HAMBURGER BEITRÄGE
ZUR MATHEMATIK

Heft 165

The Erdős-Menger conjecture for source/sink sets with disjoint closures

Reinhard Diestel

Januar 2003

The Erdős-Menger conjecture for source/sink sets with disjoint closures

Reinhard Diestel

Erdős conjectured that, given an infinite graph G and vertex sets $A, B \subseteq V(G)$, there exist a set $\mathcal{P}$ of disjoint A–B paths in G and an A–B separator X 'on' $\mathcal{P}$, in the sense that X consists of a choice of one vertex from each path in $\mathcal{P}$. We prove the conjecture for vertex sets A and B that have disjoint closures in the usual topology on graphs with ends. The result can be extended by allowing A, B and X to contain ends as well as vertices.

1. Introduction

The following conjecture of Erdős is perhaps the main open problems in infinite graph theory:

Erdős-Menger Conjecture. *For every graph $G = (V, E)$ and any two sets $A, B \subseteq V$ there is a set $\mathcal{P}$ of disjoint A–B paths in G and an A–B separator X consisting of a choice of one vertex from each of the paths in $\mathcal{P}$.*

The conjecture appears in print first in Nash-Williams's 1967 survey [11] on infinite graphs, although it seems to be considerably older. It was proved by Aharoni for countable graphs [3], and by Aharoni, Nash-Willliams and Shelah [2, 6] for bipartite graphs G with bipartition (A, B). As shown by Aharoni [1], the bipartite result implies the conjecture for rayless graphs. The current state of the art, including further partial results by other authors, is described in Aharoni [4].

Our main result in this paper is the following:

Theorem 1.1. *Every graph G satisfies the Erdős-Menger conjecture for all vertex sets A and B that have disjoint closures in $|G|$.*

Here, $|G|$ denotes the topological space usually associated with G and its ends, to be defined formally in Section 2. Although Theorem 1.1 is most naturally stated in these terms, it can easily be rephrased without formally referring to $|G|$: the sets A and B have disjoint closures in $|G|$ if and only if $A \cap B = \emptyset$

and every infinite path in G can be separated from A or from B by a finite set of vertices.

In [8] the Erdős-Menger conjecture has been generalized to sets A and B that may include ends as well as vertices (in which case the paths in $\mathcal{P}$ may be rays or double rays between these ends or vertices, and the separator X may also contain ends from A or B), and proved in this more general form for countable G. Theorem 1.1, too, generalizes in this way:

Theorem 1.2. *Every graph $G = (V, E, \Omega)$ satisfies the Erdős-Menger conjecture for all sets $A, B \subseteq V \cup \Omega$ that have disjoint closures in $|G|$.*

(Here, V and Ω denote the set of vertices and ends of G, respectively. The precise definitions of A–B paths and A–B separators for arbitrary sets $A, B \subseteq V \cup \Omega$ are what one expects; see [8].)

Thus, formally, Theorem 1.1 is just a special case of Theorem 1.2. In the interest of readability, however, we shall prove Theorem 1.1 directly, and merely sketch its extension to ends. This extension, though not short, is not difficult given the main result of [5] and the techniques from [8], and the main focus of this paper is intended as a contribution towards the Erdős-Menger conjecture itself.

2. Terminology and basic tools

The basic terminology we use is that of [7] – except that most of our graphs will be infinite, and $|G|$ will denote a certain topological space associated with a graph G, not its order. Our graphs are simple and undirected, but the result we prove can easily be adapted to directed graphs.

An infinite path that has a first but no last vertex is a *ray*; a path with neither a first nor a last vertex is a *double ray*. The subrays of a ray are its *tails*. Any union of a ray R and infinitely many disjoint finite paths ending on R but otherwise disjoint from R is a *comb* with *back* R; the starting vertices of those paths are the *teeth* of the comb. (Note that the paths may be trivial, ie. the teeth of a comb may lie on its back.)

