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TFT CONSTRUCTION OF RCFT CORRELATORS II: UNORIENTED WORLD SHEETS

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Abstract

A full rational CFT, consistent on all orientable world sheets, can be constructed from the underlying chiral CFT, i.e. a vertex algebra, its representation category $\mathcal{C}$, and the system of chiral blocks, once we select a symmetric special Frobenius algebra A in the category $\mathcal{C}$ [I]. Here we show that the construction of [I] can be extended to unoriented world sheets by specifying one additional datum: a reversion σ on A – an isomorphism from the opposed algebra of A to A that squares to the twist. A given full CFT on oriented surfaces can admit inequivalent reversions, which give rise to different amplitudes on unoriented surfaces, in particular to different Klein bottle amplitudes.

We study the classification of reversions, work out the construction of the annulus, Möbius strip and Klein bottle partition functions, and discuss properties of defect lines on non-orientable world sheets. As an illustration, the Ising model is treated in detail.

manche meinen
lechts und rinks
kann man nicht
velwechsern.
werch ein illtum!

Ernst Jandl [J]

Contents

1 Introduction and summary

Understanding conformal field theory on unoriented surfaces is not necessarily regarded as a fundamental goal in the analysis of CFT. But there is at least one crucial application where, together with world sheets having a boundary, non-orientable world sheets appear naturally: type I string theory. Accordingly, it is not surprising that the investigation of CFT on unoriented surfaces started out, and for some time was largely confined to, studies in string theory; see e.g. [1–12]. However, apart from dealing with inherently string theoretic aspects, such as the Klein bottle and Möbius strip projections [6], these studies eventually also led to a better understanding of CFT on non-orientable surfaces in its own right. For instance, it was realised [7] that it is necessary to have a left-right symmetric bulk spectrum, and concepts like the P-matrix which relates the open and closed string channels of the Möbius strip amplitude [13] and the crosscap constraint [11] were introduced. (The crosscap state itself – that is, the conformal block for a one-point correlation function on the crosscap $\mathbb{RP}^2$ – was also described in [14, 15].) One crucial insight [16, 17, 10] was that starting from a given CFT defined on orientable world sheets, in particular its torus and annulus partition functions, there can be several inequivalent ways to extend it to non-orientable surfaces, giving rise to distinct Klein bottle and Möbius amplitudes. Fixing this freedom was termed the 'choice of Klein bottle projection'.

The knowledge gathered in this research finally made it possible to proceed to a more intrinsic study of rational CFT on unoriented surfaces. This development was pushed forward in particular by the Rome [18–22] and Amsterdam [23–28] groups. In the latter works the role of simple currents for the construction of different Klein bottle projections was emphasised. These fields are particularly important for the study of non-orientable surfaces; for example, all known Klein bottle amplitudes are related to the standard one by a simple current. In the orientable case simple currents appear in connection with symmetry breaking boundary conditions [29, 30]; combining the results for orientable and non-orientable world sheets finally led to universal formulas for the boundary and crosscap coefficients for all (symmetric) simple current modifications of the charge conjugation invariant of any rational CFT [31].

Meanwhile the interest in these matters has increased a lot – for reviews as well as further references see [32, 33]. To name just a few results, let us mention that various properties of the Y-tensor (which appears in the Möbius strip and Klein bottle amplitudes) were uncovered [19, 22, 23, 34–37], that the standard Klein bottle coefficients were seen to coincide with Frobenius-Schur indicators [23, 36], and that polynomial equations relating the annulus, Möbius and Klein bottle coefficients were derived [38]. Many more results have been obtained in the study of specific (classes of) models, ranging from the critical Ising model [18, 39] and free boson orbifolds [40] to WZW [19, 20, 41–44] and coset [45, 37] models, permutation orbifolds [46], and $N=2$ superconformal theories [47, 48].

The amount of information obtained so far is certainly impressive. But one should also be aware of the fact that results were typically obtained by solving only a relatively small number of necessary consistency conditions, involving only the most basic correlation functions such as the crosscap states and the Möbius strip and Klein bottle amplitudes. In the present paper we address rational CFT on arbitrary unoriented world sheets within a general approach that allows for a uniform construction of arbitrary correlation functions and which is guaranteed to satisfy all locality, modular invariance and factorisation constraints. Thus the resulting correlators are automatically physical, unlike e.g. the ones obtained by solving only the modular invariance constraint for the torus or the NIM-rep property for the annulus.

Summary of the construction of [I]

The starting point of the construction of correlation functions in a full rational CFT that was initiated in [49, I] is an underlying chiral CFT: a chiral algebra (which contains the Virasoro algebra), its representations, and the associated vector spaces $\mathcal{H}(S)$ of conformal blocks for every closed Riemann surface S. A correlator C of the full CFT on a world sheet X is a specific element of the space $\mathcal{H}(\hat{X})$ of conformal blocks on the complex double $\hat{X}$ of X. The central idea [50, 51] is to use tools from three-dimensional topological field theory (TFT) to describe this vector $C \in \mathcal{H}(\hat{X})$. The TFT in question is constructed from the category $\mathcal{C}$ of representations of the chiral algebra, which is a modular tensor category.

From a given chiral CFT one can, in general, construct several inequivalent full CFTs; well-known examples are the A-D-E series of minimal models and the free boson compactified at different radii. In the construction of [49, I] this is reflected by the choice of a *symmetric special Frobenius algebra* A in $\mathcal{C}$. Accordingly, for the construction of correlators the TFT methods are complemented by non-commutative algebra in tensor categories.

To apply the TFT tools we need to fix a three-manifold M_X and a ribbon graph R in M_X. M_X is the *connecting manifold* [52, 50]; the world sheet X is naturally embedded in M_X (in fact, X is a retract of M_X) and the boundary $\partial\mathrm{M}_X$ is the complex double $\hat{X}$. To the data M_X and R the TFT associates a particular element of $\mathcal{H}(\partial\mathrm{M}_X)$; this is the specific vector that gives the correlator C on the world sheet X.

To construct the ribbon graph $R \subset \mathrm{M}_X$ one chooses a (dual) triangulation of X. Along the edges of the triangulation ribbons are placed that are labelled by the algebra A, while the trivalent vertices are constructed from the multiplication of A. Thinking of the world sheet X as embedded in M_X, the part R_A of the ribbon graph R built in this way lies entirely in X. The ribbons have an orientation as a two-manifold; when X is oriented, then there is a canonical choice for the orientation of R_A, namely the one induced by the embedding $R_A \subset \mathrm{M}_X$. As shown in [I], the correlator $C \in \mathcal{H}(\hat{X})$ obtained by this construction is independent of the chosen triangulation of X. However, for a general algebra A, C does depend on the orientation of the world sheet X.

Summary of results

To build correlators on unoriented (or even non-orientable) world sheets one needs an additional structure on the algebra A that ensures independence of the (local) orientation of the world sheet. For the particular case of two-dimensional *topological* field theory, for which the relevant tensor category $\mathcal{C}$ is the category $\mathcal{V}ect$ of complex vector spaces (so that A is an algebra in the ordinary sense), it was found in [53, 54] that what is required is an isomorphism of A to its opposed algebra that squares to the identity.

Algebras for which an involutive isomorphism to the opposed algebra exists have been discussed in the literature under the name 'algebras with an involution'. However, in the setting of two-dimensional *conformal* field theory, the braiding in the category $\mathcal{C}$ is, in general, not symmetric. As a consequence, in the genuinely braided case one deals with a whole family of algebras $A^{(n)}$, labelled by integers, rather than just an algebra and its opposed algebra; the twist θ_A turns out to provide an algebra isomorphism from $A^{(n)}$ to $A^{(n+2)}$.

It turns out that the notion of algebras with involution generalises to the braided setting as follows. For $A \equiv A^{(0)}$, the opposed algebra $A_{\rm op} \equiv A^{(-1)}$, whose multiplication is obtained from the multiplication in A by composition with the inverse braiding, must be isomorphic, as an algebra, to A, and furthermore this isomorphism must square to the twist. That is, we need an algebra isomorphism $\sigma \in {\rm Hom}(A_{\rm op}, A)$ such that $\sigma \circ \sigma = \theta_A$. Such an isomorphism will be called a *reversion* on A, and a symmetric special Frobenius algebra with reversion will be called a *Jandl algebra*, see definition 2.1. The basic result of this paper is:

> *Given a chiral CFT and a Jandl algebra – a symmetric special Frobenius algebra with reversion – one can construct a full CFT defined on all world sheets, including unoriented world sheets which may have a boundary.*

As already mentioned, to be able to formulate the CFT on unoriented world sheets the bulk spectrum must be left-right symmetric, i.e. the matrix describing the torus partition function must be symmetric, $Z^{\rm t} = Z$. Now $Z = Z(A)$ is uniquely determined by A, and as shown in [I] one has $Z(A_{\rm op}) = Z(A)^{\rm t}$. Thus $Z(A)^{\rm t} = Z(A)$ is indeed a *necessary* condition for the existence of a reversion on A. The existence of a reversion, in turn, gives a precise *sufficient* condition ensuring that one can formulate the CFT on unoriented world sheets as well.

From the point of view of type II string theory, a closed string background is characterised by a full CFT on closed oriented world sheets. Given such a background, one can consider several D-brane configurations. This is reflected in the fact that Morita equivalent algebras give rise to the same full CFT defined on oriented world sheets. Thus in the oriented case a full CFT is associated to a whole Morita class $\mathcal{S}$, not just a single algebra $A \in \mathcal{S}$. Once we have chosen a particular D-brane configuration by picking a representative $A \in \mathcal{S}$ we can try to find an orientifold projection to describe a type I background. This corresponds to finding reversions on A, and hence in this sense a Jandl algebra encodes a type I background. Schematically we have

$$\begin{array}{rcl} \text{choice of Morita class } \mathcal{S} \text{ of algebras} & \longleftrightarrow & \text{choice of closed string background} \\ \text{choice of representative } A \text{ in } \mathcal{S} & \longleftrightarrow & \text{choice of D-brane configuration} \\ \text{choice of reversion } \sigma \text{ on } A & \longleftrightarrow & \text{choice of orientifold projection} \end{array}$$

From the string perspective it is intuitively clear that the number of possible orientifold projections does depend on the choice of D-brane configuration. This rephrases another crucial insight of this paper:

> *The number of possible reversions does depend on the choice of representative in a Morita class of symmetric special Frobenius algebras.*

This result immediately poses the question of classification of reversions; this issue will be addressed in section 2.5. The outcome is that every Morita class $\mathcal{S}$ contains a *finite* number of representatives A_i with reversions $\sigma_{i,j}$ such that every reversion σ on any $A \in \mathcal{S}$ can be related to one of the Jandl algebras $(A_i, \sigma_{i,j})$, in a way described in proposition 2.16. In CFT terminology, this means that a reversion on an arbitrary D-brane configuration in a given background can be related to a reversion on either an elementary D-brane or else a superposition of two distinct elementary D-branes with the same boundary field content.

This result is in accordance with the geometric interpretation of D-branes and orientifold planes in string theory. Either an elementary D-brane is already invariant with respect to reflection about the orientifold plane, or it is not, in which case one must take the configuration consisting of the D-brane and its image under the reflection about the orientifold plane. We want to stress, however, that our classification of reversions results from an intrinsic CFT computation which does not rely on the existence of any space-time interpretation of the CFT. Thus our analysis shows that the geometric picture retains some validity in the deep quantum regime.

Also recall from [I] that boundary conditions are described by left A-modules. The existence of a reversion σ on A allows us to associate to any left A-module M another left A-module M^σ, and thereby to define a (σ-dependent) conjugation C^σ on left A-modules, i.e. on the space of boundary conditions (see definition 2.6 and proposition 2.8). Note that, in contrast, the structure of a symmetric special Frobenius algebra alone does not allow one to endow the category of left A-modules with a conjugation.

The prescription for the three-manifold M_X and ribbon graph that describe a correlation function is given, for the case without field insertions, in section 3.1. Three examples are worked out in detail: the annulus (section 3.2), Möbius strip (section 3.3) and Klein bottle (section 3.4) partition functions. For the annulus, both boundaries are chosen to have the same orientation. The corresponding annulus coefficients, giving the multiplicities of open string states, are written with two lower indices, A_{kMN}; we show that they are symmetric in the boundary conditions M and N, and that the coefficients A_{0MN} coincide with the boundary conjugation matrix (equation (3.23) and lemma 3.3). The ribbon graphs for the three partitions functions are presented in the figures (3.22), (3.37) and (3.55), respectively. The coefficients $\mathrm{mö}_{kR}$ of the Möbius strip amplitude are shown to be integers and satisfy $\frac{1}{2}\,(\mathrm{mö}_{kR}+\mathrm{A}_{kRR})\in\mathbb{Z}_{\ge 0}$ (theorem 3.5). The latter property is necessary for a consistent interpretation in type I string theory, where this combination of annulus and Möbius strip counts the number of open string states. Similarly, for the Klein bottle coefficients K_k it is shown that $\mathrm{K}_k\in\mathbb{Z}$ and $\frac{1}{2}\,(Z_{kk}+\mathrm{K}_k)\in\mathbb{Z}_{\ge 0}$, where Z_{ij} are the coefficients of the torus partition function (theorem 3.7).

Another important constraint is the consistency of the Möbius strip and Klein bottle amplitudes in the crossed channel, where in both cases only closed strings propagate. Note that all these constraints are automatically satisfied on general grounds, because the amplitudes constructed with the present method are consistent with factorisation. Nevertheless we investigate the crossed channels for Möbius strip and Klein bottle explicitly; in particular we establish the relations

$$\mathrm{mö}_{kR} = \frac{1}{S_{00}} \sum_{l,\alpha,\beta} P_{kl}\, S^A_{R,l\alpha}\, g^{\bar{l}\,l}_{\alpha\beta}\, \gamma^\sigma_{l\beta} \qquad \text{and} \qquad \mathrm{K}_k = \frac{1}{S_{00}} \sum_{l,\alpha,\beta} S_{kl}\, \gamma^\sigma_{l\alpha}\, g^{\bar{l}\,l}_{\alpha\beta}\, \gamma^\sigma_{l\beta} \tag{1.1}$$

(see equations (3.119), (3.120) and sections 3.5 – 3.7). This is done for several reasons. First, the details of the general proof of factorisation have not been published yet, and it is illustrative to discuss these particular cases separately. Second, the relations (1.1) are used to actually compute $\mathrm{mö}_{kR}$ and K_k. This is done by giving explicit expressions for the constituents S^A (formulas (3.106) and (3.108)), γ^σ ((3.103), (3.110) and (3.112)) and g (formula (3.118)). The latter expressions involve only the defining data of the modular tensor category (the braiding and fusing data R and F), the Jandl algebra (the product m and the reversion σ), the A-modules (the representation morphisms ρ), and a basis μ in the space of local morphisms (definition 3.9 and equation (3.104)).

We also discuss defect lines on unoriented world sheets (section 3.8). Here a new feature arises. Suppose a conformal defect is wrapping a non-orientable cycle on the world sheet, that is, a cycle whose neighbourhood has the topology of a Möbius strip. For a generic defect this is only possible when inserting a disorder field, because the two ends of a defect wrapping a non-orientable cycle generally cannot be joined in a conformally invariant manner. While a conformal defect can be deformed continously without changing the value of the correlator, the position of the disorder field insertion must remain fixed, i.e. the insertion of a disorder field marks a point on the world sheet. This way one can single out an interesting subclass of conformal defects, namely those which can wrap a non-orientable cycle without the need to insert any disorder field. A defect Y belongs to this subclass iff $\mathrm{Hom}_{A|A}(Y^s,Y)$ has nonzero dimension, see the end of section 3.8 and equation (3.159). (Of course Y^s depends on the choice of reversion.)

Finally we work out two examples in detail. First, the classification of reversions is carried out for the case $\mathcal{C}=\mathcal{V}ect$, illustrating the simplifications that occur when working with algebras in the ordinary sense (section 4.1). Second, as simplest non-trivial CFT model, the critical Ising model is treated in great detail (section 4.2). The defining data of the modular category $\mathcal{C}$ are presented, and the symmetric special Frobenius algebras are classified and shown to form a single Morita class. Then all reversions are classified; it is found that all reversions can be related to two fundamental ones. The boundary conjugation matrix and the Möbius strip and Klein bottle amplitudes are computed. On the basis of these a lattice interpretation is offered for the two different reversions.

In four appendices we provide further details. We display a collection of moves on ribbon graphs which are useful in the evaluation of various invariants (appendix A.1), provide the slightly lengthy proof of the algorithm for finding reversions (appendix A.2), comment on the recursion formula (3.131) for computing invariants obtained by glueing two solid tori (appendix A.3), and give an auxiliary calculation for the classification of reversions in the Ising model that is omitted in the main text (appendix A.4).

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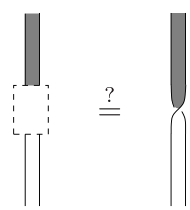
2 Jandl algebras

To be able to deal with non-orientable world sheets, an additional structure on algebras – a reversion – must be introduced. To see why this structure is indeed needed, let us have a look at the prescription on how to construct the ribbon graph for an oriented world sheet and point out what is needed to extend that prescription to unoriented world sheets.

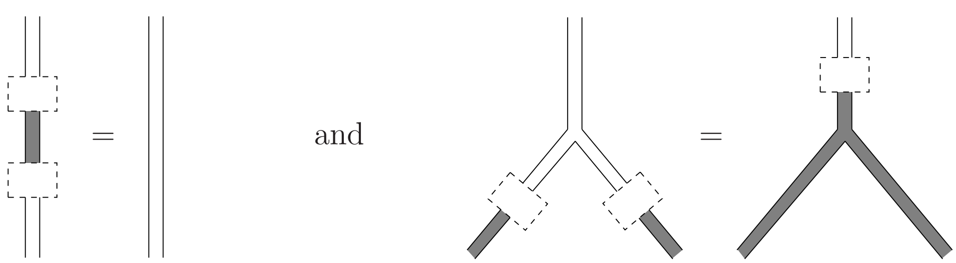
Recall from section I:5.1 that in the oriented case we choose a dual triangulation of the world sheet and then place A-ribbons along its edges. These ribbons must be oriented as two-manifolds. In a graphical calculus the two possible orientations are indicated by drawing the ribbon with two different colours, say with a white and a black side, respectively. For the case of oriented world sheets, in [I] we have chosen the orientation inherited from the orientation of the world sheet. For non-orientable world sheets, it is still possible to choose a dual triangulation, but after doing so we face two problems. First, we do not know how to orient the ribbons. Of course, we can still choose local orientations. But every non-orientable surface contains a cycle that, after fattening, gives a ribbon with the topology of the Möbius strip. As a consequence, we are inevitably faced with the task to join two A-ribbons with opposite orientations, or in other words, we must join the white side of an A-ribbon to the black side of another A-ribbon. A priori it is unclear how this should be done; let us symbolise this unknown operation by a dashed box and refer to it as a *reversing move*. The simplest manipulation one could carry out at a reversing move is the purely geometric operation of just performing a "half-twist", i.e. rotating the ribbon by an angle of π, according to



(2.1)

Already at this point, it is apparent that there is too much arbitrariness in this prescription: it is not clear that turning the ribbon in the opposite direction will give an equivalent result.

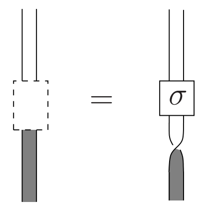
To determine what properties the reversing move must possess in order to avoid such ambiguities, we must analyze the situation in more detail. Let us choose, right from the start, at every vertex of the triangulation a local orientation of the world sheet. This also fixes, around that vertex, what is to be considered as the white and the black side of the ribbon. Independence of the choices of local orientation and of the triangulation can then be seen to impose the conditions



(2.2)

on the reversing move. The second of these equalities ensures that two different choices of the local orientation at the vertex give equivalent results, while the first equality allows us to cancel two reversing moves.

One can quickly convince oneself that the first ansatz (2.1) of simply turning the ribbons by an angle π does not obey these conditions [1]. Instead a more general ansatz is required. To this end we include in the reversing move a suitable 'compensating' morphism $\sigma \in \mathrm{Hom}(A, A)$, i.e. every half-twist is accompanied by the morphism σ, according to



(2.3)

The conditions (2.2) then translate into statements about the morphism σ – they imply that σ is a *reversion* on A, see definition 2.1 below (one also needs proposition 2.4).

As a matter of fact, the reasoning above just repeats what has already been done in [53] for two-dimensional lattice topological field theory. A 2d TFT is defined by an algebra over $\mathbb{C}$, i.e. an algebra in $\mathcal{V}ect_{\mathbb{C}}$. The line of arguments just described lead the authors of [53] to introduce an involutive algebra anti-homomorphism σ, i.e. a linear map satisfying $\sigma(ab) = \sigma(b)\,\sigma(a)$ and $\sigma \circ \sigma = id$.

2.1 Reversions

For general modular tensor categories the only modification comes from the fact that the twist and the braiding can be non-trivial. This suggests the

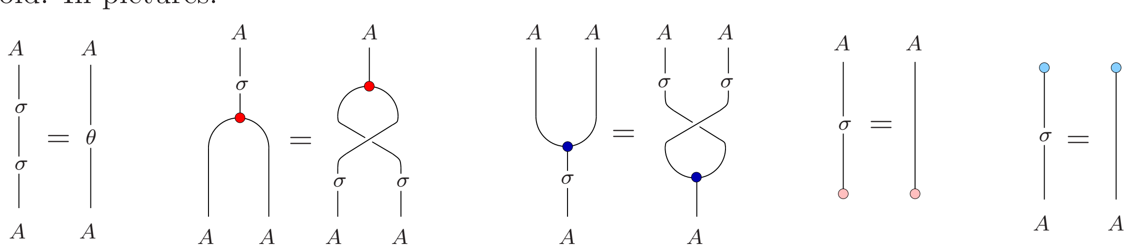
Definition 2.1:
(i) A *reversion* on an algebra $A = (A, m, \eta)$ is an endomorphism $\sigma \in \mathrm{Hom}(A, A)$ that is an algebra anti-homomorphism and squares to the twist, i.e.

$$\sigma \circ \eta = \eta\,, \qquad \sigma \circ m = m \circ c_{A,A} \circ (\sigma \otimes \sigma)\,, \qquad \sigma \circ \sigma = \theta_A\,. \tag{2.4}$$

If the algebra A is also a coalgebra, $A = (A, m, \eta, \Delta, \varepsilon)$, then we demand that in addition

$$\varepsilon \circ \sigma = \varepsilon \qquad \text{and} \qquad \Delta \circ \sigma = (\sigma \otimes \sigma) \circ c_{A,A} \circ \Delta \tag{2.5}$$

hold. In pictures:



(2.6)

(ii) The quadruple $A = (A, m, \eta, \sigma)$ consisting of an algebra and a reversion is called an *algebra with reversion*.

(iii) A symmetric special Frobenius algebra with reversion will also be called a *Jandl algebra*.

[1] For an interpretation of half-twists in the context of quantum groups see remark 7.2 of [55].

Recall from corollary I:3.19 and proposition I:3.20 that from a given algebra A one can obtain a whole series of algebras $A^{(n)}$ by composing the multiplication with some power $c_{A,A}^n$ of the braiding. The twist θ_A provides algebra isomorphisms from $A^{(n)}$ to $A^{(n+2)}$. With respect to this property, finding a reversion σ on A amounts to taking a square root of the twist:

Proposition 2.2:
Let A be an algebra and n an integer. A morphism $\sigma \in \mathrm{Hom}(A,A)$ is a reversion for A if and only if it is an algebra morphism from $A^{(n)}$ to $A^{(n+1)}$ and squares to θ_A.
An analogous statement holds if A is in addition a coalgebra.

Proof:
Let $A = (A, m, \eta)$ be an algebra. By definition,

$$A^{(n)} = (A, m \circ (c_{A,A})^n, \eta)\,. \tag{2.7}$$

That $\sigma \in \mathrm{Hom}(A,A)$ is an algebra morphism from $A^{(n)}$ to $A^{(n+1)}$ squaring to the twist thus means

$$\sigma \circ m^{(n)} = m^{(n+1)} \circ (\sigma \otimes \sigma)\,, \qquad \sigma \circ \eta^{(n)} = \eta^{(n+1)} \qquad \text{and} \qquad \sigma \circ \sigma = \theta_A\,. \tag{2.8}$$

By functoriality of the braiding we have $c_{A,A} \circ (\sigma \otimes \sigma) = (\sigma \otimes \sigma) \circ c_{A,A}$. Hence by substituting the definition (2.7) into (2.8) one sees that (2.4) and (2.8) are indeed equivalent.

If in addition A is coalgebra with coproduct Δ and counit ε, then the coproduct and counit of $A^{(n)}$ are given by $\Delta^{(n)} = (c_{A,A})^{-n} \circ \Delta$ and $\varepsilon^{(n)} = \varepsilon$. Then by the same reasoning as above one checks the equivalence of

$$(\sigma \otimes \sigma) \circ \Delta^{(n)} = \Delta^{(n+1)} \circ \sigma \qquad \text{and} \qquad \varepsilon^{(n)} = \varepsilon^{(n+1)} \circ \sigma \tag{2.9}$$

with the conditions (2.5). ✓

If we are given two reversions σ_1 and σ_2 of an algebra A, the above proposition implies that $\omega = \sigma_1^{-1} \circ \sigma_2$ is an algebra automorphism of A. Conversely, all possible reversions for a given algebra A can be obtained by composing a single reversion σ_0 with suitable algebra automorphisms ω of A. Note, however, that while the first two properties in (2.4) for $\sigma := \omega \circ \sigma_0$ being a reversion follow directly from the fact that ω is an algebra morphism, the requirement $\sigma \circ \sigma = \theta_A$ places an additional condition on ω. Writing $\theta_A = \sigma_0 \circ \sigma_0$ yields the following result.

Proposition 2.3:
Let A be an algebra with reversion σ_0 and $\omega \in \mathrm{Hom}(A,A)$ be an automorphism of A. Then $\sigma = \omega \circ \sigma_0$ is a reversion on A if and only if

$$\omega \circ \sigma_0 \circ \omega = \sigma_0\,. \tag{2.10}$$

Not all reversions obtained from algebra automorphisms obeying (2.10) should be treated as distinct. Let A_1 and A_2 be two algebras and $f \in \mathrm{Hom}(A_1, A_2)$ be an algebra isomorphism. If A_1 and A_2 have reversions σ_1 and σ_2, then f is an isomorphism of algebras with reversion if $f \circ \sigma_1 = \sigma_2 \circ f$. In particular two reversions (A, σ) and $(A, \omega \circ \sigma)$ on the same algebra are isomorphic if there exists an automorphism f of A such that $\omega \circ \sigma = f \circ \sigma \circ f^{-1}$.

If A is a symmetric special Frobenius algebra, the coproduct and counit are determined through the multiplication. This reduces the number of conditions σ has to satisfy to be a reversion on A. In fact, due to the proposition below it is enough to check that

$$\sigma \circ \sigma = \theta_A \tag{2.11}$$

and

$$\sigma \circ m = m \circ c_{A,A} \circ (\sigma \otimes \sigma)\,. \tag{2.12}$$

These are precisely the conditions that correspond to the properties displayed graphically in picture (2.2).

Proposition 2.4:
Let A be an algebra, and $\sigma \in \mathrm{Hom}(A,A)$ satisfy the relation (2.12). Then σ also obeys

$$\sigma \circ \eta = \eta\,. \tag{2.13}$$

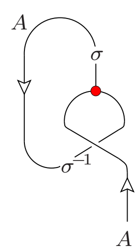
If in addition σ has the property (2.11) and A is symmetric special Frobenius, then σ is a reversion on A, i.e. we also have

$$\varepsilon = \varepsilon \circ \sigma\,, \qquad (\sigma \otimes \sigma) \circ \Delta = c_{A,A}^{-1} \circ \Delta \circ \sigma\,. \tag{2.14}$$

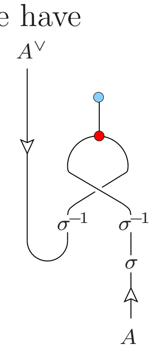
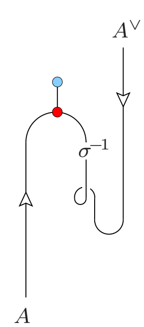
Proof:
The equality $\sigma \circ \eta = \eta$ is obtained as

$$\sigma \circ \eta = m \circ (\sigma \otimes id_A) \circ (\eta \otimes \eta) = \sigma \circ m \circ c_{A,A}^{-1} \circ (\eta \otimes \sigma^{-1}) \circ \eta = \eta\,, \tag{2.15}$$

where the second step uses (2.12).
To show that $\varepsilon \circ \sigma = \varepsilon$ we use that, by lemma I:3.11, the counit ε of a symmetric Frobenius algebra is of the form $\varepsilon = \beta\, \varepsilon_\natural$ for some $\beta \in \mathbb{C}^\times$, with the morphism $\varepsilon_\natural$ as defined in (I:3.46). It follows that

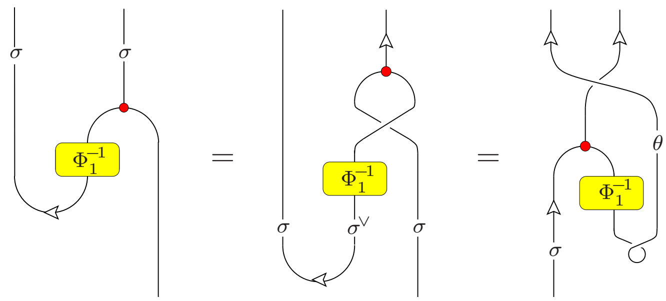
$\varepsilon \circ \sigma \;=\; \beta\, \varepsilon_\natural \circ \sigma \;=\; \beta$  $\;=\; \beta\, \varepsilon_\rho \;=\; \varepsilon\,.$ (2.16)

Here the second step uses (2.12). In the third step σ^{-1} is cancelled against σ and the A-loop is moved to the right side of the ingoing A-ribbon; the two resulting twists in the A-loop cancel, so that one is left with ε_ρ as defined in (I:3.46). The last step uses lemma I:3.9(ii) which states that $\varepsilon_\rho = \varepsilon_\natural$.
Finally we must check that $(\sigma \otimes \sigma) \circ \Delta = c_{A,A}^{-1} \circ \Delta \circ \sigma$. To this end we use the morphism $\Phi_1 \in \mathrm{Hom}(A, A^\vee)$ in (I:3.33). We have

$\Phi_1 \circ \sigma \;=$  $=$  $=\; \sigma^\vee \circ \Phi_1\,.$ (2.17)

In the first step the definition of Φ_1 is inserted and $\varepsilon = \varepsilon \circ \sigma^{-1}$ together with (2.12) is used. The second step is a deformation of the A-ribbon. In the last step the twist is absorbed via property (2.11), i.e. $\sigma^{-1} \circ \theta_A = \sigma$. We are then left with $\sigma^\vee \circ \Phi_2$ (the dual $f^\vee$ of a morphism f has been defined in (I:2.8) and (I:2.12)), and finally we can use that $\Phi_2 = \Phi_1$ because A is symmetric (definition I:3.4). Now from the expression (I:3.42) for the coproduct Δ we have

$$(\sigma \otimes \sigma) \circ \Delta \;=$$



$$= \; (c_{A,A})^{-1} \circ \Delta \circ \sigma \,, \tag{2.18}$$

where the second step combines the result (2.17) with property (2.12). In the third step the two σ's are combined to a twist θ_A via (2.11) and the leftmost A-ribbon is moved to the right, resulting in another twist. Then the two twists cancel and one recognises another expression in the list (I:3.42) for the coproduct (recall $\Phi_1 = \Phi_2$). ✓

Remark 2.5:

(i) According to proposition I:5.3, the matrices Z_{ij} that give the torus partition functions of the full CFTs for A and $A_{\rm op} \equiv A^{(-1)}$ are related by $Z(A_{\rm op}) = Z(A)^{\rm t}$. Since the morphism σ provides an algebra isomorphism between $A_{\rm op}$ and A, it follows that $Z(A) = Z(A)^{\rm t}$ is a necessary condition for A to admit a reversion.

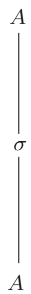
(ii) Recall from theorem I:3.6(i) that an algebra (A, m, η) can have, up to isomorphism, only a single symmetric special Frobenius structure. The qualifiers 'symmetric', 'special' and 'Frobenius' contain thus no additional information, but rather indicate a restriction on the class of algebras one can use.
For reversions the situation is different. While the existence of a reversion further restricts the class of algebras (see point (i)), a given symmetric special Frobenius algebra can allow for several inequivalent reversions. Section 4 will provide examples for this phenomenon. To construct a conformal field theory on unoriented surfaces, one must thus choose *both* a (symmetric special Frobenius) algebra *and* a reversion on that algebra; different reversions on the same algebra can give inequivalent conformal field theories.

(iii) There can exist several distinct morphisms $\sigma_{(i)}$ that turn a given symmetric special Frobenius algebra into a Jandl algebra. An important point to notice is that the number of such morphisms is, in general, *not* the same for every member of a given Morita class of algebras. Again we will meet examples in section 4. (We even cannot exclude the possibility that some representatives of the Morita class allow for reversions while others cannot carry any reversion at all, though we are not aware of an example of this phenomenon.)

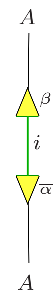
2.2 Expressing reversions in a basis

When introducing bases for the vector spaces $\mathrm{Hom}(U_i, A)$ and $\mathrm{Hom}(A, U_i)$, with U_i any simple object, the morphism σ is encoded in a collection of matrices $\sigma(i)$. The conditions (2.11) and (2.12) can then be formulated as relations among those matrices. Let us describe this in detail.

As an object of $\mathcal{C}$ the algebra A is a direct sum $A \cong \bigoplus_{i\in\mathcal{I}} n_i\, U_i$ with some multiplicities $\langle i,A\rangle = n_i \in \mathbb{Z}_{\geq 0}$, where $\{U_i \,|\, i\in\mathcal{I}\}$ is a (finite) set of representatives for the isomorphism classes of irreducible objects of $\mathcal{C}$. We choose a basis $\{\alpha\}$ in the morphism space $\mathrm{Hom}(U_i, A)$ and a dual basis $\{\bar\alpha\} \subset \mathrm{Hom}(A, U_i)$, as in (I:3.4). In this basis the multiplication on A is given by numbers $m_{a\alpha,b\beta}^{c\gamma;\delta}$ as in (I:3.7). Similarly, the reversion σ is then encoded in matrices $\sigma(i)_\alpha^{\ \beta}$ via



$$= \sum_{i\in\mathcal{I}} \sum_{\alpha=1}^{\langle i,A\rangle} \sum_{\beta=1}^{\langle i,A\rangle} \sigma(i)_\alpha^{\ \beta} \qquad (2.19)$$



where we use the short-hand $\langle X,Y\rangle \equiv \dim_{\mathbb{C}} \mathrm{Hom}(X,Y)$. Inserting (2.19) into (2.11) and (2.12) we get the conditions

$$\begin{aligned} &\sum_{\beta=1}^{\langle i,A\rangle} \sigma(i)_\alpha^{\ \beta}\, \sigma(i)_\beta^{\ \gamma} = \delta_{\alpha,\gamma}\, \theta_i \qquad \text{and} \\ &\sum_{\rho=1}^{\langle k,A\rangle} m_{i\alpha,j\beta}^{k\rho;\delta}\, \sigma(k)_\rho^{\ \gamma} = \sum_{\mu=1}^{\langle i,A\rangle} \sum_{\nu=1}^{\langle j,A\rangle} \sigma(i)_\alpha^{\ \mu}\, \sigma(j)_\beta^{\ \nu} \sum_{\varepsilon=1}^{N_{ji}^{\ k}} m_{j\nu,i\mu}^{k\gamma;\varepsilon}\, \mathsf{R}^{(i\,j)k}_{\varepsilon\,\delta} \end{aligned} \qquad (2.20)$$

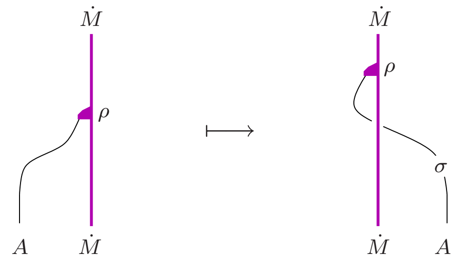
for the matrices $\sigma(i)$, where R are the braiding matrices which express the braiding after a choice of bases, see (I:2.41). If we restrict to the particular subclass of modular tensor categories for which $N_{ij}^{\ k} \in \{0,1\}$ and to algebras with multiplicities $\langle i,A\rangle \in \{0,1\}$, the equations (2.20) simplify considerably:

$$\sigma(i)^2 = \theta_i \qquad \text{and} \qquad m_{ij}^k\, \sigma(k) = \sigma(i)\, \sigma(j)\, m_{ji}^k\, \mathsf{R}^{(i\,j)k}\,. \qquad (2.21)$$

(Thus $\sigma(i)$ is a square root of the number θ_i; note that its value does not depend on any gauge choices.) However, the simplifying assumption $\langle i,A\rangle \in \{0,1\}$ is very restrictive. Indeed, one finds that certain types of reversions can only be realised on algebras having $\langle i,A\rangle > 1$ for some simple objects U_i (in fact this happens already for $\mathfrak{su}(2)_k$ WZW models with k odd).

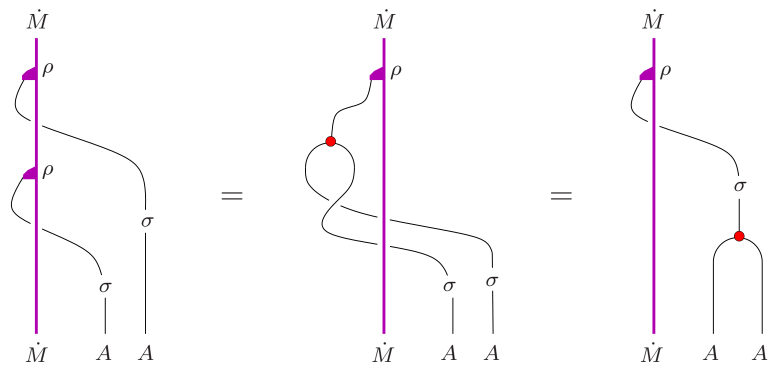
2.3 Conjugate modules

The reversion σ can be combined with a braiding to turn left modules into right modules. This is achieved via the mapping



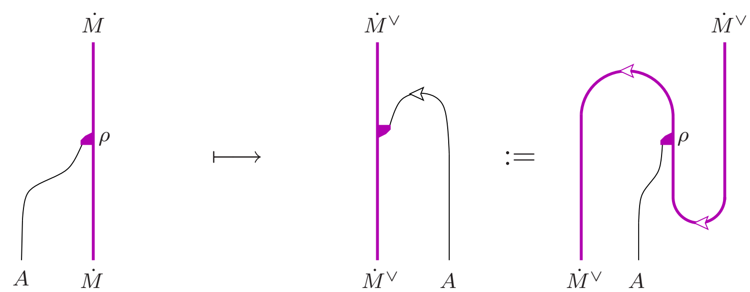
(2.22)

Indeed one quickly verifies that the representation property for right modules is satisfied:



(2.23)

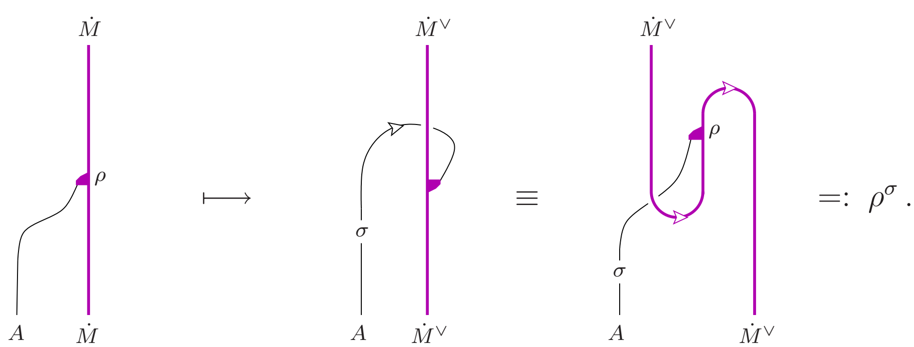
There is also another way to obtain a right module, denoted by $M^\vee$, given a left module M, which does not involve a reversion, but rather the duality, see (I:4.60):



(2.24)

However, while (2.22) defines a right module structure on the same object $\dot M$, (2.24) defines it on the dual object $\dot M^\vee$.

Combining (2.22) and (2.24) we obtain a map from left modules to left modules:



(2.25)

This suggests the following definition.

Definition 2.6:
Let A be an algebra. For a left A-module $M = (\dot{M}, \rho)$ the *conjugate* (left) *module* with respect to a reversion σ on A is

$$M^\sigma := (\dot{M}^\vee, \rho^\sigma) \tag{2.26}$$

with ρ^σ given in (2.25).

M^σ is again a left A-module. Note that $M^{\sigma\sigma}$ is in general not equal to M, just like $U^{\vee\vee}$ is in general not equal to U as an object of $\mathcal{C}$. But $M^{\sigma\sigma}$ and M are isomorphic as left A-modules. To see this recall that an isomorphism δ_U between U and $U^{\vee\vee}$ can be given with the help of the left and right dualities (cf. (I:2.13) and chapter 2.2 of [56]):

$$\delta_U = (id_{U^{\vee\vee}} \otimes d_U) \circ (\tilde{b}_{U^\vee} \otimes id_U) \in \mathrm{Hom}(U, U^{\vee\vee}) \,. \tag{2.27}$$

The morphisms δ_U satisfy $\delta_{U\otimes V} = \delta_U \otimes \delta_V$ and $\delta_{U^\vee} = ((\delta_U)^\vee)^{-1}$. It can be checked by direct substitution (being careful not to mix up the left and right dualities) that an isomorphism between $M^{\sigma\sigma}$ and M is provided by $\delta_{\dot{M}}$, i.e. we have

$$\rho = \delta_{\dot{M}}^{-1} \circ \rho^{\sigma\sigma} \circ (id_A \otimes \delta_{\dot{M}}) \,. \tag{2.28}$$

As a word of warning, let us recall that [57] with suitable assumptions on the algebra A it can happen that the category of left A-modules has the structure of a tensor category with duality. In that case, the conjugate module as introduced in definition 2.6 is, however, not necessarily equal to the dual module in the sense of this duality.

The following proposition lists some further properties of conjugate modules.

Proposition 2.7:
Let A be a Jandl algebra, with reversion σ.

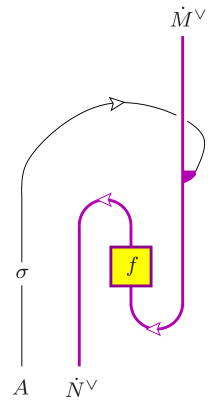
(i) The mapping $M \mapsto M^\sigma$ and $f \mapsto f^\vee$ furnishes a contravariant endofunctor of $\mathcal{C}_A$. Thus for any two A-modules M, N, the mapping $f \mapsto f^\vee$ furnishes an isomorphism $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_A(N^\sigma, M^\sigma)$.

(ii) The square of this endofunctor is equivalent to the identity functor, so that $\mathrm{Hom}_A(M, N^\sigma) \cong \mathrm{Hom}_A(N, M^\sigma)$.

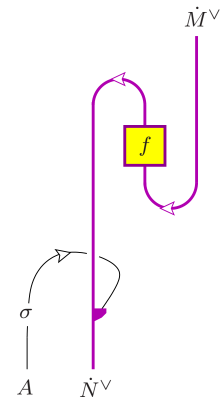
(iii) For induced modules one has $(\mathrm{Ind}_A(U))^\sigma \cong \mathrm{Ind}_A(U^\vee)$ as left A-modules. In particular, $A^\sigma \cong A$.

Proof:
(i) Given a morphism $f \in \mathrm{Hom}_A(M,N)$ we must verify that $f^\vee \in \mathrm{Hom}_A(N^\sigma, M^\sigma)$. Writing out the definitions, this amounts to proving that

$\rho^\sigma \circ (id_A \otimes f^\vee)$ =



=



$= f^\vee \circ \rho^\sigma \,.$ (2.29)

This equation can be seen to hold by deforming the $\dot{M}$- and $\dot{N}$-ribbons to move f close to the representation morphism ρ and using that f is in $\mathrm{Hom}_A(M,N)$. That this mapping is linear and invertible is obvious.

(ii) We have already seen above that $M^{\sigma\sigma} \cong M$ via the morphism δ_U (2.27). To check the statement for morphisms, we note that $f \mapsto f \circ \delta_{\dot{N}}$ provides an isomorphism from $\mathrm{Hom}_A(N^{\sigma\sigma}, M)$ to $\mathrm{Hom}_A(N,M)$. Combining this with the isomorphism from (i) gives

$$\mathrm{Hom}_A(M, N^\sigma) \cong \mathrm{Hom}_A(N^{\sigma\sigma}, M^\sigma) \cong \mathrm{Hom}_A(N, M^\sigma)\,. \tag{2.30}$$

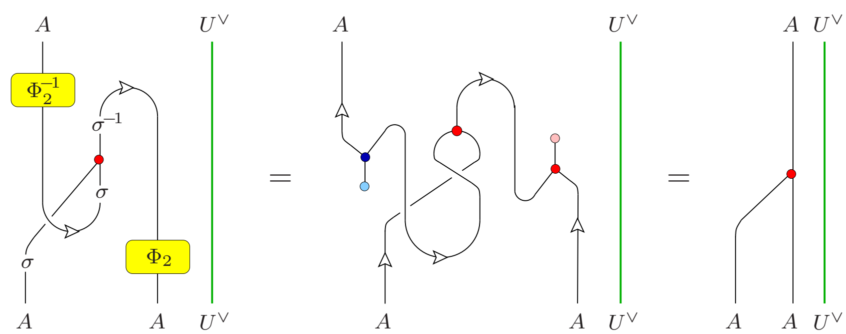
(iii) Consider the morphism $\varphi \in \mathrm{Hom}(A \otimes U^\vee, U^\vee \otimes A^\vee)$ given by

$$\varphi := (c_{U^\vee, A^\vee})^{-1} \circ (\Phi_2 \otimes id_{U^\vee}) \circ (\sigma^{-1} \otimes id_{U^\vee})\,. \tag{2.31}$$

Its inverse is easily computed to be

$$\varphi^{-1} = (\sigma \otimes id_{U^\vee}) \circ (\Phi_2^{-1} \otimes id_{U^\vee}) \circ c_{U^\vee, A^\vee} \tag{2.32}$$

(as A is symmetric, the morphism Φ_2, defined in (I:3.33), is equal to Φ_1, see above; Φ_2^{-1} is given explicitly in (I:3.35)). Denote now by ρ the representation morphism of $\mathrm{Ind}_A(U^\vee)$. To establish that $(\mathrm{Ind}_A(U))^\sigma$ is isomorphic to $\mathrm{Ind}_A(U^\vee)$ it is enough to show that $\rho = \varphi^{-1} \circ \rho^\sigma \circ (id_A \otimes \varphi)$. This equality, in turn, can be seen as follows.

$\varphi^{-1} \circ \rho^\sigma \circ (id_A \otimes \varphi) \;=$  (2.33)

In the first step (2.17) together with the explicit expression for φ and φ^{-1} is inserted. The $U^\vee$-ribbon can then be disentangled, resulting in the first picture. The second picture is obtained by using (2.12) and the explicit form of Φ_2 and Φ_2^{-1}. The last step amounts to a deformation of the A-ribbons and then associativity, Frobenius, unit and counit property.

The special case $A^\sigma \cong A$ is obtained when setting $U = \mathbf{1}$. ✓

The mapping $M \mapsto M^\sigma$ gives rise to an involution on the set $\mathcal{J} = \{M_\kappa \,|\, \kappa = 1, \dots, |\mathcal{J}|\}$ of representatives of isomorphism classes of simple A-modules. Some properties of this involution are collected in the following

Proposition 2.8:

(i) For M_ρ a simple left A-module, the left A-module $(M_\rho)^\sigma$ is simple, too.

Further, setting

$$C^\sigma_{\kappa\rho} := \dim_{\mathbb{C}} \mathrm{Hom}_A(M_\kappa, (M_\rho)^\sigma)\,, \tag{2.34}$$

we have

(ii) $(M_\kappa)^\sigma \cong \bigoplus_{\rho \in \mathcal{J}} C^\sigma_{\kappa\rho} M_\rho$ as left A-modules.

(iii) The matrix $C^\sigma = (C^\sigma_{\kappa\rho})$ is symmetric and squares to the unit matrix.

(iv) C^σ is a permutation matrix of order two.

(v) If A is haploid, then it is simple as a module over itself, $A \cong M_{\kappa_\circ}$ for some $\kappa_\circ \in \mathcal{J}$, and $C^\sigma_{\kappa_\circ\rho} = \delta_{\kappa_\circ,\rho}$.

Proof:
(i) The statement that $(M_\rho)^\sigma$ is simple is equivalent to $\dim_{\mathbb{C}} \mathrm{Hom}_A((M_\rho)^\sigma, (M_\rho)^\sigma) = 1$. The latter, in turn, is an immediate consequence of M_ρ being simple and proposition 2.7(i).

(ii) Since the category $\mathcal{C}_A$ of left A-modules is semisimple, every A-module M decomposes as an A-module as $M \cong \bigoplus_{\kappa\in\mathcal{J}} \langle M_\kappa, M\rangle_A M_\kappa$, abbreviating $\langle M,N\rangle_A \equiv \dim_{\mathbb{C}} \mathrm{Hom}_A(M, N)$ as introduced in (I:4.5). When M is simple, this yields (ii).

(iii) Symmetry of $C^\sigma_{\kappa\rho}$ is a consequence of proposition 2.7(i).
Specialising the general relation $\mathrm{Hom}_A(M, N) \cong \bigoplus_{\lambda\in\mathcal{J}} \mathrm{Hom}_A(M, M_\lambda) \otimes \mathrm{Hom}_A(M_\lambda, N)$ to the case $M = (M_\kappa)^\sigma$ and $N = (M_\rho)^\sigma$ and comparing dimensions results in $\delta_{\kappa,\rho} = \sum_{\lambda\in\mathcal{J}} C^\sigma_{\kappa\lambda} C^\sigma_{\lambda\rho}$.

(iv) Since $(M_\rho)^\sigma$ is again simple, it is isomorphic to precisely one M_κ with $\kappa \in \mathcal{J}$. It follows that C^σ is a permutation, which by (iii) has order two.

(v) follows directly from proposition 2.7(ii). ✓

Remark 2.9:
(i) Recall that simple A-modules correspond to elementary boundary conditions. Thus the matrix C^σ is precisely the *boundary conjugation matrix* that was introduced, based on the idea of "complex charges" [13], in [21] (see also [28]). Also recall that the A-modules give rise to a NIMrep of the fusion rules of the category $\mathcal{C}$; the additional structure provided by the matrix C^σ has been termed an 'S-NIMrep' in [28]. We will return to this point in the context of annulus coefficients in section 3.2, where we also give an explicit formula for $C^\sigma_{\kappa\rho}$ in terms of the reversion σ on A.

(ii) Proposition 2.7(ii) implies that for *induced* modules the effect of conjugation is independent of σ. As a consequence, a characteristic feature of a reversion σ is its action on submodules of induced modules. Concretely, suppose that

$$\mathrm{Ind}_A(U_k) \cong M_1^{(k)} \oplus \cdots \oplus M_{n_k}^{(k)} \tag{2.35}$$

is the decomposition of the induced module $\mathrm{Ind}_A(U_k)$ into simple submodules. Because of $(U_k)^\vee \cong U_{\bar k}$, for any $i = 1, 2, \dots, n_k$, we then have $(M_i^{(k)})^\sigma \cong M_j^{(\bar k)}$ for some j. These conditions restrict the possible actions of σ on simple modules. For example, for $k = \bar k$ every σ only permutes the simple submodules of $\mathrm{Ind}_A(U_k)$ amongst themselves.

(iii) When A is not haploid, then it is not simple as a module over itself, and boundary conjugation acts, in general, non-trivially on the simple submodules of A.

In order to discuss defect lines on non-orientable world sheets we need a notion of conjugation also for bimodules. In fact we will make use of three different types of conjugation. Our conventions for A-B-bimodules are given in definition I:4.5. An A-B-bimodule X is a triple $X = (X, \rho, \tilde\rho)$, where ρ and $\tilde\rho$ are a left action of A and a commuting right action of B on X, respectively. The following result is a straightforward consequence of the definition of σ and of a left/right action. It can be established by moves analogous to (2.23); we omit the details of the proof.

