

**SPECIAL COHOMOGENEITY ONE METRICS WITH  $Q^{1,1,1}$   
OR  $M^{1,1,0}$  AS PRINCIPAL ORBIT**

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ABSTRACT. We classify all cohomogeneity one manifolds with principal orbit  $Q^{1,1,1} = SU(2)^3/U(1)^2$  or  $M^{1,1,0} = (SU(3) \times SU(2))/(SU(2) \times U(1))$  whose holonomy is contained in  $\text{Spin}(7)$ . Various metrics with different kinds of singular orbits can be constructed by our methods. It turns out that the holonomy of our metrics is automatically  $SU(4)$  and that they are asymptotically conical. Moreover, we investigate the smoothness of the metrics at the singular orbit.

1. INTRODUCTION

The subject of this article are eight-dimensional manifolds with special holonomy. These manifolds are not only of interest for purely mathematical reasons, but they are also studied in super string theory (cf. Acharya, Gukov [1], Cvetič et al. [9],[11],[12], Gukov, Sparks [18]). Our aim is to classify all metrics of a special type whose holonomy is contained in  $\text{Spin}(7)$ . Explicit metrics with holonomy  $\text{Spin}(7)$  or  $SU(4)$  are hard to construct. This problem becomes a lot of easier if we assume that the metric is preserved by a cohomogeneity one action. In this situation, our task is equivalent to solving a system of ordinary differential equations. Examples of cohomogeneity one metrics with holonomy  $\text{Spin}(7)$  or  $SU(4)$  can be found in Bazaikin [3], Bryant, Salamon [7], Cvetič et al. [9], [10],[11],[12], and Herzog, Klebanov [19].

If we assume that the group which acts with cohomogeneity one is compact, the number of possible principal orbits is countable [24]. Furthermore, the space of all homogeneous  $G_2$ -structures on a fixed principal orbit is finite-dimensional. This fact makes it possible to obtain partial classification results. In this article, we assume that the principal orbit is of a certain kind. First, we consider coset spaces of type  $SU(2)^3/U(1)^2$  which are denoted by  $Q^{k,l,m}$ . The indices  $k$ ,  $l$ , and  $m$  describe the embedding of  $U(1)^2$  into  $SU(2)^3$ . Second, we investigate quotients of type  $(SU(3) \times SU(2))/(SU(2) \times U(1))$  where the semisimple part of  $SU(2) \times U(1)$  is embedded into the first factor

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of  $SU(3) \times SU(2)$ . These spaces are denoted by  $M^{k,l,0}$ . If the third index is non-zero, we obtain a space which is covered by  $M^{k,l,m}$ . The meaning of the indices  $k$  and  $l$  will be explained in Section 5. At the singular orbit, a space of cohomogeneity one may have a singularity. In this article, we allow orbifold singularities but exclude all other ones. Our results can be summed up as follows:

**Theorem 1.** *Let  $(M, \Omega)$  be a cohomogeneity one  $Spin(7)$ -manifold whose principal orbits are of type  $Q^{k,l,m}$ . In this situation, the following statements are true:*

- (1) *The principal orbits are  $SU(2)^3$ -equivariantly diffeomorphic to  $Q^{1,1,1}$ .*
- (2) *The metric  $g$  which is associated to  $\Omega$  has holonomy  $SU(4)$ .*
- (3) *If  $M$  has a singular orbit, it has to be  $S^2 \times S^2$  or  $S^2 \times S^2 \times S^2$ . Any  $SU(2)^3$ -invariant metric on the singular orbit can be uniquely extended to a complete cohomogeneity one metric with holonomy  $SU(4)$ . For any choice of the singular orbit and its metric,  $(M, g)$  is asymptotically conical. If the singular orbit is  $S^2 \times S^2$ ,  $g$  is a smooth metric. In the other case, the metric cannot be smooth at the singular orbit.*

**Theorem 2.** *Let  $(M, \Omega)$  be a cohomogeneity one  $Spin(7)$ -orbifold whose principal orbit is of type  $M^{k,l,0}$ . In this situation, the following statements are true:*

- (1) *The principal orbit is  $SU(3) \times SU(2)$ -equivariantly diffeomorphic to  $M^{1,1,0}$ .*
- (2) *The metric  $g$  which is associated to  $\Omega$  has holonomy  $SU(4)$ .*
- (3) *If  $M$  has a singular orbit, it has to be  $S^2$ ,  $\mathbb{C}\mathbb{P}^2$ , or  $S^2 \times \mathbb{C}\mathbb{P}^2$ . Any  $SU(3) \times SU(2)$ -invariant metric on the singular orbit can be uniquely extended to a complete cohomogeneity one metric with holonomy  $SU(4)$ . For any choice of the singular orbit and its metric,  $(M, g)$  is asymptotically conical. If the singular orbit is  $S^2$  or  $S^2 \times \mathbb{C}\mathbb{P}^2$ ,  $g$  cannot be a smooth orbifold metric near the singular orbit. In the second case,  $g$  is always smooth.*

The metrics which we construct are also described in Cvetic et al. [9],[12] and Herzog, Klebanov [19]. These works are based on methods by Berard-Bergery [4], Page, Pope [23], and Stenzel [25]. Nevertheless, our proofs that there are no further metrics of the above kind and that the holonomy automatically reduces to  $SU(4)$  are new results. In particular, we prove that we cannot deform the metrics from the literature into metrics with holonomy  $Spin(7)$  without losing the  $SU(2)^3$ - or  $SU(3) \times SU(2)$ -symmetry. Moreover, the smoothness of the metrics at the singular orbit and the global shape of our manifolds are studied in detail.

This article is organized as follows: In Section 2 and 3, we collect some facts on metrics with exceptional holonomy and on cohomogeneity one manifolds.

The metrics with principal orbit  $Q^{k,l,m}$  are studied in Section 4 and those with principal orbit  $M^{k,l,0}$  in Section 5.

## 2. METRICS WITH HOLONOMY $\text{Spin}(7)$

In this section, we will collect some general facts on  $\text{Spin}(7)$ -manifolds. Later on, we have to deal not only with the  $\text{Spin}(7)$ -structure on the manifold but also with the induced  $G_2$ -structure on the principal orbits. Therefore, we first introduce the group  $G_2$ :

**Definition 2.1.** Let  $\mathbb{O}$  be the division algebra of the *octonions*. An  $\mathbb{R}$ -linear non-zero map  $\phi : \mathbb{O} \rightarrow \mathbb{O}$  satisfying  $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$  for all  $x, y \in \mathbb{O}$  is called an *automorphism of  $\mathbb{O}$* . The group of all automorphisms of  $\mathbb{O}$  we call  $G_2$  and its Lie algebra  $\mathfrak{g}_2$ .

**Lemma 2.2.** (1) *Any automorphism of  $\mathbb{O}$  fixes 1 and leaves its orthogonal complement  $\text{Im}(\mathbb{O})$  invariant. This yields an irreducible seven-dimensional representation of  $G_2$ , which we call the standard representation of  $G_2$ .*

(2) *The group which is generated by the left multiplications with unit octonions is isomorphic to  $\text{Spin}(7)$ . Its Lie algebra we denote by  $\mathfrak{spin}(7)$ . The action of this algebra on  $\mathbb{O}$  is equivalent to the spinor representation of  $\mathfrak{so}(7)$ .*

We rewrite the canonical basis  $(1, i, j, k, \epsilon, i\epsilon, j\epsilon, k\epsilon)$  of  $\mathbb{O}$  as  $(e_0, \dots, e_8)$ . Furthermore, we define the one-forms  $dx^i$  by  $dx^i(e_j) := \delta_j^i$  and  $dx^{i_1 \dots i_k}$  as  $dx^{i_1} \wedge \dots \wedge dx^{i_k}$ . The three-form

$$(1) \quad \omega := dx^{123} + dx^{145} - dx^{167} + dx^{246} + dx^{257} + dx^{347} - dx^{356}$$

is invariant under  $G_2$  and the four-form

$$(2) \quad \begin{aligned} \Omega := & dx^{0123} + dx^{0145} - dx^{0167} + dx^{0246} + dx^{0257} + dx^{0347} - dx^{0356} \\ & - dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567} . \end{aligned}$$

is invariant under  $\text{Spin}(7)$ . Up to constant multiples, the above forms are the only elements of  $\wedge^3 \text{Im}(\mathbb{O})^*$  ( $\wedge^4 \mathbb{O}^*$ ) with these properties.  $\Omega$  and  $\omega$  are related by the formula

$$(3) \quad \Omega = *\omega + dx^0 \wedge \omega ,$$

where  $*$  is the Hodge star with respect to the canonical metric on  $\text{Im}(\mathbb{O})$ . We are now able to define the notion of a  $G_2$ - ( $\text{Spin}(7)$ -)structure:

- Definition 2.3.** (1) Let  $M$  be a seven-dimensional manifold and  $\omega$  be a three-form on  $M$ . We assume that for any  $p \in M$  there exists an open neighborhood  $U$  of  $p$  and a local frame  $(X_i)_{1 \leq i \leq 7}$  on  $U$  with the following property:  $\omega$  has with respect to  $(X_i)_{1 \leq i \leq 7}$  the same coefficients as the three-form (1) with respect to  $(e_i)_{1 \leq i \leq 7}$ . In this situation,  $\omega$  is called a  $G_2$ -structure on  $M$ .
- (2) Let  $M$  be an eight-dimensional manifold and  $\Omega$  be a four-form on  $M$ . We assume that for any  $p \in M$  there exists an open neighborhood  $U$  of  $p$  and a local frame  $(X_i)_{0 \leq i \leq 7}$  on  $U$  with the following property:  $\Omega$  has with respect to  $(X_i)_{0 \leq i \leq 7}$  the same coefficients as the four-form (2) with respect to  $(e_i)_{0 \leq i \leq 7}$ . In this situation,  $\Omega$  is called a  $Spin(7)$ -structure on  $M$ .

Alternatively, we could have defined a  $G_2$ - ( $Spin(7)$ -)structure as a certain principal bundle with structure group  $G_2$  ( $Spin(7)$ ). On any manifold with a  $G_2$ - ( $Spin(7)$ -)structure there exist an *associated metric and orientation*, which depend on  $\omega$  ( $\Omega$ ) only. The  $G_2$ - ( $Spin(7)$ -)structures can be divided according to their intrinsic torsion into 16 classes (cf. Fernández, Gray [16] for further details). For our considerations, we only need the following of those classes:

- Definition 2.4.** (1) A  $G_2$ -structure  $\omega$  is called *nearly parallel* if there exists a  $\lambda \in \mathbb{R} \setminus \{0\}$  such that  $d\omega = \lambda * \omega$ . " $*$ " denotes the Hodge star operator with respect to the associated metric and orientation.
- (2) A  $G_2$ -structure  $\omega$  is called *cosymplectic* if  $d * \omega = 0$ .
- (3) A  $Spin(7)$ -structure  $\Omega$  is called *parallel* if  $\nabla^g \Omega = 0$ , where  $g$  is the associated metric and  $\nabla^g$  is the Levi-Civita connection of  $g$ . In this situation, we call  $(M, \Omega)$  a  $Spin(7)$ -manifold.

If  $\Omega$  is a  $Spin(7)$ -structure,  $\nabla^g \Omega = 0$  is equivalent to  $d\Omega = 0$  (cf. Fernández [17]). Since it is more convenient to work with the latter equation in practical situations, we will often use it instead of  $\nabla^g \Omega = 0$ .  $Spin(7)$ -manifolds have the following interesting properties:

**Theorem 2.5.** (Cf. Bonan [5] and Wang [26].)

- (1) *The associated metric is Ricci-flat.*
- (2) *The holonomy of the metric is contained in  $Spin(7)$ .*

*Remark 2.6.* Since  $SU(4)$  can be embedded into  $Spin(7)$ , manifolds whose holonomy is contained in  $SU(4)$  are a special case of  $Spin(7)$ -manifolds. The holonomy of a metric on an eight-dimensional manifold is a subgroup of  $SU(4)$  if and only if it is Ricci-flat and Kähler.

## 3. GEOMETRICAL STRUCTURES OF COHOMOGENEITY ONE

In this section, we motivate why it is promising to assume that the metric is invariant under a cohomogeneity one action and give a short introduction into the issue of Spin(7)-manifolds of cohomogeneity one.

Since there are certain non-linear restrictions on  $\Omega$ ,  $d\Omega = 0$  should be viewed as a non-linear partial differential equation. This makes it extraordinary difficult to construct examples of Spin(7)-manifolds. The first local examples have been constructed by Bryant [6], the first complete ones by Bryant and Salamon [7], and the existence of compact manifolds with holonomy Spin(7) has been proven by Joyce [21]. If  $\Omega$  is preserved by a large group acting on the manifold, the equation  $d\Omega = 0$  simplifies. The optimal case would be where the group action is transitive. Unfortunately, there are no interesting homogeneous examples:

**Theorem 3.1.** *(See Alekseevskii and Kimelfeld [2].) Any homogeneous Ricci-flat metric is necessary flat.*

The next case which is natural to consider is where the group acts by cohomogeneity one:

- Definition 3.2.**
- (1) Let  $(M, g)$  be a connected Riemannian manifold with an isometric action by a Lie group  $G$ . An orbit  $\mathcal{O}$  of this action is called a *principal orbit* if there is an open subset  $U$  of  $M$  with the following properties:  $\mathcal{O} \subseteq U$  and  $U$  is  $G$ -equivariantly diffeomorphic to  $\mathcal{O} \times V$ , where  $V \subseteq \mathbb{R}^n$  is an open set.
  - (2) In the above situation,  $\dim V$  (or equivalently  $\dim M - \dim \mathcal{O}$ ) is called the *cohomogeneity of the  $G$ -action on  $M$* .
  - (3) A Spin(7)-manifold  $(M, \Omega)$  is called *of cohomogeneity one* if there exists a cohomogeneity one action which preserves  $\Omega$  (and thus the associated metric).

Since any two principal orbits are  $G$ -equivariantly diffeomorphic, the cohomogeneity is a well-defined number. Any principal orbit can be identified with a coset space  $G/H$ . Moreover, any non-principal orbit can be identified with a coset space  $G/K$  and after conjugation we can assume that  $H \subseteq K \subseteq G$ . The group  $K$  cannot be chosen arbitrarily:

**Theorem 3.3.** *(See Mostert [22].) Let  $(M, g)$  be a Riemannian manifold with an isometric cohomogeneity one action by a Lie group  $G$ . Furthermore, let all principal orbits be  $G$ -equivariantly diffeomorphic to  $G/H$  with  $H \subseteq G$ . We assume that there exists a non-principal orbit which we identify with  $G/K$  where  $H \subseteq K \subseteq G$ . In this situation,  $K/H$  is diffeomorphic to a sphere.*

*Remark 3.4.* (1) If  $K/H$  is a quotient of a sphere by a finite group,  $M$  is not a manifold, but still an orbifold. Since we are also interested

in Spin(7)-manifolds with "nice" singularities, we will include such spaces into our considerations, too. On the following pages, we will assume that  $M$  is a manifold. Nevertheless, we can easily modify our statements such that they are valid in the orbifold-case, too.

- (2) Since the volume of  $K/H$  shrinks to zero as we approach the singular orbit, we will refer to  $K/H$  as the *collapsing sphere*.