Two rays in a graph $G = (V, E)$ are *equivalent* if no finite set of vertices separates them in G. The corresponding equivalence classes of rays are the *ends* of G; the set of these ends is denoted by $\Omega = \Omega(G)$, and G together with its ends is referred to as $G = (V, E, \Omega)$. (The grid, for example, has one end, the double ladder has two, and the binary tree has continuum many.) We shall endow our graphs G, complete with vertices, edges and ends, with a standard topology to be defined below. (When G is locally finite, this is its "Freudenthal compactification".) This topological space will be denoted by $|G|$, and the closure in $|G|$ of a subset X will be written as $\overline{X}$. See [9] for more background on ends and this topology.

To define $|G|$, we start with G viewed as a 1-complex. Then every edge is homeomorphic to the real interval $[0,1]$, the basic open sets around an inner point being just the open intervals on the edge. The basic open neighbourhoods of a vertex x are the unions of half-open intervals $[x,z)$, one from every edge $[x,y]$ at x; note that we do not require local finiteness here.

For $\omega \in \Omega$ and any finite set $S \subseteq V$, the graph $G - S$ has exactly one component $C = C(S,\omega)$ that contains a tail of every ray in ω. We say that ω *belongs* to C. Write $\Omega(S,\omega)$ for the set of all ends of G belonging to C, and $E(S,\omega)$ for the set of all edges of G between S and C. Now let $|G|$ be the point set $V \cup \Omega \cup \bigcup E$ endowed with the topology generated by the open sets of the 1-complex G and all sets of the form

$$\widehat{C}(S,\omega) := C(S,\omega) \cup \Omega(S,\omega) \cup E'(S,\omega)\,,$$

where $E'(S,\omega)$ is any union of half-edges $(x,y] \subset e$, one for every $e \in E(S,\omega)$, with $x \in \mathring{e}$ and $y \in C$. (So for each end ω, the sets $\widehat{C}(S,\omega)$ with S varying over the finite subsets of V are the basic open neighbourhoods of ω.) This is the standard topology on graphs with ends. With this topology, $|G|$ is a Hausdorff space in which every ray converges to the end that contains it. $|G|$ is easily seen to be compact if and only if every vertex has finite degree.

A *subgraph* $G' = (V', E')$ of G will be viewed topologically as just the point set $V' \cup \bigcup E'$, without any ends. Then the closure $\overline{G'}$ of this set in $|G|$ may contain some ends of G, which should not be confused with ends of G'.

We now list a few easy or well-known lemmas that we shall need in our proofs. Let us start with two observations about the Erdős-Menger conjecture itself. The first is that we may assume A and B to be disjoint:

Lemma 2.1. *If $G' := G - (A \cap B)$ satisfies the Erdős-Menger conjecture for $A' := A \setminus B$ and $B' := B \setminus A$, then G satisfies the conjecture for A and B.*

Proof. Let X' be an A'–B' separator on a set $\mathcal{P}'$ of A'–B' paths in G'. Then $X' \cup (A \cap B)$ is an A–B separator on the set $\mathcal{P}' \cup \{\,(x) \mid x \in A \cap B\,\}$ of A–B paths in G, where (x) denotes the trivial path with vertex x. □

We shall also need the following special case of the Erdős-Menger conjecture, which can be reduced to finite graphs [10] and is covered by the results in [5].

Lemma 2.2. *The Erdős-Menger conjecture holds for A and B in G if every set of disjoint A–B paths in G is finite.*

Our next two lemmas are standard tools for infinite graphs.

Lemma 2.3. *Let $R \subseteq G$ be a ray, with end ω say, and $X \subseteq V$. Then $\omega \in \overline{X}$ if and only if G contains a comb with back R and teeth in X.*

Proof. If $\omega \notin \overline{X}$, then ω has a neighbourhood $\widehat{C}(S,\omega)$ in $|G|$ that avoids X. As $R \in \omega$, R has a tail in C. Then all the infinitely many disjoint paths that start on this tail and end in X have to pass through the finite set S, a contradiction.

Conversely, if $\omega \in \overline{X}$ then every $C = C(S,\omega)$ meets both R and X, and we can construct the desired comb inductively by taking as S the (finite) union of the X–R paths already chosen, and finding a new X–R path in C. □

A proof of the following lemma can be found in [9].