Proposition 2.10:
Let $X = (\dot{X}, \rho, \tilde{\rho})$ be an A-B-bimodule. Then

$$X^s = (\dot{X}, \rho^s, \tilde{\rho}^s) \qquad \text{and} \qquad X^v = (\dot{X}^\vee, \rho^v, \tilde{\rho}^v) \tag{2.36}$$

with

$$\begin{aligned} \rho^s &:= \tilde{\rho} \circ c_{BX} \circ (\sigma_B \otimes id_X)\,, & \tilde{\rho}^s &:= \rho \circ c_{XA} \circ (id_X \otimes \sigma_A)\,, \\ \rho^v &:= (\tilde{d}_B \otimes id_{\dot{X}^\vee}) \circ (id_B \otimes (\tilde{\rho})^\vee)\,, & \tilde{\rho}^v &:= (id_{\dot{X}^\vee} \otimes d_A) \circ ((\tilde{\rho})^\vee \otimes id_A)\,, \end{aligned} \tag{2.37}$$

are B-A-bimodules, and

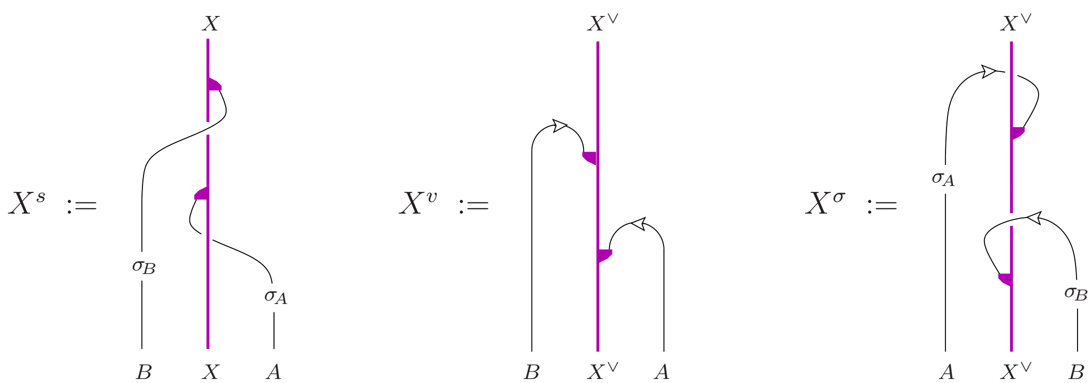
$$X^\sigma = (\dot{X}^\vee, \rho^\sigma, \tilde{\rho}^\sigma) \tag{2.38}$$

with

$$\rho^\sigma := (\tilde{d}_A \otimes id_{\dot{X}^\vee}) \circ (\sigma_A \otimes (\rho \circ c_{XA})^\vee)\,, \qquad \tilde{\rho}^\sigma := (id_{\dot{X}^\vee} \otimes d_B) \circ ((\tilde{\rho} \circ c_{BX})^\vee \otimes \sigma_B) \tag{2.39}$$

is an A-B-bimodule.

Pictorially, the three types of conjugation look as follows:



(2.40)

In relation with defect lines on non-orientable surfaces one is lead to investigate the conjugate module X^s in more detail. The following proposition tells us how this conjugation acts on α-induced bimodules (recall the definition of the latter from section I:5.4).

Proposition 2.11:
For U an object of $\mathcal{C}$, there are the isomorphisms

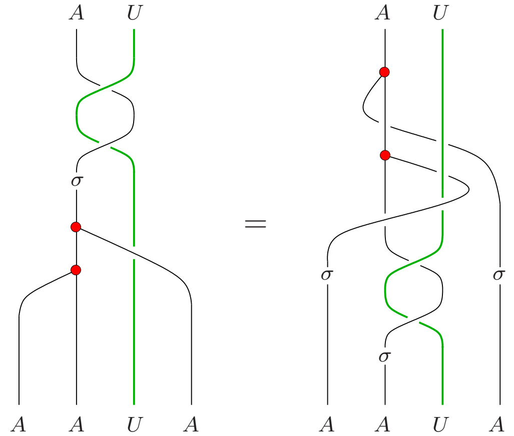
$$(\alpha^+(U))^s \cong \alpha^-(U) \qquad \text{and} \qquad (\alpha^-(U))^s \cong \alpha^+(U)\,. \tag{2.41}$$

of A-A-bimodules

Proof:
Let us demonstrate $(\alpha^+(U))^s \cong \alpha^-(U)$; the second identity can be seen in a similar fashion. Consider the morphism $\psi := c_{UA} \circ c_{AU} \circ (\sigma \otimes id) \in \mathrm{Hom}(A \otimes U, A \otimes U)$. Clearly, ψ is invertible.

Further, denoting by ρ and $\tilde{\rho}$ the left/right representation of A for $\alpha^-(U)$, the following moves establish that ψ commutes with the representation morphisms of $\alpha^\pm(U)$:

$$\psi \circ \tilde{\rho} \circ (\rho \otimes id_A) \;=$$



$$= \; \tilde{\rho}^s \circ (\rho^s \otimes id_A) \circ (id_A \otimes \psi \otimes id_A) \,. \tag{2.42}$$

Thus we have $\psi \in \mathrm{Hom}_{A|A}(\alpha^-(U), (\alpha^+(U))^s)$, showing that indeed the two bimodules are isomorphic. ✓

2.4 Reversions on Morita equivalent algebras

When studying the relationship between algebras and conformal field theory, it is natural to introduce a concept of CFT-equivalence for algebras. Loosely speaking:

> *Two algebras are CFT-equivalent iff the associated full conformal field theories have the same correlators, up to a suitable relabelling of fields, boundary conditions and defects.*

It was already noted in [I] that when considering the class of CFTs defined on oriented world sheets, two (symmetric special Frobenius) algebras are CFT-equivalent if they are Morita equivalent. However, as was already mentioned above, the number of possible reversions for an algebra is not invariant under Morita equivalence. Thus when considering CFTs defined on unoriented world sheets and Jandl algebras, CFT-equivalence of algebras will amount to a certain refinement of the notion of Morita equivalence.

The purpose of this section is to provide a sufficient criterion, given in proposition 2.16, for when a reversion on an algebra A induces a reversion on a Morita equivalent algebra B. We expect that two simple Jandl algebras are CFT-equivalent if they are Morita equivalent and if the construction of proposition 2.16 is applicable. We will return to this issue in future work.

Suppose we are given a symmetric special Frobenius algebra A and a left A-module M. In this section we will in addition assume that A is a *simple* algebra, which means that $Z(A)_{00} = 1$. If this is not the case, A can be written as a direct sum of several simple algebras, which corresponds to the superposition of several CFTs, see the discussion at the end of section I:3.2.

Below we will several times make use of subobjects that are defined as images of idempotents. The conventions we use are summarised in the following definition.

Definition 2.12 :
Let $P \in \mathrm{Hom}(U,U)$ be an idempotent, $P \circ P = P$. A triple (S,e,r), consisting of an object $S \in \mathcal{O}bj(\mathcal{C})$ and morphisms $e \in \mathrm{Hom}(S,U)$ and $r \in \mathrm{Hom}(U,S)$, is called a *retract* of P iff

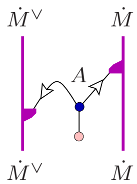
$$e \circ r = P \qquad \text{and} \qquad r \circ e = id_S \,. \tag{2.43}$$

The subobject S of U is then called the *image* of P.

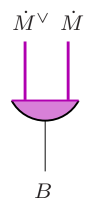
Since by assumption the modular tensor category $\mathcal{C}$ is in particular semisimple and abelian, morphisms e and r satisfying (2.43) exist for every idempotent P. Note further that $P\circ e = e\circ r\circ e = e$. Together with a similar argument for r, this leads to the relations

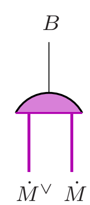
$$P\circ e = e \qquad \text{and} \qquad r\circ P = r\,. \tag{2.44}$$

Given an A-module M, consider the idempotent $P_{\otimes A}\in \mathrm{Hom}(M^\vee{\otimes}M, M^\vee{\otimes}M)$ given by

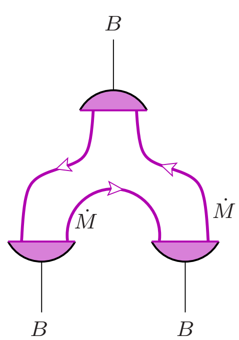
$P_{\otimes A}$:=  (2.45)

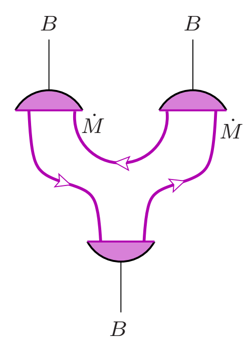
That this in indeed an idempotent follows from the same moves as in (I:5.127). The image of this idempotent is a subobject B of $M^\vee{\otimes}M$, isomorphic to $M^\vee{\otimes}_A M$. The graphical notation we will use for e and r is

e := 

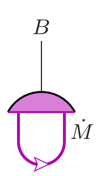
r :=  (2.46)

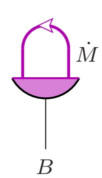
Proposition 2.13:
Let A be a simple symmetric special Frobenius algebra, M an A-module, and let B the image of the idempotent $P_{\otimes A}$ in (2.45). The object B together with the morphisms $m, \eta, \Delta, \varepsilon$ given by

m := 

Δ := $\dfrac{\dim(A)}{\dim(\dot M)}$ 

(2.47)

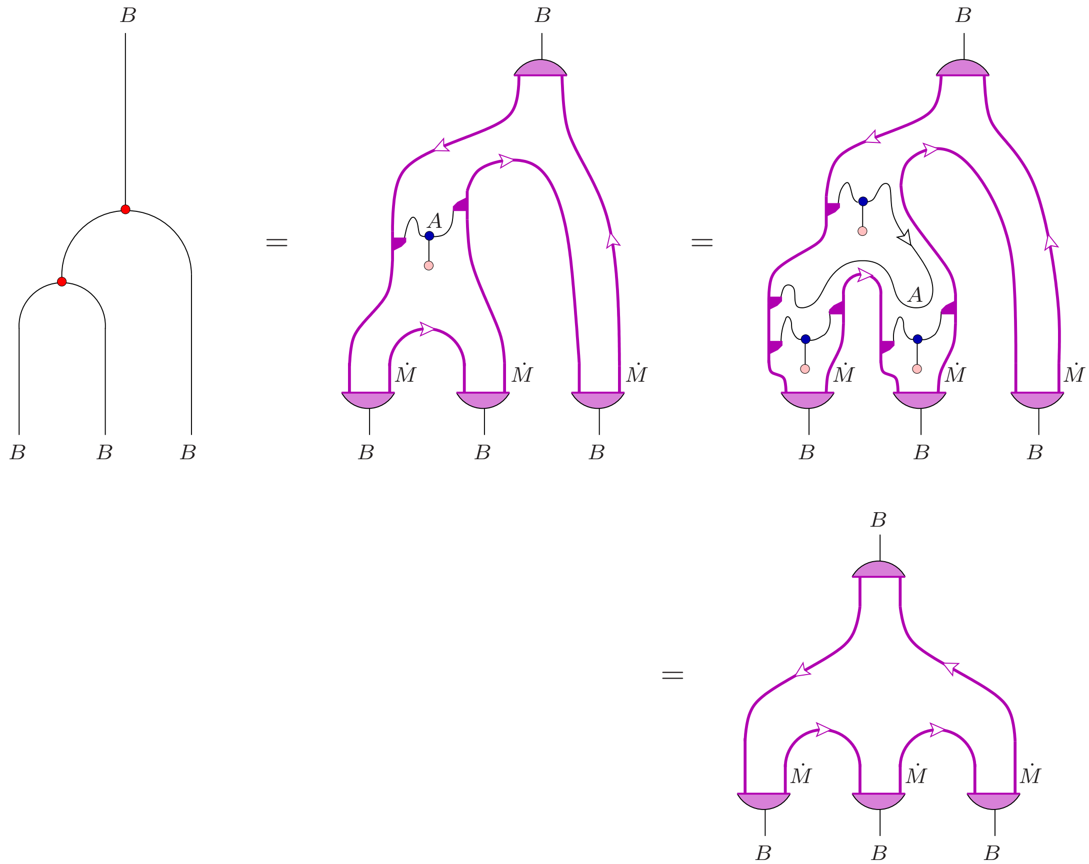
η := 

ε := $\dfrac{\dim(\dot M)}{\dim(A)}$ 

is a simple symmetric special Frobenius algebra.

Proof:
As an example, let us demonstrate that B is associative, special and simple; we omit the details

of the analogous arguments for the unit, counit, co-associativity, Frobenius and symmetry properties. Consider the moves

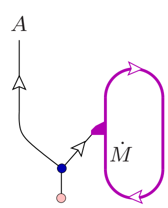


(2.48)

In the first step the definition (2.47) for the multiplication on B is inserted as well as the relation (2.43) to get the idempotent $P_{\otimes A}$. In the second step the two leftmost e's have been replaced by $P_{\otimes A} \circ e$; using a series of moves involving the representation property of the module M as well as the properties of A, the upper A ribbon can be attached with both of its ends to the left M-ribbon. Using the representation property once more, together with the fact that A is symmetric and special, the A-loop can be removed altogether; this has been done in step three. Comparison with the 'mirrored' analogue of the above manipulations then shows that B is associative.

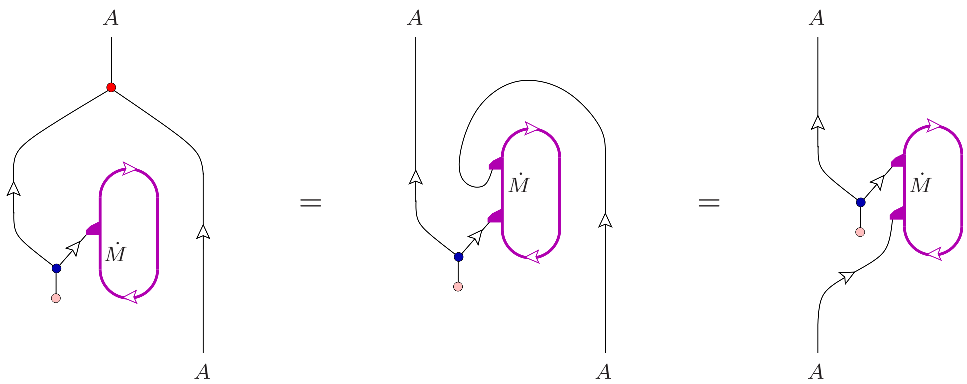
Specialness is a bit harder to see. Define the morphism $\phi \in \mathrm{Hom}(\mathbf{1}, A)$ by

ϕ :=



(2.49)

The morphism ϕ has the property $m\circ(\phi\otimes id) = m\circ(id\otimes\phi)$. To see this, note the following moves on $m\circ(\phi\otimes id)$:



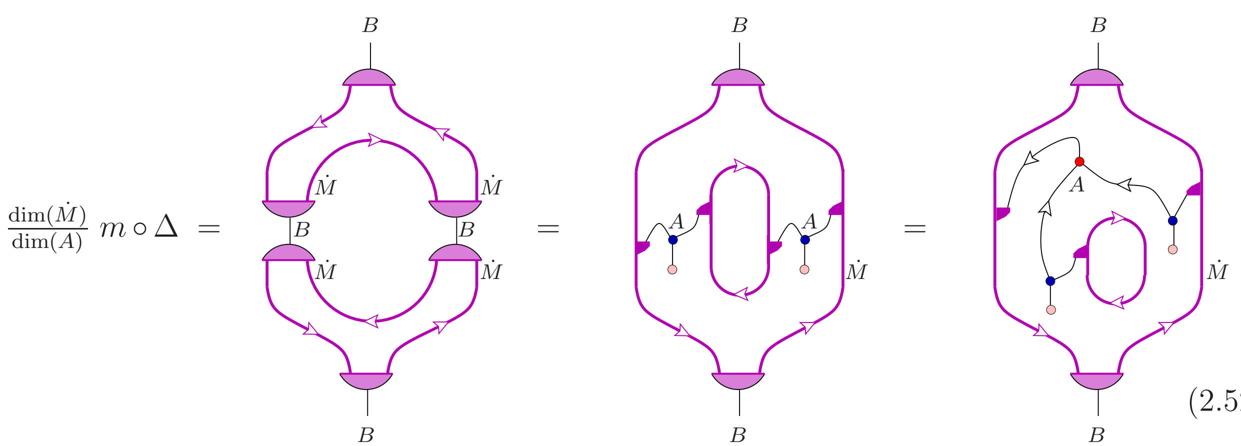
(2.50)

The first step uses the various associativity properties and symmetry of A, as well as the representation property of the module M. In the second step the ribbon graph is deformed to take the upper representation morphism around the annular M-ribbon. By a similar set of moves one can arrive at the right hand side of (2.50) from $m\circ(id\otimes\phi)$, thus establishing the stipulated equality.

As in section I:3.4 we denote the vector space $\mathrm{Hom}(\mathbf{1}, A)$ by $A_{\rm top}$. This space has a distinguished subspace denoted by $\mathrm{cent}_A(A_{\rm top})$ (see definition I:3.15); the observation above just tells us that $\phi\in\mathrm{cent}_A(A_{\rm top})$. On the other hand, the unit η of A is always in $\mathrm{cent}_A(A_{\rm top})$. Now by theorem I:5.1(iii) the dimension of this space is equal to $Z(A)_{00}$. In our case A is simple, so that $Z(A)_{00} = 1$ and hence $\phi = \lambda\eta$ for some complex number λ. Composing both sides with the counit ε we find that $\lambda = \dim(\dot M)/\dim(A)$, i.e.

$$\phi = \frac{\dim(\dot M)}{\dim(A)}\,\eta\,. \tag{2.51}$$

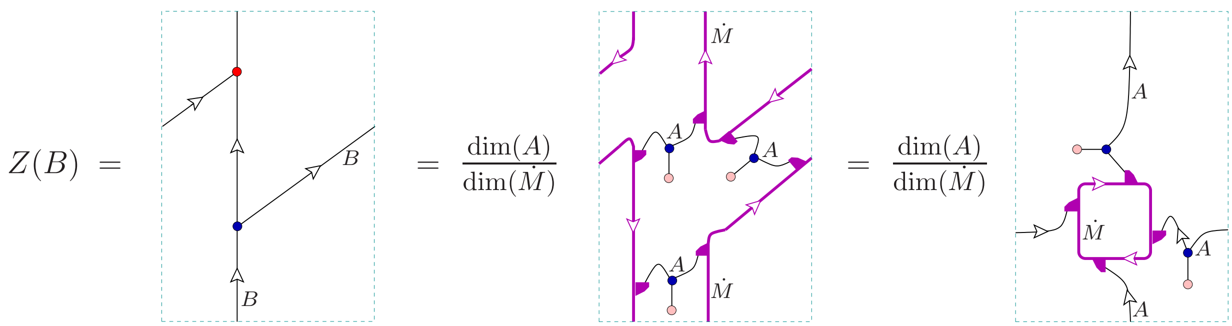
After these prelimiaries we can show that



(2.52)

In the first step the definitions (2.47) are substituted and in the second step the combinations $e\circ r$ have been replaced by $P_{\otimes A}$ from (2.45). The combination of the two A-ribbons attached to the closed M-loop in the middle figure above can be deformed to correspond to the right hand side of (2.50). The third step amounts to replacing it by the lhs of (2.50). Finally using (2.51) we can get rid of the annular M-ribbon altogether and are left with $\dim(\dot M)/\dim(A)$ times the

combination $r \circ P_{\otimes A} \circ e$ which is equal to id_B. The second part of the specialness property, i.e. $\varepsilon \circ \eta = \dim(B)$, follows from the identity (I:3.49), applied to the algebra B. Indeed, composing (I:3.49) with η and using $m \circ \Delta = id_B$, which was established above, gives $\varepsilon \circ \eta$ on the right and $\dim(B)$ on the left hand side, respectively.
Thus B is special. Next we treat simplicity of B, i.e. show that $Z(B)_{00} = 1$. It is a general principle (to be explained in detail in a separate publication) that Morita equivalent algebras lead to identical CFTs on oriented surfaces. In particular we have $Z(A) = Z(B)$ when A and B are Morita equivalent; it is this more general equality that will be checked in the sequel.
Consider the ribbon graph for $Z(B)$, as given in figure (I:5.24) with all A-ribbons replaced by B's:



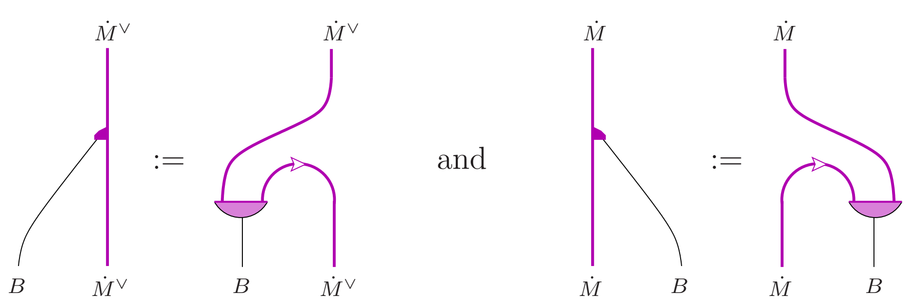
(2.53)

Here in the second step the definition of m and Δ as in (2.47) is substituted and the morphisms e and r are pulled around a cycle of the torus so as to combine them to $P_{\otimes A}$'s. In the second figure note that the top horizontal A-ribbon can be removed with a move similar to the one indicated in the second-to-last figure of (2.48). The third equality amounts to shrinking the M-ribbon and shifting the fundamental domain of the torus such that the M-loop is in the center of the figure. Finally, making use of the representation property and the relation (2.51) above, the ribbon graph on the right hand side of (2.53) can be transformed into that of $Z(A)$. ✓

So far we have established that $B = M^\vee \otimes_A M$ is again a simple symmetric special Frobenius algebra. To establish Morita equivalence of A and B we need A-B-bimodules ${}_AM_B$ and ${}_BM_A$ such that

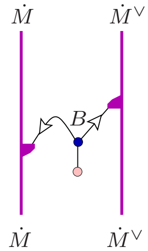
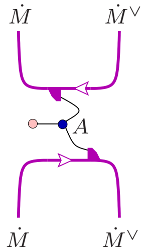
$$B \cong {}_BM_A \otimes_A {}_AM_B \qquad \text{and} \qquad A \cong {}_AM_B \otimes_B {}_BM_A \tag{2.54}$$

as B-B- and A-A-bimodules, respectively. Below we will show that the left A-module M is also a right B-module and that $M^\vee$ is a left B-module. Then it is checked that indeed $A \cong M \otimes_B M^\vee$. The left and right module structures on $M^\vee$ and M are simply given by

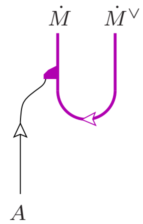
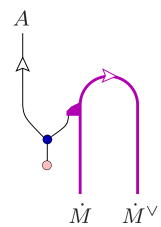


(2.55)

It is easy to check that these indeed fulfill the representation properties (I:4.2). The idempotent $P_{\otimes B}$ on $M \otimes M^\vee$ whose image is $M \otimes_B M^\vee$ takes the form

$P_{\otimes B} \;=$  $= \dfrac{\dim(A)}{\dim(\dot M)}$  (2.56)

To establish that the image of $P_{\otimes B}$ is indeed A we give explicitly the morphisms $e_B \in \mathrm{Hom}(A, M \otimes M^\vee)$ and $r_B \in \mathrm{Hom}(M \otimes M^\vee, A)$:

$e_B \;=$  $\qquad r_B \;=\; \dfrac{\dim(A)}{\dim(\dot M)}$  (2.57)

One can check that these obey the relations (2.43) and (2.44), with A replaced by B. Note that all this relies on A being simple, for we need the relation (2.51) to remove the M-loops that occur in the calculation. One can further verify that the algebra structure on $M \otimes_B M^\vee$ agrees with that of A. Altogether we have established the following

Theorem 2.14 :
Let A be a simple symmetric special Frobenius algebra in $\mathcal{C}$. Given a left A-module M that is not a zero object, set $B := M^\vee \otimes_A M$. The morphisms (2.47) turn B into a simple symmetric special Frobenius algebra such that $A \cong M \otimes_B M^\vee$ and $B \cong M^\vee \otimes_A M$ as A-A- and B-B-bimodules, respectively. In particular, A and B are Morita equivalent.

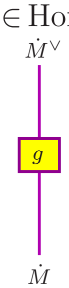
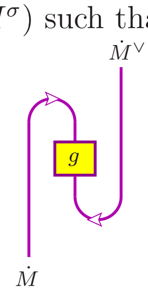
As an aside, note that composing the identities (I:3.49) with the unit η, replacing all A's by B's and recalling that B is special, we find the relation $\varepsilon \circ \eta = \dim(B)$ for the unit and counit of B. On the other hand, starting from the definitions (2.47) we find $\varepsilon \circ \eta = (\dim(\dot M))^2/\dim(A)$. This leads to

Corollary 2.15 :
If two simple symmetric special Frobenius algebras A and B are related by the isomorphisms (2.54) with ${}_BM_A = ({}_AM_B)^\vee$ then, with $M = {}_AM_B$,

$$\dim(A)\dim(B) = (\dim(\dot M))^2 \,. \qquad (2.58)$$

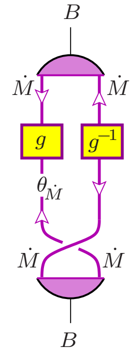
Let us now turn to the question whether a reversion on A gives rise to a reversion on B.

Proposition 2.16 :
Let A be a simple Jandl algebra, with reversion σ. Let M be a left A-module and $B = M^\vee \otimes_A M$. Suppose there is an isomorphism $g \in \mathrm{Hom}_A(M, M^\sigma)$ such that

 $= \varepsilon_g$  (2.59)

Then $\varepsilon_g \in \{\pm 1\}$, and $\tilde{\sigma}_g \in \mathrm{Hom}(B,B)$ given by [2]

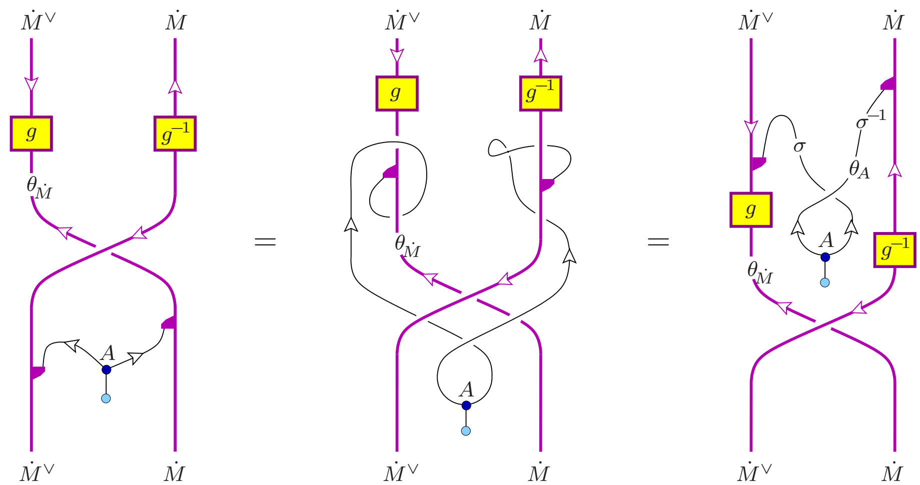
$$\tilde{\sigma}_g \;:=\; \varepsilon_g \qquad (2.60)$$



is a reversion on B.

Proof:
For the sake of this proof we abbreviate $\tilde{\sigma} \equiv \tilde{\sigma}_g$. That ε_g takes values in $\{\pm 1\}$ follows immediately from applying (2.59) twice and deforming the resulting ribbon graph.
Let $s \in \mathrm{Hom}(M^\vee \otimes M, M^\vee \otimes M)$ be given by figure (2.60) with r and e at the top and bottom removed, such that $\tilde{\sigma} = r \circ s \circ e$. The moves below establish the identity $s \circ P_{\otimes A} = P_{\otimes A} \circ s$:

$$\varepsilon_g \, s \circ P_{\otimes A} \;= \qquad = \qquad = \qquad (2.61)$$



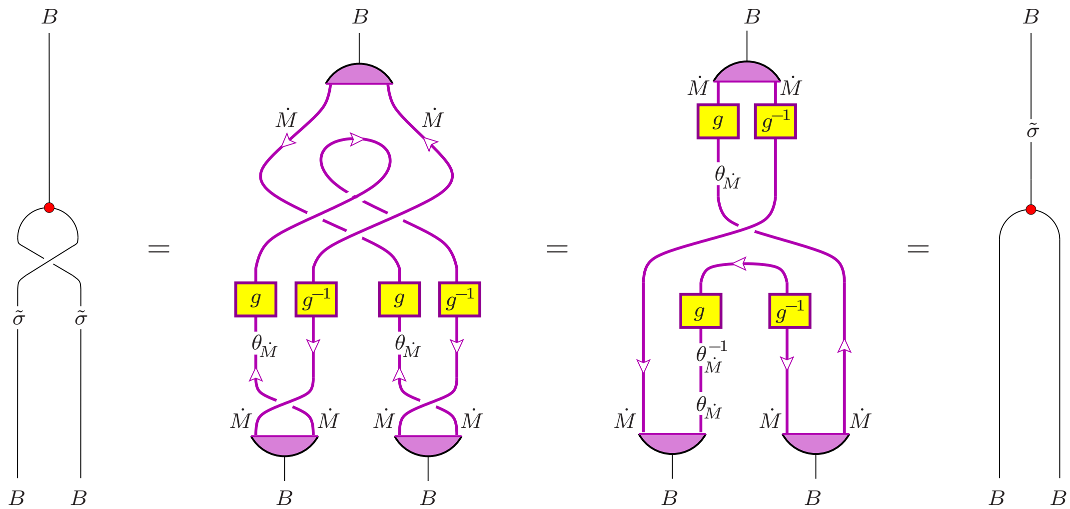
The second equality is just a deformation of the ribbon graph. In the third equality we use that $g \in \mathrm{Hom}_A(M, M^\sigma)$, which allows us to take the representation morphisms past g and g^{-1}. Recall that the representation of A on M^σ is of the form (2.25). The right-most graph in (2.61) is equal to $\varepsilon_g \, P_{\otimes A} \circ s$: the twist θ_A on the A-ribbon combines with σ^{-1} to σ so that the braiding in front of the coproduct can be removed with the help of (2.5).
It then follows that

$$\tilde{\sigma} \circ \tilde{\sigma} = r \circ s \circ e \circ r \circ s \circ e = r \circ s \circ P_{\otimes A} \circ s \circ e = r \circ P_{\otimes A} \circ s \circ s \circ e \,. \qquad (2.62)$$

Further we have $r \circ P_{\otimes A} = r$, and one quickly checks that $s \circ s = \theta_{M^\vee \otimes M}$. Functoriality of the twist thus implies $\tilde{\sigma} \circ \tilde{\sigma} = r \circ \theta_{M^\vee \otimes M} \circ e = r \circ e \circ \theta_B$. The identity $r \circ e = id_B$ then gives the desired relation $\tilde{\sigma} \circ \tilde{\sigma} = \theta_B$.

[2] To avoid confusion about the labelling, let us recall some of the conventions of [I] for the pictorial representation of morphisms and ribbon graphs. Labels for objects of $\mathcal{C}$ that are placed above and below the ends of a ribbon denote the target and the source of the relevant morphism, respectively. In contrast, when a label for an object is placed *next* to a ribbon, it indicates the label, or 'color', of that ribbon (see section I:2.3). Thus e.g. a ribbon with an upwards-pointing arrow labelled by X represents the identity morphism in $\mathrm{Hom}(X,X)$, while a ribbon with a downwards-pointing arrow labelled by X represents the identity morphism in $\mathrm{Hom}(X^\vee, X^\vee)$.

To obtain $\tilde\sigma \circ m = m \circ c_{B,B} \circ (\tilde\sigma \otimes \tilde\sigma)$ one can use the following equalities:



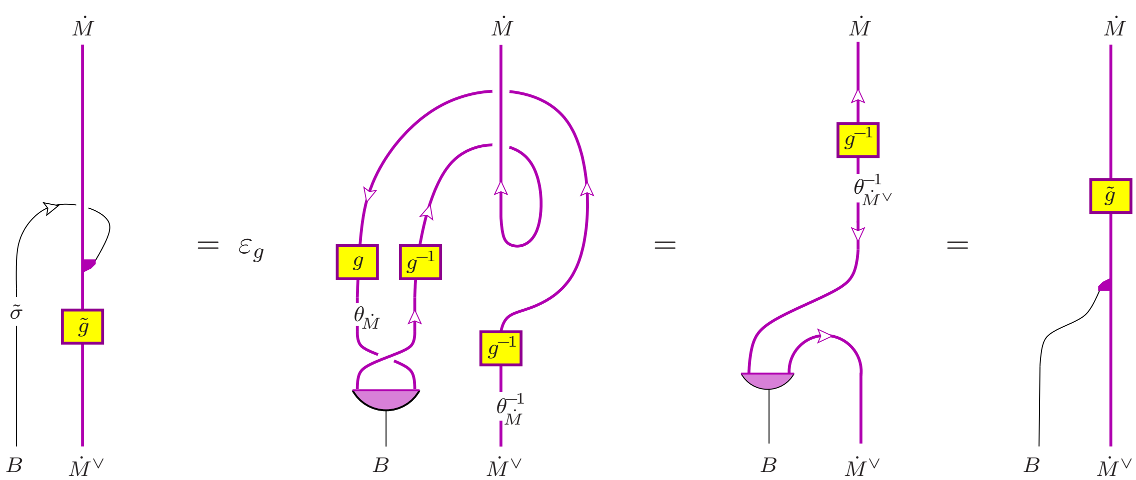
(2.63)

In the first step the definitions (2.47) and (2.60) for the multiplication and $\tilde\sigma$ are substituted. The second step is just a deformation of the ribbon graph. The see the last equality, replace the morphism g in the bottom part of the graph by the right hand side of (2.59); then it cancels against g^{-1}, and the factor ε_g combines with the resulting ribbon graph to $\tilde\sigma \circ m$.
In view of proposition 2.4, we have now established that $\tilde\sigma$ is a reversion on the symmetric special Frobenius algebra B. ✓

Proposition 2.16 allows us to construct a reversion $\tilde\sigma_g$ on $B = M^\vee \otimes_A M$ if we are given an isomorphism $g \in \mathrm{Hom}_A(M, M^\sigma)$ with property (2.59). In the remainder of this section we illustrate that this construction can be inverted, i.e. from $(B, \tilde\sigma_g)$ one can recover (A, σ):

Proposition 2.17:
Applying the prescription (2.60) twice yields the identity map, i.e. $\tilde{\tilde\sigma} = \sigma$.

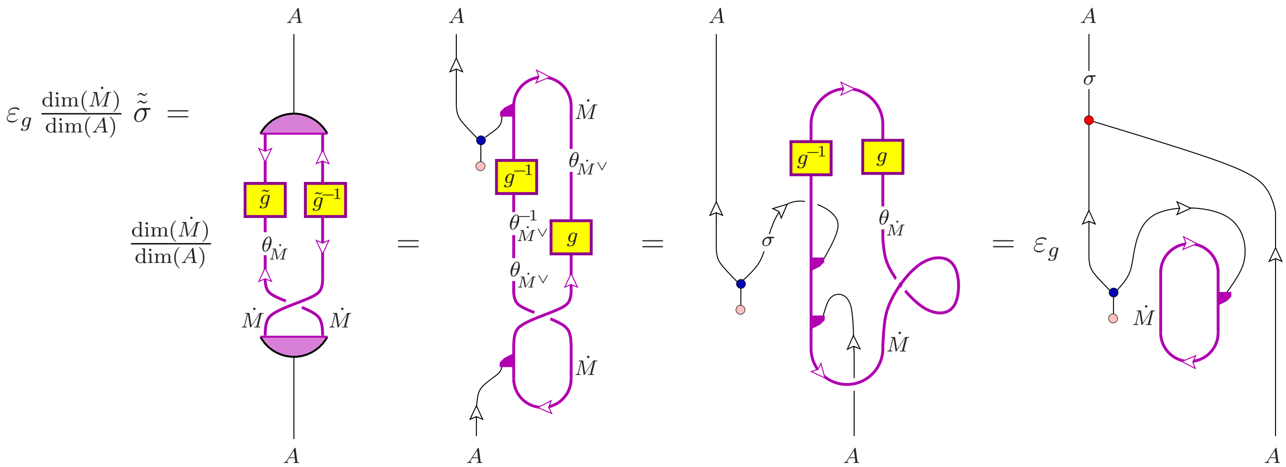
Proof:
Let us again abbreviate $\tilde\sigma \equiv \tilde\sigma_g$. Recall from (2.55) that $M^\vee$ is a left B-module and that $A \cong M \otimes_B M^\vee$, with A realised as a retract of $M \otimes M^\vee$ according to (2.57). Consider the morphism $\tilde g := g^{-1} \circ \theta^{-1}_{\dot M^\vee}$. The equalities



(2.64)

imply that $\tilde{g} \in \mathrm{Hom}_B(M^\vee, (M^\vee)^{\tilde{\sigma}})$. Here in the first step we substitute the definitions (2.55), (2.60) and use (2.61) to cancel the idempotent $P_{\otimes A}$ against e. In the second step (2.59) is inserted to combine a g and a g^{-1} and to remove the prefactor ε_g, and functoriality of the twist is used.

The morphism $\tilde{g}$ satisfies the analogue of (2.59), with all g's replaced by $\tilde{g}$ and $\varepsilon_g = \varepsilon_{\tilde{g}}$. This can be seen by composing both sides of (2.59) with θ and comparing their inverses. We can now apply proposition 2.16 to $(B, \tilde{\sigma})$ and $\tilde{g}$. This gives rise to a reversion $\tilde{\tilde{\sigma}}$ on A given by



(2.65)

In the second step the morphisms e_B and r_B from (2.57) are written out. The fourth equality follows when moving σ past the coproduct (with moves similar to (2.17)), canceling g against g^{-1} with the help of (2.59), cancelling the two twists on $\dot{M}$ and using the representation property on ρ_M. The $\dot{M}$-loop can be flipped from the right to the left (as is easily reproduced with actual ribbons, see also (3.87) below) so that the A-ribbon now is attached to the outside of the loop. In the right-most figure of (2.65) one can recognise the morphism ϕ given in (2.49). Since A is taken to be simple we can apply (2.50). This produces a factor $\dim(\dot{M})/\dim(A)$ and leaves us with $\tilde{\tilde{\sigma}} = \sigma$, i.e. we have indeed recovered the original reversion on A. ✓

2.5 On the classification of reversions

As already mentioned, the number of possible reversions is not constant on a Morita class $\mathcal{S}$ of simple symmetric special Frobenius algebras. Given one such algebra A one can pass to 'larger and larger' representatives $M^\vee \otimes_A M$ of its class $\mathcal{S}$ by taking reducible A-modules M with an arbitrarily large number of simple submodules. A priori it is very hard to control the number of solutions to the polynomial equations (2.20) that a reversion has to fulfill. However, it turns out that it is actually possible to describe all reversions on a given Morita class $\mathcal{S}$ by solving the polynomial equations (2.20) only for a finite number of 'small' representatives of $\mathcal{S}$.

The algorithm can be conveniently summarised in the language of *module categories* [58,59]. The category of left modules over an algebra in a tensor category $\mathcal{C}$ has the structure of a module category $\mathcal{M}$ over $\mathcal{C}$. Two algebras give rise to equivalent module categories precisely if they are Morita equivalent. Module categories therefore allow one to formulate statements that are invariant under Morita equivalence in an invariant way. More concretely, every algebra in a given Morita class is isomorphic to the *internal End* $\underline{\mathrm{End}}(M)$ of some object M of $\mathcal{M}$. In our algorithm for finding representatives for Jandl algebras, it is sufficient to restrict M to be either a simple object of $\mathcal{M}$ or else the direct sum of two non-isomorphic simple objects, $M \cong M_1 \oplus M_2$, that have the property that their internal Ends are isomorphic as opposite algebras, $\underline{\mathrm{End}}(M_1) \cong \underline{\mathrm{End}}(M_2)_{\mathrm{op}}$. Reversions need to be classified only on all algebras $\underline{\mathrm{End}}(M)$ with M of the form just described.

An explicit realisation of this algorithm is formulated in the following

Prescription:

1) Choose a haploid representative A of the Morita class $\mathcal{S}$.

2) Determine a complete set of (representatives of isomorphism classes of) simple modules $\{M_\kappa \,|\, \kappa \in \mathcal{J}\}$ of A.

The next step consists of two parts:

3a) For each of the algebras $B^{\mathrm{a}}_\kappa := M^\vee_\kappa \otimes_A M_\kappa$ with $\kappa \in \mathcal{J}$, find all solutions to the requirements (2.20) that a reversion must satisfy.

3b) For each of the algebras $B^{\mathrm{b}}_{\kappa\kappa'} := (M_\kappa \oplus M_{\kappa'})^\vee \otimes_A (M_\kappa \oplus M_{\kappa'})$ with $\kappa, \kappa' \in \mathcal{J}$, $\kappa \neq \kappa'$ and $M^\vee_\kappa \otimes_A M_\kappa \cong M^\vee_{\kappa'} \otimes_A M_{\kappa'}$ as objects in $\mathcal{C}$, find all solutions to (2.20) that act as a permutation $(\alpha, \beta) \mapsto (\beta, \alpha)$ on $\mathbb{C} \oplus \mathbb{C} \cong B^{\mathrm{b}}_{\kappa\kappa',\mathrm{top}}$.

We denote by $\mathcal{B} = \{(B_x, \sigma_x) \,|\, x \in S\}$, for some index set S, the set of all algebras with reversion (up to isomorphism) constructed in steps 3a) and 3b). Then all other reversions on $\mathcal{S}$ can be linked to one already appearing in $\mathcal{B}$ in the following manner:

4) Assume that σ_C is a reversion on some $C \in \mathcal{S}$. Then there exist $(B_x, \sigma_x) \in \mathcal{B}$, together with a left B_x-module N and a morphism $g \in \mathrm{Hom}_{B_x}(N, N^{\sigma_x})$ fulfilling (2.59) with $\varepsilon_g \in \{\pm 1\}$, such that (C, σ_C) is isomorphic as a Jandl algebra to $(N^\vee \otimes_{B_x} N, \tilde{\sigma}_g)$ with $\tilde{\sigma}_g$ defined as in (2.60).

Remark 2.18:

(i) In short, the non-linear equations (2.20) need to be solved only for a finite number of cases, namely for the algebras B^{a}_κ and $B^{\mathrm{b}}_{\kappa\kappa'}$. For all other algebras in $\mathcal{S}$ only a linear problem must be solved. Furthermore, these particular representatives of $\mathcal{S}$ are 'small', so that this task is much easier to fulfill than for a generic representative. Of particular convenience is that the algebra A can be taken to be haploid (by proposition A.4, this is always possible). The algorithm could be easily adapted to non-haploid representatives, but the concrete calculations would then become more involved.

(ii) In CFT terms the above procedure can be interpreted as follows. A full CFT is constructed from the algebra of boundary fields on a given D-brane. In order to define the full CFT on non-orientable surfaces, the algebra must be equipped with a reversion. Now all distinct ways to define the full CFT on non-orientable surfaces can already be found by considering reversions on D-branes which are either elementary or else a superposition of two distinct elementary D-branes that have the same boundary field content.

(iii) Note that in steps 1) – 4) one is only concerned with reversions on *simple* symmetric special Frobenius algebras. In general the algebra could be a direct sum

$$A \;\cong\; A_1 \oplus A_2 \oplus \cdots \oplus A_n \tag{2.66}$$

of simple algebras A_k in the sense of proposition I:3.21. In the category of vector spaces treated in section 4.1 below, this corresponds to having a direct sum of several full matrix algebras. It does not add much complication to extend the treatment to non-simple algebras. The reversion σ either maps a component algebra A_k to itself, or else it maps A_k to the component $A_{\pi(k)}$ with $\pi(k) \neq k$ for some order two permutation π. In the first case σ must be of the form described in the algorithm. In the second case, we may suppose that $k < \pi(k)$. Then as will be shown in appendix A.2, with an appropriate choice of basis σ acts like the identity from A_k to $A_{\pi(k)}$ and like the twist from $A_{\pi(k)}$ to A_k. More specifically, let $U \in \mathcal{O}bj(\mathcal{C})$ be isomorphic to A_k and $A_{\pi(k)}$ as an object of $\mathcal{C}$; then there are isomorphisms $\varphi_1 \in \mathrm{Hom}(U, A_k)$ and $\varphi_2 \in \mathrm{Hom}(U, A_{\pi(k)})$ such that $\varphi_2^{-1} \circ \sigma \circ \varphi_1 = id_U$ and $\varphi_1^{-1} \circ \sigma \circ \varphi_2 = \theta_U$. (That θ_U appears is required in order that σ squares to the twist.)
As discussed in remark I:5.4(i) and at the end of section I:3.2, a direct sum $A \cong A_1 \oplus A_2$ of two algebras corresponds to the direct sum of the two associated CFTs; this may be written symbolically as $\mathrm{CFT} = \mathrm{CFT}_1 + \mathrm{CFT}_2$. If $A_2 \cong A_{1,\mathrm{op}}$, then there exists a reversion on A that exchanges the two subalgebras. On the CFT side this implies, for instance, that transporting a field in CFT_1 around the circumference of a Möbius strip, one obtains a field in CFT_2 and vice versa.

The statements implicit in the steps 1) – 4) above are proven in appendix A.2. Here we just summarise the strategy of the proof. First, the structure of the $\mathbb{C}$-algebra A_{top} is analyzed. It turns out to be a semisimple algebra over $\mathbb{C}$ with an involution (an involutive anti-algebra homomorphism) σ_{top}. Further, one sees that idempotents in A_{top} give rise to idempotents on the whole algebra A. If A is simple, then the image of A under such an idempotent turns out to be Morita equivalent to A. This provides a direct way to show that every simple symmetric special Frobenius algebra is Morita equivalent to a haploid algebra.

For a similar reasoning in the case of Jandl algebras, we use idempotents in A_{top} that are left invariant by the involution σ_{top}. The image of A under such an idempotent is then equivalent, as a Jandl algebra, to A. Using the classification of semisimple complex algebras with involution, one arrives at the alternatives 3a) and 3b) in the description above. An additional argument, which uses Morita equivalence of full matrix algebras over $\mathbb{C}$, finally excludes direct sums of two isomorphic simple modules.

3 Partition functions on unoriented world sheets

In this section we present the construction of the connecting three-manifold and the embedded ribbon graph for unoriented world sheets X without field insertions. X may be non-orientable and may have a non-empty boundary. The Möbius strip and the Klein bottle will be treated explicitly.

For oriented world sheets the construction of the connecting manifold and the ribbon graph has already been achieved in section I:5.1. The procedure given below restricts to the one of section I:5.1 in the oriented case. There is, however, an important difference. The construction in section I:5.1 works for any symmetric special Frobenius algebra A; it requires both an orientable world sheet *and* the specification of an orientation. The construction below, on the other hand, requires again a symmetric special Frobenius algebra A, but in addition the specification of a reversion on A. On the other hand, if A admits a reversion at all, then the correlation functions of the associated conformal field theory on any oriented world sheet do not depend on the actual choice of orientation, provided that the change of orientation is accompanied by an appropriate conjugation on boundary conditions and fields.

The labelling of boundary segments is slightly different from the oriented case, too. Instead of associating an individual left A-module to a boundary segment, we specify an equivalence class as follows. We consider the pairs[3] (M, or_1) consisting of a left A-module M and an orientation or_1 of the boundary segment, which in the cases studied here – no field insertions – is topologically an S^1. Boundary conditions are such pairs modulo the equivalence relation

$$(M, \mathrm{or}_1) \sim (M', \mathrm{or}_1') \quad \text{if either} \quad \mathrm{or}_1' = \mathrm{or}_1 \text{ and } M' \cong M \\ \text{or} \quad \mathrm{or}_1' = -\mathrm{or}_1 \text{ and } M' \cong M^\sigma . \tag{3.1}$$

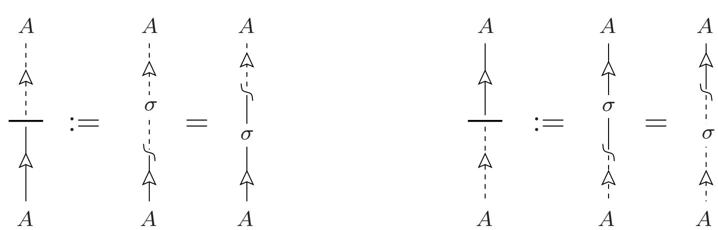
Thus the set of all boundary conditions is the quotient

$$\mathcal{B} = \{\, (M, \mathrm{or}_1) \,\} \,/\!\sim\; =: \{\, [M, \mathrm{or}_1] \,\} . \tag{3.2}$$

When displaying pictures we still like to draw lines instead of ribbons, so we introduce some more notation to keep track of the orientation and half-twists of the ribbons. The ‘white’ side of a ribbon will still be drawn as a solid line, while for the opposite, ‘black’, side we use a dashed line. In other words, ribbons are parallel to the paper plane, and if a ribbon is drawn as a solid line, then its orientation agrees with that of the paper plane (i.e. the standard orientation of $\mathbb{R}^2$), while if it is drawn as a dashed line, then it has the opposite orientation. Half-twists will be drawn as follows:


etc. (3.3)

Half-twists occur usually in combination with the morphism σ. It is therefore convenient to abbreviate this combination, as we already have done in (2.2); to save space, in the sequel we also use the notation

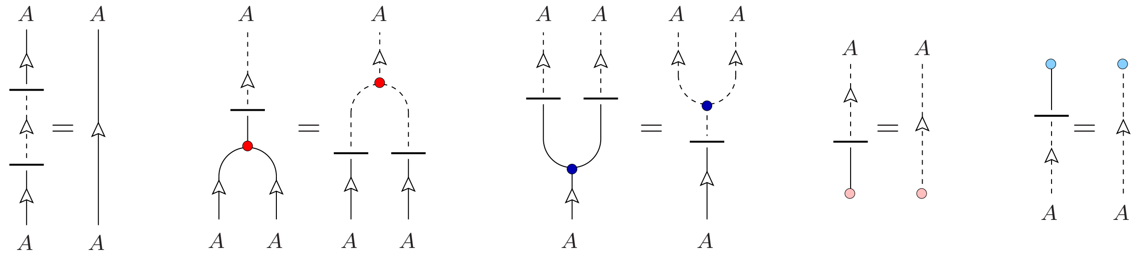


(3.4)

for the reversing move that we introduced in (2.2). Also recall that the reversing move is not a morphism; instead, it is to be understood as an abbreviation for a piece of ribbon graph embedded in a three-manifold. Further, when the coupon representing a morphism has only dashed lines attached to it, it is implied that we are also looking at the black side of the coupon.

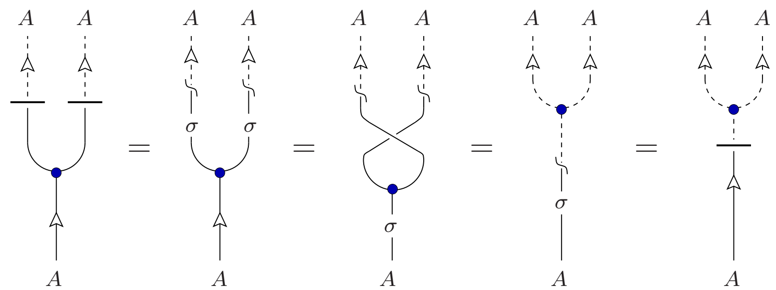
[3] The subscript ‘1’ in or_1 reminds us that or_1 refers to the orientation of a one-dimensional manifold. Below we also encounter orientations or_2 of surfaces.

The reversing move (3.4) satisfies



(3.5)

as well as the corresponding relations with all 2-orientations reversed. The properties (3.5) are direct consequences of (2.4) and (2.5). For example,



(3.6)

3.1 The connecting manifold and the ribbon graph

As input for the construction we first must specify the rational CFT we are considering. This is done by giving the modular tensor category $\mathcal{C}$ which encodes the chiral data, plus a symmetric special Frobenius algebra A in $\mathcal{C}$ and a reversion σ on A. Second, we must specify the world sheet X for which the correlator is to be computed. Each connected component of the boundary ∂X must be labelled by a class $[M, \mathrm{or}_1] \in \mathcal{B}$.

