HALF-FLAT STRUCTURES ON PRODUCTS OF THREE-DIMENSIONAL LIE GROUPS

FABIAN SCHULTE-HENGESBACH

ABSTRACT. We classify six-dimensional Lie groups which admit a left-invariant half-flat SU(3)-structure and which split in a direct product of three-dimensional factors. Moreover, a complete list of those direct products is obtained which admit a left-invariant half-flat SU(3)-structure such that the three-dimensional factors are orthogonal. Similar classification results are proved for left-invariant half-flat $SL(3,\mathbb{R})$-structures on direct products with either definite and orthogonal or isotropic factors.

1. Introduction

An SU(3)-structure (g, J, ω, Ψ) on a six-dimensional manifold M consists of a Riemannian metric g, an orthogonal almost complex structure J, the fundamental two-form $\omega = g(\,.\,, J\,.\,)$ and a complex-valued $(3,0)$-form Ψ of constant length. If furthermore the exterior system

$$d(\omega \wedge \omega) = 0 \,, \qquad d(\mathrm{Re}\Psi) = 0 \,,$$

is satisfied, the SU(3)-structure is called *half-flat*. This notion was introduced in [ChSa] where SU(3)-structures are classified by irreducible components of the intrinsic torsion. The main motivation for studying half-flat SU(3)-structures is their close relation to parallel G_2-structure via the Hitchin flow. On the one hand, a parallel G_2-structure on a seven-manifold induces a half-flat SU(3)-structure on every oriented hypersurface. On the other hand, half-flat SU(3)-structures on a compact six-manifold M can be embedded in a seven-manifold with parallel G_2-structure as follows. Given a (global) solution of the Hitchin flow on an interval I which is a half-flat SU(3)-structure at a time t_0, there is a parallel G_2-structure on $M \times I$, [H1]. In fact, the proof is generalised to non-compact six-manifolds in [CLSS].

Another motivation for the study of half-flat SU(3)-structures comes from string theory and supergravity which discusses them as candidates for internal spaces of compactifications in the presence of background fluxes ([GLMW] or, more recently, [GLM] and references therein).

In the mathematical literature, half-flat SU(3)-structures have been studied intensively on nilmanifolds. For instance, a classification under different additional assumptions is obtained in [CF], [ChSw] and [CT]. Very recently, the classification of nilmanifolds admitting invariant half-flat SU(3)-structures without any further restrictions has been obtained in [C]. Apart from the nilpotent case, examples and constructions of half-flat SU(3)-structures can be found in [TV] and [AFFU]. The Ricci curvature of a half-flat SU(3)-structure is computed in [BV] and [AC].

In this article, we ask the question which direct products of two three-dimensional Lie groups admit a left-invariant half-flat SU(3)-structure. There are 12 isomorphism classes of three-dimensional Lie algebras (see tables 1 and 2), if we count the two Bianchi classes which depend on a continuous parameter as three classes characterised by the property that the parameter can be deformed continuously without leaving the class. Thus, we have to consider $78 = \binom{13}{2}$ classes of direct sums in total after reducing the problem to the Lie algebra as usual.

Initially, we tried to find a classification by a direct proof which avoids the verification of the existence or non-existence case by case. However, this was only successful when we asked for the existence of a half-flat SU(3)-structure (g, J, ω, Ψ) such that the two factors are orthogonal with respect to the metric g. The result is that exactly 15 classes admit such an SU(3)-structure, 11 of

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which are unimodular and comply with a regular pattern, whereas the remaining four do not seem to share many properties. Given, that the additional assumption is rather strong and the proof, which is presented in section 3, is already quite technical, an answer to the initial question with this method cannot be expected. However, an advantage of the assumption of a Riemannian product is the fact that the curvature is completely determined by the Ricci tensors of the three-dimensional factors and that the possible Ricci tensors of left-invariant metrics on three-dimensional Lie groups are classified in [M]. Furthermore, we remark that a basis is introduced in Lemma 3.1 which is well adapted to an almost Hermitian structure on a Riemannian product of three-manifolds which could be useful beyond the framework of half-flat structures.

A completely different method is used in [C] for classifying the nilmanifolds admitting an arbitrary half-flat SU(3)-structure. An obstruction to the existence of a half-flat SU(3)-structure is introduced in terms of the cohomology of a double complex which can be constructed on most of the nilpotent Lie algebras. In our situation, such a double complex can be constructed if and only if both Lie groups are solvable. However, as the methods of homological algebra turn out not to be advantageous for our problem, we prove a simplified version of the obstruction condition in section 4.1. This obstruction is applied directly to 41 isomorphism classes of direct sums in section 4.2. Two classes resist the obstruction, although they do not admit a half-flat structure either, which is proved individually by finding refined obstruction conditions. The remaining 35 direct sums, including all unimodular direct sums and all non-solvable direct sums, admit a half-flat SU(3)-structure which is proved by giving one explicit example in each case in the appendix. We point out that the products of unimodular three-dimensional Lie groups are particularly interesting since they admit co-compact lattices, [RV].

In fact, the most time-consuming part of the classification was the construction of examples of half-flat structures for the 20=35-15 classes which do not admit an "orthogonal" half-flat SU(3)-structure. The construction essentially relies on the fact that a left-invariant half-flat SU(3)-structure is defined by a pair $(\omega, \rho) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ of stable forms which satisfy

$$\omega \wedge \rho = 0, \quad d\omega^2 = 0, \quad d\rho = 0 \tag{1.1}$$

and which induce a Riemannian metric. Working in a basis with fixed Lie bracket, which determines the exterior derivative completely in the left-invariant case, two of the equations are quadratic and one is linear in the coefficients of ω and ρ. For each case separately in a standard basis, large families of solutions of the equations (1.1) can be constructed with the help of a computer algebra system, for instance Maple. However, even after Maple was taught to compute the induced metric, finding a solution inducing a positive definite metric required a certain persistence, in particular for the non-unimodular direct sums. We remark that in each case, all solutions of (1.1) in a small neighbourhood of the constructed example give rise to a, in most cases rather large, family of half-flat SU(3)-structures since the condition that the metric is positive definite is open.

The stable form formalism in dimension six is due to Hitchin, [H1], [H2], and is explained in section 2.2. The formalism suggests to consider also half-flat $\mathrm{SU}(p,q)$-structures, $p+q=3$, with pseudo-Riemannian metrics or even half-flat $\mathrm{SL}(3,\mathbb{R})$-structures where the almost complex structure is replaced by an almost para-complex structure which is involutive instead of anti-involutive. In fact, all these structures are described by a pair of stable forms satisfying (1.1). An analogue of the Hitchin flow relates such structures with indefinite metrics to G_2^*-structures which is elaborated in [CLSS]. More details are recalled in the preliminary section 2.

In section 5, we give an obstruction to the existence of half-flat $\mathrm{SU}(p,q)$-structures for arbitrary signature which is stronger than the obstruction established before and applies to 15 classes. Apart from giving an example of a Lie algebra admitting a half-flat $\mathrm{SU}(1,2)$-structure, but no half-flat SU(3)-structure, we abstain from completing the classification in the indefinite case since it would involve constructing approximately 62=78-15-1 explicit examples of half-flat $\mathrm{SU}(1,2)$-structures.

In section 6, we turn to the para-complex case of $\mathrm{SL}(3,\mathbb{R})$-structures. Again, we give an example of a Lie algebra admitting a half-flat $\mathrm{SL}(3,\mathbb{R})$-structure, but no half-flat $\mathrm{SU}(p,q)$-structure for any signature. Furthermore, we consider half-flat $\mathrm{SL}(3,\mathbb{R})$-structures on direct sums such that the summands are mutually orthogonal, as before, and with the additional assumption, that the metric restricted to each summand is definite. It turns out that the proof of the classification of "orthogonal" half-flat SU(3)-structures in section 3 generalises with some sign modifications and we end up with the same list of 15 Lie algebras. Finally, we consider half-flat $\mathrm{SL}(3,\mathbb{R})$-structure such that both summands are isotropic. The straightforward result is that such a structure is admitted on a direct sum of three-dimensional Lie algebras if and only if both summands are unimodular.

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2. Preliminaries

2.1. $\mathrm{SU}(p,q)$**-structures and** $\mathrm{SL}(m,\mathbb{R})$**-structures.** Let $M = M^{2m}$ be an even-dimensional manifold. An *almost pseudo-Hermitian structure* (g,J,ω) on M consists of a pseudo-Riemannian metric g, an orthogonal almost complex structure J and a two-form $\omega = g(\,.\,,J\,.)$, called the fundamental two-form. In order to prevent confusion, we point out that many authors define the fundamental two-form with the opposite sign. An almost pseudo-Hermitian structure is equivalent to the reduction of the frame bundle of M to $\mathrm{U}(p,q)$, $p+q=m$, where $(2p,2q)$ is the signature of the metric g. A further reduction to $\mathrm{SU}(p,q)$, i.e. an SU*(p,q)-structure*, is given by a non-trivial complex $(m,0)$-form Ψ of constant length.

Locally, there is always a pseudo-orthonormal frame $\{\eta_1,\dots,\eta_{2m}\}$ such that $J\eta_i = \eta_{i+m}$ and $\sigma_i = g(\eta_i,\eta_i) = \pm 1$ for $i=1,\dots,m$ and

$$\omega = -\sum_{i=1}^{m} \sigma_i\, \eta^{i(i+m)} \quad , \qquad \Psi = (\eta^1 + \mathrm{i}\,\eta^{1+m}) \wedge \ldots \wedge (\eta^m + \mathrm{i}\,\eta^{2m}),$$

where upper indices denote dual basis vectors and η^{ij} stands for the wedge product of η^i and η^j.

Similarly, an *almost para-Hermitian structure* (g,J,ω) on M consists of a neutral metric g, an anti-orthogonal para-complex structure J and the fundamental two-form $\omega = g(\,.\,,J\,.\,)$. We recall that an almost para-complex structure J is a section in $\mathrm{End}(TM)$ such that $J^2 = \mathrm{id}_{TM}$ and the ±1-eigenbundles $TM^{\pm}$ with respect to J have dimension m. An almost para-Hermitian structure is equivalent to a $\mathrm{GL}(m,\mathbb{R})$-structure where $\mathrm{GL}(m,\mathbb{R})$ acts reducibly on $T_pM = T_pM^+ \oplus T_pM^-$ for all $p \in M$. A recent survey on para-complex geometry is for instance contained in [AMT].

We denote by C the para-complex numbers $a + \mathrm{e}b$, $\mathrm{e}^2 = 1$, $a,b \in \mathbb{R}$, and by $\Omega^{k,l}M$ the bi-grading induced by the decomposition of the para-complexification $TM \otimes C$ into the $\pm\mathrm{e}$-eigenspaces of J. In analogy to the almost pseudo-Hermitian case, an $(m,0)$-form Ψ of constant *non-zero* length defines a reduction of the structure group from $\mathrm{GL}(m,\mathbb{R})$ to $\mathrm{SL}(m,\mathbb{R})$.

Stressing the similarity to the almost pseudo-Hermitian situation, we can choose a local pseudo-orthonormal frame $\{\eta_1,\dots,\eta_{2m}\}$ such that $J\eta_i = \eta_{i+m}$ and $g(\eta_i,\eta_i) = -g(\eta_{i+m},\eta_{i+m}) = 1$ for $i=1,\dots,m$ and moreover,

$$\omega = -\sum_{i=1}^{m} \eta^i \wedge \eta^{i+m} \quad , \qquad \Psi = (\eta^1 + \mathrm{e}\,\eta^{1+m}) \wedge \ldots \wedge (\eta^m + \mathrm{e}\,\eta^{2m}).$$

Alternatively, a local frame $\{\xi_1,\dots,\xi_{2m}\}$ can always be chosen such that

$$\begin{aligned} g &= 2\sum_{i=1}^{m} \xi^i \cdot \xi^{i+m}, \qquad J\xi_i = \xi_i,\ J\xi_{i+m} = -\xi_{i+m} \quad \text{for } i=1,\dots,m, \\ \omega &= -\sum_{i=1}^{m} \xi^i \wedge \xi^{i+m}, \quad \Psi = \sqrt{2}\,\{\,(\xi^{1\dots m} + \xi^{(m+1)\dots 2m}) + \mathrm{e}(\xi^{1\dots m} - \xi^{(m+1)\dots 2m})\,\}\,. \end{aligned} \tag{2.1}$$

We will need the following formula, which is easily verified in the given local frames.

Lemma 2.1. *On an almost pseudo-Hermitian or almost para-Hermitian manifold* (M^{2m},g,J,ω)*, the identity*

$$\alpha \wedge J^*\beta \wedge \omega^{m-1} \;=\; \frac{1}{m} g(\alpha,\beta)\omega^m \tag{2.2}$$

holds for all one-forms α, β.