Lemma 2.4. *Assume that G is connected, and let $U \subseteq V$ be an infinite set of vertices. Then G contains either a comb with $|U|$ teeth in U or a subdivided star with $|U|$ leaves in U. (Note that if U is uncountable then the latter holds.)*

3. Proof of Theorem 1.1

The basic idea for the proof of Theorem 1.1 is to reduce the problem to rayless graphs, an early result of Aharoni [1]:

Lemma 3.1. (Aharoni 1983)
The Erdős-Menger conjecture holds for all graphs containing no infinite path.

We shall eliminate the infinite paths in our given graph G in three steps. In the first two steps we eliminate the rays whose ends lie in $\overline{A}$ and $\overline{B}$, respectively, and in the third step we eliminate any remaining rays.

The first step consists of the following reduction lemma applied with $H := G$ and $U := A$ and $W := B$.

Lemma 3.2. *Let $H = (V, E, \Omega)$ be a graph, and let $U, W \subseteq V$ be such that $\overline{U} \cap \overline{W} = \emptyset$. Then there exist a subgraph $H' = (V', E', \Omega')$ of H containing W, and a set $U' \subseteq V'$ with $\Omega' \cap \overline{U'} = \emptyset$ (where the closure $\overline{U'}$ is taken in $|H'|$), such that the Erdős-Menger conjecture holds for U and W in H if it holds for U' and W in H'.*

After this first step, it remains to prove the Erdős-Menger conjecture for $A' := U'$ and $B = W$ in $G' := H'$. By Lemma 2.1, we may assume that $A' \cap B = \emptyset$. Since $\Omega' \cap \overline{A'} = \emptyset$ in $|G'|$ as a result of the first application of the lemma, we then have $\overline{A'} \cap \overline{B} = \emptyset$. We may thus apply the lemma again with $H := G'$ and $U := B$ and $W := A'$, to obtain a subgraph $G'' = (V'', E'', \Omega'')$ of G' that contains A' and a set $U' =: B'$ such that $\Omega'' \cap \overline{B'} = \emptyset$.

Note that also $\Omega'' \cap \overline{A'} = \emptyset$ in $|G''|$. For by Lemma 2.3 this is equivalent to the non-existence of a comb in G'' with teeth in A'. As any such comb would also lie in G', its existence would likewise imply $\Omega' \cap \overline{A'} \neq \emptyset$ in $|G'|$, a contradiction.

To this graph G'' we then apply the following lemma as our third reduction step (setting $H := G''$ and $U := A'$ and $W := B'$):

Lemma 3.3. *Let $H = (V, E, \Omega)$ be a graph, and let $U, W \subseteq V$ be such that $\Omega \cap (\overline{U} \cup \overline{W}) = \emptyset$. Then H has a rayless subgraph $H' \subseteq H$ containing $U \cup W$ such that the Erdős-Menger conjecture for U and W holds in H if it does in H'.*

Since the Erdős-Menger conjecture does hold in H' by Lemma 3.1, this completes the proof of Theorem 1.1.

It remains to prove Lemmas 3.2 and 3.3.

Proof of Lemma 3.2. Our first aim is to construct a subgraph $H^* \subseteq H$ such that

(i) $\Omega \cap \overline{U} \cap \overline{H^*} = \emptyset$ in $|H|$;

(ii) $W \subseteq V(H^*)$;

(iii) for every component C of $H - H^*$, its set $S_C := N_H(C)$ of neighbours in H^* cannot be linked to $U_C := U \cap (V(C) \cup S_C)$ by infinitely many disjoint paths in $H_C := H[\,V(C) \cup S_C\,]$.

Our desired graph $H' \subseteq H$ will be a supergraph of H^*.