The correlator for the world sheet X is an element of the space of conformal blocks on the double $\hat{X}$ of X. The double is the orientation bundle modulo an equivalence relation:

$$\hat{X} = \mathrm{Or}(X) \,/ \sim \qquad \text{with} \quad (x, \mathrm{or}_2) \sim (x, -\mathrm{or}_2) \ \text{ iff } \ x \in \partial X\,. \tag{3.7}$$

We denote the points of $\hat{X}$ as equivalence classes $[x, \mathrm{or}_2]$.

The three-dimensional topological field theory associated to the modular tensor category $\mathcal{C}$ (see sections I:2.3 and I:2.4 for a short overview and references) provides us with a state space $\mathcal{H}(\hat{X})$ and, for any three-manifold M_X with boundary $\hat{X}$ and embedded ribbon graph, a map $Z(\mathrm{M}_X, \emptyset, \hat{X})\colon \mathbb{C} \to \mathcal{H}(\hat{X})$. The crucial point of the construction is to find a (universal) prescription for M_X such that the image of $1 \in \mathbb{C}$ under the map Z is precisely the correlation function we are searching.

As a three-manifold, M_X is the *connecting manifold* for X, defined as [52, 50, 51]

$$\mathrm{M}_X := \hat{X} \times [-1,1] \,/ \sim \qquad \text{with} \quad ([x, \mathrm{or}_2], t) \sim ([x, -\mathrm{or}_2], -t)\,. \tag{3.8}$$

Again we denote equivalence classes by square brackets, i.e. write $[x, \mathrm{or}_2, t]$ for points of M_X. By construction, the boundary of M_X is indeed just the double $\hat{X}$, and there is a natural embedding $\iota_X\colon X \to \mathrm{M}_X$ as well as a projection $\pi_X\colon \mathrm{M}_X \to X$, given by

$$\iota_X(x) = [x, \pm\mathrm{or}_2, 0]\,, \qquad \pi_X([x, \mathrm{or}_2, t]) = x\,. \tag{3.9}$$

Both choices of sign in the definition of ι_X refer to the same equivalence class. In fact, M_X is nothing else than a 'thickening' of the world sheet X. Conversely, X is a retract of M_X, so that going from X to M_X does not introduce any new homotopical information.

To turn the double $\hat{X}$ of the world sheet into an extended surface we still need to fix a Lagrangian subspace $\lambda_X \subset H_1(\hat{X},\mathbb{R})$, see e.g. section IV.4.1 of [60] for details. We take it to be the kernel of the inclusion homomorphism $H_1(\partial \mathrm{M}_X,\mathbb{R}) \to H_1(\mathrm{M}_X,\mathbb{R})$. In short, λ_X is spanned by all cycles in $H_1(\hat{X},\mathbb{R})$ which are contractible within M_X.

Remark 3.1:
An alternative definition of the Lagrangian subspace is given in lemma 3.5 of [51]. Recall that X can be regarded as the quotient of $\hat{X}$ by the orientation reversing involution $\sigma\colon \hat{X} \to \hat{X}$ (not to be confused with some reversion on an algebra) that acts as $[x,\mathrm{or}_2] \mapsto [x,-\mathrm{or}_2]$, and that there is an induced action σ_* of this involution on $H_1(\hat{X},\mathbb{R})$. In [51] the Lagrangian subspace is taken to be the eigenspace of σ_* to the eigenvalue -1; we denote this space by $\lambda_-(X)$.
That the two definitions are equivalent, i.e. that $\lambda_-(X) = \lambda_X$, can be seen as follows. Choose a basis of cycles in $\hat{X}$ that generates $H_1(\hat{X},\mathbb{Z})$. Since the action σ_* is induced by the action of σ on cycles, in this basis σ_* is given by a matrix with integral entries. It follows that we can also find vectors $\{v_i^-\}$ with integral entries that form a basis of the eigenspace $\lambda_-(X)$ in $H_1(\hat{X},\mathbb{R})$. Let a be one of the basis vectors. By construction we have $\sigma_*(a) = -a$. Let the closed curve[4] $\gamma\colon [0,1] \to \hat{X}$ be a representative of a, and denote $\tilde{\gamma}(s) := \sigma(\gamma(1-s))$. Owing to $\sigma_*(a) = -a$ the union $\gamma \sqcup \tilde{\gamma}$ is a representative of the cycle $2a$.
Let us show that $\gamma \sqcup \tilde{\gamma}$ is contractible when embedded in M_X. The connecting manifold M_X may be rewritten as

$$\mathrm{M}_X = \hat{X} \times [0,1]/\sim \qquad \text{with} \qquad (x,0) \sim (\sigma(x),0)\,. \tag{3.10}$$

Via the embedding $\iota\colon \hat{X} \to \mathrm{M}_X$ given by $x \mapsto (x,1)$ we get two curves $\gamma_1(s) = (\gamma(s),1)$ and $\tilde{\gamma}_1(s) = (\tilde{\gamma}(s),1)$. Denoting by ι_* the embedding $H_1(\hat{X},\mathbb{Z}) \to H_1(\mathrm{M}_X,\mathbb{Z})$ induced by ι, it follows that $\gamma_1 \sqcup \tilde{\gamma}_1$ is a representative of $\iota_*(2a)$. In fact, $\{\gamma_t \sqcup \tilde{\gamma}_t\}$ with $\gamma_t(s) = (\gamma(s),t)$ and $\tilde{\gamma}_t(s) = (\tilde{\gamma}(s),t)$ provides us with a whole family of representatives of $\iota_*(2a)$. Now due to the identification in (3.10) we have $\gamma_0(s) = \tilde{\gamma}_0(1-s)$. Thus $\gamma_0 \sqcup \tilde{\gamma}_0$ is contractible in M_X, and hence $\iota_*(2a) = 2\iota_*(a) = 0$. Finally, we obtain $H_1(\,\cdot\,,\mathbb{R})$ by tensoring $H_1(\,\cdot\,,\mathbb{Z})$ with $\otimes_{\mathbb{Z}}\mathbb{R}$, and thus in $H_1(M_X,\mathbb{R})$ we indeed have $\iota_*(a) = 0$. It follows that the basis elements v_i^- of $\lambda_-(X)$ are all contained in λ_X. Since all Lagrangian subspaces have equal dimension, this implies that $\lambda_-(X) = \lambda_X$.

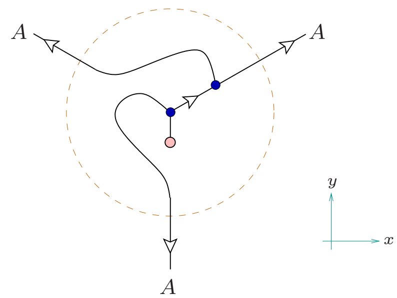
The ribbon graph embedded in M_X is constructed in several steps, which involve a number of arbitrary choices. We first list all the steps and show afterwards that the outcome is independent of all the choices made, i.e. they all lead to equivalent ribbon graphs.

- Choose a representative (M,or_1) in each of the equivalence classes $[M,\mathrm{or}_1]$ that label the boundary components of ∂X – Choice #1.
- Choose a dual triangulation of X – Choice #2.
- Choose a local orientation around each vertex of the triangulation that lies in the interior of X – Choice #3.

[4] In case that a is represented by a union of several curves the proof works analogously.

- Each component of the boundary ∂X is already oriented by the choice of (M, or_1) made above. Around each vertex of the triangulation that lies on ∂X, choose the orientation or_2 induced by or_1 and the inward pointing normal (the local orientation then agrees with the one around a boundary point on the upper half plane $\mathrm{Im}(z) \geq 0$).

- Think of the triangulated world sheet with local orientations as embedded in M_X according to (3.9). At each vertex in the interior place the (piece of) ribbon graph

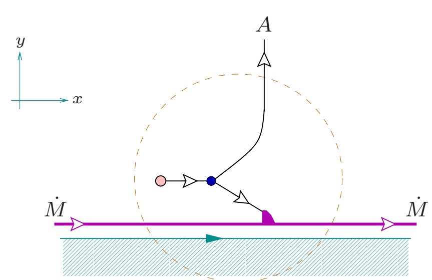


(3.11)

  such that its orientation agrees with the local orientation or_2 around that vertex. The x-y-coordinates indicated in (3.11) encode the orientation by comparing it to the standard orientation of $\mathbb{R}^2$. The graph (3.11) can be inserted in three distinct ways, obtained by rotating the picture by an angle $\pm 2\pi/3$ – Choice #4.

- On a boundary edge with chosen representative (M, or_1) place an M-ribbon such that the orientation of its core agrees with or_1 and the orientation of its surface agrees with the local orientation on X given by or_1 and the inward pointing normal.

- At each vertex on a boundary component with chosen representative (M, or_1) place the ribbon graph



(3.12)

  in such a way that the orientation of bulk and boundary agree with that around the vertex. The arrow on the boundary indicates its orientation. (There is only a single way to place the graph (3.12).)

- Two vertices v, v' connected by an edge of the triangulation have the same orientation if $\mathrm{or}_2(v)$, transported along the edge, agrees with $\mathrm{or}_2(v')$. Note that whether two vertices possess the same orientation or not in general depends not only on the vertices, but also on the particular edge that connects them.

  At an edge in the interior of X which connects two vertices of the same orientation (with respect to that edge) place the ribbon graph (3.13 a) such that the orientation at either end agrees with that around the vertices. At an edge in the interior of X which connects two vertices of opposite orientation place the ribbon graph (3.13 b) again such that the orientations agree. [5]

(3.13)

In both cases there are two possibilities to place the corresponding graph – Choices #5a and #5b.

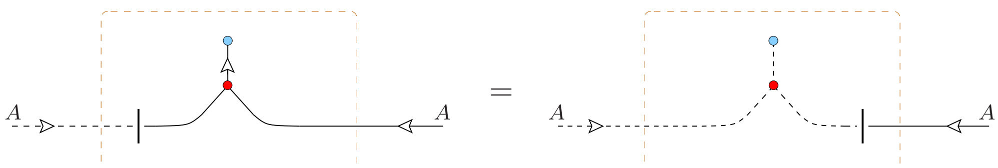
This completes the prescription to construct the ribbon graph embedded in M_X. We now proceed to show independence of the various choices involved.

■ Choices #4, #5a:

Independence of these two choices was already shown in equations (I:5.8) and (I:5.9).

■ Choice #5b:

The ribbon graph is independent of this choice if



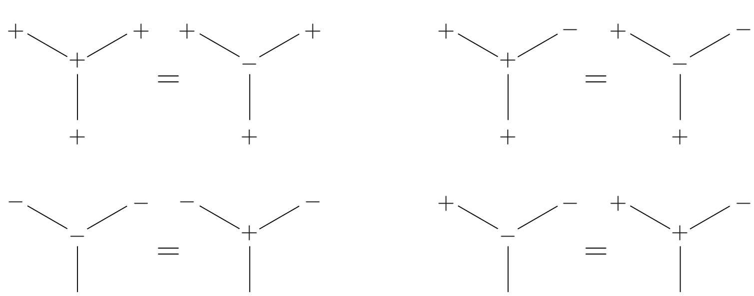
(3.14)

But this equality is an immediate consequence of the properties (3.5) of the reversing move (3.4).

■ Choice #3:

Let us show that reversing the orientation of a vertex in the bulk yields equivalent ribbon graphs.

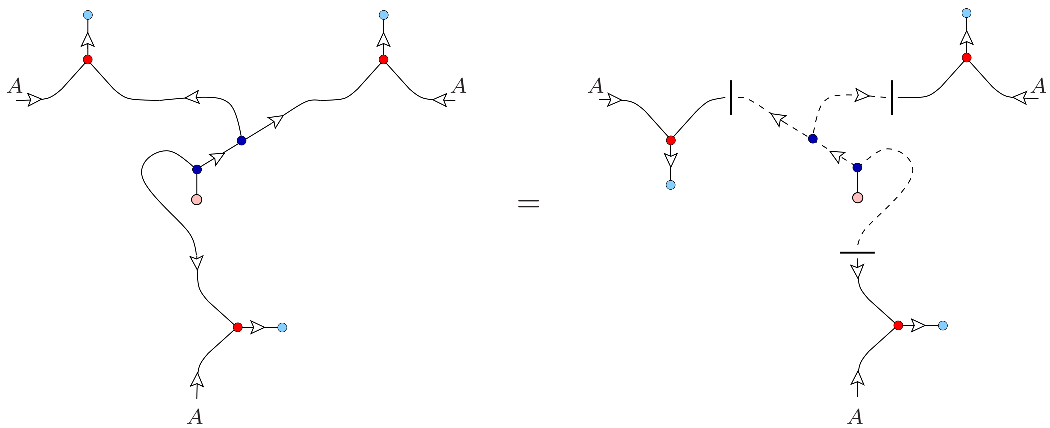
Pick a vertex V in the bulk and denote its orientation by $+$. The vertex V is joined by edges of the triangulation to three other vertices (some of which may lie on the boundary). Depending on the orientation of the surrounding vertices, there are a priori four equalities to be shown, each of which amounts to reversing the orientation of V:



(3.15)

[5] The second of the graphs (3.13) differs slightly from (2.3) because the construction is formulated such that an edge in the interior of X has two incoming A-ribbons, rather than one incoming and one outgoing.

In fact of these four relations only two are independent. The equations arranged in a column in (3.15) are the same; they are just looked at from the two different sides of the paper plane. Let us demonstrate the top left equality in (3.15) as an example. Choosing one of the three ways to insert the bulk vertex (due to independence of choice #4 it does not matter which one) and one of the two ways to put the ribbon graphs for the edges (choices #5a, #5b above) the equality amounts to



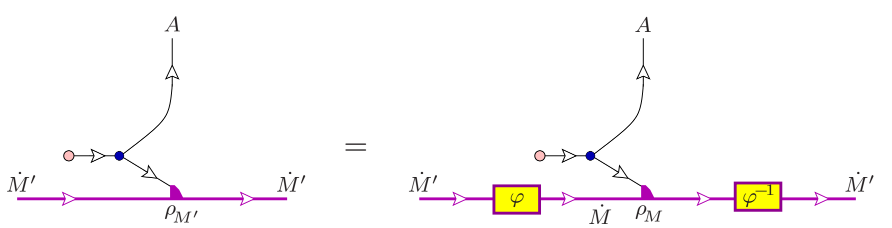
(3.16)

This equality, in turn, is again an immediate consequence of the properties (3.5).

■ Choice #1:

We must show that the choice of different representatives from the classes labelling boundary components leads to equivalent ribbon graphs. Pick a boundary component D. Let (M,or_1) and (M',or_1') be two representatives of the class $[M,\mathrm{or}_1]$ labelling D. By the definition (3.1) of the equivalence relation there are then two possibilites:

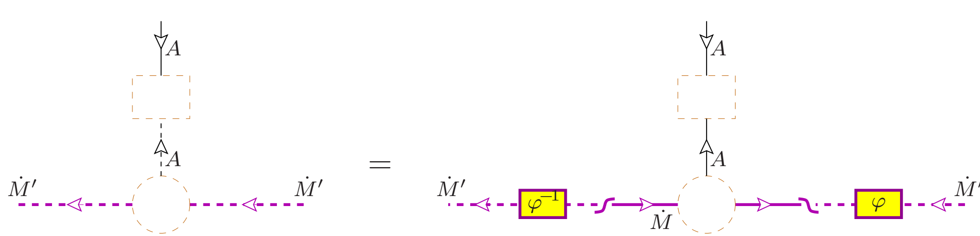
(i) $\mathrm{or}_1 = \mathrm{or}_1'$. Then $M' \cong M$. Let $\varphi \in \mathrm{Hom}_A(M',M)$ be an invertible intertwiner, i.e. $\rho_{M'} = \varphi^{-1} \circ \rho_M \circ (id_A \otimes \varphi)$. The corresponding equality between ribbon graphs looks as



(3.17)

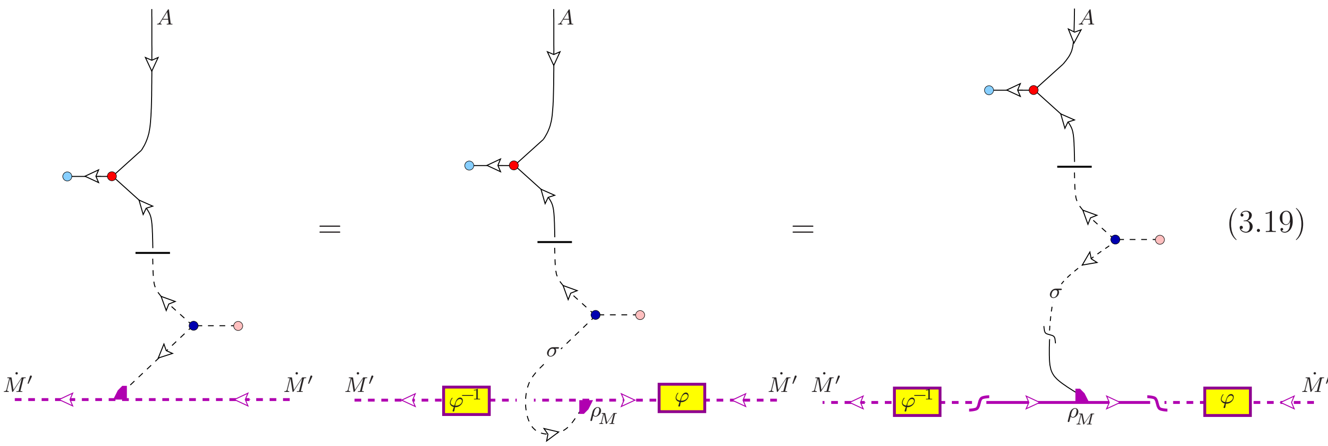
Starting from the ribbon graph obtained with the representative (M',or_1'), this identity can be substituted at each each vertex on D. Since D is topologically a circle, each φ will then cancel against the φ^{-1} from a neighbouring vertex. In the end what is left is the ribbon graph for the representative (M,or_1).

(ii) $\mathrm{or}_1 = -\mathrm{or}_1'$. Then $M' \cong M^\sigma$. Let $\varphi \in \mathrm{Hom}_A(M',M^\sigma)$ be an invertible intertwiner, i.e. $\rho_{M'} = \varphi^{-1} \circ \rho_M^\sigma \circ (id_A \otimes \varphi)$. We would like to show the identity



(3.18)

respectively – depending on the orientation of the vertex the outgoing A-ribbon connects to – the corresponding identity with the outgoing A-ribbon drawn dashed. Below we demonstrate (3.18); the other identity can be seen similarly. If we substitute the equality (3.18) at every vertex on D, the half-twists as well as the morphisms φ, φ^{-1} from neighbouring vertices cancel, thus transforming the ribbon graph resulting from choosing the representative (M', or_1') into the one for (M, or_1).
To establish the relation (3.18) consider the moves



(3.19)

The left hand side of (3.19) is obtained by substituting the definitions (3.12) and (3.13) into the left hand side of (3.18). Then in the first step the intertwiner between M' and M^σ is inserted, and the representation morphism ρ_M^σ is expressed through ρ_M as in (2.25). The second step amounts to a 180°-rotation of the segment of the ribbon graph around ρ_M. On the right hand side of (3.19) the combination of σ and half-twist can be replaced by the reversing move (3.4). Using (3.5) the two reversing moves cancel; the resulting ribbon graph is equal to the right hand side of (3.18).

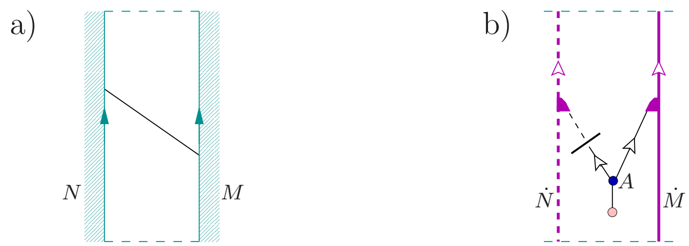
■ Choice #2:
Independence of the triangulation follows by showing that the so-called *fusion* and *bubble* moves lead to equivalent ribbon graphs, see (I:5.10). In the oriented case this amounts to showing the identities (I:5.11) and (I:5.12). Since independence of the local orientation around the bulk vertices and the boundary components has already been established above, we are free to choose the orientations in such a way that they coincide with those used in the proof for the oriented case, so that one can copy that proof literally. (The proof is straightforward, but a bit lengthy, and accordingly has not been written down in detail in [I].)

This establishes the independence of the construction of all the choices involved. In the sequel, after reconsidering the annulus we will apply the construction to the Möbius strip and Klein bottle.

3.2 The annulus revisited

The annulus coefficients A_{kMN} with two lower indices are those appearing in the open string partition function in type I string theory. Here we will obtain these numbers from considering an annulus whose boundary components possess the same orientation, as indicated in figure (3.20 a) below. This should be contrasted with the construction in the orientable case that was given in section I:5.8. In the context studied there, the two boundaries of the annulus automatically have opposite orientations. But apart from this aspect the construction is very similar, so we can go through it rather quickly.

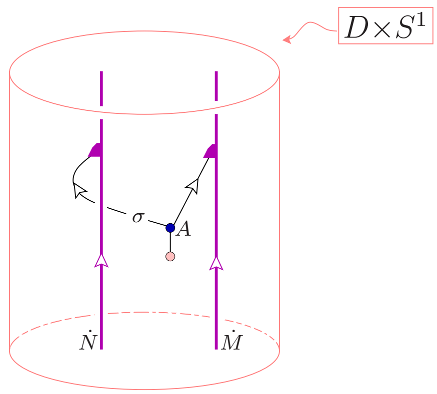
Recall from section I:5.8 that the double $\hat{X}$ of an annulus X is a torus, and the connecting manifold M_X is a solid torus. Let the two boundary components of X be labelled by the equivalence classes $[M, \mathrm{or}_1]$ and $[N, \mathrm{or}_1']$. The orientations or_1 and or_1' of the boundary components are as in figure (3.20 a), which also indicates the triangulation we use:



(3.20)

Figure (3.20 b) shows the resulting ribbon graph. It is understood that the world sheet is embedded in M_X; the ribbon graph is then obtained inserting the elements (3.12) and (3.13). Turning the N-ribbon in (3.20 b) by 180° leads to the following three-manifold and embedded ribbon graph:

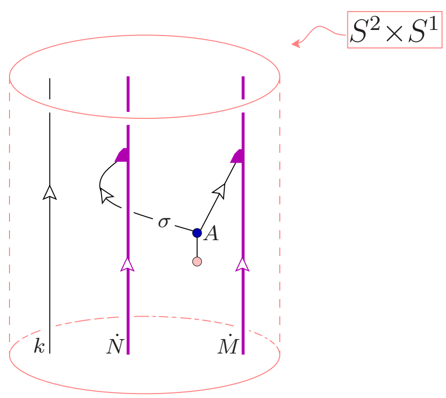
A_{MN} =



$\in \mathcal{H}(\emptyset; \mathrm{T})$. (3.21)

Here $\mathcal{H}(\emptyset; \mathrm{T})$ is the space of zero-point conformal blocks on the torus, see section I:5.2. Expanding (3.21) in terms of characters $|\chi_k; \mathrm{T}\rangle$ as in (I:5.118) we obtain the annulus coefficients:

A_{kMN} =



(3.22)

Note that upon moving the M- and N-ribbons past each other, rotating the M-ribbon by 360° and using (2.17), it follows that

$$\mathrm{A}_{kMN} = \mathrm{A}_{kNM} \, , \tag{3.23}$$

i.e. the annulus coefficients with two lower boundary indices are symmetric. This is in contrast to the situation for the annulus coefficients with one lower and one upper boundary index, which according to (I:5.122) satisfy $\mathrm{A}_{kM}^{\ \ N} = \mathrm{A}_{\bar kN}^{\ \ M}$. Also, comparing the graphs (3.22) and (2.25) to the graph (I:5.119) for $\mathrm{A}_{kM}^{\ \ N}$ yields the identity (compare [28])

$$\mathrm{A}_{kMN} = \mathrm{A}_{kM}^{\ \ N^\sigma} . \tag{3.24}$$

As in the oriented case, we can cut the three-manifold (3.22) along an S^2 so as to obtain a ribbon graph in $S^2 \times [0,1]$. The graphical representation of the resulting cobordism is the same as in (3.22), except that we no longer identify top and bottom. Let us denote the linear map associated to this cobordism by

$$Q_{kMN}: \quad \mathcal{H}(k,\dot N,\dot M;S^2) \to \mathcal{H}(k,\dot N,\dot M;S^2) . \tag{3.25}$$

Q_{kMN} is a projector, so that upon gluing top and bottom of the cobordism $S^2 \times [0,1]$ back together results in the relation

$$\mathrm{A}_{kMN} = \mathrm{tr}_{\mathcal{H}(k,\dot N,\dot M;S^2)}\, Q_{kMN} \in \mathbb{Z}_{\geq 0} . \tag{3.26}$$

The projector property can be shown in much the same way as in (I:5.127), the only difference being that we also must use the equality (2.12). The trace computes the dimension of the image of the projector and hence is a non-negative integer. We will see that the projector Q_{kMN} also enters in the computation of the Möbius strip amplitude.

Remark 3.2:
Note that in writing (3.22) and (3.26) we have implicitly made use of the functoriality axiom (I:2.56). For example, in (3.22) functoriality reads

$$\langle \chi_k ; \mathrm{T} | \mathrm{A}_{MN} = \kappa^m \, Z(\mathrm{A}_{kMN}, \emptyset, \emptyset)\, 1 \tag{3.27}$$

with $\kappa = S_{0,0} \sum_{j\in\mathcal{I}} \theta_j^{-1} \mathrm{dim}(U_j)^2$. This leads to the result (3.22) provided that the factor κ^m arising form the framing anomaly is equal to 1. Now recall from remark 3.1 that the Lagrangian subspace in $\mathrm{T} = \partial \mathrm{A}_{MN}$ is spanned by the cycle contractible in A_{MN}. The same holds for $\partial M^-_{\chi_k}$ (see section 3.7 below and appendix A.3). Then the argument leading to (A.71) is applicable, and using it one can show that the two Maslov indices determining m are equal to zero.

For the same reason there will not appear any anomaly factor κ^m in the equations (3.37) for the Möbius strip, (3.55) for the Klein bottle, (3.127), (3.128) and (3.135) for the Möbius strip in the crossed channel, and (3.145) for the crossed-channel Klein bottle. With a similar reasoning one can show that the anomaly m vanishes in the relations expressed by (3.26), as well as in (3.38) and (3.57) below, i.e. the factor κ^m that a priori appears in these formulas is equal to 1. In general one has the trace formula (I:2.58).

The boundary conjugation matrix C^σ defined in proposition 2.8 coincides with the one described in [21,28], i.e. it can be expressed in terms of annulus coefficients with two lower indices. This is stated in

Lemma 3.3 :
The boundary conjugation matrix (2.34) satisfies

$$C^{\sigma}_{MN} = \mathrm{A}_{0MN} \,. \tag{3.28}$$

Proof:
In the present approach this relation emerges as follows. According to formula (3.24), interpreting the annulus coefficients $\mathrm{A}_{kM}^{\;\;N}$ as dimensions of spaces of A-morphisms as in (I:5.135) yields

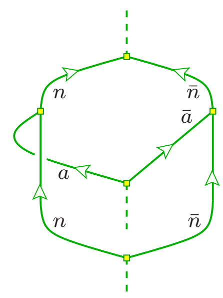
$$\mathrm{A}_{kMN} = \dim_{\mathbb{C}} \mathrm{Hom}_A(M \otimes U_k, N^{\sigma}) \,. \tag{3.29}$$

Taking now M and N to be simple A-modules and setting $k=0$, comparison with the definition (2.34) of C^{σ} gives (3.28). ✓

The equality (3.28) shows that it is indeed natural to use C^{σ} and its inverse to lower and raise the indices of annulus partition functions.

Just like in [I] we will present more explicit formulas only for the special case that $N_{ij}^{\;k} \in \{0,1\}$ and $\langle U_k, A\rangle_A \in \{0,1\}$ for all $i,j,k \in \mathcal{I}$. The evaluation of the invariant A_{0MN} is analogous to the computation of $\mathrm{A}_{kM}^{\;\;N}$ in (I:5.144); we find

$$\mathrm{A}_{0MN} \;=\; \sum_{a,n,\alpha,\beta} \rho^{N\,n\alpha}_{a,n\alpha}\, \rho^{M\,\bar n\beta}_{\bar a,\bar n\beta}\, \sigma(a)\, \Delta^{a\bar a}_0 \tag{3.30}$$

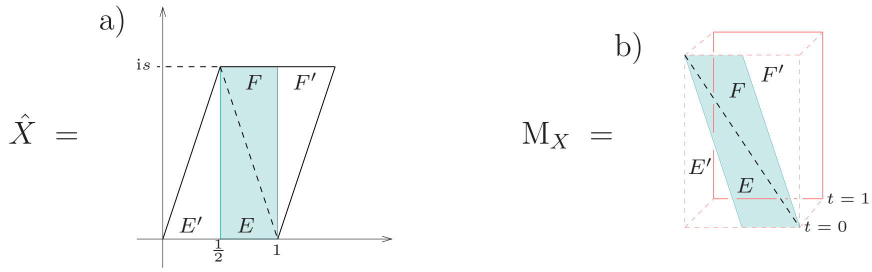


Here the a-summation runs over simple subobjects of A, the n-summation over simple subobjects U_n of the simple module N such that $U_n^{\vee}$ is a subobject of M, and the labels α, β are multiplicity labels for the occurrence of U_n in N and $U_n^{\vee}$ in M, respectively. It follows that

$$C^{\sigma}_{MN} = \sum_{a,n,\alpha,\beta} \rho^{N\,n\alpha}_{a,n\alpha}\, \rho^{M\,\bar n\beta}_{\bar a,\bar n\beta}\, \sigma(a)\, \Delta^{a\bar a}_0\, \mathsf{R}^{(n\,a)n}\, \mathsf{G}^{(n\,a\,\bar a)\,n}_{n\,0}\, \mathsf{F}^{(n\,\bar a\,\bar n)\,0}_{\bar n\,n} \tag{3.31}$$

3.3 The Möbius strip

Like any world sheet X, the Möbius strip is obtained from its complex double $\hat{X}$ by dividing out an anticonformal involution σ. The double $\hat{X}$ is a torus with complex structure parametrised by $\tau = \frac{1}{2} + \mathrm{i}s$ with $s \in \mathbb{R}_{\geq 0}$. The involution acts as $\sigma(z) = 1 - \bar{z}$. Using that z is identified with $z{+}1$ and $z{+}\tau$ it follows that the fixed points of σ are $\frac{1}{2} + \mathrm{i}\mathbb{R}$ and $1 + \mathrm{i}\mathbb{R}$, and that a fundamental domain X for the action of σ on $\hat{X}$ is given by the shaded region in figure (3.32 a):

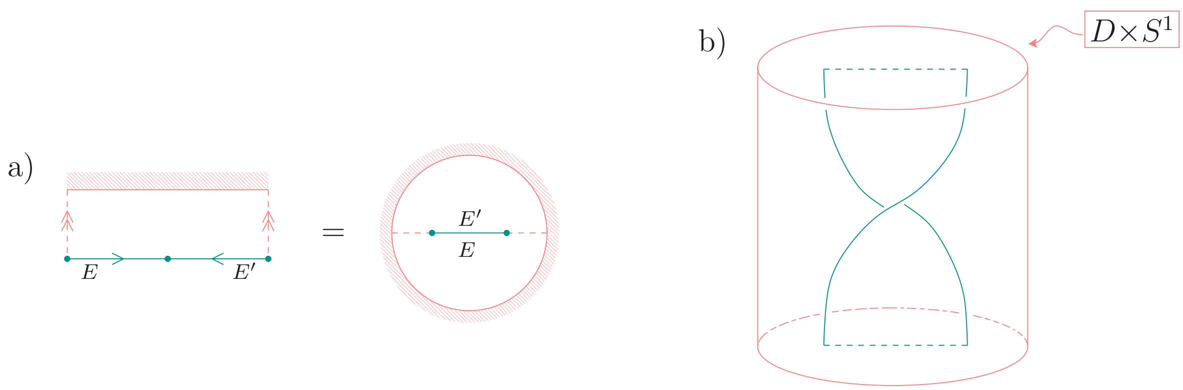


(3.32)

It is easily checked that X indeed has the topology of a Möbius strip.

The connecting manifold M_X is constructed according to (3.8): $\mathrm{M}_X = \hat{X} \times [-1,1]\,/{\sim}$. This is drawn in (3.32 b), where for convenience the quadrangle defining the torus as in (3.32 a) is drawn with right angles – which we may do because when applying TFT arguments we are interested in M_X only as a topological manifold. The top and bottom rectangles of (3.32 b), as well as the left and right side, are identified. Figure (3.32 b) also shows a fundamental domain for the action of the relation $\sim$. It consists of $\hat{X} \times [0,1]$, together with an identification of points on $X \times \{0\}$: points in the triangle E are identified with their reflections in E', and points in F with their reflections in F'.

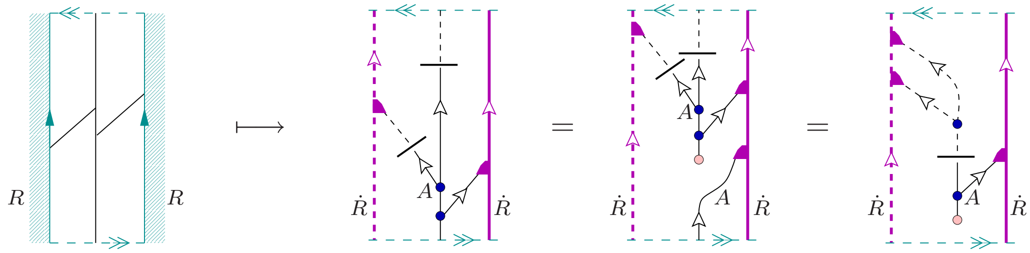
For a horizontal section of figure (3.32 b) the identification yields the topology of a disk. For the bottom side of (3.32 b), this is illustrated in figure (3.33 a):



(3.33)

As we move upwards in (3.32 b), the identification fixed points move half a period along the horizontal direction; this results in figure (3.33 b). The two-manifold drawn in M_X is thus the image of the embedding $\iota_X\colon X \to \mathrm{M}_X$ of the Möbius strip as in (3.9).

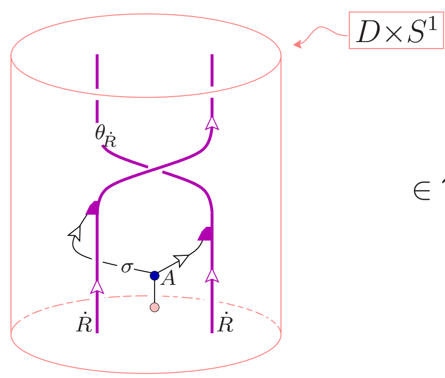
Next we construct the ribbon graph embedded in M_X. Let the boundary condition be $[R, \mathrm{or}_1]$, where the boundary orientation or_1 is 'up' in (3.32 a). As a triangulation to start with we choose one that is slightly more complicated than necessary, in order to illustrate how the moves (3.5) simplify the resulting ribbon graph:



(3.34)

The left hand side shows the starting point – the chosen triangulation of the Möbius strip, together with a representative (R, or_1) to label the boundary. In the other pictures the world sheet is regarded as being embedded in M_X. The ribbon graph in the second picture is obtained by choosing the orientation of the paper plane at the two interior vertices and inserting the graphs (3.11), (3.12) and (3.13), as well as removing some unit and counit morphisms via the Frobenius property. The first equality is a consequence of the representation property of ρ_R and the Frobenius property. To obtain the second equality, one takes the bottom A-ribbon of the third picture through the anti-periodic identification of the top and bottom. In the last picture one can use the representation property on the dashed A-ribbons, together with specialness of A; the resulting ribbon graph, with the Möbius strip embedded in $D \times S^1$ as in (3.33 b), is

$$\mathrm{Mö}_R \;=$$



$$\in \mathcal{H}(\emptyset; \mathrm{T}) \qquad (3.35)$$

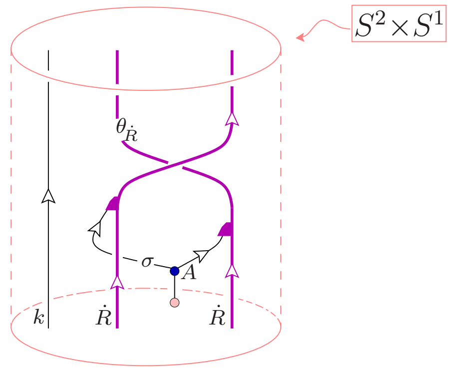
Here the twist $\theta_{\dot R}$ comes from the two half-twists of the boundary ribbon that are implicit in the identification of the boundary segments in figure (3.34).

The ribbon invariant $\mathrm{Mö}_R$ can be expanded in terms of the standard basis of $\mathcal{H}(\emptyset; \mathrm{T})$ that was given in section I:5.2:

$$\mathrm{Mö}_R = \sum_{l \in \mathcal{I}} \mathrm{Mö}_{lR}\, |\chi_l; \mathrm{T}\rangle \qquad (3.36)$$

To extract the coefficients $\mathrm{Mö}_{kR}$ we compose both sides of (3.36) with the dual basis $\langle \chi_k; \mathrm{T}|$. On the right hand side this yields $\delta_{k,l}$, while on the left hand side it amounts to gluing the manifold (I:5.18) to $\mathrm{Mö}_R$. The result is a ribbon graph in the closed three-manifold $S^2 \times S^1$ whose invariant gives the coefficients $\mathrm{Mö}_{kR}$ of the Möbius amplitude (there is no framing anomaly, see remark 3.2):

$$\text{Mö}_{kR} \ =$$



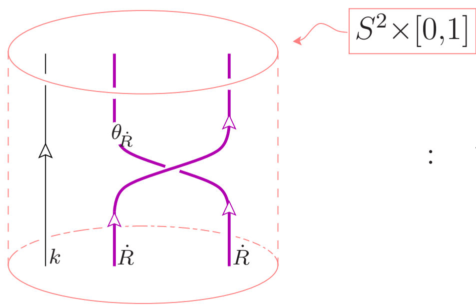
(3.37)

To establish some properties of the numbers Mö_{kR} it is helpful rewrite them as a trace:

$$\text{Mö}_{kR} = \text{tr}_{\mathcal{H}(k,R,R;S^2)}(T_{kR}\, Q_{kRR})\,, \tag{3.38}$$

where Q_{kRR} is the linear map defined in (3.25) and T_{kR} is given by

$$T_{kR} \ =$$



$$: \quad \mathcal{H}(k,R,R;S^2) \,\rightarrow\, \mathcal{H}(k,R,R;S^2) \tag{3.39}$$

The linear maps T and Q fulfill the relations given in

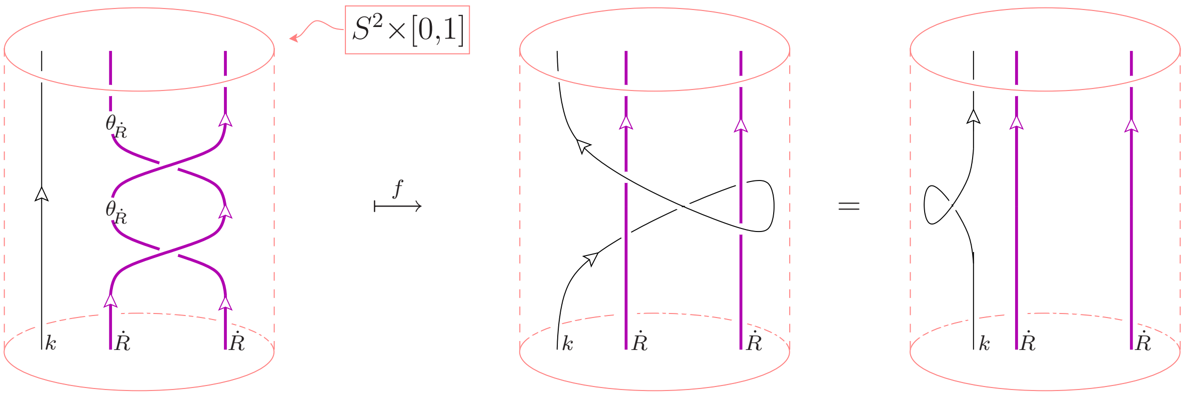
Lemma 3.4:
The linear maps T_{kR} and Q_{kRR} defined in (3.25) and (3.39) satisfy $Q_{kRR}\, Q_{kRR} = Q_{kRR}$ as well as

$$T_{kR}\, T_{kR} = \theta_k\, id_{\mathcal{H}(k,R,R;S^2)} \qquad \text{and} \qquad T_{kR}\, Q_{kRR} = Q_{kRR}\, T_{kR}\,. \tag{3.40}$$

It follows in particular that T_{kR} and Q_{kRR} can be diagonalised simultaneously.

Proof:
The first property was already established in section 3.2. The first of the relations (3.40) can be seen as follows. It is one of the basic properties of a topological field theory that when two cobordisms are related by a homeomorphism that restricts to the identity on the boundary, then the linear maps assiged to the cobordisms are identical. The following figure shows two

cobordisms from S^2 to S^2 that are related in this manner:

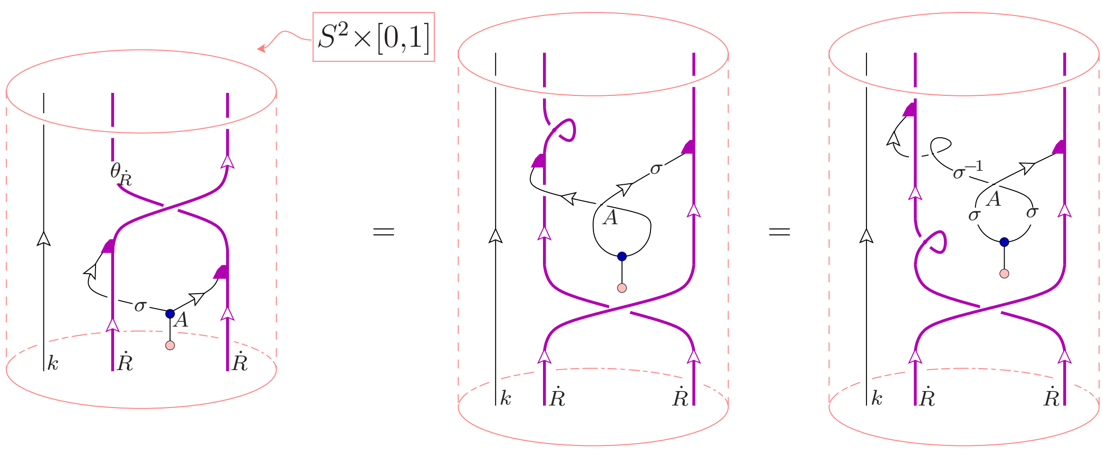


(3.41)

If we take the vertical interval to be $[0, 2\pi]$ and denote the vertical position by φ, the map f corresponds to a clockwise rotation of the horizontal S^2's by the angle φ around the central point in the drawing (3.41). (This is best visualised with actual ribbons; the pictures above have already been converted back to blackboard framing). The equality on the right hand side of (3.41) follows by wrapping the k-ribbon around 'infinity' on the horizontal S^2. Now the cobordism on the right hand side of (3.41) is indeed equal to $\theta_k id$.

That the second relation in (3.40) holds is already clear from the fact that the two sides of the equality just correspond to different choices for drawing the horizontal triangulation line on the embedded Möbius strip in figure (3.33 b). Let us nonetheless check it explicitly; consider the moves

$T_{kR}\,Q_{kRR} =$



(3.42)

In the first step the horizontal A-ribbon is moved past the braiding of the two R-ribbons. In the second step the representation morphism on the left R-ribbon is taken through the twist, which leads to a twist θ_A on the A-ribbon, and an identity $id_A = \sigma^{-1} \circ \sigma$ is inserted. On the right hand side, the two σ morphisms above the coproduct cancel, by (2.5), against the braiding of the A-ribbons, and the combination $\theta_A \circ \sigma^{-1}$ equals σ. Implementing these identities leaves us with the ribbon graph for $Q_{kRR}\,T_{kR}$.

Finally, Q_{kRR} can be diagonalised because it is a projector, and T_{kR} can be diagonalised because it squares to a multiple of the identity. Due to $T_{kR}\,Q_{kRR} = Q_{kRR}\,T_{kR}$, the diagonalisation of both maps can be achieved simultaneously. ✓

It turns out that the numbers $\mathrm{Mö}_{kR}$ are not integral. But they are integral up to known phases. To see this, let us choose once and for all a square root of the phases θ_k which specify the action of the twist on simple objects U_k, i.e. choose numbers

$$t_k \qquad \text{such that} \qquad t_0 = 1\,, \quad {t_k}^2 = \theta_k \quad \text{and} \quad t_k = t_{\bar k}\,. \tag{3.43}$$

If the modular tensor category is the representation category of a chiral algebra (which is the case we are interested in), then each object U_k comes with a conformal weight Δ_k and one can take

$$t_k = \exp(-\pi \mathrm{i} \Delta_k)\,. \tag{3.44}$$

The numbers

$$\mathrm{mö}_{kR} := \mathrm{Mö}_{kR}/t_k \tag{3.45}$$

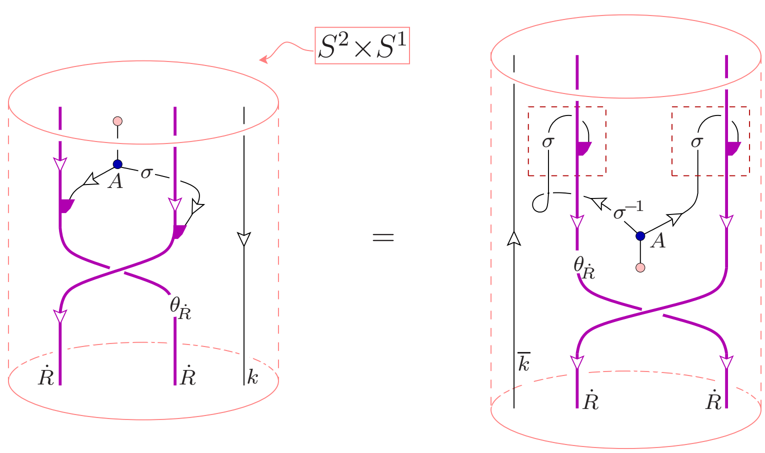
are indeed integers:

Theorem 3.5 :
The numbers $\mathrm{mö}_{kR}$ defined in (3.45) satisfy

$$\begin{aligned} &\text{(i)} \quad \mathrm{mö}_{kR} \in \mathbb{Z}\,, \\ &\text{(ii)} \quad \mathrm{mö}_{kR} = \mathrm{mö}_{\bar k R^\sigma}\,, \\ &\text{(iii)} \quad \tfrac12\,(\mathrm{mö}_{kR} + \mathrm{A}_{kRR}) \in \{0,1,\dots,\mathrm{A}_{kRR}\}\,. \end{aligned} \tag{3.46}$$

Proof:
(i) According to (3.38) the coefficient $\mathrm{Mö}_{kR}$ is the trace of the linear map $T_{kR}\,Q_{kRR}$. Let $\{v_i\}$ be a common eigenbasis of Q_{kRR} and T_{kR} (which exists by lemma 3.4). The eigenvalues λ_i^T of T_{kR} in this basis square to θ_k, and hence $\lambda_i^T = \pm t_k$, while the eigenvalues λ_i^Q of the projector Q_{kRR} are either zero or one. As a consequence the combinations $\lambda_i^T \lambda_i^Q/t_k$ take values in $\{0,\pm1\}$, and hence $\mathrm{mö}_{kR} = \mathrm{Mö}_{kR}/t_k = \sum_i \lambda_i^T \lambda_i^Q/t_k$ is indeed an integer.

(ii) We show that $\mathrm{Mö}_{kR} = \mathrm{Mö}_{\bar k R^\sigma}$, which owing to $t_k = t_{\bar k}$ in (3.43) implies (ii). Let us turn the ribbon graph (3.37) in $S^2 \times S^1$ whose invariant is $\mathrm{Mö}_{kR}$ upside down. We then have



(3.47)

which is derived by the following manipulations. First the downwards pointing k-ribbon is replaced by an upwards directed $\bar k$-ribbon, which is then moved to the left side of the R-ribbons. Then symmetry in the form (I:3.35) is used on the coproduct so as to move the counit below the coproduct. Next a pair $\sigma \circ \sigma^{-1}$ is inserted on the A-ribbon that joins the left R-ribbon. Now the part of the ribbon graph lying within the dashed boxes can be recognised from (2.25) as the representation morphism for R^σ. The twist θ_A combines with σ^{-1} to σ. Using periodicity in the vertical direction we can shift the figure such that the braiding is placed above the A-ribbon. The resulting graph is precisely that of $\mathrm{Mö}_{\bar{k}R^\sigma}$.

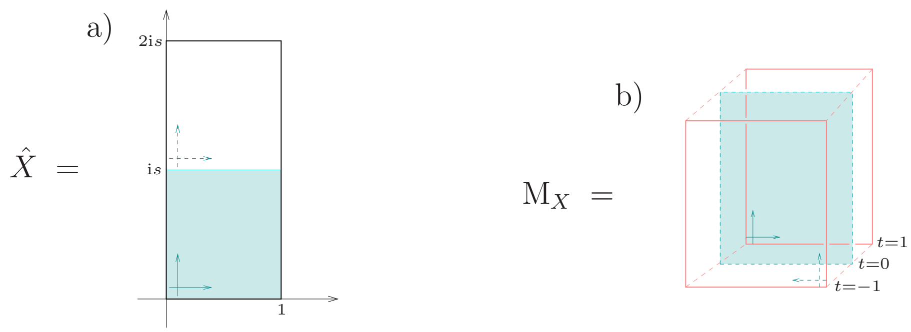
(iii) Equations (3.26) and (3.38) imply that

$$\tfrac{1}{2}\,(\mathrm{mö}_{kR} + \mathrm{A}_{kRR}) = \mathrm{tr}\left[\tfrac{1}{2}\,(T_{kR}/t_k + id)\, Q_{kRR}\right]. \tag{3.48}$$

Owing to lemma 3.4 the combination $\frac{1}{2}\,(T_{kR}/t_k + id_{\mathcal{H}(k,R,R;S^2)})$ is a projector and commutes with Q_{kRR}. Thus the product appearing in the trace (3.48) is a projector as well, and hence its trace equals the dimension of its image and is thus a non-negative integer. Furthermore the image of the product of the projectors is contained in the image of Q_{kRR}, so that $\frac{1}{2}(\mathrm{mö}_{kR}+\mathrm{A}_{kRR}) \le \mathrm{A}_{kRR}$. ✓

3.4 The Klein bottle

The double $\hat{X}$ of the Klein bottle is a torus with modular paramenter $\tau = 2\mathrm{i}s$, as indicated in figure (3.49 a):



(3.49)

The anticonformal involution on $\hat{X}$ is given by $\sigma(z) = 1 + \mathrm{i}s - \bar{z}$. The quotient $X = \hat{X}/\sigma$ has indeed the topology of a Klein bottle; a fundamental domain for the action of σ on $\hat{X}$ is indicated by the shaded region in figure (3.49 a).

The connecting manifold $\mathrm{M}_X = \hat{X} \times [-1,1]\,/\sim$, with $(z,t) \sim (\sigma(z), -t)$, is presented in figure (3.49 b). What is shown is a fundamental domain

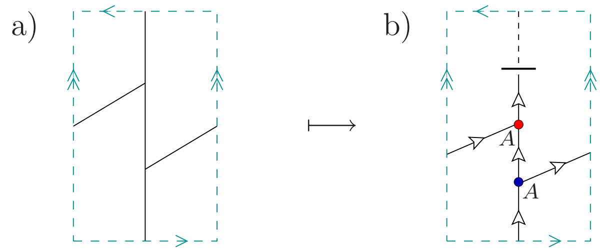
$$\mathrm{M}_X = [0,1] \times [0,s] \times [-1,1]\,/\sim\,, \tag{3.50}$$

where now the equivalence relation $\sim$ identifies the left and right side of (3.49 b) periodically, while top and bottom rectangles are identified with a 180° rotation. Explicitly,

$$(0,y,t) \sim (1,y,t) \qquad \text{and} \qquad (x,0,t) \sim (1{-}x, s, -t)\,. \tag{3.51}$$

The small perpendicular arrows '⮡' in figures (3.49 a) and (3.49 b) indicate how the shaded and unshaded regions in figure (3.49 a) for the double $\hat{X}$ are to be identified with the boundary of M_X. The shaded plane in (3.49 b) shows where the embedding $\iota_X\colon X \to \mathrm{M}_X$ places the world sheet inside M_X.

