Geometric structures on Lie groups with flat bi-invariant metric

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Abstract

Let $L \subset V = \mathbb{R}^{k,l}$ be a maximally isotropic subspace. It is shown that any simply connected Lie group with a bi-invariant flat pseudo-Riemannian metric of signature (k, l) is 2-step nilpotent and is defined by an element $\eta \in \Lambda^3 L \subset \Lambda^3 V$. If η is of type $(3, 0)+(0, 3)$ with respect to a skew-symmetric endomorphism J with $J^2 = \epsilon Id$, then the Lie group $\mathcal{L}(\eta)$ is endowed with a left-invariant nearly Kähler structure if $\epsilon = -1$ and with a left-invariant nearly para-Kähler structure if $\epsilon = +1$. This construction exhausts all complete simply connected flat nearly (para-)Kähler manifolds. If $\eta \neq 0$ has rational coefficients with respect to some basis, then $\mathcal{L}(\eta)$ admits a lattice Γ, and the quotient $\Gamma \setminus \mathcal{L}(\eta)$ is a compact inhomogeneous nearly (para-)Kähler manifold. The first non-trivial example occurs in six dimensions.

Contents

Introduction

A pseudo-Riemannian manifold (M, g) endowed with a skew-symmetric almost complex structure J is called nearly Kähler if the Levi-Civita covariant derivative DJ is skew-symmetric, that is $(D_X J)X = 0$ for all $X \in TM$. Nearly Kähler manifolds with a positive definite metric are by now well studied, see [N] and references therein. Replacing the equation $J^2 = -Id$ by $J^2 = Id$ one arrives at the definition of nearly para-Kähler manifold, see [IZ]. This generalises the notion of a para-Kähler (or bi-Lagrangian) manifold. Such manifolds occur naturally in super-symmetric field theories over Riemannian rather than Lorentzian space-times, see [CMMS]. In [IZ] Ivanov and Zamkovoy ask for examples of Ricci-flat nearly para-Kähler manifolds in six dimensions with $DJ \neq 0$. In this paper we will give a classification of flat nearly para-Kähler manifolds. In particular, we will show that there exists a compact six-dimensional such manifold with $DJ \neq 0$.

It is noteworthy that flat nearly para-Kähler manifolds M provide also solutions of the so-called tt*-equations, see [S] and references therein. As a consequence, they give rise to a (para-)pluriharmonic map from M into the pseudo-Riemannian symmetric space $SO_0(n,n)/GL(n)$.

Let V be a pseudo-Euclidian vector space and $\eta \in \Lambda^3 V$. Contraction with η defines a linear map $\Lambda^2 V^* \to V$. The image of that map is denoted by Σ_η and is called the support of η. In the first section we will show that any 3-vector $\eta \in \Lambda^3 V$ with isotropic support defines a simply connected 2-step nilpotent Lie group $\mathcal{L}(\eta)$ with a flat bi-invariant pseudo-Riemannian metric h of the same signature as V. We prove that this exhausts all simply connected Lie groups with a flat bi-invariant metric, see Theorem 2. After completion of our article, Oliver Baues, has kindly informed us about the paper [W], which already contains a version of that result.

It is shown that the groups $(\mathcal{L}(\eta), h)$ admit a lattice $\Gamma \subset \mathcal{L}(\eta)$ if η has rational coefficients with respect to some basis and that the quotient $M(\eta, \Gamma) := \Gamma \setminus \mathcal{L}(\eta)$ is a *flat compact homogeneous pseudo-Riemannian manifold*, see Corollary 3. Compact homogeneous flat pseudo-Riemannian manifolds were recently classified in independent work by Baues, see [B]. It follows from this classification that the above examples exhaust all compact homogeneous flat pseudo-Riemannian manifolds.

Assume now that $\dim V$ is even and that we fix $J \in \mathfrak{so}(V)$ such that $J^2 = -Id$ or $J^2 = Id$. We denote the corresponding left-invariant endomorphism field on the group $\mathcal{L}(\eta)$ again by J. Assume that $\eta \in \Lambda^3 V$ has isotropic support and satisfies, in addition,

$$\{\eta_X, J\} := \eta_X J + J\eta_X = 0 \quad \text{for all} \quad X \in V,$$

or, equivalently, that η has type $(3,0) + (0,3)$. Then $(\mathcal{L}(\eta), h, J)$ is a flat nearly Kähler manifold if $J^2 = -Id$ and a flat nearly para-Kähler manifold if $J^2 = Id$. This follows from the results of [CS] for the former case and is proven in the second section of this paper for the latter case, see Theorem 3. Moreover it is shown that any complete simply connected flat nearly (para-)Kähler manifold is of this form, see Corollary 6 and [CS]. To sum up, we have shown that any simply connected complete flat nearly (para-)Kähler manifold is a Lie group $\mathcal{L}(\eta)$ with a left-invariant nearly (para-)Kähler structure and bi-invariant metric. Conversely, it follows from unpublished work of Paul-Andi Nagy and the first author that a Lie group with a left-invariant nearly (para-)Kähler structure and bi-invariant metric is necessarily flat and is therefore covered by one of our groups $\mathcal{L}(\eta)$. The proof of this statement uses the unicity of the connection with totally skew-symmetric torsion preserving the nearly (para-)Kähler structure and the Jacobi identity.

Suppose now that $\Gamma \subset \mathcal{L}(\eta)$ is a lattice. Then the almost (para-)complex structure J on the group $\mathcal{L}(\eta)$ induces an almost (para-) complex structure J on the compact manifold $M = M(\eta, \Gamma) = \Gamma \backslash \mathcal{L}(\eta)$. Therefore (M, h, J) is a compact nearly (para-)Kähler manifold. However, the (para-)complex structure is not $\mathcal{L}(\eta)$-invariant, unless $\eta = 0$. Moreover, for $\eta \neq 0$, (M, h, J) is an inhomogeneous nearly (para-) Kähler manifold, that is, it does not admit any transitive group of automorphisms of the nearly (para-)Kähler structure. Since J is not right-invariant, this follows from the fact that $\mathrm{Isom}_0(M)$ is obtained from the action of $\mathcal{L}(\eta)$ by right-multiplication on M, see Corollary 3. The first such non-trivial flat compact nearly para-Kähler nilmanifold $M(\eta) = \Gamma \setminus \mathcal{L}(\eta)$ is six-dimensional and is obtained from a non-zero element $\eta \in \Lambda^3 V^+ \cong \mathbb{R}$, where $V^+ \subset V = \mathbb{R}^{3,3}$ is the $+1$-eigenspace of J.