We define H^* by transfinite ordinal recursion, as a limit $H^* = \bigcap_{\alpha \leqslant \alpha^*} H_\alpha$ of a well-ordered descending family of subgraphs H_α indexed by ordinals. Let $H_0 := H$, and for non-zero limits α let $H_\alpha := \bigcap_{\beta < \alpha} H_\beta$. For successor ordinals $\alpha + 1$ we first check whether $\Omega \cap \overline{U} \cap \overline{H_\alpha} = \emptyset$ in $|H|$, in which case we put $\alpha =: \alpha^*$ and terminate the recursion with $H^* = H_\alpha$. Otherwise pick $\omega_\alpha \in \Omega \cap \overline{U} \cap \overline{H_\alpha}$, and let S_α be a finite set of vertices such that $\widehat{C}(S_\alpha, \omega_\alpha)$ is an open neighbourhood of ω_α in $|H|$ that does not meet W. (Such a set S_α exists, as $\overline{U} \cap \overline{W} = \emptyset$ by assumption.) Put $C_\alpha := C(S_\alpha, \omega_\alpha)$, and let $H_{\alpha+1} := H_\alpha - C_\alpha$.

For any vertex $v \in H - H^*$ we record as $\alpha(v) := \min\{\, \alpha \mid v \in C_\alpha \,\}$ the 'time it was deleted'. Note that, as $\omega_\alpha \in \overline{H_\alpha}$, we have $C_\alpha \cap H_\alpha \neq \emptyset$ for every α, so the recursion terminates. Let us write $\mathcal{C}$ for the set of components of $H - H^*$.

H^* satisfies (i) because $H^* = H_{\alpha^*}$, and (ii) by the choice of the S_α and C_α. To prove (iii), let a component $C \in \mathcal{C}$ be given. Suppose there is an infinite family $P_i = s_i \ldots u_i$ ($i \in \mathbb{N}$) of disjoint S_C–U_C paths in H_C. Let us show that Lemma 2.4 yields a comb in H_C with teeth in $\{\, s_i \mid i \in \mathbb{N} \,\} \subseteq S_C$. If not, then H_C contains an infinite subdivided star with leaves in this set; let v be its centre and $\alpha := \alpha(v)$. As $v \in C_\alpha$ but $S_C \subseteq V(H^*) \subseteq V(H_{\alpha+1}) \subseteq V(H - C_\alpha)$, the finite set S_α separates v in H from the leaves of this star, a contradiction. So H_C contains the desired comb; let $\omega \in \Omega$ denote the end of its back. Then every basic open neighbourhood $\widehat{C}(S, \omega)$ of ω contains infinitely many s_i, and hence also infinitely many P_i and their endvertices in U. Therefore $\omega \in \overline{U}$ as well as $\omega \in \overline{S_C} \subseteq \overline{H^*}$ in $|H|$, and thus $\Omega \cap \overline{U} \cap \overline{H^*} \neq \emptyset$ contradicting (i). This completes the proof of (iii).

To expand H^* to our desired subgraph H', we now consider the components of $H - H^*$ separately. For every $C \in \mathcal{C}$, there exist in H_C a finite set $\mathcal{P}_C$ of S_C–U_C paths and an S_C–U_C separator X_C on $\mathcal{P}_C$ (by (iii) and Lemma 2.2).

Let $\mathcal{D}_C$ denote the set of all the components of $H_C - X_C$ that meet U_C, and put $\mathcal{D} := \bigcup_{C \in \mathcal{C}} \mathcal{D}_C$. Then let

$$H' := H - \bigcup \mathcal{D} \quad \text{and} \quad U' := (U \cap V') \cup \bigcup_{C \in \mathcal{C}} X_C\,.$$

Let us show that $\Omega' \cap \overline{U'} = \emptyset$ in $|H'|$. If not, then by Lemma 2.3 there is a comb K' in H' with teeth in U'; let R be its back. Using the paths in $\bigcup_{C \in \mathcal{C}} \mathcal{P}_C$ (more precisely, their segments between X_C and U_C), we can extend K' to a comb K in H with back R and teeth in U. Since every infinite subset of $V(K)$ has the end of R in its closure, our condition (i) implies that K meets H^* in only finitely many vertices. We may thus assume that $K \subseteq C$ for some $C \in \mathcal{C}$. As R is also the back of $K' \subseteq H'$, we thus have $R \subseteq C \cap H'$. But the finite set X_C separates $C \cap H'$ from U_C in H_C, and hence the back of K from its teeth (a contradiction).