For the rest of this section, let  $(M, \Omega)$  be a Spin(7)-manifold of cohomogeneity one.  $G$ ,  $H$ , and  $K$  shall denote the same groups as above. We will assume from now on that  $G$  is compact. We denote the Lie algebras of  $G$ ,  $H$ , and  $K$  by  $\mathfrak{g}$ ,  $\mathfrak{h}$ , and  $\mathfrak{k}$ . Beside Theorem 3.3 there are further restrictions on the shape of  $M$ :

It is a well-known fact (cf. Mostert [22]) that  $M/G$  is homeomorphic to  $S^1$ ,  $\mathbb{R}$ ,  $[0, 1]$ , or  $[0, \infty)$ . In the first case,  $M$  would have an infinite fundamental group and the holonomy thus would be a subgroup of  $G_2$ . We will therefore exclude this case. In the second case,  $M$  would contain a line. It would follow from the Cheeger-Gromoll splitting theorem that  $M$  is a Riemannian product of  $\mathbb{R}$  and another space. This case we will not consider, too. In the third case,  $M$  would be compact. Since its Ricci curvature would be non-positive, all Killing vector fields would be parallel and commute with each other. The principal orbit thus would be a flat torus. Since in that situation,  $(M, g)$  would be flat, too, we can assume that  $M/G$  is  $[0, \infty)$ . The point 0 corresponds to a non-principal orbit and all other orbits are principal ones.

If  $K/H = S^0 = \mathbb{Z}_2$ , the non-principal orbit is called *exceptional*. In that case,  $M$  would be twofold covered by a space  $\widetilde{M}$  with  $\widetilde{M}/G = \mathbb{R}$ . We will therefore restrict ourselves to the case where there is exactly one non-principal orbit with  $\dim K - \dim H > 1$ . An orbit which satisfies this requirement is called a *singular orbit*.

The pull-back of the inclusion  $\iota$  of a principal orbit into  $M$  yields a four-form  $\iota^*(\Omega)$  on any principal orbit. Its Hodge-dual with respect to the restricted metric and a fixed orientation is a three-form which can be proven to be a  $G_2$ -structure  $\omega$ . From the equation  $d\Omega = 0$  it follows that  $d*\omega = 0$ . Conversely, any  $G$ -invariant cosymplectic  $G_2$ -structure on  $G/H$  can be extended to a parallel Spin(7)-structure on  $G/H \times (-\epsilon, \epsilon)$ :

**Theorem 3.5.** *Let  $G/H$  be a seven-dimensional homogeneous space which carries a  $G$ -invariant cosymplectic  $G_2$ -structure  $\tilde{\omega}$ . Then there exists an  $\epsilon > 0$  and a one-parameter family  $(\omega_t)_{t \in (-\epsilon, \epsilon)}$  of  $G$ -invariant three-forms on  $G/H$  such that the initial value problem*

$$(4) \quad \frac{\partial}{\partial t} * \omega_t = d_{G/H} \omega_t$$

$$(5) \quad \omega_0 = \tilde{\omega}$$

has a unique solution on  $G/H \times (-\epsilon, \epsilon)$  with the following properties:

- (1)  $\omega_t$  is a  $G_2$ -structure on  $G/H \times \{t\}$ .
- (2)  $d_{G/H} * \omega_t = 0$ .
- (3)  $*\omega_t$  is in the same cohomology class in  $H^4(M, \mathbb{R})$  as  $*\tilde{\omega}$ .

In the above formula,  $\frac{\partial}{\partial t}$  denotes the Lie derivative in  $t$ -direction. The index  $G/H$  of  $d$  emphasizes that we consider the exterior derivative on  $G/H \times \{t\}$  instead of  $G/H \times (-\epsilon, \epsilon)$ . In the above situation, the four-form  $\Omega := *\omega + dt \wedge \omega$  is a  $G$ -invariant parallel  $Spin(7)$ -structure on  $G/H \times (-\epsilon, \epsilon)$ .

Conversely, let  $\Omega$  be a parallel  $Spin(7)$ -structure preserved by a cohomogeneity one action of a compact Lie group  $G$ . The isotropy group of the  $G$ -action on the principal orbit we denote by  $H$ . We identify the union of all principal orbits  $G$ -equivariantly with  $G/H \times I$ , where the interval  $I$  is parameterized by arclength. In this situation, the  $G_2$ -structures on the principal orbits are cosymplectic and satisfy equation (4).

- Remark 3.6.*
- (1) The above theorem has been proven by Hitchin [20] in a more general context: In [20],  $G/H$  is replaced by a seven-dimensional compact manifold which carries a cosymplectic  $G_2$ -structure but is not necessarily homogeneous.
  - (2) If  $\omega$  is nearly parallel, the maximal solution of (4) describes a cone over  $G/H$ . More precisely, the space of cohomogeneity one is isometric to a cone over  $G/H$  which carries the metric associated to  $\omega$ .
  - (3) Since  $(M, \Omega)$  is of cohomogeneity one, the equation  $\frac{\partial}{\partial t} * \omega_t = d\omega_t$  is equivalent to a system of ordinary differential equations and thus easier to handle than the equation  $d\Omega = 0$  in the general situation. Since we assume that  $M$  has at least one singular orbit, we try to fix the initial conditions on a singular orbit  $G/K$ . Since  $\dim G/K < 7$ , the differential equations will degenerate at the singular orbit.

Before we can make the initial value problem (4), (5) explicit, we have to fix a principal and a singular orbit. The possible principal orbits are exactly those homogeneous spaces  $G/H$  which admit at least one  $G$ -invariant cosymplectic  $G_2$ -structure  $\omega$ . From the  $G$ -invariance of  $\omega$  it follows that the isotropy action of  $H$  on the tangent space has to be equivalent to the action of a subgroup of  $G_2$  on  $\text{Im}(\mathbb{O})$ . Conversely, there always exists a  $G$ -invariant  $G_2$ -structure on  $G/H$  if  $H$  has the above property:  $G$  can be considered as an  $H$ -bundle over  $G/H$ . Its extension to a principal bundle with structure group  $G_2$  is the  $G_2$ -structure we search for. We sum up our observations to the following lemma:

**Lemma 3.7.** *Let  $G/H$  be a homogeneous space such that  $G$  acts effectively on  $G/H$  and let  $p \in G/H$  be arbitrary. We identify  $H$  with its isotropy representation on  $T_p G/H$  and  $G_2$  with its standard representation.  $G/H$*

admits a  $G$ -invariant  $G_2$ -structure if and only if there exists a vector space isomorphism  $\varphi : T_p G/H \rightarrow \text{Im}(\mathbb{O})$  such that  $\varphi H \varphi^{-1} \subseteq G_2$ .

For our calculations, we not only need the existence of one particular  $G$ -invariant cosymplectic  $G_2$ -structure. More precisely, we need a set of  $G_2$ -structures such that the flow equation  $\frac{\partial}{\partial t} * \omega_t = d\omega_t$  does not leave this set. The set of all  $G$ -invariant cosymplectic  $G_2$ -structures clearly satisfies this condition. On the principal orbits which we consider in this article we are able to describe that set explicitly. We will see that it is not difficult to classify all  $G$ -invariant metrics on  $G/H$ . A Riemannian metric on a seven-dimensional manifold together with an orientation is the same as an  $SO(7)$ -structure. We thus can replace the problem of finding all  $G$ -invariant  $G_2$ -structures by the following simpler one:

**Problem 3.8.** Let  $\mathcal{G}$  be an arbitrary  $G$ -invariant  $SO(7)$ -structure on  $G/H$ . Find all  $G$ -invariant  $G_2$ -structures on  $G/H$  whose extension to a principal bundle with structure group  $SO(7)$  is  $\mathcal{G}$ .

This problem can be solved by purely algebraic methods: Since  $G$  acts transitively on  $G/H$ , a  $G$ -invariant  $G_2$ -structure  $\omega$  is determined by a basis of a tangent space  $T_p G/H$  which can be identified with the basis  $(i, j, k, \epsilon, i\epsilon, j\epsilon, k\epsilon)$  of  $\text{Im}(\mathbb{O})$ . Any other  $G$ -invariant  $G_2$ -structure can be identified with another basis or equivalently with a linear map  $\phi : T_p G/H \rightarrow T_p G/H$  which maps the first basis into the second one. The condition that  $\omega$  is  $G$ -invariant translates into  $H\phi = \phi H$ , where  $H$  is as before identified with its isotropy representation on  $T_p G/H$ . If we identify  $T_p G/H$  with  $\text{Im}(\mathbb{O})$ , this condition translates into  $\phi \in \text{Norm}_{GL(7)} H := \{\phi \in GL(7) \mid \phi H \phi^{-1} = H\}$ . The group of all  $\phi$  which leave the extension of the  $G_2$ -structure to an  $SO(7)$ -structure invariant is  $\text{Norm}_{SO(7)} H$ . Since the  $G_2$ -structure is stabilized by  $\text{Norm}_{G_2} H$ , the set we search for in Problem 3.8, can be described as  $\text{Norm}_{SO(7)} H / \text{Norm}_{G_2} H$ :

**Lemma 3.9.** *Let  $G/H$  be a seven-dimensional homogeneous space. We assume that  $G$  acts effectively and that  $G/H$  admits a  $G$ -invariant  $G_2$ -structure. The space of all  $G$ -invariant  $G_2$ -structures on  $G/H$  which have a fixed associated metric and orientation is  $\text{Norm}_{SO(7)} H$ -equivariantly diffeomorphic to:*

$$(6) \quad \text{Norm}_{SO(7)} H / \text{Norm}_{G_2} H .$$

*In particular, this space does not depend on the choice of the  $G$ -invariant metric and the orientation.*

By solving  $d*\omega = 0$  we are able to parameterize the space of all  $G$ -invariant cosymplectic  $G_2$ -structures on  $G/H$  and to transform the equation  $\frac{\partial}{\partial t} * \omega_t = d\omega_t$  into an explicit system of ordinary differential equations. We assume



that we have obtained a solution of those equations which has a singular orbit. In that situation, we are not done yet, since we have to prove that the metric  $g$  and the four-form  $\Omega$  can be smoothly extended from the union of all principal orbits to the singular orbit. A set of necessary and sufficient smoothness conditions for arbitrary tensor fields of cohomogeneity one can be found in Eschenburg, Wang [14]. Before we can state the theorem of Eschenburg and Wang, we have to introduce some notation.

The principal orbit  $G/H$  is a sphere bundle over the singular orbit  $G/K$ . There is an appropriate parameterize of  $G/K$ , a so called *tubular parameterize*, which is a disc bundle over  $G/K$ . The tangent space of any point of  $G/K$  can therefore be splitted into a horizontal and a vertical part. As a  $K$ -module, the horizontal part is the same as the complement  $\mathfrak{p}$  of  $\mathfrak{k} \subseteq \mathfrak{g}$  with respect to an  $\text{Ad}_K$ -invariant metric. The orbits of the  $K$ -action on the vertical part  $\mathfrak{p}^\perp$  are, except  $\{0\}$ , spheres of type  $K/H$ . Let  $\rho$  be a  $G$ -invariant tensor field with values in a vector bundle  $\mathcal{B}$  which is defined everywhere on the tubular parameterize except on  $G/K$ . Its extension to the singular orbit is determined by its values at an arbitrary point of  $G/K$  if it exists.

On any cohomogeneity one manifold there exists a geodesic  $\gamma$  which intersects all orbits perpendicularly. We assume that  $\gamma(0) \in G/K$  and that  $\gamma$  is parameterized by arclength. The action of  $K$  on such a geodesic generates a fiber of the disc bundle. Therefore, it suffices to consider  $\rho$  along  $\gamma$  only. Any metric whose holonomy is contained in  $\text{Spin}(7)$  is Einstein. Since any Einstein metric is analytic [13], we assume that  $\rho$  is a power series with respect to the parameter of  $\gamma$ . The  $m^{\text{th}}$  derivative of  $\rho$  in the vertical direction can be considered as a map, which assigns to a tuple  $(v_1, \dots, v_m) \in \mathfrak{p}^\perp$  an element

$$(7) \quad \frac{\partial^m}{\partial v_1 \dots \partial v_m} \Big|_{\gamma(0)} \rho$$

of  $\mathcal{B}_p$ . This map can be extended to a map  $\psi_m : S^m(\mathfrak{p}^\perp) \rightarrow \mathcal{B}_p$ , where  $S^m$  denotes the symmetric power. Since  $\rho$  is analytic in the above sense, it is determined by the  $\psi_m$ . If  $\rho$  has a smooth extension to the singular orbit, the  $\psi_m$  have to be  $K$ -equivariant. This necessary condition is in fact sufficient, too:

**Theorem 3.10.** (Cf. Eschenburg, Wang [14].) *Let  $(M, g)$  be Riemannian manifold with an isometric action of cohomogeneity one by a Lie group  $G$ . We assume that this action has a singular orbit. The isotropy group of the  $G$ -action at the singular orbit will be denoted by  $K$ . Let  $\mathcal{B} \subseteq \otimes^{s_1} TM \otimes \otimes^{s_2} T^*M$  be a vector bundle over  $M$  whose fibers at the singular orbit are  $K$ -equivariantly isomorphic to a  $K$ -module  $B$ . Let  $r : (0, \varepsilon) \rightarrow B$ , where  $\varepsilon > 0$ , be a real analytic map with Taylor expansion  $\sum_{m=1}^{\infty} r_m t^m$ . By the construction which we have described above, we can identify  $r$  with a tensor*

field  $\rho$ . This tensor field is defined on a tubular neighborhood of the singular orbit, but not on the singular orbit itself.  $\rho$  is well-defined and has a smooth extension to the singular orbit if and only if

$$(8) \quad r_m \in \iota_m(W_m) \quad \forall m \in \mathbb{N}_0.$$

In the above formula,  $W_m$  is the space of all  $K$ -equivariant maps:

$$(9) \quad W_m := \{P : S^m(\mathfrak{p}^\perp) \rightarrow B \mid P \text{ is linear and } K\text{-equivariant}\}$$

and  $\iota_m$  is the evaluation map

$$(10) \quad \begin{aligned} \iota_m : W_m &\rightarrow B \\ \iota_m(P) &:= P(\gamma'(0)). \end{aligned}$$

The metrics  $g$  which we consider in this article have no "mixed coefficients", i.e.

$$(11) \quad g \in S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp).$$

In order to check the smoothness of  $g$ , it suffices to consider the subspaces

$$(12) \quad \begin{aligned} W_m^h &:= \{P : S^m(\mathfrak{p}^\perp) \rightarrow S^2(\mathfrak{p}) \mid P \text{ is linear and } K\text{-equivariant}\} \quad \text{and} \\ W_m^v &:= \{P : S^m(\mathfrak{p}^\perp) \rightarrow S^2(\mathfrak{p}^\perp) \mid P \text{ is linear and } K\text{-equivariant}\} \end{aligned}$$

of  $W_m$ . We describe certain elements of  $W_m^h$  and  $W_m^v$ . This description yields a sufficient smoothness condition for the metric which can be easily checked. Let  $q^h \in S^2(\mathfrak{p})$  be  $K$ -invariant. We denote the derivation in the direction of  $\gamma'(0)$  by  $\frac{\partial}{\partial t}$ . Let  $n := \dim K - \dim H$  and let  $(e_1, \dots, e_n)$  be an orthonormal basis of  $\mathfrak{p}^\perp$  with respect to a  $K$ -invariant metric  $q^v$  on  $\mathfrak{p}^\perp$ . Since  $K/H$  has to be a distance sphere,  $q^v$  is unique up to a constant factor, which is fixed by  $\|\frac{\partial}{\partial t}\| = 1$ .  $\frac{\partial}{\partial t}$  extends by the action of  $K$  to a vector field in the radial direction and we have

$$(13) \quad \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} = e_1 \otimes e_1 + \dots + e_n \otimes e_n.$$

We define

$$(14) \quad \begin{aligned} \phi_{2m}^h &: S^{2m}(\mathfrak{p}^\perp) \rightarrow S^2(\mathfrak{p}) \\ \phi_{2m}^h((e_1 \otimes e_1 + \dots + e_n \otimes e_n)^{\otimes m}) &:= q^h. \end{aligned}$$

On the orthogonal complement of  $(e_1 \otimes e_1 + \dots + e_n \otimes e_n)^{\otimes m}$ ,  $\phi_{2m}^h$  shall vanish.  $\phi_{2m}^h$  is obviously  $K$ -equivariant. Theorem 3.10 tells us that the restriction of the metric to  $S^2(\mathfrak{p})$  is smooth if all odd derivatives vanish and the even derivatives are described by a map of type  $\phi_{2m}^h$ . We are now going to interpret this fact geometrically. On the union of all principal orbits we have

$$(15) \quad g = g_t + dt^2,$$

where  $g_t$  is a  $t$ -dependent  $G$ -invariant metric on  $G/H$ . The tangent space of  $G/H$  can be identified with the complement  $\mathfrak{m}$  of  $\mathfrak{h}$  in  $\mathfrak{g}$ . This identification allows us to consider  $\mathfrak{p}$  as a subset of  $T_p G/H$ . Our statement on  $\phi_{2m}^h$  simply means that the horizontal part of the metric is smooth if it is invariant under the action of  $K$  on  $S^2(\mathfrak{p})$  and if  $t \mapsto g_t(v, v)$  is an even analytic function for all  $v \in \mathfrak{p}$ .