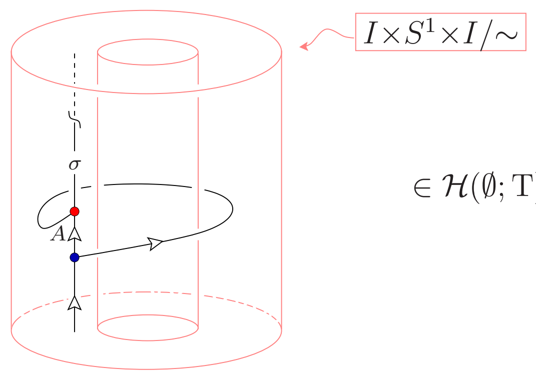
The next step is to choose a triangulation of the Klein bottle X. We take the one displayed in (3.52 a):



(3.52)

Like for the Möbius strip we choose the orientation of the paper plane at the two vertices and insert the graphs (3.11), (3.13). The resulting ribbon graph can be simplified, leading to (3.52 b). Picture (3.52 b) is to be understood as embedded in M_X like in (3.49 b), with the third direction suppressed. Drawing also the latter, the three-manifold and ribbon graph representing the Klein bottle amplitude are given by

$\mathrm{K} =$



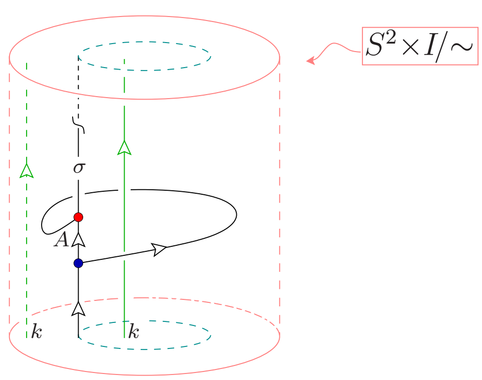
$\in \mathcal{H}(\emptyset; \mathrm{T})$ (3.53)

In this picture top and bottom are identified via $(r,\phi)_{\rm top} \sim (1/r, -\phi)_{\rm bottom}$, where r and ϕ are radial and angular coordinates in the horizontal plane and I is an interval. Note that with this identification the boundary of (3.53) is indeed a single torus. As usual, the coefficients K_k occuring in the expansion

$$\mathrm{K} = \sum_{k\in\mathcal{I}} \mathrm{K}_k \, |\chi_k; \mathrm{T}\rangle \tag{3.54}$$

of K in a basis are obtained by taking the inner product with $\langle\chi_k; \mathrm{T}|$. On the right hand side this corresponds to gluing the solid torus (I:5.18) to the boundary of K:

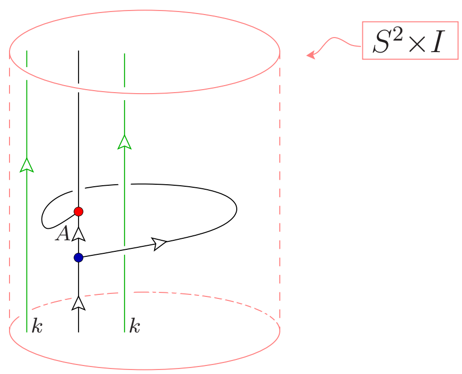
$\mathrm{K}_k =$

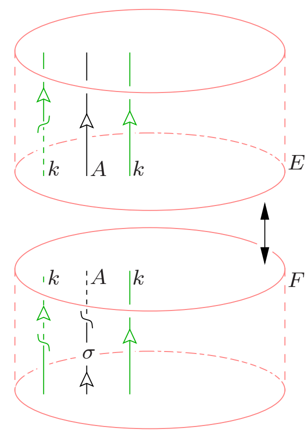


(3.55)

The picture shows the closed three-manifold that is obtained from $I\times S^2$ via the identification $(r,\phi)_{\rm top} \sim (1/r, -\phi)_{\rm bottom}$. The dashed circles indicate the intersection of the embedded world sheet with the boundary S^2's of $I\times S^2$; they are placed at $r\,{=}\,1$. With this identification the U_k-ribbon that hits the top-S^2 in the interior of the circle has its orientation as a two-manifold reversed and is identified with the U_k-ribbon hitting the bottom-S^2 outside the circle (which accordingly is to be drawn as a dashed line).

To proceed we would like to rewrite (3.55) as a trace over $\mathcal{H}(k,A,k;S^2)$. The corresponding linear map can be written as the composition of two pieces. The first is the specialisation to $i\,{=}\,j$ of the map P_{ij} from (I:5.37), which was used in (I:5.38) to express the coefficients Z_{ij} of the torus partition function as a trace. The second map, to be denoted by G_k, implements the gluing map identifying top and bottom in (3.55):

$P_{kk}\ =$ 

$G_k\ =$ 

(3.56)

In the ribbon graph for G_k the two intermediate horizontal S^2's (labelled E, F) are identified via $(r,\phi)_E\sim(1/r,-\phi)_F$. (Here again according to remark 3.2 the framing anomaly vanishes.)

Both P_{kk} and G_k are endomorphisms of $\mathcal{H}(k,A,k;S^2)$. Gluing the top-S^2 of the ribbon graph for P_{kk} to the bottom of G_k using the identity map and vice versa, one arrives precisely at the closed three-manifold (3.55). Thus

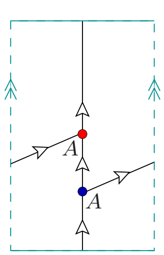
$$\mathrm{K}_k = \mathrm{tr}_{\mathcal{H}(k,A,k;S^2)}\, G_k\, P_{kk}\,. \tag{3.57}$$

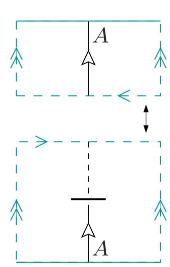
Properties of the number K_k can be established along the same lines as for the Möbius strip. We start with

Lemma 3.6:
The linear maps G_k and P_{kk} defined in (3.56) satisfy $P_{kk}\,P_{kk}=P_{kk}$ as well as

$$G_k\,G_k = id_{\mathcal{H}(k,A,k;S^2)} \qquad \text{and} \qquad G_k\,P_{kk}=P_{kk}\,G_k\,. \tag{3.58}$$

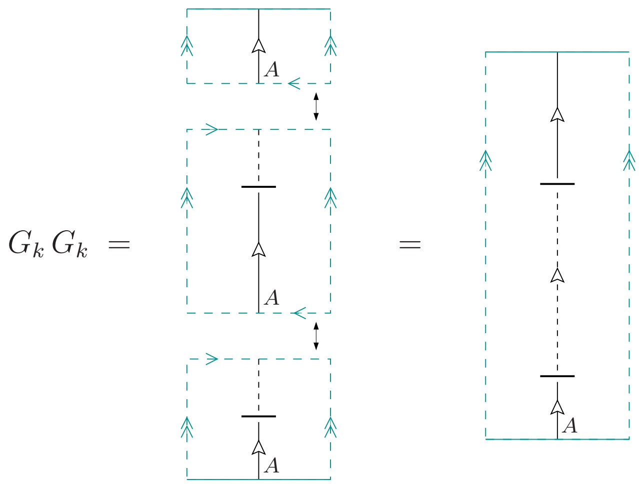
Proof:
That P_{kk} is a projector was already shown in (I:5.39). In the manipulations establishing (3.58) we will display only the cylinder at $r\,{=}\,1$ of the ribbon graphs (3.56), or in other words, only the A-ribbons; the k-ribbons just play the role of spectators. The abbreviations for P_{kk} and G_k obtained this way look as

$P_{kk}\ =$ 

$G_k\ =$ 

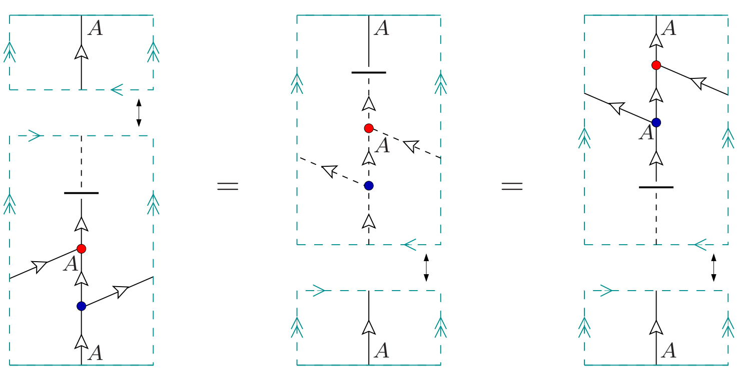
(3.59)

That G_k squares to the identity map follows from



(3.60)

The second equality can be understood as a homeomorphism of three-manifolds which acts as the identity on the top and bottom parts of the first picture, and as $(r,\varphi)\to(1/r,-\varphi)$ in the middle (recall that the r-direction is suppressed). This changes the two gluing maps in the first picture to identity maps, so that the second picture can be drawn as a single piece. The two reversing moves on the right hand side of (3.60) cancel by (3.5), leaving the identity map on $\mathcal{H}(k,A,k;S^2)$.
To see the second equality in (3.58), consider the moves



(3.61)

The left hand side equals $G_k P_{kk}$. In the first step the horizontal section along which the two pieces of the three-manifold are glued together is moved downwards. The identification $(r,\varphi)\to(1/r,-\varphi)$ causes a reflection in the picture and a change of the 2-orientation of the A-ribbon. The second step uses (3.5) to move the reversing move through the A-ribbon that winds around the cylinder. Upon moving the reversing move through the identification plane as well and applying a move like (I:5.25) so as to reverse the direction of the horizontal A-ribbon, one arrives at the ribbon graph for $P_{kk}G_k$. ✓

It is now straightforward to establish the crucial properties of the numbers K_k. They are summarised in

Theorem 3.7:
The numbers K_k defined in (3.55) fulfill

$$\begin{aligned} &\text{(i)} && \mathrm{K}_k \in \mathbb{Z}\,, \\ &\text{(ii)} && \mathrm{K}_k = \mathrm{K}_{\bar k}\,, \\ &\text{(iii)} && \tfrac{1}{2}\,(Z_{kk} + \mathrm{K}_k) \in \{0,1,\dots,Z_{kk}\}\,. \end{aligned} \tag{3.62}$$

Proof:
(i) is an immediate consequence of lemma 3.6. The linear maps G_k and P_{kk} commute, are diagonalisable and have integer eigenvalues. Thus the trace of their product is an integer.

(ii) can be seen analogously to theorem 3.5(ii). Start with drawing the three-manifold and embedded ribbon graph of (3.55) upside down, then use the move (I:5.25) to reverse the direction of all A-ribbons, and finally move the combination of σ and half-twist (the reversing move) through the identification. The resulting graph is that for $\mathrm{K}_{\bar k}$.

To see (iii) recall formula (I:5.38) and write

$$\frac{1}{2}\,(Z_{kk} + \mathrm{K}_k) = \mathrm{tr}_{\mathcal{H}(k,A,k;S^2)} \left[\frac{1}{2}\,(G_k + id_{\mathcal{H}(k,A,k;S^2)})\,P_{kk}\right]. \tag{3.63}$$

Lemma 3.6 shows that $(G_k + id)/2$ is a projector that commutes with P_{kk}. Thus their product is a projector, too, and its trace is a non-negative integer. The upper bound Z_{kk} follows since the image of the product is contained in the image of P_{kk}. ✓

Remark 3.8:
(i) For the charge conjugation modular invariant, i.e. for the Jandl algebra $A = \mathbf{1}$ (having reversion $\sigma = id_{\mathbf{1}}$), a TFT demonstration of the assertions of theorems 3.5 and 3.7 was already given in [51].

(ii) In unoriented closed and open string theory, the properties (3.46(iii)) and (3.62(iii)) involving the four partition functions Z, K, A and Mö arise as consistency conditions. The particular combinations appearing in these relations determine the spectra of (orientifold-projected) open and closed string states, respectively, and therefore must be non-negative integers.

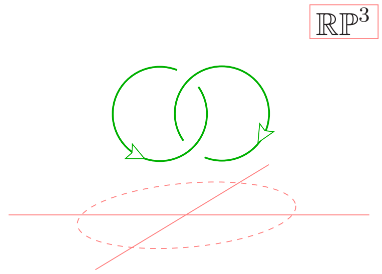
3.5 About ribbons in $\mathbb{RP}^3$

We now present the basic invariant of the three-manifold $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$. Using this invariant, arbitrary ribbon graphs in $\mathbb{RP}^3$ can easily be expressed as ribbon graphs in S^3. This will be instrumental in the evaluation of the crossed channel amplitudes for the Klein bottle and the Möbius strip. Other quantities arising in the computation of these amplitudes will be introduced in section 3.6.

To represent $\mathbb{RP}^3$ we take the upper half of $\mathbb{R}^3$ modulo an identification of points on its boundary:

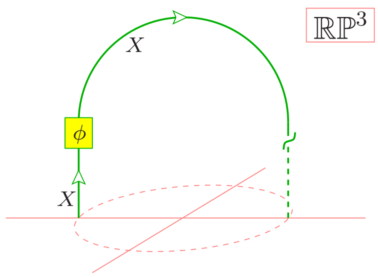
$$\mathbb{RP}^3 = \{(x,y,z) \in \mathbb{R}^3 \cup \{\infty\} \mid z \ge 0\} /\!\sim\,, \qquad (x,y,0) \sim \tfrac{-1}{x^2+y^2}\,(x,y,0)\,. \tag{3.64}$$

Similar to the pictorial conventions for ribbon graphs in S^3, $S^2 \times S^1$ and $D^2 \times S^1$ that we introduced in (I:5.13), ribbon graphs in $\mathbb{RP}^3$ will be drawn as indicated in the example of the Hopf link as follows:

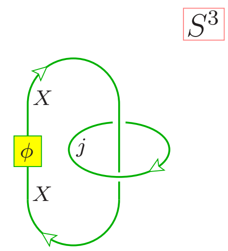


(3.65)

Here the dashed circle indicates the unit circle $x^2 + y^2 = 1$ in the x-y plane. The three-manifold $\mathbb{RP}^3$ can be obtained from surgery on the unknot in S^3 with framing -2, as is described in detail in section 4.6 and appendix B of [51]. It follows that



$$= \; S_{00}\, \kappa^{-1} \sum_{j \in \mathcal{I}} S_{0j}\, \theta_j^{-2}$$

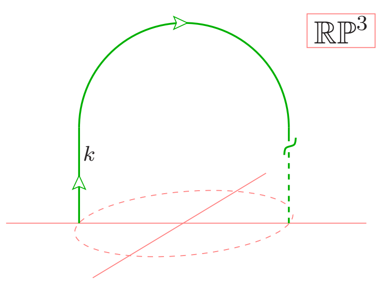


(3.66)

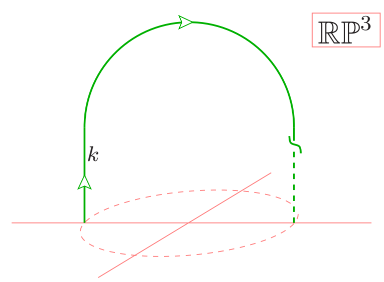
for every $\phi \in \mathrm{Hom}(X,X)$. Here the number κ is the *charge* of the modular tensor category $\mathcal{C}$, which (see section I:2.4) is given by $\kappa = S_{00} \sum_{k \in \mathcal{I}} \theta_k^{-1} (\dim(U_k))^2$. In terms of the central charge c of the CFT, κ is expressed as $\kappa = \mathrm{e}^{2\pi \mathrm{i} c/8}$.

Note that the half-twist that is present in the graph in $\mathbb{RP}^3$ on the left hand side of (3.66) must be chosen precisely as indicated; it cannot be replaced by the opposite half-twist. As a check of this statement, let us study the particular case that $\phi = id_X$ with $X = U_k$, and that the three-dimensional TFT is unitary in the sense of [60]. Such TFTs have the property that orientation reversal of a three-manifold results in complex conjugation of its invariant, see sections II.5.4 and IV.11 of [60]. We consider the two ribbon graphs

$\Gamma_+ \; :=$

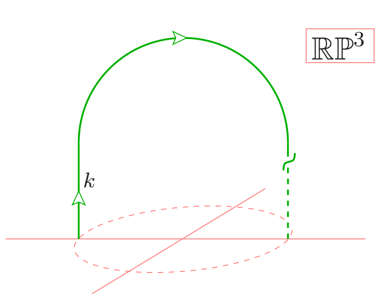


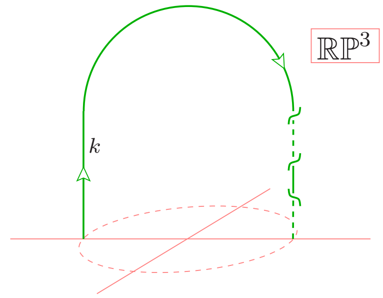
and $\quad \Gamma_- \; :=$



(3.67)

These graphs can be mapped to one another by an orientation reversing homeomorphism, so that by assumption the associated invariants satisfy $(\Gamma_+)^* = \Gamma_-$. Furthermore,

$\Gamma_+ \;=$ 

$=$ 

$= \; \theta_k \, \Gamma_- \,.$ (3.68)

These relations may be rewritten as

$$(\Gamma^*_{\epsilon)} = \Gamma_{-\epsilon} \qquad \text{and} \qquad \Gamma_\epsilon = (\theta_k)^\epsilon \, \Gamma_{-\epsilon} \tag{3.69}$$

for $\epsilon = \pm$.

Now recalling the definition of S_{ij} as the invariant of the Hopf link in S^3 (see (I:2.22)), in the situation at hand the ribbon graph on the right hand side of (3.66) is given by $s_{kj} = S_{kj}/S_{00}$, and hence according to (3.66) we have

$$\Gamma_\epsilon = \kappa^{-1} \sum_{j\in\mathcal{I}} S_{0j} \theta_j^{-2} S_{kj} \tag{3.70}$$

with $\epsilon = +$. As a consistency check, we now show by an independent argument that the index ϵ in formula (3.70) cannot be an arbitrary sign, but must be $\epsilon = +$. The matrices S and T representing the modular group satisfy $(ST)^3 = C = S^2$ with C the charge conjugation matrix. Since C commutes with S and T this implies $S^{-1}T^2S^{-1} = CT^{-1}ST^{-2}ST^{-1}$. Substituting $T_k = \kappa^{-1/3}\theta_k^{-1}$ into the $0\bar{k}$-component of this relation yields

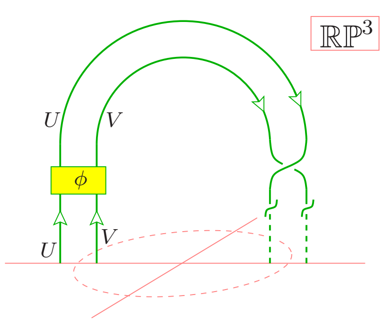
$$\kappa^{-1} \sum_{j\in\mathcal{I}} S_{0j} \theta_j^{-2} S_{jk} = \kappa \sum_{j\in\mathcal{I}} S_{0j} \theta_j^{2} S_{j\bar{k}} \theta_k \,, \tag{3.71}$$

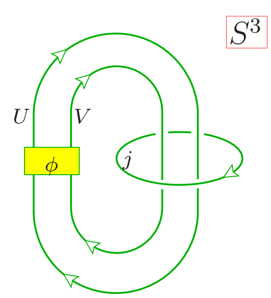
which when combined with (3.70) leads to

$$(\Gamma_\epsilon)^* = (\kappa^{-1} \sum_{j\in\mathcal{I}} S_{0j} \theta_j^{-2} S_{jk})^* = \kappa \sum_{j\in\mathcal{I}} S_{0j} \theta_j^{2} S_{j\bar{k}} = \theta_k^{-1} \kappa^{-1} \sum_{j\in\mathcal{I}} S_{0j} \theta_j^{-2} S_{jk} = \theta_k^{-1} \Gamma_\epsilon \,. \tag{3.72}$$

Comparison with (3.69) thus shows that indeed $\epsilon = +$, as implied by (3.66).

For the computations in the sequel it will be convenient to display separately the form that (3.66) takes when $X = U \otimes V$:



$= \; S_{00} \kappa^{-1} \sum_{j\in\mathcal{I}} S_{0j} \, \theta_j^{-2}$ 

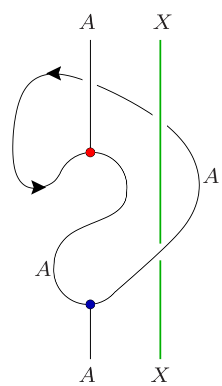
(3.73)

We would also like to point out an effect that arises when manipulating ribbon graphs in $\mathbb{RP}^3$ in the presentation (3.65) and that will play a role in some of the calculations below. Recall that the ribbons touch the x-y-plane indeed not in a single point, as suggested by the pictures in blackboard framing convention, but in a little arc. When moving the pair of arcs at which a ribbon touches the x-y-plane one must take into account the identification (3.64). In general this introduces additional twists on the ribbons which are hard to see in the blackboard framing convention. For example, for the left hand side of (3.66) one has

(3.74)

3.6 The ingredients S^A, $\tilde{S}^A$ and Γ^σ

In the closed string channel of annulus, Möbius strip and Klein bottle partition functions, specific ribbon graphs S^A, $\tilde{S}^A$ and Γ^σ appear as basic constituents. Before introducing them, we recall the definition

P_X :=


(3.75)

(see (I:5.34)) of the projector $P_X \in \mathrm{Hom}(A{\otimes}X, A{\otimes}X)$. This projector allows to select interesting subspaces $\mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X, Y)$ (already defined in section I:5.5) and $\mathrm{Hom}^{\mathrm{loc}}(X, A{\otimes}Y)$ of $\mathrm{Hom}(A{\otimes}X, Y)$ and $\mathrm{Hom}(X, A{\otimes}Y)$, respectively:

Definition 3.9:
The subspaces of *local morphisms* in $\mathrm{Hom}(A{\otimes}X, Y)$ and $\mathrm{Hom}(X, A{\otimes}Y)$ are

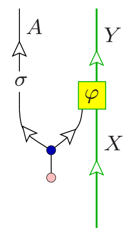
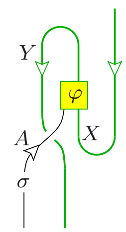
$$\begin{aligned}\mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X, Y) &:= \{\, \varphi \in \mathrm{Hom}(A{\otimes}X, Y) \,|\, \varphi \circ P_X = \varphi \,\}\,,\\ \mathrm{Hom}^{\mathrm{loc}}(X, A{\otimes}Y) &:= \{\, \psi \in \mathrm{Hom}(X, A{\otimes}Y) \,|\, P_Y \circ \psi = \psi \,\}\,.\end{aligned} \tag{3.76}$$

We will also need two linear maps that can be restricted to bijections between the subspaces (3.76).

Definition 3.10:
We define two linear maps

$$\begin{aligned} \Lambda_X^Y :\quad & \mathrm{Hom}(A{\otimes}X, Y) \rightarrow \mathrm{Hom}(X, A{\otimes}Y) \qquad \text{and} \\ \Omega_X^Y :\quad & \mathrm{Hom}(A{\otimes}X, Y) \rightarrow \mathrm{Hom}(A{\otimes}Y^\vee, X^\vee) \end{aligned} \tag{3.77}$$

by

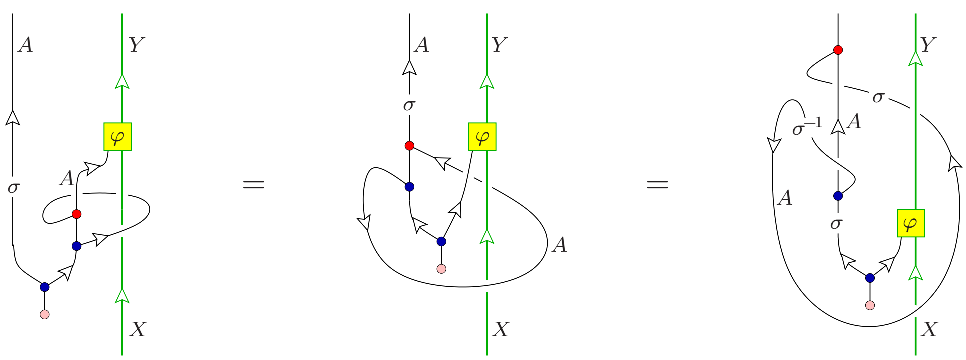
$\Lambda_X^Y(\varphi) \;:=$  and $\Omega_X^Y(\varphi) \;:=$  (3.78)

for $\varphi \in \mathrm{Hom}(A{\otimes}X, Y)$.

Proposition 3.11:
The maps Λ_X^Y and Ω_X^Y (3.78) restrict to bijections

$$\begin{aligned} \Lambda_X^Y :\quad & \mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X, Y) \xrightarrow{\cong} \mathrm{Hom}^{\mathrm{loc}}(X, A{\otimes}Y)\,, \\ \Omega_X^Y :\quad & \mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X, Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{loc}}(A{\otimes}Y^\vee, X^\vee)\,. \end{aligned} \tag{3.79}$$

Proof:
For any $\varphi \in \mathrm{Hom}(A{\otimes}X, Y)$ the equalities

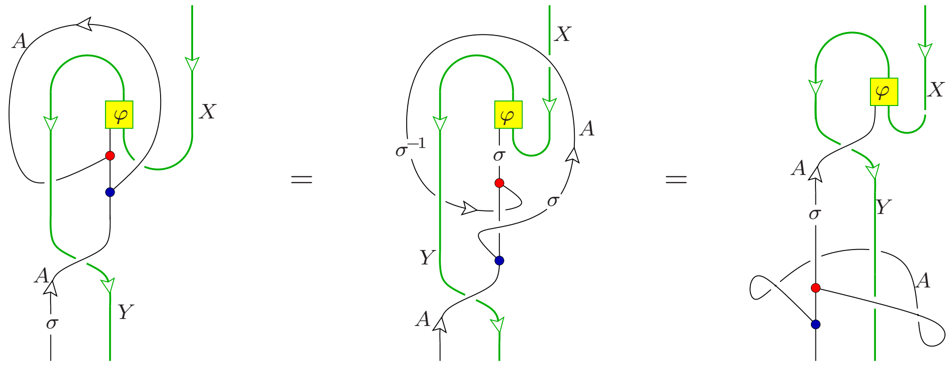
 (3.80)

hold. The left hand side is equal to $\Lambda_X^Y(\varphi \circ P_X)$. In the second step σ is taken through the multiplication and comultiplication. The resulting σ and σ^{-1} on the A-loop cancel; afterwards the A-loop can be deformed into the projector P_Y (the move involves two twists θ_A and θ_A^{-1}, which cancel). Altogether we find

$$\Lambda_X^Y(\varphi \circ P_X) = P_Y \circ \Lambda_X^Y(\varphi)\,. \tag{3.81}$$

In particular, if φ is already in $\mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X, Y)$, then $\Lambda_X^Y(\varphi)$ lies in $\mathrm{Hom}^{\mathrm{loc}}(A{\otimes}X, Y)$.

Next consider, again for any $\varphi \in \mathrm{Hom}(A{\otimes}X, Y)$, the moves



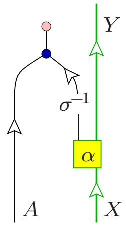
(3.82)

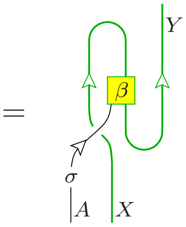
Moving the A-loop down behind the morphism φ on the left hand side one checks that the left hand side equals $\Omega_X^Y(\varphi \circ P_X)$. On the right hand side the two twists on the A-loop cancel and the resulting morphism is $\Omega_X^Y(\varphi) \circ P_{Y^\vee}$, so that

$$\Omega_X^Y(\varphi \circ P_X) = \Omega_X^Y(\varphi) \circ P_{Y^\vee} \,. \tag{3.83}$$

The equalities (3.81) and (3.83) establish that the Λ_X^Y and Ω_X^Y indeed map local morphisms to local morphisms as claimed in (3.79).

It remains to be shown that Λ_X^Y and Ω_X^Y are invertible. To this end define the maps $\tilde\Lambda_X^Y$ and $\tilde\Omega_X^Y(\beta)$ by

$\tilde\Lambda_X^Y(\alpha) \;:=$ 

and $\tilde\Omega_X^Y(\beta) \;:=$ 

(3.84)

for $\alpha \in \mathrm{Hom}^{\mathrm{loc}}(X, A{\otimes}Y)$ and $\beta \in \mathrm{Hom}_{\mathrm{loc}}(A{\otimes}Y^\vee, X^\vee)$. An argument similar to the one given above for Λ_X^Y and Ω_X^Y shows that they restrict to maps

$$\begin{aligned} \tilde\Lambda_X^Y :\;\; & \mathrm{Hom}^{\mathrm{loc}}(X, A{\otimes}Y) \to \mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X, Y)\,, \\ \tilde\Omega_X^Y :\;\; & \mathrm{Hom}_{\mathrm{loc}}(A{\otimes}Y^\vee, X^\vee) \to \mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X, Y)\,. \end{aligned} \tag{3.85}$$

Finally one easily verifies that Λ_X^Y and $\tilde\Lambda_X^Y$, respectively Ω_X^Y and $\tilde\Omega_X^Y$, are left and right inverse to one another. ✓

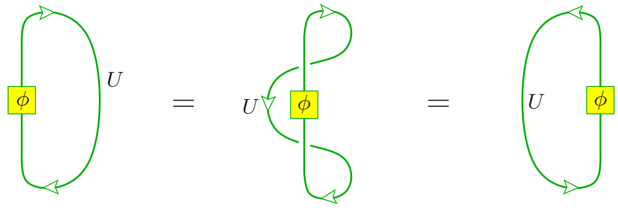
Note that while the definition of the morphism $\tilde\Omega_X^Y(\beta)$ in (3.84) looks similar to that of $\Omega_{Y^\vee}^{X^\vee}(\beta)$, the former results in a morphism in $\mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X, Y)$ while the latter gives an element of $\mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X^{\vee\vee}, Y^{\vee\vee})$. The two maps can be related via an isomorphism

$$\tilde\Omega_X^Y(\beta) = \delta_Y^{-1} \circ \Omega_{Y^\vee}^{X^\vee}(\beta) \circ (id_A{\otimes}\delta_X)\,, \tag{3.86}$$

with $\delta_U \in \mathrm{Hom}(U, U^{\vee\vee})$ as defined above (I:2.13). When evaluating $\tilde\Omega_X^Y$ in a basis, this can give rise to additional sign factors.

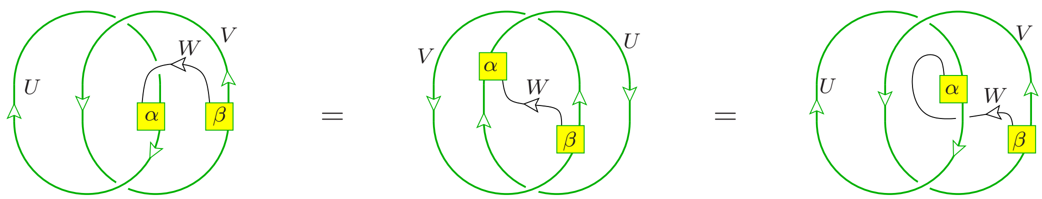
In the manipulations below we will use two specific moves on several occasions. It is helpful to list them explicitly:

Lemma 3.12:
(i) For every object U, any endomorphism $\phi \in \mathrm{Hom}(U,U)$ has equal left and right trace,



(3.87)

(ii) For any three objects U, V, W and morphisms α, β (with source and range as indicated in the figure) we have



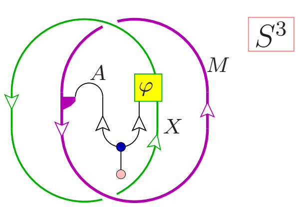
(3.88)

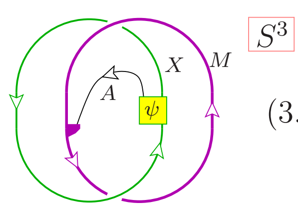
Proof:
(i) The two twists θ_U and θ_U^{-1} introduced on the U-ribbon in the middle figure cancel.
(ii) The left and right figures are obtained from the middle one by keeping β and the V-ribbon fixed and rotating the coupon that represents the morphism α (dragging along also the attached ribbons) by $\pm 180°$. ✓

We call the move (3.87) the *trace flip* on U, and the moves (3.88) the clockwise (right equality) and counter-clockwise (left equality) *Hopf turn* of U. That left and right traces coincide means that a modular tensor category $\mathcal{C}$ is in particular spherical.

We now present the definition of S^A, $\tilde{S}^A$ and Γ^σ.

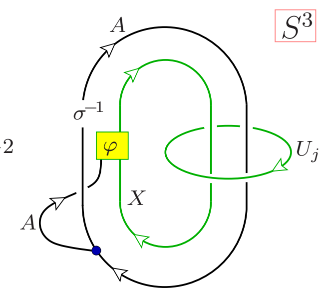
Definition 3.13:
For X an object, M a left A-module, $\varphi \in \mathrm{Hom}(A \otimes X, X)$ and $\psi \in \mathrm{Hom}(X, A \otimes X)$ we set

$S^A(M;X,\varphi) \;:=\; S_{00}$ 

$\tilde{S}^A(X,\psi;M) \;:=\; S_{00}$ 

(3.89)

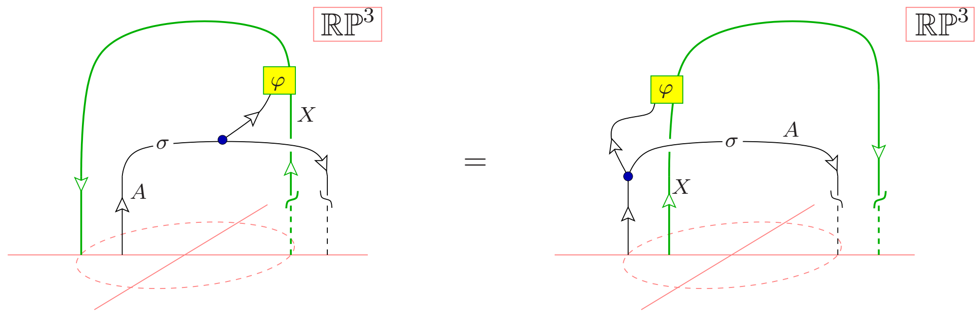
and

$$\Gamma^\sigma(X,\varphi) \;:=\; S_{00}\kappa^{-1} \sum_{j\in\mathcal{I}} S_{0j}\, \theta_j^{-2}$$ 

(3.90)

Alternatively one can represent the invariant $\Gamma^\sigma(X,\varphi)$ directly in $\mathbb{RP}^3$; for example one has

$\Gamma^\sigma(X,\varphi)$ =



(3.91)

The equality of the first and second graph in (3.91) amounts to moving the point at which the core of the left-most X-ribbon touches the x-y-plane counter-clockwise along the dashed circle for half a circumference. Due to the identification this moves the right-most intersection point of the X-ribbon with the horizontal plane counter-clockwise as well. In this manipulation, we must keep in mind the effect mentioned at the end of section 3.5. Afterwards the X-ribbon is displaced slightly to the right, while the A-ribbon is shifted to the left.
The equality of (3.91) and (3.90) is obtained by applying (3.73): To bring the second graph in (3.91) to the form (3.73), lift the A-ribbon above the X-ribbon. One also must reverse the half-twist on the A-ribbon, giving rise to a twist θ_A^{-1}, which combines with σ to σ^{-1}.

The relation of (3.89) to the similar quantities (I:5.97) and (I:5.132) is via a choice of basis, as will be discussed in more detail at the end of this section.

Proposition 3.14:
For X an object, $\varphi \in \mathrm{Hom}(A\otimes X,X)$, $\psi \in \mathrm{Hom}(X,A\otimes X)$ and M a left A-module, the quantities S^A, $\tilde S^A$ and Γ^σ defined in (3.89) and (3.90) fulfill

$$\text{(i)}\qquad S^A(M^\sigma;X,\varphi) = \tilde S^A(X,\Lambda_X^X(\varphi);M)\,, \tag{3.92}$$

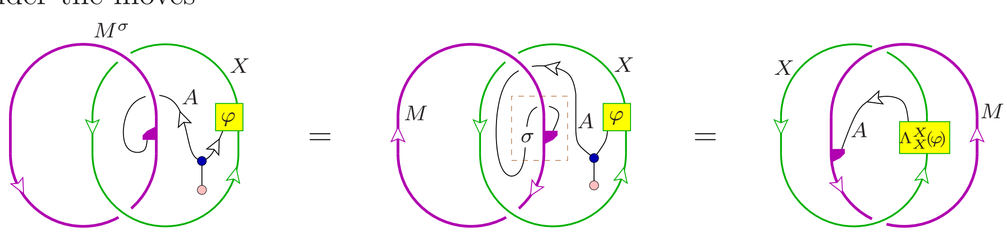
$$\text{(ii)}\qquad S^A(M^\sigma;X,\varphi) = S^A(M;X^\vee,\Omega_X^X(\varphi))\,, \tag{3.93}$$

$$\text{(iii)}\qquad \Gamma^\sigma(X,\varphi) = \Gamma^\sigma(X^\vee,\Omega_X^X(\varphi))\,, \tag{3.94}$$

$$\text{(iv)}\qquad S^A(M;X,\varphi) = S^A(M;X,\varphi\circ P_X)\,, \qquad \Gamma^\sigma(X,\varphi) = \Gamma^\sigma(X,\varphi\circ P_X)\,,$$
$$\tilde S^A(X,\psi;M) = \tilde S^A(X,P_X\circ\psi;M)\,. \tag{3.95}$$

Proof:
For the proof of (i) – (iii) we use the trace flip (3.87) and the Hopf turn (3.88).

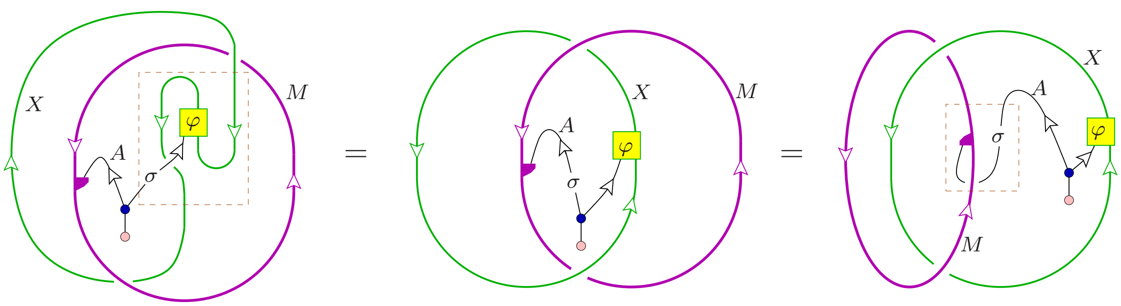
(i) Consider the moves



(3.96)

The left hand side is obtained from $S^A(M^\sigma; X, \varphi)$ by an counter-clockwise Hopf turn on M^σ. In the first step the definition (2.25) of ρ^σ is substituted and the second step amounts to a trace flip on M as well as insertion of the definition (3.78) of $\Lambda_X^X(\varphi)$. The right hand side equals $\tilde{S}^A(X, \Lambda_X^X(\varphi); M)$.

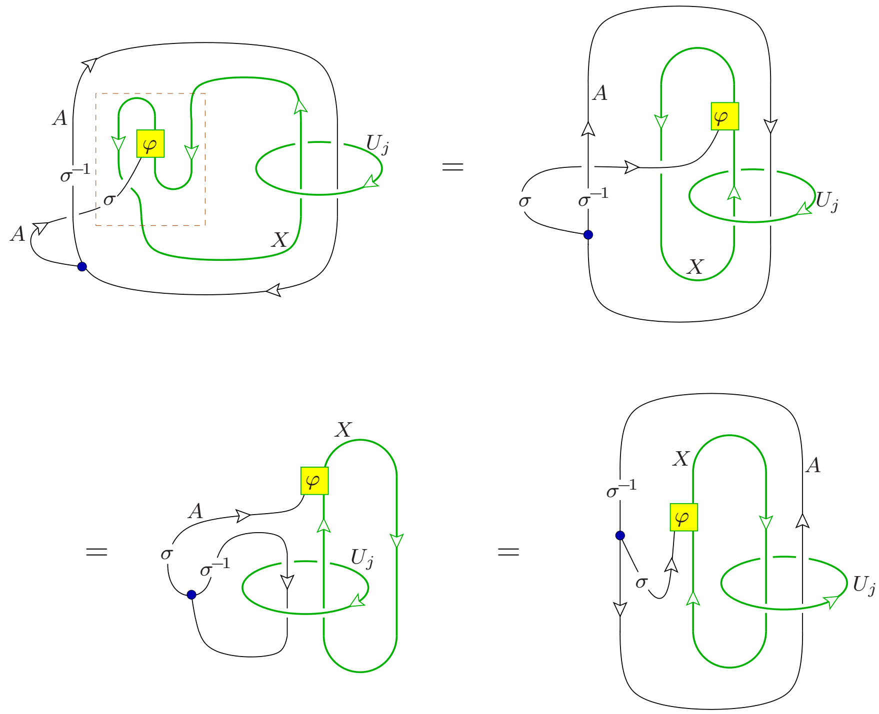
(ii) Consider the moves



(3.97)

The left hand side shows the definition of $S^A(\,M; X^\vee, \Omega_X^X(\varphi))$. The first step amounts to a clockwise Hopf turn of X followed by a trace flip on X. Also, in the first step the identity (2.17) is used to move σ past the coproduct. The second step is a clockwise Hopf turn of M. By definition (2.25) of ρ^σ the right hand side is equal to $S^A(M^\sigma; X, \varphi)$.

(iii) Consider the moves

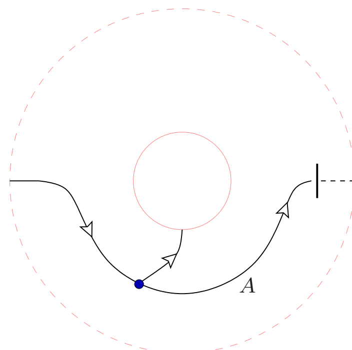


(3.98)

The first step is a deformation of the X-ribbon. The second step consists of a trace flip on X combined with a deformation of the A-ribbon. In the third step the A-loop is rotated by 180° clockwise around its center and then a trace flip is executed on it. Afterwards one can use the identity (I:5.25) to turn around the direction of the A-ribbon. This also changes the product into a coproduct. Finally the morphism σ^{-1} is moved clockwise around the A-loop; upon use of (2.5) this cancels the σ in front of φ and introduces the correct braiding to get (3.90). When multiplied with $\kappa^{-1} S_{00} S_{0j} \theta_j^{-2}$ and summed over $j \in \mathcal{I}$, the left hand side gives $\Gamma^\sigma(X^\vee, \Omega_X^X(\varphi))$, while the right hand side yields (after reversing the direction of the U_j-ribbon, which is allowed owing to the j-summation) $\Gamma^\sigma(X,\varphi)$.

(iv) The two equalities involving S^A and $\tilde{S}^A$ can be demonstrated by moves similar to the ones in (I:5.99). To see the equality for Γ^σ one can start from the first representation of $\Gamma^\sigma(X,\varphi)$ in (3.91). To simplify the pictures we will only show a two-dimensional slice of $\mathbb{RP}^3$ that does not intersect the X-ribbon. In the representation (3.64), (3.65) of $\mathbb{RP}^3$ the slice is the set $D := \{\, (x,y,z) \,|\, x^2+y^2=1, z\in[0,1] \,\}$. The set D can be mapped to a crosscap with a hole, which we will identify with the set $C_\circ = \{x\in\mathbb{R}^2 \,|\, |x|\in[1/2,1]\}/{\sim}$ with $x\sim -x$ iff $|x|=1$.

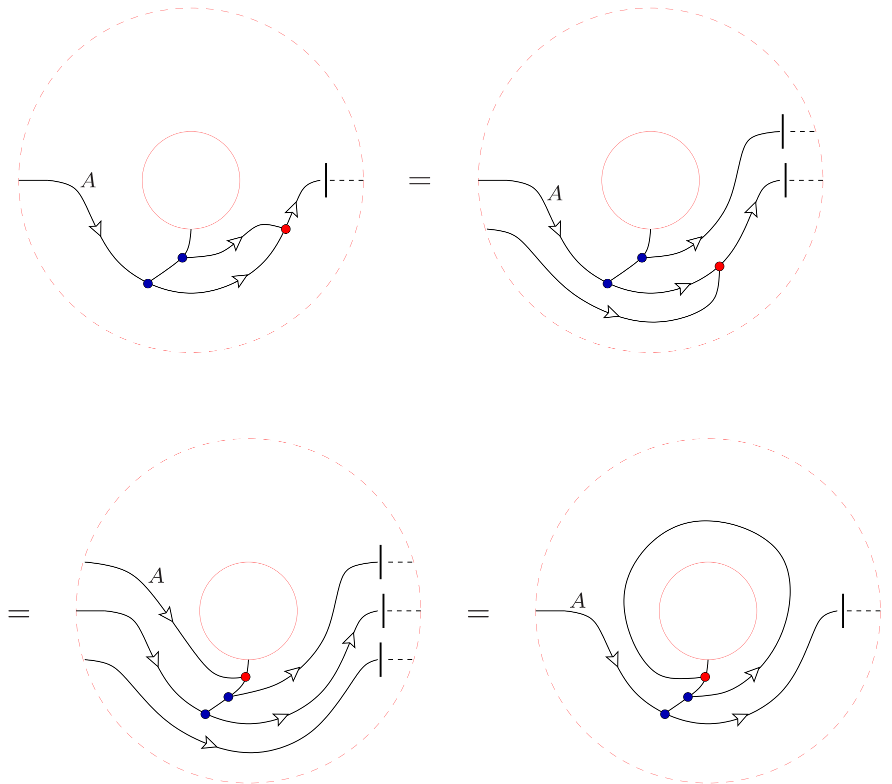
In (3.91) we can deform all the A-ribbons to lie within the set D, except for the final part of the A-ribbon that is attached to the coupon for the morphism φ; after mapping D to $C_\circ$ we then obtain



(3.99)

Here the dashed circle corresponds to the dashed circle in (3.91), and opposite points on this circle are identified. The A-ribbon ending on the solid circle in the center continues to the morphism φ in the full picture (3.91). In addition, the morphism σ and the half-twist in (3.91) have been combined to the reversing move (3.4).

In the representation (3.99), consider now the following moves on the A-ribbons



(3.100)

The left hand side is equal to (3.99) by coassociativity and specialness. In the first step the multiplication morphism is taken past the reversing move using (3.5) and is then moved further along the A-ribbon, using also the Frobenius property to move it past the comultiplication. The second step is similar to the previous one. In the third step the lower segment of the A-ribbon is taken through the dotted circle, reemerging at the top in the way indicated. Now drawing the right hand side of (3.100) in the full picture (3.91), we see that what we achieved is precisely to insert a projector P_X in front of φ. ✓

As already pointed out in section I:5.4 and will be described in detail elsewhere, bulk fields are labelled by local morphisms in the spaces $\mathrm{Hom}(A{\otimes}U_j, U_i^\vee)$. For example, by lemma I:5.6 the dimension of the local subspace is exactly Z_{ij}. The quantities S^A, $\tilde S^A$ and Γ^σ are related to one-point functions on the disk and on the crosscap. Proposition 3.14(iv) implies that when we choose an eigenbasis of the projector P_{U_j}, only 'physical' labels give nonzero results. Such a basis and its dual were introduced in (I:5.55) and (I:5.56), denoting the bases of $\mathrm{Hom}(A{\otimes}U_l, U_{\bar k})$ and $\mathrm{Hom}(U_{\bar k}, A{\otimes}U_l)$ by $\{\mu_\alpha^{kl}\}$ and $\{\bar\mu_\alpha^{kl}\}$, respectively. Let us order the basis vectors such that the subsets

$$\{\mu_\alpha^{kl}\,|\,\alpha=1,\ldots,Z_{kl}\} \qquad \text{and} \qquad \{\bar\mu_\alpha^{kl}\,|\,\alpha=1,\ldots,Z_{kl}\} \tag{3.101}$$

are dual bases of the subspaces $\mathrm{Hom}_{\mathrm{loc}}(A{\otimes}U_l, U_{\bar k})$ and $\mathrm{Hom}^{\mathrm{loc}}(U_{\bar k}, A{\otimes}U_l)$, respectively. With these choices the relation between S^A and $\tilde S^A$ from (3.89) and the quantities given in (I:5.97) and (I:5.132) is

$$S^A_{\kappa,p\alpha} = S^A(M_\kappa; U_p, \mu_\alpha^{\bar pp}) \qquad \text{and} \qquad \tilde S^A_{p\alpha,\kappa} = \tilde S^A(U_p, \bar\mu_\alpha^{\bar pp}; M_\kappa)\,. \tag{3.102}$$

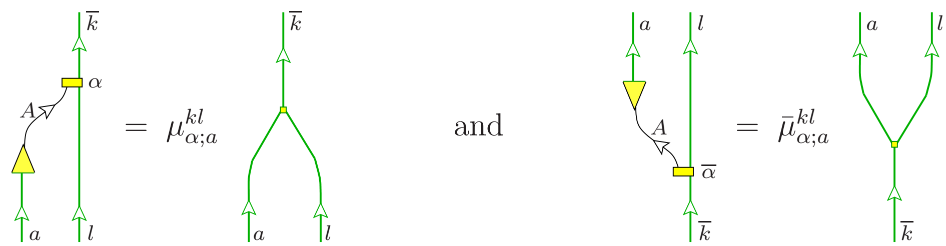
It was shown in propositions I:5.16 and I:5.17 that the matrices $S^A_{\kappa,p\alpha}$ and $\tilde S^A_{p\alpha,\kappa}$ are each others' inverses. Further we set

$$\Gamma^\sigma_{p\alpha} := \Gamma^\sigma(U_p, \mu^{\bar p p}_\alpha) \qquad \text{and} \qquad \gamma^\sigma_{p\alpha} := t_p^{-1}\Gamma^\sigma_{p\alpha} \tag{3.103}$$

with t_p the numbers introduced in (3.43). The combinations $\gamma^\sigma_{p\alpha}$ are introduced because the crossed channel amplitudes look a bit simpler when expressed in terms of these combinations. They are also used frequently in the literature, since they arise when working with phase rotated characters $\hat\chi_k(\tau) = \mathrm{e}^{\pi\mathrm{i}(\Delta_k - c/24)}\chi_k(\tau)$, as is convenient for the Möbius strip amplitude (see e.g. [61]).

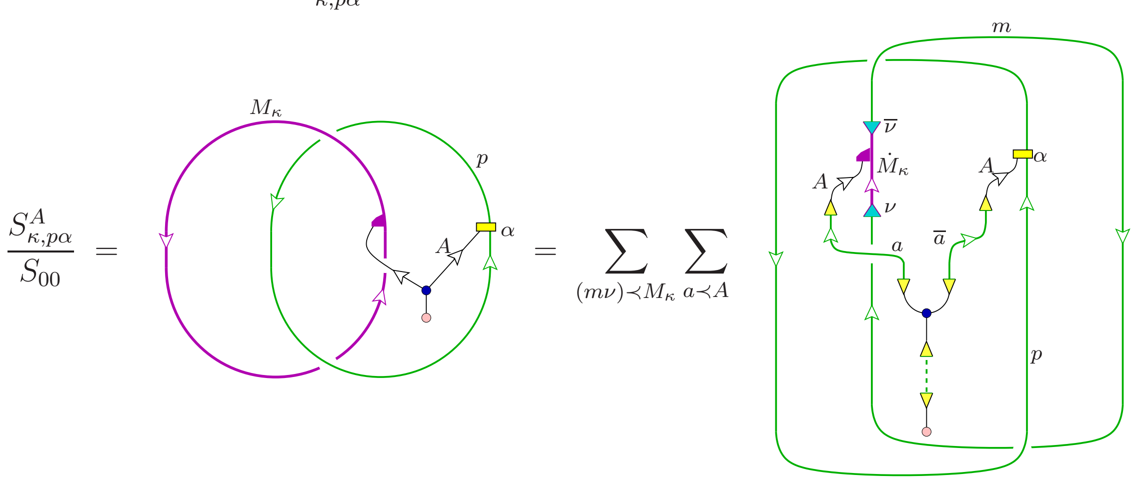
The quantitites $S^A_{\kappa,p\alpha}$, $\tilde S^A_{p\alpha,\kappa}$ and $\Gamma^\sigma_{p\alpha}$ can be expressed explicitly in terms of fusing and braiding matrices. For brevity, as in paper I we present these expressions only for the case that both ${N_{ij}}^k \in \{0,1\}$ and $\dim_{\mathbb{C}} \mathrm{Hom}(U_k, A) \in \{0,1\}$ for all $i,j,k \in \mathcal{I}$. Also recall that when one of the labels i,j,k is 0, the choice of basis elements in $\mathrm{Hom}(U_i \otimes U_j, U_k)$ is prescribed by formula (I:2.33).

