic metrics $d_{\varepsilon'}$ have that property and all boundaries are doubling metric spaces. But it need not to be the case that $\dim_A(\partial G, d_\varepsilon) = \dim_A(\partial G, d_{\varepsilon'})$. Thus it remains open, if there is another dimension concept, perhaps the topological dimension—recall that $\dim(X) \leq \dim_A(X)$ for all metric spaces by Remark 3.3—, invariant under changing the metric d_h such that the upper bound of distinct rays to one boundary point depends only on that dimension.

Any graph G has *bounded growth at some scale* if there are constants r, R with $R > r > 0$ and $N \in \mathbb{N}$ such that every ball of radius less than R can be covered by N balls of radius less than r.

Bonk and Schramm [2, Theorem 9.2] proved the following theorem about hyperbolic graphs with bounded growth at some scale.

Theorem 5.3. *Let X be a locally finite hyperbolic graph with bounded growth at some scale. The hyperbolic boundary ∂G is doubling and has finite Assouad dimension.* □

This immediately proves the following corollary.

Corollary 5.4. *Let G be a locally finite hyperbolic graph with bounded growth at some scale. Then there exists an $n \in \mathbb{N}$ and a rooted spanning tree T of G with the following properties:*

(i) Every ray in T converges to a point in the boundary of G;

(ii) for every boundary point η of G there is a ray converging to η;

(iii) for every boundary point η of G there are at most n distinct rays in T starting at the root of T and converging to η. □

The most important examples of graphs with bounded growth at some scale are graphs with bounded degree. These are in particular all almost transitive graphs and thus all Cayley-graphs.

The canonical map $\partial T \to \partial G$, where T is a spanning tree of G like in Theorem 1.1, exists and thus we immediately get the following corollary.

Corollary 5.5. *Let G be a locally finite hyperbolic graph whose boundary ∂G has finite Assouad dimension. Then there exists a rooted spanning tree T of G such that the canonical map $\partial T \to \partial G$ has at most M preimages of each boundary point of G, where M is a constant depending onlyon the Assouad dimension of ∂G.* □

We can also replace the assumption on the Assouad dimension of the hyperbolic boundary in the previous corollary by bounded growth at some scale and get:

Corollary 5.6. *Let G be a locally finite hyperbolic graph with bounded growth at some scale. Then there exists a rooted spanning tree T of G such that the canonical map $\partial T \to \partial G$ has at most M preimages of each boundary point of G, where M is a constant depending only on the bounded growth.* □

A direct consequence is Theorem 1.3.

6 Geodetic spanning trees in hyperbolic graphs

In this section we will prove Theorem 1.4. Before we are able to prove the theorem, we first have to prove some propositions.

Proposition 6.1. *Let G be a locally finite hyperbolic graph, T a rooted geodetic spanning tree of G and S a finite set of vertices. Let Z be the set of limit points of all rays in a connected component C of $T - S$. Then Z is a closed subset of ∂G.*

Proof. Let z be a boundary point of G with $z \in \overline{Z}$. We have to show $z \in Z$. So let $(\eta_i)_{i\in\mathbb{N}}$ be an infinite sequence in Z of boundary points converging to z. Let π_i be a geodetic ray from the root of T to η_i with only finitely many vertices outside C. Since G is locally finite, so is T. Hence there exists a ray π in T such that each edge of T lies in infinitely many of the rays π_i. Since T is a geodetic tree, π has exactly one limit point η. Furthermore, π has also only finitely many vertices outside C, and thus η is an element of Z. Using the Gromov-product we see that π converges towards the limit point of $(\eta_i)_{i\in\mathbb{N}}$ and hence $z = \eta \in Z$. □

Proposition 6.2. *Let G be a locally finite hyperbolic graph. Let $\mathcal{U}$ be a finite open cover of ∂G. Every rooted geodetic spanning tree T has the following property:*

($**$) *There is a finite set of vertices S such that for every connected component C of $T - S$ there is a $U \in \mathcal{U}$ such that each ray in C converges to an $u \in U$.*

Proof. Let us suppose that there is no finite set S of vertices fulfilling ($**$). Thus for very finite set S of vertices there is a set Z of limit points of rays in one connected component C of $T - S$ such that there is no $U \in \mathcal{U}$ with $Z \subseteq U$. Hence we have to extend S by at least one vertex s from C. Since we have to make this infinitely often, the Infinity-Lemma 2.12 gives us a ray π in T from that we have to take infinitely many vertices. Let η be the limit point of π in ∂G. Then there exists a $U \in \mathcal{U}$ with $\eta \in U$. Since U is open, there is a k such that every boundary point ν of G with $(\eta, \nu) \geq k$ lies in U. But then there is a vertex x with distance at most $k + 3\delta$ such that for every boundary point μ, that is contained in the set of limit points of rays in that component of $T - x$ that contains the infinite part of π, there is $(\eta, \mu) \geq k$ and hence the set of those boundary points is a subset of U. Thus we only used finitely many of the vertices of π. □