For the vertical part of the metric, we define analogous maps  $\phi_{2m}^v : S^{2m}(\mathfrak{p}^\perp) \rightarrow S^2(\mathfrak{p}^\perp)$ . If  $m = 0$ ,  $\phi_{2m}^v$  has to assign  $q^v$  to  $1 \in \mathbb{R}$ . We translate the uniqueness of  $q^v$  into a geometrical condition on  $g$ : Since the vertical part of  $g$  has to coincide with  $q^v$  up to  $0^{th}$  order, the metric on the collapsing sphere  $K/H$  has to approach the metric of a round sphere with radius  $t$ . The length of any great circle on the collapsing sphere has to be  $2\pi t + O(t^2)$  for small  $t$ . We denote this length by  $\ell(t)$  and have finally found the smoothness condition  $\ell'(0) = 2\pi$ .

Next, we construct the  $\phi_{2m}^v$  for  $m \geq 1$ . Without loss of generality, let  $e_1 = \gamma'(0)$ .  $\text{span}(e_2, \dots, e_n)$  is an  $H$ -module and can be identified with the tangent space of the collapsing sphere at  $\gamma(t)$ . There is a one-to-one correspondence between the  $H$ -invariant symmetric bilinear forms on  $\text{span}(e_2, \dots, e_n)$  and the  $K$ -invariant sections of  $S^2(T^*K/H)$ . The second derivative of the metric in the direction  $e_1$  has to be such a bilinear form, which we denote by  $q_1^v$ . By the action of  $K$  we can transform  $e_1$  into any other direction in  $\mathfrak{p}^\perp$ . We therefore define the  $K$ -equivariant map  $\phi_2^v$  by

$$(16) \quad \phi_2^v((L_{k*}e_1) \otimes (L_{k*}e_1)) := L_k^* q_1^v \quad \forall k \in K.$$

In the above formula,  $L_{k*}$  denotes the push-forward of the left-multiplication by  $k$  and  $L_k^*$  the pull-back. Analogously, to the horizontal case, we define

$$(17) \quad \phi_{2m}^v \left( (e_1 \otimes e_1 + \dots + e_n \otimes e_n)^{\otimes m-1} \vee (L_{k*}e_1) \otimes (L_{k*}e_1) \right) := L_k^* q_m^v \quad \forall k \in K,$$

where  $\vee$  denotes the symmetrized tensor product and the  $q_m^v$  are arbitrary  $H$ -invariant symmetric bilinear forms on  $\text{span}(e_2, \dots, e_n)$ . The fact that the maps  $\phi_{2m}^v$  are for any choice of  $q_m^v$   $K$ -equivariant makes a statement

on the even derivatives of the vertical part of the metric. If we write our metric as  $g_t + dt^2$ , the restriction of  $g_t$  to the vertical directions describes the metric on a shrinking sphere. In other words, we write the metric in "polar coordinates" rather than in "Euclidean" ones. A  $2m^{\text{th}}$  derivative of the metric on  $\mathfrak{p}^\perp$  at the origin therefore corresponds to a  $(2m + 1)^{\text{st}}$  derivative of the vertical part of  $g_t$ . Theorem 3.10 states that the vertical part of the metric is smooth if all even derivatives of  $g_t$  vanish and all odd ones are described by maps of type  $\phi_{2m}^v$ . More explicitly, the vertical part is smooth if

- (1) the restriction of  $g_t$  to  $\text{span}(e_2, \dots, e_n)$  is  $H$ -invariant for all  $t$ ,
- (2) the values of  $\frac{\partial}{\partial t}|_{t=0} g_t(e_i, e_j)$  for  $i, j \in \{2, \dots, n\}$  make  $\ell'(0) = 2\pi$  for any great circle on  $K/H$ ,
- (3) and  $\sqrt{g_t(v, v)}$  is for all  $v \in \mathfrak{p}$  an even analytic function.

The smoothness conditions which we have proven on the previous pages will be sufficient for the purpose of this article. At the end of this section, we study the question if the four-form  $\Omega$  has a smooth extension to the singular orbit. We assume that we have already proven that the metric satisfies the smoothness conditions. First, we consider the case where the holonomy of the metric on the union of all principal orbits  $M^0$  is  $\text{Spin}(7)$ . Since the metric is smooth, the holonomy of  $(M, g)$  equals  $\text{Spin}(7)$ , too. Therefore, there exists a unique smooth  $\text{Spin}(7)$ -structure  $\tilde{\Omega}$  on  $M$ , whose associated metric is  $g$ .  $\Omega$  and  $\tilde{\Omega}$  coincide on  $M^0$ . This observation proves that  $\tilde{\Omega}$  is a smooth extension of  $\Omega$  to the singular orbit.

For our considerations, the case of holonomy  $SU(4)$  is more important. Before we can prove the smoothness of  $\Omega$  in this case, we need the following lemma:

- Lemma 3.11.** (1) *Let  $M$  be an eight-dimensional manifold which carries a parallel  $SU(4)$ -structure  $\mathfrak{G}$ . We denote the space of all parallel  $\text{Spin}(7)$ -structures on  $M$  which are an extension of  $\mathfrak{G}$  and have the same extension to an  $SO(8)$ -structure as  $\mathfrak{G}$  by  $\mathcal{S}$ . Any connected component of  $\mathcal{S}$  is diffeomorphic to a circle.*
- (2) *Let  $M$  be an eight-dimensional manifold which carries a one-parameter family  $\mathcal{S}$  of parallel  $\text{Spin}(7)$ -structures. Moreover, let the extension of all  $\text{Spin}(7)$ -structures to an  $SO(8)$ -structure be the same and let  $\mathcal{S}$  be diffeomorphic to a circle. Then, there also exists a parallel  $SU(4)$ -structure on  $M$ .*

*Proof.* Since all  $G$ -structures in the lemma are parallel, it suffices to consider the situation at a single point. Moreover, we can identify the tangent space of  $M$  with  $\mathbb{R}^8$  and the groups  $SU(4)$ ,  $\text{Spin}(7)$ , and  $SO(8)$  with their real eight-dimensional irreducible representations. We search for matrices  $A \in GL(8, \mathbb{R})$  such that conjugation with  $A$  leaves  $SU(4)$  and  $SO(8)$  invariant but changes  $\text{Spin}(7)$ . Since conjugation by a multiple of the identity matrix

leaves any group invariant, we can restrict ourselves to  $A \in SL(8, \mathbb{R})$ . By algebraic arguments, we see that set of all  $A$  which leave  $SU(4)$  and  $SO(8)$  invariant is a group  $G$  with identity component  $U(4)$ . The subgroup of  $G$  which also leaves  $\text{Spin}(7)$  invariant has  $SU(4)$  as identity component. The connected components of  $\mathcal{S}$  thus are diffeomorphic to  $U(4)/SU(4) \cong S^1$ .

In the situation of the second part of the lemma, the holonomy has to be a proper subgroup of  $\text{Spin}(7)$ . All we have to prove is that the holonomy is contained in  $SU(4)$ . The only case where this is not true is where the holonomy equals  $G_2$ . If the holonomy was  $G_2$ , there would be a parallel  $G_2$ -structure  $\omega$  and a parallel one-form  $\alpha$  on  $M$ . Any parallel  $\text{Spin}(7)$ -structure in  $\mathcal{S}$  would be given by  $*\omega + \lambda\alpha \wedge \omega$  for a  $\lambda \in \mathbb{R} \setminus \{0\}$ . Since  $\mathcal{S}$  has no subset which is diffeomorphic to a circle, we have obtained a contradiction.  $\square$

*Remark 3.12.* If we drop the word "parallel" and replace the circle  $\mathcal{S}$  by the sections of a certain circle bundle over  $M$ , the statement of Lemma 3.11 remains true.

We now assume that the holonomy of  $M^0$  is  $SU(4)$  and that we have checked the smoothness conditions for  $g$ . By the same arguments as above, we can prove the existence of a unique smooth  $SU(4)$ -structure on all of  $M$ . It follows from Lemma 3.11 that there exists a certain family  $\mathcal{S}$  of  $\text{Spin}(7)$ -structures on  $M$ . On  $M^0$ ,  $\Omega$  has to coincide with one of them. Since all elements of  $\mathcal{S}$  are smooth at the singular orbit,  $\Omega$  has a smooth extension to the singular orbit. With help of the facts which we have collected in this section we are now able to construct explicit examples of  $\text{Spin}(7)$ -manifolds.

#### 4. METRICS WITH PRINCIPAL ORBIT $Q^{k,l,m}$

In this section, we assume that the parallel  $\text{Spin}(7)$ -structure is preserved by a cohomogeneity one action of  $SU(2)^3$ . For our calculations, we will represent the elements of  $SU(2)^3$  by triples of complex  $2 \times 2$ -matrices. Since all principal orbits are seven-dimensional, the isotropy group at a point of a principal orbit is isomorphic to  $U(1)^2$ . There are infinitely many non-conjugate embeddings of  $U(1)^2$  into  $SU(2)^3$ . We will describe them in detail and check if the coset space  $SU(2)^3/U(1)^2$  admits an  $SU(2)^3$ -invariant  $G_2$ -structure. In order to do this, we first describe the one-dimensional subalgebras of  $3\mathfrak{su}(2)$ . Up to conjugation, any such subalgebra is embedded by a map  $i_{k,l,m} : \mathfrak{u}(1) \rightarrow 3\mathfrak{su}(2)$  with  $k, l, m \in \mathbb{Z}$  and

$$(18) \quad i_{k,l,m}(ix) := \left( \begin{pmatrix} ikx & 0 \\ 0 & -ikx \end{pmatrix}, \begin{pmatrix} ilx & 0 \\ 0 & -ilx \end{pmatrix}, \begin{pmatrix} imx & 0 \\ 0 & -imx \end{pmatrix} \right).$$

Without loss of generality, we can assume that  $(k, l, m)$  are coprime. Furthermore, we can even restrict ourselves to non-negative values of  $k, l$ , and  $m$ : Let  $\phi_P : SU(2)^3 \rightarrow SU(2)^3$  be defined by  $\phi_P(Q) := PQP^{-1}$ , where

$$(19) \quad P := \left( \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right) \in SU(2)^3.$$

$\phi_P$  maps  $i_{k,l,m}(\mathfrak{u}(1))$  into  $i_{-k,l,m}(\mathfrak{u}(1))$ . By another choice of  $P$ , we can map  $i_{k,l,m}(\mathfrak{u}(1))$  into  $i_{k,-l,m}(\mathfrak{u}(1))$  or  $i_{k,l,-m}(\mathfrak{u}(1))$ . Therefore, our restriction to the case where  $k, l, m \geq 0$  is justified. It is also possible to choose  $P$  as a permutation of the three components of  $(\mathbb{C}^2)^3$ . Since the group which is generated by the corresponding  $\phi_P$  acts on  $(k, l, m)$  by permutations, too, we can finally assume that  $k \geq l \geq m \geq 0$ .

We continue describing the embeddings of  $U(1)^2$  into  $SU(2)^3$ . The one-dimensional subalgebras  $i_{k,l,m}(\mathfrak{u}(1))$  together span a Cartan subalgebra of  $3\mathfrak{su}(2)$ , which we denote throughout this section by  $3\mathfrak{u}(1)$ . The equation  $q(X, Y) := -\text{tr}(XY)$ , where  $X, Y \in 3\mathfrak{su}(2)$ , defines a biinvariant metric  $q$  on  $SU(2)^3$ . The  $q$ -orthogonal complement of  $i_{k,l,m}(\mathfrak{u}(1)) \subseteq 3\mathfrak{u}(1)$  we denote by  $2\mathfrak{u}(1)_{k,l,m}$  and the Lie subgroup of  $SU(2)^3$  whose Lie algebra is  $2\mathfrak{u}(1)_{k,l,m}$  by  $U(1)_{k,l,m}^2$ . The quotient  $SU(2)^3/U(1)_{k,l,m}^2$  is called  $Q^{k,l,m}$ . Since  $U(1)^2$  can be mapped by a conjugation inside any maximal torus of  $SU(2)^3$ , any quotient  $SU(2)^3/U(1)^2$  is  $SU(2)^3$ -equivariantly diffeomorphic to a  $Q^{k,l,m}$ .