It remains to show that the Erdős-Menger conjecture holds for U and W in H if it holds for U' and W in H'. Assume the latter, and let $\mathcal{P}'$ be a set of disjoint U'–W paths in H' with a U'–W separator X on it. Let $\mathcal{P}$ be obtained from $\mathcal{P}'$ by appending to every $P \in \mathcal{P}'$ whose first vertex u' in U' lies in $U' \setminus U$, and hence in some X_C, the X_C–U_C segment of the path in $\mathcal{P}_C$ containing u'. These segments will be disjoint for different u', because different $C \in \mathcal{C}$ are disjoint and the paths in $\mathcal{P}_C$ are disjoint for each C. (We remark that u' may lie on X_C for several C if $u' \in H^*$, so the choice of C may not be unique.)

Thus, $\mathcal{P}$ is a set of disjoint U–W paths in H, and X consists of a choice of one vertex from each path in $\mathcal{P}$. It remains to show that X separates U from W in H. So let Q be a U–W path in H. If $Q \subseteq H'$ then its first vertex lies in $U \cap V' \subseteq U'$, so Q links U' to W in H' and hence meets X. Suppose then that Q has a vertex in $H - H'$, and let z be its last such vertex. Then the component D of $H - H'$ containing z is an element of $\mathcal{D}_C$ for some $C \in \mathcal{C}$, so $N_H(D) = X_C \subseteq U'$. As $W \subseteq V'$ and hence $W \cap D = \emptyset$, the vertex z is not the last vertex of Q. But the vertex x following z on Q lies in H', and hence in $X_C \subseteq U'$. So xQ joins U' to W in H' and hence meets X. □

For our proof of Lemma 3.3 we need the following lemma of Stein [12]. Let T be a finite set of vertices in a graph J. A *T-path*, for the purpose of this paper, is any path whose endvertices lie in T, whose inner vertices lie outside T, and which has at least one inner vertex. Paths $P_1, \dots, P_k$ are said to be *disjoint outside* some given $Q \subseteq J$ if $P_i \cap P_j \subseteq Q$ whenever $i \neq j$.

Lemma 3.4. *Let J be a graph, let $T \subseteq V(J)$ be finite, and let $k \in \mathbb{N}$. Then J has a finite subgraph J' containing T such that for every T-path $Q = s \dots t$ in J that meets $J - J'$ there are k distinct T-paths from s to t in J' that are disjoint outside Q.*

A proof of Lemma 3.4 can be found in [8].

Proof of Lemma 3.3. As in the proof of Lemma 3.2, we start by constructing a subgraph $H^* \subseteq H$. This time, we require that H^* satisfy the following conditions:

(i) $\Omega \cap \overline{H^*} = \emptyset$ in $|H|$;

(ii) $U \cup W \subseteq V(H^*)$;

(iii) for every component C of $H - H^*$, its set $S_C := N_H(C)$ of neighbours in H^* is finite.

Again, our desired graph $H' \subseteq H$ will be a supergraph of H^*.

We define H^* recursively as before, putting $H_0 := H$ and $H_\alpha := \bigcap_{\beta<\alpha} H_\beta$ for non-zero limits α. For successor ordinals $\alpha+1$ we check whether $\Omega \cap \overline{H_\alpha} = \emptyset$ in $|H|$, in which case we put $\alpha =: \alpha^*$ and terminate the recursion with $H^* = H_\alpha$. Otherwise we pick $\omega_\alpha \in \Omega \cap \overline{H_\alpha}$ and an open neighbourhood $\widehat{C}(S_\alpha, \omega_\alpha)$ of ω_α in $|H|$ that avoids $U \cup W$, which exists as $\Omega \cap (\overline{U} \cup \overline{W}) = \emptyset$ by assumption. We finally let $C_\alpha := C(S_\alpha, \omega_\alpha)$ and $H_{\alpha+1} := H_\alpha - C_\alpha$.

For vertices $v \in H - H^*$ put $\alpha(v) := \min\{\, \alpha \mid v \in C_\alpha \,\}$. Write $\mathcal{C}$ for the set of components of $H - H^*$, and let $H_C := H[\,V(C) \cup S_C\,]$ for each $C \in \mathcal{C}$.