Proposition 6.3. *Let G be a locally finite hyperbolic graph. Let T be a rooted geodetic spanning tree of G such that there exists an $m \in \mathbb{N}$ and such that for every $\eta \in \partial G$ there are at most m distinct rays in T starting in the root of T and converging to η. Let $\mathcal{U}$ be a finite open cover of ∂G, let S be a finite set of vertices as in Lemma 6.2, let C_i be the infinite components of $T - S$, let Z_i be the set of all limit points of rays in C_i, and lLet $\mathcal{Z}$ be the set of all Z_i. Then there exists an $\varepsilon > 0$ such that $B_\varepsilon(\eta)$ intersects with at most m elements of $\mathcal{Z}$ non-trivially.*

Proof. Since there are only m distinct r-η-rays for every $\eta \in \partial G$, each η is contained in at most m different elements of $\mathcal{Z}$.

Let us assume that the Proposition does not hold. Then there is an infinite sequence of boundary points $(\eta_i)_{i\in\mathbb{N}}$ and an infinite sequence of real numbers $(\varepsilon_i)_{i\in\mathbb{N}}$ that converges to 0 such that every ε_i-neighbourhood of η_i intersects with at least $m + 1$ elements of $\mathcal{Z}$ non-trivially. The sequence of the boundary points has an accumulation point η as ∂G is compact. We may assume that the

sequence converges towards η. Since $\mathcal{Z}$ has only finitely many elements, there is a Z_{i_1} which intersects with an infinite subsequence non-trivially. Because each ε_i-neighbourhood of η_i intersects with $m+1$ elements of $\mathcal{Z}$ non-trivially, we additionally find $Z_{i_2}, \ldots, Z_{i_{m+1}} \in \mathcal{Z}$ such that each of them intersects with infinitely many $B_{\varepsilon_i}(\eta_i)$ non-trivially. Hence η lies in the closure of all $Z_{i_1}, \ldots, Z_{i_{m+1}}$. Since the sets Z_{i_j} are closed by Lemma 6.1, η lies in all of them. But this contradicts the fact that any $\mu \in \partial G$ lies in at most m elements of $\mathcal{Z}$. □

Now we are able to proof Theorem 1.4.

Proof of Theorem 1.4. Let $\mathcal{U}$ be a critical open cover of ∂G. We know that ∂G is compact and hence we may assume that $\mathcal{U}$ is finite. Additionally, we may assume that there is an m such that T contains at most m distinct rays from the root to each $\eta \in \partial G$ since otherwise the theorem trivially holds. By Proposition 6.2 there is a set $\mathcal{Z}$ of closed subsets of ∂G such that for every $Z \in \mathcal{Z}$ there is a $U \in \mathcal{U}$ with $Z \subseteq U$ and by Proposition 6.3 there is an $\varepsilon > 0$ such that $B_\varepsilon(\eta)$ intersects with only m elements of $\mathcal{Z}$ non-trivially for every $\eta \in \partial G$. Let us define for every $Z \in \mathcal{Z}$ a set Z' that consists of Z and every ε-neighbourhood of all $\eta \in Z$. Then Z' is an open set. Let U be in $\mathcal{U}$ with $Z \subseteq U$, and let Z'' be $Z' \cap U$. Then Z'' is an open set, too. Let $\mathcal{V}$ be the set of all the Z'' for $Z \in \mathcal{Z}$. By construction, $\mathcal{V}$ is an open cover of ∂G and also a refinement of $\mathcal{U}$. Thus every $\eta \in \partial G$ lies in at most $n+1$ elements of $\mathcal{V}$. Since $\mathcal{U}$ is critical, there is an $\eta \in \partial G$ that lies in exactly $n+1$ elements of $\mathcal{V}$. By the construction of the Z'' each boundary point lies in at most $2m$ of the sets as it is in at most m elements of $\mathcal{Z}$ and as at most m elements of $\mathcal{Z}$ intersects with $B_\varepsilon(\eta)$ non-trivially. Thus we get that $2m \geq n+1$ and hence the theorem is proved. □

We immediately get the following corollary of Theorem 1.4.

Corollary 6.4. *Let G be a locally finite hyperbolic graph whose boundary has infinite topological dimension. Then for every rooted geodetic spanning tree T there is no $n \in \mathbb{N}$ such that for every boundary point $\eta \in \partial G$ there are at most n distinct rays in T starting at the root and converging to η.* □

In contrast to the upper bound of arbitrary spanning trees, the lower bound of distinct rays to one boundary point in geodetic spanning trees is best possible and we showed the non-existence of an upper bound in section 4. So we have determined the best upper and lower bounds in this case.

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