2.2. **Stable forms in dimension six.** A p-form on a vector space is called *stable* if its orbit under $\mathrm{GL}(V)$ is open [H1]. We will frequently use the properties of stable forms in dimension six and recall the basic facts omitting the proofs which can be found in [H2] and [CLSS].

Let V be a six-dimensional oriented vector space and κ the canonical isomorphism

$$\kappa\,:\,\Lambda^5V^*\to V\otimes\Lambda^6V^*,\ \xi\mapsto X\otimes\nu\quad\text{with } X\lrcorner\,\nu=\xi.$$

For every three-form $\rho\in\Lambda^3V^*$, one can define

$$K_\rho(v)=\ \kappa((v\lrcorner\,\rho)\wedge\rho)\quad\in V\otimes\Lambda^6V^*, \tag{2.3}$$
$$\lambda(\rho)=\ \tfrac{1}{6}\mathrm{tr}\,K_\rho^2\quad\in(\Lambda^6V^*)^{\otimes 2}, \tag{2.4}$$
$$\phi(\rho)=\ \sqrt{\lambda(\rho)}\quad\in\Lambda^6V^*, \tag{2.5}$$

where the positively oriented square root is chosen. In fact, the three-form ρ is stable if and only if $\lambda(\rho)\neq 0$. For a stable three-form ρ, we define

$$J_\rho=\ \tfrac{1}{\phi(\rho)}K_\rho\quad\in\mathrm{End}(V), \tag{2.6}$$

which is a complex structure if $\lambda(\rho)<0$ and a para-complex structure for $\lambda(\rho)>0$. Moreover, the form $\rho+\mathrm{i}J_\rho^*\rho$, or $\rho+\mathrm{e}J_\rho^*\rho$, respectively, is a (3,0)-form with respect to J_ρ.

Lemma 2.2. *The (para-)complex structure J_ρ induced by a stable three-form ρ acts on one-forms by the formula*

$$J_\rho^*\alpha(v)\,\phi(\rho)\quad=\quad\alpha\wedge(v\lrcorner\,\rho)\wedge\rho,\qquad v\in V,\ \alpha\in V^*. \tag{2.7}$$

Proof. The formula follows directly from the definition since we have

$$\alpha\wedge(v\lrcorner\,\rho)\wedge\rho\overset{(2.3)}{=}\alpha\wedge\kappa^{-1}(K_\rho(v))\overset{(2.6)}{=}\alpha\wedge(J_\rho(v)\lrcorner\,\phi(\rho))=\alpha(J_\rho v)\,\phi(\rho)=J_\rho^*\alpha(v)\,\phi(\rho)$$

for all $v\in V$ and all $\alpha\in V^*$. □

A two-form $\omega\in L^2V^*$ in dimension six is stable if and only if it is non-degenerate, i.e.

$$\phi(\omega)=\frac{1}{6}\omega^3\neq 0.$$

A pair $(\omega,\rho)\in\Lambda^2V^*\times\Lambda^3V^*$ of stable forms is called *compatible* if

$$\omega\wedge\rho=0\quad\Longleftrightarrow\quad\omega(\,.\,,J_\rho\,.)=-\omega(J_\rho\,.\,,\,.) \tag{2.8}$$

and *normalised* if

$$\phi(\rho)=\pm 2\phi(\omega)\quad\Longleftrightarrow\quad J_\rho^*\rho\wedge\rho=\pm\frac{2}{3}\,\omega^3. \tag{2.9}$$

The choice of the sign $\pm$ in the normalisation condition determines in particular the orientation which is needed to uniquely define $\phi(\rho)$ and the induced (para-)complex structure J_ρ. A compatible and normalised pair induces a pseudo-Euclidean metric

$$g=g_{(\omega,\rho)}=\varepsilon\,\omega(\,.\,,J_\rho\,.). \tag{2.10}$$

By compatibility, the induced (para-)complex structure J_ρ is (anti-)orthogonal with respect to this induced metric and the stabiliser of a compatible and normalised pair is

$$\mathrm{Stab}_{\mathrm{GL}(V)}(\rho,\omega)\cong\begin{cases}\mathrm{SU}(p,q),\ p+q=3, & \text{if }\lambda(\rho)<0,\\ \mathrm{SL}(3,\mathbb{R})\,, & \text{if }\lambda(\rho)>0.\end{cases}$$

In particular, the conventions are chosen such that $\phi(\rho)=+2\phi(\omega)$ if the induced metric is positive definite which is in fact the motivation for the sign convention $\omega=g(\,.\,,J\,.)$.

2.3. **Half-flat structures.** Let M be a six-manifold. We call $\mathrm{SU}(p,q)$-structures, $p+q=3$, and $\mathrm{SL}(3,\mathbb{R})$-structures defined by tensors (g,J,ω,Ψ) *normalised* if

$$\mathrm{Im}\Psi \wedge \mathrm{Re}\Psi = \pm\frac{2}{3}\,\omega^3.$$

This can always be achieved by rescaling the length of Ψ which is constant and non-zero by definition. In fact, the local frames given in section 2.1 are already normalised. Furthermore, we call a p-form ρ on a manifold stable if ρ_p is stable on T_pM for all $p \in M$. With this terminology, the discussion of stable forms in dimension six can be applied to $\mathrm{SU}(p,q)$-structures (g,J,ω,Ψ), $p+q=3$ and $\mathrm{SL}(3,\mathbb{R})$-structures as follows.

Proposition 2.3. *Let M be a six-manifold.*

(i) There is a one-to-one correspondence between normalised $\mathrm{SU}(p,q)$-structures (g,J,ω,Ψ), $p+q=3$, on M and pairs $(\omega,\rho) \in \Omega^2 M \times \Omega^3 M$ of stable forms which are everywhere compatible and normalised and satisfy $\lambda(\rho_p) < 0$ for all $p \in M$.

(ii) There is a one-to-one correspondence between normalised $\mathrm{SL}(3,\mathbb{R})$-structures (g,J,ω,Ψ) on M and pairs $(\omega,\rho) \in \Omega^2 M \times \Omega^3 M$ of stable forms which are everywhere compatible and normalised and satisfy $\lambda(\rho_p) > 0$ for all $p \in M$.

An $\mathrm{SU}(p,q)$-structure, $p+q=3$, or an $\mathrm{SL}(3,\mathbb{R})$-structure, defined by a pair of forms (ω,ρ), is called *half-flat* if

$$d\rho = 0\,, \qquad d\omega^2 = 0. \tag{2.11}$$

We will speak of a *half-flat structure* if the sign of J^2 and the signature of g are not important.

Moreover, left-invariant half-flat structures (ω,ρ) on Lie groups G are in one-to-one correspondence with pairs $(\omega,\rho) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ of stable forms on the corresponding Lie algebra $\mathfrak{g}$ satisfying the exterior system (2.11) and $\omega \wedge \rho = 0$. Therefore, we denote a pair $(\omega,\rho) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ with these properties as a *half-flat structure on a Lie algebra* and the existence problem of left-invariant half-flat structures on Lie groups reduces to the existence of half-flat structures on Lie algebras.

2.4. **Three-dimensional Lie algebras.** Let $\mathfrak{g}$ be the Lie algebra of an n-dimensional real Lie group G. Identifying $\mathfrak{g}$ with the Lie algebra of left-invariant vector fields on G, the formula

$$d\alpha(X,Y) = -\alpha([X,Y]), \quad \alpha \in \mathfrak{g}^*\,,\ X,Y \in \mathfrak{g},$$

shows that the exterior derivative of G restricted to left-invariant one-forms contains the same information as the Lie bracket. Since the Jacobi identity is equivalent to $d^2 = 0$, we have a complex $(\Lambda^*\mathfrak{g}^*, d)$. Its cohomology $H^*(\mathfrak{g})$ is the Chevalley-Eilenberg or Lie algebra cohomology for the trivial representation.

Recall that a Lie algebra $\mathfrak{g}$ is called *unimodular* if the trace of the adjoint representation ad_{X} vanishes for all $X \in \mathfrak{g}$.

Lemma 2.4. *The following conditions are equivalent for an n-dimensional Lie algebra.*

(i) $\mathfrak{g}$ is unimodular

(ii) All $(n-1)$-forms on $\mathfrak{g}$ are closed.

(iii) $H^n(\mathfrak{g}) = \mathbb{R}$

(iv) Let $\{c_{ij}^k\}$ denote the structure constants with respect to a basis $\{\mathrm{e}^i\}$ of $\mathfrak{g}^*$ which are defined by $d\mathrm{e}^k = \sum_{i<j} c_{ij}^k \mathrm{e}^{ij}$. Then, it holds $\sum_{k=1}^n c_{k,m}^k = 0$ for $1 \le m \le n$.

(v) The associated connected Lie groups G are unimodular, i.e. the Haar measure of G is bi-invariant.

Unimodularity is a necessary condition for the existence of a co-compact lattice, see for instance [M], in dimension three it is also sufficient. Indeed, the closed three-manifolds of the form $\Gamma\backslash G$ where G is a Lie group with lattice Γ are classified in [RV]. Since a direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of Lie algebras is unimodular if and only if both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular, a direct product $G_1 \times G_2$ of three-dimensional Lie groups admits a co-compact lattice if and only if it is unimodular.

Lemma 2.5. *Let $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the direct sum of two Lie algebras of dimension three. Moreover, let ω be a non-degenerate two-form in $\Lambda^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2)^* = \Lambda^2\mathfrak{g}_1^* \oplus (\mathfrak{g}_1^* \otimes \mathfrak{g}_2^*) \oplus \Lambda^2\mathfrak{g}_2^*$ such that the projections of ω on $\Lambda^2\mathfrak{g}_1^*$ and $\Lambda^2\mathfrak{g}_2^*$ vanish. Then ω^2 is closed if and only if both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular.*

Proof. Since $\omega \in \mathfrak{g}_1^* \otimes \mathfrak{g}_2^*$ is non-degenerate, we can always choose bases $\{\mathrm{e}^i\}$ of $\mathfrak{g}_1^*$ and $\{\mathrm{f}^i\}$ of $\mathfrak{g}_2^*$ such that $\omega = \sum_{j=1}^3 \mathrm{e}^j \mathrm{f}^j$. Therefore, we have

$$\omega^2 = -2\sum_{i<j} \mathrm{e}^{ij}\mathrm{f}^{ij} \quad \Rightarrow \quad -\frac{1}{2}\,d\omega^2 = \sum_{i<j} d\,(\mathrm{e}^{ij}) \wedge \mathrm{f}^{ij} + \sum_{i<j} \mathrm{e}^{ij} \wedge d\,(\mathrm{f}^{ij}).$$

By Lemma 2.4, both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular if and only if all two-forms e^{ij} and f^{ij} are closed. Since the sum is a direct sum of Lie algebras, the assertion follows immediately. □

In the following chapter, we need to determine in which isomorphism class a given three-dimensional Lie algebra lies. All information we need, including proofs, can be found in [M]. We summarise the results in two propositions. Recall that a Euclidean cross product in dimension three is determined by a scalar product and an orientation.

Proposition 2.6 (Unimodular case)**.** *Let $\mathfrak{g}$ be a three-dimensional Lie algebra and choose a scalar product and an orientation.*

(a) There is a uniquely defined endomorphism L of $\mathfrak{g}$ such that $[u, v] = L(u \times v)$.
(b) The Lie algebra $\mathfrak{g}$ is unimodular if and only if L is self-adjoint.
(c) If $\mathfrak{g}$ is unimodular, the isomorphism class of $\mathfrak{g}$ is characterised by the signs of the eigenvalues of L. It can be achieved that there is at most one negative eigenvalue of L by possibly changing the orientation.