1 A class of flat pseudo-Riemannian Lie groups

Let $V = (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the standard pseudo-Euclidian vector space of signature (k, l), $n = k + l$. Using the (pseudo-Euclidian) scalar product we shall identify $V \cong V^*$ and $\Lambda^2 V \cong \mathfrak{so}(V)$. These identifications provide the inclusion $\Lambda^3 V \subset V^* \otimes \mathfrak{so}(V)$. Using it we consider a three-vector $\eta \in \Lambda^3 V$ as an $\mathfrak{so}(V)$-valued one-form. Further we denote by $\eta_X \in \mathfrak{so}(V)$ the evaluation of this one-form on a vector $X \in V$. The support of $\eta \in \Lambda^3 V$ is defined by

$$\Sigma_\eta := \mathrm{span}\{\eta_X Y \mid X, Y \in V\} \subset V. \tag{1.1}$$

Theorem 1 *Each*

$$\eta \in \mathcal{C}(V) := \{\eta \in \Lambda^3 V \,|\Sigma_\eta \textit{ (totally) isotropic}\} = \bigcup_{L \subset V} \Lambda^3 L$$

defines a 2-step nilpotent simply transitive subgroup $\mathcal{L}(\eta) \subset \mathrm{Isom}(V)$, where the union runs over all maximal isotropic subspaces. The subgroups $\mathcal{L}(\eta)$, $\mathcal{L}(\eta') \subset \mathrm{Isom}(V)$ associated to $\eta, \eta' \in \mathcal{C}(V)$ are conjugated if and only if $\eta' = g \cdot \eta$ for some element of $g \in O(V)$.

Proof: It is easy to see that any three-vector $\eta \in \Lambda^3 V$ satisfies $\eta \in \Lambda^3 \Sigma_\eta$, cf. [CS] Lemma 7. This implies the equation $\mathcal{C}(V) = \bigcup_{L \subset V} \Lambda^3 L$. Let an element $\eta \in \mathcal{C}(V)$ be given. One can easily show that Σ_η is isotropic if and only if the endomorphisms $\eta_X \in \mathfrak{so}(V)$ satisfy $\eta_X \circ \eta_Y = 0$ for all $X, Y \in V$, cf. [CS] Lemma 6. The 2-step nilpotent group

$$\mathcal{L}(\eta) := \left\{ g_X := \exp \begin{pmatrix} \eta_X & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} + \eta_X & X \\ 0 & 1 \end{pmatrix} \,\middle|\, X \in V \right\}$$

acts simply transitively on $V \cong V \times \{1\} \subset V \times \mathbb{R}$ by isometries:

$$\begin{pmatrix} \mathbb{1} + \eta_X & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} X \\ 1 \end{pmatrix}.$$

Let us check that $\mathcal{L}(\eta)$ is a group: Using $\eta_X \circ \eta_Y = 0$ we obtain

$$\begin{aligned} g_X \cdot g_Y &= \begin{pmatrix} \mathbb{1} + \eta_X & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} + \eta_Y & Y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} + \eta_X + \eta_Y + \eta_X \eta_Y & X + Y + \eta_X Y \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1} + \eta_{X+Y} & X + Y + \eta_X Y \\ 0 & 1 \end{pmatrix} = g_{X+Y+\eta_X Y}. \end{aligned}$$

In the last step we used $\eta_{\eta_X Y} = 0$, which follows from $\langle \eta_{\eta_X Y} Z, W \rangle = \langle \eta_Z W, \eta_X Y \rangle$ for all $X, Y, Z, W \in V$. Next we consider $\eta, \eta' \in \mathcal{C}(V)$, $g \in O(V)$. The computation

$$g\mathcal{L}(\eta)g^{-1} = \left\{ \begin{pmatrix} \mathbb{1} + g\eta_X g^{-1} & gX \\ 0 & 1 \end{pmatrix} \,\middle|\, X \in V \right\} = \left\{ \begin{pmatrix} \mathbb{1} + g\eta_{g^{-1}Y} g^{-1} & Y \\ 0 & 1 \end{pmatrix} \,\middle|\, Y \in V \right\}$$

shows that $g\mathcal{L}(\eta)g^{-1} = \mathcal{L}(\eta')$ if and only if $\eta'_X = (g \cdot \eta)_X = g\, \eta_{g^{-1}X}\, g^{-1}$ for all $X \in V$. $\square$

Let $\mathcal{L} \subset \mathrm{Isom}(V)$ be a simply transitive group. Pulling back the scalar product on V by the orbit map

$$\mathcal{L} \ni g \mapsto g0 \in V \tag{1.2}$$

yields a left-invariant flat pseudo-Riemannian metric h on $\mathcal{L}$. A pair $(\mathcal{L}, h)$ consisting of a Lie group $\mathcal{L}$ and a flat left-invariant pseudo-Riemannian metric h on $\mathcal{L}$ is called a flat pseudo-Riemannian Lie group.

Theorem 2

(i) The class of flat pseudo-Riemannian Lie groups $(\mathcal{L}(\eta), h)$ defined in Theorem 1 exhausts all simply connected flat pseudo-Riemannian Lie groups with bi-invariant metric.

(ii) A Lie group with bi-invariant metric is flat if and only if it is 2-step nilpotent.