As before, H^* clearly satisfies (i) and (ii). To prove (iii), consider any component $C \in \mathcal{C}$. If S_C is infinite, then H_C contains a comb with teeth in S_C (as before). But then the back of this comb has its end in $\overline{H^*}$, contradicting (i). Therefore S_C is finite, as claimed.

To expand H^* to our desired subgraph H', we again consider the components of $H - H^*$ separately. For each $C \in \mathcal{C}$, denote by H'_C the graph J' which Lemma 3.4 returns on input $J := H_C$ and $k := |S_C|$. We then define

$$H' := H^* \cup \bigcup_{C \in \mathcal{C}} H'_C\,.$$

Let us show that H' is rayless. Suppose there is a ray R in H', say with end $\omega \in \Omega$. Since H' contains from every component C of $H - H^*$ only (part of) the finite subgraph H'_C, R must have infinitely many vertices in H^*. But then ω lies in the closure in $|H|$ of this set of vertices and hence in $\overline{H^*}$, contrary to (i).

It remains to show that the Erdős-Menger conjecture holds for U and W in H if it does so in H'. Suppose there exist in H' a set $\mathcal{P}$ of disjoint U–W paths and a U–W separator X on $\mathcal{P}$. As $H' \subseteq H$, it suffices to show that X also separates U from W in H. Suppose not, and let Q be a U–W path in $H - X$. As Q starts and ends in H', and every segment of Q outside H' lies in some $C \in \mathcal{C}$, we can find a sequence of internally disjoint segments sQt of Q, each with all its inner vertices in some $C \in \mathcal{C}$ (and at least one of these outside H') and its endvertices s, t in S_C, such that the union of these segments contains

$Q - H'$. Our aim is to replace each of these segments $sQt \subseteq H_C$ with an S_C-path P_{st} from s to t in H'_C that avoids X: this will turn Q into a connected subgraph of $H' - X$ that contains both the starting vertex of Q in U and its endvertex in W, contradicting our assumption that X separates U from W in H'.

For our choice of P_{st}, Lemma 3.4 offers $k = |S_C|$ different paths that are disjoint outside sQt. Since Q avoids X, we can thus find P_{st} as desired if we can show that X has fewer than k vertices in C. But every $x \in X \cap V(C)$ lies on a path $P_x \in \mathcal{P}$ that links U to W, and hence by (ii) has at least two vertices in S_C. As these P_x are disjoint for different x, X has at most $|S_C|/2 < k$ vertices in C. □

4. Sketch of a proof of Theorem 1.2

The proof of Theorem 1.2 is basically a combination of the proof of Theorem 1.1 with some special techniques developed in [8]. Assuming familiarity with [8], we describe in this section which difficulties arise when one adapts the proof of Theorem 1.1 to ends, and how to deal with these difficulties. Our description amounts to a sketch of a proof of Theorem 1.2 that should allow any reader to reconstruct the details.

One formal problem with the proof of Theorem 1.2 is that the ends of the subgraphs G' and G'' resulting from the first two reduction steps in our proof of Theorem 1.1 are never, formally, ends of G. Thus if B contains ends as well as vertices, it is not formally possible to require that B be contained in G' (as we do require in the first reduction step). What we shall prove instead is that

Every ray of an end $\omega \in B$ has a tail in G', and all such tails (for fixed ω) belong to the same end ω' of G'. (4.1)

We then have a map $\omega \mapsto \omega'$ from $B \cap \Omega$ to Ω', which is clearly injective. Replacing any end ω in B with its image ω' in Ω', we may then require of G' that $V' \cup \Omega'$ should contain this amended set B. This problem does not arise in the second reduction step, because A' will consist of vertices only.

Alternatively, it would be possible to avoid considering ends of G' and of G'' altogether, and instead work with the closures of these subgraphs in $|G|$. This results in other formal complications.

Let us briefly address how the lemmas in Section 2 have to be adapted. Lemma 2.1 remains unchanged. Lemma 2.2 will be replaced by the main result from [5], which implies that the Erdős-Menger conjecture with ends holds whenever the source set A is countable. (This will be used in H_C with S_C as the source set.) Lemma 2.3 adapts to ends in that the teeth of a comb and the elements of X may now be either ends or vertices. Lemma 2.4 remains unchanged, but we shall also need its uncountable version now (which always gives a subdivided star).