	Bianchi type	Eigenvalues of L	Standard Lie bracket
$\mathfrak{su}(2) \cong \mathfrak{so}(3)$	IX	(+,+,+)	$de^1 = e^{23},\ de^2 = e^{31},\ de^3 = e^{12}$
$\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(1,2)$	VIII	(+,+,-)	$de^1 = e^{23},\ de^2 = e^{31},\ de^3 = e^{21}$
$\mathfrak{e}(2)$	VII_0	(+,+,0)	$de^2 = e^{31},\ de^3 = e^{12}$
$\mathfrak{e}(1,1)$	VI_0	(+,-,0)	$de^2 = e^{31},\ de^3 = e^{21}$
$\mathfrak{h}_3$	II	(+,0,0)	$de^3 = e^{12}$
$\mathbb{R}^3$	I	(0,0,0)	abelian

Table 1. Three-dimensional unimodular Lie algebras

Recall that the unimodular kernel of a Lie algebra $\mathfrak{g}$ is the kernel of the Lie algebra homomorphism

$$\mathfrak{g} \to \mathbb{R}\ ,\ X \mapsto \mathrm{tr}(\mathrm{ad_X}).$$

Proposition 2.7 (Non-unimodular case)**.** *Let $\mathfrak{g}$ be a non-unimodular three-dimensional Lie algebra.*

(a) The unimodular kernel $\mathfrak{u}$ of $\mathfrak{g}$ is two-dimensional and abelian.
(b) Let $X \in \mathfrak{g}$ such that $\mathrm{tr}(\mathrm{ad_X}) = 2$ and let $\tilde{L} : \mathfrak{u} \to \mathfrak{u}$ be the restriction of $\mathrm{ad_X}$ to the unimodular kernel $\mathfrak{u}$. If $\tilde{L}$ is not the identity map, the isomorphism class of $\mathfrak{g}$ is characterised by the determinant D of $\tilde{L}$.

We remark that all three-dimensional Lie algebras are solvable except for $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$ which are simple. The three-dimensional Heisenberg algebra $\mathfrak{h}_3$ represents the only non-abelian nilpotent isomorphism class. The two Lie algebras $\mathfrak{e}(2)$ and $\mathfrak{e}(1,1)$ correspond to the groups of rigid motions of the Euclidean plane $\mathbb{R}^2$ and of the Minkowskian plane $\mathbb{R}^{1,1}$, respectively. The names

	Bianchi type	D	Standard Lie bracket
$\mathfrak{r}_2 \oplus \mathbb{R}$	III	0	$de^2 = e^{21}$
$\mathfrak{r}_3$	IV	1 (and $\tilde{L} \neq \mathrm{id}$)	$de^2 = e^{21} + e^{31}$, $de^3 = e^{31}$
$\mathfrak{r}_{3,1}$	V	1 (and $\tilde{L} = \mathrm{id}$)	$de^2 = e^{21}$, $de^3 = e^{31}$
$\mathfrak{r}_{3,\mu}$ $(-1 < \mu < 0)$	VI	$D = \frac{4\mu}{(\mu+1)^2} < 0$	$de^2 = e^{21}$, $de^3 = \mu e^{31}$
$\mathfrak{r}_{3,\mu}$ $(0 < \mu < 1)$	VI	$0 < D = \frac{4\mu}{(\mu+1)^2} < 1$	$de^2 = e^{21}$, $de^3 = \mu e^{31}$
$\mathfrak{r}'_{3,\mu}$ $(\mu > 0)$	VII	$D = 1 + \frac{1}{\mu^2} > 1$	$de^2 = \mu e^{21} + e^{13}$, $de^3 = e^{21} + \mu e^{31}$

Table 2. Three-dimensional non-unimodular Lie algebras

for the non-unimodular Lie algebras are taken from [GOV] and the Bianchi types are defined in the original classification by Bianchi from 1898, [B1], see [B2] for an English translation.

3. Classification of direct sums admitting a half-flat SU(3)-structure such that the summands are orthogonal

A Hermitian structure on a $2m$-dimensional Euclidean vector space (V, g) is given by an orthogonal complex structure J. As before, we define the fundamental two-form by $\omega = g(\,.\,, J\,.\,)$. The following Lemma is crucial for the proof of the first classification result.

Lemma 3.1. *Let (V_1, g_1) and (V_2, g_2) be three-dimensional Euclidean vector spaces and let (g, J, ω) be a Hermitian structure on the orthogonal product $(V_1 \oplus V_2, g = g_1 + g_2)$. There are orthonormal bases $\{e_1, e_2, e_3\}$ of V_1 and $\{f_1, f_2, f_3\}$ of V_2 which can be joined to an orthonormal basis of $V_1 \oplus V_2$ such that*

$$\omega = a\, e^{12} + \sqrt{1-a^2}\, e^1 f^1 + \sqrt{1-a^2}\, e^2 f^2 + e^3 f^3 - a\, f^{12} \tag{3.1}$$

for a real number a with $-1 < a \leq 1$.

Proof. Let $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ be orthonormal bases of V_1 and V_2, respectively. The group $O(3) \times O(3)$ acts transitively on the pairs of orthonormal bases. Let Ω be the Gram matrix of the two-form ω with respect to our basis. Writing the upper right block of Ω as a product of an orthogonal and positive semi-definite matrix and acting with an appropriate pair of orthogonal matrices, we find an orthonormal basis and nine real parameters such that

$$\Omega = \begin{pmatrix} 0 & y_1 & y_2 & x_1 & 0 & 0 \\ -y_1 & 0 & y_3 & 0 & x_2 & 0 \\ -y_2 & -y_3 & 0 & 0 & 0 & x_3 \\ -x_1 & 0 & 0 & 0 & z_1 & z_2 \\ 0 & -x_2 & 0 & -z_1 & 0 & z_3 \\ 0 & 0 & -x_3 & -z_2 & -z_3 & 0 \end{pmatrix}$$

with $x_i \geq 0$ for all i and $\det(\Omega) \neq 0$.

Since $\omega = g(\,.\,, J\,.\,)$, the matrix Ω with respect to an orthonormal basis has to be a complex structure, i.e. $\Omega^2 = -\mathbb{1}$, where $\mathbb{1}$ denotes the identity matrix. In our basis, Ω^2 is

$$\begin{pmatrix} -y_1^2 - y_2^2 - x_1^2 & -y_2 y_3 & y_1 y_3 & 0 & y_1 x_2 + x_1 z_1 & y_2 x_3 + x_1 z_2 \\ -y_2 y_3 & -y_1^2 - y_3^2 - x_2^2 & -y_1 y_2 & -y_1 x_1 - x_2 z_1 & 0 & y_3 x_3 + x_2 z_3 \\ y_1 y_3 & -y_1 y_2 & -y_2^2 - y_3^2 - x_3^2 & -y_2 x_1 - x_3 z_2 & -y_3 x_2 - x_3 z_3 & 0 \\ 0 & -y_1 x_1 - x_2 z_1 & -y_2 x_1 - x_3 z_2 & -x_1^2 - z_1^2 - z_2^2 & -z_2 z_3 & z_1 z_3 \\ y_1 x_2 + x_1 z_1 & 0 & -y_3 x_2 - x_3 z_3 & -z_2 z_3 & -x_2^2 - z_1^2 - z_3^2 & -z_1 z_2 \\ y_2 x_3 + x_1 z_2 & y_3 x_3 + x_2 z_3 & 0 & z_1 z_3 & -z_1 z_2 & -x_3^2 - z_2^2 - z_3^2 \end{pmatrix}.$$

We end up with a set of 18 quadratic equations (and one inequality) and determine all solutions modulo the action of $O(3) \times O(3)$ and an exchange of the summands.

On the one hand, assume $y_i = 0$ for all i. It follows that all equations are satisfied if and only if $x_i = 1$ and $z_i = 0$ for all i. In this case, the two-form ω is in the normal form (3.1) with $a = 0$.

On the other hand, assume that one of the y_i is different from zero, say $a := y_1 \neq 0$ without loss of generality. Inspecting the first two terms of the third line of Ω^2, we observe $y_2 = y_3 = 0$. Since $x_i \geq 0$, the first three elements on the diagonal enforce $x_3 = 1$, $x_1 = x_2 = \sqrt{-a^2+1}$ and $|a| \leq 1$. But $x_3 = 1$ and $y_2 = y_3 = 0$ imply that $z_2 = z_3 = 0$ due to row 3, terms 4 and 5. If $|a| < 1$ and thus $x_1 = x_2 > 0$, the term in row 1 and column 5 enforces $z_1 = -a$. Obviously, all equations are satisfied and ω is in the normal form (3.1). Finally, if $|a| = 1$, we have immediately $x_1 = x_2 = 0$ and $|z_1| = 1$ and all equations are satisfied again. Since changing the signs of the base vectors e_1 and f_1 is an orthogonal transformation which does not change x_3, we can obtain the normal form (3.1) for $a = 1$. Since we found all solutions to the 18 equations and the two-form ω is non-degenerate for all values of a, the Lemma is proven. □

We call the Hermitian structure of type I if it admits a basis with $a = 0$ and of type II if it admits a basis with $a \neq 0$.

Theorem 3.2. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum of three-dimensional Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.*

The Lie algebra $\mathfrak{g}$ admits a half-flat SU(3)*-structure such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are mutually orthogonal and such that the underlying Hermitian structure is of type I if and only if*

(i) $\mathfrak{g}_1 = \mathfrak{g}_2$ *and both are unimodular or*

(ii) $\mathfrak{g}_1$ *is non-abelian unimodular and* $\mathfrak{g}_2$ *abelian or vice versa.*

Moreover, the Lie algebra $\mathfrak{g}$ admits a half-flat SU(3)*-structure such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are mutually orthogonal and such that the underlying Hermitian structure is of type II if and only if the pair $(\mathfrak{g}_1, \mathfrak{g}_2)$ or $(\mathfrak{g}_2, \mathfrak{g}_1)$ is contained in the following list:*

$$
\begin{aligned}
&(\mathfrak{e}(1,1), \mathfrak{e}(1,1)), && \\
&(\mathfrak{e}(2), \mathbb{R} \oplus \mathfrak{r}_2), && \\
&(\mathfrak{su}(2), \mathfrak{r}_{3,\mu}) && \textit{for } 0 < \mu \leq 1, \\
&(\mathfrak{sl}(2,\mathbb{R}), \mathfrak{r}_{3,\mu}) && \textit{for } -1 < \mu < 0.
\end{aligned}
$$

Proof. Given an arbitrary (almost) Hermitian structure (g, J, ω) such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal, we can use Lemma 3.1 and choose an orthonormal basis $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\{e_1, e_2, e_3\}$ spans $\mathfrak{g}_1$, $\{f_1, f_2, f_3\}$ spans $\mathfrak{g}_2$ and

$$\omega = a\,\mathrm{e}^{12} + \sqrt{1-a^2}\,\mathrm{e}^1\mathrm{f}^1 + \sqrt{1-a^2}\,\mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3 - a\,\mathrm{f}^{12} \tag{3.2}$$

for a real number a with $-1 < a \leq 1$. The reductions from U(3) to SU(3) are parameterised by the space of complex-valued $(3,0)$-forms $\Psi = \psi + i\phi$ which is complex one-dimensional. We remark that, working on a vector space, the length normalisation of the $(3,0)$-form is not important for the existence question. The Lie bracket of the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is encoded in the 18 structure constants of $\mathfrak{g}_1$ and $\mathfrak{g}_2$:

$$d\mathrm{e}^i = c^i_{j,k}\mathrm{e}^{jk} \quad \text{and} \quad d\mathrm{f}^i = c^{i+3}_{j+3,k+3}\mathrm{f}^{jk} \quad \text{with i,j,k} \in \{1,2,3\}.$$

Therefore, our ansatz includes 21 parameters consisting of 18 structure constants, two real parameters defining an arbitrary SU(3) reduction and the parameter a. Our strategy is to find all solutions of the equations defining half-flatness

$$d\omega^2 = 0 \quad \text{and} \quad d\psi = 0$$

and the Jacobi identity $d^2 = 0$ and to determine the isomorphism classes of the solutions if necessary.