Proof: (i) The group $\mathcal{L}(\eta)$ associated to a three-vector $\eta \in \mathcal{C}(V)$ is diffeomorphic to $\mathbb{R}^n$ by the exponential map. We have to show that the flat pseudo-Riemannian metric h on $\mathcal{L}(\eta)$ is bi-invariant. The Lie algebra of $\mathcal{L}(\eta)$ is identified with the vector space V endowed with the Lie bracket

$$[X, Y] := \eta_X Y - \eta_Y X = 2\eta_X Y, \quad X, Y \in V. \tag{1.3}$$

The left-invariant metric h on $\mathfrak{L}(\eta)$ corresponds to the scalar product $\langle\cdot,\cdot\rangle$ on V. Since $\eta \in \Lambda^3 V$, the endomorphisms $\eta_X = \frac{1}{2} ad_X$ are skew-symmetric. This shows that h is bi-invariant.

Conversely, let $(V, [\cdot,\cdot])$ be the Lie algebra of a pseudo-Riemannian Lie group of dimension n with bi-invariant metric h. We can assume that the bi-invariant metric corresponds to the standard scalar product $\langle\cdot,\cdot\rangle$ of signature (k,l) on V. Let us denote by $\eta_X \in \mathfrak{so}(V)$, $X \in V$, the skew-symmetric endomorphism of V which corresponds to the Levi-Civita covariant derivative D_X acting on left-invariant vector fields. From the bi-invariance and the Koszul formula we obtain that $\eta_X = \frac{1}{2} ad_X$ and, hence, $R(X,Y) = -\frac{1}{4} ad_{[X,Y]}$ for the curvature. The last formula shows that h is flat if and only if the Lie group is 2-step nilpotent. This proves (ii). To finish the proof of (i) we have to show that, under this assumption, η is completely skew-symmetric and has isotropic support. The complete skew-symmetry follows from $\eta_X = \frac{1}{2} ad_X$ and the bi-invariance. Similarly, using the bi-invariance, we have

$$4\langle \eta_X Y, \eta_Z W\rangle = \langle [X,Y],[Z,W]\rangle = -\langle Y,[X,[Z,W]]\rangle = 0,$$

since the Lie algebra is 2-step nilpotent. This shows that Σ_η is isotropic. $\square$

Corollary 1 *With the above notations, let $L \subset V$ be a maximally isotropic subspace. The correspondence $\eta \mapsto \mathfrak{L}(\eta)$ defines a bijection between the points of the orbit space $\Lambda^3 L/GL(L)$ and isomorphism classes of pairs $(\mathfrak{L}, h)$ consisting of a simply connected Lie group $\mathfrak{L}$ endowed with a flat bi-invariant pseudo-Riemannian metric h of signature (k,l).*

Corollary 2 *Any simply connected Lie group $\mathfrak{L}$ with a flat bi-invariant metric h of signature (k,l) contains a normal subgroup of dimension $\geq \max(k,l) \geq \frac{1}{2}\dim V$ which acts by translations on the pseudo-Riemannian manifold $(\mathfrak{L}, h) \cong \mathbb{R}^{k,l}$.*

Proof: Let $\mathfrak{a} := ker(X \mapsto \eta_X) \subset V$ be the kernel of η. Then $\mathfrak{a} = \Sigma_\eta^\perp$ is co-isotropic and defines an Abelian ideal $\mathfrak{a} \subset \mathfrak{l} := Lie\,\mathfrak{L} \cong V \cong \mathbb{R}^{k,l}$. The corresponding normal subgroup $A \subset \mathfrak{L} = \mathfrak{L}(\eta)$ is precisely the subgroup of translations. So we have shown that $\dim A \geq \max(k,l) \geq \frac{1}{2}\dim V$. $\square$

Remarks 1) The number $\dim \Sigma_\eta$ is an isomorphism invariant of the groups $\mathfrak{L} = \mathfrak{L}(\eta)$, which is independent of the metric. We will denote it by $s(\mathfrak{L})$. Let $L_3 \subset L_4 \subset \cdots \subset L$ be a filtration, where $\dim L_j = j$ runs from 3 to $\dim L$. The invariant $\dim \Sigma_\eta$ defines a decomposition of $\Lambda^3 L/GL(L)$ as a union

$$\{0\} \cup \bigcup_{j=3}^{\dim L} \Lambda^3_{reg} L_j/GL(L_j),$$

where $\Lambda^3_{reg}\mathbb{R}^j \subset \Lambda^3\mathbb{R}^j$ is the open subset of 3-vectors with j-dimensional support. The points of the stratum $\Lambda^3_{reg} L_j/GL(L_j) \cong \Lambda^3_{reg}\mathbb{R}^j/GL(j)$ correspond to isomorphism classes of pairs $(\mathfrak{L}, h)$ with $s(\mathfrak{L}) = j$.

2) Since in the above classification Σ_η is isotropic, it is clear that a flat (or 2-step nilpotent) bi-invariant metric on a Lie group is indefinite, unless $\eta = 0$ and the group is Abelian.

It follows from Milnor's classification of Lie groups with a flat left-invariant Riemannian metric [Mi] that a 2-step nilpotent Lie group with a flat left-invariant Riemannian metric is necessarily Abelian.

Since a nilpotent Lie group with rational structure constants has a (co-compact) lattice [Ma], we obtain:

Corollary 3 *The groups $(\mathfrak{L}(\eta), h)$ admit lattices $\Gamma \subset \mathfrak{L}(\eta)$, provided that η has rational coefficients with respect to some basis. $M = M(\eta, \Gamma) := \Gamma \setminus \mathfrak{L}(\eta)$ is a* flat compact homogeneous pseudo-Riemannian manifold*. The connected component of the identity in the isometry group of M is the image of the natural group homomorphism π from $\mathfrak{L}(\eta)$ into the isometry group of M.*