We now discuss how to adapt Lemma 3.2 and its proof. The statement of the lemma changes in two respects. First, we shall have to contract rather than delete some of the components $D \in \mathcal{D}$, so the reduced graph H' will be a minor rather than a subgraph of H. (Vertices of H that are neither deleted nor affected by the contraction will be viewed as vertices of H'.) Second, as the set $W \subseteq V \cup \Omega$ cannot be required to lie in $V' \cup \Omega'$ (as explained above), it has to be replaced by a set $W' \subseteq V' \cup \Omega'$ whose position in $|H'|$ reflects the position of W in $|H|$.

Lemma 4.2. *Let $H = (V, E, \Omega)$ be a graph, and let $U, W \subseteq V \cup \Omega$ be such that $\overline{U} \cap \overline{W} = \emptyset$. Then there exist a minor $H' = (V', E', \Omega')$ of H and sets $U', W' \subseteq V' \cup \Omega'$ that satisfy the following conditions:*

- (a) *$\Omega' \cap \overline{U'} = \emptyset$ (in particular, $U' \subseteq V'$);*
- (b) *if $\Omega \cap \overline{W} = \emptyset$ then $\Omega' \cap \overline{W'} = \emptyset$ (and in particular, $W' \subseteq V'$);*
- (c) *the Erdős-Menger conjecture holds for U and W in H if it holds for U' and W in H'.*

Condition (b) ensures that the gain of the first application of the lemma, that $\Omega' \cap \overline{A'} = \emptyset$, is preserved in its second application (where $W := A'$).

To prove Lemma 4.2, we start by constructing a subgraph H^* of H exactly as in the proof of Lemma 3.2. As before, the termination rule for the contruction of H^* ensures that

(i) $\Omega \cap \overline{U} \cap \overline{H^*} = \emptyset$ in $|H|$.

Now consider a component C of $H - H^*$. Its set of neighbours S_C in H^* must be countable, as otherwise Lemma 2.4 would give us an infinite (even uncountable) subdivided star in H_C with leaves in S_C, with the same contradiction as before (as its centre v should be separated from S_C by the finite set $S_{\alpha(v)}$). The fact that S_C is countable enables us, by the main result of [5] (which establishes the Erdős-Menger conjecture with ends for any graph G in which A and B can be countably separated), to find in $H_C =: (V_C, E_C, \Omega_C)$ a set $\mathcal{P}_C$ of disjoint S_C–U_C paths with an S_C–U_C separator X_C on it, for any set $U_C \subseteq V_C \cup \Omega_C$.

But how should we define U_C now that U may contain ends of H? The answer is the same as with B and H' before: there is a natural way in which Ω_C 'contains' the ends in $\Omega \cap U$ that have a ray in H_C. Indeed, if one ray R of an end $\omega \in \Omega \cap U$ has a tail in H_C then so does every other such ray R', and these tails of R and R' are equivalent in H_C. For since R and R' are equivalent in H, there are infinitely many disjoint paths in H between their tails. But only finitely many of these paths can meet S_C, since they would otherwise form a comb with R or R' that has its teeth in $S_C \subseteq H^*$ but whose back lies in $\omega \in U$, contradicting (i). So we may take as U_C the set $U \cap V_C$ together with the set of those ends of H_C whose rays lie in $U \cap \Omega$.