Type I: Assume first that $a = 0$. Due to Lemma (2.5), the first half-flat equation $d\omega^2 = 0$ is satisfied if and only if both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular. It remains to solve the second half-flat equation for unimodular summands. Since we have $J(f_i) = e_i$ in our basis for $a = 0$, the dual vectors satisfy $\mathrm{e}^i \circ J = \mathrm{f}^i$. Therefore, the complex-valued form

$$\begin{aligned}\Psi_0 &= \psi_0 + \mathrm{i}\phi_0 = (\mathrm{e}^1 - \mathrm{i}\mathrm{e}^1 \circ J) \wedge (\mathrm{e}^2 - \mathrm{i}\mathrm{e}^2 \circ J) \wedge (\mathrm{e}^3 - \mathrm{i}\mathrm{e}^3 \circ J) \\ &= \mathrm{e}^{123} - \mathrm{e}^1\mathrm{f}^{23} - \mathrm{e}^2\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{i}(\mathrm{f}^{123} - \mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{23}\mathrm{f}^1)\end{aligned}$$

is a $(3,0)$-form with respect to J. By multiplying Ψ_0 with a non-zero complex number $\xi_1 + \mathrm{i}\xi_2$, we obtain all $(3,0)$-forms. Their real part is $\psi = \xi_1\psi_0 - \xi_2\phi_0$. Considering that all two-forms on both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are closed, we compute the exterior derivative of ψ:

$$\begin{aligned}d\psi = &-(\xi_1 c^1_{1,2} - \xi_2 c^6_{5,6})\,\mathrm{e}^{12}\mathrm{f}^{23} - (\xi_1 c^1_{2,3} - \xi_2 c^4_{5,6})\,\mathrm{e}^{23}\mathrm{f}^{23} - (\xi_1 c^1_{3,1} - \xi_2 c^5_{5,6})\,\mathrm{e}^{31}\mathrm{f}^{23} \\ &-(\xi_1 c^2_{1,2} - \xi_2 c^6_{6,4})\,\mathrm{e}^{12}\mathrm{f}^{31} - (\xi_1 c^2_{2,3} - \xi_2 c^4_{6,4})\,\mathrm{e}^{23}\mathrm{f}^{31} - (\xi_1 c^2_{3,1} - \xi_2 c^5_{6,4})\,\mathrm{e}^{31}\mathrm{f}^{31} \\ &-(\xi_1 c^3_{1,2} - \xi_2 c^6_{4,5})\,\mathrm{e}^{12}\mathrm{f}^{12} - (\xi_1 c^3_{2,3} - \xi_2 c^4_{4,5})\,\mathrm{e}^{23}\mathrm{f}^{12} - (\xi_1 c^3_{3,1} - \xi_2 c^5_{4,5})\,\mathrm{e}^{31}\mathrm{f}^{12}.\end{aligned}$$

If ξ_1 or ξ_2 is zero we have obviously $d\psi = 0$ if and only if one of the summands is abelian. By Lemma (2.4), the unimodularity of $\mathfrak{g}_2$ is equivalent to $c^6_{6,4} = -c^5_{5,4}$, $c^6_{6,5} = -c^4_{4,5}$ and $c^5_{5,4} = -c^6_{6,4}$. Therefore, if both ξ_1 and ξ_2 are different from zero, $d\psi$ vanishes if and only if the structure constants of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ coincide up to the scalar $\frac{\xi_1}{\xi_2}$ and therefore $\mathfrak{g}_1 = \mathfrak{g}_2$. This comprises all solutions under the assumption $a = 0$.

Type II: Assume now that the U(3)-structure satisfies $a \neq 0$. To improve readability, the abbreviation $b := \sqrt{1 - a^2}$ is introduced.

With this notation, we compute

$$\begin{aligned}\frac{1}{2}\omega^2 &= a\,\mathrm{e}^{123}\mathrm{f}^3 - a\,\mathrm{e}^3\mathrm{f}^{123} - \mathrm{e}^{12}\mathrm{f}^{12} - b\,\mathrm{e}^{13}\mathrm{f}^{13} - b\,\mathrm{e}^{23}\mathrm{f}^{23} \\ \frac{1}{2}d(\omega^2) &= (c^4_{4,6} + c^5_{5,6} - ac^3_{1,2})\quad \mathrm{e}^{12}\mathrm{f}^{123} + (bc^6_{5,6} - bc^4_{4,5} - ac^3_{1,3})\quad \mathrm{e}^{13}\mathrm{f}^{123} \\ &\quad - (ac^3_{2,3} + bc^5_{4,5} + bc^6_{4,6})\quad \mathrm{e}^{23}\mathrm{f}^{123} + (c^1_{1,3} + c^2_{2,3} - ac^6_{4,5})\quad \mathrm{e}^{123}\mathrm{f}^{12} \\ &\quad - (bc^1_{1,2} - bc^3_{2,3} + ac^6_{4,6})\quad \mathrm{e}^{123}\mathrm{f}^{13} - (ac^6_{5,6} + bc^2_{1,2} + bc^3_{1,3})\quad \mathrm{e}^{123}\mathrm{f}^{23}.\end{aligned}$$

We reduce our ansatz to the space of solutions of $d\omega^2 = 0$ by substituting

$$\begin{aligned}c^2_{2,3} &= ac^6_{4,5} - c^1_{1,3} \quad , \quad c^3_{2,3} = b^2 c^1_{1,2} - abc^5_{4,5} \quad , \quad c^3_{1,3} = -b^2 c^2_{1,2} - abc^4_{4,5} \quad , \\ c^5_{5,6} &= ac^3_{1,2} - c^4_{4,6} \quad , \quad c^6_{5,6} = b^2 c^4_{4,5} - abc^2_{1,2} \quad \text{and} \quad c^6_{4,6} = -b^2 c^5_{4,5} - abc^1_{1,2}.\end{aligned}$$

In our basis, we have $\mathrm{e}^1 \circ J = b\mathrm{f}^1 + a\mathrm{e}^2$, $\mathrm{e}^3 \circ J = \mathrm{f}^3$ and $\mathrm{f}^2 \circ J = -b\mathrm{e}^2 + a\mathrm{f}^1$. Using this, we compute a (3,0)-form $\Psi_0 = \psi_0 + \mathrm{i}\phi_0$ with

$$\begin{aligned}\psi_0 &= +b\,\mathrm{f}^{123} - b\,\mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^{13}\mathrm{f}^2 - \mathrm{e}^{23}\mathrm{f}^1 + a\,\mathrm{e}^1\mathrm{f}^{13} + a\,\mathrm{e}^2\mathrm{f}^{23} \\ \phi_0 &= -b\,\mathrm{e}^{123} + b\,\mathrm{e}^3\mathrm{f}^{12} - \mathrm{e}^2\mathrm{f}^{13} + \mathrm{e}^1\mathrm{f}^{23} - a\,\mathrm{e}^{13}\mathrm{f}^1 - a\,\mathrm{e}^{23}\mathrm{f}^2\end{aligned}$$

In the following, we work with the real part $\psi = \xi_1\psi_0 - \xi_2\phi_0$ of an arbitrary (3,0)-form. By possibly changing the roles of the two summands, we can assume that ξ_1 is non-zero and we normalise our (3,0)-form such that $\xi_1 = 1$. The exterior derivative of ψ is, after inserting the above substitutions,

$$\begin{aligned}d\psi = &\; ab\,c^6_{4,5}\,\mathrm{e}^{123}\mathrm{f}^3 - \xi_2 ab\,c^3_{1,2}\,\mathrm{e}^3\mathrm{f}^{123} - b\,(c^6_{4,5} + \xi_2\,c^3_{1,2})\,\mathrm{e}^{12}\mathrm{f}^{12} \\ &+ (-\xi_2 ab\,c^1_{1,2} - a^2 b\,c^2_{1,2} - a^3 c^4_{4,5} + \xi_2 a^2 c^5_{4,5})\,\mathrm{e}^1\mathrm{f}^{123} \\ &+ (a^2 b\,c^1_{1,2} - \xi_2 ab\,c^2_{1,2} - \xi_2 a^2 c^4_{4,5} - a^3 c^5_{4,5})\,\mathrm{e}^2\mathrm{f}^{123} \\ &+ (\xi_2 a^3 c^1_{1,2} - a^2 c^2_{1,2} + ab\,c^4_{4,5} + \xi_2 a^2 b\,c^5_{4,5})\,\mathrm{e}^{123}\mathrm{f}^1 \\ &+ (a^2 c^1_{1,2} + \xi_2 a^3 c^2_{1,2} - \xi_2 a^2 b\,c^4_{4,5} + ab\,c^5_{4,5})\,\mathrm{e}^{123}\mathrm{f}^2 \\ &+ (a(2-a^2)\,c^1_{1,2} + \xi_2\,c^2_{1,2} + b^3\,c^5_{4,5})\,\mathrm{e}^{12}\mathrm{f}^{13} \\ &+ (-\xi_2\,c^1_{1,2} + a(2-a^2)\,c^2_{1,2} - b^3\,c^4_{4,5})\,\mathrm{e}^{12}\mathrm{f}^{23} \\ &+ (+\xi_2 b^3\,c^2_{1,2} \quad \xi_2 a(2-a^2)\,c^4_{4,5} + c^5_{4,5})\,\mathrm{e}^{13}\mathrm{f}^{12} \\ &+ (-\xi_2 b^3\,c^1_{1,2} - c^4_{4,5} + \xi_2 a(2-a^2)\,c^5_{4,5})\,\mathrm{e}^{23}\mathrm{f}^{12}\end{aligned}$$

$$
\begin{aligned}
&+ (a\, c^1_{1,3} + \xi_2\, c^2_{1,3} + \xi_2 a\, c^4_{4,6} + c^5_{4,6})\, \mathrm{e}^{13}\mathrm{f}^{13} \\
&+ (\xi_2 a^2 c^3_{1,2} - a\, c^1_{1,3} - \xi_2 c^1_{2,3} - \xi_2 a\, c^4_{4,6} - c^4_{5,6} + a^2 c^6_{4,5})\, \mathrm{e}^{23}\mathrm{f}^{23} \\
&+ (a\, c^3_{1,2} - \xi_2\, c^1_{1,3} + a\, c^2_{1,3} - c^4_{4,6} + \xi_2 a\, c^4_{5,6})\, \mathrm{e}^{13}\mathrm{f}^{23} \\
&+ (- \xi_2\, c^1_{1,3} + a\, c^1_{2,3} - c^4_{4,6} + \xi_2 a\, c^5_{4,6} + \xi_2 a\, c^6_{4,5})\, \mathrm{e}^{23}\mathrm{f}^{13}.
\end{aligned}
$$

We need to determine all solutions of the coefficient equations of $d\psi = 0$. First of all, we observe that the variables $c^1_{1,2}$, $c^2_{1,2}$, $c^4_{4,5}$ and $c^5_{4,5}$ are subject to eight linear equations and claim that there is no non-trivial solution of this linear system. Indeed, the determinant of the four by four coefficient matrix of the first four equations is $a^4(a^2\xi_2^2+1)(a^2+\xi_2^2)(a^2+b^2)^2 = a^4(a^2\xi_2^2+1)(a^2+\xi_2^2)$ and thus never vanishes for $a \neq 0$. To deal with the remaining eight structure constants, subject to seven equations, we treat three cases separately.

(a) Assume first that $b \neq 0$ and $\xi_2 \neq 0$, i.e. $0 < |a| < 1$. Obviously, we have $c^3_{1,2} = 0$ and $c^6_{4,5} = 0$ by the vanishing of the first three coefficients. Moreover, applying easy row transformations to the remaining four equations, we observe that it holds necessarily $c^2_{1,3} = c^1_{2,3}$ and $c^5_{4,6} = c^4_{5,6}$. Considering this, $d\psi = 0$ is finally satisfied if and only if

$$s := c^4_{4,6} = \frac{a\,(\xi_2^2-1)\,c^4_{5,6} - (a^2+\xi_2^2)\,c^1_{1,3}}{\xi_2(a^2+1)}\,, \quad t := c^1_{2,3} = -\frac{(\xi_2^2 a^2+1)\,c^4_{5,6} + a(1-\xi_2^2)\,c^1_{1,3}}{\xi_2(a^2+1)}.$$

Applying all substitutions, the set of solutions of the two half-flat equations is parameterised by the four parameters a, ξ_2,

$$p := c^1_{1,3} \quad \text{and} \quad q := c^4_{5,6}.$$

In order to determine the isomorphism class of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ for all solutions, we apply Propositions 2.6 and 2.7. We choose orientations on $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $e_1 \times e_2 = -e_3$ and $e_4 \times e_5 = -e_6$. Let $L_{\mathfrak{g}_1}$ and $L_{\mathfrak{g}_2}$ denote the matrices representing the endomorphisms defined in Proposition 2.6 with respect to our bases. On the set of solutions, they simplify to

$$L_{\mathfrak{g}_1} = \begin{pmatrix} t & -p & 0 \\ -p & -t & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad L_{\mathfrak{g}_2} = \begin{pmatrix} q & -s & 0 \\ -s & -q & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Jacobi identity is already satisfied. Both $L_{\mathfrak{g}_1}$ and $L_{\mathfrak{g}_2}$ are symmetric and in consequence, both summands are unimodular. The eigenvalues of $L_{\mathfrak{g}_1}$ and $L_{\mathfrak{g}_2}$ are $\{0, \pm\sqrt{p^2+t^2}\}$ and $\{0, \pm\sqrt{s^2+q^2}\}$. Hence, if $p \neq 0$ or $q \neq 0$, the Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is isomorphic to $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ with two remaining parameters $\xi_2 \neq 0$ and $0 < |a| < 1$. If $p = 0$ and $q = 0$, the Lie algebra is abelian.

