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CORRESPONDENCES OF RIBBON CATEGORIES

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Abstract

Much of algebra and representation theory can be formulated in the general framework of tensor categories. The aim of this paper is to further develop this theory for braided tensor categories. Several results are established that do not have a substantial counterpart for symmetric tensor categories. In particular, we exhibit various equivalences involving categories of modules over algebras in ribbon categories. Finally we establish a correspondence of ribbon categories that can be applied to, and is in fact motivated by, the coset construction in conformal quantum field theory.

Contents

1 Introduction

In this paper we study equivalences involving categories of modules over algebras in ribbon categories. Our main results are Theorem 5.20 and Theorem 7.6. To motivate these results and clarify their relevance, we start by looking at a classical analogue: correspondences.

1.1 Correspondences

Correspondences are often needed to express relations between mathematical objects of the same type. For instance, in algebraic geometry they enter in the definition of rational maps. A more recent application is the construction of an action of the Heisenberg algebra on the cohomology of Hilbert schemes of points on surfaces. In the present paper, we introduce a generalisation of correspondences in the setting of braided tensor categories, which turns out to provide a powerful tool for the study of such categories.

Correspondences deal with classes of mathematical objects for which a Cartesian product is defined. For definiteness, let us consider finite groups. A correspondence of two groups G_1 and G_2 is a subgroup R of the product group $G_1 \times G_2$,

$$R \leq G_1 \times G_2 \,. \tag{1.1}$$

Suppose now that the representation theories of the groups G_1 and R are known. One could then be tempted to formulate the following dream: A correspondence (1.1) might allow us to express the category $\mathcal{R}ep(G_2)$ of (finite-dimensional complex) representations of G_2 in terms of the representation categories $\mathcal{R}ep(G_1)$ and $\mathcal{R}ep(R)$.

Obviously, in this generality our dream is entirely unrealistic – just take G_1 and R to be trivial. To assess the feasibility of the dream in more general categories than representation categories of finite groups, it is helpful to reformulate the correspondence (1.1) in the spirit of the Tannaka-Krein philosophy, i.e. to express statements about groups entirely in terms of their representation categories rather than in terms of the groups themselves. One advantage of this point of view is the following. Once the statements are translated to a category-theoretic setup, one can try to relax some of the properties of the representation category so as to arrive at analogous statements applying to categories that appear in other contexts, e.g. as representation categories of quantum groups, of vertex algebras, or of precosheaves of von Neumann algebras, and that, in turn, have important applications in quantum field theory.

Our starting point, i.e. the correspondence (1.1), is easily reformulated in category-theoretic language. The representation category of the product group is simply the product of the two representation categories, $\mathcal{R}ep(G_1{\times}G_2) \cong \mathcal{R}ep(G_1) \boxtimes \mathcal{R}ep(G_2)$.[1] The correspondence R is, by definition, a subgroup of $G_1 \times G_2$; a category-theoretic analogue of the notion of subgroup is known ([26]; for earlier discussions compare also [46, 36]): There is a bijection between subgroups H of a group G and commutative algebras in the tensor category $\mathcal{R}ep(G)$. The commutative algebra in $\mathcal{R}ep(G)$ that is associated to H is given by the

[1] For a precise definition of the relevant notion of product tensor category, see Section 6.1.

space $\mathbb{C}(G/H)$ of functions on the homogeneous space G/H. The category $\mathcal{R}ep(G)_{\mathbb{C}(G/H)}$ of $\mathbb{C}(G/H)$-modules in $\mathcal{R}ep(G)$ is equivalent to $\mathcal{R}ep(H)$,

$$\mathcal{R}ep(G)_{\mathbb{C}(G/H)} \cong \mathcal{R}ep(H)\,. \tag{1.2}$$

Our dream can thus be stated more precisely as follows. Suppose we are given two tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$ and a commutative semisimple algebra A_R in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$. Denote by $\mathcal{C}_R$ an appropriate tensor category of A_R-modules. Then we might attempt to express $\mathcal{C}_2$ in terms of $\mathcal{C}_1$ and $\mathcal{C}_R$, as the category of modules over a commutative algebra B in a tensor category $\mathcal{C}$ that is derived from $\mathcal{C}_1$ and $\mathcal{C}_R$ only.

In the particular case that G_1 and R are trivial, our dream would amount to the statement that $\mathcal{R}ep(G_2)$ is equivalent to the representation category of a commutative semisimple algebra over $\mathbb{C}$, which clearly cannot be true for any non-abelian group G_2. More explicitly, in this specific situation the data involved in the correspondence are, in category-theoretic language, the tensor category $\mathcal{R}ep(G_2)$ and the commutative algebra $\mathbb{C}(G_2)$ of functions on G_2, seen as an algebra in $\mathcal{R}ep(G_2)$. Since all irreducible representations of G_2 appear as subrepresentations of $\mathbb{C}(G_2)$, this algebra has trivial representation theory:

$$\mathcal{R}ep(G_2)_{\mathbb{C}(G_2)} \cong \mathcal{V}ect_{\mathbb{C}}\,. \tag{1.3}$$

It is therefore all the more remarkable that there do exist situations in which our dream *can* be realised. It involves a generalisation of algebra and representation theory to tensor categories that are not necessarily symmetric, but are still braided. Among such categories there are, in particular, the modular tensor categories. The interest in modular tensor categories comes e.g. from the fact that such a category contains the data needed for the construction of a three-dimensional topological quantum field theory. These categories arise in many interesting applications; for example, the representation categories of certain vertex algebras are modular tensor categories.

Modular tensor categories are distinguished by a non-degeneracy property of the braiding; in particular, the braiding is "maximally non-symmetric". This makes it apprehensible that in contrast to the classical case above, in which all involved tensor categories are symmetric, such categories can indeed provide a realisation of our dream.

1.2 Frobenius algebras

Many aspects of the representation theory of rings or algebras can be generalised to the general setting of tensor categories [38]. In any tensor category one has the notions of an associative algebra with unit and its modules and bimodules. Similarly one can define coalgebras. A particularly interesting class are algebras A that are also coalgebras such that the coproduct is a bimodule morphism from A to the A-bimodule $A \otimes A$. Such algebras are called *Frobenius algebras*, because Frobenius algebras in the modular tensor category of finite-dimensional vector spaces over some field are just ordinary Frobenius algebras. Frobenius algebras in more general tensor categories have recently attracted attention in several different contexts (see e.g. [26, 20, 17, 37, 34, 18]).

In contrast to bialgebras (such as Hopf algebras), Frobenius algebras can be defined in tensor categories that are not necessarily braided. In a braided category, however, their theory becomes much richer. One then has the notion of a commutative algebra and, more generally, of center(s) of an algebra. The present paper aims at developing the theory of Frobenius algebras in such a setting. It turns out to be helpful to impose a few additional requirements, both on the algebra and on the category. In particular, we assume that the braided tensor category in question is additive, $\Bbbk$-linear (with $\Bbbk$ some field), as well as sovereign – it has a left and a right duality that coincide as functors from $\mathcal{C}$ to the opposed category; a braided sovereign tensor category is also known as a *ribbon* category. Other requirements imposed on the category will be given in the body of the paper; the setting is summarised in declaration 2.10.

The additional properties of the algebra are that it is a *special* and *symmetric* Frobenius algebra, see definition 2.22. (To ensure the existence of various images needed in our constructions, we also assume that the algebra is what we call centrally split, see definition 3.1 and declaration 3.2.) Symmetric Frobenius algebras in the category of vector spaces appear e.g. in the study of group algebras and thus play a central role in representation theory. It is worth noting that in a braided setting, a commutative Frobenius algebra is not necessarily symmetric. The specialness property of the Frobenius algebra A implies, in particular, [26, 20] that when the category $\mathcal{C}$ is semisimple then the category of left A-modules is semisimple as well.

1.3 Local modules and local induction

In this paper we study symmetric special Frobenius algebras A in ribbon categories $\mathcal{C}$. Given such an algebra, there are three other categories one should consider: The category $\mathcal{C}_A$ of left A-modules, the analogous category of right A-modules, and the category $\mathcal{C}_{A|A}$ of A-bimodules. The tensor product $B_1 \otimes_A B_2$ of bimodules endows $\mathcal{C}_{A|A}$ with the structure of a tensor category.

The braiding of $\mathcal{C}$ allows to construct two tensor functors [28, 37]

$$\alpha_A^{\pm}: \quad \mathcal{C} \rightarrow \mathcal{C}_{A|A}\,, \tag{1.4}$$

known as α-induction (see Definition 2.21). In Definition 3.3 we introduce two endofunctors

$$E_A^{l/r}: \quad \mathcal{C} \rightarrow \mathcal{C}\,. \tag{1.5}$$

We show in Proposition 3.6 that if right-adjoint functors $(\alpha^{\pm})^{\dagger}$ to (1.4) exist, then the endofunctors (1.5) are the compositions $E_A^l = (\alpha_A^+)^{\dagger} \circ \alpha_A^-$ and $E_A^r = (\alpha_A^-)^{\dagger} \circ \alpha_A^+$. For commutative algebras, the two functors $E_A^{l/r}$ coincide (see Proposition 3.8 (iv)); in this case we suppress the index l or r.

A basic principle in this paper is to try to lift a given functor F: $\mathcal{C} \rightarrow \mathcal{D}$ between two tensor categories $\mathcal{C}$ and $\mathcal{D}$ to a functor from the category $\mathcal{C}$-*Alg* of algebras in $\mathcal{C}$ to the category $\mathcal{D}$-*Alg* of algebras in $\mathcal{D}$, or even to a functor between the respective categories $\mathcal{C}$-*Frob* and $\mathcal{D}$-*Frob* of Frobenius algebras. For the functors $E_A^{l/r}$ both lifts turns out to be possible. This result, established in Proposition 3.8(i), is non-trivial because $E_A^{l/r}$ are not necessarily tensor functors. By abuse of notation, we use the same symbol for the resulting endofunctors of $\mathcal{C}$-*Alg* and of $\mathcal{C}$-*Frob* as for the underlying endofunctors of $\mathcal{C}$, i.e. we write

$$E_A^{l/r}: \quad \mathcal{C}\text{-}Alg \to \mathcal{C}\text{-}Alg \tag{1.6}$$

as well as $E_A^{l/r}$: $\mathcal{C}$-*Frob* $\to \mathcal{C}$-*Frob*.

The images of the endofunctors (1.5) carry additional structure. To describe it we need two additional ingredients: a braided version of the concept of the center of an algebra and the concept of local modules. First, the braiding allows one to generalise the notion of a center of an algebra A, and for a general braiding one obtains in fact two different centers $C_l(A)$ and $C_r(A)$, known as the *left* and the *right center* of A, respectively. After adapting, in Definition 2.31, their description in [45, 37] to the present setting, we show in Proposition 2.37 that the centers of a symmetric special Frobenius algebra carry the structure of commutative symmetric Frobenius algebras. In a braided category there is also a notion of the tensor product $A \otimes B$ of two algebras A and B. It enters e.g. in the definition [45] of the Brauer group of the category. Remarkably, in the braided setting the tensor product of two commutative algebras is not necessarily commutative. (Thus it is not natural to restrict one's attention to commutative algebras.) In Proposition 3.14(i) we compute the centers of $A \otimes B$; they can be expressed in terms of the endofunctors (1.6), namely

$$C_l(A{\otimes}B) \cong E_A^l(C_l(B)) \qquad \text{and} \qquad C_r(A{\otimes}B) \cong E_B^r(C_r(A)) \tag{1.7}$$

as Frobenius algebras.

The category of left modules over a *commutative* algebra A in $\mathcal{V}ect_{\mathbb{C}}$ is again a tensor category. In order to generalise this fact to a braided setting, a refinement is necessary, and this refinement makes use of the second ingredient of our construction – the concept of *dyslectic* [40] or *local* module. A module M over a commutative special Frobenius algebra A in a ribbon category is local iff the representation morphism commutes with the twist (see Proposition 3.17), so that the twist on M is a morphism in $\mathcal{C}_A$. The resulting generalisation of the classical statement is given in Proposition 3.21, which follows [40] and [26]: The category $\mathcal{C}_A$ of left modules over a commutative symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ has a natural full subcategory – the category $\mathcal{C}_A^{\ell oc}$ of local left A-modules – that is again a tensor category, and in fact, unlike e.g. the category of A-bimodules, even a ribbon category.

With these results at hand, we proceed to show, in Proposition 4.1, that every object in the image of the endofunctors $E_A^{l/r}$ has a natural structure of a local $C_l(A)$-module, respectively of a local $C_r(A)$-module. Thus the functors $E_A^{l/r}$ give rise to two functors

$$\ell\text{-Ind}_A^{l/r}: \quad \mathcal{C} \to \mathcal{C}_{C_{l/r}(A)}^{\ell oc}, \tag{1.8}$$

which we call *local induction* functors (Definition 4.3). (Again, for commutative algebras, the two functors coincide, and we shall then suppress the index l or r, i.e. just write ℓ-Ind$_A$.)

However, in contrast to ordinary induction, local induction is not a tensor functor. In the tensor categories $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$ we have the notion of an algebra; it turns out (Proposition 4.14) that the local induction functors can be extended to functors between categories of algebras, too, i.e. (again abusing notation)

$$\ell\text{-Ind}_A^{l/r}: \quad \mathcal{C}\text{-}Alg \rightarrow \mathcal{C}^{\ell oc}_{C_{l/r}(A)}\text{-}Alg\,. \tag{1.9}$$

All this structure enters the following result about successive local inductions. Let A and B be two commutative symmetric special Frobenius algebras in $\mathcal{C}$. Then $\mathcal{C}_A^{\ell oc}$ is again a tensor category, and $\ell\text{-Ind}_A(B)$ is a commutative algebra in that category. It thus makes sense to consider the tensor category of local $\ell\text{-Ind}_A(B)$-modules in $\mathcal{C}_A^{\ell oc}$. In Proposition 4.16 we show that this category can also be obtained as the category of local modules over some commutative algebra in $\mathcal{C}$, and that this algebra is in fact just $E_A(B)$:

$$(\mathcal{C}_A^{\ell oc})^{\ell oc}_{\ell\text{-Ind}_A(B)} \cong \mathcal{C}^{\ell oc}_{E_A(B)}\,. \tag{1.10}$$

If in addition A is simple and $E_A(B)$ is special, then this is even an equivalence of ribbon categories. (An algebra is called simple iff it is simple as a bimodule over itself, see Definition 2.26.)