Proof: First we remark that the bi-invariant metric h induces an $\mathfrak{L}(\eta)$-invariant metric on the homogeneous space $M = \Gamma \setminus \mathfrak{L}(\eta)$. We shall identify the group Γ with a subgroup of the isometry group of $\widetilde{M} := (\mathfrak{L}(\eta), h)$ using the action of Γ on $\mathfrak{L}(\eta)$ by left-multiplication. Let G be the connected component of the identity in the isometry group of $\widetilde{M}$. It is clear that any element of G which commutes with the action of Γ induces an isometry of M. Therefore we have a natural homomorphism $Z_G(\Gamma) \to \mathrm{Isom}(M)$ from the centraliser $Z_G(\Gamma)$ of Γ in G into $\mathrm{Isom}(M)$. In particular, the connected group $Z_G(\Gamma)_0$ is mapped into $\mathrm{Isom}_0(M)$. Conversely, the action of $\mathrm{Isom}_0(M)$ on M can be lifted to the action of a connected Lie subgroup $H \subset G$ on $\widetilde{M}$, which maps cosets of Γ to cosets of Γ. The latter property implies that H normalises the subgroup $\Gamma \subset \mathrm{Isom}(\widetilde{M})$. Since Γ is discrete and G is connected, we can conclude that H is a subgroup of the centraliser $Z_G(\Gamma)$ of Γ in G. As H is connected, we obtain $H \subset Z_G(\Gamma)_0$. By the previous argument, we have also the opposite inclusion $Z_G(\Gamma)_0 \subset H$ and, hence, $H = Z_G(\Gamma)_0$. Now the statement about the isometry group of M follows from the fact that the centraliser in G of the left-action of $\Gamma \subset \mathfrak{L}(\eta)$ on $\mathfrak{L}(\eta)$ is precisely the right-action of $\mathfrak{L}(\eta)$ on $\mathfrak{L}(\eta)$, since $\Gamma \subset \mathfrak{L}(\eta)$ is Zariski-dense, see [R] Theorem 2.1. In fact, this shows that H coincides with the group $\mathfrak{L}(\eta)$ acting by right-multiplication on $\widetilde{M} = \mathfrak{L}(\eta)$ and that $\mathrm{Isom}_0(M)$ coincides with $\mathfrak{L}(\eta)$ acting by right-multiplication on $M = \Gamma \setminus \mathfrak{L}(\eta)$. $\square$

Example 1 *We consider $V = (\mathbb{R}^{3,3}, \langle \cdot, \cdot \rangle)$ and a basis $(e_1, e_2, e_3, f_1, f_2, f_3)$ such that $\langle e_i, f_j \rangle = \delta_{ij}$ and $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$. Then the three-vector $\eta := f_1 \wedge f_2 \wedge f_3 \in \wedge^3 V$ has isotropic support $\Sigma_\eta = span\{f_1, f_2, f_3\}$. The non-vanishing components of the Lie bracket defined by (1.3) are*

$$[e_1, e_2] = 2f_3, [e_2, e_3] = 2f_1, [e_3, e_1] = 2f_2.$$

We have seen above that the bi-invariant metric h was obtained by pulling back the scalar product $\langle \cdot, \cdot \rangle$ by the orbit map (1.2) *which identifies $\mathfrak{L}(\eta)$ with V via $\mathfrak{L}(\eta) \ni g_X \mapsto g_X 0 = X \in V$. The inverse map is $V \ni X \mapsto g_X \in \mathfrak{L}(\eta)$. This identifies the pseudo-Riemannian manifolds $(\mathfrak{L}(\eta), h)$ and $(V, \langle \cdot, \cdot \rangle)$. In consequence the isometry group of $\mathfrak{L}(\eta)$ is isomorphic to the full affine pseudo-orthogonal group operating by $g_X \mapsto g_{AX+v}$ with $A \in O(V)$ and $v \in V$. Next we consider the lattice*

$$\Gamma := \{g_Y | Y \in \mathbb{Z}^6\},$$

where $\mathbb{Z}^6 \subset V$ is the lattice of integral vectors with respect to the basis $(e_1, e_2, e_3, f_1, f_2, f_3)$. An element $g_Y \in \Gamma$ operates from the left on $\mathcal{L}(\eta) \cong V$ as

$$X \mapsto (\mathbb{1} + \eta_Y)X + Y.$$

Let us determine the centraliser of this Γ-action in the isometry group of $\mathcal{L}(\eta)$. A short calculation shows that an affine isometry (A, v) with linear part $A \in O(V)$ and translational part $v \in V$ belongs to the centraliser of Γ if and only if

$$[\eta_Y, A]X + \eta_Y v - AY + Y = 0$$

for all $X \in V$, $Y \in \mathbb{Z}^6$. For $X = 0$ we get $AY = \eta_Y v + Y = (\mathbb{1} - \eta_v)Y$ and, hence, $A = \mathbb{1} - \eta_v$. This shows that the affine transformation (A, v) corresponds to the right action of the element g_v, which obviouly belongs to the centraliser. Therefore, in this example, we have proven by direct calculation that the centraliser in the isometry group of $\mathcal{L}(\eta)$ of Γ acting by left-multiplication on $\mathcal{L}(\eta)$ is precisely the group $\mathcal{L}(\eta)$ acting by right-multiplication. This fact was proven for arbitrary groups $\mathcal{L}(\eta)$ and lattices Γ in the proof of Corollary 3.

2 Flat nearly para-Kähler manifolds

In this section we give a constructive classification of flat nearly para-Kähler manifolds and show that such manifolds provide a class of examples for the flat Lie groups discussed in section 1. The structure of the section is as follows. In the first subsection we give a short introduction to para-complex geometry. For more information the reader is referred to [CMMS]. The second part discusses nearly para-Kähler manifolds and derives some consequences of the flatness. In the third subsection we give a local classification which relates a flat nearly para-Kähler manifold to an element of a certain subset $\mathcal{C}_\tau(V)$ of the cone $\mathcal{C}(V) \subset \wedge^3 V$ defined in Theorem 1. The structure of $\mathcal{C}_\tau(V)$ is studied in the last subsection and global classification results are derived.

2.1 Para-complex geometry

The idea of para-complex geometry is to replace the complex structure J satisfying $J^2 = -Id$ on a (finite) dimensional vector space V by a para-complex structure τ satisfying $\tau^2 = Id$ and to require that the two eigenspaces of τ, i.e. $V^\pm := \ker(Id \mp \tau)$, have the same dimension. A para-complex vector space (V, τ) is a vector space endowed with a para-complex structure. Para-complex, para-Hermitian and para-Kähler geometry was first studied in [L]. We invite the reader to consult [CFG] or the more recent article [AMT] for a survey on this subject.