(b) Now assume $b \neq 0$ and $\xi_2 = 0$. In this case, the equations simplify considerably and the only solution of $d\psi = 0$ is given by

$$c^6_{4,5} = 0\,, \quad c^4_{4,6} = ac^1_{2,3}\,, \quad c^5_{4,6} = -ac^1_{1,3}\,, \quad c^4_{5,6} = -ac^1_{1,3}\,, \quad c^3_{1,2} = -c^2_{1,3} + c^1_{2,3}.$$

As before, we rename the remaining parameters

$$p := c^2_{1,3} \quad , \quad q := c^1_{2,3} \quad \text{and} \quad r := c^1_{1,3},$$

and have a closer look at

$$L_{\mathfrak{g}_1} = \begin{pmatrix} q & -r & 0 \\ -r & -p & 0 \\ 0 & 0 & -p+q \end{pmatrix} \quad \text{and} \quad L_{\mathfrak{g}_2} = \begin{pmatrix} -ar & -aq & 0 \\ -ap & ar & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Again, the Jacobi identity is already satisfied. The first summand is always unimodular, the second summand is unimodular if and only if $p = q$. If $p = q$, both matrices are of the same type as in case (a) and $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is isomorphic to $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ or abelian.

It remains to apply Proposition 2.7 to identify the isomorphism class of the solutions with $p \neq q$. Without changing the isomorphism class, we can normalise such that $p = q + 1$. We need to find a vector $X \in \mathfrak{g}_2$ with $\mathrm{tr}(\mathrm{ad_X}) = 2$. Since $\mathrm{tr}(\mathrm{ad_{f_3}}) = \mathrm{c}^4_{4,6} + \mathrm{c}^5_{5,6} = -\mathrm{a}$, we choose

$X = -\frac{2}{a} f_3$. The unimodular kernel $\mathfrak{u}$ is spanned by f_1 and f_2 and the restriction of ad_X on $\mathfrak{u}$ is represented by the matrix

$$\tilde{L}_{\mathfrak{g}_2} = \begin{pmatrix} -2q & 2r \\ 2r & 2(q+1) \end{pmatrix} \qquad \text{with} \quad D = \det(\tilde{L}_{\mathfrak{g}_2}) = -4(q(q+1) + r^2) \leq 1.$$

If $\tilde{L}_{\mathfrak{g}_2}$ is not the identity matrix, the value of D determines the isomorphism class of $\mathfrak{g}_2$. However, the corresponding class of the unimodular summand $\mathfrak{g}_1$ varies with the value of D. In fact, with $r^2 = -q(q+1) - \frac{1}{4}D$, the eigenvalues of $L_{\mathfrak{g}_1}$ are -1 and $-\frac{1}{2} \pm \frac{1}{2}\sqrt{1-D}$. Comparing with the lists in chapter 1, we find the remaining classes listed in the theorem.

(c) The last case to be discussed is $b = 0$ which corresponds to $a = 1$. Now, the equation $d\psi = 0$ is equivalent to

$$c^1_{2,3} = -\xi_2 c^4_{5,6} + \xi_2 c^1_{1,3} + c^4_{4,6}\,, \quad c^3_{1,2} = \xi_2 c^1_{1,3} + c^4_{4,6} - \xi_2 c^4_{5,6} - c^2_{1,3},$$
$$c^5_{4,6} = -\xi_2 c^2_{1,3} - \xi_2 c^4_{4,6} - c^1_{1,3}\,, \quad c^6_{4,5} = \xi_2 c^4_{4,6} + \xi_2 c^2_{1,3} + c^4_{5,6} + c^1_{1,3}.$$

Considering these substitutions, the Jacobi identity is satisfied if and only if

$$\xi_2 c^4_{4,6} + \xi_2 c^2_{1,3} + c^4_{5,6} + c^1_{1,3} = 0 \quad \text{or} \quad \xi_2 c^1_{1,3} + c^4_{4,6} - \xi_2 c^4_{5,6} - c^2_{1,3} = 0.$$

Writing down the matrices $L_{\mathfrak{g}_1}$ and $L_{\mathfrak{g}_2}$ for both cases, it is easy to see that they are of the same form as in case (b). Therefore, the possible isomorphism classes of Lie algebras are exactly the same as in case (b).

Since we have discussed all solutions of the half-flat equations, the theorem is proved. □

4. Classification of direct sums admitting a half-flat SU(3)-structure

4.1. Obstructions to the existence of half-flat SU(3)-structures. In this section, we establish an obstruction to the existence of half-flat SU(3)-structures on Lie algebras following the idea of [C, Theorem 2].

We denote by Z^p the space of closed p-forms on a Lie algebra and by W^0 the annihilator of a subspace W.

Lemma 4.1. *Let $\mathfrak{g}$ be a six-dimensional Lie algebra and $\mathfrak{g}^* = V \oplus W$ a (vector space) decomposition such that V is two-dimensional and such that*

$$Z^3 \quad \subset \quad \Lambda^2 V \wedge W \oplus V \wedge \Lambda^2 W. \tag{4.1}$$

Then, the subspace V is J_ρ-invariant for all closed stable three-forms ρ.

Proof. Let $\rho \in Z^3$ be stable and $\alpha \in V$. Since $\dim V = 2$, the assumption (4.1) implies for all $v \in V^0$

$$v \lrcorner \rho \in \Lambda^2 V \oplus V \wedge W\,, \qquad \alpha \wedge \rho \in \Lambda^3 V \wedge W \oplus \Lambda^2 V \wedge \Lambda^2 W.$$

Therefore, it holds

$$0 = \alpha \wedge (v \lrcorner \rho) \wedge \rho \overset{(2.7)}{=} J^*_\rho \alpha(v)\, \phi(\rho)$$

for all $v \in V^0$ and, by the definition of the annihilator V^0, the subspace V is J_ρ-invariant. □

Proposition 4.2. *Let $\mathfrak{g}$ be a six-dimensional Lie algebra and $\mathfrak{g}^* = V \oplus W$ a decomposition such that V is two-dimensional and such that*

$$Z^3 \quad \subset \quad \Lambda^2 V \wedge W \quad \oplus V \wedge \Lambda^2 W, \tag{4.2}$$
$$Z^4 \quad \subset \quad \Lambda^2 V \wedge \Lambda^2 W \oplus V \wedge \Lambda^3 W. \tag{4.3}$$

Then, the subspace V is isotropic and J_ρ-invariant for every half-flat structure (ω, ρ). In particular, the Lie algebra $\mathfrak{g}$ does not admit a half-flat SU(3)-structure.

Proof. Suppose that (ω,ρ) is a half-flat structure on $\mathfrak{g}$, in particular $\rho \in Z^3$ and $\omega^2 \in Z^4$ by definition. By Lemma 4.1, the subspace V is J_ρ-invariant. Thus, the assumption (4.3) and $\dim V = 2$ imply that

$$0 = \alpha \wedge J_\rho^* \beta \wedge \omega^2 \overset{(2.2)}{=} \frac{1}{3}\, g(\alpha,\beta)\,\omega^3$$

for all $\alpha, \beta \in V$ and V has to be an isotropic subspace of $\mathfrak{g}^*$. This is of course impossible for definite metrics and there cannot exist a half-flat SU(3)-structure. □

Definition 4.3. Let $\mathfrak{g}$ be a Lie algebra. A decomposition $\mathfrak{g}^* = V \oplus W$ is called a *coherent splitting* if

$$dV \subset \Lambda^2 V, \tag{4.4}$$

$$dW \subset \Lambda^2 V \oplus V \wedge W. \tag{4.5}$$

Remark 4.4. The definition can be reformulated into an equivalent dual condition:

$$(4.4) \iff 0 = d\sigma(X,.) = -\sigma([X,.]) \quad \text{for all } X \in V^0, \sigma \in V \iff [V^0,\mathfrak{g}] \subset V^0,$$

$$(4.5) \iff 0 = d\sigma(X,Y) = -\sigma([X,Y]) \quad \text{for all } X,Y \in V^0, \sigma \in W \iff [V^0,V^0] \subset W^0.$$

In other words, a coherent splitting corresponds to a decomposition of $\mathfrak{g}$ into an abelian ideal and a vector space complement.

As elaborated in [C], a coherent splitting with $\dim V = 2$ allows the introduction of a double complex such that the obstruction conditions (4.2), (4.3) can be formulated in terms of the cohomology of this double complex. However, in the situation we are interested in, it turns out to be more practical to avoid homological algebra. Indeed, the verification of the obstruction conditions can be simplified as follows.

Lemma 4.5. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum of three-dimensional Lie algebras.*

(i) Let $\alpha_1 \in \mathfrak{g}_1^*$ and $\alpha_2 \in \mathfrak{g}_2^*$ be one-forms defining $V = \mathrm{span}(\alpha_1,\alpha_2)$. Then $\mathfrak{g}^* = V \oplus W$ is a coherent splitting for any complement W of V if and only if the two one-forms α_i are closed and satisfy

$$\mathrm{im}(d : \mathfrak{g}_i^* \to \Lambda^2 \mathfrak{g}_i^*) \subset \alpha_i \wedge \mathfrak{g}_i^* \qquad \textit{for both } i. \tag{4.6}$$

(ii) If both summands are non-abelian, every *coherent splitting with* $\dim V = 2$ *is defined by closed one-forms $\alpha_1 \in \mathfrak{g}_1^*$ and $\alpha_2 \in \mathfrak{g}_2^*$ satisfying* (4.6).

(iii) There exists a coherent splitting with $\dim V = 2$ *on $\mathfrak{g}$ if and only if $\mathfrak{g}$ is solvable.*

(iv) If $\mathfrak{g}$ is unimodular, there is no decomposition $\mathfrak{g}^* = V \oplus W$ with two-dimensional V satisfying both obstruction conditions (4.2) and (4.3).

Proof. (i) Since both the exterior algebras $\Lambda^* \mathfrak{g}_i^*$ are d-invariant, the condition (4.4) is satisfied if and only if both generators are closed and (4.5) is equivalent to (4.6).

(ii) Assume that both summands $\mathfrak{g}_i$ are not abelian and let a coherent splitting be defined by an abelian four-dimensional ideal V^0 and a complement. In consequence, both the intersection $V^0 \cap \mathfrak{g}_i$ and the projection of V^0 on $\mathfrak{g}_i$ are abelian subalgebras of $\mathfrak{g}_i$ for both i and thus at most two-dimensional. Since a one-dimensional intersection $V^0 \cap \mathfrak{g}_i$ would require the projection on the other summand to be three-dimensional, it follows that the intersections $V^0 \cap \mathfrak{g}_i$ have to be exactly two-dimensional. Equivalently, the two-dimensional space V is generated by two one-forms $\alpha_1 \in \mathfrak{g}_1^*$ and $\alpha_2 \in \mathfrak{g}_2^*$. Now, the assertion follows from part (i).

(iii) On the one hand, if $\mathfrak{g}$ is not solvable, one of the summands has to be simple, say $\mathfrak{g}_1$. However, the intersection of a four-dimensional abelian ideal with $\mathfrak{g}_1$ would be zero or $\mathfrak{g}_1$, both of which is not possible since $\dim \mathfrak{g} = 6$ and since $\mathfrak{g}_1$ is not abelian. On the other hand, inspecting the list of standard bases in tables 1 and 2 reveals that any three-dimensional solvable Lie algebra $\mathfrak{h}$ contains a closed one-form α such that $\mathrm{im}\, d \subset \alpha \wedge \mathfrak{h}^*$. Therefore, if $\mathfrak{g}$ is solvable, i.e. both summands are solvable, a coherent splitting exists by part (i).

(iv) Assume that $\mathfrak{g}$ is unimodular and let W be an arbitrary four-dimensional subspace of $\mathfrak{g}^*$. It suffices to show that there always exists a closed three-form with non-zero projection on $\Lambda^3 W$ *or* a closed four-form with non-zero projection on $\Lambda^4 W$. If the projection of W on one of the summands $\mathfrak{g}_i$ is surjective, every non-zero element of $\Lambda^3\mathfrak{g}_i^*$ is closed and has non-zero projection on $\Lambda^3 W$. Otherwise, the image of the projection of W on either of the summands has to be two-dimensional for dimensional reasons. In this case, there is always a closed four-form with non-zero projection on $\Lambda^4 W$ since all four-forms in $\Lambda^2\mathfrak{g}_1^* \wedge \Lambda^2\mathfrak{g}_2^*$ are closed by unimodularity. This finishes the proof of the lemma.