The multiplication m and comultiplication Δ of the algebra A are expressed in this basis according to (I:3.7) and (I:3.82). The reversion on A is given in terms of numbers $\sigma(a)$ introduced in formula (2.19). For the representation morphism ρ of an A-module M, the expansion in a basis looks like in (I:4.61). As a final ingredient, the local morphisms in $\mathrm{Hom}(A \otimes U_l, U_{\bar k})$ and its dual are given in a basis as



(3.104)

The link invariant for $S^A_{\kappa,p\alpha}$ can now be evaluated as



(3.105)

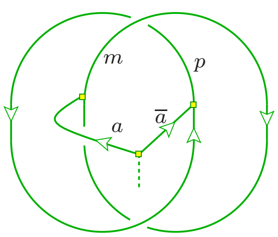
where the first figure is related to the definition of S^A in (3.89) by a Hopf turn (3.88) of M_κ. In the second step we have executed a trace flip on M_κ and inserted the bases (I:3.4) and (I:4.21) so as to decompose A and M_κ into simple objects. Recalling also the normalisation convention (I:3.79) for the basis of $\mathrm{Hom}(\mathbf{1}, A)$ and $\mathrm{Hom}(A, \mathbf{1})$, as well as the notation $a \prec A$ indicating that

U_a is a simple subobject of A (see (I:3.78)) and the analogous notation $(m\nu)\prec M$ in which ν labels possible multiplicities of U_m in $\dot{M}$, we find

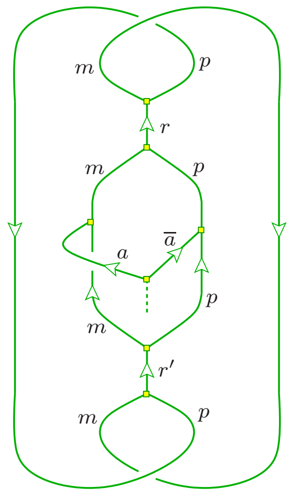
$$S^A_{\kappa,p\alpha} = \sum_{a\prec A}\,\sum_{(m\nu)\prec M_\kappa} \rho^{\kappa\, m\nu}_{a,m\nu}\, \Delta^{a\bar a}_0\, \mu^{\bar p p}_{\alpha;\bar a}\, I_S(p,a,m) \tag{3.106}$$

with $I_S(p,a,m)$ given by

$I_S(p,a,m)/S_{00} \;:=$

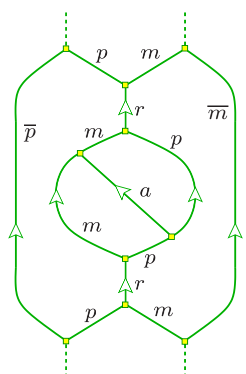


$= \sum_{r,r'}$



(3.107)

$$= \sum_r \mathsf{R}^{(m\,p)r}\mathsf{R}^{(p\,m)r}\mathsf{R}^{-\,(m\,a)m}\,\dim(U_p)\,\dim(U_m)\,\mathsf{F}^{(a\,\bar a\,p)\,p}_{p\,0}$$



In the first step two complete bases of intermediate simple objects $U_r, U_{r'}$ are inserted. Since a morphism from U_r to $U_{r'}$ is zero unless $r = r'$, this reduces to a single sum over r. In the second step the definitions (I:2.41) of R and the formula (I:2.60) for F as well as the relation (I:2.35) between duality and basis morphisms are inserted. In the last graph in (3.107) the definition (I:2.40) of $\mathsf{G}\equiv\mathsf{F}^{-1}$ can be substituted. The resulting graph already appeared in a similar calculation in (I:2.63). Altogether we find

$$I_S(p,a,m) = \frac{S_{0p}\,S_{0m}}{S_{00}}\,\mathsf{R}^{-\,(m\,a)m}\,\mathsf{F}^{(a\,\bar a\,p)\,p}_{p\,0} \sum_r \frac{\theta_r}{\theta_m\theta_p}\,\mathsf{G}^{(m\,a\,p)\,r}_{m\,p}\,\mathsf{G}^{(\bar p\,p\,m)\,m}_{0\,r}\,\mathsf{F}^{(\bar p\,p\,m)\,m}_{r\,0}\,. \tag{3.108}$$

The analogous calculation for $\tilde S^A_{p\alpha,\kappa}$ and $\Gamma^\sigma_{p\alpha}$ yields

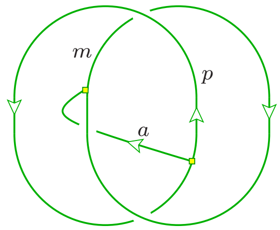
$$\tilde S^A_{p\alpha,\kappa} = \sum_{a\prec A}\,\sum_{(m\nu)\prec M_\kappa} \rho^{\kappa\, m\nu}_{a,m\nu}\, \bar\mu^{\bar p p}_{\alpha;a}\, I_{\tilde S}(p,a,m) \tag{3.109}$$

and

$$\Gamma^{\sigma}_{p\alpha} = \sum_{a,b\prec A} [\sigma(a)]^{-1}\, \Delta^{ba}_{a}\, \mu^{\bar p p}_{\alpha;b}\, I_{\Gamma}(p,a,b) \tag{3.110}$$

with constants $I_{\tilde S}(p,a,m)$ and $I_{\Gamma}(p,a,b)$ that can be evaluated in a way similar to (3.107), yielding

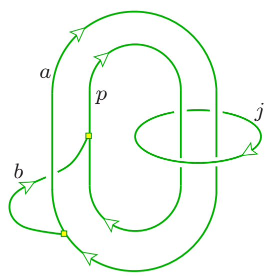
$$I_{\tilde S}(p,a,m) \;=\; S_{00}$$



$$= \frac{S_{0p}\, S_{0m}}{S_{00}}\, \mathsf{R}^{(m\,a)m} \sum_r \frac{\theta_r}{\theta_m \theta_p}\, \mathsf{G}^{(m\,a\,p)\,r}_{m\,p}\, \mathsf{G}^{(\bar p\,p\,m)\,m}_{0\,r}\, \mathsf{F}^{(\bar p\,p\,m)\,m}_{r\,0} \tag{3.111}$$

and

$$I_{\Gamma}(p,a,b) \;=\; \kappa^{-1} S_{00} \sum_j S_{0j}\, \theta_j^{-2}$$



$$= \kappa^{-1}\, \mathsf{R}^{-\,(b\,a)a} \sum_{j,r} S_{0j}\, \theta_j^{-2} S_{jr}\, \mathsf{F}^{(a\,b\,p)\,r}_{p\,a}\,. \tag{3.112}$$

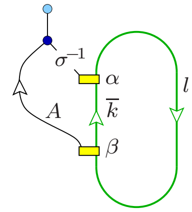
Here a summation over a complete basis of morphisms with intermediate object U_r is inserted in the two $U_a{\otimes}U_p$-ribbons. Substituting (I:2.37) gives rise to the F-matrix element. What remains is a Hopf link for a U_r- and a U_j-ribbon, whose invariant is $s_{j,r}$.

For the computations in the next section we also need to express the inverse of the map Λ^Y_X defined in (3.78) in a basis. We define the matrix $g^{kl}_{\alpha\beta}$ via

$$(\Lambda^{l}_{\bar k})^{-1}(\bar\mu^{kl}_{\alpha}) = \sum_{\beta} g^{kl}_{\alpha\beta}\, \dim(U_k)\, \mu^{\bar l \bar k}_{\beta}\,. \tag{3.113}$$

As a consequence of the bijections (3.79), if $\bar\mu^{kl}_{\alpha}$ is local, then so is the left hand side of (3.113), and thus in this case the sum on the right hand side can be restricted to local $\mu^{\bar l\bar k}_{\beta}$. Also by proposition 3.11, the map $(\Lambda^{l}_{\bar k})^{-1}$ is invertible on the local morphisms, so that $g^{kl}_{\alpha\beta}$ is invertible as a matrix in α,β. Composing both sides of (3.113) with $\bar\mu^{kl}_{\beta}$ and using the description (3.84) of the inverse of Λ^Y_X we find

$$\dim(U_k)\, \dim(U_l)\, g^{kl}_{\alpha\beta} \;=\; \tag{3.114}$$



Taking in this graph the morphism $\bar\mu_\alpha^{kl}$ around the U_l-loop results in a graph that (using symmetry of A, as well as (2.17)) can be deformed into the one for $\dim(U_l)\dim(U_k)\, g^{\overline{l}\overline{k}}_{\beta\alpha}$. Thus we get the relation

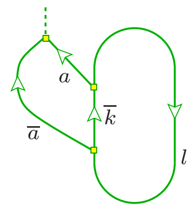
$$g^{kl}_{\alpha\beta} = g^{\bar l\,\bar k}_{\beta\alpha}\,, \tag{3.115}$$

and in particular

$$g^{\bar k k}_{\alpha\beta} = g^{\bar k k}_{\beta\alpha}\,. \tag{3.116}$$

Evaluating the invariant (3.114) in the case $N_{ij}^{\ \,k} \in \{0,1\}$ and $\langle U_k,A\rangle \in \{0,1\}$ we find

$$g^{kl}_{\alpha\beta} \;=\; \sum_{a\prec A} \frac{\dim(A)}{\dim(U_k)\dim(U_l)}\, m_{\bar a\,a}^{\ \ 0}\, \frac{1}{\sigma(a)}\, \bar\mu^{kl}_{\alpha,a}\, \bar\mu^{\bar l\,\bar k}_{\beta,\bar a} \tag{3.117}$$



(The factor $\dim(A)$ appears when inserting a basis of morphisms (I:3.4) in front of the counit and using (I:3.79) with $\beta_{\mathbf{1}} = \dim(A)$.) With the help of the identities (I:2.60) this can be rewritten as

$$g^{kl}_{\alpha\beta} = \sum_{a\prec A} \dim(A)\, \frac{m_{\bar a\,a}^{\ \ 0}}{\sigma(a)}\, \bar\mu^{kl}_{\alpha,a}\, \bar\mu^{\bar l\,\bar k}_{\beta,\bar a}\, \frac{\mathsf{G}^{(\bar a\,a\,l)\,l}_{0\,\bar k}}{\dim(U_k)}\,. \tag{3.118}$$

3.7 Crossed channel for Möbius strip and Klein bottle

In the two previous sections we have gathered the necessary ingredients to compute the coefficients of the Möbius strip and Klein bottle amplitudes in the crossed channel. The purpose of this section is to establish compatibility of the Möbius strip and Klein bottle amplitudes with the crossed channel, which amounts to the two identities

$$\text{mö}_{kR} = \frac{1}{S_{00}} \sum_{l\in\mathcal{I}} \sum_{\alpha,\beta=1}^{Z_{\bar l,l}} P_{kl}\, S^A_{R,l\alpha}\, g^{\bar l\,l}_{\alpha\beta}\, \gamma^\sigma_{l\beta} \tag{3.119}$$

and

$$\mathrm{K}_k = \frac{1}{S_{00}} \sum_{l\in\mathcal{I}} \sum_{\alpha,\beta=1}^{Z_{\bar l,l}} S_{kl}\, \gamma^\sigma_{l\alpha}\, g^{\bar l\,l}_{\alpha\beta}\, \gamma^\sigma_{l\beta}\,. \tag{3.120}$$

Here P is the P-matrix of [10]: Defining the matrix $\hat T$ by

$$\hat T_{ij} := \delta_{i,j}\, \theta_i^{-1} \tag{3.121}$$

and choosing a square root of $\hat T$ according to $(\hat T^{1/2})_{kl} := \delta_{kl} t_k^{-1}$ with t_k as in (3.43), the P-matrix is given by $P = \kappa^{-1}\hat P$, where

$$\hat P_{ij} = (\hat T^{1/2} S \hat T^2 S \hat T^{1/2})_{ij} \tag{3.122}$$

and κ is the charge of $\mathcal{C}$, i.e. the number described after (3.66).

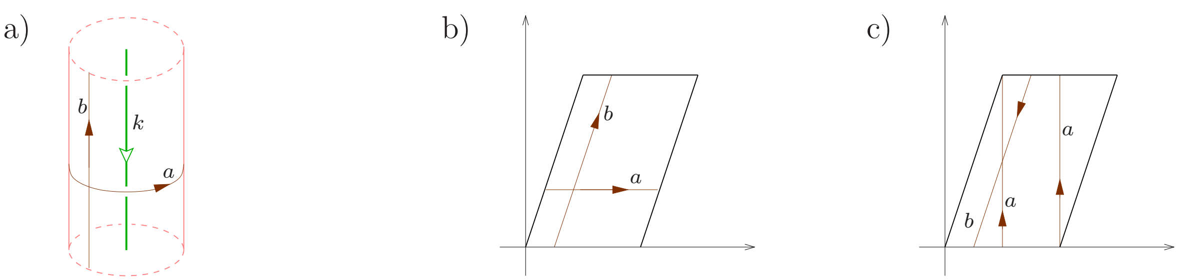
The identities (3.119) and (3.120) reflect the fact that in the crossed channel the Möbius amplitude Mö_R can be expressed as the inner product of a boundary state and a crosscap state, while the Klein bottle amplitude K is given by the inner product of two crosscap states. This amounts to a consistency condition between the closed string spectrum and the coefficients mö_{kR} and K_k which is satisfied when the identities (3.119) and (3.120) hold [38, 28]. In our approach the relations (3.119) and (3.120) are deduced, rather than postulated, i.e. the consistency between closed string spectrum and Möbius strip and Klein bottle amplitudes is satisfied automatically. Also recall that the annulus coefficients provide us with a NIM-rep of the fusion rules; NIM-reps which obey the consistency condition just mentioned, as well as the integrality properties (3.46(iii)) and (3.62(iii)), have been termed 'U-NIMreps' in [28].

In section 3.3 we have obtained the three-manifold and ribbon graph Mö_R for the Möbius strip amplitude. We denote the boundary $\partial_+\text{Mö}_R$ by Y; by construction, Y is a torus.

A convenient basis in the space $\mathcal{H}(\emptyset;Y)$ of zero-point conformal blocks on the Möbius strip with boundary condition Y can be selected as follows. We start from the basis $|\chi_k;\text{T}\rangle$ in (I:5.15). Denoting the three-manifold in (I:5.15) by $M^+_{\chi_k}$ (in section I:5.2, the notation M_1 was used instead), by definition $|\chi_k;\text{T}\rangle = Z(M^+_{\chi_k},\emptyset,\text{T})1$. Similarly, for the dual basis one chooses the three-manifold $M^-_{\chi_k}$ as in (I:5.18) (where it was called M_2) and puts $\langle\chi_k;\text{T}| = Z(M^-_{\chi_k},\text{T},\emptyset) \in \mathcal{H}(\emptyset;\text{T})^*$. The basis in $\mathcal{H}(\emptyset;\text{T})$ can be transported to $\mathcal{H}(\emptyset;Y)$ by specifying a homeomorphism $f\colon \text{T}\to Y$. Let $C_Y := Y\times[0,1]$ be the cylinder over Y. For a homeomorphism g: $\partial_+M_2 \to \partial_-M_1$, denote by $M_1 \overset{g}{\hookleftarrow} M_2$ the three-manifold obtained by glueing ∂_+M_2 to ∂_-M_1 using g. Then $Z(C_Y \overset{f}{\hookleftarrow} M^+_{\chi_k},\emptyset,Y)1$, with k ranging over $\mathcal{I}$, provides a basis of $\mathcal{H}(\emptyset;Y)$.

An essential point below will be to compare two bases obtained by different choices of f. On the torus T fix a homology basis of cycles a, b such that the a-cycle is contractible in $M^+_{\chi_k}$ and the b-cycle is the cycle that is obtained when moving the U_k-ribbon in $M^+_{\chi_k}$ to the boundary and reversing the direction. In terms of the picture (I:5.15), the a-cycle winds counter-clockwise horizontally around the torus, while the b-cycle runs vertically from bottom to top, as is indicated in figure (3.123 a) below.

We now introduce two homeomorphisms $f_{a,b}\colon \text{T}\to Y$, whose homotopy class is determined via the images of the a- and b-cycle in Y; figure (3.123 b) indicates the action of f_a, while (3.123 c) shows the action of f_b:



(3.123)

We denote the two corresponding bases of $\mathcal{H}(\emptyset;Y)$ by

$$|a_k;Y\rangle := Z(C_Y \overset{f_a}{\hookleftarrow} M^+_{\chi_k},\emptyset,Y)1 \qquad \text{and} \qquad |b_k;Y\rangle := Z(C_Y \overset{f_b}{\hookleftarrow} M^+_{\chi_k},\emptyset,Y)1\,, \tag{3.124}$$

respectively. This way we get two expansions of the vector $Z(\text{Mö}_R,\emptyset,Y)1 \in \mathcal{H}(\emptyset;Y)$ (as before, by abuse of notation we will use the symbol Mö_R also for this vector),

$$\text{Mö}_R = \sum_{l\in\mathcal{I}} \text{Mö}_{lR}\,|a_l;Y\rangle = \sum_{l\in\mathcal{I}} \widetilde{\text{Mö}}_{lR}\,|b_l;Y\rangle\,. \tag{3.125}$$

The basis $\{|a_l;Y\rangle\}$ is chosen such that the coefficients $\mathrm{M\ddot{o}}_{lR}$ appearing here are precisely those given in formula (3.37). The coefficients $\widetilde{\mathrm{M\ddot{o}}}_{lR}$ in the basis $\{|b_l;Y\rangle\}$ will be computed below.

First, however, let us derive the relation between the two expansions. One quickly checks that the bases dual to (3.124) are given by

$$\langle a_k;Y| = Z(M^-_{\chi_k} \overset{f_a^{-1}}{\hookleftarrow} C_Y, Y, \emptyset) \qquad \text{and} \qquad \langle b_k;Y| = Z(M^-_{\chi_k} \overset{f_b^{-1}}{\hookleftarrow} C_Y, Y, \emptyset)\,, \tag{3.126}$$

respectively. It then follows from (3.125) that

$$\mathrm{M\ddot{o}}_{kR} = \sum_{l\in\mathcal{I}} \langle a_k;Y|b_l;Y\rangle\, \widetilde{\mathrm{M\ddot{o}}}_{lR}\,. \tag{3.127}$$

By construction, the inner products $\langle a_k;Y|b_l;Y\rangle$ are ribbon invariants:

$$\langle a_k;Y|b_l;Y\rangle = Z(M^-_{\chi_k} \overset{f_a^{-1}\circ f_b}{\hookleftarrow} M^+_{\chi_l}, \emptyset, \emptyset)1\,. \tag{3.128}$$

More generally, let us represent the a-cycle by $\binom{1}{0}$ and the b-cycle by $\binom{0}{1}$, and let $g\colon T\to T$ be an invertible homeomorphism acting on the homology basis as $\left(\begin{smallmatrix} p & q \\ r & s\end{smallmatrix}\right)$, with $ps-qr=1$. Then we are interested in the matrix $M(g)$ with entries

$$M(g)_{kl} = M\Big[\left(\begin{smallmatrix} p & q \\ r & s\end{smallmatrix}\right)\Big]_{kl} := Z(M^-_{\chi_k} \overset{g}{\hookleftarrow} M^+_{\chi_l}, \emptyset, \emptyset)1\,. \tag{3.129}$$

The matrices M form an $|\mathcal{I}|$-dimensional projective representation of the modular group $\mathrm{SL}(2,\mathbb{Z})$. One way to compute them is to combine

$$M\Big[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)\Big]_{kl} = S_{kl} \qquad \text{and} \qquad M\Big[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)\Big]_{kl} = \hat{T}_{kl} = \delta_{kl}\,\theta_k^{-1} \tag{3.130}$$

with the recursion relations

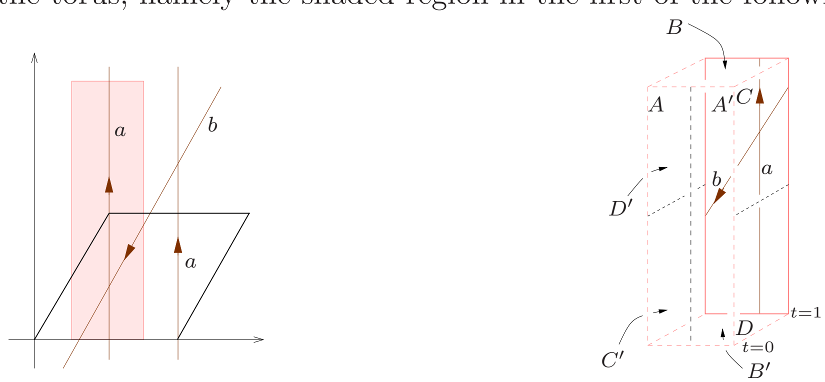
$$\begin{aligned} M\Big[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\Big]_{kl} &= \textstyle\sum_r \hat{T}_{kr}\, M\Big[\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\Big]_{rl}\,, \\ M\Big[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\Big]_{kl} &= \kappa^{-\mathrm{sign}(ac)} \textstyle\sum_r S_{kr}\, M\Big[\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\Big]_{rl}\,. \end{aligned} \tag{3.131}$$

A detailed derivation of these relations is given in appendix A.3.

Note that the composition $f_a^{-1}\circ f_b$ in (3.128) takes the cycle b to $-b$ and a to $-a{+}2b$. Applying (3.131) to (3.128) we thus compute

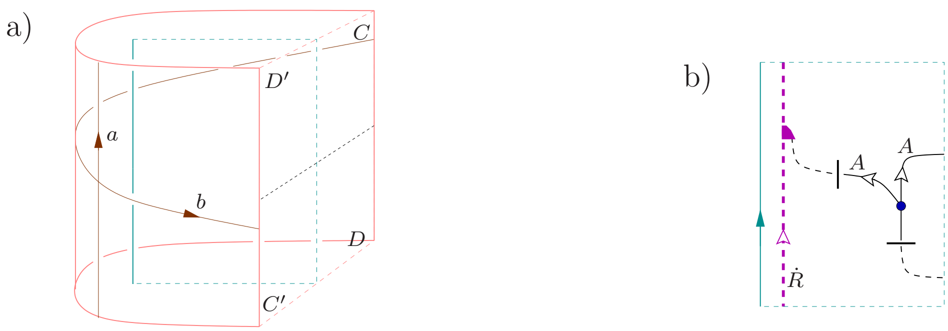
$$\langle a_k;Y|b_l;Y\rangle = M\left[\left(\begin{smallmatrix} -1 & 0 \\ 2 & -1\end{smallmatrix}\right)\right]_{kl} = \kappa^{-1}(S\hat{T}^2S)_{kl}\,. \tag{3.132}$$

After working out the matrix describing the change of basis in (3.127) we proceed to evaluate the link invariant $\widetilde{\mathrm{M\ddot{o}}}_{kR}$. To construct the three-manifold we start from a different fundamental domain for the torus, namely the shaded region in the first of the following pictures:



(3.133)

The connecting manifold $\mathrm{Mö}_R$ is constructed as in (3.8). A fundamental domain of $\mathrm{Mö}_R$ is shown in the second of the pictures (3.133). This picture is to be understood as follows. The top and bottom faces are to be identified periodically, $B \sim B'$; among the lateral faces one must identify $C \sim C'$ and $D \sim D'$; on the face at $t=0$ the left half A is to be identified with the right half A' via reflection about the dashed line. The face at $t=1$ is the actual boundary of $\mathrm{Mö}_R$ (a torus), and we have indicated on it the images of the a- and b-cycles. After implementing the identification $A \sim A'$ explicitly, $\mathrm{Mö}_R$ is described by the first of the following pictures:

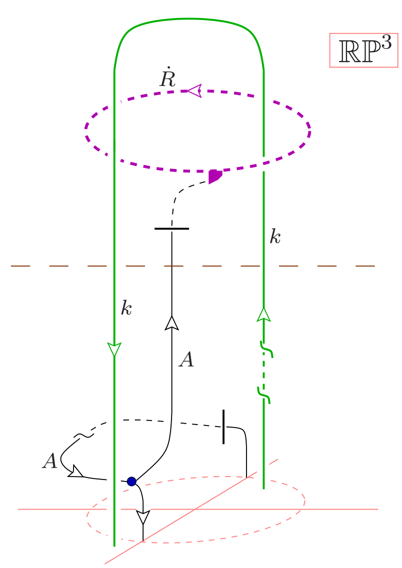


(3.134)

Here again the a- and b-cycles are drawn on the boundary $\partial_+\mathrm{Mö}_R$; the dashed lines show where the world sheet is to be embedded in $\mathrm{Mö}_R$, with the left vertical line indicating its boundary. Let us stress that this picture describes the same three-manifold as (3.33), though this is not immediately obvious from the pictures. The second figure in (3.134) represents the world sheet, i.e. the Möbius strip. Here top and bottom are identified while for the right boundary – thinking of it as an S^1 of circumference 2π – we have $\phi \sim \phi+\pi$, i.e. a crosscap. This picture also contains the ribbon graph to be embedded in $\mathrm{Mö}_R$; one can convince oneself that this ribbon graph is equivalent to the one in (3.34).

To obtain the invariant for $\widetilde{\mathrm{Mö}}_{kR}$ via (3.125), we must evaluate $\langle b_k; Y|\mathrm{Mö}_R$. In terms of the left figure in (3.134), this amounts to glueing the solid torus (I:5.18) along the boundary in accordance with the location of the cycles. The resulting three-manifold is an $\mathbb{RP}^3$. In order to reproduce the ribbon graph including all twists it is helpful to draw ribbons instead of the reduced blackboard framing notation. One finds

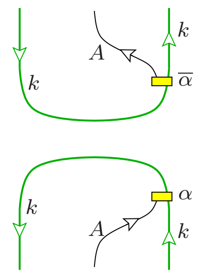
$\widetilde{\mathrm{Mö}}_{kR} \;=$



(3.135)

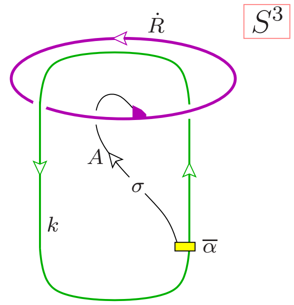
In the representation of $\mathbb{RP}^3$ that we use here, horizontal sections (like the one indicated by the long dashes) correspond to S^2's. To proceed we cut (3.135) along the dashed line and insert the identity

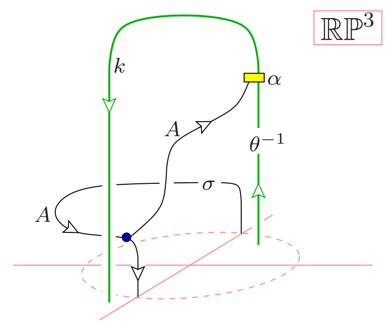
$$id_{\mathcal{H}((k,-),A,k;S^2)} \;=\; \frac{1}{S_{00}\dim(U_k)} \sum_\alpha \qquad (3.136)$$



This is an identity of endomorphisms of $\mathcal{H}((k,-),A,k;S^2)$. The right hand side consists of two three-manifolds, which are solid three-balls: one of them contains the top component of the ribbon graph, and the other one the bottom component. The boundary of this (disconnected) three-manifold is given by $S^2 \sqcup S^2$. As usual, we implicitly apply the functor $(Z,\mathcal{H})$, so that the right hand side is an endomorphism of $\mathcal{H}((k,-),A,k;S^2)$, too. The factor $S_{00}\dim(U_k)$ is the invariant of an S^3 with an embedded U_k-loop, which appears when one verifies that (3.136) obeys $id \circ id = id$. This leads to the following decomposition of the invariant (3.135):

$$\widetilde{\text{Mö}}_{kR} \;=\; \frac{1}{S_{00}\dim(U_k)} \sum_\alpha \qquad (3.137)$$





By comparison with the results (3.92), (3.89) and (2.25), the ribbon graph in S^3 can be recognised to be $\tilde{S}^A(U_k, \bar{\mu}_\alpha^{\bar{k}k}; R^\sigma)$. The ribbon graph in $\mathbb{RP}^3$ can be deformed so as to become equal to $\theta_k^{-1}\Gamma^\sigma(U_k, \mu_\alpha^{\bar{k}k})$. To see this, rotate the points at which the A-ribbon touches the identification plane by 90° counter-clockwise, and the U_k-ribbons by the same amount clockwise, keeping in mind the subtlety (3.74). Up to the twist θ_k^{-1}, one then indeed obtains the first picture in (3.91).

With the help of (3.92) and (3.113), the $\tilde{S}^A$-factor can be rewritten as

$$\tilde{S}^A(U_k, \bar{\mu}_\alpha^{\bar{k}k}; R^\sigma) = S^A((R^\sigma)^\sigma; U_k, (\Lambda_k^k)^{-1}(\bar{\mu}_\alpha^{\bar{k}k})) = \sum_\beta \dim(U_k)\, g_{\alpha\beta}^{\bar{k}k}\, S^A((R^\sigma)^\sigma; U_k, \mu_\beta^{\bar{k}k}) \,. \quad (3.138)$$

Recalling from (2.28) that $(R^\sigma)^\sigma \cong R$ as A-modules, it follows that altogether we have

$$\widetilde{\text{Mö}}_{kR} = \frac{1}{S_{00}} \sum_{\alpha,\beta} g_{\alpha\beta}^{\bar{k}k}\, S^A_{R,k\beta}\, \theta_k^{-1}\, \Gamma^\sigma_{k\alpha} \,. \qquad (3.139)$$

The relation between Mö_{kR} and $\widetilde{\text{Mö}}_{kR}$ thus becomes

$$\text{Mö}_{kR} = \sum_l \kappa^{-1}(S\hat{T}^2S)_{kl}\, \widetilde{\text{Mö}}_{lR} = \frac{1}{S_{00}} \sum_{l,\alpha,\beta} \kappa^{-1}\, t_k\, \hat{P}_{kl}\, t_l\, \theta_l^{-1}\, S^A_{R,l\alpha}\, g_{\alpha\beta}^{\bar{l}l}\, \Gamma^\sigma_{l\beta} \,. \qquad (3.140)$$

Using the definitions (3.45) and (3.103) and the relations (3.121) and (3.122), the above equation gives the relation (3.119). The summation range for α and β in (3.119) (and likewise in (3.120)) follows from proposition 3.14(iv) together with the convention (3.101).

Let us now turn to the corresponding calculation for the Klein bottle, which we present in less detail. The cobordism K for the Klein bottle amplitude was constructed in (3.53). Denote the boundary $\partial_+\mathrm{K}$ by Y, which is again a 2-torus. The expansion (3.54) uses the basis $|a_k;Y\rangle \in \mathcal{H}(\emptyset;Y)$ that is obtained by transporting the basis $|\chi_k;T\rangle$ from $\mathcal{H}(\emptyset;T)$ using a map $f_a\colon T \to Y$ acting on cycles as indicated in the figure on the left hand side of (3.141), while the basis $|b_k;Y\rangle$ which we want to expand in below is given by transporting $|\chi_k;T\rangle$ with a map $f_b\colon T \to Y$ acting on cycles as in the figure on the right hand side of (3.141):

(3.141)

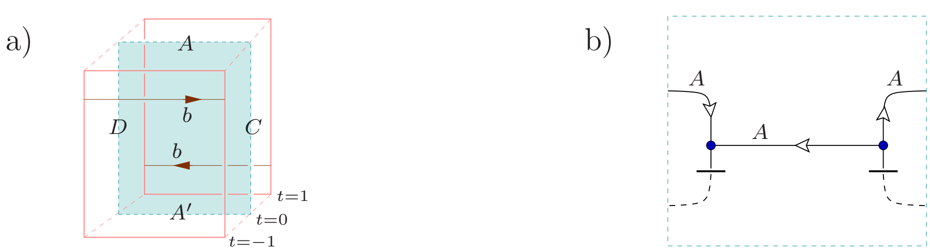
The two expansions of $\mathrm{K} \in \mathcal{H}(\emptyset;Y)$ read

$$\mathrm{K} = \sum_{l\in\mathcal{I}} \mathrm{K}_l \, |a_l;Y\rangle = \sum_{l\in\mathcal{I}} \tilde{\mathrm{K}}_l \, |b_l;Y\rangle \,. \tag{3.142}$$

This defines implicitly the coefficients $\tilde{\mathrm{K}}_l$ which we will evaluate as a link invariant. Before doing so, note that by the same argument as above the coefficients K_k and $\tilde{\mathrm{K}}_l$ are related by the matrix $M[\,\cdot\,]_{kl}$. But this time the two bases differ just by an S-transformation, so that

$$\mathrm{K}_k = \sum_{l\in\mathcal{I}} M\Big[\Big(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\Big)\Big]_{kl} \tilde{\mathrm{K}}_l = \sum_{l\in\mathcal{I}} S_{kl} \, \tilde{\mathrm{K}}_l \,. \tag{3.143}$$

In the left figure of (3.141) the shaded area indicates the base that was used in (3.49) to give a fundamental domain of the connecting manifold K. Here we use a different realisation of the same manifold K, with the shaded region in the right figure of (3.141) as a base, above which we take the intervals $[-1,1]$. The resulting manifold is shown in the first of the following figures:

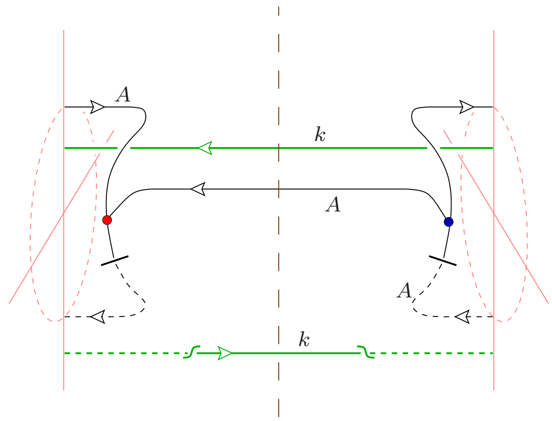


(3.144)

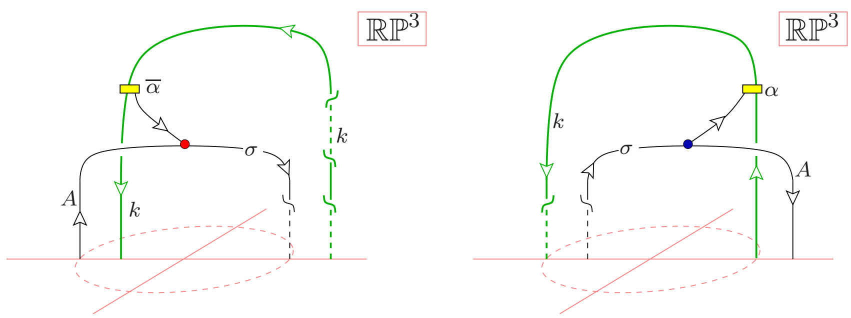
In this figure, the top and bottom faces are identified periodically, $A \sim A'$. On the face C points are identified via $(x{=}1/2, y, t) \sim (x{=}1/2, y{+}s, -t)$, and similarly on the face D, $(x{=}0, y, t) \sim (x{=}0, y{+}s, -t)$. The front and back faces of the figure constitute the boundary Y of K. The position of the b-cycle from the right figure of (3.141) is indicated. The second of the figures

(3.144) shows the world sheet; the place at which it is embedded in the manifold in the first figure is indicated by the shaded square, which is the plane $t=0$. Also shown in the second figure is the ribbon graph that one has on the world sheet; it can be seen to be equivalent to the one in (3.52).

The ribbon graph for the coefficient $\tilde{\mathrm{K}}_k = \langle b_k; Y | \mathrm{K}$ is obtained by glueing the graph for $\langle \chi_k; \mathrm{T}|$ in (I:5.18) to the boundary Y of K as prescribed by the cycles in the left figure of (3.144). One finds

$\tilde{\mathrm{K}}_k \; =$  (3.145)

This three-manifold drawn here has S^2's as vertical sections and is bounded by two crosscaps. At the dashed vertical line we now insert the identity (3.136); this leads to a decomposition into products of invariants of graphs in $\mathbb{RP}^3$:

$$\tilde{\mathrm{K}}_k \;=\; \frac{1}{S_{00}\dim(U_k)} \sum_\alpha$$  (3.146)

The second of these $\mathbb{RP}^3$-invariants has the same form as the second graph in (3.91), hence it is equal to $\Gamma^\sigma(U_k, \mu_\alpha^{\bar{k}k})$. Also, after substituting the definitions (3.78) and (3.84), by comparison with the second graph in (3.91) the first $\mathbb{RP}^3$-invariant is found to be equal to

$$\theta_k^{-1}\Gamma^\sigma(U_k^\vee, \Omega_k^k(\Lambda_k^{k\,-1}(\bar\mu_\alpha^{\bar{k}k}))) = \theta_k^{-1}\Gamma^\sigma(U_k, \Lambda_k^{k\,-1}(\bar\mu_\alpha^{\bar{k}k})) = \theta_k^{-1}\dim(U_k) \sum_\beta g_{\alpha\beta}^{\bar{k}k}\Gamma^\sigma(U_k, \mu_\beta^{\bar{k}k})\,. \quad (3.147)$$

The first equality follows by (3.94), while in the second the expansion (3.113) is inserted. Putting these results together we arrive at

$$\tilde{\mathrm{K}}_k = \frac{\theta_k^{-1}}{S_{00}} \sum_{\alpha,\beta} \Gamma^\sigma_{k\alpha}\, g^{\bar{k}k}_{\alpha\beta}\, \Gamma^\sigma_{k\beta}\,. \quad (3.148)$$

Together with (3.103) and (3.143) this establishes (3.120).

Remark 3.15:
The notation in (3.119) and (3.120) has been chosen so as to resemble formulas (2.14) and (2.15) of [28]. More explicitly, the quantities appearing in [28] – to be decorated with a hat – are related to the ones appearing here as follows:

$$\hat\Gamma_{m\alpha} = x_m \gamma^\sigma_{m\alpha}\,, \qquad \hat g^m_{\alpha\beta} = (S_{00} x_m^2)^{-1} g^{\bar m m}_{\alpha\beta} \qquad \text{and} \qquad \hat B_{(m\alpha)a} = x_m S^A_{a,m\alpha} \tag{3.149}$$

for some normalisation constants x_m; we can choose the bases (3.101) such that $x_m = 1$. The identities proven in the setting of this paper can be employed to reproduce some of the expressions in [28]. For example (2.19) of [28] follows from writing out (3.94) in a basis, with $\hat\Omega = \Omega^{-1}$. Furthermore, since S^A and $\tilde S^A$ are each other's inverse, (2.3) in [28] forces $\tilde S^A_{j\alpha,b} = S_{0j}\, \hat B^*_{(j\alpha)b}$. Finally, comparing (2.8) in [28] to a combination of (3.92) and (3.93) yields $\hat C = 1/S_{00}\,\Omega g$, in agreement with (2.9) and (2.10) of [28]. In particular (3.115) shows that $\hat g$ is indeed symmetric.

3.8 Defect lines on non-orientable surfaces

One may think of a defect as a prescription for how to join two world sheets along their boundaries (or two boundary components of a single world sheet) by some kind of boundary condition. The joint boundary component constitutes the one-dimensional defect. The theories on the two sides of the defect need in general not be the same. Indeed, one could even interpret a physical world sheet boundary as a defect linking the world sheet theory to the trivial theory with only a single state. In the CFT context defects have been studied in several situations, see e.g. [62–73], and for a lattice based analysis also [74–76].

The defects we study here are not of the most general kind. First of all, we require the defect to be conformally invariant, i.e. correlators involving the stress tensors T or $\bar T$ vary smoothly as we move the insertion point across a defect. This implies that the generators L_m, $\bar L_m$ of infinitesimal conformal transformations commute with the defect. As a consequence, the defect can be deformed continuously without changing the value of the correlation function. In this sense conformal defects are *tensionless*.

The defects we consider form a subclass of conformal defects. We are working with a given chiral algebra and describe all conformal defects that join two CFTs having this chiral algebra in common, requiring also that the currents in the chiral algebra commute with the defect. Thus these defects are *rational conformal defects*. Rational conformal defects with the same CFT on either side have been studied in [68–70, 73].

Defects on orientable world sheets have been discussed in [I], see specifically sections I:4.4, I:5.10 and remark I:5.19(ii). Below we discuss defects with the same CFT on either side on surfaces that are not necessarily orientable. We will see that wrapping defects around a non-orientable cycle [6] singles out a special subclass of defects. These defects can be wrapped around such cycles without the need to insert a defect changing field, i.e. they can still be moved freely on the world sheet.

The treatment of defect lines parallels that of boundary conditions in section 3.1. For the moment, we take the defect line to be the embedding of a circle S^1 into the world sheet such

[6] By a non-orientable cycle we mean a cycle on the world sheet that does not possess an orientable neighbourhood.

that it wraps an orientable cycle; non-orientable cycles will be treated afterwards.[7] Such defects are labelled by equivalence classes of triples $(X, \mathrm{or}_1, \mathrm{or}_2)$ where X is an A-A-bimodule, or_1 is an orientation of the defect line, and or_2 is an orientation of a neighbourhood of the defect in the world sheet X. The equivalence relation is given by

$$(X, \mathrm{or}_1, \mathrm{or}_2) \sim (X', \mathrm{or}_1', \mathrm{or}_2') \;\text{ iff }\; \begin{cases} \text{either} & \mathrm{or}_1' = \mathrm{or}_1 \;\text{ and }\; \mathrm{or}_2' = \mathrm{or}_2 \;\text{ and }\; X' \cong X\,, \\ \quad\text{or} & \mathrm{or}_1' = \mathrm{or}_1 \;\text{ and }\; \mathrm{or}_2' = -\mathrm{or}_2 \;\text{ and }\; X' \cong X^s\,, \\ \quad\text{or} & \mathrm{or}_1' = -\mathrm{or}_1 \;\text{ and }\; \mathrm{or}_2' = \mathrm{or}_2 \;\text{ and }\; X' \cong X^v\,, \\ \quad\text{or} & \mathrm{or}_1' = -\mathrm{or}_1 \;\text{ and }\; \mathrm{or}_2' = -\mathrm{or}_2 \;\text{ and }\; X' \cong X^\sigma\,. \end{cases} \tag{3.150}$$

As we will see below, graphs describing correlators with defect lines that are identified by this equivalence relation are equivalent as ribbon graphs. We denote the set of all defect line labels by

$$\mathcal{D} := \{\, (X, \mathrm{or}_1, \mathrm{or}_2)\,\} /\!\sim \;=: \{\, [X, \mathrm{or}_1, \mathrm{or}_2]\,\}\,. \tag{3.151}$$

The starting point of the construction of the ribbon graph and three-manifold for a correlator with defect lines is a world sheet X with a set of marked non-intersecting circles, such that the cycles wrapped by the circles are orientable. The boundary components of X are labelled by elements of $\mathcal{B}$ (see (3.2)), while the circles are labelled by elements of $\mathcal{D}$.

The three-manifold M_X is constructed precisely as in section 3.1. The construction of the ribbon graph is identical to the construction given there, apart from the following modification:

- Choose a representative $(X, \mathrm{or}_1, \mathrm{or}_2)$ for each equivalence class $[X, \mathrm{or}_1, \mathrm{or}_2]$ labelling a circular defect line – Choice #1′.
- The dual triangulation of X is chosen in such a way that (just like the boundary components) the defect circles are part of the triangulation – Choice #2′.
- At each vertex of the triangulation that lies on on a defect circle labelled by $[X, \mathrm{or}_1, \mathrm{or}_2]$, place one of the (pieces of) ribbon graphs

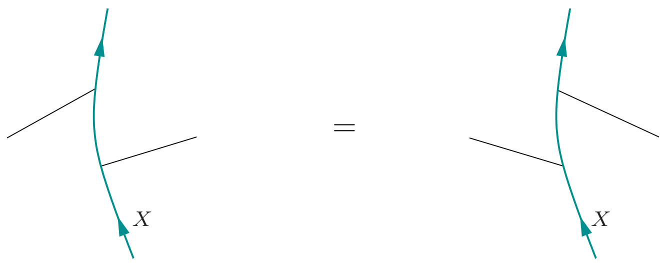
(3.152)

depending on to which side of the defect the triangulation edge lies. The representative $(X, \mathrm{or}_1, \mathrm{or}_2)$ chosen before gives an orientation or_1 to the defect circle and or_2 to a neighbourhood of the circle. The ribbons are placed such that the orientation of the ribbon matches or_2 and such that the orientation of the core of the X-ribbon matches or_1.

[7] Using TFT tools we can also describe two defect lines joining into a single one or, more generally, networks of defect lines embedded in the world sheet. In the present paper such more general situations will not be considered.

No further modifications to the prescription in section 3.1 are needed. Let us show that also after these modifications the correlator is independent of the choices made.

■ Choice #2′:
Independence of the triangulation in the presence of defect lines can be seen in much that same way as was the case for choice #2 in section 3.1. The crucial additional ingredient is that the left and right action of the algebra A on a bimodule X commute with each other. This allows to move two vertices of the triangulation that lie on the defect past each other, as indicated in the following picture:

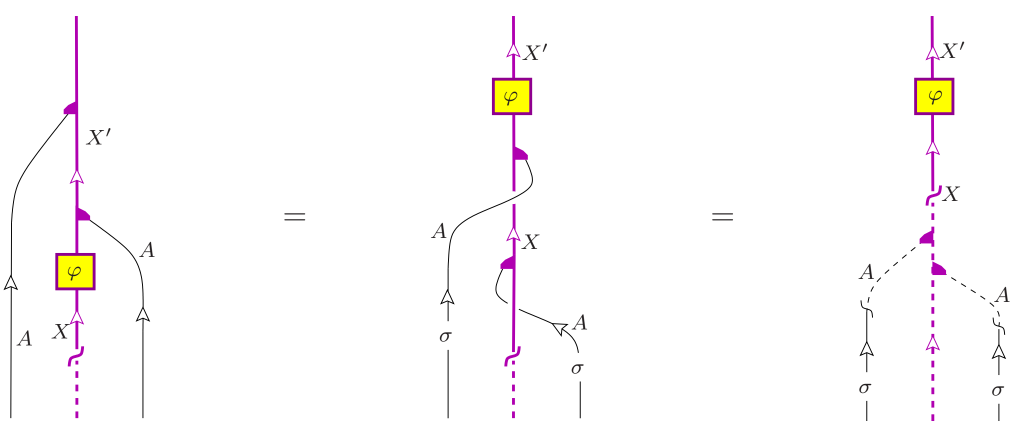


(3.153)

■ Choice #1′:
Consider a defect circle labelled by the equivalence class $[X, \mathrm{or}_1, \mathrm{or}_2]$. We must show that any two representatives $(X, \mathrm{or}_1, \mathrm{or}_2)$ and $(X', \mathrm{or}_1', \mathrm{or}_2')$ of this class lead to equivalent ribbon graphs. Analogously to choice #1 in section 3.1 this can be established by treating one by one the cases arising in the equivalence relation (3.150). Supposing that the ribbon graph has been constructed for the representative $(X', \mathrm{or}_1', \mathrm{or}_2')$, we thus proceed as follows.

(i) If $\mathrm{or}_1' = \mathrm{or}_1$ and $\mathrm{or}_2' = \mathrm{or}_2$, then $X' \cong X$.
Choose an isomorphism $\varphi \in \mathrm{Hom}_{A|A}(X, X')$ and insert the identity $id_{X'} = \varphi \circ \varphi^{-1}$ on the X'-ribbon. By definition, φ commutes with the left and right action of A on X'; therefore we can move φ around the defect circle until it reaches its inverse. Using $\varphi^{-1} \circ \varphi = id_X$, it follows that the net effect of these moves is that we have replaced the annular X'-ribbon by an X-ribbon with the same orientation of core and surface.

(ii) If $\mathrm{or}_1' = \mathrm{or}_1$ and $\mathrm{or}_2' = -\mathrm{or}_2$, then $X' \cong X^s$.
Choose an isomorphism $\varphi \in \mathrm{Hom}_{A|A}(X^s, X')$ and insert $id_{X'} = \varphi \circ \varphi^{-1}$ on the X'-ribbon. Then use the moves

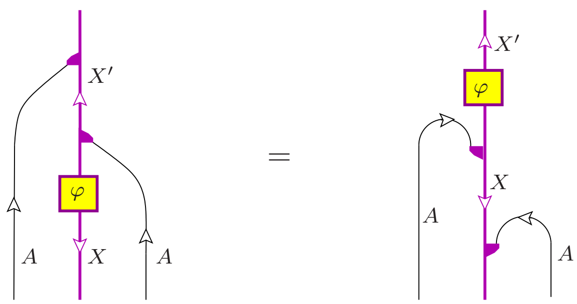


(3.154)

to take φ around the defect circle. Here we used the fact φ is an intertwiner and substituted the definition (2.40) of the action of A on X^s and then rotated the X-ribbon clockwise by 180°.

The combination of half-twist and σ in the graph on the right hand side can be recognised as the reversing move (3.4). Combining finally φ with φ^{-1} to id_X, we have replaced the X'-ribbon by an X-ribbon with opposite orientation of the surface and the same orientation of the core, as required.

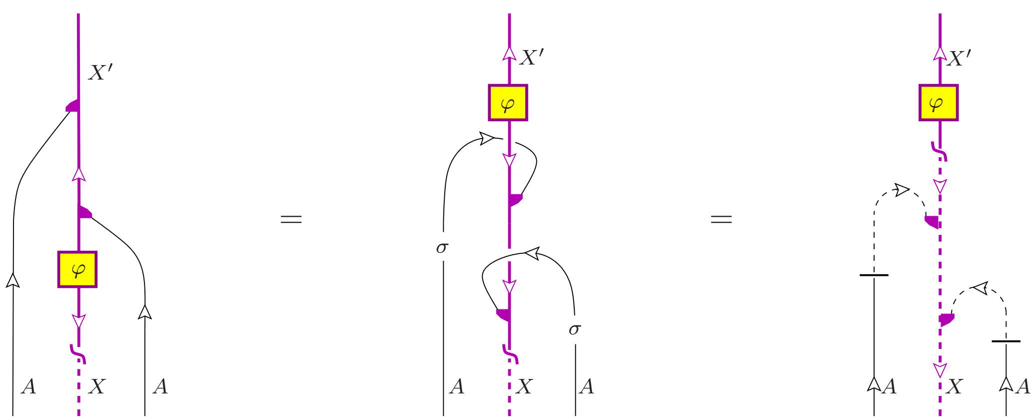
(iii) If $\mathrm{or}'_1 = -\mathrm{or}_1$ and $\mathrm{or}'_2 = \mathrm{or}_2$, then $X' \cong X^v$.
Choose an isomorphism $\varphi \in \mathrm{Hom}_{A|A}(X^v, X')$ and follow the same procedure as in point (ii), but this time using the moves



(3.155)

in place of (3.154). Then taking φ around the defect circle replaces the X'-ribbon by an X-ribbon with opposite orientation of the core.

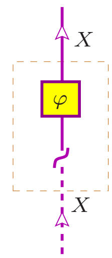
(iv) If $\mathrm{or}'_1 = -\mathrm{or}_1$ and $\mathrm{or}'_2 = -\mathrm{or}_2$, then $X' \cong X^\sigma$.
Choose an isomorphism φ in $\mathrm{Hom}_{A|A}(X^\sigma, X')$ and take it around the defect circle, using the equalities



(3.156)

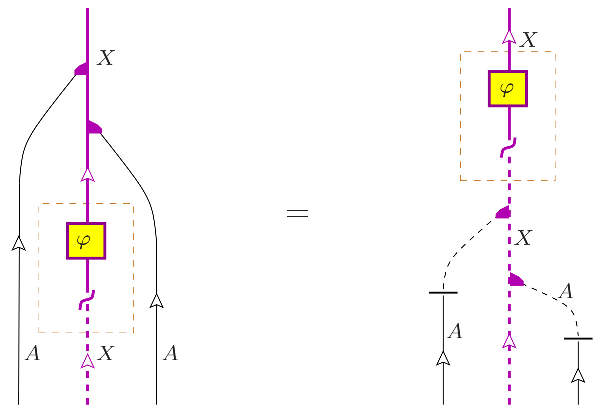
Again this results in the ribbon graph obtained for the representative $(X, \mathrm{or}_1, \mathrm{or}_2)$.