Definition 1 *An* almost para-complex structure *τ on a smooth manifold M is an endomorphism field $\tau \in \Gamma(End(TM), p \mapsto \tau_p$, such that τ_p is a para-complex structure on T_pM for all points $p \in M$. A manifold endowed with an almost para-complex structure is called an* almost para-complex manifold.

An almost para-complex structure is called integrable *if its eigendistributions* $T^{\pm}M := \ker(Id \mp \tau)$ *are both integrable. A manifold endowed with an integrable almost para-complex structure is called a* para-complex manifold.

We remark that the obstruction to integrability (cf. Proposition 1 of [CMMS]) of an almost para-complex structure is the Nijenhuis tensor of τ, which is the tensor field defined by

$$N_\tau(X,Y) := [X,Y] + [\tau X, \tau Y] - \tau[X, \tau Y] - \tau[\tau X, Y],$$

for all vector fields X, Y on M.

Definition 2 *Let* (V, τ) *be a para-complex vector space. A* para-Hermitian scalar product g *on* (V,τ) *is a pseudo-Euclidian scalar product, such that* $\tau^* g(\cdot,\cdot) = g(\tau\cdot, \tau\cdot) = -g(\cdot,\cdot)$. *A* para-Hermitian vector space *is a para-complex vector space endowed with a para-Hermitian scalar product. The pair* (τ, g) *is called* para-Hermitian structure *on the vector space* V.

The next two examples give two frequently used models of para-Hermitian structures:

Example 2 *Let us consider the vector space* $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ *and denote by* $e_i^+ = e_i \oplus 0$ *and* $e_i^- = 0 \oplus e_i$ *its standard basis. Its standard para-complex structure is given by* $\tau e_i^{\pm} = \pm e_i^{\pm}$. *A para-Hermitian scalar product* g *is given by* $g(e_i^{\pm}, e_j^{\pm}) = 0$ *and* $g(e_i^{\pm}, e_j^{\mp}) = \delta_{ij}$. *We call the pair* (τ, g) *the* standard para-Hermitian structure *of* $\mathbb{R}^{2n}$.

Example 3 *We denote by* $C = \mathbb{R}[e] \cong \mathbb{R} \oplus \mathbb{R}$, $e^2 = 1$, *the ring of para-complex numbers. Consider the real vector space* $C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$ *with standard basis given by* $(e_1, \dots, e_n, f_1, \dots, f_n)$, *where* $f_i = ee_i$ *and its standard para-complex structure which is defined by* $\tau e_i = f_i$ *and* $\tau f_i = e_i$. *Then we can define a para-Hermitian scalar product by* $g(e_i, e_j) = -g(f_i, f_j) = \delta_{ij}$ *and* $g(e_i, f_j) = 0$. *We denote this pair* (τ, g) *the* standard para-Hermitian structure *of* C^n.

The decomposition of the cotangent bundle $T^*M = (T^*M)^+ \oplus (T^*M)^-$ with respect to the dual para-complex structure induces a bi-grading on the bundle of exterior forms $\Lambda^k T^*M = \oplus_{k=p+q} \Lambda^{p,q} T^*M$. An element of $\Lambda^{p,q}T^*M$ will be called of type (p,q). The corresponding decomposition on differential forms is denoted by $\Omega^k(M) = \oplus_{k=p+q} \Omega^{p,q}(M)$.

Definition 3 *An* almost para-Hermitian manifold (M, τ, g) *is an almost para-complex manifold* (M, τ) *which is endowed with a pseudo-Riemannian metric* g *which is* para-Hermitian, *i.e. it satisfies* $\tau^* g(\cdot,\cdot) = g(\tau\cdot, \tau\cdot) = -g(\cdot,\cdot)$.

Note that the condition on the metric to be para-Hermitian forces it to have split signature (n, n).

2.2 Basic facts and results about nearly para-Kähler manifolds

The notion of a *nearly para-Kähler manifold* was recently introduced by Ivanov and Zamkovoy [IZ].

Definition 4 *An almost para-Hermitian manifold (M, τ, g) is called* nearly para-Kähler *manifold, if its Levi-Civita connection D satisfies the equation*

$$(D_X \tau)Y = -(D_Y \tau)X, \quad \forall X, Y \in \Gamma(TM). \tag{2.1}$$

A nearly para-Kähler manifold is called strict*, if $D\tau \neq 0$.*

Like for a nearly Kähler manifold there exists a canonical para-hermitian connection with totally skew-symmetric torsion.

Proposition 1 *[Prop. 5.1 in [IZ]] Let (M, τ, g) be a nearly para-Kähler manifold. Then there exists a unique connection ∇ with totally skew-symmetric torsion T^∇ (i.e. $g(T^\nabla(\cdot,\cdot),\cdot)$ is a three-form) satisfying $\nabla g = 0$ and $\nabla \tau = 0$.*

More precisely, this connection is given by

$$\nabla_X Y = D_X Y - \eta_X Y \text{ with } \eta_X Y = -\frac{1}{2}\tau(D_X \tau)Y \text{ and } X, Y \in \Gamma(TM) \tag{2.2}$$

and in consequence the torsion is

$$T^\nabla = -2\eta \tag{2.3}$$

and one has $\{\eta_X, \tau\} = 0$ for all vector fields X. In the same reference [IZ] Theorem 5.3 it is shown that, as in the nearly Kähler case, the torsion of ∇ is parallel, i.e.

$$\nabla \eta = 0 \text{ and } \nabla(D\tau) = 0. \tag{2.4}$$

Proposition 2 *Let (M, g, τ) be a flat nearly para-Kähler manifold, then*

1) *$\eta_X \circ \eta_Y = 0$ for all X, Y,*

2) *$D\eta = \nabla \eta = 0$.*

Proof: On a nearly para-Kähler manifold one has the identity

$$R^D(X,Y,Z,W) + R^D(X,Y,\tau Z,\tau W) = g((D_X\tau))Y, (D_Z\tau)W),$$

cf. [IZ] Proposition 5.2. For a flat nearly para-Kähler manifold it follows

$$g((D_X\tau)Y, (D_Z\tau)W) = 0 \quad \forall X, Y, Z, W. \tag{2.5}$$

With this identity and $D\tau \circ \tau = -\tau \circ D\tau$ we obtain

$$0 = g((D_X\tau)Y, (D_Z\tau)W) \;\; = \;\; -g((D_Z\tau)(D_X\tau)Y, W) = 4g(\eta_Z \circ \eta_X Y, W).$$