Our next step is to choose a basis  $(e_1, \dots, e_9)$  of  $3\mathfrak{su}(2)$ . In order to do this, we first define

$$(20) \quad \sigma_1 := \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \sigma_2 := \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_3 := \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

With this notation, we further define

$$(21) \quad \begin{array}{ll} e_1 := (\sigma_1, 0, 0) & e_2 := (\sigma_2, 0, 0) \\ e_3 := (0, \sigma_1, 0) & e_4 := (0, \sigma_2, 0) \\ e_5 := (0, 0, \sigma_1) & e_6 := (0, 0, \sigma_2) \\ e_7 := (k\sigma_3, l\sigma_3, m\sigma_3) & e_8 := (l\sigma_3, -k\sigma_3, 0) \\ e_9 := (mk\sigma_3, ml\sigma_3, -(k^2 + l^2)\sigma_3) & \end{array}$$

The tangent space of  $Q^{k,l,m}$  can be identified with the  $q$ -orthogonal complement  $\mathfrak{m}$  of  $2\mathfrak{u}(1)_{k,l,m}$  in  $3\mathfrak{su}(2)$ .  $\mathfrak{m}$  is spanned by  $(e_1, \dots, e_7)$  and  $2\mathfrak{u}(1)_{k,l,m}$  is spanned by  $(e_8, e_9)$ .  $e_8$  and  $e_9$  act on the tangent space by the isotropy representation of  $2\mathfrak{u}(1)_{k,l,m}$ . This action can be described by the commutator of matrices from  $2\mathfrak{u}(1)_{k,l,m}$  and  $\mathfrak{m}$ . We recall that we want  $Q^{k,l,m}$  to admit an  $SU(2)^3$ -invariant  $G_2$ -structure. According to Lemma 3.7, this is the case if and only if the isotropy representation of  $U(1)_{k,l,m}^2$  is equivalent to the action of a Cartan subalgebra of  $G_2$  on  $\text{Im}(\mathbb{O})$ . Since  $U(1)_{k,l,m}^2$  is connected, it suffices to consider the action of its Lie algebra. The isotropy action of  $2\mathfrak{u}(1)_{k,l,m}$  on  $\mathfrak{m}$  yields the following subalgebra of  $\mathfrak{gl}(\mathfrak{m})$ :

$$(22) \quad \left\{ \left( \begin{array}{c|c|c|c|c} \boxed{0} & x & & & \\ \hline -x & 0 & & & \\ \hline & & \boxed{0} & y & \\ \hline & & -y & 0 & \\ \hline & & & & \boxed{0} & z \\ \hline & & & & -z & 0 \\ \hline & & & & & & \boxed{0} \end{array} \right) \left| \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right. \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \quad kx + ly + mz = 0,$$

where the matrix representation is with respect to the basis  $(e_1, \dots, e_7)$ . We equip  $\mathbb{R}^7$  with the action of the Cartan subalgebra

$$(23) \quad \left\{ \left( \begin{array}{c|c|c|c|c} \boxed{0} & & & & \\ \hline & \boxed{0} & x & & \\ \hline & -x & 0 & & \\ \hline & & & \boxed{0} & y \\ \hline & & & -y & 0 \\ \hline & & & & & \boxed{0} & z \\ \hline & & & & & -z & 0 \end{array} \right) \left| \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right. \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \quad x + y - z = 0$$

of  $\mathfrak{g}_2$ . By comparing the weights of both actions, we see that  $Q^{k,l,m}$  admits an  $SU(2)^3$ -invariant  $G_2$ -structure if and only if  $\{k, l, m\} \in \{-1, 1\}$ . Without loss of generality, we assume that  $k = l = m = 1$ .

Our next task is to describe the set of all  $SU(2)^3$ -invariant cosymplectic  $G_2$ -structures on  $Q^{1,1,1}$ . We split this problem into three subproblems: First, we classify all  $SU(2)^3$ -invariant metrics on  $Q^{1,1,1}$ , second we classify all invariant  $G_2$ -structures whose associated metric and orientation are fixed ones, and third, we prove which of them are cosymplectic. For any  $SU(2)^3$ -invariant metric  $g$  on  $Q^{1,1,1}$  there exists a  $U(1)_{1,1,1}^2$ -equivariant,  $q$ -symmetric, positive definite endomorphism  $\varphi$  of  $\mathfrak{m}$ , which is defined by  $q(\varphi(X), Y) = g(X, Y)$ . Conversely, any such  $\varphi$  yields an  $SU(2)^3$ -invariant metric on  $Q^{1,1,1}$ .  $\mathfrak{m}$  splits into the following irreducible  $2\mathfrak{u}(1)_{1,1,1}$ -submodules:

$$(24) \quad \begin{array}{ll} V_1 & := \text{span}(e_1, e_2) \\ V_2 & := \text{span}(e_3, e_4) \\ V_3 & := \text{span}(e_5, e_6) \\ V_4 & := \text{span}(e_7) \end{array}$$

In order to describe the invariant metrics, we have to check if any pair of the above  $2\mathfrak{u}(1)_{1,1,1}$ -modules is equivalent. It is easy to see that  $V_1$ ,  $V_2$ , and  $V_3$  are pairwise inequivalent, since on any pair of those spaces either the one-dimensional Lie algebra generated by  $e_8$  or the Lie algebra generated by  $e_9$  acts with different weights.  $V_4$  cannot be equivalent to one of the other modules, since it is of lower dimension. We conclude with help of

Schur's lemma that the  $SU(2)^3$ -invariant metrics  $g$  on  $Q^{1,1,1}$  are precisely those which satisfy

$$(25) \quad g = a^2(e^1 \otimes e^1 + e^2 \otimes e^2) + b^2(e^3 \otimes e^3 + e^4 \otimes e^4) \\ + c^2(e^5 \otimes e^5 + e^6 \otimes e^6) + f^2 e^7 \otimes e^7$$

with  $a, b, c, f \in \mathbb{R} \setminus \{0\}$ . The most generic  $G_2$ -structure on  $Q^{1,1,1}$  can be described conveniently by a three-form whose coefficients are odd powers of  $a, b, c$ , and  $f$ . Therefore, we allow those parameters to take negative values although this does not change the metric. If  $g$  is a cohomogeneity one metric with principal orbit  $Q^{1,1,1}$ ,  $a, b, c$ , and  $f$  turn into functions which are defined on the interval  $M/SU(3)$ .

Our next step is to describe the set of all homogeneous  $G_2$ -structures on  $Q^{1,1,1}$  whose extension to an  $SO(7)$ -structure is a fixed one. On the following pages,  $2\mathfrak{u}(1)$  will denote the Cartan subalgebra (23) of  $\mathfrak{g}_2$ . The maximal torus of  $G_2$  whose Lie algebra is  $2\mathfrak{u}(1)$  we will denote by  $U(1)^2$ . As we have proven in Lemma 3.9, the set we search for is

$$(26) \quad \text{Norm}_{SO(7)}U(1)^2 / \text{Norm}_{G_2}U(1)^2.$$

First, we describe the numerator of the above quotient. The Lie algebra of  $\text{Norm}_{SO(7)}U(1)^2$  is

$$(27) \quad \text{Norm}_{\mathfrak{so}(7)}2\mathfrak{u}(1) := \{x \in \mathfrak{so}(7) \mid \text{ad}_x(2\mathfrak{u}(1)) \subseteq 2\mathfrak{u}(1)\}.$$

From now on  $3\mathfrak{u}(1)$  denotes the Cartan subalgebra

$$(28) \quad \left\{ \left( \begin{array}{cccc} \boxed{0} & & & \\ & \boxed{0} & a & \\ & -a & \boxed{0} & \\ & & & \boxed{0} & b \\ & & & -b & \boxed{0} \\ & & & & & \boxed{0} & c \\ & & & & & -c & \boxed{0} \end{array} \right) \mid a, b, c \in \mathbb{R} \right\}$$

of  $\mathfrak{so}(7)$ . Obviously,  $3\mathfrak{u}(1) \subseteq \text{Norm}_{\mathfrak{so}(7)}2\mathfrak{u}(1)$ . We will prove that the other inclusion is also satisfied. For any  $x \in \text{Norm}_{\mathfrak{so}(7)}2\mathfrak{u}(1)$  we have

$$(29) \quad [x, z] \in 2\mathfrak{u}(1) \quad \forall z \in 2\mathfrak{u}(1).$$



Since the Killing form  $\kappa$  of  $\mathfrak{so}(7)$  is associative, it follows that for any  $y \in 2\mathfrak{u}(1)$

$$(30) \quad \kappa([x, z], y) = \kappa(x, [z, y]) = 0$$

and therefore  $[x, z] = 0$ . This observation proves that the normalizer equals the centralizer

$$(31) \quad C_{\mathfrak{so}(7)} 2\mathfrak{u}(1) := \{x \in \mathfrak{so}(7) | \text{ad}_x(2\mathfrak{u}(1)) = \{0\}\}.$$

For the following considerations, we complexify the Lie algebras  $\mathfrak{so}(7)$  and  $2\mathfrak{u}(1)$  and return to the real case later on.  $x$  has a Cartan decomposition

$$(32) \quad x = x_h + \sum_{\alpha \in \Phi} \mu_\alpha g_\alpha \quad \text{with} \quad x_h \in 3\mathfrak{u}(1) \otimes \mathbb{C}, \mu_\alpha \in \mathbb{C}, g_\alpha \in L_\alpha,$$

where  $\Phi$  is the root system of  $\mathfrak{so}(7, \mathbb{C})$  and  $L_\alpha$  is the eigenspace to the eigenvalue  $\alpha : 3\mathfrak{u}(1) \otimes \mathbb{C} \rightarrow \mathbb{C}$  of  $\text{ad}_{3\mathfrak{u}(1) \otimes \mathbb{C}}$ . Let  $z \in 2\mathfrak{u}(1) \otimes \mathbb{C}$  be arbitrary. Applying  $\text{ad}_z$  to (32) yields the following equation:

$$(33) \quad \text{ad}_z(x) = \sum_{\alpha \in \Phi} \mu_\alpha \alpha(z) g_\alpha.$$

We want to prove that  $x$  cannot be a non-zero element of the orthogonal complement of  $3\mathfrak{u}(1) \otimes \mathbb{C}$ . If there exists an  $\alpha \in \Phi$  with  $\alpha(2\mathfrak{u}(1) \otimes \mathbb{C}) = 0$ , then  $[z, g_\alpha] = \alpha(z)g_\alpha = 0$  for all  $z \in 2\mathfrak{u}(1) \otimes \mathbb{C}$ . In that situation, we could choose  $x$  as  $g_\alpha$ . Conversely, we assume that there is no such  $\alpha$  and choose  $(3\mathfrak{u}(1) \otimes \mathbb{C})^\perp \setminus \{0\} \ni x = \sum_{\alpha \in \Phi} \mu_\alpha g_\alpha$  arbitrarily. If we choose a  $z$  with  $\alpha(z) \neq 0$  for an  $\alpha$  with  $\mu_\alpha \neq 0$ , we have  $\text{ad}_z(x) \neq 0$ . In that case,  $\text{Norm}_{\mathfrak{so}(7, \mathbb{C})} 2\mathfrak{u}(1) \otimes \mathbb{C}$  would be  $3\mathfrak{u}(1) \otimes \mathbb{C}$ . We therefore have to answer the question if there is an  $\alpha$  with  $\alpha(2\mathfrak{u}(1) \otimes \mathbb{C}) = 0$ .

In order to do this, we have to take a closer look at the root system of  $\mathfrak{so}(7, \mathbb{C})$ . Let  $L_1$  be the element of the Cartan subalgebra (28) with  $a = 1$  and  $b = c = 0$ . Analogously, let  $L_2$  be given by  $b = 1, a = c = 0$ , and  $L_3$  by  $c = 1$  and  $a = b = 0$ . We denote the dual of  $L_j$  with respect to the Killing form by  $\theta_j$ . The root system of  $\mathfrak{so}(7, \mathbb{C})$  is:

$$(34) \quad \{\pm\theta_j | 1 \leq j \leq 3\} \cup \{\pm\theta_j \pm \theta_k | 1 \leq j < k \leq 3\}.$$

The Cartan subalgebra  $2\mathfrak{u}(1) \otimes \mathbb{C}$  of  $\mathfrak{g}_2^{\mathbb{C}}$  is the plane which is orthogonal to  $L_1 + L_2 - L_3$ . The  $\alpha \in \Phi$  which vanish on  $2\mathfrak{u}(1) \otimes \mathbb{C}$  are precisely those which are multiples of  $\theta_1 + \theta_2 - \theta_3$ . Since there is no root of  $\mathfrak{so}(7, \mathbb{C})$

with this property, we have proven that indeed  $\text{Norm}_{\mathfrak{so}(7, \mathbb{C})}(2\mathfrak{u}(1) \otimes \mathbb{C}) = 3\mathfrak{u}(1) \otimes \mathbb{C}$ . By passing to the compact real form of  $\mathfrak{so}(7, \mathbb{C})$ , we can conclude that  $\text{Norm}_{\mathfrak{so}(7)}2\mathfrak{u}(1) = 3\mathfrak{u}(1)$ .

Let  $U(1)^3$  be the maximal torus of  $SO(7)$  with Lie algebra  $3\mathfrak{u}(1)$ . Our next step is to describe the discrete group  $\Gamma := (\text{Norm}_{SO(7)}U(1)^2)/U(1)^3$ . The group  $(\text{Norm}_{SO(7)}U(1)^3)/U(1)^3$  is isomorphic to the Weyl group  $\mathcal{W}_{\mathfrak{so}(7)}$  of  $\mathfrak{so}(7)$ . We prove that  $\text{Norm}_{SO(7)}U(1)^2 \subseteq \text{Norm}_{SO(7)}U(1)^3$ . Let us assume that there is an element  $h$  of  $SO(7)$  such that  $\text{Ad}_h$  leaves  $2\mathfrak{u}(1)$  invariant, but does not leave  $3\mathfrak{u}(1)$  invariant. Then  $3\mathfrak{u}(1)$  and  $\text{Ad}_h(3\mathfrak{u}(1))$  are two distinct Cartan subalgebras whose intersection is  $2\mathfrak{u}(1)$ . In this situation, the centralizer of  $2\mathfrak{u}(1)$  is at least four-dimensional which is not the case.  $\Gamma$  therefore is a subgroup of  $\mathcal{W}_{\mathfrak{so}(7)}$ . More precisely, it is the subgroup of  $\mathcal{W}_{\mathfrak{so}(7)}$  which leaves the plane  $2\mathfrak{u}(1) \subseteq 3\mathfrak{u}(1)$  invariant.

In order to describe  $\Gamma$  explicitly, we introduce some facts on  $\mathcal{W}_{\mathfrak{so}(7)}$ . The Weyl group of  $\mathfrak{so}(7)$  is isomorphic to  $\mathbb{Z}_2^3 \rtimes S_3$ , which is of order 48. The first factor of  $\mathcal{W}_{\mathfrak{so}(7)}$  acts by changing the signs of the  $\theta_i$ . The second factor of the Weyl group consists of the permutations of  $\{\theta_1, \theta_2, \theta_3\}$ .  $2\mathfrak{u}(1)$  is the plane of all  $x \in 3\mathfrak{u}(1)$  satisfying:

$$(35) \quad \theta_1(x) + \theta_2(x) - \theta_3(x) = 0.$$

By replacing  $L_3$  by  $-L_3$ , we can change this equation into:

$$(36) \quad \theta_1(x) + \theta_2(x) + \theta_3(x) = 0.$$

The subgroup of  $\mathcal{W}_{\mathfrak{so}(7)}$  which leaves this equation invariant is generated by the permutations and the simultaneous change of all signs.  $\Gamma$  thus is the direct product  $\mathbb{Z}_2 \times S_3$ , which is isomorphic to the Dieder group  $D_6$ . All in all, we have proven:

$$(37) \quad \text{Norm}_{SO(7)}U(1)^2 = U(1)^3 \rtimes D_6.$$

Next, we have to determine  $\text{Norm}_{G_2}U(1)^2$ . Since  $2\mathfrak{u}(1)$  is a Cartan subalgebra of  $\mathfrak{g}_2$ ,  $\text{Norm}_{G_2}U(1)^2/U(1)^2$  is the Weyl group  $\mathcal{W}_{\mathfrak{g}_2}$  of  $\mathfrak{g}_2$ . It is known that  $\mathcal{W}_{\mathfrak{g}_2}$  is isomorphic to  $D_6$ , too.  $\text{Norm}_{G_2}U(1)^2$  therefore is a semidirect product  $U(1)^2 \rtimes D_6$ . There is the following exact sequence:

$$(38) \quad \begin{array}{c} \pi_0(\text{Norm}_{G_2}U(1)^2) \xrightarrow{\pi_0(i)} \pi_0(\text{Norm}_{SO(7)}U(1)^2) \\ \xrightarrow{\pi_0(\pi)} \pi_0(\text{Norm}_{SO(7)}U(1)^2/\text{Norm}_{G_2}U(1)^2) \rightarrow \{0\}, \end{array}$$

where  $\pi_0(i)$  is induced by the inclusion of  $\text{Norm}_{G_2}U(1)^2$  into  $\text{Norm}_{SO(7)}U(1)^2$  and  $\pi_0(\pi)$  by the projection map.  $\text{Norm}_{G_2}U(1)^2$  and  $\text{Norm}_{SO(7)}U(1)^2$  have both 12 connected components. If we were able to prove that  $\pi_0(i)$  is surjective, we could conclude that  $\text{Norm}_{SO(7)}U(1)^2/\text{Norm}_{G_2}U(1)^2$  is connected. Since  $\pi_0(\text{Norm}_{SO(7)}U(1)^2)$  and  $\pi_0(\text{Norm}_{G_2}U(1)^2)$  are both finite, we can prove the injectivity instead. Let  $x \in \text{Norm}_{G_2}U(1)^2$  with  $x \notin U(1)^2$  be arbitrary. Then  $x = \alpha x_0$  with  $\alpha \in \mathcal{W}_{\mathfrak{g}_2} \setminus \{e\}$  and  $x_0 \in U(1)^2$ . Since  $\alpha$  acts non-trivially on the dual of the Cartan subalgebra  $2\mathfrak{u}(1)$ , it cannot be an element of the maximal torus  $U(1)^3$  of  $SO(7)$ . Therefore,  $x$  is not an element of the identity component of  $\text{Norm}_{SO(7)}U(1)^2$  and  $\pi_0(i)$  is thus injective. All in all, we have proven that  $\text{Norm}_{SO(7)}U(1)^2/\text{Norm}_{G_2}U(1)^2$  is connected. More precisely, it is a group which is isomorphic to  $U(1)$ .