The next statement – Theorem 5.20 – is the first main result of this paper: Provided that the left and right centers $C_l(A)$ and $C_r(A)$ of a symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ (which are symmetric Frobenius by Proposition 2.37) are also special, the categories of local modules over $C_l(A)$ and $C_r(A)$ are equivalent as ribbon categories,

$$\mathcal{C}^{\ell oc}_{C_l(A)} \cong \mathcal{C}^{\ell oc}_{C_r(A)}\,. \tag{1.11}$$

Moreover, there is in addition a ribbon equivalence of these categories to a certain subcategory of α-induced A-bimodules, the category $\mathcal{C}^0_{A|A}$ of *ambichiral* A-bimodules, introduced in Definition 5.6.

The equivalence (1.11) can, in general, not be extended to an equivalence of the respective categories of all modules (as module categories over $\mathcal{C}$) – the left center and the right center are not necessarily Morita equivalent.

It is instructive to see how the results (1.7), (1.10) and (1.11) simplify for a symmetric tensor category $\mathcal{C}$, in which the braiding obeys $c_{U,V}^{-1} = c_{V,U}$, for all objects U, V. This includes in particular the 'classical' situation that $\mathcal{C}$ is the category $\mathcal{V}ect_{\Bbbk}$ of finite-dimensional vector spaces over a field $\Bbbk$, as well as the category of finite-dimensional super vector spaces. In this case, the notions of left and right center coincide, there is only a single center $C(A)$. The relations (1.7) then reduce to the statement that the center of the tensor product of two algebras is the tensor product of the centers, $C(A{\otimes}B) \cong C(A) \otimes C(B)$.

Furthermore, in a symmetric tensor category *all* modules over a commutative special Frobenius algebra are local. The result (1.10) thus simplifies to a simple statement about the induction with respect to the tensor product of two commutative algebras A and B: $(\mathcal{C}_A)_{\text{Ind}_A(B)} \cong \mathcal{C}_{A\otimes B}$.

Finally, there is only a single α-induction $\alpha_A = \alpha_A^+ = \alpha_A^-$, and the two endofunctors $E_A^{l/r}$ of $\mathcal{C}$ just amount to tensoring objects with $C(A)$ and morphisms with $id_{C(A)}$. The two functors $\ell\text{-Ind}_A^{l/r}$ coincide as well, and are induction to modules over $C(A)$. Therefore, in a symmetric tensor category, our first main result (1.11) becomes a tautology – in other words, (1.11) is a theorem of 'braided algebra' without substantial classical analogue.

1.4 Correspondences and the trivialisation of ribbon categories

Before we can discuss the category-theoretic generalisation of correspondences, we must still find an appropriate generalisation to the braided setting of the relation (1.3), i.e. of the fact that the category $\mathcal{R}ep(G)$ of representations of a group G contains a commutative special symmetric Frobenius algebra $A = \mathbb{C}(G)$ such that $\mathcal{R}ep(G)_A^{\ell oc} \cong \mathcal{V}ect_{\mathbb{C}}$. We call an algebra A in $\mathcal{C}$ with the property that $\mathcal{C}_A^{\ell oc} \cong \mathcal{V}ect_{\Bbbk}$ a *trivialising algebra* for $\mathcal{C}$.

Requiring the existence of a trivialising algebra is too restrictive for the applications we have in mind. We rather need the following more general concept (Definition 6.4): We call a ($\Bbbk$-linear) ribbon category $\mathcal{C}$ *trivialisable* iff there exist a ribbon category $\mathcal{C}'$ and a commutative symmetric special Frobenius algebra T in $\mathcal{C} \boxtimes \mathcal{C}'$ such that the category of local T-modules is trivial,

$$(\mathcal{C}\boxtimes\mathcal{C}')_T^{\ell oc} \;\cong\; \mathcal{V}ect_{\Bbbk} \,. \tag{1.12}$$

An important class of braided tensor categories are the modular tensor categories, which play a key role in various applications. In Proposition 6.23 we show that every modular tensor category is trivialisable, with $\mathcal{C}' = \overline{\mathcal{C}}$ the tensor category dual to $\mathcal{C}$.

Combining all these results finally allows us to obtain a category-theoretic generalisation of the correspondence (1.1). Suppose that a ribbon category $\mathcal{C}_3$ is equivalent to the category of local A-modules in the product of two ribbon categories $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e. that the correspondence takes the form

$$\mathcal{C}_3 \cong (\mathcal{C}_1\boxtimes\mathcal{C}_2)_A^{\ell oc} \,, \tag{1.13}$$

where $\mathcal{C}_2$ is trivialisable with trivialising algebra T in $\mathcal{C}_2\boxtimes\mathcal{C}_2'$. The dream spelt out in the beginning then amounts to expressing $\mathcal{C}_1$ as the category of local modules over a commutative special Frobenius algebra in $\mathcal{C}_3 \boxtimes \mathcal{C}_2'$. We shall indeed show (Proposition 7.1) that, quite generally, it is possible to express a category of local modules over a certain commutative algebra in $\mathcal{C}_1$ in terms of $\mathcal{C}_3$ and $\mathcal{C}_2'$:

$$(\mathcal{C}_1)^{\ell oc}_{\ell\text{-Ind}_{\mathbf{1}\otimes T}(A\otimes\mathbf{1})} \;\cong\; (\mathcal{C}_3\boxtimes\mathcal{C}_2')^{\ell oc}_{\ell\text{-Ind}_{A\otimes\mathbf{1}}(\mathbf{1}\otimes T)} \,. \tag{1.14}$$

($A\otimes\mathbf{1}$ and $\mathbf{1}\otimes T$ are algebras in $\mathcal{C}_1\boxtimes\mathcal{C}_2\boxtimes\mathcal{C}_2'$, and $\mathbf{1}$ denotes the tensor unit of the respective category; the product $\boxtimes$ of tensor categories is associative, see Remark 6.6.)

Moreover, the situation simplifies considerably when we make the following restrictions. First, we demand that the category $\mathcal{C}_2$ is modular; second, we require that the algebra A

in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ has the property that the only subobject of A of the form $U \times \mathbf{1}$ is $\mathbf{1} \times \mathbf{1}$. Then the commutative algebra $\ell\text{-Ind}_{\mathbf{1}\otimes T}(A\otimes\mathbf{1})$ in $\mathcal{C}_1$ is the tensor unit, so that (1.14) reduces to

$$\mathcal{C}_1 \;\cong\; (\mathcal{C}_3 \boxtimes \overline{\mathcal{C}}_2)_B^{\ell oc} \tag{1.15}$$

with $B = \ell\text{-Ind}_{A\otimes\mathbf{1}}(\mathbf{1} \otimes T)$. This result – Theorem 7.6 – is arguably the strongest possible realisation of our dream. We stress that only in a braided setting such an effect can happen: It is the non-triviality of the braiding that is responsible for getting the *locally* induced algebra $\ell\text{-Ind}_{\mathbf{1}\otimes T}(A \otimes \mathbf{1})$ so small.

1.5 Applications in quantum field theory

$\mathbb{C}$-linear tensor categories have played a prominent role in quantum field theory, especially in connection with the general analysis of superselection rules and of quantum statistics [12, 13]. In two- and three-dimensional quantum field theory they have become an indispensable tool for studying braid statistics and quantum symmetries. The analysis presented in this paper is primarily inspired by problems in two-dimensional conformal field theory and string theory and has grown out of the results presented in [17, 18]. Concrete applications of our results, in particular of (1.15), to conformal field theory form the subject of a forthcoming paper. Here we just give an indication of what some of these applications consist in.

First consider the case that $\mathcal{C}_3$ is trivial, $\mathcal{C}_3 \cong \mathcal{V}ect_{\mathbb{C}}$. This case can e.g. be realised through certain conformal embeddings of directs sums $\hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_2$ of untwisted affine Lie algebras into an untwisted affine Lie algebra $\hat{\mathfrak{g}}$. The relevant tensor categories are the categories $\mathcal{C}_i = \mathcal{C}(\mathfrak{g}_i, k_i)$ of integrable representations of the affine Lie algebras $\hat{\mathfrak{g}}_1$ and $\hat{\mathfrak{g}}_2$ with specified values $k_{1,2}$ of the level; as representations for $\hat{\mathfrak{g}}$ one must take the integrable representations at some level k, and require that the category of those representations is equivalent to $\mathcal{V}ect_{\mathbb{C}}$, which is the case for $\mathfrak{g} = E_8$ at level $k = 1$. These are modular tensor categories, and the embedding of $\hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_2$ into $\hat{\mathfrak{g}}$ provides us with a simple commutative symmetric special Frobenius algebra A in their product $\mathcal{C}_1 \boxtimes \mathcal{C}_2$. Our result (1.14) then asserts that a category of local modules in $\mathcal{C}_1$ is equivalent to a category of local modules in the category $\overline{\mathcal{C}}_2$ dual to $\mathcal{C}_2$. If, in addition, the conditions are met that the only subobject of A of the form $U \times \mathbf{1}$ is $\mathbf{1} \times \mathbf{1}$ and the only subobject of A of the form $\mathbf{1} \times U$ is $\mathbf{1} \times \mathbf{1}$, then the categories $\mathcal{C}_1$ and $\overline{\mathcal{C}}_2$ are equivalent; this happens e.g. for those conformal embeddings in $\hat{\mathfrak{g}} = E_8^{(1)}$ for which

$$(\mathfrak{g}_1, \mathfrak{g}_2) \;=\; (A_2, E_6) \quad \text{or} \quad (A_1, E_7) \quad \text{or} \quad (F_4, G_2) \tag{1.16}$$

and, in each case, $k_1 = k_2 = 1$. The corresponding equivalences of modular tensor categories are known. On the other hand, the two conditions are not met for the conformal embedding into $E_8^{(1)}$ of $A_2^{(1)} \oplus A_1^{(1)}$ with $k_1 = 6$ and $k_2 = 16$. In this case only categories of local modules, in fact so-called simple current extensions, for the two categories are equivalent.

The second application we have in mind concerns coset conformal field theories. In these theories one starts from the representation categories $\mathcal{C}(\mathfrak{g}, k)$ and $\mathcal{C}(\mathfrak{h}, k')$ for a pair of untwisted affine Lie algebras for which $\mathfrak{h} \subset \mathfrak{g}$, and desires to understand the representation category of the commutant of the conformal vertex algebra associated to $(\mathfrak{h}, k')$ in the conformal vertex algebra associated to $(\mathfrak{g}, k)$. The results of the present paper will form an essential ingredient of a universal description of these representation categories, including, in particular, the so-called maverick coset theories. A discussion of this application is beyond the scope of this introduction; it will appear in a separate publication.

1.6 Relation to earlier work

The methods and results of this paper owe much to work that has been done within two lines of development: the study of algebras in tensor categories, and alpha induction for nets of subfactors. Algebras in symmetric tensor categories already played an important role in Deligne's characterisation of Tannakian categories (see e.g. [41, 10]). They were studied in much detail by Pareigis (see e.g. [38, 39]), who also introduced the concept of local (dyslectic) modules of a commutative algebra in a braided tensor category [40]. More recently, commutative algebra and local modules in semisimple braided tensor categories were e.g. studied in the context of conformal field theory and quantum subgroups in [26], in relation to weak Hopf algebras in [37], and in connection with Morita equivalence for tensor categories in [34]. The algebras relevant in the conformal field theory context are symmetric special Frobenius algebras [20, 17, 18]; those encoding properties of conformal field theory on surfaces with boundary are, generically, non-commutative. It is also worth mentioning that while bi- or Hopf algebras in braided tensor categories (for a review, see [31]) do not play a role in this context, they are indeed important for other applications in quantum field theory, see e.g. [25].

The concept of α-induction (see Definition 2.21) was invented in [28] in the framework of the C^*-algebraic approach to quantum field theory (see e.g. [12, 13]). α-induction was further developed in [47] and in a series of papers by Böckenhauer, Evans and Kawahigashi (see e.g. [5, 7, 8, 6]), in particular applying it to the construction of subfactors associated to modular invariants, and it was formulated in purely categorical form (and, unlike in the quantum field theory and subfactor context, without requiring that one deals with a *-category) in [37]. Also in the study of subfactors Frobenius algebras arise naturally, in the guise of 'Q-systems' [27, 29]. Indeed, every Q-system is a symmetric special *-Frobenius algebra [15], and the product and coproduct, and unit and counit, respectively, are *'s of each other (the Frobenius property can then in fact be derived from the other properties). For instance, the trivialising algebra defined in Lemma 6.19 corresponds to the Q-system that is associated to the canonical endomorphism of a subfactor, see Proposition 4.10 of [28].

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2 Algebras in tensor categories

2.1 Tensor categories

Let $\mathcal{C}$ be a category. We denote the class of its objects by $\mathrm{Obj}(\mathcal{C})$ and the morphism sets by $\mathrm{Hom}(U,V)$, for U,V in $\mathrm{Obj}(\mathcal{C})$; we will often abbreviate endomorphism sets $\mathrm{Hom}(U,U)$ by $\mathrm{End}(U)$. In this paper we will be concerned with categories that come with the following additional structure. First, they are small ($\mathrm{Obj}(\mathcal{C})$ is a set), they are additive (so that, in particular, they have direct sums) and their morphism sets are vector spaces over the ground field $\Bbbk$. Second, most often they are tensor categories. By invoking the coherence theorems, tensor categories will be assumed to be strict; we denote the associative tensor product by '$\otimes$', both for objects and for morphisms, and the tensor unit by $\mathbf{1}$. Third, most of the categories we will be interested in are *ribbon* categories; this includes as a special subclass the *modular* tensor categories.