This shows $\eta_X \circ \eta_Y = 0$ for all X, Y and finishes the proof of part 1.).
2.) With two vector fields X, Y we calculate

$$\begin{aligned}(D_X\eta)_Y &= D_X(\eta_Y) - \eta_{D_X Y} \overset{D=\nabla+\eta}{=} \nabla_X(\eta_Y) + [\eta_X, \eta_Y] - \eta_{D_X Y} \\ &= (\nabla_X \eta)_Y + \eta_{[\nabla_X Y - D_X Y]} + [\eta_X, \eta_Y] = (\nabla_X\eta)_Y - \eta_{\eta_X Y} + [\eta_X, \eta_Y] \\ &\overset{(2.1)}{=} (\nabla_X\eta)_Y + \eta.\eta_X Y + [\eta_X, \eta_Y] \overset{1.)}{=} (\nabla_X \eta)_Y \overset{(2.4)}{=} 0.\end{aligned}$$

This is part 2). □

2.3 Local classification of flat nearly para-Kähler manifolds

We consider (C^n, τ_{can}) endowed with the standard τ_{can}-anti-invariant pseudo-Euclidian scalar product g_{can} of signature (n, n).

Let (M, g, τ) be a flat nearly para-Kähler manifold. Then there exists for each point $p \in M$ an open set $U_p \subset M$ containing the point p, a connected open set U_0 of C^n containing the origin $0 \in C^n$ and an isometry $\Phi : (U_p, g) \tilde{\to} (U_0, g_{can})$, such that in $p \in M$ we have $\Phi_* \tau_p = \tau_{can} \Phi_*$. In other words, we can suppose, that locally M is a connected open subset of C^n containing the origin 0 and that $g = g_{can}$ and $\tau_0 = \tau_{can}$. Summarizing Proposition 1 and 2 we obtain the next Corollary.

Corollary 4 *Let $M \subset C^n$ be an open neighborhood of the origin endowed with a nearly para-Kähler structure (g, τ) such that $g = g_{can}$ and $\tau_0 = \tau_{can}$. The $(1,2)$-tensor*

$$\eta := -\frac{1}{2}\tau D\tau$$

defines a constant three-form on $M \subset C^n = \mathbb{R}^{n,n}$ given by $\eta(X, Y, Z) = g(\eta_X Y, Z)$ and satisfying

(i) $\eta \in \mathcal{C}(V)$, *i.e.* $\eta_X \, \eta_Y = 0, \quad \forall X, Y,$

(ii) $\{\eta_X, \tau_{can}\} = 0, \quad \forall X.$

The rest of this subsection is devoted to the local classification result. In subsection 2.4 we study the structure of the subset of $\mathcal{C}(V)$ given by the condition (ii) in more detail and give global classification results. The converse statement of Corollary 4 is given in the next lemma.

Lemma 1 *Let η be a constant three-form on an open connected set $M \subset C^n$ of 0 satisfying (i) and (ii) of Corollary 4. Then there exists a unique para-complex structure τ on M such that*

a) $\tau_0 = \tau_{can}$,

b) $\{\eta_X, \tau\} = 0, \quad \forall X,$

c) $D\tau = -2\tau\eta,$

where D is the Levi-Civita connection of the pseudo-Euclidian vector space C^n.
Let $\nabla := D - \eta$ and assume b) then c) is equivalent to

c)' $\nabla\tau = 0.$

Furthermore, this para-complex structure τ is skew-symmetric with respect to g_{can}.

Proof: One proves the equivalence of c) and c)' by an easy computation.
Let us show the uniqueness: Given two almost para-complex structures satisfying a)-c) we deduce $(\tau - \tau')_0 = 0$ and $\nabla\tau = \nabla\tau' = 0$. This shows $\tau \equiv \tau'$. To show the existence we define

$$\tau \;=\; \exp\left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\tau_{can} \overset{(i)}{=} \left(Id + 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\tau_{can}, \tag{2.6}$$

where x^i are linear coordinates of $C^n = \mathbb{R}^{n,n} = \mathbb{R}^{2n}$ and $\partial_i = \frac{\partial}{\partial x^i}$.
Claim: τ defines a para-complex structure which satisfies a)-c).
a) From $x^i(0) = 0$ we obtain $\tau_0 = \tau_{can}$.
b) Follows from the definition of τ (cf. equation (2.6)) and the properties (i) and (ii).
c) One computes

$$D_{\partial_j}\tau \;=\; 2\exp\left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\eta_{\partial_j}\tau_{can} \overset{(ii)}{=} \underbrace{-2\exp\left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\tau_{can}}_{\tau}\,\eta_{\partial_j} \;=\; -2\tau\,\eta_{\partial_j}.$$

It holds $\tau = \tau_{can} + \left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\tau_{can}$, where $\{\eta_{\partial_i}, \tau_{can}\} = 0$ and η_{∂_i} is g-skew-symmetric. This implies that τ is g-skew-symmetric. It remains to prove $\tau^2 = Id$.

$$\begin{aligned}\tau^2 \;&=\; \left(Id + 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\tau_{can}\left(Id + 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\tau_{can}\\ &=\; \left(Id + 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\left(Id - 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right) \;=\; \left[Id - 4\left(\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)^2\right] \overset{(i)}{=} Id.\end{aligned}$$

This finishes the proof of the lemma. □

Theorem 3 *Let η be a constant three-form on a connected open set $U \subset C^n$ containing the origin 0 which satisfies (i) and (ii) of Corollary 4. Then there exists a unique almost para-complex structure*

$$\tau = \exp\left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\tau_{can} \tag{2.7}$$

on U such that a) $\tau_0 = \tau_{can}$, and b) $M(U,\eta) := (U, g = g_{can}, \tau)$ is a flat nearly para-Kähler manifold. Any flat nearly para-Kähler manifold is locally isomorphic to a flat nearly para-Kähler manifold of the form $M(U,\eta)$.