For our considerations we need to describe the action of  $\text{Norm}_{SO(7)}U(1)^2/\text{Norm}_{G_2}U(1)^2$  on  $\mathfrak{m}$ . We define

$$(39) \quad T := \left\{ \left( \begin{array}{cccc} \boxed{R_\theta} & & & \\ & \boxed{R_\theta} & & \\ & & \boxed{R_\theta} & \\ & & & \boxed{1} \end{array} \right) =: T_\theta \mid \theta \in \mathbb{R} \right\},$$

where  $R_\theta$  denotes the rotation in the plane around the angle  $\theta$  and the matrix representation of  $T_\theta$  is with respect to the basis  $(e_1, \dots, e_7)$  of  $\mathfrak{m}$ .  $T$  commutes with the action of  $U(1)_{1,1,1}^2$  and the intersection  $T \cap U(1)_{1,1,1}^2$  is discrete. Furthermore,  $T_\theta$  is orthogonal with respect to  $g$  and orientation preserving. Since  $U(1)_{1,1,1}^2$  preserves any  $SU(2)^3$ -invariant  $G_2$ -structure  $\omega$  and  $\text{rank } G_2 = 2$ , the action of  $T$  cannot preserve  $\omega$ . All in all, the set of all invariant  $G_2$ -structures with the same associated metric and orientation as  $\omega$  is generated by  $T$ . Let  $S$  be the following subgroup of  $\text{Norm}_{SU(2)^3}U(1)^2$ :

$$(40) \quad \left\{ \left( \left( \begin{array}{cc} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{array} \right), \left( \begin{array}{cc} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{array} \right), \left( \begin{array}{cc} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{array} \right) \right) =: S_\theta \mid \theta \in \mathbb{R} \right\}$$

Conjugation by  $S_\theta$  is a well-defined diffeomorphism of  $Q^{1,1,1}$ . Moreover, it is an orientation preserving isometry. Its differential acts as  $T_\theta$  on  $\mathfrak{m}$ . Our set of  $G_2$ -structures can therefore be obtained by an isometric action of  $S$  on  $\omega$ . The simultaneous action of an  $S_\theta$  on all principal orbits can be extended to an isometry  $\Phi_\theta$  of the cohomogeneity one manifold  $M$ . We assume that  $d*\omega = 0$  and denote the extension of  $\omega$  to a parallel  $\text{Spin}(7)$ -structure by  $\Omega$ . Since  $\Phi_\theta^*\Omega$  is parallel if  $\Omega$  is parallel, it suffices to consider only one cosymplectic  $\omega$  on the principal orbit instead of the whole one-parameter family generated by  $S$ .

We fix an arbitrary  $2\mathfrak{u}(1)_{1,1,1}$ -invariant metric  $g$  on  $\mathfrak{m}$ , which has to be of type (25). Furthermore, we assume that  $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 1$ . Our aim is to construct a basis  $(f_i)_{0 \leq i \leq 7}$  of the tangent space  $\mathfrak{m} \oplus \text{span}(\frac{\partial}{\partial t})$  which yields an  $SU(2)^3$ -invariant  $\text{Spin}(7)$ -structure with  $g$  as associated metric. We can change the orientation of  $M$  by replacing  $t$  by  $-t$ . Therefore, we do not have to take care of orientation issues. Let  $\psi : \text{Im}(\mathbb{O}) \rightarrow \mathfrak{m}$  be the isomorphism which maps the  $G_2$ -structure on  $\text{Im}(\mathbb{O})$  to the tangent space of  $Q^{1,1,1}$ .  $\psi$  has to preserve the inner product of both spaces. Moreover, it has to turn the Cartan subalgebra (23) into the isotropy representation of  $2\mathfrak{u}(1)_{1,1,1}$ . If all these conditions are satisfied,  $(f_i)_{1 \leq i \leq 7}$  can be chosen as  $(\psi(i), \dots, \psi(k\epsilon))$ . By choosing  $f_0 := \frac{\partial}{\partial t}$ , we obtain a cohomogeneity one  $\text{Spin}(7)$ -structure. A possible basis of this kind is:

$$(41) \quad \begin{aligned} f_0 &:= \frac{\partial}{\partial t} & f_1 &:= \frac{1}{f}e_7 & f_2 &:= \frac{1}{a}e_1 & f_3 &:= \frac{1}{a}e_2 \\ f_4 &:= \frac{1}{b}e_3 & f_5 &:= \frac{1}{b}e_4 & f_6 &:= \frac{1}{c}e_6 & f_7 &:= \frac{1}{c}e_5 \end{aligned}$$

Since the group  $S$  acts isometrically and transitively on the set of all  $G_2$ -structures with a fixed extension to an  $SO(7)$ -structure, this is the most general basis which we have to consider. The basis (41) yields the following  $\text{Spin}(7)$ -structure:

$$(42) \quad \begin{aligned} \Omega &= abc f e^{1357} - abc f e^{1467} - abc f e^{2367} - abc f e^{2457} \\ &\quad - a^2 b^2 e^{1234} - a^2 c^2 e^{1256} - b^2 c^2 e^{3456} \\ &\quad - a^2 f e^{127} \wedge dt - b^2 f e^{347} \wedge dt - c^2 f e^{567} \wedge dt \\ &\quad - abc e^{136} \wedge dt - abc e^{145} \wedge dt - abc e^{235} \wedge dt + abc e^{246} \wedge dt \end{aligned}$$

Let  $X$  and  $Y$  be left-invariant vector fields on  $SU(2)^3$  and  $\alpha$  be a left-invariant one-form. We have  $d\alpha(X, Y) = -\alpha([X, Y])$ . With help of the anti-derivation property of  $d$  we can compute the exterior derivatives of the pull-backs  $\pi^*\omega$  and  $\pi^* * \omega$  of  $\omega$  and  $*\omega$  to  $SU(2)^3$ . This enables us to calculate  $d\Omega$ . We finally see that  $d\Omega = 0$  is equivalent to:

$$(43) \quad \begin{aligned} a' &= -\frac{1}{6} \frac{f}{a} \\ b' &= -\frac{1}{6} \frac{f}{b} \\ c' &= -\frac{1}{6} \frac{f}{c} \\ f' &= \frac{1}{6} \frac{f^2}{a^2} + \frac{1}{6} \frac{f^2}{b^2} + \frac{1}{6} \frac{f^2}{c^2} - 3 \end{aligned}$$

Our next step is to solve the above system. The initial conditions for (43) shall be  $a(0) = a_0$ ,  $b(0) = b_0$ ,  $c(0) = c_0$ , and  $f(0) = f_0$ . Since we have:

$$(44) \quad a'a = b'b = c'c = -\frac{1}{6}f,$$

the functions  $a^2$ ,  $b^2$ , and  $c^2$  are up to a constant summand the same. We introduce a function  $F$  with  $F' = f$ . By requiring  $F(0) = 0$ , we make  $F$  unique. From the above equation, it follows that

$$(45) \quad a^2 - a_0^2 = b^2 - b_0^2 = c^2 - c_0^2 = -\frac{1}{3}F.$$

We rewrite the fourth equation of (43):

$$(46) \quad f' = -\frac{1}{6} \frac{f^2}{\frac{1}{3}F - a_0^2} - \frac{1}{6} \frac{f^2}{\frac{1}{3}F - b_0^2} - \frac{1}{6} \frac{f^2}{\frac{1}{3}F - c_0^2} - 3.$$

Since there is only one principal orbit,  $f$  does not change its sign and  $F$  is injective. The function  $\tilde{f}$  with  $\tilde{f}(t) := f(F^{-1}(t))$  satisfies the equation:

$$(47) \quad \tilde{f}\tilde{f}' = -\frac{\tilde{f}^2}{2t - 6a_0^2} - \frac{\tilde{f}^2}{2t - 6b_0^2} - \frac{\tilde{f}^2}{2t - 6c_0^2} - 3.$$

This equation is linear in  $f^2$  and can be solved explicitly. By variation of constants, we obtain the following solution of our initial value problem:

$$(48) \quad \tilde{f}(t)^2 = \frac{f_0^2}{\left(1 - \frac{t}{3a_0^2}\right)\left(1 - \frac{t}{3b_0^2}\right)\left(1 - \frac{t}{3c_0^2}\right)} - \frac{3}{(t - 3a_0^2)(t - 3b_0^2)(t - 3c_0^2)} \int_0^t (s - 3a_0^2)(s - 3b_0^2)(s - 3c_0^2) ds.$$

The equations (45) and (48) describe the metric completely. Since we are in the lucky situation to have explicit solutions of the equations of the holonomy reduction, we are able to describe the global shape of the metric. The function  $\tilde{f}$  is of type  $c\sqrt{t} + O(1)$  for large values of  $t$  and a  $c \in \mathbb{R} \setminus \{0\}$ . We insert the definition of  $\tilde{f}$  and obtain:

$$(49) \quad f(t) = c\sqrt{F(t)} + O(1).$$

From this equation we can deduce that  $f(t) = \frac{c^2}{2}t + O(1)$  and from (45) it follows that  $a$ ,  $b$ , and  $c$  approach linear functions, too. This proves that

the metric is asymptotically conical and in particular complete. We make the ansatz  $a(t) = a_1 t, \dots, f(t) = f_1 t$  and obtain  $a_1^2, b_1^2, c_1^2 = \frac{1}{8}$  and  $f_1 = \frac{3}{4}$ . These numbers determine the metric on the base of the cone which our cohomogeneity one metric approaches. Since the cone has holonomy  $\text{Spin}(7)$ ,  $(a_1, b_1, c_1, f_1)$  describes a nearly parallel  $G_2$ -structure on  $Q^{1,1,1}$ .

We now determine the holonomy of our metrics. The diffeomorphisms  $\Phi_\theta^*$  preserve the metric and orientation but not the  $\text{Spin}(7)$ -structure. The set of all  $\text{Spin}(7)$ -structures which we obtain by the action of  $S$  is diffeomorphic to a circle. According to Lemma 3.11, the holonomy is contained in  $SU(4)$ . If the holonomy is not all of  $SU(4)$ , it is either a subgroup of  $SU(3)$  or of  $Sp(2)$ . In the first case, there exists a parallel vector field  $X$  on  $M$ . The holonomy bundle is invariant under the isometry group of  $M$ . We can therefore assume that  $X$  is  $SU(2)^3$ -invariant. This is the case if and only if  $X = c_1 \frac{1}{f} e_7 + c_2 \frac{\partial}{\partial t}$ , where  $c_1$  and  $c_2$  depend on  $t$  only. Since the length of  $X$  is constant and  $\nabla_{\frac{\partial}{\partial t}} X = 0$ ,  $c_1$  and  $c_2$  have to be constant, too. If  $c_1 \neq 0$ ,  $f$  is also a constant non-zero function. It follows from (45), that  $a^2, b^2$ , and  $c^2$  are either all strictly increasing or strictly decreasing. In any case, the right hand side of the fourth equation of (43) cannot vanish. This is a contradiction to  $f$  being constant. If  $X$  was a multiple of  $\frac{\partial}{\partial t}$ , we would have  $a' = b' = c' = f' = 0$ , which is impossible, too.

If the holonomy was a subgroup of  $Sp(2)$ , there would exist three linearly independent Kähler forms on  $M$ . For similar reasons as above, any Kähler form  $\eta$  on  $M$  has to be  $SU(2)^3$ -invariant. The two-forms

$$(50) \quad e^{12}, \quad e^{34}, \quad e^{56}, \quad e^7 \wedge dt$$

span the space of all  $SU(2)^3$ -invariant two-forms on  $M$ . The Kähler-form thus has to satisfy:

$$(51) \quad \eta = \epsilon_1 a^2 e^{12} + \epsilon_2 b^2 e^{34} + \epsilon_3 c^2 e^{56} + \epsilon_4 f e^7 \wedge dt \quad \text{with} \quad \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{-1, 1\}.$$

For the exterior derivative of  $\eta$ , we obtain:

$$(52) \quad d\eta = \frac{1}{3} \epsilon_4 f e^{12} \wedge dt + \epsilon_1 2a'a e^{12} \wedge dt + \frac{1}{3} \epsilon_4 f e^{34} \wedge dt + \epsilon_2 2b'b e^{34} \wedge dt \\ + \frac{1}{3} \epsilon_4 f e^{56} \wedge dt + \epsilon_3 2c'c e^{56} \wedge dt.$$

$d\eta = 0$  yields the following equations:

$$(53) \quad \epsilon_4 f = -6\epsilon_1 a'a = -6\epsilon_2 b'b = -6\epsilon_3 c'c.$$

By comparing (53) with the system (43), we see that  $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \pm 1$  and thus:

$$(54) \quad \eta = \pm(a^2 e^{12} + b^2 e^{34} + c^2 e^{56} + f e^7 \wedge dt).$$

Since there are only two linearly dependent Kähler forms on  $M$ , the holonomy is all of  $SU(4)$ . We finally investigate the behaviour of the metric near the singular orbit. The possible singular orbits of a cohomogeneity one manifold with principal orbit  $Q^{1,1,1}$  are classified by the following lemma:

**Lemma 4.1.** *Let  $U(1)_{1,1,1}^2 \subseteq SU(2)^3$  be chosen as at the beginning of this section. For reasons of convenience, we denote this subgroup by  $U(1)^2$  and its Lie algebra by  $2\mathfrak{u}(1)$ . Furthermore, let  $K$  with  $U(1)^2 \subsetneq K \subseteq SU(2)^3$  be a closed, connected subgroup. We denote the Lie algebra of  $K$  by  $\mathfrak{k}$ . In this situation,  $\mathfrak{k}$  and  $K$  can be found in the table below. Furthermore,  $K/U(1)^2$  and  $SU(2)^3/K$  satisfy the following topological conditions:*

$\mathfrak{k}$	$K$	$K/U(1)^2$	$SU(2)^3/K$
$3\mathfrak{u}(1)$	$U(1)^3$	$\cong S^1$	$\cong S^2 \times S^2 \times S^2$
$2\mathfrak{u}(1) \oplus \mathfrak{su}(2)$	$U(1)^2 \times SU(2)$	$\cong S^3$	$\cong S^2 \times S^2$
$\mathfrak{u}(1) \oplus 2\mathfrak{su}(2)$	$U(1) \times SU(2)^2$	$\not\cong S^5/\Gamma$	$\cong S^2$
$3\mathfrak{su}(2)$	$SU(2)^3$	$= Q^{1,1,1} \not\cong S^7/\Gamma$	

In the above table,  $\Gamma$  denotes an arbitrary discrete subgroup of  $O(6)$  or  $O(8)$ .