□

Lemma 4.6. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum of three-dimensional Lie algebras and let $\mathfrak{g}^* = V \oplus W$ be a coherent splitting such that $V = \mathrm{span}(\alpha_1, \alpha_2)$ is defined by closed one-forms $\alpha_1 \in \mathfrak{g}_1^*$ and $\alpha_2 \in \mathfrak{g}_2^*$ satisfying* (4.6). *Then, the obstruction conditions* (4.2) *and* (4.3) *are equivalent to the condition that d is injective when restricted to $\Lambda^3 W$ and $\Lambda^4 W$.*

Proof. The injectivity of d on $\Lambda^3 W$ and $\Lambda^4 W$ is obviously necessary for (4.2) and (4.3). With the assumptions, it is also sufficient since the coherent splitting satisfies $dW \subset V \wedge W$ and $dV = 0$ such that the images of $\Lambda^3 W$ and $\Lambda^4 W$ are linearly independent from the images of the complements $\Lambda^2 V \wedge W \oplus V \wedge \Lambda^2 W$ and $\Lambda^2 V \wedge \Lambda^2 W \oplus V \wedge \Lambda^3 W$, respectively. □

4.2. **The classification.** Using the obstruction established in the previous section, we obtain the following classification result.

Theorem 4.7. *A direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of three-dimensional Lie algebras admits a half-flat* SU(3)*-structure if and only if*

(i) $\mathfrak{g}$ is unimodular or
(ii) $\mathfrak{g}$ is not solvable or
(iii) $\mathfrak{g}$ is isomorphic to $\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ or $\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$.

Proof. A standard basis of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ will always denote the union of a standard basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}_1$ and a standard basis $\{f_1, f_2, f_3\}$ of $\mathfrak{g}_2$ as defined in tables 1 and 2. For all Lie algebras admitting a half-flat SU(3)-structure, such a structure is explicitly given in a standard basis in the appendix. We remark that most examples are constructed exploiting the stable form formalism and with computer support. In the following, we prove the non-existence of half-flat SU(3)-structures on the remaining Lie algebras.

In most of the cases, the obstructions of section 4.1 can be applied directly.

Lemma 4.8. *The Lie algebra $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ and all Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1$ solvable and $\mathfrak{g}_2$ one of the algebras $\mathfrak{r}_3$, $\mathfrak{r}_{3,\mu}$, $0 < |\mu| \leq 1$, $\mathfrak{r}'_{3,\mu}$, $\mu > 0$, do not admit a half-flat* SU(3)*-structure.*

Proof. We want to apply the obstruction established in Proposition 4.2 and, given any of the Lie algebras $\mathfrak{g}$ in question, we define a decomposition

$$V = \mathrm{span}\{e^1, f^1\}, \quad W = \mathrm{span}\{e^2, e^3, f^2, f^3\},$$

in a standard basis of $\mathfrak{g}^*$. By Lemma 4.6 it suffices to show that this is a coherent splitting such that the restrictions $d_{|\Lambda^3 W}$ and $d_{|\Lambda^4 W}$ are injective. In fact, the coherence can be verified directly by comparing the conditions of Lemma 4.5, (i), with the standard bases of the solvable three-dimensional Lie algebras.

If $\mathfrak{g}_2$ is one of the algebras $\mathfrak{r}_3$, $\mathfrak{r}_{3,\mu}$, $0 < |\mu| \leq 1$ or $\mathfrak{r}'_{3,\mu}$, $\mu > 0$, the standard bases satisfy

$$df^2 \neq 0, \nexists\, c \in \mathbb{R} : df^3 = c\, df^2, \; df^{23} \neq 0.$$

Thus, considering again that the exterior algebras $\Lambda^*\mathfrak{g}_i^*$ of the summands are d-invariant, the image

$$d(\Lambda^3 W) = \mathrm{span}\{d(e^{23}f^2), \, d(e^{23}f^3), \, d(e^2 f^{23}), \, d(e^3 f^{23})\}$$

is four-dimensional and the image

$$d(\Lambda^4 W) = \mathrm{span}\{d(e^{23}f^{23})\}$$

is one-dimensional. The same restrictions are injective for $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathfrak{r}_2 \oplus \mathbb{R}$, since in this case $d\mathrm{e}^{23} \neq 0$ and $d\mathrm{f}^{23} \neq 0$. This finishes the proof. □

The obstruction theory cannot be applied directly to the two remaining Lie algebras, although they admit coherent splittings and we have to deal with them separately.

Lemma 4.9. *The Lie algebra* $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ *does not admit a half-flat* $\mathrm{SU}(3)$*-structure. Furthermore, there is no decomposition* $\mathfrak{g}^* = V \oplus W$ *with two-dimensional* V *satisfying the obstruction condition* (4.2).

Proof. We start by proving the second assertion. Let $W \subset \mathfrak{g}^*$ be an arbitrary four-dimensional subspace. It suffices to show that there is always a closed three-form with non-zero projection on $\Lambda^3 W$. If the projection of W on one of the summands $\mathfrak{g}_i^*$ is surjective, a generator of $\Lambda^3 \mathfrak{g}_i^*$ is closed and has non-zero projection on $\Lambda^3 W$. For dimensional reasons, the only remaining possibility is that both projections have two-dimensional image in W. However, since all two-forms in $\Lambda^2 \mathfrak{h}_3^*$ are closed and the kernel of d is two-dimensional on $\mathfrak{r}_2 \oplus \mathbb{R}$, there is necessarily a closed three-form in $\Lambda^2 \mathfrak{h}_3^* \wedge (\mathfrak{r}_2 \oplus \mathbb{R})^*$ with non-zero projection on $\Lambda^3 W$. Therefore, the obstruction condition (4.2) is never satisfied.

However, we can prove that there is no half-flat SU(3)-structure by refining the idea of the obstruction condition as follows. Suppose that (ρ, ω) is a half-flat SU(3)-structure, i.e. $\rho \in Z^3$ and $\sigma = \frac{1}{2}\omega^2 \in Z^4$ and let $\{\mathrm{e}_1, \ldots, \mathrm{f}_3\}$ denote a standard basis of $\mathfrak{h}_3 \oplus \mathfrak{r}_2 \oplus \mathbb{R}$. We claim that

$$\mathrm{f}^1 \wedge J_\rho^* \mathrm{f}^1 \wedge \sigma = 0$$

which suffices to prove the non-existence since f_1 would be isotropic by (2.2). First of all, an easy calculation reveals that

$$\mathrm{f}_1 \wedge \sigma \in \mathrm{span}\{\mathrm{f}^1\mathrm{e}^{12}\mathrm{f}^{23},\ \mathrm{f}^1\mathrm{e}^{123}\mathrm{f}^3\}$$

for an arbitrary closed four-form σ. Thus, it remains to show that $J_\rho^* \mathrm{f}^1$ has no component along e^3 and f^2 or equivalently that

$$J_\rho^* \mathrm{f}^1(v)\phi(\rho) \overset{(2.7)}{=} \mathrm{f}^1 \wedge (v \lrcorner \rho) \wedge \rho$$

vanishes for $v \in \{\mathrm{e}_3, \mathrm{f}_2\}$. This assertion is straightforward to verify for an arbitrary closed three-form ρ. □

For the last remaining Lie algebra, we apply a different argument.

Lemma 4.10. *The Lie algebra* $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathbb{R}^3$ *does not admit a closed stable form* ρ *with* $\lambda(\rho) < 0$*, in particular it does not admit a half-flat* $\mathrm{SU}(3)$*-structure. Furthermore, there is no decomposition* $\mathfrak{g}^* = V \oplus W$ *with two-dimensional* V *which satisfies the obstruction condition* (4.2).

Proof. Suppose that ρ is a closed stable form inducing a complex or a para-complex structure J_ρ. Let $\{\mathrm{e}_1, \mathrm{e}_2\}$ be a basis of $\mathfrak{r}_2$ such that $d\mathrm{e}^2 = \mathrm{e}^{21}$. Since ρ is closed, there are a one-form $\beta \in (\mathbb{R}^4)^*$, a two-form $\gamma \in \Lambda^2(\mathbb{R}^4)^*$ and a three-form $\delta \in \Lambda^3(\mathbb{R}^4)^*$, such that

$$\rho = \mathrm{e}^{12} \wedge \beta + \mathrm{e}^1 \wedge \gamma + \delta.$$

Therefore, we have

$$K_\rho(\mathrm{e}_2) = \kappa((\mathrm{e}_2 \lrcorner \rho) \wedge \rho) = \kappa(-\mathrm{e}^1 \wedge \beta \wedge \delta)$$

with $\beta \wedge \delta \in \Lambda^4(\mathbb{R}^4)^*$. However, this implies that $J(\mathrm{e}_2)$ is proportional to e_2 by (2.6) which is only possible if $\lambda(\rho) > 0$ and the first assertion is proven.

In order to prove the second assertion, it suffices to show that for every four-dimensional subspace $W \subset \mathfrak{g}^*$, there is a closed three-form with non-zero projection on $\Lambda^3 W$. This follows immediately from the observation that $\dim\,(\ker d) = 5$ which implies that $\dim\,(\ker d \cap W) \geq 3$ for every four-dimensional subspace W. □

The lemma finishes the proof of the theorem as all possible direct sums according to the classification of three-dimensional Lie algebras have been considered. □

We remark that the lemmas 4.9 and 4.10 give two examples of solvable Lie algebras which show that the condition of [C, Theorem 5], which characterises six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structure, cannot be generalised without further restrictions to solvable Lie algebras.

5. Half-flat SU(1, 2)-structures on direct sums

In this section, we describe some interesting observations concerning the existence of half-flat $SU(p, q)$-structures, $p + q = 3$, with indefinite metrics on direct sums $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of three-dimensional Lie algebras. It suffices to consider SU(1, 2)-structures after possibly multiplying the metric by minus one.

First of all, the obstruction condition of Proposition 4.2 does not apply since isotropic subspaces are of course possible for metrics of signature (2, 4). For instance, the Lie algebra $\mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ does admit a half-flat SU(1, 2)-structure but no half-flat SU(3)-structure. Indeed, the structure defined in the standard basis by

$$\begin{aligned}
\rho &= -\mathrm{e}^{123} - \mathrm{e}^{12}\mathrm{f}^{3} - \mathrm{e}^{12}\mathrm{f}^{2} + 2\,\mathrm{e}^{13}\mathrm{f}^{3} + \mathrm{e}^{2}\mathrm{f}^{12} - \mathrm{e}^{3}\mathrm{f}^{13} + \mathrm{f}^{123},\\
\omega &= \mathrm{e}^{13} - \mathrm{e}^{1}\mathrm{f}^{2} + \mathrm{e}^{1}\mathrm{f}^{3} + \mathrm{e}^{2}\mathrm{f}^{3} - \mathrm{f}^{12},\\
g &= -(\mathrm{e}^{2})^2 - 2\,(\mathrm{f}^{3})^2 + 2\,\mathrm{e}^{1}\!\cdot\mathrm{e}^{3} + 2\,\mathrm{e}^{1}\!\cdot\mathrm{f}^{2} + 2\,\mathrm{e}^{1}\!\cdot\mathrm{f}^{3} - 2\,\mathrm{e}^{2}\!\cdot\mathrm{f}^{3} + 2\,\mathrm{e}^{3}\!\cdot\mathrm{f}^{1} + 2\,\mathrm{f}^{1}\!\cdot\mathrm{f}^{3},
\end{aligned}$$

is a half-flat SU(1, 2)-structure with $V = \mathrm{span}\{\mathrm{e}^1, \mathrm{f}^1\}$ J_ρ-invariant and isotropic.

In fact, the obstruction established in Lemma 4.10 is stronger and also shows the non-existence of a half-flat SU(1, 2)-structure on $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathbb{R}^3$. It can be generalised to the following Lie algebras.

Proposition 5.1. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a Lie algebra such that $\mathfrak{g}_1$ is one of the algebras $\mathbb{R}^3$, $\mathfrak{h}_3$ or $\mathfrak{r}_2 \oplus \mathbb{R}$ and $\mathfrak{g}_2$ is one of the algebras $\mathfrak{r}_3$, $\mathfrak{r}_{3,\mu}$, $0 < |\mu| \le 1$, $\mathfrak{r}'_{3,\mu}$, $\mu > 0$.*

Every closed three-form ρ on one of these Lie algebras $\mathfrak{g}$ satisfies $\lambda(\rho) \ge 0$. In particular, these Lie algebras do not admit a half-flat $SU(p, q)$-structure for any signature (p, q) with $p + q = 3$.

Proof. The proof is straightforward, but tedious without computer support. In a fixed basis, the condition $d\rho = 0$ is linear in the coefficients of an arbitrary three-form ρ and can be solved directly. When identifying $\Lambda^6 V^*$ with $\mathbb{R}$ with the help of a volume form ν, one can calculate the quartic invariant $\lambda(\rho) \in \mathbb{R}$, for instance in a standard basis. Carrying this out with Maple and factorising the resulting expression, we verified $\lambda(\rho) \ge 0$ for an arbitrary closed three-form on any of the Lie algebras in question.