Proof: (M, g) is a flat pseudo-Riemannian manifold. Due to Lemma 1 τ, is a skew-symmetric almost para-complex structure on M and $\tau_0 = \tau_{can}$. From Lemma 1 c) and the skew-symmetry of η it follows the skew-symmetry of $D\tau$. Therefore (M, g, τ) is a nearly para-Kähler manifold. The remaining statement follows from Corollary 4 and Lemma 1.
□

2.4 The variety $\mathcal{C}_\tau(V)$

Now we discuss the solution of (i) and (ii) of Corollary 4. In the following we shall freely identify the real vector space $V := C^n = \mathbb{R}^{n,n} = \mathbb{R}^{2n}$ with its dual V^* by means of the pseudo-Euclidian scalar product $g = g_{can}$. The geometric interpretation is given in terms of an affine variety $\mathcal{C}_\tau(V) \subset \Lambda^3 V$.

Proposition 3 *A three-form $\eta \in \Lambda^3 V^* \cong \Lambda^3 V$ satisfies (i) of Corollary 4, i.e. $\eta_X \circ \eta_Y = 0, X, Y, \in V$, if and only if there exists an isotropic subspace $L \subset V$ such that $\eta \in \Lambda^3 L \subset \Lambda^3 V$. If η satisfies (i) and (ii) of Corollary 4 then there exists a τ_{can}-invariant isotropic subspace $L \subset V$ with $\eta \in \Lambda^3 L$.*

Proof: The proposition follows from the next lemma by taking $L = \Sigma_\eta$. □

Lemma 2

1. *Σ_η is isotropic if and only if η satisfies (i) of Corollary 4. If η satisfies (ii) of Corollary 4, then Σ_η is τ_{can}-invariant.*
2. *Let $\eta \in \Lambda^3 V$. Then $\eta \in \Lambda^3 \Sigma_\eta$.*

Proof: The proof of the first part is analogous to Lemma 6 in [CS]. The second part is Lemma 7 of [CS]. □

Any three-form η on (V, τ_{can}) decomposes with respect to the grading induced by the decomposition $V = V^{1,0} \oplus V^{0,1}$ into $\eta = \eta^+ + \eta^-$ with $\eta^+ \in \Lambda^+ V := \Lambda^{2,1} V + \Lambda^{1,2} V$ and $\eta^- \in \Lambda^- V := \Lambda^{3,0} V + \Lambda^{0,3} V$.

Theorem 4 *A three-form $\eta \in \Lambda^3 V^* \cong \Lambda^3 V$ satisfies (i) and (ii) of Corollary 4 if and only if there exists an isotropic τ_{can}-invariant subspace L such that $\eta \in \Lambda^- L = \Lambda^{3,0} L + \Lambda^{0,3} L \subset \Lambda^3 L \subset \Lambda^3 V$ (The smallest such subspace L is Σ_η.).*

We need the following general Lemma.

Lemma 3 *It is*

$$\Lambda^- V = \{\eta \in \Lambda^3 V \,|\, \eta(\cdot, \tau\cdot, \tau\cdot) = \eta(\cdot,\cdot,\cdot)\} = \{\eta \in \Lambda^3 V \,|\, \{\eta_X, \tau\} = 0, \ \forall X \in V\}.$$

Proof: (of Theorem 4) By Proposition 3, the conditions (i) and (ii) of Corollary 4 imply the existence of an isotropic τ_{can}-invariant subspace $L \subset V$ such that $\eta \in \Lambda^3 L$. The last lemma shows that the condition (ii) is equivalent to $\eta \in \Lambda^- V$. Therefore $\eta \in \Lambda^3 L \cap \Lambda^- V = \Lambda^- L$. The converse statement follows from the same argument. □

Corollary 5

(i) *The conical affine variety $\mathcal{C}_\tau(V) := \{\eta \,|\, \eta$ satisfies (i) and (ii) in Corollary 4 $\} \subset \Lambda^3 V$ has the following description $\mathcal{C}_\tau(V) = \bigcup_{L \subset V} \Lambda^- L = \bigcup_{L \subset V} (\Lambda^3 L^+ + \Lambda^3 L^-)$, where the union is over all τ-invariant maximal isotropic subspaces.*

(ii) If $\dim V < 12$ *then it holds* $\mathcal{C}_\tau(V) = \Lambda^3 V^+ \cup \Lambda^3 V^-$.

(iii) Any flat nearly para-Kähler manifold M *is locally of the form* $M(U, \eta)$*, for some* $\eta \in \mathcal{C}_\tau(V)$ *and some open subset* $U \subset V$.

(iv) There are no strict flat nearly para-Kähler manifolds of dimension less than 6.

Proof: (i) follows from Theorem 4.
(ii) Let $L \subset V$ be a τ-invariant isotropic subspace. If $\dim V < 12$, then $\dim L < 6$ and, hence, either $\dim L^+ < 3$ or $\dim L^- < 3$. In the first case we have

$$\Lambda^- L = \Lambda^3 L^+ + \Lambda^3 L^- = \Lambda^3 L^- \subset \Lambda^3 V^-,$$

in the second case it is $\Lambda^- L = \Lambda^3 L^+ + \Lambda^3 L^- = \Lambda^3 L^+ \subset \Lambda^3 V^+$.
(iii) is a consequence of (i), Theorem 3 and 4.
(iv) By (iii) the strict nearly para-Kähler manifold M is locally of the form $M(U, \eta)$, which is strict if and only if $\eta \neq 0$. This is only possible for $\dim L \geq 3$, i.e. for $\dim M \geq 6$. □

Example 4 *We have the following example which shows that part (ii) of Corollary 5 fail in dimension* ≥ 12*: Consider* $(V, \tau) = (C^6, e) = \mathbb{R}^6 \oplus e\mathbb{R}^6$ *with a basis given by* $(e_1^+, \ldots, e_6^+, e_1^-, \ldots, e_6^-)$*, such that* $e_i^\pm$ *form a basis of* $V^\pm$ *with* $g(e_i^+, e_j^-) = \delta_{ij}$*. Then the form* $\eta := e_1^+ \wedge e_2^+ \wedge e_3^+ + e_4^- \wedge e_5^- \wedge e_6^-$ *lies in the variety* $\mathcal{C}_\tau(V)$.