The following fact will help us to prove the lemma:

**Lemma 4.2.** *Let  $G$  be a Lie group and  $H$  be a closed subgroup of  $G$ . Moreover, let  $T \subseteq G \times U(1)$  be isomorphic to  $U(1)$  such that  $T \not\subseteq G$ ,  $T \cap H = \{e\}$ , and  $T$  and  $H$  commute. In this situation,  $G/H$  is a  $|\Gamma|$ -fold covering of  $(G \times U(1))/(H \times T)$ , where  $\Gamma = T \cap G$ .*

*Proof.* A possible covering map is given by

$$(55) \quad \begin{aligned} \pi : G/H &\rightarrow (G \times U(1))/(H \times T) \\ \pi(gH) &:= (g, e)H \times T \end{aligned}$$

□

We are now able to prove Lemma 4.1:

*Proof.* The Lie algebra  $\mathfrak{k}$  is a  $2\mathfrak{u}(1)$ -module. Since  $\mathfrak{m}$  decomposes into pairwise inequivalent  $2\mathfrak{u}(1)$ -submodules, the modules  $V$  with  $2\mathfrak{u}(1) \subseteq V \subseteq 3\mathfrak{su}(2)$  can be easily classified. After we have checked for each  $V$  if it is closed under the Lie bracket, we obtain the  $\mathfrak{k}$  from the table of the lemma. Our statements on the topology of  $SU(2)^3/K$  can be proven with help of

the Hopf fibration  $SU(2)/U(1) \cong S^2$ . The remaining part of the proof is to check for each  $K$  if  $K/U(1)^2$  is a sphere. The first non-trivial case is where  $K \cong SU(2) \times U(1)^2$ . In this situation  $K \subseteq SU(2)^3$  is given by:

$$(56) \quad \{(A, Q_\phi, Q_\psi) | A \in SU(2), \phi, \psi \in [0, 2\pi)\},$$

where

$$Q_\phi := \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

$U(1)^2$  can be described as:

$$(57) \quad \{(Q_{-\psi-\phi}, Q_\phi, Q_\psi) | \phi, \psi \in [0, 2\pi)\}.$$

We apply Lemma 4.2 to our situation. The intersection of the two-dimensional group  $U(1)^2$  which we divide out with the semisimple part of  $K$  is trivial. Therefore, we have  $K/U(1)^2 = (SU(2) \times U(1)^2)/U(1)^2 \cong (SU(2) \times U(1))/U(1) \cong SU(2) \cong S^3$ . Next, we assume that  $K = SU(2)^2 \times U(1)$ . For similar reasons as above,  $K/U(1)^2$  can be described as  $SU(2)^2/U(1)$  with

$$(58) \quad U(1) := \{(Q_{-\phi}, Q_\phi) | \phi \in [0, 2\pi)\}.$$

$S^3 \times S^3$  is a circle bundle over  $K/U(1)^2$ . If  $K/U(1)^2$  was covered by  $S^5$ , we would have:

$$(59) \quad \dots \rightarrow \pi_3(S^1) \rightarrow \pi_3(S^3 \times S^3) \rightarrow \pi_3(S^5/\Gamma) \rightarrow \dots$$

Since the higher homotopy groups of  $S^5$  and  $S^5/\Gamma$  coincide, the above exact sequence becomes:

$$(60) \quad \dots \rightarrow \{0\} \rightarrow \mathbb{Z}^2 \rightarrow \{0\} \rightarrow \dots$$

which is impossible. We finally consider the case  $K = SU(2)^3$ .  $Q^{1,1,1}$  is a circle bundle over  $SU(2)^3/U(1)^3 = S^2 \times S^2 \times S^2$ . Therefore, we obtain the following exact sequence:

$$(61) \quad \dots \rightarrow \pi_2(S^7/\Gamma) \rightarrow \pi_2((S^2)^3) \rightarrow \pi_1(S^1) \rightarrow \pi_1(S^7/\Gamma) \rightarrow \dots,$$

which can be explicitly written as



$$(62) \quad \dots \rightarrow \{0\} \rightarrow \mathbb{Z}^3 \rightarrow \mathbb{Z} \rightarrow \Gamma \rightarrow \dots$$

Since  $\Gamma$  is finite, this exact sequence is impossible, too, and  $Q^{1,1,1}$  is not homeomorphic to  $S^7/\Gamma$ .  $\square$

Lemma (4.1) yields two sets of initial conditions at the singular orbit:

- (1) If the singular orbit is  $S^2 \times S^2 \times S^2$ , we have  $f(0) = 0$  and  $a(0), b(0), c(0) \neq 0$ .
- (2) If the singular orbit is  $S^2 \times S^2$ , we have  $f(0) = 0$  and exactly two of the three initial values  $a(0), b(0)$ , and  $c(0)$  are non-zero. Without loss of generality, we assume that  $a(0) = 0$ .

We check for each of the two cases if the metric can be smoothly extended to the singular orbit and start with the first one. According to the considerations which we have made at the end of Section 2, the metric is smooth if the following conditions are satisfied:

- (1)  $a, b, c$ , and  $f$  are analytic.
- (2) The metric on  $\mathfrak{p} = \text{span}(e_1, \dots, e_6)$  is  $U(1)^3$ -invariant for all  $t$ .
- (3)  $a, b$ , and  $c$  are even and  $f$  is odd.
- (4) The length of the collapsing circle is  $2\pi t + O(t^2)$  for small values of  $t$ .

It follows from (45) and (48) that the first condition is satisfied. Since  $\mathfrak{p}$  splits into  $V_1 \oplus V_2 \oplus V_3$  with respect to  $U(1)^3$ ,  $a(t), b(t)$ , and  $c(t)$  can be chosen arbitrarily and the second condition is satisfied. Let  $(a(t), b(t), c(t), f(t))$  be a solution of (43). It is easy to see that  $(a(-t), b(-t), c(-t), -f(-t))$  is another solution of (43), whose value at  $t = 0$  is the same. Since (43) has a unique solution for any initial metric on the singular orbit  $S^2 \times S^2 \times S^2$ , the third condition is satisfied, too.

Let  $\gamma : [0, \frac{4\pi}{3}] \rightarrow U(1)^3/U(1)_{1,1,1}^2$  be the loop with  $\gamma(\theta) := S_\theta U(1)_{1,1,1}^2$ , where  $S_\theta$  is defined by (40). As before,  $S$  is the one-dimensional subgroup of  $SU(2)^3$  which consists of all  $S_\theta$ . Since the intersection  $S \cap U(1)_{1,1,1}^2$  is isomorphic to  $\mathbb{Z}_3$ ,  $\gamma$  winds around the collapsing circle  $U(1)^3/U(1)_{1,1,1}^2$  exactly once. Moreover, we have  $|\gamma'(0)| = |e_7| = |f(t)|$  and the length of the collapsing circle thus is  $\frac{4\pi}{3}|f(t)|$ . The fourth smoothness condition is therefore equivalent to  $|f'(0)| = \frac{3}{2}$ . If we insert  $f(0) = 0$  into (43), we obtain  $f'(0) = -3$  and see that the metric has a singularity at the singular orbit. More precisely, the length of the collapsing circle is twice as long as it should be. We can think of the singularity as "the opposite" of a conical singularity. We turn our attention to the second set of initial conditions. As before, we have the following sufficient smoothness conditions:

- (1)  $a, b, c$ , and  $f$  are analytic.

- (2) The metric on  $\mathfrak{p} = \text{span}(e_3, e_4, e_5, e_6)$  is  $U(1)^2 \times SU(2)$ -invariant for all  $t$ .
- (3)  $b$  and  $c$  are even functions while  $a$  and  $f$  are odd functions.
- (4) The length of any great circle on the collapsing sphere is  $2\pi t + O(t^2)$  for small values of  $t$ .

For the similar reasons as in the previous case, the first two conditions are satisfied for any choice of the initial values  $b(0)$  and  $c(0)$ . We can also prove the third smoothness condition by the same method as before, since  $(a(t), b(t), c(t), f(t)) \mapsto (-a(-t), b(-t), c(-t), -f(-t))$  is another symmetry of the system (43). In order to check the fourth smoothness condition, we investigate the great circles on the collapsing sphere  $S^3$ . Since the metric on  $S^3$  is determined by the values of  $a$  and  $f$ , it suffices to consider the great circles in the directions of  $e_1$  and  $e_7$ . By the same arguments as before, we obtain the old condition  $|f'(0)| = \frac{3}{2}$  and  $|a'(0)| = \frac{1}{2}$  as smoothness conditions. We cannot directly insert  $a(0) = f(0) = 0$  into (43), since those equations contain terms of type  $\frac{f}{a}$ . However, we can apply l'Hôpital's rule and obtain a system of two equations which yields  $a'(0) = \pm\frac{1}{2}$  and  $f'(0) = -\frac{3}{2}$ . All in all, we have proven that the metrics with singular orbit  $S^2 \times S^2$  are smooth. We finally sum up the results of this section:

**Theorem 4.3.** *Let  $(M, \Omega)$  be a  $Spin(7)$ -manifold with a cohomogeneity one action of  $SU(2)^3$  which preserves  $\Omega$ . In this situation, the following statements are true:*

- (1) *The principal orbits are  $SU(2)^3$ -equivariantly diffeomorphic to  $Q^{1,1,1}$ .*
- (2) *The metric  $g$  which is associated to  $\Omega$  is Ricci-flat and Kähler and its holonomy is all of  $SU(4)$ .*
- (3) *The restriction of  $g$  to a principal orbit can be written as (25). The coefficient functions  $a$ ,  $b$ ,  $c$ , and  $f$  satisfy the equations (43), whose solutions are described by (45) and (48). The Kähler form is given by (54).*

*If  $M$  has a singular orbit, which has to be the case if  $(M, g)$  is complete, it is  $S^2 \times S^2 \times S^2$  or  $S^2 \times S^2$ . Any  $SU(2)^3$ -invariant metric on the singular orbit can be extended to a unique complete cohomogeneity one metric with holonomy  $SU(4)$ . For any choice of the singular orbit and its metric,  $(M, g)$  is asymptotically conical. If the singular orbit is  $S^2 \times S^2 \times S^2$ , the metric is not differentiable at the singular orbit. If the singular orbit is  $S^2 \times S^2$ , the metric is smooth at the singular orbit.*

*Remark 4.4.* The metrics with holonomy  $SU(4)$  which we have constructed have also been considered by Cvetič, Gibbons, Lü, and Pope in [12]. In that paper, the authors construct Ricci-flat Kähler-metrics on holomorphic vector bundles over a product of several Einstein-Kähler manifolds. As a special case, they obtain the same metrics as the author. The authors also prove that the metrics are non-compact and complete away from the singular

orbit. In [9], the authors investigate the same metrics in another context. Both papers are based on earlier works by Berard-Bergery [4], Page and Pope [23], and Stenzel [25]. The examples with singular orbit  $S^2 \times S^2 \times S^2$  have also been considered by Herzog and Klebanov [19]. Although the metrics which we have constructed are known, the proof that any parallel Spin(7)-manifold with a cohomogeneity one action of  $SU(2)^3$  is one of our examples is new. Our results thus prove that it is impossible to deform the Kähler examples into metrics with holonomy Spin(7) without losing the  $SU(2)^3$ -symmetry. Moreover, it follows from Remark 3.12 that any (not necessarily parallel) Spin(7)-structure of cohomogeneity one with principal orbit  $Q^{1,1,1}$  has to reduce to an  $SU(4)$ -structure. This fact has not been mentioned in the literature before, either. All in all, we hope to have introduced an interesting, more algebraic approach to the issue of special cohomogeneity one metrics with principal orbit  $Q^{1,1,1}$ .

## 5. METRICS WITH PRINCIPAL ORBIT $M^{k,l,0}$

In this section, we consider cohomogeneity one manifolds whose principal orbit is of type  $(SU(3) \times SU(2))/(SU(2) \times U(1))$ . The semisimple part  $SU(2)$  of the isotropy group shall be embedded into the first factor of  $SU(3) \times SU(2)$ . We will write any element of  $SU(3) \times SU(2)$  as a block matrix in  $GL(5, \mathbb{R})$  consisting of a  $3 \times 3$ - and a  $2 \times 2$ -matrix.

Our first step is to classify all homogeneous spaces of type  $(SU(3) \times SU(2))/(SU(2) \times U(1))$ . We use the notation of Castellani et al. [8] and Fabbri et al. [15] and denote these spaces by  $M^{k,l,0}$ . In order to explain the meaning of the three indices, we first consider more general spaces of type  $M^{k,l,m} = (SU(3) \times SU(2) \times U(1))/(SU(2) \times U(1)' \times U(1)'')$ . We define the following one-dimensional subalgebra of  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ :

$$(63) \quad \mathfrak{u}(1)'''_{k,l,m} := \left\{ \left( \begin{array}{ccc|cc|c} \frac{k}{2}ix & 0 & 0 & & & \\ 0 & \frac{k}{2}ix & 0 & & & \\ 0 & 0 & -kix & & & \\ \hline & & & -\frac{l}{2}ix & 0 & \\ & & & 0 & \frac{l}{2}ix & \\ \hline & & & & & mix \end{array} \right) \mid x \in \mathbb{R} \right\}.$$

The diagonal matrices in  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  are a Cartan subalgebra  $\mathfrak{t}$  of that algebra. With help of the above definitions, we define an embedding  $\iota_{k,l,m} : \mathfrak{su}(2) \oplus \mathfrak{u}(1)' \oplus \mathfrak{u}(1)'' \rightarrow \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ :  $\mathfrak{su}(2)$  is embedded into  $\mathfrak{su}(3)$  such that it leaves the subspace  $\mathbb{C}^2$  of  $\mathbb{C}^3$  invariant.  $\iota_{k,l,m}(\mathfrak{u}(1)' \oplus \mathfrak{u}(1)'')$  shall be the orthogonal complement of  $\mathfrak{u}(1)'''_{k,l,m} \oplus (\mathfrak{su}(2) \cap \mathfrak{t})$  in  $\mathfrak{t}$  with respect to the Killing form. The quotient of  $SU(3) \times SU(2) \times U(1)$  by the Lie subgroup whose Lie algebra is  $\iota_{k,l,m}(\mathfrak{su}(2) \oplus \mathfrak{u}(1)' \oplus \mathfrak{u}(1)'')$  is denoted by  $M^{k,l,m}$ . In fact,

any coset space of type  $(SU(3) \times SU(2) \times U(1))/(SU(2) \times U(1)' \times U(1)'')$  where  $SU(2)$  is embedded into  $SU(3)$  can be identified with an  $M^{k,l,m}$ .