As a half-flat $SU(p, q)$-structure is defined by a pair (ρ, ω) of stable forms which satisfy in particular $\lambda(\rho) < 0$ and $d\rho = 0$, such a structure cannot exist and the lemma is proven. □

We add the remark, that a result analogous to Lemma 3.1 for a pseudo-Hermitian structure of indefinite signature would involve a considerably more complicated normal form for ω. Therefore, a generalisation of the proof of Theorem 3.2 to indefinite metrics seems to be difficult.

6. Half-flat SL(3, ℝ)-structures on direct sums

Finally, we turn to the para-complex case of $SL(3, \mathbb{R})$-structures. As explained in section 2, a half-flat $SL(3, \mathbb{R})$-structure is defined by a pair (ρ, ω) of stable forms such that J_ρ is an almost para-complex structure and

$$\omega \wedge \rho = 0, \quad d\omega^2 = 0, \quad d\rho = 0.$$

As the induced metric is always neutral and $\lambda(\rho) > 0$, neither Proposition 4.2 nor Lemma 5.1 obstruct the existence of such a structure. For instance, the Lie algebra $\mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathfrak{r}_3$ does not admit a half-flat $SU(p, q)$-structure for any signature (p, q) with $p + q = 3$, but

$$\begin{aligned}
\rho &= -2\,\mathrm{e}^{12}\mathrm{f}^{3} - 2\,\mathrm{e}^{2}\mathrm{f}^{31} + \mathrm{e}^{3}\mathrm{f}^{12} - \mathrm{e}^{3}\mathrm{f}^{31} + \mathrm{f}^{123},\\
\omega &= \mathrm{e}^{13} - \mathrm{e}^{23} + \mathrm{e}^{1}\mathrm{f}^{3} + \mathrm{e}^{2}\mathrm{f}^{2} - \mathrm{e}^{3}\mathrm{f}^{1} + 2\,\mathrm{f}^{13},\\
g &= -2\,(\,\mathrm{e}^{1}\!\cdot\mathrm{e}^{3} - \mathrm{e}^{2}\!\cdot\mathrm{e}^{3} + \mathrm{e}^{1}\!\cdot\mathrm{f}^{3} + \mathrm{e}^{2}\!\cdot\mathrm{f}^{2} + \mathrm{e}^{3}\!\cdot\mathrm{f}^{1}\,),
\end{aligned}$$

is an example of a half-flat $SL(3, \mathbb{R})$-structure.

When trying to generalise Theorem 3.2 to the para-complex situation, we find an astonishingly similar result if we additionally require the metric to be definite when restricted to one of the summands. We omit the proofs which are very similar to the original ones due to the analogies explained in section 2.

Lemma 6.1. *Let* (V_1, g_1) *and* (V_2, g_2) *be Euclidean vector spaces and let* (g, J, ω) *be a para-Hermitian structure on the orthogonal product* $(V_1 \oplus V_2, g = -g_1 + g_2)$*. There are orthonormal bases* $\{e_1, e_2, e_3\}$ *of* V_1 *and* $\{f_1, f_2, f_3\}$ *of* V_2 *which can be joined to a pseudo-orthonormal basis of* $V_1 \oplus V_2$ *such that*

$$\omega = a\,\mathrm{e}^{12} + \sqrt{1+a^2}\,\mathrm{e}^1\mathrm{f}^1 + \sqrt{1+a^2}\,\mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3 + a\,\mathrm{f}^{12} \tag{6.1}$$

for a real number a.

In analogy to the Hermitian case, we call the para-Hermitian structure of type I if it admits a basis with $a = 0$ and of type II if it admits a basis with $a \neq 0$.

Theorem 6.2. *Let* $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *be a direct sum of three-dimensional Lie algebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$*.*

The Lie algebra $\mathfrak{g}$ *admits a half-flat* $\mathrm{SL}(3,\mathbb{R})$*-structure such that* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are mutually orthogonal, such that the restriction of the metric to both summands is definite and such that the underlying para-Hermitian structure is of type I if and only if*

(i) $\mathfrak{g}_1 = \mathfrak{g}_2$ *and both are unimodular or*

(ii) $\mathfrak{g}_1$ *is non-abelian unimodular and* $\mathfrak{g}_2$ *abelian or vice versa.*

Moreover, the Lie algebra $\mathfrak{g}$ *admits a half-flat* $\mathrm{SL}(3,\mathbb{R})$*-structure such that* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are mutually orthogonal, such that the restriction of the metric to both summands is definite and such that the underlying para-Hermitian structure is of type II if and only if the pair* $(\mathfrak{g}_1, \mathfrak{g}_2)$ *or* $(\mathfrak{g}_2, \mathfrak{g}_1)$ *is contained in the following list:*

$$\begin{aligned}
&(\mathfrak{e}(1,1), \mathfrak{e}(1,1)), && \\
&(\mathfrak{e}(2), \mathbb{R} \oplus \mathfrak{r}_2), && \\
&(\mathfrak{su}(2), \mathfrak{r}_{3,\mu}) && \text{for } 0 < \mu \leq 1, \\
&(\mathfrak{sl}(2,\mathbb{R}), \mathfrak{r}_{3,\mu}) && \text{for } -1 < \mu < 0.
\end{aligned}$$

If we require, instead of orthogonality, that the $\mathrm{SL}(3,\mathbb{R})$-structure is adapted to the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ in the sense that the summands $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the eigenspaces of J, we find the following interesting relation to unimodularity.

Proposition 6.3. *A direct sum* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ *of three-dimensional Lie algebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *admits a half-flat* $\mathrm{SL}(3,\mathbb{R})$*-structure* (g, J, ω, Ψ) *such that* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are the* ± 1*-eigenspaces of* J *if and only if both* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are unimodular.*

Proof. Let (g, J, ω, Ψ) be an $\mathrm{SL}(3,\mathbb{R})$-structure on $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathfrak{g}_1$ is the $+1$-eigenspace of J and $\mathfrak{g}_2$ is the -1-eigenspaces of J. Since $\psi^+ = \mathrm{Re}\Psi$ is a stable form inducing the para-complex structure J, we can choose bases $\{\mathrm{e}^i\}$ of $\mathfrak{g}_1^*$ and $\{\mathrm{f}^i\}$ of $\mathfrak{g}_2^*$ such that $\psi^+ = \mathrm{e}^{123} + \mathrm{f}^{123}$ is in the normal form 2.1. Thus, the real part ψ^+ is closed as we are dealing with a direct sum of Lie algebras. Due to the simple form of ψ^+, it is easy to verify that the relation $\omega \wedge \psi^+ = 0$ holds for an arbitrary non-degenerate ω if and only if ω has only terms in $\mathfrak{g}_1^* \otimes \mathfrak{g}_2^*$. Now we are in the situation of Lemma 2.5 and conclude that the only remaining equation $d\omega^2 = 0$ is satisfied if and only if both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular. □

7. Appendix

The following tables contain all direct sums $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of three-dimensional Lie algebras which admit a half-flat SU(3)-structure. In each case, an explicit example (ω, ρ) of a normalised half-flat SU(3)-structure including the induced Riemannian metric g is given where $\{\mathrm{e}^i\}$ is a standard basis (as defined in tables 1 and 2) of $\mathfrak{g}_1^*$ and $\{\mathrm{f}^i\}$ is a standard basis of $\mathfrak{g}_2^*$.

Table 3. Unimodular direct sums of three-dimensional Lie algebras

Lie algebra	Half-flat SU(3)-structure with $\omega = \mathrm{e}^1\mathrm{f}^1 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3$
$\mathfrak{h} \oplus \mathfrak{h}$, $\mathfrak{h}$ unimodular	$\rho = \frac{1}{2}\sqrt{2}\,\{\, \mathrm{e}^{123} - \mathrm{e}^1\mathrm{f}^{23} - \mathrm{e}^2\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{23}\mathrm{f}^1 - \mathrm{f}^{123} \,\}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$
$\mathfrak{h} \oplus \mathbb{R}^3$, $\mathfrak{h}$ unimodular	$\rho = \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{23}\mathrm{f}^1 - \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$
$\mathfrak{su}(2) \oplus \mathfrak{sl}(2,\mathbb{R})$,	$\rho = 2^{\frac{1}{4}}\,\{\, \frac{1}{2}\,\mathrm{e}^{123} + \mathrm{e}^{23}\mathrm{f}^1 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^1\mathrm{f}^{23} - \mathrm{e}^2\mathrm{f}^{31} + \mathrm{e}^3\mathrm{f}^{12} - 2\,\mathrm{f}^{123} \,\}$ $g = \sqrt{2}\,\{\, \frac{3}{2}\,(\mathrm{e}^1)^2 + \frac{3}{2}\,(\mathrm{e}^2)^2 + \frac{1}{2}\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + 3\,(\mathrm{f}^3)^2$ $+ 2\,\mathrm{e}^1\!\cdot\!\mathrm{f}^1 + 2\,\mathrm{e}^2\!\cdot\!\mathrm{f}^2 - 2\,\mathrm{e}^3\!\cdot\!\mathrm{f}^3 \,\}$
$\mathfrak{su}(2) \oplus \mathfrak{e}(2)$	$\rho = -\mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{31} + \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2\,(\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2 - 2\,\mathrm{e}^1\!\cdot\!\mathrm{f}^1$
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{e}(2)$	$\rho = -2\,\mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + 2\,(\mathrm{e}^2)^2 + 2\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2 + 2\,\mathrm{e}^2\!\cdot\!\mathrm{f}^2 - 2\,\mathrm{e}^3\!\cdot\!\mathrm{f}^3$
$\mathfrak{su}(2) \oplus \mathfrak{e}(1,1)$, $\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$	$\rho = -2\,\mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + 2\,(\mathrm{e}^2)^2 + 2\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2 + 2\,\mathrm{e}^2\!\cdot\!\mathrm{f}^2 - 2\,\mathrm{e}^3\!\cdot\!\mathrm{f}^3$
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{e}(1,1)$	$\rho = -\mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{31} + \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2\,(\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2 - 2\,\mathrm{e}^1\!\cdot\!\mathrm{f}^1$
$\mathfrak{su}(2) \oplus \mathfrak{h}_3$, $\mathfrak{e}(2) \oplus \mathfrak{h}_3$	$\rho = -\mathrm{e}^{23}\mathrm{f}^1 - \frac{5}{4}\,\mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = \frac{5}{4}\,(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{5}{4}\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + \frac{5}{4}\,(\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$ $-\mathrm{e}^1\!\cdot\!\mathrm{f}^1 - \mathrm{e}^2\!\cdot\!\mathrm{f}^2 + \mathrm{e}^3\!\cdot\!\mathrm{f}^3$
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{h}_3$, $\mathfrak{e}(1,1) \oplus \mathfrak{h}_3$	$\rho = -\mathrm{e}^{23}\mathrm{f}^1 - \frac{5}{4}\,\mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = \frac{5}{4}\,(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{5}{4}\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + \frac{5}{4}\,(\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$ $+\,\mathrm{e}^1\!\cdot\!\mathrm{f}^1 + \mathrm{e}^2\!\cdot\!\mathrm{f}^2 - \mathrm{e}^3\!\cdot\!\mathrm{f}^3$

Table 4. Solvable, non-unimodular direct sums admitting a half-flat SU(3)-structure

Lie algebra	Half-flat SU(3)-structure
$\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$	$\omega = \mathrm{e}^{12} + \mathrm{e}^3\mathrm{f}^1 - \mathrm{f}^{23}$ $\rho = \mathrm{e}^{23}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{21} + \mathrm{e}^{13}\mathrm{f}^2 - \mathrm{e}^1\mathrm{f}^{31}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$
$\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$	$\omega = -\mathrm{e}^1\mathrm{f}^3 - \mathrm{e}^3\mathrm{f}^2 + \mathrm{e}^2\mathrm{f}^1 - \mathrm{f}^{23}$ $\rho = \mathrm{e}^{23}\mathrm{f}^3 - 2\,\mathrm{e}^{31}\mathrm{f}^1 + \mathrm{e}^{12}\mathrm{f}^2 - 3\,\mathrm{e}^1\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + 2\,\mathrm{f}^{123}$ $g = 2\,(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + 2\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + 5\,(\mathrm{f}^3)^2 - 2\,\mathrm{e}^1\!\cdot\!\mathrm{f}^2 - 6\,\mathrm{e}^3\!\cdot\!\mathrm{f}^3$