Definition 2.1:
A *ribbon category* is a tensor category with the following additional structure. To every object $U\in\mathrm{Obj}(\mathcal{C})$ one assigns an object $U^\vee\in\mathrm{Obj}(\mathcal{C})$, called the (right-) dual of U, and there are three families of morphisms, [2]

$$\begin{array}{lll} \text{(Right-) Duality:} & b_U \in \mathrm{Hom}(\mathbf{1}, U\otimes U^\vee)\,, & d_U \in \mathrm{Hom}(U^\vee\otimes U, \mathbf{1})\,, \\ \text{Braiding :} & c_{U,V} \in \mathrm{Hom}(U\otimes V, V\otimes U)\,, & \\ \text{Twist :} & \theta_U \in \mathrm{Hom}(U,U) & \end{array} \tag{2.1}$$

for all $U\in\mathrm{Obj}(\mathcal{C})$, respectively for all $U,V\in\mathrm{Obj}(\mathcal{C})$, satisfying

$$\begin{array}{ll} (d_V\otimes id_{V^\vee})\circ(id_{V^\vee}\otimes b_V) = id_{V^\vee}\,, & (id_V\otimes d_V)\circ(b_V\otimes id_V) = id_V\,, \\ c_{U,V\otimes W} = (id_V\otimes c_{U,W})\circ(c_{U,V}\otimes id_W)\,, & c_{U\otimes V,W} = (c_{U,W}\otimes id_V)\circ(id_U\otimes c_{V,W})\,, \\ (g\otimes f)\circ c_{U,W} = c_{V,X}\circ(f\otimes g)\,, & \theta_V\circ f = f\circ\theta_U\,, \\ (\theta_V\otimes id_{V^\vee})\circ b_V = (id_V\otimes\theta_{V^\vee})\circ b_V\,, & \theta_{V\otimes W} = c_{W,V}\circ c_{V,W}\circ(\theta_V\otimes\theta_W) \end{array} \tag{2.2}$$

for all $U,V,W,X\in\mathrm{Obj}(\mathcal{C})$ and all $f\in\mathrm{Hom}(U,V)$, $g\in\mathrm{Hom}(W,X)$.

In a tensor category with duality, one defines the morphism dual to $f\in\mathrm{Hom}(U,V)$ by $f^\vee := (d_V\otimes id_{U^\vee})\circ(id_{V^\vee}\otimes f\otimes id_{U^\vee})\circ(id_{V^\vee}\otimes b_U)\in\mathrm{Hom}(V^\vee,U^\vee)$. A left-duality is an assignment of a left-dual object ${}^\vee U$ to each $U\in\mathrm{Obj}(\mathcal{C})$ together with a family of morphisms,

$$\text{Left-duality:} \qquad \tilde b_U \in \mathrm{Hom}(\mathbf{1}, {}^\vee U\otimes U)\,, \quad \tilde d_U \in \mathrm{Hom}(U\otimes {}^\vee U, \mathbf{1})\,, \tag{2.3}$$

that obey analogous properties as a right-duality. In a ribbon category, there is automatically also a left-duality; it can be constructed from right-duality, braiding and twist, and in

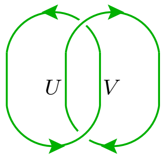
[2] The existence of a duality is often included in the definition of a tensor category. What we refer to as a tensor category is then called a *monoidal* category.

fact coincides with the right-duality both on objects and on morphisms, ${}^\vee U = U^\vee$, ${}^\vee f = f^\vee$. Tensor categories with coinciding left- and right-duality functors from $\mathcal{C}$ to $\mathcal{C}^{\rm opp}$ are called *sovereign*. Thus, every ribbon category is in particular sovereign; conversely, every braided sovereign category is a ribbon category. For a tensor category with both a left- and a right-duality, one defines left and right *traces* of an endomorphism $f \in {\rm Hom}(U,U)$ as

$$\mathrm{tr}_\mathrm{l}(f) := d_U \circ (id_{U^\vee} \otimes f) \circ \tilde b_U \,, \qquad \mathrm{tr}_\mathrm{r}(f) := \tilde d_U \circ (f \otimes id_{U^\vee}) \circ b_U \,, \tag{2.4}$$

and the left and right (quantum) dimensions of an object U as $\dim_{\mathrm{l/r}}(U) := \mathrm{tr}_{\mathrm{l/r}}(id_U)$. In a ribbon category the left and right traces coincide, i.e. ribbon categories are *spherical*. Accordingly, in a ribbon category we denote the trace just by tr.

The properties (2.2) of the braiding, twist and duality morphisms in a strict tensor category allow us to visualise them via ribbon graphs (see e.g. [21] and chapter XIV of [24]). In the sequel we will make ample use of this graphical notation. When drawing such graphs we follow the conventions set up in Section 2 of [18]. In particular, all diagrams are to be read from bottom to top and, for simplicity, we use the blackboard framing convention so that ribbons can be drawn as lines. For convenience, we have also summarised the basic structural data in an appendix; the graphs for the braiding, twist and duality morphisms are collected in appendix A.1, and the graphical transcription of the axioms (2.2) is given in appendix A.2. To give another example, the graph

$s_{U,V} :=$  (2.5)

with U,V any pair of objects of a braided tensor category with dualities, is the trace

$$s_{U,V} = \mathrm{tr}(c_{U,V} \circ c_{V,U}) = (d_V \otimes \tilde d_U) \circ [\, id_{V^\vee} \otimes (c_{U,V} \circ c_{V,U}) \otimes id_{U^\vee}] \circ (\tilde b_V \otimes b_U) \tag{2.6}$$

of the endomorphism $c_{U,V} \circ c_{V,U}$ of $V \otimes U$.

When $\mathcal{C}$ is semisimple, then we are particularly interested in simple objects. We denote by $\{U_i \,|\, i \in \mathcal{I}\}$ a collection of representatives of the isomorphism classes of (non-zero) simple objects of $\mathcal{C}$, and set

$$N_{ij}^{\ k} := \dim_\Bbbk(\mathrm{Hom}(U_i \otimes U_j, U_k)) \tag{2.7}$$

(taking values in $\mathbb{Z}_{\ge 0} \cup \{\infty\}$). Assuming that the tensor unit is simple, we take it as one of these representatives, so that $\mathcal{I} \ni 0$ with $U_0 = \mathbf{1}$. If an object U is simple, then so is its dual $U^\vee$; thus in particular for every $i \in \mathcal{I}$ there is a unique label $\bar\imath \in \mathcal{I}$ such that $U_{\bar\imath} \cong U_i^\vee$.

Definition 2.2:
A *modular tensor category* is a semisimple additive ribbon category for which the index set $\mathcal{I}$ is finite and for which the *s-matrix* $s = (s_{i,j})_{i,j \in \mathcal{I}}$ with entries

$$s_{i,j} := s_{U_i,U_j} = \mathrm{tr}(c_{U_i,U_j} \circ c_{U_j,U_i}) \tag{2.8}$$

is non-degenerate.

Instead of non-degeneracy of s, one can equivalently [9] require that, up to isomorphism, the tensor unit is the only 'transparent' simple object, i.e. that any simple object U for which $c_{V,U} \circ c_{U,V} = id_{U\otimes V}$ holds for all $V \in \mathrm{Obj}(\mathcal{C})$ satisfies $U \cong \mathbf{1}$.

The dimension of an object $U \in \mathrm{Obj}(\mathcal{C})$ is expressed through the numbers (2.5) as $\dim(U) \equiv \mathrm{tr}\, id_U = s_{U,\mathbf{1}} = s_{\mathbf{1},U}$. In a modular tensor category, the square of the matrix s is, up to a multiplicative constant, a permutation matrix,

$$(s^2)_{i,j} = \delta_{i,\bar{j}} \sum_{k\in\mathcal{I}} (\dim(U_k))^2 \,. \tag{2.9}$$

(In the physics literature, one usually considers the field of complex numbers, and instead of using s it is more conventional to work with the unitary matrix S defined as $S := S_{0,0}\, s$ with $S_{0,0} := \big[\sum_{i\in\mathcal{I}} (\dim(U_i))^2\big]^{-1/2}$.)

For later reference we quote the following criterion for a functor F to be an equivalence of categories (see e.g. Theorem IV.4.1 of [30]).

Proposition 2.3:
A functor F is an equivalence of categories if and only if F is essentially surjective (i.e. surjective up to isomorphisms) and fully faithful (i.e. bijective on morphisms).

Also note that when a functor F is an equivalence of *braided* tensor categories, then, owing to the uniqueness properties of the left and right dualities and the fact that the twist can be expressed through the dualities and the braiding, F is even an equivalence of *ribbon* categories.

We will occasionally have to deal with constructions which, just like functors, assign to each object U of a category $\mathcal{C}$ an object $F(U)$ of a category $\mathcal{D}$, and to each morphism f of $\mathcal{C}$ a morphism $F(f)$ of $\mathcal{D}$ in a manner compatible with the domain and target structure (i.e. such that $F(f) \in \mathrm{Hom}(F(U), F(V))$ for $f \in \mathrm{Hom}(U,V)$), but which are not, or are not known to be, functors. For definiteness, we will call a collection of maps that has these properties an *operation* on the category $\mathcal{C}$.

2.2 Idempotents and retracts

In order to fix our conventions and notation for subobjects and retracts we review a few notions from category theory (for more details see e.g. Sections I.5, V.7, VIII.1 and VIII.3 of [30]). For brevity, in this description we often dispense with naming the source and target objects of a morphism explicitly; the corresponding statements are meant to hold for every object for which they can be formulated at all.

A morphism e is called *monic* iff $e \circ f = e \circ g$ implies that $f = g$. A morphism r is called *epi* iff $f \circ r = g \circ r$ implies that $f = g$. A *subobject* of an object U is an equivalence class of monics $e \in \mathrm{Hom}(\,\cdot\,, U)$. Here two monics $e \in \mathrm{Hom}(S,U)$ and $e' \in \mathrm{Hom}(S',U)$ are called equivalent iff there exists an isomorphism $\varphi \in \mathrm{Hom}(S, S')$ such that $e = e' \circ \varphi$. A subobject (K, e) of U is a *kernel* of $f \in \mathrm{Hom}(U, V)$ iff $f \circ e = 0$ and for every $h \in \mathrm{Hom}(W, U)$ with $f \circ h = 0$ there exists a unique $h' \in \mathrm{Hom}(W, K)$ such that $h = e \circ h'$. If a kernel exists, it is unique up to equivalence of subobjects. Cokernels are defined by reversing all arrows. The *image* $\mathrm{Im}\, f$ of a morphism f is the kernel of the cokernel of f. It is often convenient to think of an isomorphism class of subobjects, kernels or cokernels as a single pair (S, f). This is done by selecting a definite representative of the isomorphism class, invoking the axiom of choice (recall that all categories we consider are small).

A subobject S is called *split* iff together with the monic $e \in \mathrm{Hom}(S, U)$ there also comes a morphism $r \in \mathrm{Hom}(U, S)$ such that $r \circ e = id_S$ (the letters e and r remind of 'embedding' and 'restriction'/'retract', respectively). We refer to the triple (S, e, r) as a *retract* of U (just like for subobjects, we use the term retract both for the corresponding equivalence class of such triples and for an individual representative). We use the notations $S \prec U$ and $U \succ S$ to indicate that there exists a retract (S, e, r) of U; when it is clear from the context what retract we are considering, we also use the abbreviations $e \equiv e_S \equiv e_{S \prec U}$ and $r \equiv r_S \equiv r_{U \succ S}$. In the pictorial notation we will use the following shorthands for the morphisms e, r specifying a retract:

$e =$ 

$r =$ 

(2.10)

Two retracts S, S' are called equivalent iff (S, e) and (S', e') are equivalent as subobjects and $e \circ r = e' \circ r'$.

An endomorphism $p \in \mathrm{Hom}(U, U)$ is called an *idempotent* (or a *projector*) iff $p \circ p = p$. To every retract $S = (S, e, r)$ of U there is associated an idempotent $P_S \in \mathrm{Hom}(U, U)$, namely $P_S := e \circ r$. An idempotent p is said to be *split* if, conversely, there exists a retract (S, e, r) with $p = P_S \equiv e \circ r$. Thus a split idempotent has in particular an image, $\mathrm{Im}(p) = S$, and split subobjects are precisely the images of split idempotents. Further, the retract (S, e, r) is then unique up to equivalence of retracts, and

$$e \circ r = p\,, \qquad r \circ e = id_S\,, \qquad p \circ e = e\,, \qquad r \circ p = r\,. \tag{2.11}$$

Also note that in a sovereign tensor category it follows, via the cyclicity of the trace, that $\mathrm{tr}_U(p) = \dim(\mathrm{Im}\, p)$, both for the left and the right trace, for any split idempotent p.

Lemma 2.4:

(i) For any two objects U, V and any split idempotent $p \in \mathrm{Hom}(U, U)$, there is a natural bijection between the vector space $\mathrm{Hom}(\mathrm{Im}\, p, V)$ and the subspace

$$\mathrm{Hom}_{(p)}(U, V) := \{ f \in \mathrm{Hom}(U, V) \mid f \circ p = f \} \tag{2.12}$$

of $\mathrm{Hom}(U,V)$.

(ii) For any two objects U,V and any split idempotent $q\in\mathrm{Hom}(V,V)$, there is a natural bijection between the vector space $\mathrm{Hom}(U,\mathrm{Im}\,q)$ and the subspace

$$\mathrm{Hom}^{(q)}(U,V) := \{f\in\mathrm{Hom}(U,V)\,|\,q\circ f=f\} \tag{2.13}$$

of $\mathrm{Hom}(U,V)$.

Proof:
Recall from the remarks before (2.11) that $\mathrm{Im}\,p$ is in a canonical way a retract $(\mathrm{Im}\,p,e,r)$ of U. With the help of the relations (2.11) one checks immediately that the map $\mathrm{Hom}(\mathrm{Im}\,p,V)\ni\varphi\mapsto\varphi\circ r$ maps to the correct subspace $\mathrm{Hom}_{(p)}(U,V)\subseteq\mathrm{Hom}(U,V)$ and that it has the map $\mathrm{Hom}_{(p)}(U,V)\ni\psi\mapsto\psi\circ e$ as a two-sided inverse. This establishes (i). Statement (ii) follows analogously, the relevant mappings now being $\varphi\mapsto e\circ\varphi$ and $\psi\mapsto r\circ\psi$. □

Definition 2.5:
A category $\mathcal{C}$ is called *Karoubian* (or *idempotent complete*, or *pseudo-abelian*) iff every idempotent is split.

Remark 2.6:
To every idempotent $p\in\mathrm{Hom}(U,U)$ in an additive Karoubian category there corresponds an isomorphism $U\cong\mathrm{Im}(p)\oplus\mathrm{Im}(id_U{-}p)$. All abelian categories, as well as all additive semisimple categories, are Karoubian.

Definition 2.7:
The *Karoubian envelope* (or *idempotent completion*, or *pseudo-abelian hull*) $\mathcal{C}^{\mathrm{K}}$ of a category $\mathcal{C}$ is a Karoubian category $\mathcal{C}^{\mathrm{K}}$ together with an embedding functor $K\colon\mathcal{C}\to\mathcal{C}^{\mathrm{K}}$ which is universal in the sense that every functor $F\colon\mathcal{C}\to\mathcal{D}$ to a Karoubian category $\mathcal{D}$ factors as $F=G\circ K$, with the functor $G\colon\mathcal{C}^{\mathrm{K}}\to\mathcal{D}$ unique up to isomorphism of functors.