Having described the construction of the ribbon graph for defects that wrap orientable cycles, let us now turn to the case of non-orientable cycles. Consider the Möbius strip as an example, and take the circle running parallel to the boundary in the middle of the strip. When trying to place a bimodule ribbon X along such a cycle, at some point one must join the 'white' side of the X-ribbon to its 'black' side. The most general way to do this is to pick some morphism $\varphi \in \mathrm{Hom}(X,X)$ and take



(3.157)

Choosing the other possible half-twist differs from the above by a twist, which can be absorbed into the definition of φ. However we still need to impose the requirement that different triangulations of the world sheet lead to equivalent ribbon graphs. This amounts to the property



(3.158)

Comparison with (3.154) shows that independence of triangulation requires $\varphi \in \mathrm{Hom}_{A|A}(X^s,X)$. In general there need not exist a bimodule homomorphism between X^s and X. In this case it would be necessary to insert a suitable defect field (if it exists at all) at some point of the defect so as to join the two ends of the X-ribbon, thereby pinning down the location of the defect at this point.[8] In contrast, if we can find a nonzero $\varphi \in \mathrm{Hom}_{A|A}(X^s,X)$, then a defect described by the bimodule X can be wrapped around a non-orientable cycle without fixing it at any point via insertion of a defect field, so that it can still be deformed freely on the world sheet.

Computing the dimension of the space $\mathrm{Hom}_{A|A}(X^s,X)$ amounts to computing the invariant

$$\dim_{\mathbb{C}} \mathrm{Hom}_{A|A}(X^s,Y) = Z_{00}^{X^s|Y}\,, \tag{3.159}$$

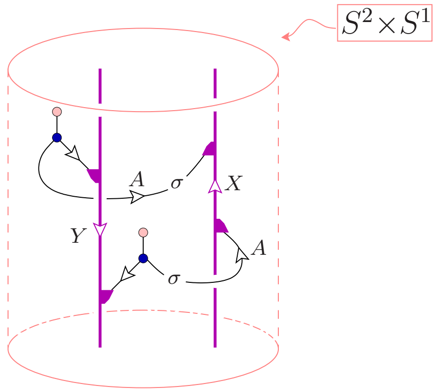
see (I:5.151) for its definition. The equality can be demonstrated by combining part (iv) of theorem I:5.23 with propositions I:4.6 and I:5.22.

In the case ${N_{ij}}^k \in \{0,1\}$ and $\langle U_k,A\rangle_A \in \{0,1\}$ for all $i,j,k \in \mathcal{I}$, the invariant $Z_{00}^{X^s|Y}$ can be evaluated as follows. First of all, inserting the definition of X^s in (2.40) into the ribbon graph

[8] Since it commutes with the Virasoro generators L_m and $\bar L_m$, a defect line can be deformed continously, on the world sheet minus the positions of field insertions, without changing the value of the correlator. But field insertions do not commute with L_m, $\bar L_m$, and hence moving a defect field insertion does in general change the value of a correlator. In this sense a defect line gets pinned down at the insertion point of a defect field.

(I:5.151) (with $k=l=0$) and slightly deforming the resulting graph gives

$$Z_{00}^{X^s|Y} \;=\; \tag{3.160}$$

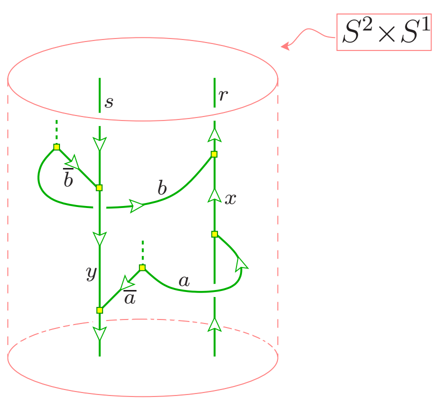


When substituting the expressions for comultiplication and the representation morphisms in a basis this leads to

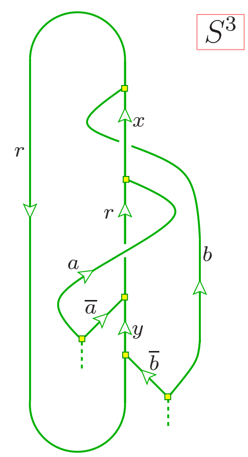
$$\begin{aligned} Z_{00}^{X^s|Y} = \sum_{a,b\prec A} & \sigma(a)\,\sigma(b)\,\Delta_0^{a\bar a}\,\Delta_0^{\bar b b} \\ & \sum_{x,r\prec X}\,\sum_{y,s\prec Y}\,\sum_{\alpha,\beta,\mu,\nu} \rho_{b,x\alpha}^{X\;r\mu}\,\tilde\rho_{a,r\mu}^{X\;x\alpha}\,\rho_{\bar a,y\beta}^{X\;s\nu}\,\tilde\rho_{\bar b,s\nu}^{X\;y\beta}\,\Gamma_{X^sY}(a,b,r,s,x,y)\,, \end{aligned} \tag{3.161}$$

where α, β, μ and ν run from 1 to $\langle U_x,X\rangle$, $\langle U_y,Y\rangle$, $\langle U_r,X\rangle$ and $\langle U_s,Y\rangle$, respectively. The invariant $\Gamma_{X^sY}(a,b,r,s,x,y)$ is defined as

$$\Gamma_{X^sY}(a,b,r,s,x,y) \;:=$$



$$= \;\frac{\delta_{rs}}{\dim(U_r)} \tag{3.162}$$



In the second equality we first apply dominance on the vertical U_s- and U_r-ribbons; only the tensor unit survives in the intermediate channel, compare the manipulations in (I:5.101). Next the three-point vertices on the left-most vertical ribbon are dragged around the trace. The resulting invariant in S^3 is easily evaluated to give

$$\Gamma_{X^sY}(a,b,r,s,x,y) = \delta_{x,y}\,\delta_{r,s}\,\mathsf{F}_{r\,0}^{(a\,\bar a\,x)\,x}\,\mathsf{G}_{x\,0}^{(r\,\bar b\,b)\,r}\,\mathsf{R}^{(a\,r)x}\,\mathsf{R}^{(x\,b)r}. \tag{3.163}$$

4 Examples for reversions

4.1 The category of complex vector spaces

The simplest example of a modular tensor category is the category $\mathcal{V}ect_{\mathbb{C}}$ of finite-dimensional complex vector spaces. It has only a single isomorphism class of simple objects, the one containing the tensor unit $\mathbf{1}\cong\mathbb{C}$. The braiding is symmetric and simply given by $c_{U,V}(x\otimes y)=y\otimes x$.

Algebras in $\mathcal{V}ect_{\mathbb{C}}$ are just ordinary algebras over the complex numbers. As we will see, special symmetric Frobenius algebras are semisimple, and hence isomorphic to direct sums of matrix algebras. Jandl algebras, in turn, are semisimple algebras with involution. This simple form allows us to find all reversions explicitly [77].

For $\mathcal{C}=\mathcal{V}ect_{\mathbb{C}}$ one has by definition (see section I:3.4) $A\cong A_{\rm top}$. Lemma A.3 then tells us that every symmetric special Frobenius algebra in $\mathcal{V}ect_{\mathbb{C}}$ is isomorphic to a direct sum of matrix algebras, and hence semisimple. Without loss of generality we may thus take

$$A=\bigoplus_{i=1}^{r}\mathrm{Mat}_{n_i}(\mathbb{C})\,. \tag{4.1}$$

The defining properties (2.11) and (2.12) of a reversion σ reduce to the statement that σ is an involutive anti-automorphism,

$$\sigma^2=id_A \qquad\text{and}\qquad \sigma(\vec X\vec Y)=\sigma(\vec Y)\,\sigma(\vec X)\quad\text{for all }\ \vec X,\vec Y\in A\,, \tag{4.2}$$

i.e. A is an algebra with involution.

It is also easy to write down a reversion on A. Let us write an element $\vec X$ of A as $\vec X=(X_1,\dots,X_r)$ and abbreviate by $\vec X^{\rm t}$ the element $(X_1^{\rm t},\dots,X_r^{\rm t})$. Then $\sigma_\circ$ defined by

$$\sigma_\circ(\vec X):=\vec X^{\rm t} \tag{4.3}$$

is a reversion on A. Furthermore, as discussed in section 2.1, any two reversions can be related by an automorphism of A,

$$\sigma=\omega\circ\sigma_\circ\,. \tag{4.4}$$

Thus the classification of all reversions on A reduces to the standard problem of finding all automorphisms of A.

To describe this classification, denote the ith simple matrix block of A by A_i and let e_i be the corresponding primitive idempotent, i.e. the unit matrix in A_i. The e_i span the center of A and obey

$$id_A=\sum_{i=1}^{r}\mathrm{e}_i \qquad\text{and}\qquad \mathrm{e}_i\,\mathrm{e}_j=\delta_{i,j}\,\mathrm{e}_i\,. \tag{4.5}$$

Furthermore, for every idempotent p in the center of A, $\mathrm{e}_i\,p=p$ implies that either $p=\mathrm{e}_i$ or $p=0$.

Let ω be an automorphism of A. It is easily verified that the set $\{\omega(\mathrm{e}_i)\,|\,i=1,\dots,r\}$ satisfies the properties (4.5). Since the set of primitive idempotents is unique, it follows that there is a permutation $\pi\in S_r$ such that $\omega(\mathrm{e}_i)=\mathrm{e}_{\pi^{-1}(i)}$. Because of $x\mathrm{e}_i=x$ for $x\in A_i$, this implies that $\omega(x)\,\mathrm{e}_{\pi^{-1}(i)}=\omega(x)$, so that $\omega(x)\in A_{\pi^{-1}(i)}$. Thus the automorphism ω induces an isomorphism

$\omega_i\colon A_i \to A_{\pi^{-1}(i)}$. This is only possible if the two matrix blocks have the same size, $n_i = n_{\pi^{-1}(i)}$, and then ω_i is an automorphism of $\mathrm{Mat}_{n_i}(\mathbb{C})$.

By the Skolem–Noether theorem (see e.g. section I.3 of [78]) all automorphisms of the simple algebra A_i are inner, $\omega_i(X) = U_i X U_i^{-1}$ for some invertible $U_i \in A_i$. The automorphism ω thus acts on a general element $\vec{X} \in A$ as

$$\omega(\vec{X}) = (U_{\pi(1)} X_{\pi(1)} U_{\pi(1)}^{-1}, \dots, U_{\pi(r)} X_{\pi(r)} U_{\pi(r)}^{-1}) \,. \tag{4.6}$$

However, while (2.12) is true for all choices of U_i in (4.6), the requirement $\sigma^2 = id_A$ gives the extra condition that

$$\sigma(\sigma(\vec{X})) = (U_{\pi(1)} (U_{\pi(\pi(1))}^{-1})^{\mathrm{t}} X_{\pi(\pi(1))} U_{\pi(\pi(1))}^{\mathrm{t}} U_{\pi(1)}^{-1}, \dots,) \tag{4.7}$$

must be equal to $\vec{X}$ for all $\vec{X} \in A$. This implies, first of all, that π is a permutation of order two. Second, it follows that the matrix $U_i^{\mathrm{t}} U_{\pi(i)}^{-1}$ commutes with all matrices X_i, for $i = 1, \dots, r$, and hence is a multiple of the identity matrix, so that $U_i^{\mathrm{t}} U_{\pi(i)}^{-1} = \lambda_i \mathbb{1}_{n_i \times n_i}$, i.e.

$$U_i^{\mathrm{t}} = \lambda_i \, U_{\pi(i)} \,. \tag{4.8}$$

Relabelling $U_{\pi(i)} \to U_i$, we conclude that all reversions on A are of the form

$$\sigma^{\pi}_{U_1,\dots,U_m}(\vec{X}) = (U_1 X_{\pi(1)}^{\mathrm{t}} U_1^{-1}, \dots, U_m X_{\pi(m)}^{\mathrm{t}} U_m^{-1}) \,, \tag{4.9}$$

where π is a permutation of order two and $U_{\pi(i)} = \lambda_i U_i^{\mathrm{t}}$ for some $\lambda_i \in \mathbb{C}$.

We would now like to bring (4.9) to a standard form by choosing an appropriate isomorphism of algebras with involution. Consider the automorphism ω of the algebra A given by

$$\omega(\vec{X}) = (V_1 X_1 V_1^{-1}, \dots, V_m X_m V_m^{-1}) \,. \tag{4.10}$$

Note that this is in general only an automorphism of algebras, but not of algebras with involution. Indeed, under ω the reversion σ changes to $\tilde{\sigma} = \omega \circ \sigma \circ \omega^{-1}$. The action of $\tilde{\sigma}$ is given by

$$\tilde{\sigma}(\vec{X}) = (V_1 U_1 V_{\pi(1)}^{\mathrm{t}} X_{\pi(1)}^{\mathrm{t}} (V_1 U_1 V_{\pi(1)}^{\mathrm{t}})^{-1}, \dots) \,. \tag{4.11}$$

Now given $i \in \{1, 2, \dots, r\}$ with $\pi(i) \neq i$, take the smaller of the two values i, $\pi(i)$. For concreteness, assume this is i. Choose $V_i = U_i^{-1}$ and $V_{\pi(i)} = \mathbb{1}$. Then on the ith component we have $\tilde{\sigma}(\vec{X})|_i = X_{\pi(i)}^{\mathrm{t}}$ while on the $\pi(i)$th component we find

$$\tilde{\sigma}(\vec{X})|_{\pi(i)} = V_{\pi(i)} U_{\pi(i)} V_i^{\mathrm{t}} X_i^{\mathrm{t}} (V_{\pi(i)} U_{\pi(i)} V_i^{\mathrm{t}})^{-1} = X_i^{\mathrm{t}} \,, \tag{4.12}$$

where we also used $U_{\pi(i)} = \lambda_i U_i^{\mathrm{t}}$.

On the other hand, for $\pi(i) = i$ the condition $U_{\pi(i)} = \lambda_i U_i^{\mathrm{t}}$ implies that $U_i = \pm U_i^{\mathrm{t}}$, and the ith component of $\tilde{\sigma}$ reads

$$\tilde{\sigma}(\vec{X})|_i = V_i U_i V_i^{\mathrm{t}} X_i^{\mathrm{t}} (V_i U_i V_i^{\mathrm{t}})^{-1} \,. \tag{4.13}$$

By an appropriate choice of V_i we can ensure that the combination $V_i U_i V_i^{\mathrm{t}}$ is either equal to the identity in $\mathrm{Mat}_{n_i}(\mathbb{C})$ or else is built out of antisymmetric $2{\times}2$ blocks along the diagonal.

We conclude that given a $\mathbb{C}$-algebra (A,σ) with involution, we can find an isomorphism $\omega \in \mathrm{Hom}(A,A)$ which is an isomorphism, as algebras with involution, of (A,σ) and $(A,\tilde\sigma)$ with $\tilde\sigma = \omega\circ\sigma\circ\omega^{-1}$ of the special form

$$\tilde\sigma = \sigma^{\pi}_{D_1,\dots,D_r}\,, \tag{4.14}$$

where π is again a permutation of order two and the $D_i \in \mathrm{Mat}_{n_i}(\mathbb{C})$ are the identity matrix whenever $\pi(i) \neq i$. For $\pi(i) = i$ the matrix D_i is either the identity matrix or the antisymmetric matrix with blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ along the diagonal.

4.2 The Ising model

In this section we treat the example of the critical Ising model in some detail. The results obtained in the section are well known; see e.g. chapter 12 of [79] and [18, 19, 62, 63, 39] for an (incomplete) list of references treating the Ising model from a conformal field theory point of view. The two different Möbius strip and Klein bottle amplitudes for the Ising model were first written down in [19]. In [80–82] the Ising model on a lattice has been analysed on the Möbius strip and Klein bottle. What is new in our presentation is the entirely systematic manner in which the various data are obtained.

Below we start by determining all haploid algebras in the category of $c = \frac{1}{2}$ Virasoro representations. Next we run through the steps 1)−3) of section 2.5 so as to find all possible reversions. Then the algebra $A = \mathbf{1}\oplus\epsilon$, which allows for two reversions, is considered. For this algebra the modules and bimodules are worked out and the Möbius and Klein bottle coefficients are computed.

Algebras and induced modules

Let $\mathcal{C}_{\mathsf{Is}}$ be the modular category given by the chiral data of the two-dimensional critical Ising model. We will define it in more detail below. For now just note that $\mathcal{C}_{\mathsf{Is}}$ has precisely three non-isomorphic simple objects $\mathbf{1}$, σ and ϵ, with fusion rules

$$\sigma\otimes\sigma \cong \mathbf{1}\oplus\epsilon\,, \qquad \sigma\otimes\epsilon \cong \sigma\,, \qquad \epsilon\otimes\epsilon \cong \mathbf{1}\,. \tag{4.15}$$

From this information alone we can already drastically reduce the number of objects in $\mathcal{C}_{\mathsf{Is}}$ that can carry the structure of a simple symmetric special Frobenius algebra. From proposition A.4 we know that each Morita class of such algebras has a haploid representative, so that for the moment we restrict our attention to the case $\langle\mathbf{1},A\rangle = 1$ (recall the notations (I:2.3) and (I:4.5)). Thus we would like to turn the object

$$A = \mathbf{1}\oplus\sigma^{\oplus n_\sigma}\oplus\epsilon^{\oplus n_\epsilon} \tag{4.16}$$

into a symmetric special Frobenius algebra.

Let us suppose that A is such an algebra, and compute the embedding structure of the induced A-modules, using the reciprocity relation $\langle \mathrm{Ind}_A(U),\mathrm{Ind}_A(V)\rangle_A = \langle U, A\otimes V\rangle$. We find the following table for the numbers $\langle \mathrm{Ind}_A(U),\mathrm{Ind}_A(V)\rangle_A$:

	$\mathbf{1}$	σ	ϵ
$\mathbf{1}$	1	n_σ	n_ϵ
σ	n_σ	$1+n_\epsilon$	n_σ
ϵ	n_ϵ	n_σ	1

(4.17)

The diagonal entries equal to 1 tell us that $\mathrm{Ind}_A(\mathbf{1})$ and $\mathrm{Ind}_A(\epsilon)$ are simple A-modules. It follows that $n_\epsilon = \langle \mathrm{Ind}_A(\mathbf{1}), \mathrm{Ind}_A(\epsilon)\rangle_A$ must be either one or zero, depending on whether the two induced modules are isomorphic or not.

(a) Suppose that $n_\epsilon = 0$. Then also $\mathrm{Ind}_A(\sigma)$ is simple, so that $\langle \mathrm{Ind}_A(\mathbf{1}), \mathrm{Ind}_A(\sigma)\rangle_A \le 1$. Suppose $n_\sigma = 1$. Then from table (4.17) we read off that both $\mathrm{Ind}_A(\mathbf{1}) \cong \mathrm{Ind}_A(\sigma)$ and $\mathrm{Ind}_A(\sigma) \cong \mathrm{Ind}_A(\epsilon)$, whereas $n_\epsilon = 0$ implies that $\mathrm{Ind}_A(\mathbf{1})$ is not isomorphic to $\mathrm{Ind}_A(\epsilon)$. This contradiction tells us that for $n_\epsilon = 0$ we also have $n_\sigma = 0$.

(b) Suppose that $n_\epsilon = 1$. Then $\mathrm{Ind}_A(\mathbf{1}) \cong \mathrm{Ind}_A(\epsilon)$ and $\mathrm{Ind}_A(\sigma) \cong M_1 \oplus M_2$ with $M_{1,2}$ non-isomorphic simple A-modules. Thus at most one of the $M_{1,2}$ can be isomorphic to $\mathbf{1}$, and inspecting again table (4.17) we deduce that $n_\sigma \le 1$.

The analysis of induced A-modules thus proves to be surprisingly restrictive. It reduces the list (4.16) to the possibilities

$$A = \mathbf{1} \qquad \text{or} \qquad A = \mathbf{1} \oplus \epsilon \qquad \text{or} \qquad A = \mathbf{1} \oplus \sigma \oplus \epsilon \,. \tag{4.18}$$

The object $\mathbf{1}$ is trivially a simple symmetric special Frobenius algebra. For $A = \mathbf{1} \oplus \epsilon$ we note that the structure constant $m_{\epsilon\epsilon}^{\mathbf{1}}$ (we use the notation introduced in section I:3.6) is merely a normalisation and can be set to one. Since $\epsilon \otimes \epsilon \cong \mathbf{1}$ it follows that the multiplication on $\mathbf{1} \oplus \epsilon$ is unique. Furthermore, since $\mathbf{1} \oplus \epsilon \cong \sigma^\vee \otimes \sigma$, and since any object of the form $U^\vee \otimes U$ can be endowed with the structure of a simple symmetric special Frobenius algebra, the multiplication on $\mathbf{1} \oplus \epsilon$ indeed exists. The third case $A = \mathbf{1} \oplus \sigma \oplus \epsilon$, on the other hand, does not allow for a special Frobenius algebra structure. But in order to see this we still need a better knowledge of the category $\mathcal{C}_{\mathsf{Is}}$.

Chiral data

Let us summarise the chiral data $\mathcal{C}_{\mathsf{Is}}$. The Virasoro central charge of the Ising model is $c = \frac{1}{2}$, so that $\kappa = \mathrm{e}^{\pi \mathrm{i}/8}$. There are three isomorphism classes of simple objects, from which we choose representatives $\mathbf{1}$, σ and ϵ. Their conformal weights and quantum dimensions are

$$\Delta_{\mathbf{1}} = 0\,, \quad \Delta_\sigma = 1/16\,, \quad \Delta_\epsilon = 1 \qquad \text{and} \qquad \dim(\mathbf{1}) = 1 = \dim(\epsilon)\,, \quad \dim(\sigma) = \sqrt{2}\,. \tag{4.19}$$

In terms of the weights Δ_k the twist and the braiding of $\mathcal{C}_{\mathsf{Is}}$ can be expressed as

$$\theta_k = \mathrm{e}^{-2\pi \mathrm{i} \Delta_k} \qquad \text{and} \qquad \mathsf{R}^{(a\,b)c} = \mathrm{e}^{-\pi \mathrm{i}(\Delta_c - \Delta_a - \Delta_b)}\,. \tag{4.20}$$

The S-matrix and the P-matrix (see before (3.122)) are given by

$$S \equiv \begin{pmatrix} S_{\mathbf{1}\mathbf{1}} & S_{\mathbf{1}\sigma} & S_{\mathbf{1}\epsilon} \\ S_{\sigma\mathbf{1}} & S_{\sigma\sigma} & S_{\sigma\epsilon} \\ S_{\epsilon\mathbf{1}} & S_{\epsilon\sigma} & S_{\epsilon\epsilon} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \qquad \text{and} \qquad P = \begin{pmatrix} \cos\frac{\pi}{8} & 0 & \sin\frac{\pi}{8} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{8} & 0 & -\cos\frac{\pi}{8} \end{pmatrix}. \tag{4.21}$$

Finally, the fusion matrices can be described as follows. We will only consider F-matrix elements allowed by fusion (all others are zero). All those $\mathsf{F}_{p\,q}^{(a\,b\,c)\,d}$ for which one or more of a, b, c, d is $\mathbf{1}$

are equal to one. The other F's read, with a suitable choice of gauge,

$$\begin{aligned} &\mathsf{F}_{\mathbf{1}\mathbf{1}}^{(\epsilon\,\epsilon\,\epsilon)\,\epsilon} = 1\,, && \mathsf{F}_{\sigma\,\sigma}^{(\sigma\,\epsilon\,\sigma)\,\epsilon} = \mathsf{F}_{\sigma\,\sigma}^{(\epsilon\,\sigma\,\epsilon)\,\sigma} = -1\,, \\ &\mathsf{F}_{\sigma\,\mathbf{1}}^{(\sigma\,\sigma\,\epsilon)\,\epsilon} = \mathsf{F}_{\sigma\,\mathbf{1}}^{(\epsilon\,\epsilon\,\sigma)\,\sigma} = 2\,, && \mathsf{F}_{\mathbf{1}\,\sigma}^{(\epsilon\,\sigma\,\sigma)\,\epsilon} = \mathsf{F}_{\mathbf{1}\,\sigma}^{(\sigma\,\epsilon\,\epsilon)\,\sigma} = \tfrac{1}{2}\,, \\ &\mathsf{F}_{x\,y}^{(\sigma\,\sigma\,\sigma)\,\sigma} = \begin{pmatrix} \mathsf{F}_{\mathbf{1}\mathbf{1}} & \mathsf{F}_{\mathbf{1}\epsilon} \\ \mathsf{F}_{\epsilon\mathbf{1}} & \mathsf{F}_{\epsilon\epsilon} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \sqrt{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} . \end{aligned} \tag{4.22}$$

The inverse matrices G are related to F via (I:2.61), which in the present case simplifies to

$$\mathsf{G}_{p\,q}^{(a\,b\,c)\,d} = \mathsf{F}_{p\,q}^{(c\,b\,a)\,d} . \tag{4.23}$$

Classification of algebras

We now show that there is no symmetric special Frobenius algebra structure on the object $A = \mathbf{1} \oplus \sigma \oplus \epsilon$. To this end we consider the associativity constraint (I:3.78) for three choices of the quadruple $(abc)d$:

$$\begin{aligned} &\text{(i)}: && (\epsilon\sigma\sigma)\epsilon && \Rightarrow && m_{\epsilon\sigma}^{\sigma}\, m_{\sigma\sigma}^{\epsilon} = m_{\sigma\sigma}^{\mathbf{1}}\, m_{\epsilon\mathbf{1}}^{\epsilon}\, \mathsf{F}_{\mathbf{1}\,\sigma}^{(\epsilon\,\sigma\,\sigma)\,\epsilon}\,, \\ &\text{(ii)}: && (\sigma\epsilon\sigma)\epsilon && \Rightarrow && m_{\sigma\epsilon}^{\sigma}\, m_{\sigma\sigma}^{\epsilon} = m_{\epsilon\sigma}^{\sigma}\, m_{\sigma\sigma}^{\epsilon}\, \mathsf{F}_{\sigma\,\sigma}^{(\sigma\,\epsilon\,\sigma)\,\epsilon}\,, \\ &\text{(iii)}: && (\sigma\sigma\epsilon)\epsilon && \Rightarrow && m_{\sigma\sigma}^{\mathbf{1}}\, m_{\mathbf{1}\epsilon}^{\epsilon} = m_{\sigma\epsilon}^{\sigma}\, m_{\sigma\sigma}^{\epsilon}\, \mathsf{F}_{\sigma\,\mathbf{1}}^{(\sigma\,\sigma\,\epsilon)\,\epsilon}\,. \end{aligned} \tag{4.24}$$

Since A is required to be special, the morphisms (I:3.80) must be invertible. This forces $m_{aa}^{\mathbf{1}}$ to be nonzero; we normalise it to $m_{aa}^{\mathbf{1}} = 1$. With the explicit values for the F's we then find from (4.24(i)) and (4.24(iii)) that $m_{\sigma\sigma}^{\epsilon} \neq 0$ and that $m_{\sigma\epsilon}^{\sigma} = m_{\epsilon\sigma}^{\sigma} \neq 0$. On the other hand, upon inserting the value of $\mathsf{F}_{\sigma\,\sigma}^{(\sigma\,\epsilon\,\sigma)\,\epsilon}$ and using that $m_{\sigma\sigma}^{\epsilon} \neq 0$, (4.24(ii)) gives $m_{\sigma\epsilon}^{\sigma} = -m_{\epsilon\sigma}^{\sigma}$, so that we have produced a contradiction. Thus there is no associative multiplication on $\mathbf{1}\oplus\sigma\oplus\epsilon$ that is also special.

The complete list of (isomorphism classes of) haploid symmetric special Frobenius algebras in $\mathcal{C}_{\mathsf{ls}}$ thus just consists of $\mathbf{1}$ and the algebra $A = \mathbf{1} \oplus \epsilon$ with multiplication $m_{\epsilon\epsilon}^{\mathbf{1}} = 1$. Moreover, since the algebra structure on A is unique, we have $A \cong \sigma^\vee \otimes \sigma$ not only as objects, but also as algebras; thus A is in the Morita class of $\mathbf{1}$.

To summarise, there is only a single Morita class of simple symmetric special Frobenius algebras in $\mathcal{C}_{\mathsf{ls}}$, and each such algebra is isomorphic to an algebra of the form $U^\vee \otimes U$ for a (not necessarily simple) object U of $\mathcal{C}_{\mathsf{ls}}$.

Classification of reversions

To classify the possible reversions, we work through the steps 1) − 3) of section 2.5.

Step 1): As seen above, there is only a single Morita class of simple symmetric special Frobenius algebras in $\mathcal{C}_{\mathsf{ls}}$. We choose the representative $A = \mathbf{1}$.

Step 2): For $A = \mathbf{1}$ the simple modules are just the simple objects of $\mathcal{C}_{\mathsf{ls}}$, i.e. we have three simple A-modules

$$M_{\mathbf{1}} = \mathbf{1}\,, \qquad M_\sigma = \sigma\,, \qquad M_\epsilon = \epsilon\,. \tag{4.25}$$

Step 3a): We must solve (2.21) for the three algebras

$$B^a_{\mathbf{1}} = \mathbf{1}\,, \qquad B^a_\sigma = \sigma^\vee \otimes \sigma \cong \mathbf{1} \oplus \epsilon\,, \qquad B^a_\epsilon = \epsilon^\vee \otimes \epsilon \cong \mathbf{1}\,. \tag{4.26}$$

On $B^a_{\mathbf{1}}$ and B^a_ϵ there is the unique choice $\sigma(\mathbf{1}) = 1$ for the reversion. For B^a_σ we set $\sigma(k) =: s_k \mathrm{e}^{-\pi \mathrm{i} \Delta_k}$ for $k \in \{\mathbf{1}, \epsilon\}$. Then the first condition in (2.21) forces $s_k \in \{\pm 1\}$, while with the special form of R in (4.20) the second condition reads

$$m_{ij}^{\;k}\, s_k = s_i\, s_j\, m_{ji}^{\;k}\,. \tag{4.27}$$

These relations are solved by $s_{\mathbf{1}} = 1$ and $s_\epsilon = \pm 1$. We have thus found two distinct reversions on B^a_σ.

Step 3b): The only case we need to investigate is $B^b_{\mathbf{1}\epsilon} = (\mathbf{1}\oplus\epsilon)^\vee \otimes (\mathbf{1}\oplus\epsilon)$. It turns out that this leads again to two possible reversions, both of which are related to the $s_\epsilon = -1$ reversion on B^a_σ via proposition 2.16. The slightly lengthy details are presented in appendix A.4.

The algebra $A = \mathbf{1}\oplus\epsilon$

We will now investigate the algebra $A = \mathbf{1}\oplus\epsilon$ in more detail. The multiplication and comultiplication (recall equation (I:3.83)) are given by

$$m_{\mathbf{11}}^{\mathbf{1}} = m_{\mathbf{1}\epsilon}^{\epsilon} = m_{\epsilon\mathbf{1}}^{\epsilon} = m_{\epsilon\epsilon}^{\mathbf{1}} = 1 \qquad \text{and} \qquad \Delta_{\mathbf{1}}^{\mathbf{11}} = \Delta_{\epsilon}^{\mathbf{1}\epsilon} = \Delta_{\epsilon}^{\epsilon\mathbf{1}} = \Delta_{\mathbf{1}}^{\epsilon\epsilon} = \tfrac{1}{2}\,, \tag{4.28}$$

respectively. The two possible reversions on A are

$$\sigma(\mathbf{1}) = 1\,, \qquad \sigma(\epsilon) = s_\epsilon/\mathrm{i} \quad \text{with} \;\; s_\epsilon = \pm 1\,. \tag{4.29}$$

Before working out the left A-modules and A-A-bimodules, let us compute the partition function to see how many of them there are. Applying formula (I:5.85) gives $Z_{ij} = \delta_{i,j}$, as expected. From theorem I:5.18 and remark I:5.19(ii) we then see that

$$\begin{aligned} &\#\{\text{isom. classes of simple } A\text{-modules}\} = \mathrm{tr}\, Z = 3 \qquad \text{and} \\ &\#\{\text{isom. classes of simple } A\text{-}A\text{-bimodules}\} = \mathrm{tr}\,(Z Z^{\mathrm{t}}) = 3\,. \end{aligned} \tag{4.30}$$

Consider the induced modules $\mathrm{Ind}_A(U)$. The dimensions $\langle \mathrm{Ind}_A(U), \mathrm{Ind}_A(V)\rangle_A$ are found to be

(4.31)

	$\mathbf{1}$	σ	ϵ
$\mathbf{1}$	1	0	1
σ	0	2	0
ϵ	1	0	1

Thus $\mathrm{Ind}_A(\mathbf{1}) \cong \mathrm{Ind}_A(\epsilon)$ and $\mathrm{Ind}_A(\sigma) \cong F^+ \oplus F^-$ with two simple modules $F^\pm$. Since as an object we have $\mathrm{Ind}_A(\sigma) \cong \sigma \oplus \sigma$, this means that the simple object σ can be turned into an A-module in two distinct ways. Let us abbreviate $\rho_a := \rho_{a,\sigma}^{F^\pm\,\sigma}$, where the latter notation is as introduced in (I:4.61). The representation property (I:4.62) then reads

$$\rho_{\mathbf{1}} = 1 \qquad \text{and} \qquad \rho_a\, \rho_b\, \mathsf{F}^{(a\,b\,\sigma)\,\sigma}_{\sigma\,c} = \delta_{c\prec A}\, m_{ab}^{c}\, \rho_c\,, \tag{4.32}$$

where $\delta_{c\prec A}=1$ if U_c is a subobject of A and $\delta_{c\prec A}=0$ else. The only non-trivial condition arises from $a=b=\epsilon$. Using the explicit values of F it reads $(\rho_\epsilon)^2=\frac{1}{2}$. Altogether we get the following result for the simple A-modules:

$$
\begin{array}{ccl}
A\text{-module} & \text{as object} & \text{representation morphisms} \\
\hline
A & \mathbf{1}\oplus\epsilon & {\rho^{A}_{a,b}}^{c}=m^c_{ab} \\
F^+ & \sigma & {\rho^{F^+}_{\mathbf{1},\sigma}}^{\sigma}=1\,,\quad {\rho^{F^+}_{\epsilon,\sigma}}^{\sigma}=\frac{1}{\sqrt{2}} \\
F^- & \sigma & {\rho^{F^-}_{\mathbf{1},\sigma}}^{\sigma}=1\,,\quad {\rho^{F^-}_{\epsilon,\sigma}}^{\sigma}=-\frac{1}{\sqrt{2}}
\end{array}
\tag{4.33}
$$

The effect of conjugation $M\mapsto M^\sigma$ on the simple modules is encoded in the boundary conjugation matrix. Evaluating (3.31) yields

$$
C^\sigma=\begin{pmatrix} C^\sigma_{F^+F^+} & C^\sigma_{F^+A} & C^\sigma_{F^+F^-} \\ C^\sigma_{AF^+} & C^\sigma_{AA} & C^\sigma_{AF^-} \\ C^\sigma_{F^-F^+} & C^\sigma_{F^-A} & C^\sigma_{F^-F^-} \end{pmatrix} = \begin{pmatrix} \delta_{s_\epsilon,1} & 0 & \delta_{s_\epsilon,-1} \\ 0 & 1 & 0 \\ \delta_{s_\epsilon,-1} & 0 & \delta_{s_\epsilon,1} \end{pmatrix}. \tag{4.34}
$$

Thus the module A is fixed under conjugation for both reversions, as already shown in general in proposition 2.7(ii), while for $s_\epsilon=-1$ the conjugation exchanges the modules F^+ and F^-.

To obtain the A-A-bimodules, in general one would have to decompose the induced $A\otimes A_{\rm op}$-left modules. In the present case it turns out to be sufficient to look at the α-induced bimodules. Let us recall the formulas (I:5.64), (I:5.65) as well as proposition 2.36 of [83]:

$$
\begin{aligned}
&\dim_{\mathbb{C}}\mathrm{Hom}_{A|A}(\alpha^-(U_k),\alpha^+(U_l)) = Z_{\bar l,k}\,, \\
&\dim_{\mathbb{C}}\mathrm{Hom}_{A|A}(\alpha^+(U_k),\alpha^+(U_l)) = \dim_{\mathbb{C}}\mathrm{Hom}(C_l(A)\otimes U_k,U_l)\,, \\
&\dim_{\mathbb{C}}\mathrm{Hom}_{A|A}(\alpha^-(U_k),\alpha^-(U_l)) = \dim_{\mathbb{C}}\mathrm{Hom}(C_r(A)\otimes U_k,U_l)\,.
\end{aligned}
\tag{4.35}
$$

The left and right centers $C_{l/r}(A)$ of A are both equal to $\mathbf{1}$. Evaluating the dimensions (4.35) for the six α-induced bimodules $\alpha^\pm(U_k)$ obtained from $U_k=\mathbf{1},\sigma,\epsilon$ gives three isomorphism classes of simple bimodules,

$$
\alpha^+(\mathbf{1})\cong\alpha^-(\mathbf{1})\,,\qquad \alpha^+(\sigma)\cong\alpha^-(\sigma)\,,\qquad \alpha^+(\epsilon)\cong\alpha^-(\epsilon)\,. \tag{4.36}
$$

The counting argument (4.30) shows that in this way we have found a representative for each isomorphism class of simple A-A-bimodules.

Recall from section 3.8 that A-A-bimodules label defect lines, and that there is a subclass of defects which can be wrapped around a non-orientable cycle without marking a point. Such defects are labelled by bimodules X with $X^s\cong X$. Combining (4.36) with proposition 2.11 we see that all defects in the Ising model are of this type.

Finally we also need bases of local morphisms $\mu^{kl}_\alpha\in\mathrm{Hom}_{\rm loc}(A\otimes U_l,U_{\bar k})$, see (3.101). The dimensions of these spaces are $Z_{kl}=\delta_{k,l}$. Now the morphism spaces $\mathrm{Hom}(A\otimes\mathbf{1},\mathbf{1})$ and $\mathrm{Hom}(A\otimes\epsilon,\epsilon)$ are already one-dimensional, so that we can choose $\mu^{\mathbf{11}}$ and $\mu^{\epsilon\epsilon}$ to be any nonzero element in the respective spaces. For the two-dimensional space $\mathrm{Hom}(A\otimes\sigma,\sigma)$ the situation is more complicated since only a one-dimensional subspace is local. To find this subspace one must determine the image of the projector P_σ as introduced in (3.75).

Expanding μ_α^{kl} and its dual as in (3.104), altogether we find the local morphisms to be given by

$$\begin{aligned} &\mu_{\mathbf{1}}^{\mathbf{11}} = \sqrt{2}\,, && \mu_{\mathbf{1}}^{\sigma\sigma} = 0\,, && \mu_{\mathbf{1}}^{\epsilon\epsilon} = -\sqrt{2}\,, \\ &\mu_{\epsilon}^{\mathbf{11}} = 0\,, && \mu_{\epsilon}^{\sigma\sigma} = \mathrm{e}^{\pi\mathrm{i}/4}\,, && \mu_{\epsilon}^{\epsilon\epsilon} = 0\,. \end{aligned} \tag{4.37}$$

The nonzero constants are normalisations and can be chosen at will. We have picked them in such a way that the S^A-matrix computed below coincides with the S-matrix. Among the vanishing constants, the only non-trivial case is $\mu_{\mathbf{1}}^{\sigma\sigma} = 0$. The corresponding dual local morphisms are

$$\begin{aligned} &\bar\mu_{\mathbf{1}}^{\mathbf{11}} = \tfrac{1}{\sqrt{2}}\,, && \bar\mu_{\mathbf{1}}^{\sigma\sigma} = 0\,, && \bar\mu_{\mathbf{1}}^{\epsilon\epsilon} = -\tfrac{1}{\sqrt{2}}\,, \\ &\bar\mu_{\epsilon}^{\mathbf{11}} = 0\,, && \bar\mu_{\epsilon}^{\sigma\sigma} = \mathrm{e}^{-\pi\mathrm{i}/4}\,, && \bar\mu_{\epsilon}^{\epsilon\epsilon} = 0\,. \end{aligned} \tag{4.38}$$

Möbius and Klein bottle amplitudes for A

The coefficients of the Möbius and Klein bottle amplitude can be obtained with the crossed channel relations (3.119) and (3.120). To this end we must compute the quantities $S^A_{R,k}$, γ_k^σ and $g^{\bar kk}$.

Let us start with $S^A_{R,k}$ as given in (3.106) and (3.108). From the chiral data for $\mathcal{C}_{\mathsf{Is}}$ one computes

$$\begin{aligned} &I_S(\mathbf{1},\mathbf{1},\mathbf{1}) = I_S(\mathbf{1},\mathbf{1},\epsilon) = I_S(\epsilon,\mathbf{1},\epsilon) = I_S(\epsilon,\mathbf{1},\mathbf{1}) = \tfrac{1}{2}\,, \\ &I_S(\mathbf{1},\mathbf{1},\sigma) = \tfrac{1}{\sqrt{2}}\,, \qquad I_S(\epsilon,\mathbf{1},\sigma) = -\tfrac{1}{\sqrt{2}}\,, \qquad I_S(\sigma,\epsilon,\sigma) = 2\,\mathrm{e}^{-\pi\mathrm{i}/4}\,, \end{aligned} \tag{4.39}$$

which together with the data derived from the algebra A results in the matrix

$$S^A \equiv \begin{pmatrix} S^A_{F+,\mathbf{1}} & S^A_{F+,\sigma} & S^A_{F+,\epsilon} \\ S^A_{A,\mathbf{1}} & S^A_{A,\sigma} & S^A_{A,\epsilon} \\ S^A_{F-,\mathbf{1}} & S^A_{F-,\sigma} & S^A_{F-,\epsilon} \end{pmatrix} = \begin{pmatrix} \frac12 & \frac{1}{\sqrt2} & \frac12 \\ \frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2} \\ \frac12 & -\frac{1}{\sqrt2} & \frac12 \end{pmatrix}. \tag{4.40}$$

With normalisations as chosen in (4.37), S^A thus coincides with the ordinary S-matrix of $\mathcal{C}_{\mathsf{Is}}$. While generically such a choice is not possible, in the present situation such a normalisation is guaranteed to exist because A is Morita equivalent to $\mathbf{1}$.

Next we turn to the crosscap coefficients, for which we need the invariants (3.112). Inserting the chiral data gives

$$I_\Gamma(\mathbf{1},\mathbf{1},\mathbf{1}) = I_\Gamma(\epsilon,\epsilon,\mathbf{1}) = \cos\tfrac{\pi}{8}\,, \qquad I_\Gamma(\mathbf{1},\epsilon,\mathbf{1}) = I_\Gamma(\epsilon,\mathbf{1},\mathbf{1}) = -\mathrm{i}\sin\tfrac{\pi}{8}\,. \tag{4.41}$$

Combining this with (3.110) and (3.103) and noting the trigonometric identities $\cos\frac{\pi}{8} + \sin\frac{\pi}{8} = \sqrt{2}\cos\frac{\pi}{8}$ and $\cos\frac{\pi}{8} - \sin\frac{\pi}{8} = \sqrt{2}\sin\frac{\pi}{8}$ results in

$$\gamma_{\mathbf{1}}^\sigma = \begin{cases} \cos\frac{\pi}{8} & \text{for } s_\epsilon = 1\,, \\ \sin\frac{\pi}{8} & \text{for } s_\epsilon = -1\,, \end{cases} \qquad \gamma_\sigma^\sigma = 0\,, \qquad \gamma_\epsilon^\sigma = \begin{cases} \sin\frac{\pi}{8} & \text{for } s_\epsilon = 1\,, \\ -\cos\frac{\pi}{8} & \text{for } s_\epsilon = -1\,. \end{cases} \tag{4.42}$$

Finally the numbers $g^{\bar kk}$ are obtained from (3.118); we get

$$g^{\mathbf{11}} = 1\,, \qquad g^{\sigma\sigma} = \tfrac{s_\epsilon}{\sqrt{2}}\,, \qquad g^{\epsilon\epsilon} = 1\,. \tag{4.43}$$

We have now gathered all ingredients to evaluate the formulas (3.119) and (3.120). A short calculation yields

$$\begin{aligned} &\mathrm{m\ddot{o}}_{\mathbf{1}A} = 1\,, \qquad &&\mathrm{m\ddot{o}}_{\sigma A} = 0\,, \qquad &&\mathrm{m\ddot{o}}_{\epsilon A} = s_\epsilon\,, \\ &\mathrm{m\ddot{o}}_{\mathbf{1}F^\pm} = \delta_{s_\epsilon,1}\,, \qquad &&\mathrm{m\ddot{o}}_{\sigma F^\pm} = 0\,, \qquad &&\mathrm{m\ddot{o}}_{\epsilon F^\pm} = \delta_{s_\epsilon,-1} \end{aligned} \tag{4.44}$$

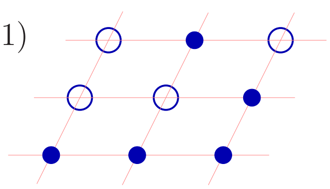
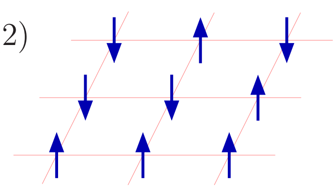
for the coefficients of the Möbius amplitude and

$$\mathrm{K}_{\mathbf{1}} = 1\,, \qquad \mathrm{K}_\sigma = s_\epsilon\,, \qquad \mathrm{K}_\epsilon = 1 \tag{4.45}$$

for the Klein bottle [19].

Lattice interpretation

The appearance of two different reversions for the Ising model can be given an intuitive lattice interpretation. Consider the Ising model realised on a 'square lattice', i.e. on a quadrangulation of the world sheet, with a 'spin' variable, which can take one of two possible values, placed at each lattice site. We would like to think of the situation in terms of two different geometric realisations of this spin variable. In the first the two values are given by a small and a large circle, respectively, while in the second they describe the direction of a spin vector orthogonal to the lattice plane:

(1)  (2)  (4.46)

This visualisation suggests two different rules for transporting the spin variable around a non-orientable cycle (compare section 3.8): if we think of the spin variable as in case (1) of (4.46), then we obtain back the same value, whereas in case (2) the two values get exchanged. As long as we are only interested in orientable surfaces, the two geometric realisations are indistiguishable, but as we will see below, if we include lattices on non-orientable surfaces, alternative (1) is described by $s_\epsilon = 1$ while (2) corresponds to $s_\epsilon = -1$.

Let us start by considering the possible boundary conditions. In the case of description (1) we have 'fixed small (s)', 'fixed large (l)' and 'free (f)' boundary conditions, while for description (2) one can take 'fixed up (u)', 'fixed down (d)' and 'free (f)'. On the Möbius strip one must choose a single boundary condition R. Equation (3.19) tells us that a segment of the Möbius strip looks like a strip with boundary conditions R and R^σ. The effect of boundary conjugation can thus be read off by 'transporting' the boundary condition half-way along the boundary of the Möbius strip. In the description (1) this has no effect, i.e.

$$s \mapsto s\,, \qquad l \mapsto l\,, \qquad f \mapsto f\,, \tag{4.47}$$

while in (2) the direction of the spin in the fixed boundary conditions is inverted,

$$u \mapsto d\,, \qquad d \mapsto u\,, \qquad f \mapsto f\,. \tag{4.48}$$

Identifying u and s with F^+, d and l with F^-, and f with A, this is precisely the behaviour encoded in formula (4.34).

Next consider the Klein bottle amplitude, with $\tau = 2\mathrm{i}L/R$,

$$\mathrm{K}(R,L) = \mathrm{K}_{\mathbf{1}}\, \chi_{\mathbf{1}}(\tau) + \mathrm{K}_\sigma\, \chi_\sigma(\tau) + \mathrm{K}_\epsilon\, \chi_\epsilon(\tau)\,. \tag{4.49}$$

This gives the partition function of the critical Ising model on a rectangular lattice of width R and length L for which the two vertical boundaries are periodically identified, while for the two horizontal boundaries antiperiodic boundary conditions are chosen. Furthermore the antiperiodic identification exchanges ↑ with ↓ in realisation (2). Configurations of low energy in (1) tend to have all spins aligned, while (2) effectively has an additional line of frustration at which the interaction between neighbouring spins is reversed. Thus configurations with long range order will tend to have a higher energy for (2) than for (1). For example, the state on the rectangular lattice with all spin values equal has zero energy for (1), but has nonzero energy for (2), originating from the interaction across the horizontal boundary. From the lattice point of view we therefore expect that

$$\mathrm{K}^{(1)} > \mathrm{K}^{(2)}\,. \tag{4.50}$$

According to (4.45), this is indeed precisely what happens, since $\mathrm{K}^{(1)} - \mathrm{K}^{(2)} = 2\chi_\sigma(\tau) > 0$.

Similar considerations can be repeated for the partition function $\mathrm{M\ddot{o}}(R,L)$ on a Möbius strip of width R and circumference L with free boundary condition. Again the realisation (2) has effectively an additional line of frustration, so that we expect $\mathrm{M\ddot{o}}^{(1)} > \mathrm{M\ddot{o}}^{(2)}$. The free boundary condition is labelled by A, and the partition function reads

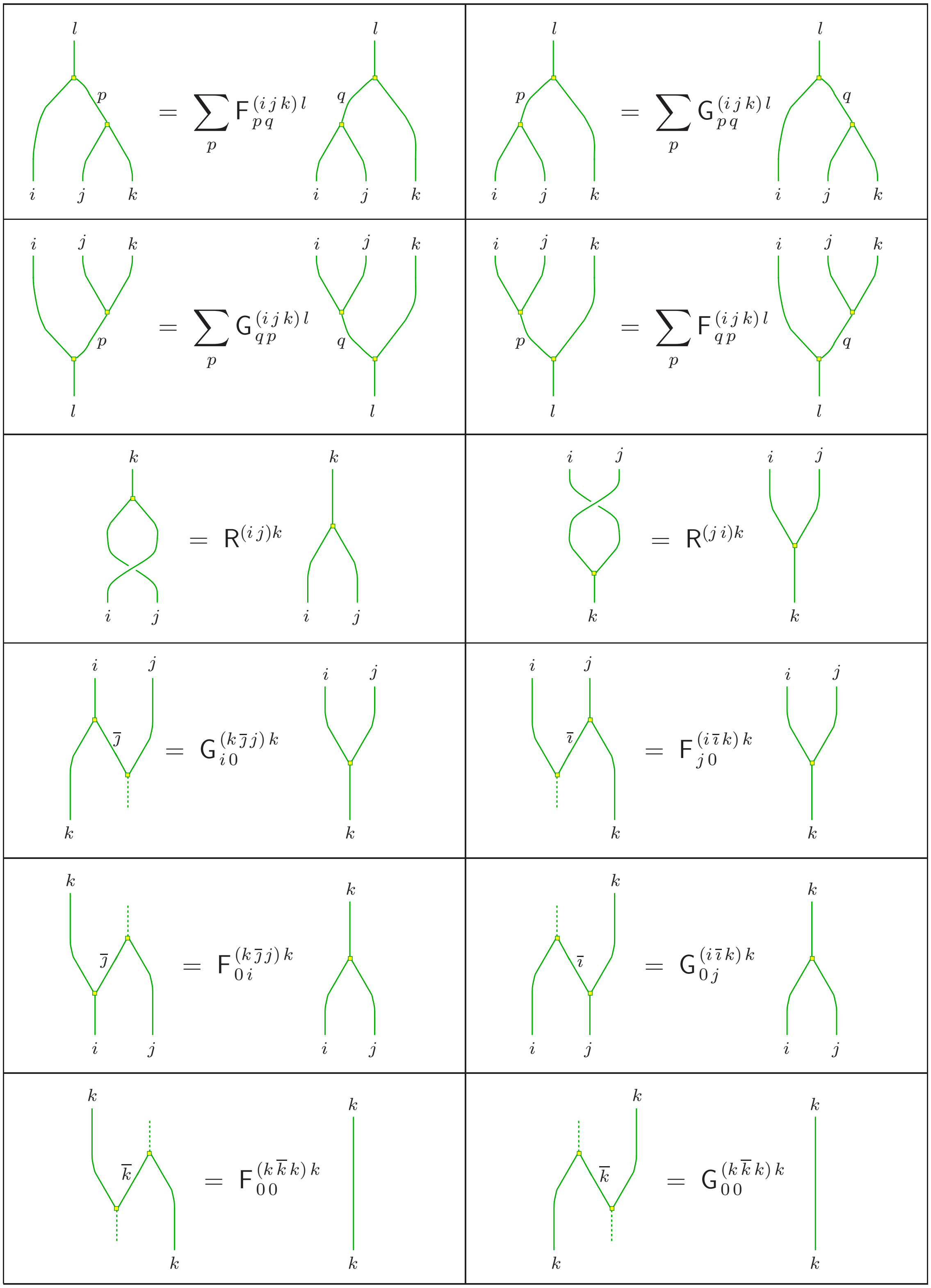
$$\mathrm{M\ddot{o}}(R,L) = \mathrm{m\ddot{o}}_{A\mathbf{1}}\, \hat\chi_{\mathbf{1}}(\tau) + \mathrm{m\ddot{o}}_{A\sigma}\, \hat\chi_\sigma(\tau) + \mathrm{m\ddot{o}}_{A\epsilon}\, \hat\chi_\epsilon(\tau)\,, \tag{4.51}$$

where $\tau = \frac{1}{2}(1+\mathrm{i}L/R)$ and $\hat\chi(\tau) = \mathrm{e}^{-\pi\mathrm{i}(\Delta - c/24)}\chi(\tau)$. Substituting the coefficients (4.44) then indeed results in $M^{(1)} - M^{(2)} = 2\hat\chi_\epsilon(\tau) > 0$.