Table 5. Direct sums which are neither solvable nor unimodular

Lie algebra	Half-flat SU(3)-structure
$\mathfrak{su}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$, $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$	$\omega = \mathrm{e}^1\mathrm{f}^1 - \mathrm{f}^{23} + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3$ $\rho = \mathrm{e}^{23}\mathrm{f}^1 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{12} - \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + 2\,(\mathrm{f}^2)^2 + (\mathrm{f}^3)^2 - 2\,\mathrm{e}^3\cdot\mathrm{f}^2$
$\mathfrak{su}(2) \oplus \mathfrak{r}_3$	$\omega = \mathrm{f}^{23} + \mathrm{e}^{23} + 2\,\mathrm{e}^1\mathrm{f}^1$ $\rho = \frac{2}{3}\,3^{\frac{3}{4}}\,\{\mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^2\mathrm{f}^{31} + \mathrm{e}^3\mathrm{f}^{31} + \mathrm{e}^2\mathrm{f}^{12}\}$ $g = \frac{2}{3}\sqrt{3}\,\{2\,(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2\,(\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$ $+ 2\,\mathrm{e}^1\cdot\mathrm{f}^1 - \mathrm{e}^2\cdot\mathrm{e}^3 + \mathrm{f}^2\cdot\mathrm{f}^3\}$
$\mathfrak{sl}(2) \oplus \mathfrak{r}_3$	$\omega = \mathrm{e}^1\mathrm{f}^1 - 2\,\mathrm{f}^{23} + \mathrm{e}^3\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^2$ $\rho = \frac{1}{3}\,\mathrm{e}^{23}\mathrm{f}^1 + 3\,\mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{31}\mathrm{f}^3 + \mathrm{e}^{12}\mathrm{f}^2 + \frac{4}{3}\,\mathrm{e}^{12}\mathrm{f}^3 - 4\,\mathrm{e}^2\mathrm{f}^{31} + \frac{7}{3}\,\mathrm{e}^3\mathrm{f}^{31} + 3\,\mathrm{e}^2\mathrm{f}^{12}$ $-\mathrm{e}^3\mathrm{f}^{12} - 26\,\mathrm{f}^{123}$ $g = 3\,(\mathrm{e}^1)^2 + \frac{4}{9}\,(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + \frac{17}{3}\,(\mathrm{f}^1)^2 + 94\,(\mathrm{f}^2)^2 + \frac{328}{9}\,(\mathrm{f}^3)^2$ $-8\,\mathrm{e}^1\cdot\mathrm{f}^1 - \frac{2}{3}\,\mathrm{e}^2\cdot\mathrm{e}^3 + \frac{34}{3}\,\mathrm{e}^2\cdot\mathrm{f}^2 + \frac{16}{9}\,\mathrm{e}^2\cdot\mathrm{f}^3 - 16\,\mathrm{e}^3\cdot\mathrm{f}^2 - \frac{34}{3}\,\mathrm{e}^3\cdot\mathrm{f}^3 + \frac{224}{3}\,\mathrm{f}^2\cdot\mathrm{f}^3$
$\mathfrak{su}(2) \oplus \mathfrak{r}_{3,\mu}$ $(0 < \mu \leq 1)$	$\omega = \frac{1}{\mu+1}\,\mathrm{e}^{12} + \mathrm{e}^3\mathrm{f}^1 - \mathrm{f}^{32}$ $\rho = \mu^{-\frac{1}{4}}(\mu+1)^{-\frac{1}{2}}\,\{\mathrm{e}^{13}\mathrm{f}^2 - \mathrm{e}^{23}\mathrm{f}^3 - \mu\,\mathrm{e}^1\mathrm{f}^{13} - \mathrm{e}^2\mathrm{f}^{12}\}$ $g = \mu^{-\frac{1}{2}}\,\{\frac{\mu}{\mu+1}\,(\mathrm{e}^1)^2 + \frac{1}{\mu+1}\,(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + \mu\,(\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + \mu\,(\mathrm{f}^3)^2\}$
$\mathfrak{sl}(2) \oplus \mathfrak{r}_{3,\mu}$ $(-1 < \mu < 0)$	$\omega = \frac{1}{\mu+1}\,\mathrm{e}^{23} + \mathrm{e}^1\mathrm{f}^1 + \mathrm{f}^{32}$ $\rho = (-\mu)^{-\frac{1}{4}}(\mu+1)^{-\frac{1}{2}}\,\{\mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^{13}\mathrm{f}^2 + \mathrm{e}^2\mathrm{f}^{12} - \mu\,\mathrm{e}^3\mathrm{f}^{13}\}$ $g = (-\mu)^{-\frac{1}{2}}\,\{(\mathrm{e}^1)^2 + \frac{1}{\mu+1}\,(\mathrm{e}^2)^2 - \frac{\mu}{\mu+1}\,(\mathrm{e}^3)^2 - \mu\,(\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 - \mu\,(\mathrm{f}^3)^2\}$
$\mathfrak{su}(2) \oplus \mathfrak{r}_{3,\mu}$ $(-1 < \mu < 0)$	$\omega = \mathrm{f}^{23} + \mathrm{e}^3\mathrm{f}^1 - \frac{\mu(2\mu+3)}{2(\mu+1)^2}\,\mathrm{e}^{23} - \mathrm{e}^1\mathrm{f}^1 + \mathrm{e}^1\mathrm{f}^3 + \frac{\mu(2\mu+3)}{2(\mu+1)^2}\,\mathrm{e}^{12} - \frac{2\mu^2+\mu-2}{2(\mu+1)^2}\,\mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3$ $\rho = -\frac{2\mu^2+3\mu+2}{2(\mu+1)^2}\,\mathrm{e}^{23}\mathrm{f}^1 - \frac{1}{\mu}\,\mathrm{e}^{23}\mathrm{f}^3 - 2\,\mathrm{e}^{13}\mathrm{f}^2 + \frac{2\mu^2+3\mu+2}{2(\mu+1)^2}\,\mathrm{e}^{12}\mathrm{f}^1$ $-\frac{1}{\mu}\,\mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^1\mathrm{f}^{13} - \mathrm{e}^3\mathrm{f}^{13} + 2\,\mathrm{e}^2\mathrm{f}^{12} + 2\,\mathrm{f}^{123}$ $g = -\frac{\mu^2+\mu+1}{\mu(\mu+1)}\,(\mathrm{e}^1)^2 - \frac{4\mu^4+20\mu^3+29\mu^2+16\mu+4}{4\mu(\mu+1)^3}\,(\mathrm{e}^2)^2 - \frac{\mu^2+\mu+1}{\mu(\mu+1)}\,(\mathrm{e}^3)^2$ $-\frac{\mu}{\mu+1}\,(\mathrm{f}^1)^2 + \frac{4+3\mu}{\mu+1}\,(\mathrm{f}^2)^2 - \frac{\mu+1}{\mu}\,(\mathrm{f}^3)^2$ $+\frac{2(\mu^2+1+3\mu)}{\mu(\mu+1)}\,\mathrm{e}^1\cdot\mathrm{e}^3 + \frac{2(\mu+2)}{\mu+1}\,\mathrm{e}^1\cdot\mathrm{f}^2 - \frac{2\mu^2+5\mu+2}{\mu(\mu+1)}\,\mathrm{e}^2\cdot\mathrm{f}^3 + \frac{2(\mu+2)}{\mu+1}\,\mathrm{e}^3\cdot\mathrm{f}^2$
$\mathfrak{sl}(2) \oplus \mathfrak{r}_{3,\mu}$ $(0 < \mu \leq 1)$	$\omega = \frac{2(2\mu+1)^{\frac{1}{2}}}{(\mu+1)^2}\,\mathrm{e}^1\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^1 + \mathrm{f}^{23} + \frac{\mu}{\mu+1}\,\mathrm{e}^{13} + \mathrm{e}^1\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3$ $\rho = 2\frac{2(2\mu+1)^{\frac{1}{2}}}{(\mu+1)^2}\,\mathrm{e}^{123} + \mathrm{e}^{23}\mathrm{f}^2 - \mathrm{e}^{13}\mathrm{f}^1 + \frac{1}{\mu}\,\mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^3\mathrm{f}^{13} + \mathrm{e}^1\mathrm{f}^{12} + \frac{\mu+1}{\mu}\,\mathrm{f}^{123}$ $g = \frac{\mu^3+11\mu^2+7\mu+1}{\mu(\mu+1)^3}\,(\mathrm{e}^1)^2 + \frac{\mu+1}{\mu}\,(\mathrm{e}^2)^2 + (2\mu+1)\,(\mathrm{e}^3)^2 + \frac{\mu+1}{\mu}\,(\mathrm{f}^1)^2 + \frac{\mu+1}{\mu^2}\,(\mathrm{f}^3)^2$ $+\frac{1+3\mu+2\mu^2}{\mu}\,(\mathrm{f}^2)^2 + \frac{6(2\mu+1)^{\frac{1}{2}}}{\mu+1}\,\mathrm{e}^1\cdot\mathrm{e}^3 + \frac{2(2\mu+1)^{\frac{1}{2}}(3\mu+1)}{\mu(\mu+1)}\,\mathrm{e}^1\cdot\mathrm{f}^2 + \frac{4(2\mu+1)}{\mu(\mu+1)^2}\,\mathrm{e}^1\cdot\mathrm{f}^3$ $+\frac{2(2\mu+1)^{\frac{1}{2}}}{\mu}\,\mathrm{e}^2\cdot\mathrm{f}^1 + (4+4\mu)\,\mathrm{e}^3\cdot\mathrm{f}^2 + \frac{2(2\mu+1)^{\frac{1}{2}}}{\mu}\,\mathrm{e}^3\cdot\mathrm{f}^3 + \frac{2(2\mu+1)^{\frac{1}{2}}}{\mu}\,\mathrm{f}^2\cdot\mathrm{f}^3$
$\mathfrak{su}(2) \oplus \mathfrak{r}'_{3,\mu}$ $(\mu > 0)$	$\omega = \mathrm{e}^2\mathrm{f}^2 - 2\mu\,\mathrm{f}^{23} + \mathrm{e}^3\mathrm{f}^3 + \mathrm{e}^1\mathrm{f}^1$ $\rho = \mathrm{e}^{23}\mathrm{f}^1 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{31} - \mu\,\mathrm{e}^3\mathrm{f}^{31} + \mu\,\mathrm{e}^2\mathrm{f}^{12} + \mathrm{e}^3\mathrm{f}^{12} + (\mu^2-1)\,\mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2\,(\mathrm{f}^1)^2 + (\mu^2+1)\,(\mathrm{f}^2)^2 + (\mu^2+1)\,(\mathrm{f}^3)^2$ $+ 2\,\mathrm{e}^1\cdot\mathrm{f}^1 + 2\mu\,\mathrm{e}^2\cdot\mathrm{f}^3 - 2\mu\,\mathrm{e}^3\cdot\mathrm{f}^2$
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}'_{3,\mu}$ $(\mu > 0)$	$\omega = \mathrm{e}^2\mathrm{f}^2 - 2\mu\,\mathrm{f}^{23} + \mathrm{e}^3\mathrm{f}^3 + \mathrm{e}^1\mathrm{f}^1$ $\rho = \frac{1}{2}\,\mathrm{e}^{23}\mathrm{f}^1 + 2\,\mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{12}\mathrm{f}^3 + 2\,\mathrm{e}^2\mathrm{f}^{31} + \mu\,\mathrm{e}^3\mathrm{f}^{31}$ $+ 2\mu\,\mathrm{e}^2\mathrm{f}^{12} - \mathrm{e}^3\mathrm{f}^{12} - (4\mu^2 + \frac{29}{4})\,\mathrm{f}^{123}$ $g = 2\,(\mathrm{e}^1)^2 + \frac{1}{2}\,(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + \frac{13}{8}\,(\mathrm{f}^1)^2 + (16\mu^2 + \frac{29}{2})\,(\mathrm{f}^2)^2 + (2\mu^2 + \frac{29}{4})\,(\mathrm{f}^3)^2$ $+ 3\,\mathrm{e}^1\cdot\mathrm{f}^1 - 5\,\mathrm{e}^2\cdot\mathrm{f}^2 - 2\mu\,\mathrm{e}^2\cdot\mathrm{f}^3 - 8\mu\,\mathrm{e}^3\cdot\mathrm{f}^2 + 5\,\mathrm{e}^3\cdot\mathrm{f}^3 - 10\mu\,\mathrm{f}^2\cdot\mathrm{f}^3$

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Department Mathematik, Universität Hamburg, Bundesstrasse 55, D-20146 Hamburg, Germany
E-mail address: `schulte-hengesbach@math.uni-hamburg.de`