Remark 2.8:
(i) In the original definition of Karoubian envelope [23] it is also assumed that the category $\mathcal{C}$ is additive, and the functors K and F are required to be additive functors. $\mathcal{C}^{\mathrm{K}}$ is then an additive category, too.

(ii) By general nonsense concerning universal properties, the Karoubian envelope is unique up to equivalence of categories. When $\mathcal{C}$ is already Karoubian, then $\mathcal{C}^{\mathrm{K}}\cong\mathcal{C}$ and $K\cong Id_{\mathcal{C}}$.

(iii) The Karoubian envelope of $\mathcal{C}$ can be realised [23] as the category whose objects are pairs $(U;p)$ of objects $U\in\mathrm{Obj}(\mathcal{C})$ and idempotents $p\in\mathrm{Hom}(U,U)$, and with morphisms

$$\mathrm{Hom}^{\mathrm{K}}((U;p),(V;q)) := \{f\in\mathrm{Hom}(U,V)\,|\,q\circ f\circ p=f\} \tag{2.14}$$

and $id^{\rm K}_{(U;p)} = p$, so that in particular $p \in {\rm Hom}^{\rm K}((U;p),(U;p))$. In this realisation the embedding functor K acts as $K(U) = (U; id_U)$ and $K(f) = f$, implying for instance that ${\rm Hom}^{\rm K}(K(U),K(V)) = {\rm Hom}(U,V)$. As a consequence, we may (and will) think of $\mathcal{C}$ as a full subcategory of $\mathcal{C}^{\rm K}$, and accordingly identify $U \in {\rm Obj}(\mathcal{C})$ with $(U; id_U) \in {\rm Obj}(\mathcal{C}^{\rm K})$. Further, when q is any idempotent in ${\rm Hom}^{\rm K}((U;p),(U;p))$, we have $q \circ p = q = p \circ q$, implying that ${\rm Im}(q) \cong (U;q)$, independently of p.

(iv) Various properties of $\mathcal{C}$ are naturally inherited by $\mathcal{C}^{\rm K}$ (compare e.g. [4]):

a) If $\mathcal{C}$ is tensor, then $\mathcal{C}^{\rm K}$ becomes a tensor category by setting $f \otimes^{\rm K} g := f \otimes g$ and

$$\mathbf{1}^{\rm K} := K(\mathbf{1}) \qquad \text{and} \qquad (U;p) \otimes^{\rm K} (V;q) := (U{\otimes}V; p{\otimes}q) \,. \tag{2.15}$$

b) If a tensor category $\mathcal{C}$ is braided, then a braiding for the tensor category $\mathcal{C}^{\rm K}$ is given by

$$c^{\rm K}_{(U;p),(V;q)} := (q \otimes p) \circ c_{U,V} \,. \tag{2.16}$$

c) If a tensor category $\mathcal{C}$ has a left duality, then a left duality for the tensor category $\mathcal{C}^{\rm K}$ is given by $(U,p)^\vee := (U^\vee, p^\vee)$ and

$$d^{\rm K}_{(U;p)} := d_U \circ (id_{U^\vee} \otimes p) \qquad \text{and} \qquad b^{\rm K}_{(U;p)} := (p \otimes id_{U^\vee}) \circ b_U \,. \tag{2.17}$$

An analogous statement holds for a right duality.

d) If a braided tensor category $\mathcal{C}$ with duality has a twist, then a twist for $\mathcal{C}^{\rm K}$ is given by $\theta^{\rm K}_{(U;p)} := p \circ \theta_U$. It follows in particular that when $\mathcal{C}$ is ribbon, then $\mathcal{C}^{\rm K}$ carries a natural structure of ribbon category as well.

Further, dimensions in $\mathcal{C}^{\rm K}$ are given by

$${\rm dim}^{\rm K}((U;p)) = {\rm tr}(p) \,. \tag{2.18}$$

(v) By the observation in Remark 2.6 it thus follows that for every idempotent $p \in {\rm Hom}(V,V)$ in an additive Karoubian ribbon category one has ${\rm dim}(V) = {\rm dim}({\rm Im}(p)) + {\rm dim}({\rm Im}(id_V{-}p))$. In particular, if all dimensions are non-negative real numbers, then ${\rm dim}(U) \le {\rm dim}(V)$ if U is a retract of V.

Lemma 2.9:

For F: $\mathcal{C} \to \mathcal{D}$ a functor between categories $\mathcal{C}$ and $\mathcal{D}$, let $F^{\rm K}$: $\mathcal{C}^{\rm K} \to \mathcal{D}^{\rm K}$ be the functor between their Karoubian envelopes given by

$$F^{\rm K}((U;p)) := (F(U);F(p)) \qquad \text{and} \qquad F^{\rm K}(f) := F(f) \tag{2.19}$$

for objects $(U;p)$ and morphisms f of $\mathcal{C}^{\rm K}$.

(i) If F is an equivalence functor, then so is $F^{\rm K}$.

(ii) If $\mathcal{C}$ and $\mathcal{D}$ are tensor categories and F is a tensor functor, then $F^{\rm K}$ is a tensor functor, too.

(iii) If $\mathcal{C}$ and $\mathcal{D}$ are ribbon categories and F is a ribbon functor, then F^{K} is a ribbon functor, too.

Proof:
(i) is derived easily by using the criterion of Proposition 2.3 for a functor to be an equivalence.
(ii) and (iii) follow by combining the respective properties of F with the prescription given in Remark 2.8(iv) for the tensor and ribbon structure on the Karoubian envelope of a tensor and ribbon category, respectively. □

In the applications to rational conformal quantum field theory, the categories of main interest are ribbon categories that are even modular in the sense of Definition 2.2. In the present paper, also categories with much less structure play a role. However, a few basic properties (shared in particular by modular tensor categories) will generally be required below. We will not mention these properties repeatedly, but rather collect them in the

Declaration 2.10:
(i) Every category $\mathcal{C}$ is a small additive category, with all morphism sets being vector spaces over some fixed field $\Bbbk$.
Whenever a tensor category is not strict, we tacitly replace it by an equivalent strict tensor category.

(ii) Unless stated otherwise, every category is a assumed to be Karoubian.

(iii) Unless stated otherwise, the tensor unit $\mathbf{1} \in \mathrm{Obj}(\mathcal{C})$ of a tensor category $\mathcal{C}$ is simple, as well as absolutely simple, i.e. satisfies $\mathrm{End}(\mathbf{1}) = \Bbbk\, id_{\mathbf{1}}$.

For the categories from which our considerations start, all these properties are *assumptions*. On the other hand, various constructions of new categories that we deal with in this paper – taking the Karoubian envelope (introduced in Definition 2.7), the Karoubian product (see Definition 6.1(ii)), the dual (Definition 6.13), the category of modules over an algebra, and the category of local modules over a commutative symmetric special Frobenius algebra (Definition 3.20) – preserve the properties in part (i) and (ii) of the declaration; the procedures of taking the Karoubian envelope, the dual, or the Karoubian product in addition also preserve the properties stated in part (iii). Below this permanence will be mentioned only when it is non-trivial.

Definition 2.11:
For U an object in a (not necessarily Karoubian) category $\mathcal{C}$, let H be a subset of the set $\mathrm{Idem}(U)$ of idempotents in $\mathrm{End}(U)$.

(i) A *maximal idempotent in* H is a morphism $P^H_{\max} \in H$ such that

$$q \circ P^H_{\max} = q = P^H_{\max} \circ q \tag{2.20}$$

for all $q \in H$.

(ii) A *maximal retract with respect to* H is a retract of U such that P_U is a maximal idempotent in H.

Lemma 2.12:
If a set $H \subseteq \mathrm{Idem}(U)$ contains a maximal idempotent, then this maximal idempotent is unique.

Proof:
Let $P_{\max}$ and $P'_{\max}$ be two maximal idempotents in H. Then $P_{\max} \circ P'_{\max} = P'_{\max}$ by the maximality of $P_{\max}$ and $P_{\max} \circ P'_{\max} = P_{\max}$ by the maximality of $P'_{\max}$. □

Corollary 2.13:
If a maximal retract with respect to some $H \subset \mathrm{Idem}(U)$ exists, then it is unique up to isomorphism of retracts.

Lemma 2.14:
Let $H \subseteq \mathrm{Idem}(U)$ be a set of idempotents on an object U for which a maximal retract $P_{\max}$ exists and is split. Then for any split idempotent $P \in H$, the image $\mathrm{Im}(P)$ is a retract of $\mathrm{Im}(P_{\max})$.

Proof:
We realise both $\mathrm{Im}(P)$ and $\mathrm{Im}(P_{\max})$ as retracts of the object U, i.e. write $(\mathrm{Im}(P), e, r)$ as well as $(\mathrm{Im}(P_{\max}), e_{\max}, r_{\max})$. Then the morphisms $\tilde e := r_{\max} \circ e \in \mathrm{Hom}(\mathrm{Im}(P), \mathrm{Im}(P_{\max}))$ and $\tilde r := r \circ e_{\max} \in \mathrm{Hom}(\mathrm{Im}(P_{\max}), \mathrm{Im}(P))$ obey $\tilde r \circ \tilde e = id_{\mathrm{Im}(P)}$ owing to the maximality of $P_{\max}$. □

2.3 Frobenius algebras

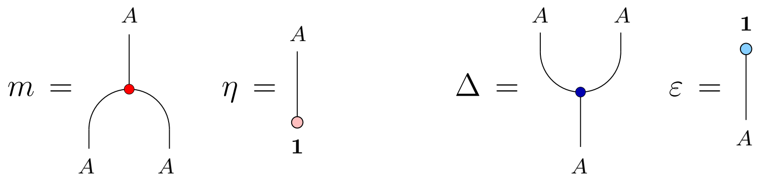
The notion of an algebra over some field $\Bbbk$ has an analogue in arbitrary tensor categories. A $\Bbbk$-algebra is then nothing but an algebra,[3] in the category-theoretic sense, in the particular tensor category $\mathcal{V}ect_\Bbbk$ of vector spaces over the field $\Bbbk$.

Definition 2.15:
An (associative) *algebra* (with unit) A in a tensor category $\mathcal{C}$ is a triple (A, m, η) consisting of an object A of $\mathcal{C}$, a multiplication morphism $m \in \mathrm{Hom}(A \otimes A, A)$ and a unit morphism $\eta \in \mathrm{Hom}(\mathbf{1}, A)$, satisfying

$$m \circ (m \otimes id_A) = m \circ (id_A \otimes m) \quad \text{and} \quad m \circ (\eta \otimes id_A) = id_A = m \circ (id_A \otimes \eta)\,. \tag{2.21}$$

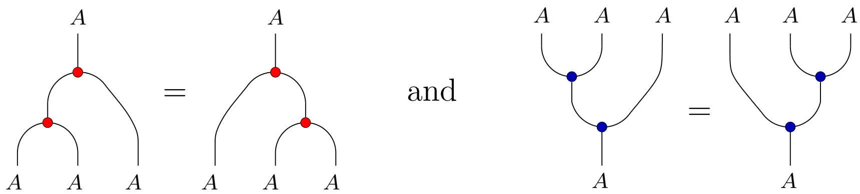
[3] In using the term 'algebra' we follow the terminology in e.g. [40, 26, 34]. In a large part of the categorical literature (see e.g. [30, 38, 44]), the term 'monoid' is used instead.

Other algebraic notions familiar from $\mathcal{V}ect_{\Bbbk}$ generalise to arbitrary tensor categories, too. In particular, a *co-algebra* in $\mathcal{C}$ is a triple (A, Δ, ε) consisting of an object A, a comultiplication $\Delta \in \mathrm{Hom}(A, A{\otimes}A)$ and a counit $\varepsilon \in \mathrm{Hom}(A, \mathbf{1})$ possessing coassociativity and counit properties that amount to 'reversing all arrows' in the associativity and unit properties (2.21). Again a pictorial notation for these morphisms is helpful; we set



(2.22)

Then e.g. the associativity of m and coassociativity of Δ look like



(2.23)

respectively.

Definition 2.16 :
A *left module* over an algebra $A \in \mathrm{Obj}(\mathcal{C})$ is a pair $M = (\dot{M}, \rho)$ consisting of an object $\dot{M}$ of $\mathcal{C}$ and a *representation morphism* $\rho \equiv \rho_M \in \mathrm{Hom}(A{\otimes}\dot{M}, \dot{M})$, satisfying

$$\rho \circ (m \otimes id_{\dot{M}}) = \rho \circ (id_A \otimes \rho) \qquad \text{and} \qquad \rho \circ (\eta \otimes id_{\dot{M}}) = id_{\dot{M}} \,. \tag{2.24}$$

By taking the A-modules as objects and the subspaces

$$\mathrm{Hom}_A(N, M) := \{ f \in \mathrm{Hom}(\dot{N}, \dot{M}) \,|\, f \circ \rho_N = \rho_M \circ (id_A{\otimes}f) \} \tag{2.25}$$

of the $\mathcal{C}$-morphisms that intertwine the A-action as morphisms, one gets the *category of left A-modules*, which we denote by $\mathcal{C}_A$. Analogously one defines right A-modules and their category. For brevity we will often refer to left A-modules just as *A-modules*. An A-module is called a *simple module* iff it is a simple object of $\mathcal{C}_A$. For $U \in \mathrm{Obj}(\mathcal{C})$, the *induced* (left) *module* $\mathrm{Ind}_A(U)$ is equal to $A{\otimes}U$ as an object in $\mathcal{C}$, with representation morphism $m \otimes id_U$; the full subcategory of $\mathcal{C}_A$ whose objects are the induced A-modules will be denoted by $\mathcal{C}_A^{\mathrm{Ind}}$. (For more details see e.g. [26, 20] and Sections 4.1 – 3 of [18].) When an A-module N is a retract, as an object of $\mathcal{C}_A$, of an A-module M, we refer to it as a *module retract* of M.

Remark 2.17 :
(i) If (A, m, η) is an algebra in a tensor category $\mathcal{C}$, then $((A; id_A), m, \eta)$ is an algebra in its Karoubian envelope $\mathcal{C}^{\mathrm{K}}$. Analogous statements hold for coalgebras, Frobenius algebras etc.

(ii) If $(\dot{M}, \rho)$ is an A-module in a tensor category $\mathcal{C}$ and $p \in \mathrm{Hom}_A(M, M)$ is a split idempotent in $\mathcal{C}_A$, then

$$(\mathrm{Im}(p), r \circ \rho \circ (id_A \otimes e)) \tag{2.26}$$

(with $e \circ r = p$ as in (2.11)) is an A-module in $\mathcal{C}$, too.