The space  $M^{k,l,m}$  is covered by  $M^{k,l,0}$  (cf. Castellani [8] for details). We will therefore assume from now on that  $m = 0$ . Since in this case the abelian factor of  $SU(3) \times SU(2) \times U(1)$  is a subgroup of  $U(1)' \times U(1)''$ ,  $M^{k,l,0}$  is diffeomorphic to a quotient of type  $(SU(3) \times SU(2))/(SU(2) \times U(1))$ . Let

$$(64) \quad P := \left( I_3, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right) \in SU(3) \times SU(2),$$

and let  $\phi_P : SU(3) \times SU(2) \rightarrow SU(3) \times SU(2)$  be defined by  $\phi_P(Q) := PQP^{-1}$ .  $\phi_P$  maps  $\mathfrak{u}_{k,l,0}(SU(2) \times U(1))$  into  $\mathfrak{u}_{k,-l,0}(SU(2) \times U(1))$ . The spaces  $M^{k,l,0}$  and  $M^{k,-l,0}$  thus are  $SU(3) \times SU(2)$ -equivariantly diffeomorphic. Since  $M^{k,l,0}$  and  $M^{-k,-l,0}$  are the same, too, we can assume without loss of generality that  $(k, l) \in \mathbb{N}_0 \times \mathbb{N}_0 \setminus \{(0, 0)\}$ .

We prove which  $M^{k,l,0}$  admit an  $SU(3) \times SU(2)$ -invariant  $G_2$ -structure. There exists a subalgebra of  $\mathfrak{g}_2$  which is isomorphic to  $\mathfrak{su}(2)$  and acts irreducibly on  $\mathbb{H}\epsilon$  and trivially on  $\text{Im}(\mathbb{H})$ . More precisely, there exists an  $\mathfrak{su}(2)$ -equivariant map from  $\mathbb{C}^2$  to  $\mathbb{H}\epsilon$  which maps the canonical basis of the real vector space  $\mathbb{C}^2$  to  $(\epsilon, i\epsilon, k\epsilon, j\epsilon)$ . Up to conjugation, there exists a unique one-dimensional subalgebra of  $\mathfrak{g}_2$  which commutes with  $\mathfrak{su}(2)$ . We thus have constructed a subalgebra which is isomorphic to  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ . By an analogous calculation as in Section 4, we see that the action of this algebra on  $\text{Im}(\mathbb{O})$  is equivalent to the isotropy action if and only if  $k = l = \pm 1$ . All in all, we have shown that the only principal orbit we have to consider is  $M^{1,1,0}$ .

We remark that the connected subgroup of  $G_2$  which has  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  as Lie algebra is not isomorphic to  $SU(2) \times U(1)$ , but to  $U(2)$ . The kernel of the isotropy representation of  $SU(2) \times U(1)$  is  $\mathbb{Z}_2$ . The group which acts effectively on the tangent space of  $M^{1,1,0}$  therefore is  $U(2)$  and there is no contradiction to Lemma 3.7.

Next, we classify the  $SU(3) \times SU(2)$ -invariant metrics and  $G_2$ -structures on  $M^{1,1,0}$ . As in the previous section, we identify the tangent space of  $M^{1,1,0}$  with the complement  $\mathfrak{m}$  of  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  in  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$  with respect to the metric  $q(X, Y) := -\text{tr}(XY)$ . We fix the following basis of  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ :

$$\begin{aligned} e_1 &:= (E_{13} - E_{31}, 0) & e_2 &:= (iE_{13} + E_{31}, 0) \\ e_3 &:= (E_{23} - E_{32}, 0) & e_4 &:= (iE_{23} + iE_{32}, 0) \\ e_5 &:= (0, E_{12} - E_{21}) & e_6 &:= (0, iE_{12} + iE_{21}) \\ e_7 &:= (\tfrac{1}{2}iE_{11} + \tfrac{1}{2}iE_{22} - iE_{33}, -\tfrac{1}{2}iE_{11} + \tfrac{1}{2}iE_{22}) & e_8 &:= (E_{12} - E_{21}, 0) \\ e_9 &:= (iE_{12} + iE_{21}, 0) & e_{10} &:= (iE_{11} - iE_{22}, 0) \\ e_{11} &:= (\tfrac{1}{3}iE_{11} + \tfrac{1}{3}iE_{22} - \tfrac{2}{3}iE_{33}, iE_{11} - iE_{22}) \end{aligned}$$

In the above table  $E_{ij}$  denotes the  $3 \times 3$ - ( $2 \times 2$ -)matrix with a "1" in the  $i$ th row and  $j$ th column. All other coefficients of  $E_{ij}$  shall be zero.  $(e_1, \dots, e_7)$  is a basis of  $\mathfrak{m}$  and  $(e_8, \dots, e_{11})$  is a basis of the isotropy algebra  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ .  $\mathfrak{m}$  splits with respect to  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  into the following irreducible submodules:

$$(65) \quad V_1 := \text{span}(e_1, e_2, e_3, e_4)$$

$$(66) \quad V_2 := \text{span}(e_5, e_6)$$

$$(67) \quad V_3 := \text{span}(e_7)$$

As in Section 4, any  $SU(3) \times SU(2)$ -invariant metric on  $M^{1,1,0}$  can be identified with a  $q$ -symmetric, positive definite,  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ -equivariant endomorphism of  $\mathfrak{m}$ . With help of Schur's Lemma, we see that the invariant metrics  $g$  are precisely those with:

$$(68) \quad g = a^2(e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4) + b^2(e^5 \otimes e^5 + e^6 \otimes e^6) + c^2(e^7 \otimes e^7)$$

with  $a, b, c \in \mathbb{R} \setminus \{0\}$ . For the same reasons as in the previous section, we allow  $a, b$ , and  $c$  to take negative values, too.

We fix an orientation and an  $SU(3) \times SU(2)$ -invariant metric  $g$  on  $M^{1,1,0}$ . Let  $\omega$  be an  $SU(3) \times SU(2)$ -invariant  $G_2$ -structure such that its associated metric is  $g$  and  $\omega \wedge * \omega$  is a positive volume form. Our aim is to describe the set  $M$  of all such  $G_2$ -structures. Any invariant  $G_2$ -structure can be described by an  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ -equivariant map  $\psi : \mathfrak{m} \rightarrow \text{Im}(\mathbb{O})$ . Let  $2\mathfrak{u}(1)$  be a Cartan subalgebra of  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ .  $\psi$  is also a  $2\mathfrak{u}(1)$ -equivariant map. The action of  $2\mathfrak{u}(1)$  on  $\mathfrak{m}$  is equivalent to the action of the Cartan subalgebra of  $\mathfrak{g}_2$  on  $\text{Im}(\mathbb{O})$  and thus to the isotropy action in Section 4.  $M$  can therefore be considered as a subset of the  $U(1)$ -orbit from that section.  $U(1)$  acts on  $\mathfrak{m}$  such that its representation is given by

$$(69) \quad T := \left\{ \left( \begin{array}{cccc} \boxed{R_\theta} & & & \\ & \boxed{R_\theta} & & \\ & & \boxed{R_\theta} & \\ & & & \boxed{1} \end{array} \right) \mid \theta \in \mathbb{R} \right\},$$

with respect to the basis  $(e_1, \dots, e_7)$ .  $R_\theta$  again denotes the rotation in the plane around an angle of  $\theta$ . Conjugation by any element of  $T$  leaves not only  $2\mathfrak{u}(1)$  but also  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  invariant. Moreover,  $T$  does not preserve  $\omega$ . Otherwise, the Lie algebra of  $T$  and  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  would generate a subalgebra of rank 3 of  $\mathfrak{g}_2$ . Therefore, the action of  $T$  generates the set of all invariant  $G_2$ -structures which have the same extension to an  $SO(7)$ -structure as  $\omega$ .

These  $G_2$ -structures can also be obtained as the pull-back of  $\omega$  by certain isometries of  $M^{1,1,0}$ . We consider the group:

$$(70) \quad S := \left\{ \left( \begin{array}{ccc|cc} \frac{i\theta}{2} & 0 & 0 & & \\ e^{\frac{i\theta}{2}} & 0 & 0 & & \\ 0 & e^{\frac{i\theta}{2}} & 0 & & \\ 0 & 0 & e^{-i\theta} & & \\ \hline & & & e^{-\frac{i\theta}{2}} & 0 \\ & & & 0 & e^{\frac{i\theta}{2}} \end{array} \right) =: S_\theta \mid \theta \in \mathbb{R} \right\}.$$

$S$  is a subgroup of  $\text{Norm}_{SU(3) \times SU(2)}(SU(2) \times U(1))$ . Conjugation by  $S_\theta$  therefore induces a well-defined diffeomorphism  $f_\theta$  of  $M^{1,1,0}$ . We prove that  $f_\theta$  is even an isometry.  $(df_\theta)_e$  is determined by the adjoint action of  $S_\theta$  restricted to  $\mathfrak{m}$ . Since  $S$  is generated by  $e_7$ , we only have to prove that  $\text{ad}_{e_7}|_{\mathfrak{m}}$  is skew-symmetric with respect to  $g$ . We obtain by a straightforward calculation:

$$(71) \quad \text{ad}_{e_7}|_{\mathfrak{m}} = \left( \begin{array}{ccc|cc|cc|c} 0 & -\frac{3}{2} & & & & & & \\ \frac{3}{2} & 0 & & & & & & \\ \hline & & 0 & -\frac{3}{2} & & & & \\ \frac{3}{2} & & 0 & 0 & & & & \\ \hline & & & & 0 & 1 & & \\ & & & & -1 & 0 & & \\ \hline & & & & & & & 0 \end{array} \right)$$

with respect to  $(e_1, \dots, e_7)$ . This endomorphism is skew-symmetric with respect to any metric of type (68). Moreover, it is orientation-preserving. Unfortunately, the above matrix does not coincide with (69). Nevertheless, it generates the same family of  $G_2$ -structures as  $T$ . The reason for this is that  $T$  and the action of  $S$  on  $\mathfrak{m}$  are both contained in  $\text{Norm}_{SO(7)}(\mathfrak{su}(2) \oplus \mathfrak{u}(1))$ , but not in  $\text{Norm}_{G_2}(\mathfrak{su}(2) \oplus \mathfrak{u}(1))$ . In fact, we could have chosen  $S$  as any connected one-dimensional subgroup of the standard maximal torus of  $SU(3) \times SU(2)$  which is not contained in  $SU(2) \times U(1)$ .

Again, let  $g$  be an arbitrary  $SU(3) \times SU(2)$ -invariant metric on  $M^{1,1,0}$ . Furthermore, we assume without loss of generality that  $\frac{\partial}{\partial t}$  is of unit length. Our next aim is to construct a basis  $(f_i)_{0 \leq i \leq 7}$  of the tangent space which yields an  $SU(3) \times SU(2)$ -invariant  $\text{Spin}(7)$ -structure such that the restriction of its associated metric to the principal orbit is  $g$ .  $(f_i)_{1 \leq i \leq 7}$  has to be orthonormal with respect to  $g$ . Moreover, it should be possible to identify the  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ -modules  $\mathfrak{m}$  and  $\text{Im}(\mathbb{O})$  with each other, such that  $(f_i)_{1 \leq i \leq 7}$  is mapped to the basis  $(i, \dots, k\epsilon)$ . Motivated by these considerations, we choose:

$$(72) \quad \begin{aligned} f_0 &:= \frac{\partial}{\partial t} & f_1 &:= \frac{1}{c}e_7 & f_2 &:= \frac{1}{b}e_6 & f_3 &:= \frac{1}{b}e_5 \\ f_4 &:= \frac{1}{a}e_1 & f_5 &:= \frac{1}{a}e_2 & f_6 &:= \frac{1}{a}e_4 & f_7 &:= \frac{1}{a}e_3 \end{aligned}$$

For similar reasons as in the previous section, this is the most general basis which we have to consider. The four-form  $\Omega$  which is determined by (72) is given by:

$$(73) \quad \begin{aligned} \Omega &= -a^4 e^{1234} + a^2 b^2 e^{1256} + a^2 b^2 e^{3456} \\ &+ a^2 b c e^{1367} - a^2 b c e^{1457} - a^2 b c e^{2357} - a^2 b c e^{2467} \\ &- a^2 b e^{135} \wedge dt - a^2 b e^{146} \wedge dt - a^2 b e^{236} \wedge dt + a^2 b e^{245} \wedge dt \\ &- a^2 c e^{127} \wedge dt - a^2 c e^{347} \wedge dt + b^2 c e^{567} \wedge dt \end{aligned}$$

After having calculated  $d\Omega$ , we are able to express the condition  $d\Omega = 0$  as a system of ordinary differential equations for the functions  $a$ ,  $b$ , and  $c$ :

$$(74) \quad \begin{aligned} \frac{a'}{a} &= \frac{3}{8} \frac{c}{a^2} \\ \frac{b'}{b} &= \frac{1}{4} \frac{c}{b^2} \\ \frac{c'}{c} &= 8 \frac{1}{c} - \frac{1}{4} \frac{c}{b^2} - \frac{3}{4} \frac{c}{a^2} \end{aligned}$$

The above system can be solved explicitly. We denote the initial values  $a(0)$ ,  $b(0)$ , and  $c(0)$  by  $a_0$ ,  $b_0$ , and  $c_0$ . Furthermore, we define  $C(t) := \int_0^t c(s) ds$ . The first two equations of (74) can be simplified to:

$$(75) \quad \begin{aligned} a^2 &= \frac{3}{4} C + a_0^2 \\ b^2 &= \frac{1}{2} C + b_0^2 \end{aligned}$$

As in Section 4, we insert the two above equations into the last one of (74). After that, we are able to deduce an equation for the function  $\tilde{c}(t) := c(C^{-1}(t))$  and find the following solution of our initial value problem:

$$(76) \quad \begin{aligned} \tilde{c}(t)^2 &= \frac{32a_0^4 b_0^2 c_0^2}{9(t + 2b_0^2)(t + \frac{4}{3}a_0^2)^2} \\ &+ \frac{8}{(t + 2b_0^2)(t + \frac{4}{3}a_0^2)^2} \int_0^t (s + 2b_0^2)(s + \frac{4}{3}a_0^2)^2 ds. \end{aligned}$$

$\tilde{c}(t)$  is  $c\sqrt{t} + O(1)$  for  $t \rightarrow \infty$  and a  $c \in \mathbb{R} \setminus \{0\}$ . We can apply the same arguments as in Section 4 and see that the metric is asymptotically conical. The cone which the metric approaches is given by

$$(77) \quad a(t)^2 = \frac{3}{4}t^2, \quad b(t)^2 = \frac{1}{2}t^2, \quad \text{and} \quad c(t) = 2t.$$

The base of the cone carries a nearly parallel  $G_2$ -structure. Conversely, any nearly parallel  $G_2$ -structure on  $M^{1,1,0}$  can be described by  $|a| = \frac{\sqrt{3}}{2}|t|$ ,  $|b| = \frac{1}{\sqrt{2}}|t|$ , and  $c = 2t$  for a  $t \in \mathbb{R} \setminus \{0\}$ .