Property (i) and the ends version of Lemma 2.3 now imply that the sets $\mathcal{P}_C$ and X_C are actually finite, just as in the proof of (iii) for Lemma 3.2. If X_C consists of vertices only, we proceed as earlier: we delete the components D of $H_C - X_C$ that contain a vertex from U_C or a ray from an end in U_C. Suppose now that X_C also contains some ends (from U_C). Then its vertices alone still separate S_C from all the vertices and ends in U_C other than those in X_C. We then delete every component D of $H_C - (X_C \cap V)$ that contains a vertex from U_C or a ray from an end in U_C, and put its set of neighbours (a subset of $X_C \cap V$) in U'. To separate off the ends in $U_C \cap X_C$, we expand $S_C \cup (X_C \cap V)$ to a finite set $T_C \subseteq V_C$ that separates the ends in X_C pairwise. Now consider an end $\omega \in X_C$. By definition of T_C, there is a unique component C_ω of $H_C - T_C$ containing a ray from ω. Put $H_\omega := H[\,V(C_\omega) \cup T_C\,]$; as $T_C \supseteq S_C$, this is a subgraph of H_C. In H_ω, apply Lemma 3.4 with $T := T_C$ and $k := |T_C| + 2$ to obtain a finite subgraph $J' =: H'_\omega$ of H_ω, and let D_ω denote the component of $H - H'_\omega$ containing a ray from ω. Contract D_ω to a single vertex, and put this vertex in U'. (This part of the proof is copied from the proof of Lemma 5.2 in [8], and is explained there in more detail.) The proof of condition (a) in Lemma 4.2 now follows Lemma 3.2: any comb with teeth in U' can be modified into a comb in H with teeth in U (which may be ends) that meets H^* in infinitely many vertices, contradicting (i).

In order to satisfy (b), we have to define W' as explained earlier in the context of condition (4.1). So W' consists of all the vertices in W, together with those ends of H' that contain a ray from an end in W. For this to make sense (and for (b) and (c) to hold) we have to prove that every end in W contains such a ray, and that any two such rays from the same end in W are equivalent in H'. These assertions follow from the fact that no ray whose end lies in W is contained in a component D that we deleted or contracted, and every such ray meets only finitely many such components D. (Otherwise it contains infinitely many vertices from U', and forms the back of a comb with teeth in U. But this cannot happen since $\overline{U} \cap W = \emptyset$.) With these precautions, any comb in H' with teeth in W' (which may be ends) is also a comb in H with teeth in W, which implies (b).

The proof of (c) is not short, but it follows exactly the proof of [8, Lemma 5.2]. This completes our sketch of the proof of Lemma 4.2.

Lemma 3.3 can be used for the proof of Theorem 1.2 unchanged.

References

1. R. Aharoni, Menger's theorem for graphs containing no infinite paths, *Europ. J. Combinatorics* **4** (1983), 201–204.
2. R. Aharoni, König's duality theorem for infinite bipartite graphs, *J. London Math. Soc.* **29** (1984), 1–12.
3. R. Aharoni, Menger's theorem for countable graphs, *J. Combin. Theory B* **31** (1987), 303–313.
4. R. Aharoni, A few remarks on a conjecture of Erdős on the infinite version of Menger's theorem, in (R.L. Graham & J. Nešetřil, eds.): *The Mathematics of Paul Erdős II*, Springer-Verlag Heidelberg 1997.
6. R. Aharoni, C.St.J.A. Nash-Willliams & S. Shelah, A general criterion for the existence of transversals, *Proc. London Math. Soc.* **47** (1983), 43–68.
5. H. Bruhn, R. Diestel & M. Stein, The Erdős-Menger conjecture with ends in arbitrary graphs with countable separators, submitted.
7. R. Diestel, *Graph Theory* (2nd edition), Springer-Verlag 2000. http://www.math.uni-hamburg.de/home/diestel/books/graph.theory/download.html
8. R. Diestel, The countable Erdős-Menger conjecture with ends, *J. Combin. Theory B* (to appear).
9. R. Diestel & D. Kühn, Graph-theoretical versus topological ends of graphs, *J. Combin. Theory B* (to appear).
10. D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig 1936.
11. C.St.J.A. Nash-Williams, Infinite graphs, a survey, *J. Combin. Theory B* **3** (1967), 40–46.
12. M. Stein, *Die Erdős-Menger'sche Vermutung für abzählbare Graphen mit Enden*, Diplomarbeit Universität Hamburg 2002.

Reinhard Diestel
Mathematisches Seminar der
Universität Hamburg
Bundesstraße 55
20146 Hamburg
Germany

diestel@math.uni-hamburg.de

Version 10 January 2003