Lemma 2.18 :
For any algebra A in a tensor category $\mathcal{C}$, the category $(\mathcal{C}_A)^{\mathrm{K}}$ is equivalent to a full subcategory of $(\mathcal{C}^{\mathrm{K}})_A$.

In particular, if $\mathcal{C}$ is Karoubian, then so is the category $\mathcal{C}_A$ of A-modules in $\mathcal{C}$.

Proof:
The first statement follows from the fact that if $M = (\dot{M}, \rho)$ is an A-module in $\mathcal{C}$ and $p \in \mathrm{Hom}_A(M, M)$ is a (not necessarily split) idempotent, then

$$((\dot{M}; p), p \circ \rho) \tag{2.27}$$

is an $(A; id_A)$-module in the Karoubian envelope $\mathcal{C}^{\mathrm{K}}$.

Since $\mathcal{C}_A$ is a full subcategory of $(\mathcal{C}_A)^{\mathrm{K}}$, the second statement is a direct consequence of the first. More explicitly, for any A-module $M = (\dot{M}, \rho)$, every idempotent $p \in \mathrm{Hom}_A(M, M)$ is in particular an idempotent in $\mathrm{Hom}(\dot{M}, \dot{M})$. Since $\mathcal{C}$ is Karoubian, there is thus a retract $(\mathrm{Im}(p), e, r)$ in $\mathcal{C}$. Defining

$$\rho_p := r \circ \rho \circ (id_A \otimes e)\,, \tag{2.28}$$

we also have $e \circ \rho_p \circ (id_A \otimes r) = p \circ \rho \circ (id_A \otimes p)$. Thus $(\mathrm{Im}(p), \rho_p) \in \mathrm{Obj}(\mathcal{C}_A)$ is a submodule of M, and hence p is split as an idempotent in $\mathcal{C}_A$. □

Remark 2.19 :
Conversely, if $((\dot{M}; p), \varrho)$ is an $(A; id_A)$-module in $\mathcal{C}^{\mathrm{K}}$, with p an idempotent that is already split in $\mathcal{C}$, then using the fact that $\varrho \in \mathrm{Hom}^{\mathrm{K}}((A; id_A) \otimes (\dot{M}; p), (\dot{M}; p))$ means (see (2.14)) that

$$p \circ \varrho \circ (id_A \otimes p) = \varrho\,, \tag{2.29}$$

one checks that

$$M_{p,\varrho} := (\mathrm{Im}(p), \varrho_p) \qquad \text{with} \qquad \varrho_p := r \circ \varrho \circ (id_A \otimes e) \tag{2.30}$$

is an A-module in $\mathcal{C}$.

Also, when the condition that the idempotent p is split is not imposed (so that $\mathrm{Im}\,(p)$ does not necessarily exist), one might be tempted to directly interpret the pair $(\dot{M}, \varrho)$

as a module. But this is not, in general, possible. While $(\dot{M},\varrho)$ does satisfy the first representation property $\varrho\circ(id_A\otimes\varrho)=\varrho\circ(m\otimes id_{\dot{M}})$, for $p\neq id_{\dot{M}}$ the second representation property fails, $\varrho\circ(\eta\otimes id_{\dot{M}})=p$.

Definition 2.20:
An A-*bimodule* is a triple $M=(\dot{M},\rho_{\rm l},\rho_{\rm r})$ such that $(\dot{M},\rho_{\rm l})$ is a left A-module, $(\dot{M},\rho_{\rm r})$ is a right A-module, and the left and right actions of A commute.

The category of A-bimodules in $\mathcal{C}$ will be denoted by $\mathcal{C}_{A|A}$. Note that in contrast to $\mathcal{C}_A$, this is always a tensor category (though not necessarily braided).

In a *braided* tensor category, for every object V the induced *left* A-module $(A\otimes V, m\otimes id_V)$ can be endowed in two obvious ways with the structure of a *right* A-module $(A\otimes V,\rho_{\rm r}^{\pm})$; the representation morphisms $\rho_{\rm r}^{\pm}\equiv\rho_{V,{\rm r}}^{\pm}\in{\rm Hom}(A\otimes V\otimes A, A\otimes V)$ are

$$\rho_{\rm r}^{+} := (m\otimes id_V)\circ(id_A\otimes c_{V,A}) \qquad\text{and}\qquad \rho_{\rm r}^{-} := (m\otimes id_V)\circ(id_A\otimes(c_{A,V})^{-1})\,, \tag{2.31}$$

respectively. These are used in

Definition 2.21:
For A an algebra in a braided tensor category $\mathcal{C}$, the functors

$$\alpha_A^{\pm}:\quad \mathcal{C}\to\mathcal{C}_{A|A} \tag{2.32}$$

of α-*induction* are defined on objects as

$$\alpha_A^{\pm}(V) := (A\otimes V, m\otimes id_V, \rho_{\rm r}^{\pm}) \tag{2.33}$$

for $V\in{\rm Obj}(\mathcal{C})$, and on morphisms as

$$\alpha_A^{\pm}(f) := id_A\otimes f\ \in{\rm Hom}(A\otimes V, A\otimes W) \tag{2.34}$$

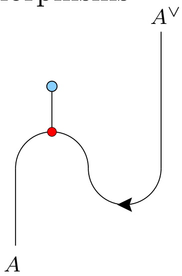
for $f\in{\rm Hom}(V,W)$.

The α-inductions $\alpha_A^{\pm}$ are indeed functors, even tensor functors, from $\mathcal{C}$ to the category $\mathcal{C}_{A|A}$ of A-bimodules. They were first studied in the theory of subfactors (see [28] and also e.g. [47, 6, 8]), and were reformulated in the form used here in [37].

We will mainly be interested in algebras with several specific additional properties, which arise e.g. in applications to conformal quantum field theory [18].

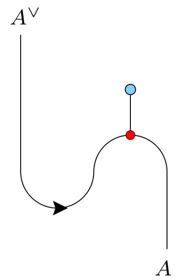
Definition 2.22:
(i) An algebra A in a tensor category with left and right dualities together with a morphism $\varepsilon\in{\rm Hom}(A,\mathbf{1})$ is called a *symmetric* algebra iff the two morphisms

$$\Phi_1 := [(\varepsilon\circ m)\otimes id_{A^\vee}]\circ(id_A\otimes b_A) = \tag{2.35}$$



and

$$\Phi_2 := [id_{A^\vee} \otimes (\varepsilon \circ m)] \circ (\tilde{b}_A \otimes id_A) = \tag{2.36}$$



in $\mathrm{Hom}(A, A^\vee)$ are equal.

(ii) A *Frobenius algebra* in a tensor category $\mathcal{C}$ is a quintuple $(A, m, \eta, \Delta, \varepsilon)$ such that (A, m, η) is an algebra in $\mathcal{C}$, (A, Δ, ε) is a co-algebra in $\mathcal{C}$, and there is the compatibility relation

$$(id_A \otimes m) \circ (\Delta \otimes id_A) = \Delta \circ m = (m \otimes id_A) \circ (id_A \otimes \Delta) \tag{2.37}$$

between the two structures.

(iii) A Frobenius algebra is called *special* iff

$$\varepsilon \circ \eta = \beta_{\mathbf{1}}\, id_{\mathbf{1}} \qquad \text{and} \qquad m \circ \Delta = \beta_A\, id_A \tag{2.38}$$

for non-zero numbers $\beta_{\mathbf{1}}$ and β_A.

Recently [34, 44], in order to emphasise the analogy with classical non-commutative ring theory (compare e.g. [22]), the term "strongly separable" was introduced for what we call "special".

For a symmetric special Frobenius algebra A one has $\beta_{\mathbf{1}}\beta_A = \dim(A)$, implying in particular that $\dim(A) \neq 0$. It is then convenient to normalise ε and Δ such that $\beta_{\mathbf{1}} = \dim(A)$ and $\beta_A = 1$. We will follow this convention unless mentioned otherwise. We also set

$$\varepsilon_\natural := d_A \circ (id_{A^\vee} \otimes m) \circ (\tilde{b}_A \otimes id_A) \in \mathrm{Hom}(A, \mathbf{1}) \tag{2.39}$$

and write $\Phi_{1,\natural}$ for the morphism that is obtained by replacing ε in the expression (2.35) by $\varepsilon_\natural$.

Remark 2.23:

(i) If A is a special Frobenius algebra then, with the normalisation $\beta_A = 1$, (A, Δ, m) is a retract of $A \otimes A$. The Frobenius property ensures that this statement holds even at the level of A-bimodules. This bears some similarity to the situation in braided tensor categories where the notion of a bi-algebra can be defined. In fact, the property of an algebra A to be a bi-algebra is equivalent to the statement that the coproduct endows A with the structure of a retract of $A \otimes A$ as an algebra, rather than as a bimodule.

(ii) When $\mathcal{C}$ is semisimple and A is special Frobenius, then the category $\mathcal{C}_A$ of left A-modules is semisimple [20].

(iii) For modules over any algebra A in a tensor category $\mathcal{C}$ a reciprocity relation holds, stating that for every left A-module M and every object U of $\mathcal{C}$ there is a canonical bijection

$$\begin{array}{rrcl} \phi_1: & \mathrm{Hom}_A(\mathrm{Ind}_A(U),M) & \xrightarrow{\cong} & \mathrm{Hom}(U,\dot{M}) \\ & f & \stackrel{\cong}{\longmapsto} & f\circ(\eta\otimes id_U) \end{array} \tag{2.40}$$

between morphism spaces in $\mathcal{C}$ and in $\mathcal{C}_A$. If A is Frobenius, then an analogous reciprocity relation also holds when the target of Hom_A is an induced module,

$$\begin{array}{rrcl} \phi_2: & \mathrm{Hom}_A(M,\mathrm{Ind}_A(V)) & \xrightarrow{\cong} & \mathrm{Hom}(\dot{M},V) \\ & g & \stackrel{\cong}{\longmapsto} & (\varepsilon\otimes id_V)\circ g\,. \end{array} \tag{2.41}$$

In other words, for an arbitrary associative algebra, the induction functor is a left adjoint functor of restriction; if the algebra carries the additional structure of a Frobenius algebra, then induction is a right adjoint of restriction as well.
The inverses of the maps (2.40) and (2.41) can also be given explicitly; they are

$$\begin{array}{l} \phi_1^{-1}(\tilde{f}) = \rho_M\circ(id_A\otimes\tilde{f}) \qquad \text{and} \\ \phi_2^{-1}(\tilde{g}) = [id_A\otimes(\tilde{g}\circ\rho_M)]\circ[(\Delta\circ\eta)\otimes id_{\dot{M}}]\,. \end{array} \tag{2.42}$$

For a proof see e.g. Propositions 4.10 and 4.11 of [20].

(iv) Given an algebra (A,m,η) with $\dim(A)\neq 0$ in a sovereign tensor category $\mathcal{C}$, morphisms Δ and ε such that $(A,m,\eta,\Delta,\varepsilon)$ is a symmetric special Frobenius algebra exist iff the morphism $\Phi_{1,\natural}$ defined after (2.39) is invertible. Further, it is always possible to normalise ε in such a way that $\varepsilon=\varepsilon_\natural$. With this normalisation of the counit the coproduct Δ is unique, and one has $\beta_{\mathbf{1}}=\dim(A)$.
For a proof see lemma 3.12 of [18].

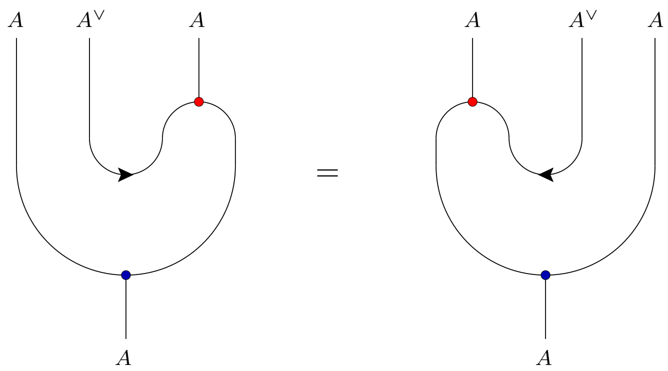
(v) For every Frobenius algebra A in a tensor category with left and right duality, the morphisms $\Phi_{1,2}$ in (2.35) and (2.36) are invertible (see lemma 3.7 of [18]), so that in particular $A\cong A^\vee$. Using $\mathrm{Hom}(U,V)\cong\mathrm{Hom}(V^\vee,U^\vee)$ it follows that for any two Frobenius algebras A, B in a tensor category with left and right duality we have

$$\mathrm{Hom}(A,B)\cong\mathrm{Hom}(B,A)\,. \tag{2.43}$$

For symmetric Frobenius algebras, there is a single distinguished isomorphism between the object underlying the algebra and its dual object, and as a consequence there is also a distinguished bijection (2.43).

(vi) If $\dim_{\Bbbk}\mathrm{Hom}(\mathbf{1},A)=d$ for a Frobenius algebra A in a tensor category with left and right duality, then $I^{(d)}:=\mathbf{1}\oplus\mathbf{1}\oplus\cdots\oplus\mathbf{1}$ (d summands) is a retract of A. Indeed, as just remarked the morphisms $\Phi_{1,2}$ are then invertible, and it is not difficult to see that one can choose $\alpha_i\in\mathrm{Hom}(\mathbf{1},A)$, $i=1,2,\dots,d$, such that $\varepsilon\circ m\circ(\alpha_i\otimes\alpha_j)=\delta_{i,j}$. (This furnishes a non-degenerate bilinear form on $\mathrm{Hom}(\mathbf{1},A)$ and thus endows $\mathrm{Hom}(\mathbf{1},A)$ with the structure of a Frobenius algebra in $\mathcal{V}ect_{\Bbbk}$.) Further, there are retracts $(\mathbf{1}, e_i, r_i)$ of $I^{(d)}$ satisfying $r_i \circ e_j = \delta_{i,j}$ and $\sum_{i=1}^{d} e_i \circ r_i = id_A$. The retract in question is then given by $(I^{(d)}, e, r)$ with $e := \sum_{i=1}^{d} \alpha_i \circ r_i$ and $r := \sum_{i=1}^{d} e_i \circ \varepsilon \circ m \circ (\alpha_i \otimes id_A)$.