Since there is an isometric  $U(1)$ -action which preserves the  $\text{Spin}(7)$ -structure, the holonomy is a subgroup of  $SU(4)$ . The existence of a parallel vector field can be excluded by the same arguments as in Section 4. The holonomy therefore is either  $SU(4)$  or contained in  $Sp(2)$ . Any  $SU(3) \times SU(2)$ -invariant two-form on  $M^{1,1,0}$  is a linear combination of:

$$(78) \quad e^{12} + e^{34}, \quad e^{56}, \quad e^7 \wedge dt.$$

Let  $\eta$  be a Kähler form on  $M$ . Analogously to the previous section,  $\eta$  has to satisfy:

$$(79) \quad \eta = \epsilon_1 a^2 e^{12} + \epsilon_1 a^2 e^{34} + \epsilon_2 b^2 e^{56} + \epsilon_3 c e^7 \wedge dt \quad \text{with} \quad \epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}.$$

The condition  $d\eta = 0$  yields:

$$(80) \quad \frac{a'}{a} = \epsilon_1 \epsilon_3 \frac{3}{8} \frac{c}{a^2} \quad \text{and} \quad \frac{b'}{b} = -\epsilon_2 \epsilon_3 \frac{1}{4} \frac{c}{b^2}.$$

If we choose  $\epsilon_1 = \epsilon_3 = 1$  and  $\epsilon_2 = -1$ , the above equations are a consequence of (74) and the Kähler form becomes:

$$(81) \quad \eta = a^2 e^{12} + a^2 e^{34} - b^2 e^{56} + c e^7 \wedge dt.$$

For any other choice of  $\epsilon_1$ ,  $\epsilon_2$ , and  $\epsilon_3$ ,  $\eta$  would change its sign or we would obtain a contradiction to (74). We thus have proven that the holonomy is all of  $SU(4)$ . As in the previous section, our next step is to classify the possible singular orbits:

**Lemma 5.1.** *Let  $SU(2) \times U(1)$  be embedded into  $SU(3) \times SU(2)$  such that the quotient  $(SU(3) \times SU(2))/(SU(2) \times U(1))$  is  $M^{1,1,0}$ . Furthermore, let  $K$  with  $SU(2) \times U(1) \subsetneq K \subseteq SU(3) \times SU(2)$  be a closed, connected subgroup. We denote the Lie algebra of  $K$  by  $\mathfrak{k}$ . In this situation,  $\mathfrak{k}$  and  $K$  can be found in the table below. Furthermore,  $K/(SU(2) \times U(1))$  and  $(SU(3) \times SU(2))/K$  satisfy the following topological conditions:*



$\mathfrak{k}$	$K$	$K/(SU(2) \times U(1))$	$(SU(3) \times SU(2))/K$
$\mathfrak{su}(2) \oplus 2\mathfrak{u}(1)$	$U(2) \times U(1)$	$\cong S^1$	$\cong \mathbb{C}\mathbb{P}^2 \times S^2$
$2\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2) \times SU(2)$	$\cong S^3$	$\cong \mathbb{C}\mathbb{P}^2$
$\mathfrak{su}(3) \oplus \mathfrak{u}(1)$	$SU(3) \times U(1)$	$\cong S^5/\mathbb{Z}_3$	$\cong S^2$
$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$	$SU(3) \times SU(2)$	$= M^{1,1,0} \not\cong S^7/\Gamma$	

In the above table,  $\Gamma \subseteq O(8)$  denotes an arbitrary discrete subgroup and the group  $\mathbb{Z}_3$  which we divide out is  $\{\lambda \text{Id}_{\mathbb{C}^3} | \lambda^3 = 1\}$ .

*Proof.* The Lie algebras  $\mathfrak{k}$  can be classified by the same methods as in the proof of Lemma 4.1.  $\mathbb{C}\mathbb{P}^2$  can be described as  $SU(3)/S(U(2) \times U(1))$ . With help of this fact, we are able to determine the topology of  $(SU(3) \times SU(2))/K$  in all three cases. In order to finish the proof, we have to check if  $K/(SU(2) \times U(1))$  is covered by a sphere. If  $K = U(2) \times U(1)$ , this is true, since the circle is the only one-dimensional, compact, connected, homogeneous space.

Next, we assume that  $K = S(U(2) \times U(1)) \times SU(2)$ .  $\mathfrak{k}$  contains the semisimple part of the isotropy algebra  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  as an ideal. We can therefore cancel one factor of type  $SU(2)$  and see that  $K/(SU(2) \times U(1))$  is covered by a space of type  $(SU(2) \times U(1))/U(1)$ . According to Lemma 4.2, the universal cover of  $K/(SU(2) \times U(1))$  is the sphere  $S^3$ . The long exact homotopy sequence

$$(82) \quad \begin{aligned} \dots &\rightarrow \pi_2(K/(SU(2) \times U(1))) \rightarrow \pi_1(SU(2) \times U(1)) \rightarrow \pi_1(K) \\ &\rightarrow \pi_1(K/(SU(2) \times U(1))) \rightarrow \{0\} \end{aligned}$$

is in our situation given by

$$(83) \quad \dots \rightarrow \{0\} \rightarrow \mathbb{Z} \xrightarrow{\pi_1(i)} \mathbb{Z} \rightarrow \pi_1(K/(SU(2) \times U(1))) \rightarrow \{0\}.$$

The morphism  $\pi_1(i)$  is induced by the inclusion  $i$  of  $SU(2) \times U(1)$  into  $K$ . The homotopy class of the circle  $\{\text{diag}(e^{it}, e^{it}, e^{-2it}, e^{3it}, e^{-3it}) | t \in [0, 2\pi]\}$  generates  $\pi_1(SU(2) \times U(1))$ . This class is mapped by  $\pi_1(i)$  to the homotopy class of  $\{\text{diag}(e^{it}, e^{it}, e^{-2it}, 0, 0) | t \in [0, 2\pi]\}$ , since  $SU(2)$  is simply connected.  $\{\text{diag}(e^{it}, e^{it}, e^{-2it}) | t \in [0, 2\pi]\}$  generates the fundamental group of  $S(U(2) \times U(1))$ .  $\pi_1(i)$  therefore is an isomorphism and  $K/(SU(2) \times U(1))$  is a sphere. We proceed to the case  $K = SU(3) \times U(1)$ : The intersection of the abelian factor of  $SU(2) \times U(1)$  with  $SU(3) \subseteq K$  is:

$$(84) \quad \left\{ \left( \begin{array}{ccc} e^{\frac{2\pi ik}{3}} & 0 & 0 \\ 0 & e^{\frac{2\pi ik}{3}} & 0 \\ 0 & 0 & e^{-\frac{4\pi ik}{3}} \end{array} \right) = e^{\frac{2\pi ik}{3}} \text{Id}_{\mathbb{C}^3} \mid k \in \mathbb{Z} \right\}.$$

We conclude with help of Lemma 4.2 that

$$(85) \quad K/(SU(2) \times U(1)) = (SU(3)/SU(2))/\mathbb{Z}_3 = S^5/\mathbb{Z}_3,$$

where  $\mathbb{Z}_3$  is generated by  $e^{\frac{2\pi i}{3}} \text{Id}_{\mathbb{C}^3}$ . We finally assume that  $M^{1,1,0}$  is diffeomorphic to  $S^7/\Gamma$ . Since  $M^{1,1,0}$  is a circle bundle over  $\mathbb{C}\mathbb{P}^2 \times S^2$ , we have the following exact sequence:

$$(86) \quad \dots \rightarrow \underbrace{\pi_2(S^7/\Gamma)}_{=\{0\}} \rightarrow \underbrace{\pi_2(\mathbb{C}\mathbb{P}^2 \times S^2)}_{=\mathbb{Z}^2} \rightarrow \underbrace{\pi_1(S^1)}_{=\mathbb{Z}} \rightarrow \dots$$

There exists no injective group homomorphism from  $\mathbb{Z}^2$  into  $\mathbb{Z}$  and thus we have proven the lemma.  $\square$

Depending on the singular orbit, we have three types of initial conditions:

- (1) If the singular orbit is  $\mathbb{C}\mathbb{P}^2 \times S^2$ , we have  $c(0) = 0$ .
- (2) If the singular orbit is  $\mathbb{C}\mathbb{P}^2$ , we have  $b(0) = c(0) = 0$ .
- (3) If the singular orbit is  $S^2$ , we have  $a(0) = c(0) = 0$ .

The other initial values are non-zero. For all three cases, we have similar sufficient smoothness conditions:

- (1)  $a$ ,  $b$ , and  $c$  are analytic.
- (2) The metric on  $\mathfrak{p}$  is  $K$ -invariant for all  $t$ .
- (3) The functions which vanish at the singular orbit are odd and the other ones are even.
- (4) If the singular orbit is  $\mathbb{C}\mathbb{P}^2 \times S^2$ , the length of the collapsing circle is  $2\pi t + O(t^2)$  for small values of  $t$ . In the other cases, the sectional curvature of the collapsing sphere has to be  $\frac{1}{t^2} + O(\frac{1}{t})$  for  $t \rightarrow 0$ .

It follows from (75) and (76) that the first condition is satisfied. We consider  $M^{1,1,0}$  as a circle bundle over  $\mathbb{C}\mathbb{P}^2 \times S^2$ . Any choice of  $a(t) \in \mathbb{R} \setminus \{0\}$  yields a multiple of the Fubini-Study metric on  $\mathbb{C}\mathbb{P}^2$ . Analogously, any  $b(t) \in \mathbb{R} \setminus \{0\}$  yields a metric with constant sectional curvature on  $S^2$ . Therefore, the second condition is satisfied, too. Since

$$(87) \quad (a(t), b(t), c(t)) \mapsto (a(-t), b(-t), -c(-t))$$

$$(88) \quad (a(t), b(t), c(t)) \mapsto (a(-t), -b(-t), -c(-t))$$

$$(89) \quad (a(t), b(t), c(t)) \mapsto (-a(-t), b(-t), -c(-t))$$

are symmetries of the system (74), the third condition is also satisfied.

We finally check the fourth condition for each of our cases separately and start with  $\mathbb{C}\mathbb{P}^2 \times S^2$  as singular orbit. Again, let  $S$  be the subgroup of

$SU(3) \times SU(2)$ , which is generated by  $e_7$ . The intersection of  $S$  and the isotropy group  $SU(2) \times U(1)$  is isomorphic to  $\mathbb{Z}_8$ . The smallest  $t > 0$  such that  $\exp(e_7 t)$  is the unit element is  $4\pi$ . By similar arguments as in Section 4, we have  $|c'(0)| = \frac{2\pi}{4\pi} \cdot 8 = 4$ . Unfortunately, it follows from (74) that  $c'(0) = 8$ . The metric thus is not smooth at the singular orbit and we have a singularity which is similar to the singularity of the metrics with principal orbit  $Q^{1,1,1}$  and singular orbit  $S^2 \times S^2 \times S^2$ .

Next, we assume that the singular orbit is  $\mathbb{C}\mathbb{P}^2$ . The collapsing sphere  $S^3$  can be identified with the Lie group  $SU(2)$ . The metric on  $S^3$  with constant sectional curvature 1 is given by  $h(X, Y) = -\frac{1}{2}\text{tr}(XY)$  for all  $X, Y \in \mathfrak{su}(2)$ . The following matrices are an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $h$ :

$$(90) \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The second and the third matrix correspond to  $e_5$  and  $e_6$ , since it is the second factor of  $SU(3) \times SU(2)$  which acts irreducibly on  $\mathfrak{p}^\perp$ . The metric on the collapsing sphere has sectional curvature  $\frac{1}{t^2} + O(\frac{1}{t})$  only if  $\frac{\partial}{\partial t} \Big|_{t=0} \sqrt{g_t(e_5, e_5)} = \frac{\partial}{\partial t} \Big|_{t=0} \sqrt{g_t(e_6, e_6)} = 1$  or equivalently if  $|b'(0)| = 1$ . Since the collapsing circle from the last case is also a great circle of  $S^3$ , we again have  $|c'(0)| = 4$  as smoothness condition. Any solution of (74) with  $b(0) = c(0) = 0$  indeed satisfies  $b'(0) \in \{-1, 1\}$  and  $c'(0) = 4$ . The reason for the different values of  $c'(0)$  in both cases is that we now have to apply l'Hôpital's rule. We remark that it suffices to restrict ourselves to the case  $b'(0) = 1$ , since we can replace  $b$  by  $-b$  without changing (74). All in all, we have proven that our metric is smooth if the singular orbit is  $\mathbb{C}\mathbb{P}^2$ .

We finally assume that the singular orbit is  $S^2$ . By similar arguments as in the previous case, we obtain  $|a'(0)| = 1$  and  $|c'(0)| = 4$  as smoothness conditions. The collapsing sphere is in fact a space form of type  $S^5/\mathbb{Z}_3$ . Nevertheless, we would obtain a smooth orbifold metric if our conditions were satisfied. The system (74) yields  $a'(0) = \pm 1$  and  $c'(0) = \frac{8}{3}$  if  $a(0) = c(0) = 0$ . Our metric therefore has a singularity. At the end of this section, we sum up our results:

**Theorem 5.2.** *Let  $(M, \Omega)$  be a  $Spin(7)$ -orbifold with a cohomogeneity one action of  $SU(3) \times SU(2)$  which preserves  $\Omega$ . We assume that all principal orbits are  $SU(3) \times SU(2)$ -equivariantly diffeomorphic to a space of type  $M^{k,l,0}$ . In this situation, the following statements are true:*

- (1) *The principal orbits are  $SU(3) \times SU(2)$ -equivariantly diffeomorphic to  $M^{1,1,0}$ .*
- (2) *The metric  $g$  which is associated to  $\Omega$  is Ricci-flat and Kähler and its holonomy is all of  $SU(4)$ .*

- (3) *The restriction of  $g$  to a principal orbit can be written as (68). The coefficient functions  $a$ ,  $b$ , and  $c$  satisfy the equations (74), whose solutions are described by (75) and (76). The Kähler form is given by (81).*

*If  $M$  has a singular orbit, which has to be the case if  $(M, g)$  is complete, it is  $\mathbb{C}\mathbb{P}^2 \times S^2$ ,  $\mathbb{C}\mathbb{P}^2$ , or  $S^2$ . Any  $SU(3) \times SU(2)$ -invariant metric on the singular orbit can be extended to a unique complete cohomogeneity one metric with holonomy  $SU(4)$ . For any choice of the singular orbit and its metric,  $(M, g)$  is asymptotically conical.*

- (1) *If the singular orbit is  $\mathbb{C}\mathbb{P}^2 \times S^2$ ,  $M$  is a manifold, but the metric  $g$  is not smooth at the singular orbit.*  
 (2) *If the singular orbit is  $\mathbb{C}\mathbb{P}^2$ ,  $M$  is a manifold and the metric is smooth.*  
 (3) *If the singular orbit is  $S^2$ ,  $M$  is an orbifold but not a manifold. At the singular orbit, the metric is not a smooth orbifold metric. Topologically,  $M$  is a  $\mathbb{C}^3/\mathbb{Z}_3$ -bundle over  $S^2$ .*

*Remark 5.3.* The Kähler metrics with singular orbit  $\mathbb{C}\mathbb{P}^2 \times S^2$  have been considered by Herzog and Klebanov in [19] and by Cvetič et al. in [9]. The examples with singular orbit  $\mathbb{C}\mathbb{P}^2$  or  $S^2$  are mentioned by the same authors in [9] and [12]. Our considerations prove that any complete cohomogeneity one orbifold metric whose principal orbits are  $M^{1,1,0}$  and whose holonomy is a subgroup of  $\text{Spin}(7)$  is one of our examples. This result is, as far as the author knows, new.

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