Theorem 5 *Any strict flat nearly para-Kähler manifold is locally a pseudo-Riemannian product* $M = M_0 \times M(U, \eta)$ *of a flat para-Kähler factor* M_0 *of maximal dimension and a flat nearly para-Kähler manifold* $M(U, \eta)$*,* $\eta \in C_\tau(V)$*, of signature* (m, m)*,* $2m = \dim M(U, \eta) \geq 6$ *such that* Σ_η *has dimension* m.

Proof: By Theorem 3 and 4, M is locally isomorphic to an open subset of a manifold of the form $M(V, \eta)$, where $\eta \in \Lambda^3 V$ has a τ_{can}-invariant and isotropic support $L = \Sigma_\eta$. We choose a τ_{can}-invariant isotropic subspace $L' \subset V$ such that $V' := L + L'$ is non-degenerate and $L \cap L' = 0$ and put $V_0 = (L + L')^\perp$. Then $\eta \in \Lambda^3 V' \subset \Lambda^3 V$ and $M(V, \eta) = M(V_0, 0) \times M(V', \eta)$. Notice that $M(V_0, 0)$ is simply the flat para-Kähler manifold V_0 and that $M(V', \eta)$ is strict of split signature (m, m), where $m = \dim L \geq 3$. □

Corollary 6 *Any simply connected nearly para-Kähler manifold with a (geodesically) complete flat metric is a pseudo-Riemannian product* $M = M_0 \times M(\eta)$ *of a flat para-Kähler factor* $M_0 = \mathbb{R}^{l,l}$ *of maximal dimension and a flat nearly para-Kähler manifold* $M(\eta) := M(V, \eta)$*,* $\eta \in C_\tau(V)$*, of signature* (m, m) *such that* Σ_η *has dimension* $m = 0, 3, 4, \ldots$.

Next we wish to describe the moduli space of (complete simply connected) flat nearly para-Kähler manifolds M of dimension $2n$ up to isomorphism. Without restriction of generality we will assume that $M = M(\eta)$ has no para-Kähler de Rham factor, which means that

$\eta \in C_\tau(V)$ has maximal support Σ_η, i.e. $\dim \Sigma_\eta = n$. We denote by $C_\tau^{reg}(V) \subset C_\tau(V)$ the open subset consisting of elements with maximal support. The group

$$G := \mathrm{Aut}(V, g_{can}, \tau_{can}) \cong GL(n)$$

acts on $C_\tau(V)$ and preserves $C_\tau^{reg}(V)$. Two nearly para-Kähler manifolds $M(\eta)$ and $M(\eta')$ are isomorphic if and only if η and η' are related by an element of the group G.

For $\eta \in C_\tau(V)$ we denote by p, q the dimensions of the eigenspaces of τ on Σ_η for the eigenvalues $1, -1$, respectively. We call the pair $(p, q) \in \mathbb{N}_0 \times \mathbb{N}_0$ the type of η. We will also say that the corresponding flat nearly para-Kähler manifold $M(\eta)$ has type (p, q). We denote by $C_\tau^{p,q}(V)$ the subset of $C_\tau(V)$ consisting of elements of type (p, q). Notice that $p + q \leq n$ with equality if and only if $\eta \in C_\tau^{reg}(V)$. We have the following decomposition

$$C_\tau^{reg}(V) = \bigcup_{(p,q)\in\Pi} C_\tau^{p,q}(V),$$

where $\Pi := \{(p, q)|p, q \in \mathbb{N}_0 \setminus \{1, 2\}, p + q = n\}$. The group $G = GL(n)$ acts on the subsets $C_\tau^{p,q}(V)$ and we are interested in the orbit space $C_\tau^{p,q}(V)/G$.

Fix a τ-invariant maximally isotropic subspace $L \subset V$ of type (p, q) and put $\Lambda^-_{reg}L := \Lambda^- L \cap C_\tau^{reg}(V) \subset C_\tau^{p,q}(V)$. The stabilizer $G_L \cong GL(L^+) \times GL(L^-) \cong GL(p) \times GL(q)$ of $L = L^+ + L^-$ in G acts on $\Lambda^-_{reg}L$.

Theorem 6 *There is a natural one-to-one correspondence between complete simply connected flat nearly para-Kähler manifolds of type (p, q), $p+q = n$, and the points of the following orbit space:*

$$C_\tau^{p,q}(V)/G \cong \Lambda^-_{reg}L/G_L \subset \Lambda^- L/G_L = \Lambda^3 L^+/GL(L^+) \times \Lambda^3 L^-/GL(L^-).$$

Proof: Consider two complete simply connected flat nearly para-Kähler manifolds M, M'. By the previous results we can assume that $M = M(\eta)$, $M' = M(\eta')$ are associated with $\eta, \eta' \in C_\tau^{p,q}(V)$. It is clear that M and M' are isomorphic if η and η' are related by an element of G. To prove the converse we assume that $\varphi : M \to M'$ is an isomorphism of nearly para-Kähler manifolds. By the results of Section 1 η defines a simply transitive group of isometries. This group preserves also the para-complex structure τ, which is ∇-parallel and hence left-invariant. This shows that M and M' admit a transitive group of automorphisms. Therefore, we can assume that φ maps the origin in $M = V$ to the origin in $M' = V$. Now φ is an isometry of pseudo-Euclidian vector spaces preserving the origin. Thus φ is an element of $O(V)$ preserving also the para-complex structure τ and hence $\varphi \in G$.

The identification of orbit spaces can be easily checked using Lemma 2 2. and the fact that any τ-invariant isotropic subspace $\Sigma = \Sigma^+ + \Sigma^-$ can be mapped onto L by an element of G. □

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