(vii) In terms of our graphical calculus, the property of an algebra to be symmetric Frobenius in essence implies that multiplications and/or comultiplications can be moved past each other in all possible arrangements. Examples for such moves are provided by the defining properties (2.23) and (2.37). Another move, which is frequently used in our calculations below (without special mentioning), is the following:



(2.44)

This identity uses both the symmetry and the Frobenius property. First note that the latter two properties imply

$$[(\varepsilon \circ m) \otimes id_{A^\vee}] \circ [id_A \otimes (b_A \circ \tilde{d}_A)] \circ [(\Delta \circ \eta) \otimes id_{A^\vee}] = id_{A^\vee} . \qquad (2.45)$$

To show (2.44), we insert this identity on the outgoing $A^\vee$-ribbon on the left hand side. Then we use the Frobenius property to convert the product m on the left hand side of (2.44) into a coproduct. The latter can then be moved past the coproduct already present in (2.44) using coassociativity. Finally the coproduct is converted back to a product using the Frobenius property once again.

The properties of an algebra A to be symmetric, special and Frobenius are all indispensable in the construction of a conformal field theory from A. Further properties of A can be important in specific applications. For us the following is the most important one:

Definition 2.24:
An algebra A in a braided tensor category is said to be *commutative* (with respect to the given braiding) iff $m \circ c_{A,A} = m$.

Note that $m \circ c_{A,A} = m$ is equivalent to $m \circ c_{A,A}^{-1} = m$. Also, while in general the category $\mathcal{C}_A$ of left A-modules is not a tensor category, a sufficient condition for $\mathcal{C}_A$ to have a tensor structure is that A is commutative. The tensor structure is not canonical, though, because in this case one can turn a left A-module into an A-bimodule in two different ways by using the braiding to define a right action of A. However, we will see later (see also [40, 26]) that for commutative A there is a full subcategory of $\mathcal{C}_A$, namely the category $\mathcal{C}_A^{\ell oc}$ of local A-modules, on which the tensor structure is canonical.

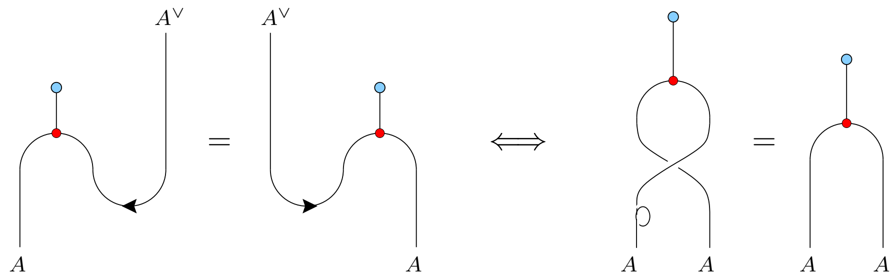
In the classical case, i.e. for algebras in the category of finite-dimensional vector spaces over a field, commutative Frobenius algebras are automatically symmetric. In a braided setting this is not true in general, but only if an additional condition is satisfied. More precisely, we have

Proposition 2.25:
(i) A commutative symmetric Frobenius algebra has trivial twist, i.e. $\theta_A = id_A$.

(ii) Conversely, every commutative Frobenius algebra with trivial twist is symmetric.

(iii) A commutative symmetric Frobenius algebra is also cocommutative.

Proof:
(i) The statement follows from the equivalence

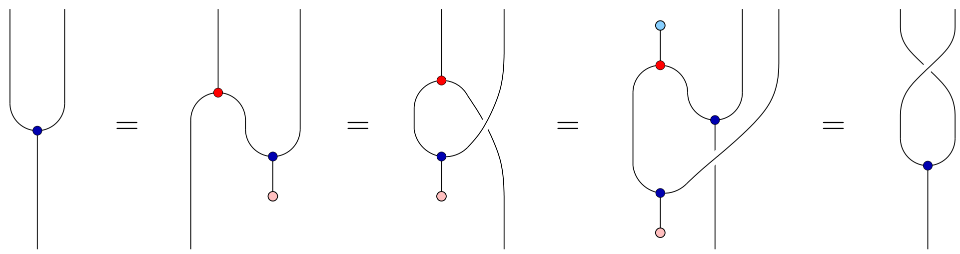


(2.46)

This is obtained by bending the outgoing $A^\vee$-ribbon via a duality morphism downwards to the left, which replaces the morphism $\Phi_{1,2}$ by $\tilde{d}_A \circ (id_A \otimes \Phi_{1,2})$.
Assuming symmetry, the left equality in (2.46) holds true. After using commutativity on the left equality, the two sides differ only by a twist. Tensoring this equality with id_A from the right and then composing with $id_A \otimes (\Delta \circ \eta)$ and using the Frobenius property finally yields $\theta_A = id_A$. Here, as well as in many arguments below, the manipulations of the graphs also involve a process of 'deforming' ribbons, making use of the defining properties of the braiding (in particular, functoriality), duality and twist.

(ii) If A is commutative and has trivial twist, the right equality in (2.46) holds, implying that A is symmetric.

(iii) is obtained by the following moves:



(2.47)

Here the first equality uses the Frobenius and unit properties, the second uses commutativity, the third the Frobenius and counit properties. The last equality is based on the symmetry property together with the result of (i). □

Definition 2.26:
An algebra A is called *simple* iff all bimodule endomorphisms of A as a bimodule over itself are multiples of the identity, i.e. $\mathrm{Hom}_{A|A}(A,A) = \Bbbk\, id_A$.

Lemma 2.27:
Let A be a simple symmetric special Frobenius algebra. Then for every left A-module M the equality

$$\tilde{d}_{\dot{M}} \circ (\rho_M \otimes id_{\dot{M}^\vee}) \circ (id_A \otimes b_{\dot{M}}) = \frac{\dim(\dot{M})}{\dim(A)}\, \varepsilon \qquad (2.48)$$

holds.

Proof:
The morphism $f := [id_A \otimes (\tilde{d}_{\dot{M}} \circ (\rho_M {\otimes} id_{\dot{M}^\vee}) \circ (id_A {\otimes} b_{\dot{M}}))] \circ \Delta$ is an A-bimodule morphism from A to A. That f is a left module morphism follows from the Frobenius property of A, while to show that it is also a morphism of right A-modules one needs A to be symmetric. Since A is simple, f is thus a multiple of id_A. Since $\varepsilon \circ f$ gives the left hand side of (2.48), the constant of proportionality is the same as on the right hand side of (2.48). This constant, in turn, immediately follows from $\varepsilon \circ f \circ \eta = \mathrm{tr}\, id_{\dot{M}}$. □

Remark 2.28:
(i) In the following an important role will be played by the dimensions of certain spaces $\mathrm{Hom}_{A|A}(M,N)$ of A-bimodule morphisms between α-induced bimodules. (Recall the Definition 2.21 of α-induction.) For any pair U,V of objects and any algebra A in a ribbon category we set

$$\tilde{Z}(A)_{U,V} := \dim_\Bbbk \left[\mathrm{Hom}_{A|A}(\alpha_A^-(V), \alpha_A^+(U))\right]. \qquad (2.49)$$

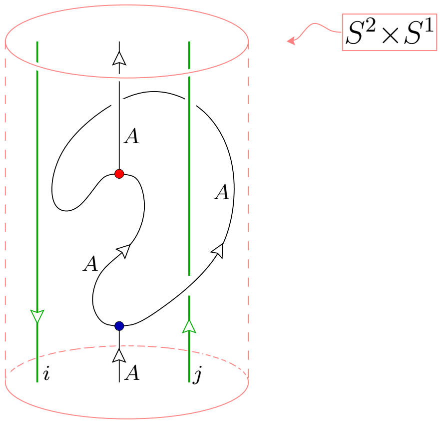
Since $A = \alpha_A^\pm(\mathbf{1})$, simplicity of A as an algebra is thus equivalent to

$$\tilde{Z}(A)_{\mathbf{1},\mathbf{1}} = 1\,. \qquad (2.50)$$

The corresponding notion with left (or right) module instead of bimodule endomorphisms is *haploidity*: A is said to be haploid [20] iff $\mathrm{Hom}_A(A,A) = \Bbbk\, id_A$, i.e. iff $\dim \mathrm{Hom}(\mathbf{1},A) = 1$. Haploidity implies simplicity, but the converse is not true; however, if A is *commutative*, then we have $\mathrm{Hom}_{A|A}(A,A) = \mathrm{Hom}_A(A,A)$, so that in this case haploid and simple are equivalent. Moreover, every simple special Frobenius algebra in a semisimple category is Morita equivalent to a haploid algebra (see the corollary in Section 3.3 of [37]).

(ii) Every modular tensor category gives rise to a three-dimensional topological field theory, and thereby to invariants of ribbon graphs in three-manifolds. (See e.g. [24] or, for a brief summary, Section 2.5 of [16].) In the three-dimensional TFT ribbons are labelled by objects of the underlying modular tensor category and coupons at which ribbons join by corresponding morphisms; our graphical notation for morphisms fits with the usual conventions for drawing the ribbon graphs. For instance, as shown in Section 5.4 of [18], when A is a symmetric special Frobenius algebra in a (semisimple) modular tensor category, then the numbers (2.49) for simple objects $U = U_i$ and $V = U_j$ coincide with the invariant

$$\tilde{Z}(A)_{ij} \;:=$$ (2.51)

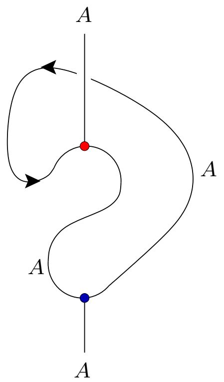


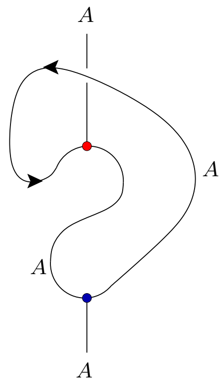
of the indicated ribbon graph in $S^2 \times S^1$. (The S^2 factor is represented by the horizontal circles, while the S^1 factor is given by the vertical direction, i.e. top and bottom of the figure are to be identified.) In Theorem 5.1 of [18] it is shown that for every $i, j \in \mathcal{I}$ the invariant $\tilde{Z}(A)_{ij} \equiv \tilde{Z}(A)_{U_i,U_j}$ is the trace of an idempotent and hence a non-negative integer. The torus partition function of the conformal field theory determined by A is given by $Z(A)_{kl} = \tilde{Z}(A)_{\bar{k}l}$. Furthermore (see Proposition 5.3 of [18] and [14]) for a modular tensor category the integers $\tilde{Z}_{ij}$ defined by (2.51) obey $\tilde{Z}(A \otimes B) = \tilde{Z}(A)\, \tilde{Z}(B)$ as a matrix equation.

2.4 Left and right centers

In this subsection, A denotes a symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$. In a braided setting, there are two different notions of center of an algebra, the left and right center; we establish some properties of the centers that we will need.

The notion of left and right center was introduced in [45] for separable algebras in abelian braided tensor categories, and in [37] for algebras in semisimple abelian ribbon categories. Here we formulate it in terms of a maximality property for idempotents; this makes it applicable even to non-Karoubian categories, provided only that the particular idempotents

P_A^l := 

and P_A^r := 

(2.52)

in $\mathrm{Hom}(A,A)$ are split. That $P_A^{l/r}$ are idempotents follows easily by using the various properties of A; for P_A^l this is described in lemma 5.2 of [18] (setting $X = \mathbf{1}$ there), and for P_A^r the argument is analogous.

These idempotents possess several nice properties. First, we have

Lemma 2.29:
The idempotents (2.52) satisfy the following relations.
(i) They trivialise the twist:

$$\theta_A \circ P_A^l = P_A^l \qquad \text{and} \qquad \theta_A \circ P_A^r = P_A^r \,. \tag{2.53}$$

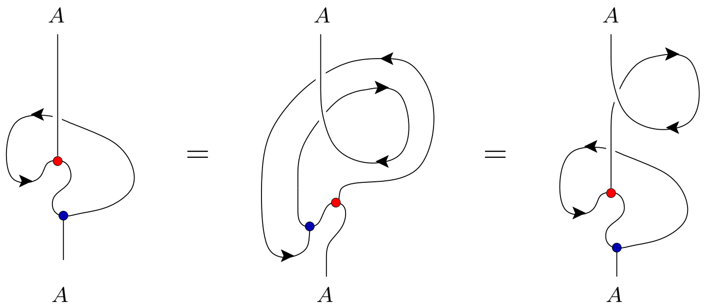
(ii) They are compatible with unit and counit:

$$P_A^{l/r} \circ \eta = \eta \,, \qquad \varepsilon \circ P_A^{l/r} = \varepsilon \,. \tag{2.54}$$

(iii) When each of the three A-ribbons forming a product or coproduct is decorated with $P_A^{l/r}$, then any one of the three idempotents can be omitted:

$$\begin{aligned} P_A^{l/r} \circ m \circ (P_A^{l/r} \otimes P_A^{l/r}) &= m \circ (P_A^{l/r} \otimes P_A^{l/r}) \\ &= P_A^{l/r} \circ m \circ (id_A \otimes P_A^{l/r}) = P_A^{l/r} \circ m \circ (P_A^{l/r} \otimes id_A) \,, \\ (P_A^{l/r} \otimes P_A^{l/r}) \circ \Delta \circ P_A^{l/r} &= (P_A^{l/r} \otimes P_A^{l/r}) \circ \Delta \\ &= (id_A \otimes P_A^{l/r}) \circ \Delta \circ P_A^{l/r} = (P_A^{l/r} \otimes id_A) \circ \Delta \circ P_A^{l/r} \,. \end{aligned} \tag{2.55}$$

Proof:
(i) The statement for P_A^l follows by the moves (the one for P_A^r is derived analogously)



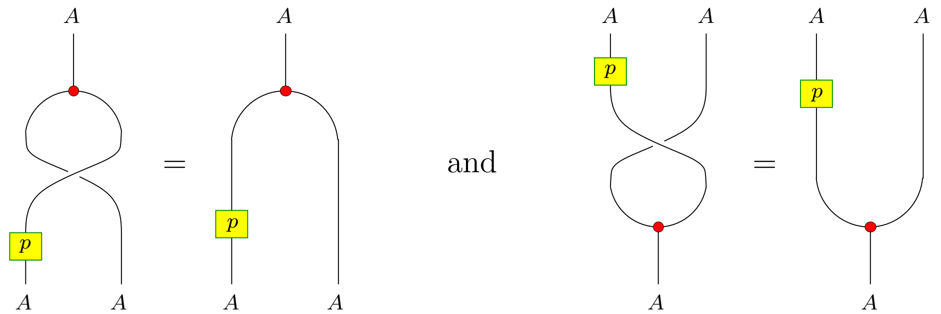
(2.56)

To get the first equality one uses the Frobenius property and then suitably drags the resulting coproduct along part of the A-ribbon. A further deformation and application of the Frobenius property then results in the second equality.