A Appendix

A.1 Fusing and braiding moves

For convenience we list here a collection of useful fusing and braiding moves.



These moves are displayed for the case that $N_{ij}^{\;k} \in \{0,1\}$. Otherwise the three-point vertices are labelled by basis elements of the respective morphism spaces, and analogous labels appear in the coefficients.

A.2 Proof of the algorithm for finding reversions

In this appendix we present a detailed proof of the algorithm described in section 2.5. First we recall from section I:3.4 that the vector space $A_{\rm top} = {\rm Hom}(\mathbf{1}, A)$ is a $\mathbb{C}$-algebra, with product

$$m_{\rm top}(a,b) := m \circ (a \otimes b) \tag{A.1}$$

for $a, b \in A_{\rm top}$, where m is the multiplication morphism on A. The reversion σ induces an involutive anti-automorphism on this $\mathbb{C}$-algebra: writing $\sigma a := \sigma \circ a$ for $a \in A_{\rm top}$, we have

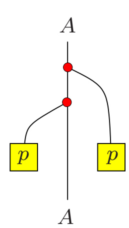
Lemma A.1:
For σ a reversion on a symmetric special Frobenius algebra A, we have $\sigma^2 a = a$ for all $a \in A_{\rm top}$, and

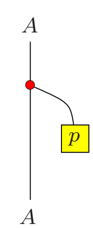
$$m_{\rm top}(\sigma a, \sigma b) = \sigma\, m_{\rm top}(b,a) \tag{A.2}$$

for all $a, b \in A_{\rm top}$.
In particular, for p an idempotent in $A_{\rm top}$, i.e. $m_{\rm top}(p,p) = p$, also σp is an idempotent, for e_i a primitive idempotent also $\sigma \mathrm{e}_i$ is a primitive idempotent, and for $a \in A_{\rm top}$ invertible, also σa is invertible.

Proof:
We have $\sigma \circ \sigma \circ a = \theta_A \circ a$ by (2.11), and this equals a owing to functoriality of the twist. The equality (A.2) follows by combining the property (2.12) of σ with the functoriality of the braiding.
For $p \in A_{\rm top}$ an idempotent, (A.2) implies $m_{\rm top}(\sigma p, \sigma p) = \sigma \circ m_{\rm top}(p,p) = \sigma \circ p$, i.e. σp is again an idempotent. Similarly, using the same arguments as after (4.5), one checks that $\{\sigma \mathrm{e}_i\}$ satisfies the defining properties of the set of primitive idempotents. Finally, for $m_{\rm top}(a,b) = 1_{A_{\rm top}} \equiv \eta$, also $m_{\rm top}(\sigma a, \sigma b) = \sigma\, m_{\rm top}(b,a) = \sigma \eta = \eta$. ✓

Now for any idempotent $p \in A_{\rm top}$ we can define two idempotents $P_B,\ P_M \in {\rm Hom}(A,A)$ as

P_B := 

and P_M := 

(A.3)

The image of the idempotent P_B can again be made into an algebra:

Lemma A.2:
For A a simple symmetric special Frobenius algebra, let $p \in A_{\rm top}$ be a nonzero idempotent and P_M, P_B as in (A.3). Denote by (M, e_M, r_M) the image of P_M and by (B, e_B, r_B) the image of P_B. Then

(i) $(B, m_B, \eta_B, \Delta_B, \varepsilon_B)$ with

$$m_B := r_B \circ m_A \circ (e_B \otimes e_B)\,, \qquad \eta_B := r_B \circ \eta_A\,,$$
$$\Delta_B := \frac{\dim(A)}{\dim(\dot{M})}\,(r_B \otimes r_B) \circ \Delta \circ e_B\,, \qquad \varepsilon_B := \frac{\dim(\dot{M})}{\dim(A)}\,\varepsilon_A \circ e_B \tag{A.4}$$

is a simple symmetric special Frobenius algebra.
(ii) The topological algebra $B_{\rm top}$ associated to B has dimension

$$\dim_{\mathbb{C}} B_{\rm top} \le \dim_{\mathbb{C}} A_{\rm top}\,, \tag{A.5}$$

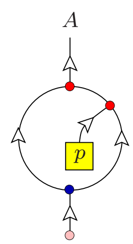
with equality if and only if p is the unit η_A of the algebra A.
(iii) M is a left A-module with the property that $B \cong M^\vee \otimes_A M$ as symmetric special Frobenius algebras. Thus B is Morita equivalent to A.

Proof:
(i) The associativity, coassociativity, unit and counit properties, as well as symmetry and the Frobenius property of B, all reduce to the corresponding properties of A by a short calculation. In these manipulations it is helpful to use the identities

$$m \circ (id_A \otimes p) \circ e_B = e_B = m \circ (p \otimes id_A) \circ e_B \tag{A.6}$$

and analogous ones for r_B, which follow directly from (2.44) and the fact that p is an idempotent. The proof of specialness of B works along the same lines as in proposition 2.13. Define the element $\phi \in A_{\rm top}$ by

ϕ :=



(A.7)

One checks that $\phi \in {\rm cent}_A(A_{\rm top})$. Since A is by assumption simple, ${\rm cent}_A(A_{\rm top})$ is one-dimensional, and hence $\phi = \xi \eta_A$ for some complex number ξ, which can be determined by composing with ε_A; one finds

$$\phi = \frac{\dim(\dot{M})}{\dim(A)}\,\eta_A\,. \tag{A.8}$$

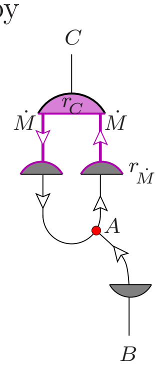
Using this identity one can verify that indeed $m_B \circ \Delta_B = id_B$.
Simplicity of B will follow from (iii) below, in combination with proposition 2.13.

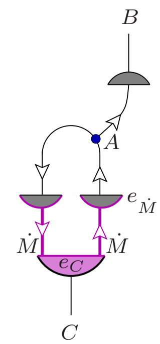
(ii) Define the linear maps φ: $B_{\rm top} \to A_{\rm top}$ and ψ: $A_{\rm top} \to B_{\rm top}$ as $\varphi(x) := e_B \circ x$ and $\psi(y) := r_B \circ y$. By the identities (2.43) we have $\psi \circ \varphi = id_{B_{\rm top}}$; thus in particular φ is injective. Next note that $\psi(\eta_A - p) = \psi(P_B \circ \eta_A) - \psi(p) = 0$, which is seen with the help of (2.44) and $P_B \circ \eta_A = p$. Now suppose there is an $x \in B_{\rm top}$ with $\varphi(x) = \eta_A - p$. Applying ψ to this equality yields $x = 0$. But $\varphi(0) = 0$, so that an x with the assumed property can exist only if $p = \eta_A$. Thus for $p \neq \eta_A$ the dimension of the image of φ is strictly smaller than the dimension of $A_{\rm top}$. Since φ is injective this implies $\dim_{\mathbb{C}} B_{\rm top} < \dim_{\mathbb{C}} A_{\rm top}$.

(iii) It is straightforward to check that

$$\rho = r_M \circ m \circ (id_A \otimes e_M) \tag{A.9}$$

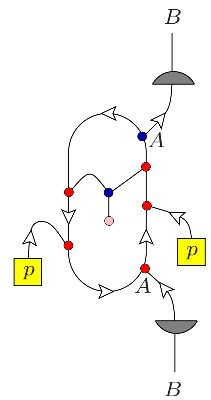
(i.e. the action of A on M by multiplication) defines a representation of A. Let $P_{\otimes A}$ be the idempotent in (2.45) and (C, e_C, r_C) be its image. By proposition 2.47, C with morphisms (2.47) is a simple symmetric special Frobenius algebra. Consider now $\varphi \in \mathrm{Hom}(B, C)$ and $\psi \in \mathrm{Hom}(C, B)$ given by

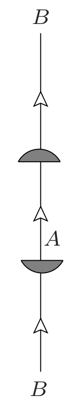
φ := 

and ψ := 

(A.10)

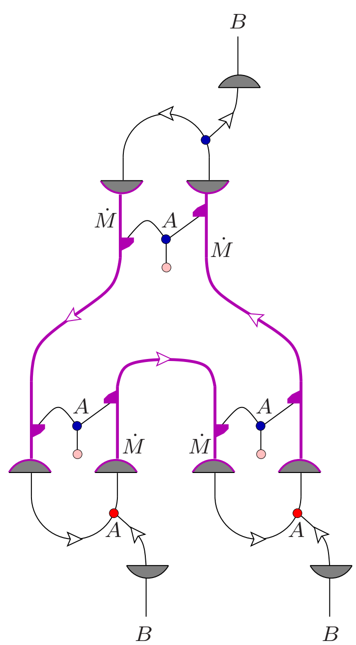
One checks that $\psi \circ \varphi = id_B$ and $\varphi \circ \psi = id_C$. For example,

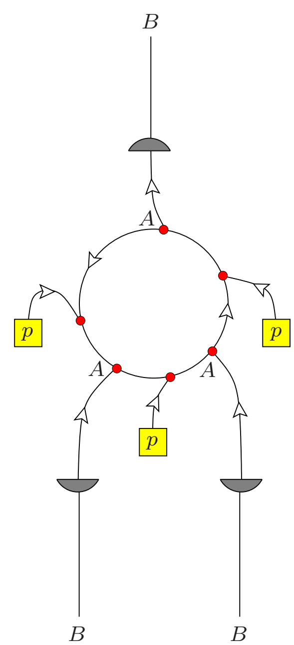
$\psi \circ \varphi$ = 

= 

= id_B . (A.11)

Here we use that by definition of the action (A.9) of A on M we have $e_M \in \mathrm{Hom}_A(M, A)$ and $r_M \in \mathrm{Hom}_A(A, M)$; also, the A-ribbons from the idempotents p are moved so as to merge directly with the A-ribbon from e_M, so that they can be removed by use of (A.6).
The morphisms φ and ψ are actually algebra isomorphisms between B and C; for instance, for the multiplication we find

$\psi \circ m_C \circ (\varphi \otimes \varphi)$ = 

= 

= m_B .

(A.12)

This establishes that indeed $B \cong M^\vee \otimes_A M$ as Frobenius algebras. ✓

It follows that, given a simple symmetric special Frobenius algebra A as in lemma A.2, we can find a Morita equivalent algebra with a smaller associated topological algebra, provided that $A_{\rm top}$ contains a non-trivial idempotent p. The following lemma tells us that the structure of $A_{\rm top}$ cannot be complicated – it can be written as a direct sum of matrix algebras.

Lemma A.3:
Let A be a special Frobenius algebra and let $\{M_\kappa \,|\, \kappa \in \mathcal{J}\}$ be a set of representatives for the simple left A-modules. Let A, regarded as a left module over itself, decompose as $\bigoplus_{\kappa\in\mathcal{J}} (M_\kappa)^{\oplus n_\kappa}$ into simple A-modules. Then

$$A_{\rm top} \;\cong\; \bigoplus_{\kappa\in\mathcal{J}} {\rm Mat}_{n_\kappa}(\mathbb{C}) \,. \tag{A.13}$$

Proof:
Define the map φ: $A_{\rm top} \to {\rm Hom}_A(A,A)$ by $\varphi(x) := m \circ (id_A \otimes x)$. The space ${\rm Hom}_A(A,A)$ is an algebra, with multiplication given by the composition. One checks that φ is an algebra anti-homomorphism

$$\varphi(xy) = \varphi(y)\,\varphi(x) \,. \tag{A.14}$$

Furthermore, $\varphi(x) \circ \eta_A = x$, implying that φ is injective. Since by reciprocity (see e.g. proposition I:4.12) we have $\dim_{\mathbb{C}} {\rm Hom}(\mathbf{1},A) = \dim_{\mathbb{C}} {\rm Hom}_A(A,A)$, it follows that φ is an isomorphism. The algebra structure of ${\rm Hom}_A(A,A)$ now follows from the identities

$$\begin{aligned} {\rm Hom}_A(A,A) \;&\cong\; \bigoplus_{\kappa,\kappa'\in\mathcal{J}} {\rm Hom}_A(M_\kappa^{\oplus n_\kappa}, M_{\kappa'}^{\oplus n_{\kappa'}}) \\ &\cong \bigoplus_{\kappa\in\mathcal{J}} {\rm Hom}_A(M_\kappa^{\oplus n_\kappa}, M_\kappa^{\oplus n_\kappa}) \cong \bigoplus_{\kappa\in\mathcal{J}} {\rm Mat}_{n_\kappa}(\mathbb{C}) \,. \end{aligned} \tag{A.15}$$

Here we substituted the assumption that A decomposes as $\bigoplus_{\kappa\in\mathcal{J}} M_\kappa^{\oplus n_\kappa}$ as an A-module. Because of ${\rm Hom}_A(M_\kappa, M_\kappa) \cong \mathbb{C}$, in the last step we obtain a direct sum of matrix algebras. In combination with φ this provides us with an algebra anti-isomorphism f from $A_{\rm top}$ to a direct sum of matrix algebras. Thus the transpose $f^{\rm t}$ is an algebra isomorphism, establishing the lemma. ✓

The only direct sum of matrix algebras that does not contain an idempotent $p \neq 0$ different from the identity is the algebra $\mathbb{C}$ itself. Thus starting from a given algebra B we can always pass to a Morita equivalent algebra A with smaller associated topological algebra, $\dim_{\mathbb{C}} A_{\rm top} < \dim_{\mathbb{C}} B_{\rm top}$, as long as $\dim_{\mathbb{C}} B_{\rm top} > 1$. Together the lemmata A.2 and A.3 amount to (see also section 3.3 of [84] and remark I:3.5(iii))

Proposition A.4:
Every simple symmetric special Frobenius algebra B is isomorphic to $M^\vee \otimes_A M$, where A is a haploid simple symmetric special Frobenius algebra, M is a left A-module, and the algebra structure on $M^\vee \otimes_A M$ is as given in (2.47).
A is haploid iff it is a simple module over itself.

So far we have learned that the Morita class of each simple symmetric special Frobenius algebra A contains at least one haploid representative. An analogous statement does, however, not hold for algebras (A, σ_A) with reversion. But proposition 2.16 provides a sufficient condition for whether σ_A can be transported to another representative in the Morita class. This can be translated into a condition on the idempotents $p \in A_{\rm top}$:

Lemma A.5:
Let A be a simple Jandl algebra, with reversion σ_A, and let $p \in A_{\rm top}$ be an idempotent such that the idempotent $\sigma_A p$ is equal to p. Let M and $B \cong M^\vee \otimes_A M$ be as in lemma A.2. Then

(i) The morphism $\sigma_B := r_B \circ \sigma_A \circ e_B$ defines a reversion on B.

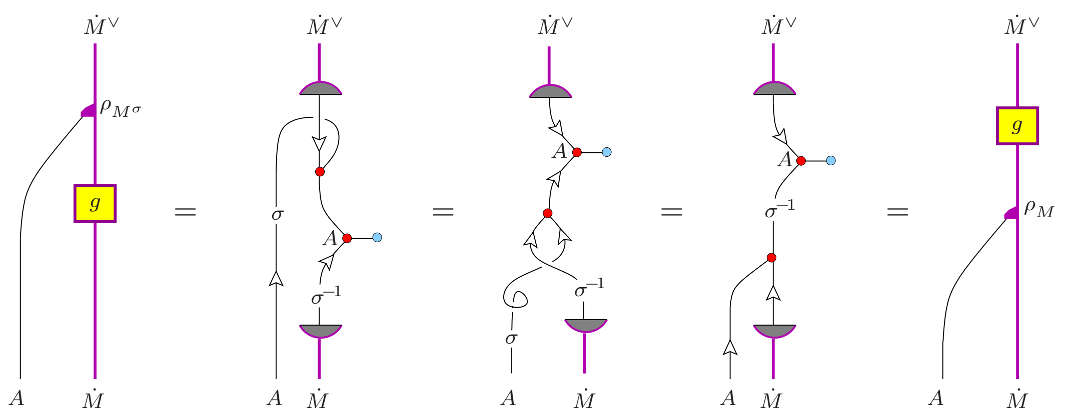
(ii) There exists a $g \in \mathrm{Hom}_A(M, M^\sigma)$ obeying (2.59) with $\varepsilon_g = 1$, such that σ_B is of the form (2.60).

Proof:
(i) By lemma A.2 we can describe B equivalently as $M^\vee \otimes_A M$ or as the image of P_B with morphisms (A.4). Note that $P_B \circ \sigma_A = \sigma_A \circ P_B$, as can be seen from properties (2.4) and $\sigma_A p = p$. Using that P_B and σ_A commute one checks that $\sigma_B \circ \sigma_B = \theta_B$ and $\sigma_B \circ m = m \circ c_{B,B} \circ (\sigma_B \otimes \sigma_B)$. Thus by proposition 2.4 σ_B is a reversion on B.

(ii) Consider the morphism $g \in \mathrm{Hom}(\dot M, \dot M^\vee)$ given by

$$g := (e_M)^\vee \circ \Phi_2 \circ \sigma_A^{-1} \circ e_M \,, \tag{A.16}$$

where (M, e_M, r_M) is the image of P_M as in lemma (A.2) and Φ_2 is taken from (I:3.33). The moves

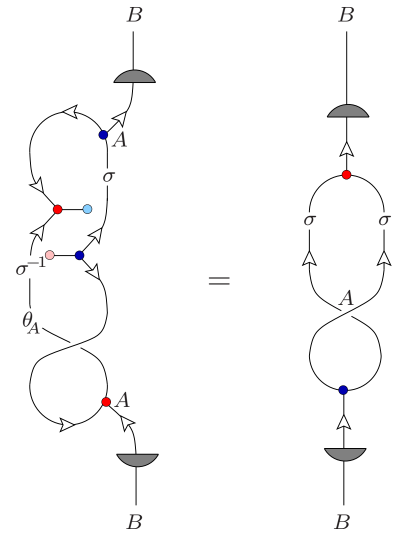


(A.17)

show that $g \in \mathrm{Hom}_A(M, M^{\sigma_A})$. Moreover, g has the properties required in proposition (2.16). Indeed, (2.59) holds with $\varepsilon_g = 1$, as can be seen by using (2.17) and $\Phi_1 = \Phi_2$ (i.e. A symmetric, see definition I:3.4).
Next we must check that σ_B equals $\tilde\sigma_g$ as defined in (2.60). To this end we use the explicit

isomorphisms (A.10) between B and $M^\vee \otimes_A M$; we have

$\psi \circ \tilde{\sigma}_g \circ \varphi =$



$= \sigma_B \, .$ (A.18)

In the first step one inserts the definitions (2.60), (A.10), (A.16) and gets rid of all idempotents using (2.61) and (A.6). The last step uses (2.5) and specialness of A, as well as the definition of σ_B in part (i) of the present lemma. ✓

For the next theorem we need more detailed information about the action of σ on the associated topological algebra A_{top}. Let us summarise the results of section 4.1 below, where the case $\mathcal{C} = \mathcal{V}ect_{\mathbb{C}}$ is treated in more detail.

Consider a direct sum $A = \bigoplus_{i=1}^{m} \mathrm{Mat}_{n_i}(\mathbb{C})$ of matrix algebras over $\mathbb{C}$. All anti-isomorphisms σ squaring to the identity on A are of the form

$$\sigma^{\pi}_{U_1,\dots,U_m}(X_1, \dots , X_m) = (U_1 X^{\mathrm{t}}_{\pi(1)} U_1^{-1}, \dots , U_m X^{\mathrm{t}}_{\pi(m)} U_m^{-1}) \tag{A.19}$$

for $X_i \in \mathrm{Mat}_{n_i}(\mathbb{C})$, where π is a permutation of order two with $n_{\pi(i)} = n_i$ and the $U_i \in \mathrm{Mat}_{n_i}(\mathbb{C})$ are invertible and satisfy $U_{\pi(i)} = \lambda_i U_i^{\mathrm{t}}$ for some $\lambda_i \in \mathbb{C}$. The linear maps ω: $A \to A$ given by

$$\omega_{V_1,\dots,V_m}(X_1, \dots , X_m) = (V_1 X_1 V_1^{-1}, \dots , V_m X_m V_m^{-1}) \tag{A.20}$$

are algebra automorphisms of A. Given any σ on A we can find an ω of the form (A.20) such that

$$\omega \circ \sigma \circ \omega^{-1} = \sigma^{\pi}_{D_1,\dots,D_m} \, , \tag{A.21}$$

where π is again a permutation of order two and the $D_i \in \mathrm{Mat}_{n_i}(\mathbb{C})$ are the identity matrix whenever $\pi(i) \neq i$. For $\pi(i) = i$ the matrix D_i is either the identity matrix or the anti-symmetric matrix with blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ along the diagonal.

Let us return to the case of a general modular tensor category $\mathcal{C}$. Below it will be instrumental to deal with an algebra automorphism of A whose restriction to A_{top} is of the form (A.20). Such an automorphism can be constructed as follows.

For A a symmetric special Frobenius algebra and M a left A-module, let $M \cong \bigoplus_{\kappa \in \mathcal{J}} M_\kappa^{\oplus n_\kappa}$ be its decomposition into the direct sum of simple A-modules, as in lemma A.3. This means that there are morphisms

$$x_{\kappa\alpha} \in \mathrm{Hom}_A(M_\kappa, M) \qquad \text{and} \qquad \bar{x}_{\kappa\alpha} \in \mathrm{Hom}_A(M, M_\kappa) \tag{A.22}$$

with $\kappa \in \mathcal{J}$ and $\alpha = 1, 2, \ldots, n_\kappa$ such that

$$\bar{x}_{\kappa\alpha} \circ x_{\kappa'\alpha} = \delta_{\kappa,\kappa'}\, \delta_{\alpha,\beta}\, id_{M_\kappa} \qquad \text{and} \qquad \sum_{\kappa\in\mathcal{J}} \sum_{\alpha=1}^{n_\kappa} x_{\kappa\alpha} \circ \bar{x}_{\kappa\alpha} = id_M \,. \tag{A.23}$$

A basis of $\mathrm{Hom}_A(M,M)$ is given by

$$e_\kappa^{\alpha\beta} := x_{\kappa\alpha} \circ \bar{x}_{\kappa\beta} \ \in \mathrm{Hom}_A(M,M) \tag{A.24}$$

with $\kappa \in \mathcal{J}$ and $\alpha, \beta = 1, 2, \ldots, n_\kappa$. The basis elements obey the composition rule

$$e_\kappa^{\alpha\beta} \circ e_{\kappa'}^{\gamma\delta} = \delta_{\kappa,\kappa'}\, \delta_{\beta\gamma}\, e_\kappa^{\alpha\delta} \,. \tag{A.25}$$

One checks that in this basis the invertible elements of $\mathrm{Hom}_A(M,M)$ are precisly those

$$f = \sum_{\kappa\in\mathcal{J}} \sum_{\alpha,\beta=1}^{n_\kappa} f^\kappa_{\alpha\beta}\, e_\kappa^{\alpha\beta} \tag{A.26}$$

for which the matrices $f^\kappa \in \mathrm{Mat}_{n_\kappa}(\mathbb{C})$ are invertible.

Invertible elements of $\mathrm{Hom}_A(M,M)$ can be used to construct algebra automorphisms of $M^\vee \otimes_A M$. The following result is a straightforward application of the definitions (2.45) and (2.47).

Lemma A.6:
For A a symmetric special Frobenius algebra and M a left A-module, denote by $(M^\vee \otimes_A M, e, r)$ the retract associated to the idempotent $P_{\otimes A}$, as in (2.45). If $f \in \mathrm{Hom}_A(M,M)$ is invertible, then

$$\omega_f := (f^{-1})^\vee \otimes_A f = r \circ ((f^{-1})^\vee \otimes f) \circ e \tag{A.27}$$

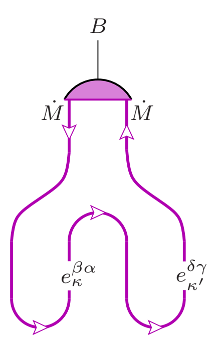
is an algebra automorphism of the symmetric Frobenius algebra $M^\vee \otimes_A M$.

Using the isomorphism $\mathrm{Hom}_A(M,M) \cong \mathrm{Hom}(\mathbf{1}, M^\vee \otimes_A M)$ one obtains a basis $\{b_\kappa^{\alpha\beta}\}$ of B_{top} from the basis (A.24) of $\mathrm{Hom}_A(M,M)$:

$$b_\kappa^{\alpha\beta} := r \circ (id_{M^\vee} \otimes e_\kappa^{\beta\alpha}) \circ \tilde{b}_M \ \in \mathrm{Hom}(\mathbf{1}, M^\vee \otimes_A M) \,. \tag{A.28}$$

Here the order of the multiplicity indices in $b_\kappa^{\alpha\beta}$ is chosen in such a way that the product of two such morphisms obeys

$$m_{\mathrm{top}}(b_\kappa^{\alpha\beta}, b_{\kappa'}^{\gamma\delta}) \ \equiv \ m \circ (b_\kappa^{\alpha\beta} \otimes b_{\kappa'}^{\gamma\delta}) \ = $$



$$= \ \delta_{\kappa,\kappa'}\, \delta_{\beta,\gamma}\, b_\kappa^{\alpha\delta} \,. \tag{A.29}$$

In fact, this basis provides us with an isomorphism φ of $\mathbb{C}$-algebras between $\bigoplus_{\kappa\in\mathcal{J}}\mathrm{Mat}_{n_\kappa}(\mathbb{C})$ and $B_{\rm top}$, given by

$$\varphi: \quad (D^1,\dots,D^{|\mathcal{J}|}) \longmapsto \sum_{\kappa\in\mathcal{J}} \sum_{\alpha,\beta=1}^{n_\kappa} D^\kappa_{\alpha\beta}\, b^{\alpha\beta}_\kappa \,. \tag{A.30}$$

Indeed it is easy to check that $\varphi(\vec{D}\vec{D}') = \varphi(\vec{D})\,\varphi(\vec{D}')$, where we use the shorthand notation $\vec{D} \equiv (D^1,\dots,D^{|\mathcal{J}|})$.

Next let us compute the action of ω_f from (A.27), with f as in (A.26), on $B_{\rm top}$. One finds

$$\omega_f\circ\varphi(\vec{D}) = \sum_{\kappa,\alpha,\beta} D^\kappa_{\alpha\beta}\, r\circ[id_{M^\vee}\otimes(f\circ e_\kappa^{\beta\alpha}\circ f^{-1})]\circ\tilde{b}_M = \varphi((\vec{f}^{-1})^{\rm t}\vec{D}\vec{f}^{\rm t}) \,. \tag{A.31}$$

Thus the action of ω_f on $B_{\rm top}$ is indeed of the form (A.20).

We have now gathered the necessary ingredients to establish the following result, which can be regarded as a generalisation [9] of the classification (A.21) from $\mathcal{V}ect_{\mathbb{C}}$ to a general modular tensor category $\mathcal{C}$.

Proposition A.7:
Let (C,σ_C) be a simple Jandl algebra in a modular tensor category $\mathcal{C}$. Then one can find A, M, B, σ_B, N and g with the following properties:

(a) $A\in\mathcal{O}bj(\mathcal{C})$ is a haploid simple symmetric special Frobenius algebra; it is Morita equivalent to C.

(b) M is a left A-module such that B is isomorphic to $M^\vee\otimes_A M$ as an algebra. M is either a simple A-module, or else it is the direct sum $M\cong M_1\oplus M_2$ of two simple A-modules $M_{1,2}$ obeying $M_1^\vee\otimes_A M_1\cong(M_2^\vee\otimes_A M_2)_{\rm op}$ as algebras. (This implies in particular that $M_1^\vee\otimes_A M_1\cong M_2^\vee\otimes_A M_2$ as objects in $\mathcal{C}$.)

(c) $\sigma_B\in\mathrm{End}(B)$ is a reversion on the algebra B, and N is a left B-module.

(d) (C,σ_C) and $(N^\vee\otimes_B N,\tilde{\sigma}_g)$ are isomorphic as Jandl algebras, where $\tilde{\sigma}_g$ is defined as in (2.60) and $g\in\mathrm{Hom}_B(N,N^{\sigma_B})$ fulfills (2.59) with $\varepsilon_g=1$.

Proof:
(i) *Construction of the idempotent p in $C_{\rm top}$*:
By proposition A.4 we can find a haploid algebra D and a D-module R such that C is isomorphic to $R^\vee\otimes_D R =: C'$. Let then $\alpha\in\mathrm{Hom}(C,C')$ be an algebra isomorphism between C and C'; it induces a reversion $\sigma'_C=\alpha\circ\sigma_C\circ\alpha^{-1}$ on C', such that $(C,\sigma_C)\cong(C',\sigma'_C)$ as algebras with reversion.
Decompose R as $R\cong\bigoplus_{\kappa\in\mathcal{J}}R_\kappa^{\otimes n_\kappa}$, with distinct simple D-modules R_κ. When applied to $C'_{\rm top}$, the construction of the isomorphism φ in (A.30) shows that $C'_{\rm top}$ is isomorphic to $\bigoplus_{\kappa=1}^{|\mathcal{J}|}\mathrm{Mat}_{n_\kappa}(\mathbb{C})$. Furthermore, since the twist is trivial on the tensor unit $\mathbf{1}$, σ'_C acts on $C'_{\rm top}$ as an anti-isomorphism that squares to the identity. Formulas (A.21) and (A.31) imply that we can

[9] What is generalised here is the case of a single matrix block, $m=1$. For a generalisation in the case $m>1$ see the discussion around equation (2.66).

find an automorphism ω_f of C' such that $\omega_f^{-1} \circ \sigma'_C \circ \omega_f$ acts as $\sigma^{\pi}_{D_1 \dots D_{|\mathcal{J}|}}$ on $C'_{\rm top}$. Let $b'^{\alpha\beta}_{\kappa}$ be a basis of $C'_{\rm top}$ constructed as in (A.28). We choose an element p' of $C'_{\rm top}$ as

$$p' := \begin{cases} \omega_f \circ b'^{11}_1 & \text{for } \; D_1 = id \; \text{ and } \; \pi(1) = 1\,, \\ \omega_f \circ (b'^{11}_1 + b'^{11}_{\pi(1)}) & \text{for } \; D_1 = id \; \text{ and } \; \pi(1) \neq 1\,, \\ \omega_f \circ (b'^{11}_1 + b'^{22}_1) & \text{if } D_1 \text{ is antisymmetric}\,. \end{cases} \tag{A.32}$$

We can transport p' back to C by defining $p := \alpha^{-1} \circ p' \in C_{\rm top}$. One verifies that in all three cases $m \circ (p \otimes p) = p$ and $\sigma_C \circ p = p$.

(ii) *Construction of the data B, σ_B, N and g*:
We can now apply lemma A.5 to C and the idempotent $p \in C_{\rm top}$. This yields an algebra with reversion (B, σ_B) and a left C-module N', as well as a morphism $g' \in {\rm Hom}_C(N', N'^{\sigma_C})$ with $\varepsilon_{g'} = 1$ such that $B = N'^{\vee} \otimes_C N'$ and σ_B is of the form (2.60). As described at the end of section 2.4 the construction of (B, σ_B) can be inverted. Thus there is a left B-module $N \cong N'^{\vee}$ and $g \in {\rm Hom}_B(N, N^{\sigma_B})$ with $\varepsilon_g = 1$ such that (C, σ_C) is isomorphic as a Jandl algebra to $(N^{\vee} \otimes_B N, \tilde\sigma_g)$.

(iii) *Construction of A and M*:
The algebra B is by definition the image of the idempotent $P_B \in {\rm Hom}(C, C)$ introduced in (A.3). It follows that $B_{\rm top}$ is isomorphic as a $\mathbb{C}$-algebra to the image of the idempotent $p \in C_{\rm top}$ from part (i) above. Moreover, σ_B restricts to an anti-isomorphism $\sigma_B|_{B_{\rm top}}$ of $B_{\rm top}$. We find that

$$(B_{\rm top}, \sigma_B|_{B_{\rm top}}) \;\cong\; \begin{cases} (\mathbb{C}, a \mapsto a) & \text{for } \; D_1 = id \; \text{ and } \; \pi(1) = 1\,, \\ (\mathbb{C} \oplus \mathbb{C}, (a,b) \mapsto (b,a)) & \text{for } \; D_1 = id \; \text{ and } \; \pi(1) \neq 1\,, \\ ({\rm Mat}_2(\mathbb{C}), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)) & \text{if } D_1 \text{ is antisymmetric}\,. \end{cases} \tag{A.33}$$

In the first case B is already haploid and we can choose $A = M = B$. Otherwise we can take the nontrivial idempotent $b^{11}_1 \in B_{\rm top}$ according to the basis (A.28). Applying lemma A.2 we obtain an algebra A and a left B-module M' such that $A \cong M'^{\vee} \otimes_B M'$. Conversely we have $B \cong M' \otimes_A M'^{\vee}$, so that we can choose $M = M'^{\vee}$. The algebra A is haploid by lemma A.2(ii) and the fact that b^{11}_1 is nontrivial.
We can construct an isomorphism between $B_{\rm top}$ and $\bigoplus_\kappa {\rm Mat}_{n_\kappa}(\mathbb{C})$ as in (A.30). One finds that in order to recover the correct algebra structure on $B_{\rm top}$, the module M must be simple in the first case of formulas (A.32) and (A.33), be of the form $M \cong M_1 \oplus M_2$ with $M_{1,2}$ distinct simple modules in the second case, and $M \cong M_1 \oplus M_1$ with M_1 simple in the third case.

(iv) *Proof of $M_1^{\vee} \otimes_A M_1 \cong (M_2^{\vee} \otimes_A M_2)_{\rm op}$*:
When $B \cong (M_1 \oplus M_2)^{\vee} \otimes_A (M_1 \oplus M_2)$ with M_1 and M_2 two simple non-isomorphic left A-modules, then the reversion σ_B acts on $B_{\rm top}$ as in the second case of (A.33). Thus we can find a basis $\{p_1, p_2\} \subset {\rm Hom}(\mathbf{1}, B)$ of $B_{\rm top}$ such that $m_{\rm top}(p_i, p_j) = \delta_{i,j} p_i$ and such that σ_B acts on this basis as $\sigma_B p_1 = p_2$, $\sigma_B p_2 = p_1$.
Denote by P_i the idempotent P_B of (A.3) with p_i in place of p and A in place of B. It is easy to check that $\sigma_B \circ P_{1,2} = P_{2,1} \circ \sigma_B$. We can choose the labelling of the idempotents P_i such that the image (S_1, e_1, r_1) of P_1 satisfies $S_1 \cong M_1^{\vee} \otimes_A M_1$ (as an object in $\mathcal{C}$), and analogously for P_2.

The objects S_1 and S_2 can be turned into algebras via lemma A.2. We would like to show that $S_1 \cong S_{2,\rm op}$. To this end we first show that the morphisms

$$f := r_2 \circ \sigma_B \circ e_1 \in \mathrm{Hom}(S_1, S_2) \qquad \text{and} \qquad g := r_1 \circ \sigma_B^{-1} \circ e_2 \in \mathrm{Hom}(S_2, S_1)\,. \tag{A.34}$$

are inverse to one another; indeed we have

$$f \circ g = r_2 \circ \sigma_B \circ P_1 \circ \sigma_B^{-1} \circ e_2 = r_2 \circ P_2 \circ e_2 = id_{S_2}\,, \tag{A.35}$$

and analogously for $g \circ f$. Next note the identities

$$f \circ m_1 = r_2 \circ \sigma_B \circ m \circ (e_1 \otimes e_1) = r_2 \circ m \circ c_{B,B} \circ (\sigma_B \otimes \sigma_B) \circ (e_1 \otimes e_1) = m_2 \circ c_{S_2,S_2} \circ (f \otimes f)\,. \tag{A.36}$$

Here $m_{1,2}$ denote the multiplications on $S_{1,2}$, and in the last step we used that $P_2 \circ \sigma = \sigma \circ P_1$. We thus see that f is an algebra isomorphism from S_1 to $S_2^{(1)} \cong S_{2,\rm op}$. ✓

We are now almost done with the proof of the algorithm stated in steps 1)−4) in section 2.5. It only remains to explain why the case that $M_1 \cong M_2$ (as A-modules) is not excluded in statement (b) of proposition A.7, but nevertheless does not occur in step 3b) of the algorithm. This is achieved by proposition A.11 below, where it is shown that this situation is Morita equivalent to an algebra with reversion of the form $M^\vee \otimes_A M$ with M a simple A-module, and hence is already treated in step 3a).

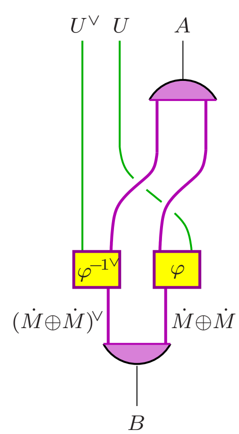
The following three lemmata prepare the proof of the proposition.

Lemma A.8:
For every left module M over a symmetric special Frobenius algebra A there is an isomorphism

$$(M \oplus M)^\vee \otimes_A (M \oplus M) \;\cong\; \mathrm{Mat}_2(\mathbb{C}) \otimes (M^\vee \otimes_A M) \tag{A.37}$$

of symmetric special Frobenius algebras. Here $\mathrm{Mat}_2(\mathbb{C})$ denotes the algebra structure on $\mathbf{1}^{\oplus 4}$ given by matrix multiplication.

Proof:
Let $U = \mathbf{1} \oplus \mathbf{1}$, $A = M^\vee \otimes_A M$ and $B = (M \oplus M)^\vee \otimes_A (M \oplus M)$. Let φ be an invertible element of $\mathrm{Hom}_A(M \oplus M, M \otimes U)$. Define the morphism $f \in \mathrm{Hom}(B, U^\vee \otimes U \otimes A)$ as

$f \;:=$  (A.38)

We endow $C := U^\vee \otimes U \otimes A$ with the algebra structure of the tensor product of the algebras $U^\vee \otimes U$ and A, see proposition I:3.22. It is a fairly straightforward application of the definitions to check that f is invertible and obeys $f\circ\eta_B=\eta_C$ and $f\circ m_B=m_C\circ(f\otimes f)$. Thus B and C are isomorphic as algebras. Since the algebra structure determines the symmetric special Frobenius structure uniquely, B and C are isomorphic also as symmetric special Frobenius algebras. ✓

Lemma A.9:
Consider the algebra $\mathrm{Mat}_2(\mathbb{C})$ equipped with the reversion σ_2 defined as

$$\sigma_2\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right). \tag{A.39}$$

The pair $(\mathrm{Mat}_2(\mathbb{C}),\sigma_2)$ is Morita equivalent to $(\mathbb{C},id)$ in the sense of proposition 2.16, with $\varepsilon_g=-1$.

Proof:
In the category of vector spaces we choose the quantities (A,σ), M and g entering in proposition 2.16 as follows. For the algebra A we take $A=\mathbb{C}$, with trivial reversion $\sigma=id_{\mathbb{C}}$. The module M is a two-dimensional vector space V on which A acts by multiplication. Let $\{e_1,e_2\}$ be a basis of V. The dual module $M^\vee$ is the dual vector space V^* with dual basis $\{e_1^*,e_2^*\}$. Since the action of A is multiplication by complex numbers and σ is the identity we have $\mathrm{Hom}(M,M^\sigma)=\mathrm{Hom}(V,V^*)$. We set

$$g:=\sum_{p,q=1}^{2}\epsilon_{pq}\,e_p^*\otimes e_q^*\equiv e_1^*\otimes e_2^*-e_2^*\otimes e_1^*\,. \tag{A.40}$$

Note that the inverse of g is $g^{-1}=\sum_{pq}\epsilon_{pq}\,e_q\otimes e_p$.
To verify that g fulfills the property (2.59) one must check that $g(e_k)=\varepsilon_g\sum_m e_k(g(e_m))\,e_m^*$ for $k=1,2$ with a suitable sign ε_g. Substituting the definition (A.40) gives $g(e_k)=\epsilon_{\bar kk}e_{\bar k}^*$ for the left hand side (where, for the sake of this proof, we use the symbol $\bar p$ to denote the index such that $\epsilon_{p\bar p}\neq 0$, i.e. $\bar p=3-p$), while for the right hand side one gets the expression

$$\varepsilon_g\sum_{m,p,q}\epsilon_{pq}((e_p^*e_k)(e_q^*e_m))\,e_m^*=\varepsilon_g\epsilon_{k\bar k}\,e_{\bar k}^*\,, \tag{A.41}$$

so that $\varepsilon_g=-1$. Thus proposition 2.16 can be applied, which provides us with a reversion $\tilde\sigma_g$ on $B=V^*\otimes V\cong\mathrm{Mat}_2(\mathbb{C})$. Computing the action of $\tilde\sigma_g$ on a basis yields

$$\tilde\sigma_g(e_p^*\otimes e_q)=(-1)\cdot g(e_q)\otimes g^{-1}(e_p^*)=\epsilon_{p\bar p}\epsilon_{q\bar q}\,e_{\bar q}^*\otimes e_{\bar p}\,. \tag{A.42}$$

This reproduces precisely the definition of σ_2 in formula (A.39). ✓

Lemma A.10:
Let A be an algebra, and let σ be a reversion on $B=\mathrm{Mat}_2(\mathbb{C})\otimes A$, with $\mathrm{Mat}_2(\mathbb{C})$ the algebra structure on $\mathbf{1}^{\oplus 4}$ given by matrix multiplication and the algebra structure on the tensor product defined according to proposition I:3.22. Assume further that σ satisfies

$$\sigma\circ(id_{\mathrm{Mat}}\otimes\eta_A)=\sigma_2\otimes\eta_A \tag{A.43}$$

with σ_2 the reversion on $\mathrm{Mat}_2(\mathbb{C})$ introduced in (A.39). Then σ can be written as

$$\sigma = \sigma_2 \otimes \sigma_A \tag{A.44}$$

for some reversion σ_A on A.

Proof:
According to the proof of lemma A.9 (and with the notations used there, in particular for $\epsilon_{pq} = -\epsilon_{qp}$ and $\bar p = 3-p$), in terms of the basis $e_{pq} \equiv e^*_p \otimes e_q$ the product and the action of σ_2 on $\mathrm{Mat}_2(\mathbb{C})$ read

$$m \circ (e_{pq} \otimes e_{rs}) = \delta_{q,r}\, e_{ps} \qquad \text{and} \qquad \sigma_2 \circ e_{pq} = \varepsilon_{p\bar p} \varepsilon_{q\bar q}\, e_{\bar q \bar p}\,. \tag{A.45}$$

Denote by $\imath^A_{i\alpha}$ a basis of $\mathrm{Hom}(U_i, A)$, as in (I:3.4). Then a basis of B is given by the morphisms $e_{pq} \otimes \imath^A_{i\alpha}$. The action of σ on this basis can be written as

$$\sigma(e_{pq} \otimes \imath^A_{i\alpha}) = \sum_{r,s} e_{rs} \otimes (\imath^A_{i\alpha})_{pq}{}^{rs} \tag{A.46}$$

for some morphisms $(\imath^A_{i\alpha})_{pq}{}^{rs} \in \mathrm{Hom}(U_i, A)$.

Next we compose the property $\sigma \circ m_B = m_B \circ c_{B,B} \circ (\sigma \otimes \sigma)$ of the reversion σ with the morphism $(e_{pq} \otimes \eta_A) \otimes (e_{rs} \otimes \imath^A_{i\alpha})$. Using that $e_{pq} \otimes \eta_A \in \mathrm{Hom}(\mathbf{1}, B)$ amounts to having trivial braiding, we find

$$\begin{aligned} & \delta_{q,r}\, \sigma(e_{ps} \otimes \imath^A_{i\alpha}) = \epsilon_{p\bar p} \epsilon_{q\bar q}\, m \circ [\sigma(e_{rs} \otimes \imath^A_{i\alpha}) \otimes (e_{\bar q \bar p} \otimes \eta_A)] \\ \Leftrightarrow\ & \delta_{q,r} \textstyle\sum_{m,n} e_{mn} \otimes (\imath^A_{i\alpha})_{ps}{}^{mn} = \epsilon_{p\bar p} \epsilon_{q\bar q} \textstyle\sum_m e_{m\bar p} \otimes (\imath^A_{i\alpha})_{rs}{}^{m\bar q} \\ \Leftrightarrow\ & \delta_{q,r} (\imath^A_{i\alpha})_{ps}{}^{mn} = \epsilon_{pn} \epsilon_{q\bar q}\, (\imath^A_{i\alpha})_{rs}{}^{m\bar q} \end{aligned} \tag{A.47}$$

Repeating the calculation with $(e_{rs} \otimes \imath^A_{i\alpha}) \otimes (e_{pq} \otimes \eta_A)$ in place of $(e_{pq} \otimes \eta_A) \otimes (e_{rs} \otimes \imath^A_{i\alpha})$ yields the analogous set of relations

$$\delta_{p,s} (\imath^A_{i\alpha})_{rq}{}^{mn} = \epsilon_{p\bar p} \epsilon_{qm}\, (\imath^A_{i\alpha})_{rs}{}^{\bar p n}\,. \tag{A.48}$$

It follows that

$$(\imath^A_{i\alpha})_{pq}{}^{rs} = \delta_{p,\bar s} \delta_{q,\bar r}\, (\imath^A_{i\alpha})_{pq}{}^{\bar q \bar p}\,. \tag{A.49}$$

Setting in addition $m = n = \bar p$ in (A.47) and $m = n = \bar q$ in (A.48) leads to

$$(\imath^A_{i\alpha})_{pp}{}^{\bar p \bar p} = \varepsilon_{p\bar p} \varepsilon_{q\bar q}\, (\imath^A_{i\alpha})_{qp}{}^{\bar p \bar q} = (\imath^A_{i\alpha})_{qq}{}^{\bar p \bar p}\,. \tag{A.50}$$

Altogether we thus have

$$(\imath^A_{i\alpha})_{pq}{}^{rs} = \epsilon_{ps} \epsilon_{qr}\, (\imath^A_{i\alpha})^\sigma \tag{A.51}$$

for some morphism $(\imath^A_{i\alpha})^\sigma \in \mathrm{Hom}(U_i, A)$. Defining the morphism $\sigma_A \in \mathrm{Hom}(A, A)$ via its action on the basis $\imath^A_{i\alpha}$ as $\sigma_A(\imath^A_{i\alpha}) := (\imath^A_{i\alpha})^\sigma$ we find that

$$\sigma(e_{pq} \otimes \imath^A_{i\alpha}) = \sum_{r,s} \epsilon_{ps} \epsilon_{qr}\, e_{rs} \otimes \sigma_A(\imath^A_{i\alpha}) = \sigma_2(e_{pq}) \otimes \sigma_A(\imath^A_{i\alpha})\,. \tag{A.52}$$

This establishes equation (A.44).

It remains to show that σ_A is a reversion on A. By the compatibility of σ with m_B we have

$$\sigma \circ m_B \circ [(e_{11} \otimes id_A) \otimes (e_{11} \otimes id_A)] = m_B \circ c_{B,B} \circ [\sigma(e_{11} \otimes id_A) \otimes \sigma(e_{11} \otimes id_A)]\,. \tag{A.53}$$

The left hand side of this identity reduces to $\sigma \circ (e_{11} \otimes m_A) = e_{22} \otimes (\sigma_A \circ m_A)$, while the right hand side can be rewritten as

$$m_B \circ c_{B,B} \circ [(e_{22} \otimes \sigma_A) \otimes (e_{22} \otimes \sigma_A)] = e_{22} \otimes [m_A \circ c_{A,A} \circ (\sigma_A \otimes \sigma_A)] \,. \tag{A.54}$$

It follows that $\sigma_A \circ m_A = m_A \circ c_{A,A} \circ (\sigma_A \otimes \sigma_A)$, as required. That the condition $\sigma_A \circ \sigma_A = \theta_A$ is satisfied as well can be seen by composing both sides of the identity $\sigma \circ \sigma = \theta_B$ with $e_{11} \otimes id_A$ and noting that $\theta_B \circ (e_{11} \otimes id_A) = e_{11} \otimes \theta_A$. ✓

Proposition A.11:
For A a symmetric special Frobenius algebra, let M be a simple left A-module and B the algebra $(M \oplus M)^\vee \otimes_A (M \oplus M)$. Suppose σ_B is a reversion on B acting on $B_{\rm top} \cong \mathrm{Mat}_2(\mathbb{C})$ according to formula (A.39). Then there exist g and $\tilde\sigma_g$ such that (B, σ_B) is Morita equivalent to $(M^\vee \otimes_A M, \tilde\sigma_g)$ in the sense of proposition 2.16.

Proof:
Abbreviate $M^\vee \otimes_A M =: C$ and $\mathrm{Mat}_2(\mathbb{C}) \otimes C =: D$. By lemma A.8 we have $B \cong D$ as algebras. Via this isomorphism the reversion σ_B induces a reversion σ on D. By assumption, σ acts on $D_{\rm top}$ as σ_2 in $(A.39)$. According to lemma A.10 we can then write $\sigma = \sigma_2 \otimes \sigma_C$ for some reversion σ_C on C.
Next we apply the construction in proposition 2.16 separately to both factors of D. That is, we take the module $V \otimes C$, and for the morphism g we choose $g = g_2 \otimes g_C$, with g_2 obtained by inverting the constrution of lemma A.9 (recall proposition 2.17) and $g_C \in \mathrm{Hom}_C(C, C^{\sigma_C})$ equal to $\Phi_2 \circ \sigma_C^{-1}$ (compare to the calculation in (A.17)). This reduces the $\mathrm{Mat}_2(\mathbb{C})$-part of D to $\mathbf{1}$ as in lemma A.9 and leaves (C, σ_C) invariant. Thus altogether (D, σ), and hence also (B, σ_B), is Morita equivalent to (C, σ_C) in the sense of proposition 2.16. ✓

This completes the proof of the classification algorithm for reversions that was described in steps 1)−4) in section 2.5.

In steps 1)−4) one is only concerned with reversions on *simple* symmetric special Frobenius algebras. In general, the algebra could be of the form (2.66), though. In the sequel we extend the construction to include also direct sums of simple algebras.

Let $e_i \in \mathrm{Hom}(A_i, A)$ and $r_i \in \mathrm{Hom}(A, A_i)$ denote the embedding and restriction for the subobject A_i of A. The e_i and r_i are algebra homomorphisms. A reversion σ on A can permute the components A_i. Thus suppose that

$$r_l \circ \sigma \circ e_k = \delta_{l,\pi(k)}\, r_{\pi(k)} \circ \sigma \circ e_k \tag{A.55}$$

for some permutation π with $\pi \circ \pi = id$. This implies in particular that $A_{\pi(k)} \cong A_{k,\rm op}$.

Suppose $\pi(k) = k$. Then σ restricts to a reversion on A_k via $r_k \circ \sigma \circ e_k$. It must therefore be in the list obtained by steps 1)−4).

Suppose $\pi(k) \neq k$. Then we can choose correlated bases for A_k and $A_{\pi(k)}$ in which σ takes a particularly simple form. Choose dual bases

$$f^k_{a,\alpha} \in \mathrm{Hom}(U_a, A_k) \qquad \text{and} \qquad \bar f^k_{a,\alpha} \in \mathrm{Hom}(A_k, U_a) \,. \tag{A.56}$$

Using these, define dual bases for the subalgebra $A_{\pi(k)}$ as

$$f_{a,\alpha}^{\pi(k)} := r_{\pi(k)} \circ \sigma \circ e_k \circ f_{a,\alpha}^k \in \mathrm{Hom}(U_a, A_{\pi(k)}) \qquad \text{and} \\ \bar{f}_{a,\alpha}^{\pi(k)} := \bar{f}_{a,\alpha}^k \circ r_k \circ \sigma^{-1} \circ e_{\pi(k)} \in \mathrm{Hom}(A_{\pi(k)}, U_a)\,. \tag{A.57}$$

Using $e_{\pi(k)} \circ r_{\pi(k)} \circ \sigma = \sigma \circ e_k \circ r_k$ the components (2.19) of the reversion σ then take the form

$$\sigma(a)_{k,\alpha}^{\;\;\,l,\beta} = \delta_{l,\pi(k)}\, \bar{f}_{a,\beta}^{\pi(k)} \circ r_{\pi(k)} \circ \sigma \circ e_k \circ f_{a,\alpha}^k = \delta_{l,\pi(k)}\, \delta_{\alpha,\beta}\,. \tag{A.58}$$

Similarly we get

$$\sigma(a)_{\pi(k),\alpha}^{\;\;\,l,\beta} = \delta_{l,k}\, \bar{f}_{a,\beta}^k \circ r_k \circ \sigma \circ e_{\pi(k)} \circ f_{a,\alpha}^{\pi(k)} = \theta_a\, \delta_{l,k}\, \delta_{\alpha,\beta}\,. \tag{A.59}$$

We can thus conclude that for all subalgebras A_k of A which are not fixed under σ, i.e. $\pi(k) \neq k$, one can always choose a basis such that the components of σ take the particular form

$$\sigma(a)_{k,\alpha}^{\;\;\,\pi(k),\beta} = \delta_{\alpha,\beta} \qquad \text{and} \qquad \sigma(a)_{\pi(k),\alpha}^{\;\;\,k,\beta} = \theta_a\, \delta_{\alpha,\beta}\,. \tag{A.60}$$

Applied to the category $\mathcal{V}ect_{\mathbb{C}}$, the simple form of σ corresponds to $D_i = id$ if $\pi(i) \neq i$ in the expression (A.21).