The statements in (ii) and (iii) are just special cases of the statements of Lemma 3.10 below – they follow from those by setting $B = \mathbf{1}$. □

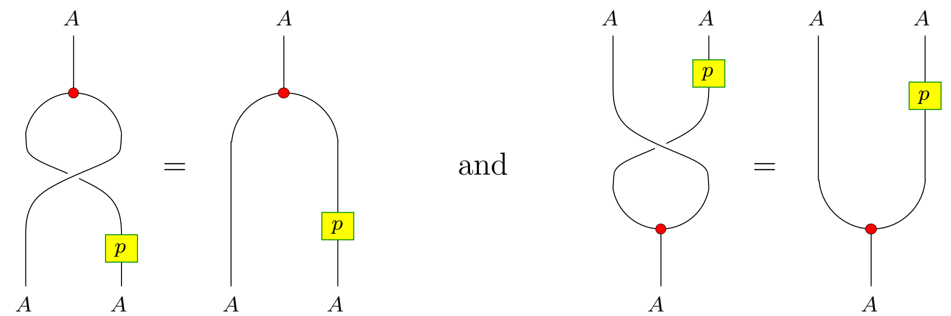
Next recall the Definition 2.11 of maximal idempotent. It turns out that P_A^l and P_A^r can be characterised as being maximal in a subset of Idem(A) that is defined by a relation involving the braiding and the product of A:

Lemma 2.30:
The subset $H_l \subseteq \mathrm{Idem}(A)$ consisting of those idempotents p in End(A) that satisfy



(2.57)

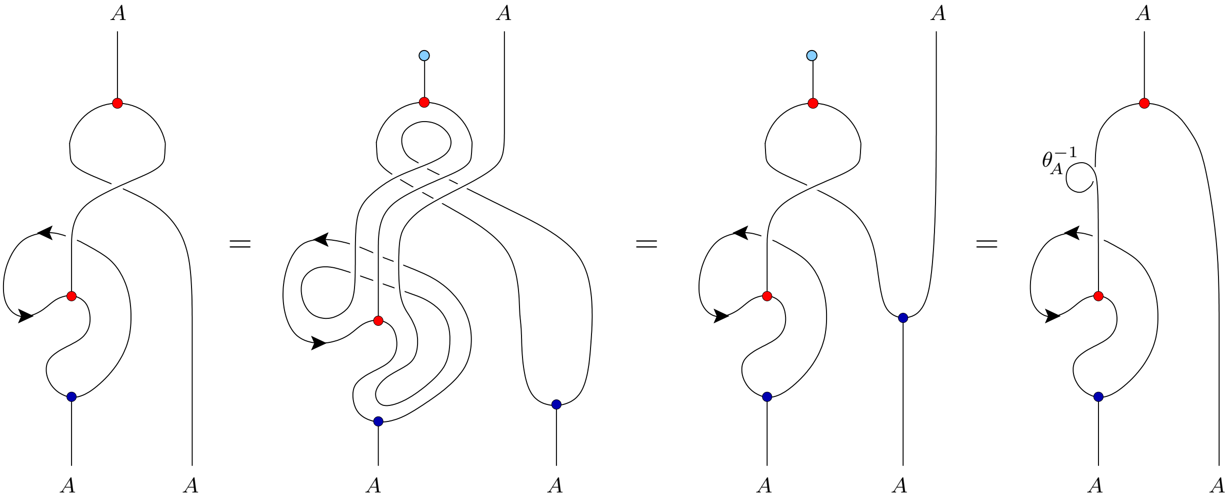
contains a maximal idempotent, and this is given by P_A^l in (2.52).
Analogously, the subset H_r of those idempotents p in End(A) satisfying



(2.58)

contains a maximal idempotent, given by P_A^r in (2.52).

Proof:
We prove the statement for P_A^l; the statement for P_A^r follows analogously.
Consider the transformations



(2.59)

In the first step the identity $id_A = (\varepsilon \otimes id_A) \circ \Delta$ is inserted in the top A-ribbon, and afterwards the coproduct Δ introduced this way is moved along a path that can easily be read off the second picture, using the Frobenius and coassociativity properties of A. The next step is just a deformation of the outgoing A-ribbon. The third step uses that A is symmetric, and then the Frobenius property.
Afterwards, according to Lemma 2.29 the twist θ_A^{-1} can be left out. This establishes that P_A^l satisfies the first of the equalities (2.57). That P_A^l satisfies the second of those equalities as well is deduced similarly, starting with an insertion of the identity $id_A = m \circ (\eta \otimes id_A)$ on the incoming A-ribbon. Together it follows that $P_A^l \in H_l$.

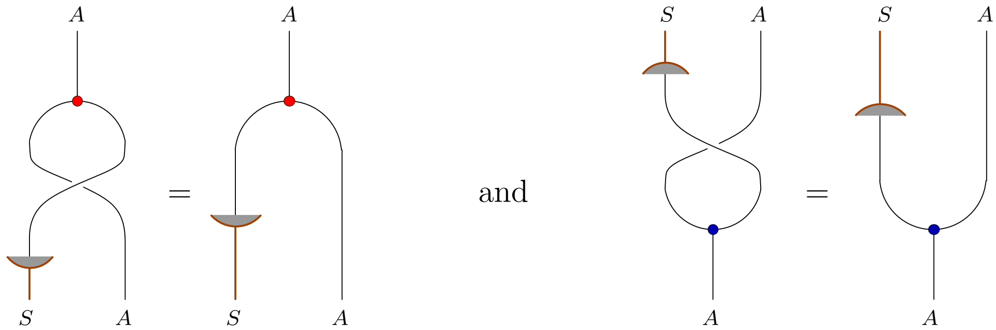
Furthermore, composing the first equality in (2.57) from the bottom with $id_A \otimes b_A$ and from the top with $(id_A \otimes \tilde{d}_A) \circ (\Delta \otimes id_{A^\vee})$ shows, upon using the symmetry, specialness and Frobenius properties, that $p = P_A^l \circ p$ for $p \in H_l$. Similar manipulations of the second equality in (2.57) show that also $p = p \circ P_A^l$ for $p \in H_l$. Thus P_A^l is maximal in H_l. □

Definition 2.31:
(i) We call the morphism P_A^l defined in (2.52) the *left central idempotent* of the symmetric special Frobenius algebra A, and P_A^r the *right central idempotent* of A.

(ii) The *left center* $C_l(A)$ of the symmetric special Frobenius algebra A is the maximal retract of A with respect to H_l.
The *right center* $C_r(A)$ of A is the maximal retract of A with respect to H_r.

According to Lemma 2.30 the left and right central idempotents $P_A^{l/r}$ are the maximal idempotents of the subsets H_l and H_r of Idem(A), respectively. It follows that the left (right) center of A exists iff the left (right) central idempotent is split; if this is the case, then by corollary 2.13, $C_{l/r}(A)$ is unique. In the sequel we will often use the short-hand notation C_l and C_r for $C_l(A)$ and $C_r(A)$, respectively. Also note that the definition of the centers involves both the algebra and the coalgebra structure of A; in place of the term center one might therefore also use the term 'Frobenius center'.

Lemma 2.32:
Any retract (S,e,r) of a symmetric special Frobenius algebra A that obeys $m\circ c_{A,A}\circ(e_{S\prec A}\otimes id_A)=m\circ(e_{S\prec A}\otimes id_A)$ and $(r_{A\succ S}\otimes id_A)\circ c^{-1}_{A,A}\circ\Delta=(r_{A\succ S}\otimes id_A)\circ\Delta$, i.e.



(2.60)

also satisfies

$$P^l_A\circ e_{S\prec A}=e_{S\prec A}\qquad\text{and}\qquad r_{A\succ S}\circ P^l_A=r_{A\succ S}\,. \tag{2.61}$$

Similarly we have

$$\left.\begin{array}{r} m\circ c_{A,A}\circ(id_A\otimes e_{S\prec A})=m\circ(id_A\otimes e_{S\prec A})\\ (id_A\otimes r_{A\succ S})\circ c^{-1}_{A,A}\circ\Delta=(id_A\otimes r_{A\succ S})\circ\Delta \end{array}\right\}\Rightarrow\left\{\begin{array}{l} P^r_A\circ e_{S\prec A}=e_{S\prec A}\\ r_{A\succ S}\circ P^r_A=r_{A\succ S}\,. \end{array}\right. \tag{2.62}$$

Proof:
Composing (2.60) from the bottom with $r\otimes id_A$ shows that the idempotent $p=e_{S\prec A}\circ r_{A\succ S}$ satisfies the first of the equalities (2.57). Analogously one shows that p also obeys the second of those equalities, and hence it is contained in H_l. Thus the relations (2.61) are implied by (2.60) together with the maximality property of P^l_A.
The implication (2.62) is derived analogously. □

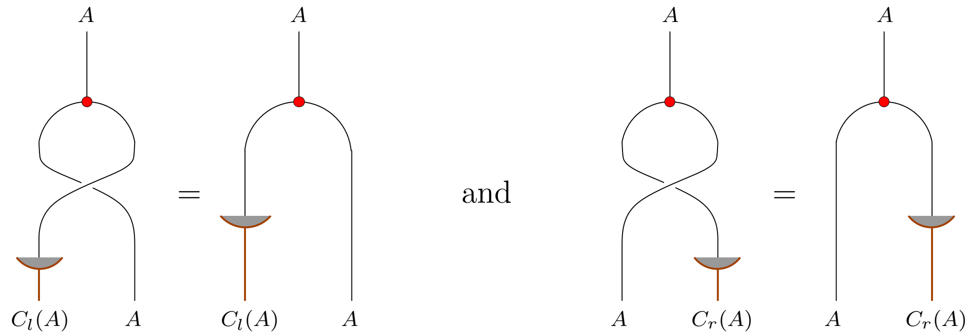
Lemma 2.33:
The left and right center of a symmetric special Frobenius algebra A have trivial twist:

$$\theta_{C_l(A)}=id_{C_l(A)}\,,\qquad\theta_{C_r(A)}=id_{C_r(A)}\,. \tag{2.63}$$

Proof:

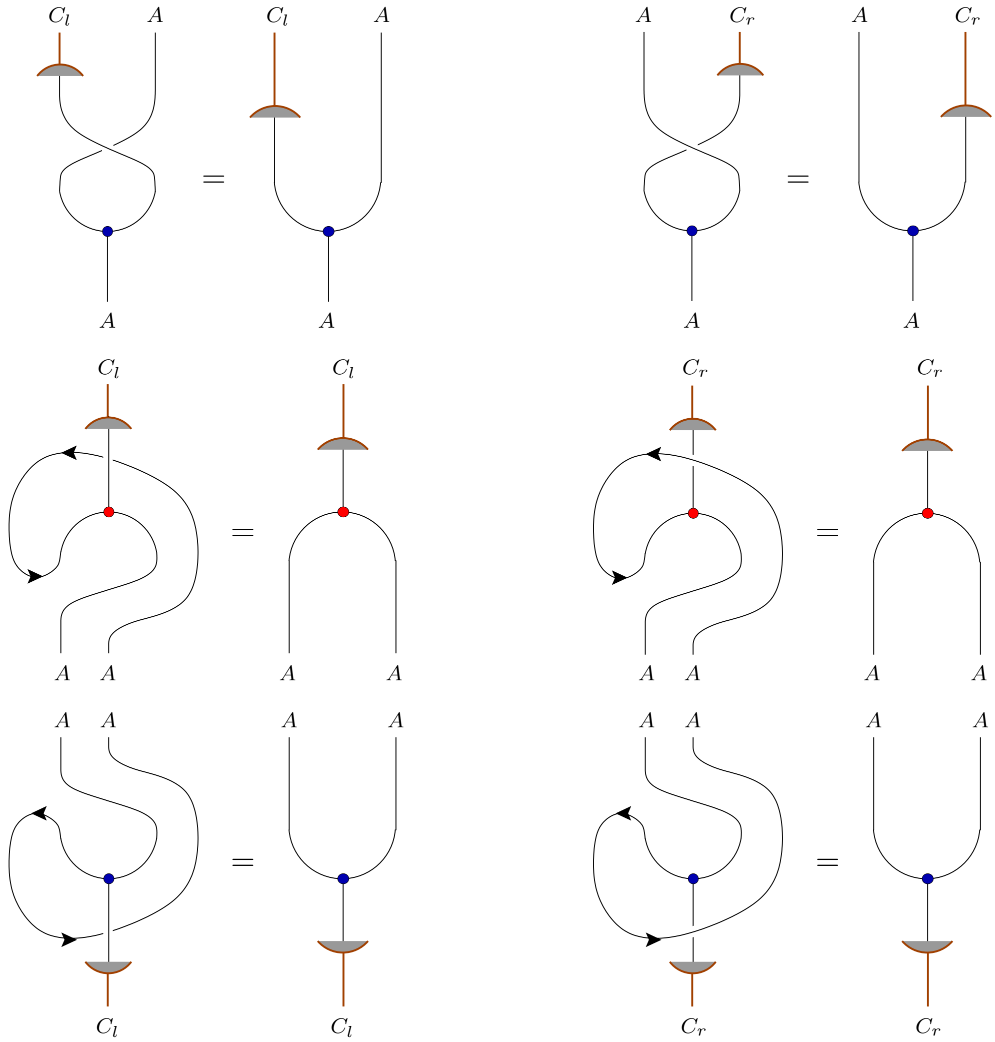
The statement follows immediately from the relations (2.53). (Conversely, (2.53) follows from (2.63) by functoriality of the twist.) □

Remark 2.34 :
As a consequence of Lemma 2.32 the centers obey



(2.64)

respectively, as well as



(2.65)

together with the 'mirrored' versions of these eight identities that are obtained by reflecting all the figures about a vertical axis. For instance, to establish the last of the equalities

(2.65), one can start with the Frobenius relation (2.37) composed with $e_{C_r\prec A}\otimes\eta$, then apply the mirrored version of (2.64) to the resulting product, and finally use the symmetry and Frobenius properties to remove the unit that was introduced in the first step.