A.3 Invariants for glueing tori

This appendix provides some details about the recursion relations (3.131). To demonstrate these relations we use the methods of [60], especially sections II.3.9, IV.3 and IV.5 (the recursion relations do not appear explicitly in [60], though). Some basic notions of three-dimensional topological field theory and references to the literature can be found in sections I:2.3 and I:2.4.

Define the extended surface T to be a torus without field insertions, with lagrangian subspace λ_T. We select a basis a, b of cycles in $H_1(T, \mathbb{R})$ with intersection number $\omega(a,b) = 1$ and take λ_T to be the span of the cocycle a.

Define the cobordism $M_{\chi_k}^+$ to be a solid torus with $\partial_+ M_{\chi_k}^+ = T$ and $\partial_- M_{\chi_k}^+ = \emptyset$ such that the a-cycle of T is contractible in $M_{\chi_k}^+$. Inside $M_{\chi_k}^+$ place a ribbon labelled by U_k running parallel to the surface T along the b-cycle and with core oriented opposite to the b-cycle, see figure (3.123 a). Similarly define $M_{\chi_k}^-$ to be a solid torus with $\partial_+ M_{\chi_k}^- = \emptyset$ and $\partial_- M_{\chi_k}^- = T$ such that the a-cycle of T is contractible in $M_{\chi_k}^-$. There is again a ribbon labelled by U_k following the b-cycle of T, but this time the orientation is the same way as the one of b. Given two cobordisms M_1, M_2 and an invertible homeomorphism g: $\partial_+ M_1 \to \partial_- M_2$, define $M_2 \overset{g}{\hookleftarrow} M_1$ to be the cobordism obtained by glueing M_1 to M_2 using g.

Identify $H_1(T, \mathbb{R})$ with $\mathbb{R}^2$ by choosing $a = (1,0)$ and $b = (0,1)$. Let g: $T \to T$ be an invertible homeomorphism, with induced action on $H_1(T, \mathbb{R})$ given by the matrix $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. The aim of this appendix is to establish recursions relations that allow us to compute the invariants

$$M \left[\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \right]_{kl} := Z(M_{\chi_k}^- \overset{g}{\hookleftarrow} M_{\chi_l}^+, \emptyset, \emptyset)\, 1\,. \tag{A.61}$$

We start by defining Maslov indices and computing them for the torus. These are needed to formulate the framing anomaly in the functoriality property of the topological field theory. The functoriality property will then provide the recursion relations (3.131).

In the description of the Maslov indices and the framing anomaly we follow closely the review in section 2.7 of [51]. Let H be a real vector space with symplectic form ω. For any triple of lagrangian subspaces $\lambda_1, \lambda_2, \lambda_3 \subset H$ define a quadratic form $q(x,y)$ on the subspace $(\lambda_1{+}\lambda_2)\cap\lambda_3$ via $q(x,y) := \omega(x_2, y)$, where $x =: x_1 + x_2$ with $x_{1,2} \in \lambda_{1,2}$. One verifies that q is independent of the choice of decomposition of x. The *Maslov index* $\mu(\lambda_1, \lambda_2, \lambda_3)$ is defined to be the signature of q; it is antisymmetric in all three arguments.

We only need Maslov indices for $H = H_1(T, \mathbb{R})$, which has dimension 2, so that the lagrangian subspaces $\lambda_{1,2,3}$ have dimension 1. We select vectors $v_1, v_2, v_3 \in H$ such that λ_i is the linear span of v_i, for $i = 1, 2, 3$. If any two of the v_i are collinear, then $\mu(\lambda_1, \lambda_2, \lambda_3) = 0$ by the antisymmetry of μ; otherwise we have

$$(\lambda_1 + \lambda_2) \cap \lambda_3 = \lambda_3 \tag{A.62}$$

and there exist $\alpha, \beta \in \mathbb{R}$ such that $v_3 = \alpha v_1 + \beta v_2$. The quadratic form q evaluated on v_3 is given by $q(v_3, v_3) = \omega(\beta v_2, v_3) = \alpha\beta\, \omega(v_2, v_1)$. Since the subspace λ_3 on which q is defined is one-dimensional, the signature of q is simply the sign of $q(v_3, v_3)$. Thus

$$\mu(\lambda_1, \lambda_2, \lambda_3) = \mathrm{sign}(\alpha\beta\, \omega(v_2, v_1))\,. \tag{A.63}$$

Now fix a basis a, b of cycles with $\omega(a, b) = 1$ and expand the vectors v_i as $v_i = x_i a + y_i b$. To determine α, β in $v_3 = \alpha v_1 + \beta v_2$ we must solve the linear system

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{A.64}$$

Noting that $\omega(v_i, v_j) = \omega(x_i a + y_i b, x_j a + y_j b) = x_i y_j - x_j y_i$ one verifies that the linear system is solved by $\alpha = -\omega(v_2, v_3)/\omega(v_1, v_2)$ and $\beta = \omega(v_1, v_3)/\omega(v_1, v_2)$. Inserting this result in (A.63), and using the convention that $\mathrm{sign}(0) = 0$, finally gives the Maslov index for the torus as

$$\mu(\lambda_1, \lambda_2, \lambda_3) = \mathrm{sign}[\,\omega(v_1, v_2)\, \omega(v_1, v_3)\, \omega(v_2, v_3)\,]\,. \tag{A.65}$$

While this formula was derived for the case that no two of the v_i are collinear, it continues to hold in the collinear case as well, since it then gives zero as required.

For any cobordism M a map N_* from the set $\Lambda(\partial_- M)$ of lagrangian subspaces of $H_1(\partial_- M, \mathbb{R})$ to $\Lambda(\partial_+ M)$ can be constructed as follows. For $\lambda \in \Lambda(\partial_- M)$ the vector $x \in H_1(\partial_+ M, \mathbb{R})$ is in $N_*\lambda$ iff there exists an $x' \in \lambda$ such that $x - x'$ is homologous to zero as a cycle in M. The map $N^*\colon \Lambda(\partial_+ M) \to \Lambda(\partial_- M)$ is defined similarly.

The functoriality axiom of the topological field theory reads

$$Z(M_2 \stackrel{f}{\leftarrow} M_1, \partial_- M_1, \partial_+ M_2) = \kappa^m\, Z(M_2, \partial_- M_2, \partial_+ M_2) \circ f_\sharp \circ Z(M_1, \partial_- M_1, \partial_+ M_1)\,, \tag{A.66}$$

with the integer m defined as the sum

$$m := \mu(f_* N_* \lambda(\partial_- M_1), f_* \lambda(\partial_+ M_1), N^* \lambda(\partial_+ M_2)) + \mu(f_* \lambda(\partial_+ M_1), \lambda(\partial_- M_2), N^* \lambda(\partial_+ M_2)) \tag{A.67}$$

of Maslov indices. The following schematic rewriting of formula (A.67) facilitates keeping track of the arguments of the Maslov indices:

$$\begin{array}{rccccccccc} & & \overbrace{\Lambda(\partial_+ M_2) \quad \xrightarrow{N^*} \quad \Lambda(\partial_- M_2)}^{M_2} & & \xleftarrow{f_*} & \overbrace{\Lambda(\partial_+ M_1) \quad \xleftarrow{N_*} \quad \Lambda(\partial_- M_1)}^{M_1} & \\ m = & \mu\,(& 3 \qquad\qquad\qquad\qquad & & & 2 \qquad\qquad\qquad 1 &) \\ + & \mu\,(& 3 \qquad\qquad\qquad 2 & & & 1 \qquad\qquad\qquad\qquad &) \end{array} \tag{A.68}$$

Both Maslov indices are evaluated in $\Lambda(\partial_- M_2)$; the numbers indicate as which argument of $\mu(\cdot,\cdot,\cdot)$ the corresponding lagrangian subspace appears.

Next we define the two sets of vectors $|\chi_k;T\rangle$ and $\langle\chi_k;T|$ as

$$\begin{aligned} |\chi_k;T\rangle &:= Z(M^+_{\chi_k},\emptyset,T)\,1 \quad \in \mathcal{H}(T;\emptyset)\,, \\ \langle\chi_k;T| &:= Z(M^+_{\chi_k},T,\emptyset) \quad\;\; \in \mathcal{H}(T;\emptyset)^*\,. \end{aligned} \tag{A.69}$$

As a warmup let us compute the composition of two of these vectors. We have

$$\delta_{k,l} = Z(M^-_{\chi_k} \overset{id}{\hookleftarrow} M^+_{\chi_l},\emptyset,\emptyset)\,1 = \kappa^m\,\langle\chi_k;T| \circ id_\sharp \circ |\chi_l;T\rangle\,. \tag{A.70}$$

The value $\delta_{k,l}$ for the invariant follows from the normalisation axiom for cobordisms of the form $X\times[0,1]$, see (I:2.55). The integer m is given by the following combination of Maslov indices:

$$\begin{array}{rcccccccc} & & \overbrace{\Lambda(\emptyset) \;\xrightarrow{N^*}\; \Lambda(T)}^{M^-_{\chi_k}} & \overset{id_*}{\longleftarrow} & \overbrace{\Lambda(T) \;\overset{N_*}{\longleftarrow}\; \Lambda(\emptyset)}^{M^+_{\chi_l}} & \\ m = & \mu\,(& 3 \qquad\qquad\quad & & 2 \qquad\qquad\quad 1 &) \\ + & \mu\,(& 3 \qquad\qquad\quad 2 & & 1 \qquad\qquad\qquad\;\; &) \end{array} \tag{A.71}$$

All six lagrangian subspaces entering in the computation of m are in fact identical and equal to λ_T, so that $m=0$. To see this first recall that λ_T is spanned by the a-cycle in T, and that $M^+_{\chi_l}$ was chosen such that the a-cycle is contractible in $M^+_{\chi_l}$. It follows that N_* maps $0\in\Lambda(\emptyset)$ to λ_T in $\Lambda(T)$. The same holds for N^*. Furthermore id_* acts as the identity on $\Lambda(T)$. Thus as expected we have

$$\langle\chi_k;T\,|\,\chi_l;T\rangle = \delta_{k,l}\,. \tag{A.72}$$

The set $\{|\chi_k;T\rangle\,|\,k\in\mathcal{I}\}$ forms a basis of $\mathcal{H}(T;\emptyset)$, while $\{\langle\chi_k;T|\,|\,k\in\mathcal{I}\}$ is a basis of $\mathcal{H}(T;\emptyset)^*$. The result (A.72) shows that these two bases are dual to each other.

Denote by C_T the cylinder $T\times[0,1]$. We have $Z(C_T,T,T)=id_{\mathcal{H}(T;\emptyset)}$ (by the normalisation axiom), so that we can write

$$Z(C_T,T,T) = \sum_{r\in\mathcal{I}} |\chi_r;T\rangle\,\langle\chi_r;T|\,. \tag{A.73}$$

Let f_1, f_2 be two invertible homeomorphisms from T to T. The two three-manifolds $M^-_{\chi_k} \overset{f_1\circ f_2}{\hookleftarrow} M^+_{\chi_l}$ and $M^-_{\chi_k} \overset{f_1}{\hookleftarrow} C_T \overset{f_2}{\hookleftarrow} M^+_{\chi_l}$ are homeomorphic. Employing functoriality we obtain the series of equalities

$$\begin{aligned} Z(M^-_{\chi_k} \overset{f_1\circ f_2}{\hookleftarrow} M^+_{\chi_l},\emptyset,\emptyset) \;&=\; Z(M^-_{\chi_k} \overset{f_1}{\hookleftarrow} C_T \overset{f_2}{\hookleftarrow} M^+_{\chi_l},\emptyset,\emptyset) \\ &= \kappa^{m_1}\,Z(M^-_{\chi_k},T,\emptyset)\circ f_{1\sharp}\circ Z(C_T \overset{f_2}{\hookleftarrow} M^+_{\chi_l},\emptyset,T) \\ &= \kappa^{m_1+m_2}\,\langle\chi_k;T|\,f_{1\sharp}\circ Z(C_T,T,T)\circ f_{2\sharp}\circ Z(M^+_{\chi_l},\emptyset,T) \\ &= \kappa^{m_1+m_2}\,\textstyle\sum_{r\in\mathcal{I}}\langle\chi_k;T|f_{1\sharp}|\chi_r;T\rangle\,\langle\chi_r;T|f_{2\sharp}|\chi_l;T\rangle \\ &= \kappa^{m_1+m_2-\tilde m_1-\tilde m_2}\,\textstyle\sum_{r\in\mathcal{I}} Z(M^-_{\chi_k} \overset{f_1}{\hookleftarrow} M^+_{\chi_r},\emptyset,\emptyset)\,Z(M^-_{\chi_r} \overset{f_2}{\hookleftarrow} M^+_{\chi_l},\emptyset,\emptyset)\,. \end{aligned} \tag{A.74}$$

The integers m_1, m_2, $\tilde{m}_1$, $\tilde{m}_2$ are given by the following combinations of Maslov indices: First,

$$\begin{array}{rlccccl} & & \overbrace{\Lambda(\emptyset) \xrightarrow{N^*} \Lambda(T)}^{M^-_{\chi_k}} & \xleftarrow{(f_1)_*} & \overbrace{\Lambda(T) \xleftarrow{N_*} \Lambda(\emptyset)}^{C_T \overset{f_2}{\hookleftarrow} M^+_{\chi_l}} & & \\ m_1 = & \mu\,(& 3 \qquad\qquad & & 2 \qquad\qquad 1 &) & \\ + & \mu\,(& 3 \qquad\qquad 2 & & 1 \qquad\qquad &) & \end{array} \tag{A.75}$$

The first Maslov index reads $\mu((f_1 f_2)_*\lambda_T,\, (f_1)_*\lambda_T,\, \lambda_T)$; the second index is zero owing to $N^*0 = \lambda_T$, as already discussed in (A.71). Second,

$$\begin{array}{rlccccl} & & \overbrace{\Lambda(T) \xrightarrow{N^*} \Lambda(T)}^{C_T} & \xleftarrow{(f_2)_*} & \overbrace{\Lambda(T) \xleftarrow{N_*} \Lambda(\emptyset)}^{M^+_{\chi_l}} & & \\ m_2 = & \mu\,(& 3 \qquad\qquad & & 2 \qquad\qquad 1 &) & \\ + & \mu\,(& 3 \qquad\qquad 2 & & 1 \qquad\qquad &) & \end{array} \tag{A.76}$$

For the cylinder C_T the map N^* acts as identity on $\Lambda(T)$, so that the first index vanishes because of $N^*\lambda_T = \lambda_T$; the second index is zero as well, since $N_*0 = \lambda_T$, again as in (A.71). Finally,

$$\begin{array}{rlccccl} & & \overbrace{\Lambda(\emptyset) \xrightarrow{N^*} \Lambda(T)}^{M^-_{\chi_k}} & \xleftarrow{(f_1)_*} & \overbrace{\Lambda(T) \xleftarrow{N_*} \Lambda(\emptyset)}^{M^+_{\chi_r}} & & \\ \tilde{m}_1 = & \mu\,(& 3 \qquad\qquad & & 2 \qquad\qquad 1 &) & \\ + & \mu\,(& 3 \qquad\qquad 2 & & 1 \qquad\qquad &) & \\ & & \overbrace{\Lambda(\emptyset) \xrightarrow{N^*} \Lambda(T)}^{M^-_{\chi_r}} & \xleftarrow{(f_2)_*} & \overbrace{\Lambda(T) \xleftarrow{N_*} \Lambda(\emptyset)}^{M^+_{\chi_l}} & & \\ \tilde{m}_2 = & \mu\,(& 3 \qquad\qquad & & 2 \qquad\qquad 1 &) & \\ + & \mu\,(& 3 \qquad\qquad 2 & & 1 \qquad\qquad &) & \end{array} \tag{A.77}$$

All four Maslov indices appearing in $\tilde{m}_1$ and $\tilde{m}_2$ are zero, since all of them have two coinciding arguments, due to either $N_*0 = \lambda_T$ or $N^*0 = \lambda_T$. Altogether we obtain

$$Z(M^-_{\chi_k} \overset{f_1 \circ f_2}{\hookleftarrow} M^+_{\chi_l}, \emptyset, \emptyset) = \kappa^{\mu((f_1 f_2)_*\lambda_T,\, (f_1)_*\lambda_T,\, \lambda_T)} \sum_{r \in \mathcal{I}} Z(M^-_{\chi_k} \overset{f_1}{\hookleftarrow} M^+_{\chi_r}, \emptyset, \emptyset)\, Z(M^-_{\chi_r} \overset{f_2}{\hookleftarrow} M^+_{\chi_l}, \emptyset, \emptyset)\,. \tag{A.78}$$

The next step in obtaining the recursion relation is the computation of the two invariants

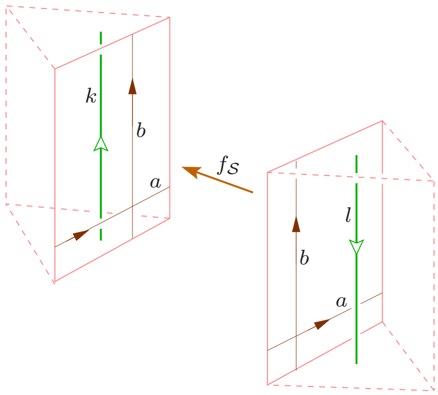
$$R(\mathcal{S})_{kl} := Z(M^-_{\chi_k} \overset{f_{\mathcal{S}}}{\hookleftarrow} M^+_{\chi_r}, \emptyset, \emptyset) \qquad \text{and} \qquad R(\mathcal{T})_{kl} := Z(M^-_{\chi_k} \overset{f_{\mathcal{T}}}{\hookleftarrow} M^+_{\chi_r}, \emptyset, \emptyset)\,. \tag{A.79}$$

Here $\mathcal{S}$ and $\mathcal{T}$ are the matrices

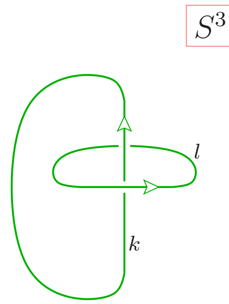
$$\mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{A.80}$$

while $f_{\mathcal{S}}\colon T \to T$ is an invertible homeomorphism such that the induced action on $H_1(T,\mathbb{R})$ is given by the matrix $\mathcal{S}$ (in the same basis that was used in (A.61)), and similarly for $f_{\mathcal{T}}\colon T \to T$. We have

$R(\mathcal{S})_{kl}$ =



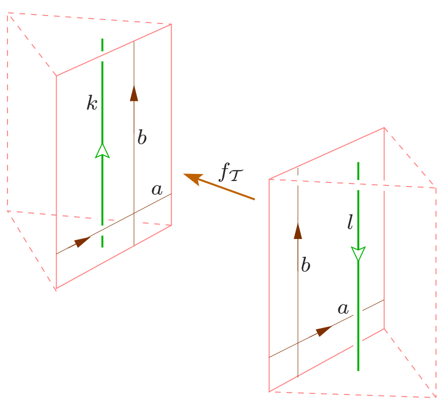
$= S_{00}$



$= S_{kl}\,.$ (A.81)

In the first step the two tori $M^+_{\chi_l}$ and $M^-_{\chi_k}$ are drawn as wedges of which the left and right dashed sides, as well as top and bottom, are identified. The map $f_{\mathcal{S}}$ glues the a-cycle to b and the b-cycle to $-a$. This results in the three-manifold S^3 with ribbons as displayed on the right hand side. Similarly, for $R(\mathcal{T})$ we find

$R(\mathcal{T})_{kl}$ =



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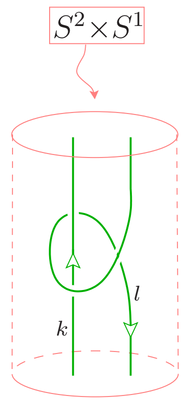

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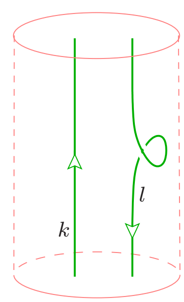


$$= \delta_{k,l}\theta_l^{-1} = \hat{T}_{kl}\,. \qquad \text{(A.82)}$$

The second step is most easily seen by drawing actual ribbons instead of using the blackboard framing convention. The third step amounts to taking the U_l-ribbon around the horizontal S^2 so that (in this description of $S^2 \times S^1$) it no longer wraps around the U_k-ribbon.

Finally we must compute the two Maslov indices

$$\mu_{\mathcal{S}} = \mu((f_{\mathcal{S}}\circ f)_*\lambda_T, (f_{\mathcal{S}})_*\lambda_T,\, \lambda_T) \qquad \text{and} \qquad \mu_{\mathcal{T}} = \mu((f_{\mathcal{T}}\circ f)_*\lambda_T, (f_{\mathcal{T}})_*\lambda_T,\, \lambda_T)\,, \qquad \text{(A.83)}$$

where f is taken to induce the action $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ on $H_1(T,\mathbb{R})$. We obtain

$$(f_{\mathcal{S}}\circ f)_*\lambda_T = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}-\gamma\\ \alpha\end{pmatrix}. \qquad \text{(A.84)}$$

Similarly,

$$(f_{\mathcal{T}}\circ f)_*\lambda_T = \begin{pmatrix}\alpha+\gamma\\ \gamma\end{pmatrix}, \qquad (f_{\mathcal{S}})_*\lambda_T = \begin{pmatrix}0\\ 1\end{pmatrix}, \qquad (f_{\mathcal{T}})_*\lambda_T = \begin{pmatrix}1\\ 0\end{pmatrix}. \qquad \text{(A.85)}$$

In these formulas, as well as in the sequel, we abuse notation by using a vector to describe both an element in $H_1(T,\mathbb{R})$ and, via its linear span, an element of $\Lambda(T)$. Using the result (A.65) we find

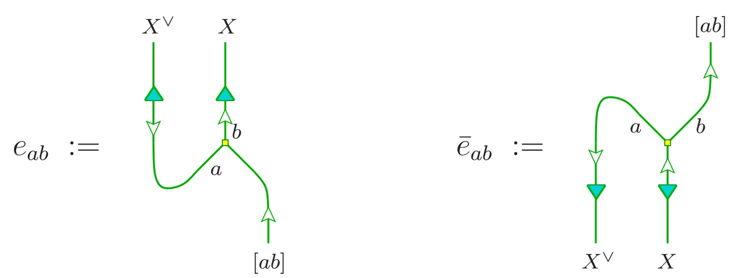
$$\begin{aligned} \mu_{\mathcal{S}} &= \mu(-\gamma a{+}\alpha b, b, a) = \mathrm{sign}(\omega(-\gamma a{+}\alpha b, b)\omega(-\gamma a{+}\alpha b, a)\omega(b,a)) = -\mathrm{sign}(\alpha\gamma)\,, \\ \mu_{\mathcal{T}} &= \mu((\alpha{+}\gamma)a{+}\gamma c, a, a) = 0\,. \end{aligned} \tag{A.86}$$

Evaluating (A.78) for $f_1 = f_{\mathcal{S},\mathcal{T}}$ and substituting (A.81), (A.82) as well as (A.86) results in the relations (3.131).

A.4 More on reversions in the Ising model

In this appendix we give some details on how to find the reversions on the algebra $(\mathbf{1}{\oplus}\epsilon)^\vee{\otimes}(\mathbf{1}{\oplus}\epsilon)$ in step 3b) of our prescription for the classification of reversions. We will then see how these reversions are related to the ones on $A = \mathbf{1}{\oplus}\epsilon$.

Set $X := \mathbf{1}{\oplus}\epsilon$. For step 3b) we would like to find reversions on $B := X^\vee{\otimes}X$ that act on a basis of B_{top} via permutation of the two basis vectors. Define the morphisms



(A.87)

where $a, b, \in \{\mathbf{1}, \epsilon\}$ and $[ab] \cong a{\otimes}b$ is a short-hand standing for $[\mathbf{11}] = [\epsilon\epsilon] = \mathbf{1}$ and $[\mathbf{1}\epsilon] = [\epsilon\mathbf{1}] = \epsilon$. We introduce the alternative labelling

$$e_1^{\mathbf{1}} := e_{\mathbf{11}}\,, \qquad e_2^{\mathbf{1}} := e_{\epsilon\epsilon}\,, \qquad e_1^{\epsilon} := e_{\mathbf{1}\epsilon}\,, \qquad e_2^{\epsilon} := e_{\epsilon\mathbf{1}}\,, \tag{A.88}$$

and similarly for $\bar e^a_\alpha$ and $\bar e_{ab}$. Note that as an object $B \cong \mathbf{1}{\oplus}\mathbf{1}{\oplus}\epsilon{\oplus}\epsilon$. With the labelling above the morphisms e^a_α thus provide bases for the spaces $\mathrm{Hom}(a,B)$ and $\bar e^a_\alpha$ for $\mathrm{Hom}(B,a)$. One can check that the bases e^a_α and $\bar e^a_\alpha$ are dual to each other.

Recall that the multiplication on $X^\vee{\otimes}X$ is given by (2.47) (with A set to $\mathbf{1}$ and M set to X). Thus to work out the multiplication on B in the basis (A.88), following (I:3.7) we must compute the constants $m_{a\alpha,b\beta}^{\;\;c\gamma}$ in

$$\bar e^c_\gamma \circ m \circ (e^a_\alpha \otimes e^b_\beta) = m_{a\alpha,b\beta}^{\;\;c\gamma}\,\lambda_{(ab)c}\,, \tag{A.89}$$

where $\lambda_{(ab)c}$ refers to the basis element chosen in $\mathrm{Hom}(a{\otimes}b,c)$ as in (I:2.29). With $\bar\beta \equiv 3{-}\beta$ this yields

$$m_{a\alpha,b\beta}^{\;\;c\gamma} = N_{ab}^{\;\;c}\cdot\begin{cases}\delta_{\alpha\gamma}\delta_{\alpha\beta} & \text{for} \;\; a = \mathbf{1}\,,\\ \delta_{\alpha\gamma}\delta_{\alpha\bar\beta} & \text{for} \;\; a = \epsilon\,.\end{cases} \tag{A.90}$$

Next, the reversion evaluated in a basis takes the form (2.19),

$$\bar e^a_\beta \circ \sigma \circ e^a_\alpha = \sigma(a)_\alpha^{\;\beta}\, id_{U_a}\,. \tag{A.91}$$

The matrices $\sigma(a)_\alpha^{\ \beta}$ are required to obey the constraints (2.20). Thus we must find solutions to the 2×2 matrix equations

$$\sigma(\mathbf{1})\,\sigma(\mathbf{1}) = 1\!\!1_{2\times 2}\,, \qquad \sigma(\epsilon)\,\sigma(\epsilon) = -1\!\!1_{2\times 2} \tag{A.92}$$

as well as the conditions

$$\sum_{\rho=1,2} m_{a\alpha,b\beta}^{c\rho}\,\sigma(c)_\rho^{\ \gamma} = \sum_{\mu,\nu=1,2} \sigma(a)_\alpha^{\ \mu}\,\sigma(a)_\beta^{\ \nu}\,m_{b\nu,a\mu}^{c\gamma}\,\mathsf{R}^{(a\,b)c}\,. \tag{A.93}$$

Step 3b) of the classification algorithm demands $\sigma(\mathbf{1})$ to act as a permutation; we thus set

$$\sigma(\mathbf{1})_\alpha^{\ \beta} = \delta_{\beta,\bar\alpha}\,. \tag{A.94}$$

In order to solve (A.93) we evaluate the condition for all choices a, b, c allowed by fusion. A short calculation shows that with (A.94) and (A.90), the condition (A.93) is identically fulfilled for $(ab)c = (\mathbf{11})\mathbf{1}$, while the other cases give the following restrictions:

$$\begin{array}{llll} (A) & (ab)c = (\mathbf{1}\epsilon)\epsilon & \Longrightarrow & \delta_{\alpha\beta}\,\sigma(\epsilon)_\beta^{\ \gamma} = \delta_{\alpha\gamma}\,\sigma(\epsilon)_\beta^{\ \alpha}\,, \\ (B) & (ab)c = (\epsilon\mathbf{1})\epsilon & \Longrightarrow & \delta_{\alpha\bar\beta}\,\sigma(\epsilon)_\alpha^{\ \gamma} = \delta_{\gamma\bar\beta}\,\sigma(\epsilon)_\alpha^{\ \bar\beta}\,, \\ (C) & (ab)c = (\epsilon\epsilon)\mathbf{1} & \Longrightarrow & \delta_{\alpha\bar\beta}\delta_{\alpha\bar\gamma} = -\sigma(\epsilon)_\alpha^{\ \bar\gamma}\,\sigma(\epsilon)_\beta^{\ \gamma}\,. \end{array} \tag{A.95}$$

From (A) the matrix $\sigma(\epsilon)$ can be deduced to be diagonal, $\sigma(\epsilon)_\alpha^{\ \beta} = \delta_{\alpha\beta}\,\sigma(\epsilon)_\alpha^{\ \alpha}$. Then condition (B) is identically fulfilled, while condition (C) is equivalent to $\sigma(\epsilon)_\alpha^{\ \alpha}\,\sigma(\epsilon)_{\bar\alpha}^{\ \bar\alpha} = -1$. Thus (A)–(C) are equivalent to $\sigma(\epsilon)$ being of the form

$$\sigma(\epsilon) = \begin{pmatrix} a & 0 \\ 0 & -a^{-1} \end{pmatrix}. \tag{A.96}$$

Relation (A.92) restricts this further to $a = \pm\mathrm{i}$. Thus altogether we find two possible reversions on B, given by $\sigma^{(\pm)}(\mathbf{1})_\alpha^{\ \beta} = \delta_{\beta\bar\alpha}$ and

$$\sigma^{(+)}(\epsilon) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \qquad \sigma^{(-)}(\epsilon) = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{A.97}$$

It turns out that both of these reversions are related to the reversion on A given in (4.29) with $s_\epsilon = -1$ via propostion 2.16. Let us see how this comes about.

We are going to apply propostion 2.16 for the choices $A = \sigma^\vee\otimes\sigma$, $M = \sigma^\vee\otimes X$ and $B = X^\vee\otimes X$. The action of A on M is defined as

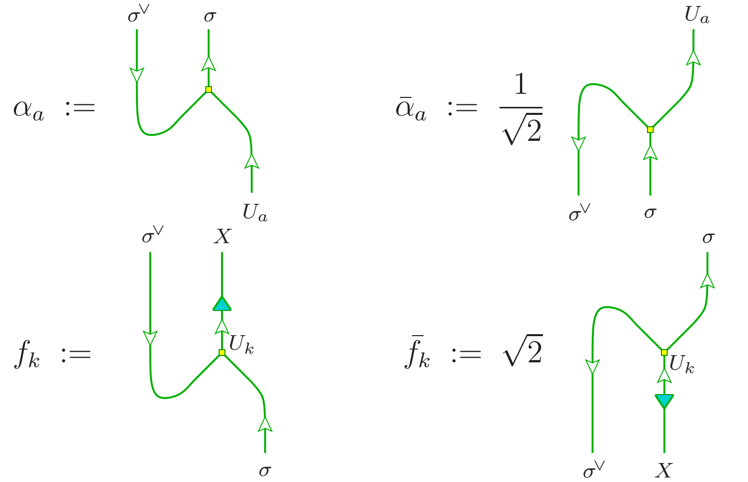
$$\rho := id_{\sigma^\vee}\otimes\tilde d_\sigma\otimes id_X \in \mathrm{Hom}(\sigma^\vee\otimes\sigma\otimes\sigma^\vee\otimes X, \sigma^\vee\otimes X)\,. \tag{A.98}$$

Let us first check that indeed B is isomorphic to $M^\vee\otimes_A M$. To this end we introduce morphisms $e\in\mathrm{Hom}(B, M^\vee\otimes M)$ and $r\in\mathrm{Hom}(M^\vee\otimes M, B)$ by

$$e := id_{X^\vee}\otimes b_\sigma\otimes id_X \quad , \quad r := \tfrac{1}{\dim(\sigma)}\,id_{X^\vee}\otimes\tilde d_\sigma\otimes id_X\,. \tag{A.99}$$

One verifies that e and r are indeed embedding and restriction morphisms as drawn in (2.46), i.e. satisfy $r\circ e = id_B$ and $e\circ r = P_{\otimes A}$ as in (2.45).

For the following calculations we need to introduce two bases in addition to (A.87), one for the morphism spaces $\mathrm{Hom}(U_a, A)$ and one for $\mathrm{Hom}(\sigma, \dot{M})$, together with their duals. We choose



(A.100)

Note that we have $m^{\mathbf{1}}_{\epsilon\epsilon} = \alpha_{\mathbf{1}} \circ m_A \circ (\alpha_\epsilon \otimes \alpha_\epsilon) = 2$, i.e. the basis α_a is normalised slightly different from (4.28).

Next we would like to describe all module morphisms $g \in \mathrm{Hom}_A(M, M^\sigma)$. Since these form a subspace of $\mathrm{Hom}(\dot{M}, \dot{M}^\vee)$ we can expand g as

$$g = \sum_{k,l \in \{\mathbf{1},\epsilon\}} G_{kl} \, (\lambda_{(\sigma\sigma)\mathbf{1}} \otimes (\bar{f}_l)^\vee) \circ (\bar{f}_k \otimes b_\sigma) \,. \tag{A.101}$$

For g to be an intertwiner for the A-action we need

$$(f_l)^\vee \circ g \circ \rho \circ (\alpha_k \otimes f_j) = (f_l)^\vee \circ \rho^\sigma \circ (id_A \otimes g) \circ (\alpha_k \otimes f_j) \tag{A.102}$$

to hold for all values of j, k, l. Here ρ is given by (A.98) and ρ^σ is defined in (2.25). A short calculation shows that (A.102) is equivalent to

$$G_{jl} \, [(-1)^{\delta_{k,\epsilon}\delta_{j,\epsilon}} - s_k \, (-1)^{\delta_{k,\epsilon}\delta_{l,\epsilon}}] = 0 \qquad \text{for all} \quad j,k,l \in \{\mathbf{1},\epsilon\} \,, \tag{A.103}$$

where s_k characterises the reversion σ on A, i.e. $s_{\mathbf{1}} = 1$ and $s_\epsilon = \pm 1$ as in (4.29). Thus every $g \in \mathrm{Hom}_A(M, M^\sigma)$ can be written as in (A.101) with

$$G = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ for } s_\epsilon = 1 \qquad \text{and} \qquad G = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \text{ for } s_\epsilon = -1 \,, \tag{A.104}$$

respectively, for some $a, b \in \mathbb{C}$. In order to apply proposition 2.16 the morphism g has to fulfill (2.59) for some ε_g. In terms of the matrix G one finds that this is equivalent to $G = \varepsilon_g G^{\mathrm{t}}$. This allows for three classes of solutions:

$$\begin{aligned} s_\epsilon = 1 : \qquad & G = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \varepsilon_g = 1 \,, \\ s_\epsilon = -1 : \qquad & G = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad \varepsilon_g = 1 \qquad \text{or} \qquad G = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad \varepsilon_g = -1 \,. \end{aligned} \tag{A.105}$$

We have now fulfilled the conditions to apply proposition 2.16 and compute the reversion $\tilde{\sigma}_g$ resulting from these three classes of g. In order to do so we need the inverse of the morphism g. We can expand $g^{-1} \in \mathrm{Hom}(M^\vee, M)$ in a basis similar to (A.101),

$$g^{-1} = \sum_{k,l} \tilde{G}_{k,l} \, (d_\sigma \otimes f_l) \circ ((f_k)^\vee \otimes \Upsilon^{(\sigma\sigma)\mathbf{1}}) \,, \tag{A.106}$$

with $\Upsilon^{(\sigma\sigma)\mathbf{1}} \in \mathrm{Hom}(\mathbf{1}, \sigma\otimes\sigma)$ the basis element dual to $\lambda_{(\sigma\sigma)\mathbf{1}}$, as in (I:2.29). The matrix $\tilde{G}$ is inverse to G up to a constant, $\tilde{G} = \sqrt{2}\, G^{-1}$.

The morphism $\tilde{\sigma}_g$ in (2.60) turns out to be independent of the actual values of $a, b \in \mathbb{C}^\times$ in (A.105), so that we may choose $a, b = 1$. With this choice we obtain $\tilde{G}_{kl} = \sqrt{2}\varepsilon_g G_{kl}$, for all three cases in (A.105). Evaluating (2.60), with some effort one finds

$$\begin{aligned} \bar{e}_{\mathbf{11}} \circ \tilde{\sigma}_g \circ e_{\mathbf{11}} &= (g_{\mathbf{11}})^2 \,, \qquad & \bar{e}_{\mathbf{11}} \circ \tilde{\sigma}_g \circ e_{\epsilon\epsilon} &= (g_{\epsilon\mathbf{1}})^2 \,, \\ \bar{e}_{\epsilon\epsilon} \circ \tilde{\sigma}_g \circ e_{\mathbf{11}} &= (g_{\mathbf{1}\epsilon})^2 \,, \qquad & \bar{e}_{\epsilon\epsilon} \circ \tilde{\sigma}_g \circ e_{\epsilon\epsilon} &= (g_{\epsilon\epsilon})^2 \end{aligned} \tag{A.107}$$

for the elements $\tilde{\sigma}_g(\mathbf{1})_\alpha^{\ \beta}$ and

$$\begin{aligned} \bar{e}_{\mathbf{1}\epsilon} \circ \tilde{\sigma}_g \circ e_{\mathbf{1}\epsilon} &= -\mathrm{i}\, g_{\mathbf{1}\epsilon}\, g_{\epsilon\mathbf{1}} \,, \qquad & \bar{e}_{\mathbf{1}\epsilon} \circ \tilde{\sigma}_g \circ e_{\epsilon\mathbf{1}} &= -2\mathrm{i}\, g_{\mathbf{11}}\, g_{\epsilon\epsilon} \,, \\ \bar{e}_{\epsilon\mathbf{1}} \circ \tilde{\sigma}_g \circ e_{\mathbf{1}\epsilon} &= -\tfrac{\mathrm{i}}{2}\, g_{\mathbf{11}}\, g_{\epsilon\epsilon} \,, \qquad & \bar{e}_{\epsilon\mathbf{1}} \circ \tilde{\sigma}_g \circ e_{\epsilon\mathbf{1}} &= -\mathrm{i}\, g_{\mathbf{1}\epsilon}\, g_{\epsilon\mathbf{1}} \end{aligned} \tag{A.108}$$

for the elements $\tilde{\sigma}_g(\varepsilon)_\alpha^{\ \beta}$. Substituting the three possible forms of g in (A.105), one finds that only the two cases with $s_\varepsilon = -1$ act as a permutation on B_{top}. Moreover, in these two cases the matrix $\tilde{\sigma}_g(\varepsilon)_\alpha^{\ \beta}$ reproduces precisely the two choices in (A.97).

In the case of the Ising model, the two reversions obtained in step 3b) are thus not new, in the sense that via proposition 2.16 they are both related to the reversion (4.29) on A with $s_\epsilon = -1$.

References

[I] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators I: Partition functions*, Nucl. Phys. B 646 (2002) 353

[J] E. Jandl, *Lichtung*, in: *Laut und Luise* (Reclam Verlag, Stuttgart 1996) (For translation(s) into English, see [85].)

[1] C.P. Burgess and T.R. Morris, *Open and unoriented strings à la Polyakov*, Nucl. Phys. B 291 (1987) 256

[2] C.G. Callan, C. Lovelace, C.R. Nappi, and S.A. Yost, *Adding holes and crosscaps to the superstring*, Nucl. Phys. B 293 (1987) 83

[3] J.P. Rodrigues, *Nonorientable one loop amplitudes for the bosonic open string: electrostatics on a Möbius strip*, J. Math. Phys. 28 (1987) 2269

[4] J. Polchinski and Y. Cai, *Consistency of open superstring theories*, Nucl. Phys. B 296 (1988) 91

[5] S.K. Blau, M. Clements, S. Della Pietra, S. Carlip, and V. Della Pietra, *The string amplitude on surfaces with boundaries and crosscaps*, Nucl. Phys. B 301 (1988) 285

[6] M. Bianchi and A. Sagnotti, *The partition function of the $SO(8192)$ bosonic string*, Phys. Lett. B 211 (1988) 407

[7] M. Bianchi and A. Sagnotti, *Open strings and the relative modular group*, Phys. Lett. B 231 (1989) 389

[8] P. Hořava, *Strings on world sheet orbifolds*, Nucl. Phys. B 327 (1989) 1989

[9] J.P. Rodrigues and A.J. van Tonder, *Spin structures for Riemann surfaces with boundaries and crosscaps*, Phys. Lett. B 217 (1989) 85

[10] M. Bianchi and A. Sagnotti, *Twist symmetry and open-string Wilson lines*, Nucl. Phys. B 361 (1991) 519

[11] D. Fioravanti, G. Pradisi, and A. Sagnotti, *Sewing constraints and non-orientable strings*, Phys. Lett. B 321 (1994) 349

[12] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and D-manifolds*, Phys. Rev. D 54 (1996) 1667

[13] M. Bianchi and A. Sagnotti, *On the systematics of open-string theories*, Phys. Lett. B 247 (1990) 517

[14] N. Ishibashi, *The boundary and crosscap states in conformal field theories*, Mod. Phys. Lett. A 4 (1989) 251

[15] T. Onogi and N. Ishibashi, *Conformal field theories on surfaces with boundaries and crosscaps*, Mod. Phys. Lett. A 4 (1989) 161 [*ibid.* 4 (1989) 885, *Erratum*]

[16] N. Marcus and A. Sagnotti, *Group theory from "quarks" at the ends of strings*, Phys. Lett. B 188 (1987) 58

[17] A. Sagnotti, *Open strings and their symmetry groups*, in: *Nonperturbative Quantum Field Theory*, G. 't Hooft, A. Jaffe, G. Mack, P.K. Mitter, and R. Stora, eds. (Plenum Press, New York 1988), p. 521

[18] M. Bianchi, G. Pradisi, and A. Sagnotti, *Planar duality in the discrete series*, Phys. Lett. B 273 (1991) 389

[19] G. Pradisi, A. Sagnotti, and Ya.S. Stanev, *Planar duality in $SU(2)$ WZW models*, Phys. Lett. B 354 (1995) 279

[20] G. Pradisi, A. Sagnotti, and Ya.S. Stanev, *The open descendants of non-diagonal $SU(2)$ WZW models*, Phys. Lett. B 356 (1995) 230

[21] G. Pradisi, A. Sagnotti, and Ya.S. Stanev, *Completeness conditions for boundary operators in 2D conformal field theory*, Phys. Lett. B 381 (1996) 97

[22] A. Sagnotti and Ya.S. Stanev, *Open descendants in conformal field theory*, Fortschritte d. Phys. 44 (1996) 585

[23] L.R. Huiszoon, A.N. Schellekens, and N. Sousa, *Klein bottles and simple currents*, Phys. Lett. B 470 (1999) 95

[24] L.R. Huiszoon and A.N. Schellekens, *Crosscaps, boundaries and T-Duality*, Nucl. Phys. B 583 (2000) 705

[25] L.R. Huiszoon, A.N. Schellekens, and N. Sousa, *Open descendants of non-diagonal invariants*, Nucl. Phys. B 575 (2000) 401

[26] A.N. Schellekens, *Open strings, simple currents and fixed points*, preprint hep-th/0001198

[27] L.R. Huiszoon, *D-branes and O-planes in string theory – an algebraic approach*, Ph.D. thesis (Amsterdam 2002)

[28] N. Sousa and A.N. Schellekens, *Orientation matters for NIMreps*, Nucl. Phys. B 653 (2003) 339

[29] J. Fuchs and C. Schweigert, *Orbifold analysis of broken bulk symmetries*, Phys. Lett. B 447 (1999) 266

[30] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries I. General theory*, Nucl. Phys. B 558 (1999) 419

[31] J. Fuchs, L.R. Huiszoon, A.N. Schellekens, C. Schweigert, and J. Walcher, *Boundaries, crosscaps and simple currents*, Phys. Lett. B 495 (2000) 427

[32] Ya.S. Stanev, *Two dimensional conformal field theory on open and unoriented surfaces*, preprint hep-th/0112222

[33] C. Angelantonj and A. Sagnotti, *Open strings*, Phys. Rep. 371 (2002) 1

[34] L.A. Borisov, M.B. Halpern, and C. Schweigert, *Systematic approach to cyclic orbifolds*, Int. J. Mod. Phys. A 13 (1998) 125

[35] T. Gannon, *Integers in the open string*, Phys. Lett. B 473 (2000) 80

[36] P. Bantay, *Frobenius–Schur indicators, the Klein-bottle amplitude, and the principle of orbifold covariance*, Phys. Lett. B 488 (2000) 207

[37] I. Brunner and K. Hori, *Notes on orientifolds of rational conformal field theories*, preprint hep-th/0208141

[38] A.N. Schellekens and Ya.S. Stanev, *Trace formulas for annuli*, J. High Energy Phys. 0112 (2001) 012

[39] S.-i. Yamaguchi, *The critical Ising model on a Möbius strip*, Progr. Theor. Phys. 97 (1997) 703

[40] A.N. Schellekens and N. Sousa, *Open descendents of* $U(2N)$ *orbifolds at rational radii*, Int. J. Mod. Phys. A 16 (2001) 3659

[41] I. Brunner, *On orientifolds of WZW models and their relation to geometry*, J. High Energy Phys. 0201 (2002) 007

[42] L.R. Huiszoon, K. Schalm, and A.N. Schellekens, *Geometry of WZW orientifolds*, Nucl. Phys. B 624 (2002) 219

[43] C. Bachas, N. Couchoud, and P. Windey, *Orientifolds of the 3-sphere*, J. High Energy Phys. 0112 (2001) 003

[44] M.B. Halpern and C. Helfgott, *Twisted open strings from closed strings: The WZW orientation orbifolds*, preprint hep-th/0306014

[45] Y. Hikida, *Orientifolds of SU(2)/U(1) WZW models*, J. High Energy Phys. 0211 (2002) 035

[46] Z. Kadar, *The torus and the Klein bottle amplitude of permutation orbifolds*, Phys. Lett. B 484 (2000) 289

[47] I. Brunner and K. Hori, *Orientifolds and mirror symmetry*, preprint hep-th/0303135

[48] L.R. Huiszoon and K. Schalm, *BPS orientifold planes from crosscap states in Calabi–Yau compactifications*, preprint hep-th/0306091

[49] J. Fuchs, I. Runkel, and C. Schweigert, *Conformal correlation functions, Frobenius algebras and triangulations*, Nucl. Phys. B 624 (2002) 452

[50] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert, *Conformal boundary conditions and three-dimensional topological field theory*, Phys. Rev. Lett. 84 (2000) 1659

[51] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert, *Correlation functions and boundary conditions in RCFT and three-dimensional topology*, Compos. Math. 131 (2002) 189

[52] P. Hořava, *Chern–Simons gauge theory on orbifolds: open strings from three dimensions*, J. Geom. and Phys. 21 (1996) 1

[53] V. Karimipour and A. Mostafazadeh, *Lattice topological field theory on non-orientable surfaces*, J. Math. Phys. 38 (1997) 49

[54] A. Alexeevski and S. Natanzon, *Non-commutative extensions of two-dimensional topological field theories and Hurwitz numbers for real algebraic curves*, preprint math.GT/0202164

[55] A.A. Kirillov, *On an inner product in modular tensor categories*, J. Amer. Math. Soc. 9 (1996) 1135

[56] B. Bakalov and A.A. Kirillov, *Lectures on Tensor Categories and Modular Functors* (American Mathematical Society, Providence 2001)

[57] J. Fuchs and C. Schweigert, *Category theory for conformal boundary conditions*, Fields Institute Commun. 39 (2003) 25

[58] B. Pareigis, *Non-additive ring and module theory I. General theory of monoids*, Publ. Math. Debrecen 24 (1977) 189

[59] J. Bernstein, *Tensor categories*, preprint q-alg/9501032

[60] V.G. Turaev, *Quantum Invariants of Knots and 3-Manifolds* (de Gruyter, New York 1994)

[61] G. Pradisi, *Open superstrings*, Nuovo Cim. 112B (1997) 467

[62] M. Oshikawa and I. Affleck, *Defect lines in the Ising model and boundary states on orbifolds*, Phys. Rev. Lett. 77 (1996) 2604

[63] M. Oshikawa and I. Affleck, *Boundary conformal field theory approach to the critical two-dimensional Ising model with a defect line*, Nucl. Phys. B 495 (1997) 533

[64] R.M. Konik and A. LeClair, *Purely transmitting defect field theories*, Nucl. Phys. B 538 (1999) 587

[65] A. LeClair and A.W.W. Ludwig, *Minimal models with integrable local defects*, Nucl. Phys. B 549 (1999) 546

[66] M. Jeng and A.W.W. Ludwig, *Random defect lines in conformal minimal models*, Nucl. Phys. B 594 (2001) 685

[67] G.M.T. Watts, *On the boundary Ising model with disorder operators*, Nucl. Phys. B 596 (2001) 513

[68] V.B. Petkova and J.-B. Zuber, *Generalised twisted partition functions*, Phys. Lett. B 504 (2001) 157

[69] V.B. Petkova and J.-B. Zuber, *Conformal boundary conditions and what they teach us*, in: *Non-perturbative QFT Methods and Their Applications*, Z. Horváth and L. Palla, eds. (World Scientific, Singapore 2001), p. 1

[70] R. Coquereaux and G. Schieber, *Twisted partition functions for ADE boundary conformal field theories and Ocneanu algebras of quantum symmetries*, J. Geom. and Phys. 42 (2002) 216

[71] C. Bachas, J. de Boer, R. Dijkgraaf, and H. Ooguri, *Permeable conformal walls and holography*, J. High Energy Phys. 0206 (2002) 027

[72] T. Quella and V. Schomerus, *Symmetry breaking boundary states and defect lines*, J. High Energy Phys. 0206 (2002) 028

[73] R. Coquereaux and M. Huerta, *Torus structure on graphs and twisted partition functions for minimal and affine models*, J. Geom. and Phys. 48 (2003) 580

[74] C.H.O. Chui, C. Mercat, W.P. Orrick, and P.A. Pearce, *Integrable lattice realizations of conformal twisted boundary conditions*, Phys. Lett. B 517 (2001) 429

[75] U. Grimm, *Spectrum of a duality-twisted Ising quantum chain*, J. Phys. A 35 (2002) L25

[76] C.H.O. Chui, C. Mercat, and P.A. Pearce, *Integrable lattice realizations of conformal twisted boundary conditions*, J. Phys. A 36 (2003) 2623

[77] W. Scharlau, *Quadratic and Hermitian Forms* (Springer Verlag, New York 1985)

[78] B. Farb and R.K. Dennis, *Noncommutative Algebra* (Springer Verlag, New York 1993)

[79] P. di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory* (Springer Verlag, New York 1996)

[80] K. Kaneda, and Y. Okabe, *Finite-size scaling for the Ising model on the Möbius strip and the Klein bottle*, Phys. Rev. Lett. 86 (2001) 2134

[81] W.T. Lu and F.Y. Wu, *Ising model on nonorientable surfaces: Exact solution for the Möbius strip and the Klein bottle*, Phys. Rev. E 63 (2001) 026107

[82] C.H. Otto Chui and P.A. Pearce, *Finitized conformal spectra of the Ising model on the Klein bottle and Möbius strip*, J. Stat. Phys. 107 (2002) 1167

[83] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Correspondences of ribbon categories*, preprint math.CT/0309465

[84] V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups 8 (2003) 177

[85] R. Waldrop (ed.), *Reft and Light: Poems by Ernst Jandl With Multiple Versions by American Poets* (Burning Deck Press, Providence 2000)