In the Definition (2.49) of the numbers $\tilde{Z}(A)_{U,V}$ the two different α-inductions were used. The corresponding morphism spaces for α-inductions of the same type turn out to be related to the centers of A. To see this we first need

Lemma 2.35:
Let A be a symmetric Frobenius algebra in a ribbon category $\mathcal{C}$ and $U,V\in\mathrm{Obj}(\mathcal{C})$. Then for any $\varphi^+\in\mathrm{Hom}_{A|A}(\alpha_A^+(U),\alpha_A^+(V))$ and any $\varphi^-\in\mathrm{Hom}_{A|A}(\alpha_A^-(U),\alpha_A^-(V))$ we have

$$(P_A^l\otimes id_V)\circ\varphi^+=\varphi^+\circ(P_A^l\otimes id_U)\qquad\text{and}\qquad(P_A^r\otimes id_V)\circ\varphi^-=\varphi^-\circ(P_A^r\otimes id_U)\,.\quad(2.66)$$

Proof:
Using functoriality of the braiding and the fact that A is symmetric Frobenius one easily rewrites the morphism $P_A^{l/r}\otimes id_U$ in such a way that it involves the left and right action of A on the bimodule $\alpha_A^{\pm}(U)$. Since $\varphi^{\pm}$ is a morphism of bimodules, these actions of A can thus be passed through $\varphi^{\pm}$ (using again also functoriality of the braiding). Afterwards one follows the steps used in rewriting $P_A^{l/r}\otimes id_U$ in reverse order, resulting in $P_A^{l/r}\otimes id_V$. □

Using this lemma, we deduce the following relation with the centers of A.

Proposition 2.36:
For any symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ and any two objects $U,V\in\mathrm{Obj}(\mathcal{C})$ there are natural bijections

$$\mathrm{Hom}(C_l(A)\otimes U,V)\;\cong\;\mathrm{Hom}_{A|A}(\alpha_A^+(U),\alpha_A^+(V))\cong\mathrm{Hom}(U,C_l(A)\otimes V)\qquad(2.67)$$

and

$$\mathrm{Hom}(C_r(A)\otimes U,V)\;\cong\;\mathrm{Hom}_{A|A}(\alpha_A^-(U),\alpha_A^-(V))\cong\mathrm{Hom}(U,C_r(A)\otimes V)\,.\qquad(2.68)$$

Proof:
We prove the first bijection in (2.67), the proof of the others being analogous.
Let us abbreviate $C_l(A)=C$ as well as $e_{C_l(A)\prec A}=e$ and $r_{A\succ C_l(A)}=r$. Consider the mappings Φ: $\mathrm{Hom}(C\otimes U,V)\to\mathrm{Hom}(A\otimes U,A\otimes V)$ and Ψ: $\mathrm{Hom}_{A|A}(\alpha_A^+(U),\alpha_A^+(V))\to\mathrm{Hom}(C\otimes U,V)$ defined by

$$\begin{aligned}\Phi(\varphi)&:=(id_A\otimes\varphi)\circ[\,((id_A\otimes r)\circ\Delta))\otimes id_U]\;,\\ \Psi(\psi)&:=(\varepsilon\otimes id_V)\circ\psi\circ(e\otimes id_U)\,.\end{aligned}\qquad(2.69)$$

It is not difficult to check that for any $\varphi\in\mathrm{Hom}(C\otimes U,V)$, $\Phi(\varphi)$ intertwines both the left and the right action of A on α_A^+-induced bimodules, and hence the image of Φ lies actually in $\mathrm{Hom}_{A|A}(\alpha_A^+(U),\alpha_A^+(V))$.
Furthermore, Φ and Ψ are two-sided inverses of each other. That $\Psi\circ\Phi(\varphi)=\varphi$ is seen

by just applying the defining property of the counit and then using $r\circ e = id_C$, while to establish $\Phi\circ\Psi(\psi)=\psi$, one invokes Lemma 2.35 to move the idempotent $e\circ r = P_A^l$ arising from the composition past ψ and then uses $\varepsilon\circ P_A^l = \varepsilon$ (Lemma 2.29(ii)). □

It follows that in case a right-adjoint functor $(\alpha_A^\pm)^\dagger$ exists, the composition of $\alpha_A^\pm$ with its adjoint functor is nothing but ordinary induction with respect to $C_{l/r}(A)$, followed by restriction to $\mathcal{C}$.

We are now in a position to establish

Proposition 2.37:
The left and right centers $C_{l/r}$ of a symmetric special Frobenius algebra A inherit natural structure as a retract of A. More precisely, we have:

(i) C_l and C_r are commutative symmetric Frobenius algebras in $\mathcal{C}$.

(ii) If, in addition, A is simple, then C_l and C_r are simple, too.

(iii) If $C_{l/r}$ is simple, then it is special iff $\dim(C_{l/r}) \neq 0$.

Proof:
(i) We set

$$\begin{aligned} m_C &:= r_C\circ m\circ(e_C\otimes e_C)\,, \qquad & \Delta_C &:= \zeta^{-1}\,(r_C\otimes r_C)\circ\Delta\circ e_C\,, \\ \eta_C &:= r_C\circ\eta\,, & \varepsilon_C &:= \zeta\,\varepsilon\circ e_C\,, \end{aligned} \tag{2.70}$$

for some $\zeta\in\Bbbk^\times$, where $C\equiv C_{l/r}$, and with $e_C\equiv e_{C\prec A}$, and $r_C\equiv r_{A\succ C}$ the embedding and restriction morphisms, respectively, for C as a retract of A. That is, for the product and the unit on C we take the restriction of the product on A, whereas the coproduct and the counit are only fixed up to some invertible scalar.
That η_C and ε_C satisfy the (co-)unit properties follows from the corresponding properties of A, by Lemma 2.29(ii). The (co-)associativity of m_C and Δ_C as well as the Frobenius property are checked with the help of Lemma 2.29(iii). Thus $(C_{l/r}, m_C, \eta_C, \Delta_C, \varepsilon_C)$ are indeed Frobenius algebras.
That C_l is commutative is seen by composing the first of the equalities (2.64) with $id_{C_l}\otimes e_{C_l}$ from below and with r_{C_l} from above. Commutativity of C_r follows analogously. Further, commutativity together with triviality of the twist (lemma 2.33) imply that C is symmetric.

(ii) It follows from (2.67), with $U=V=\mathbf{1}$, and simplicity of A that C is haploid, and hence in particular simple.

(iii) The first specialness property holds independently of the value of the dimension of C: with the help of (2.54) one finds $\varepsilon_C\circ\eta_C = \zeta\,\dim(A)$, which is non-zero.
Denote by $\varepsilon_{C,\natural}\in\mathrm{Hom}(A,\mathbf{1})$ the morphism defined as in equation (2.39), but with the Frobenius algebra C in place of A. Since C is commutative and simple, it is also haploid, and hence this morphism must be a multiple of ε_C. The constant of proportionality can be determined by composing the equality with η; the result is

$$\varepsilon_{C,\natural} = \frac{\dim(C)}{\dim(A)}\,\zeta^{-1}\,\varepsilon_C\,. \tag{2.71}$$

It follows that $\varepsilon_{C,\natural}$ and ε_C are non-zero multiples of each other iff $\dim(C)\neq 0$. On the other hand, equality of of $\varepsilon_{C,\natural}$ and ε_C up to a non-zero constant is equivalent to specialness of the symmetric Frobenius algebra C; see lemma 3.11 of [18].
(Note that we recover our usual normalisation convention for special Frobenius algebras by fixing the scalar factor ζ in (2.70) to $\zeta = \dim(C)/\dim(A)$.) □

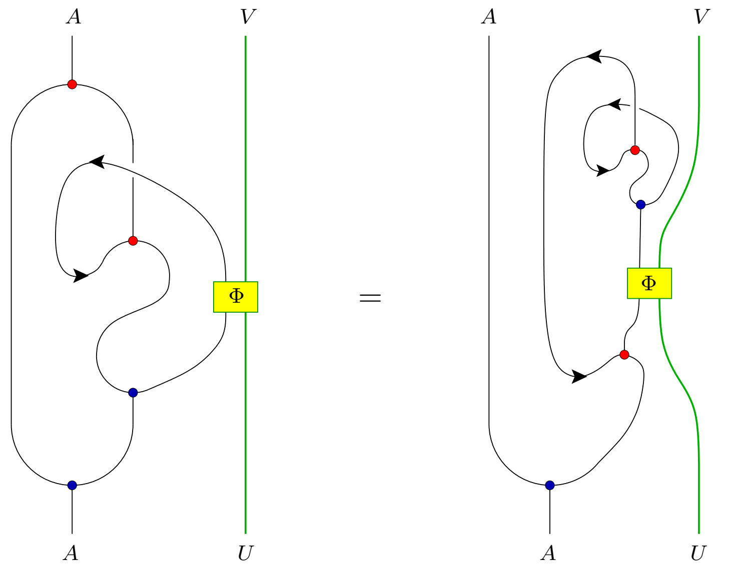
Remark 2.38:
(i) Part (ii) of the proposition generalises the classic result that the center of a simple $\mathbb{C}$-algebra is just given by $\mathbb{C}$.

(ii) Alternatively, symmetry of $C_{l/r}$ follows by combining symmetry of A with the identity $\varepsilon \circ P_A^{l/r} = \varepsilon$ (Lemma 2.29(ii)). As a consequence, triviality of the twist of $C_{l/r}$ (Lemma 2.33) or, equivalently, Lemma 2.29(i), can also be deduced by combining Proposition 2.37 with Lemma 2.29(ii).

(iii) In the proof of Proposition 2.37(iii) above, as well as at several other places below, we use conventions and results from [18]. In [18], which builds on earlier studies in [16] and [20], the relevant categories are assumed to be abelian and semisimple. The proofs of those results from [18] that are employed in this paper are, however, easily adapted to the present setting.

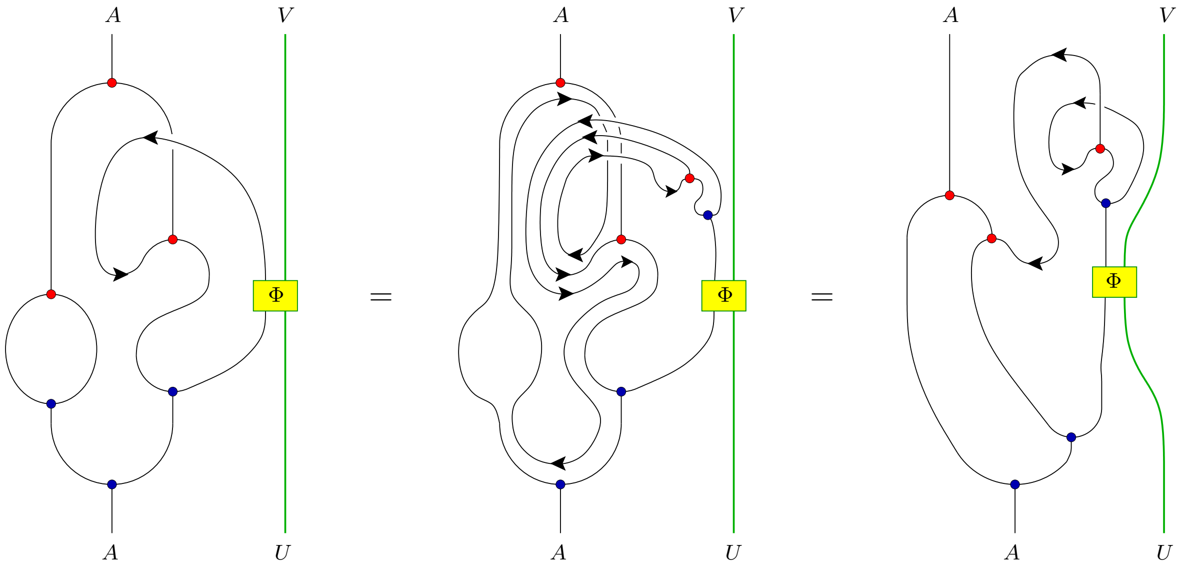
We close this section with another helpful result, to be used later on, in which the central idempotents (2.52) arise. We present the formula with P_A^l; an analogous formula with P_A^r holds in which the braiding on the left hand side is replaced by the opposite braiding.

Lemma 2.39:
For A a symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$, U and V objects of $\mathcal{C}$, and $\Phi \in \mathrm{Hom}(A{\otimes}U, A{\otimes}V)$ the following identity holds:



(2.72)

Proof:
Consider the following manipulations.



(2.73)

Here in the first step the coproduct and product in the left A-loop are dragged apart, using that A is Frobenius, along the A-ribbons until they result in the coproduct and product above Φ in the middle picture. The second step is a deformation of the A-ribbon that connects that coproduct and product, using also the properties (2.53) and (2.57) of the left central idempotent.
The left hand sides of the equations (2.73) and of (2.72) are equal owing to specialness of A, and their right hand sides are equal because A is special Frobenius. Thus (2.72) follows from (2.73). □

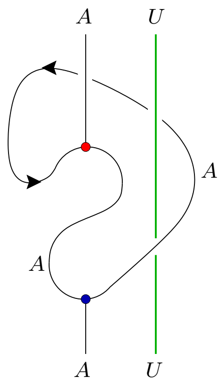
3 Local modules

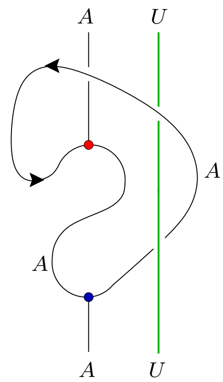
3.1 Endofunctors related to α-induction

One interesting aspect of symmetric special Frobenius algebras A in a ribbon category $\mathcal{C}$ is that they allow us to construct functors to the categories of modules over the left and right center of A, respectively, which are similar to the induction functor from $\mathcal{C}$ to the category of A-modules. We call these functors *local induction* functors. The construction makes use of certain endofunctors of $\mathcal{C}$ which are associated to A.

For these endofunctors to exist, the symmetric special Frobenius algebra must have an additional property. To motivate this property, recall from Section 2.4 that for the left and right center of A to exist, the central idempotents $P_A^{l/r}$ defined in (2.52) must be split. The construction of the endofunctors makes use of similar endomorphisms for each object

U of $\mathcal{C}$, namely of the morphisms

$$P^l_A(U) \;:=\;$$ 

$$P^r_A(U) \;:=\;$$ 

(3.1)

in $\mathrm{Hom}(A{\otimes}U,A{\otimes}U)$. It is easily verified that, just like $P^{l/r}_A \equiv P^{l/r}_A(\mathbf{1})$ these endomorphisms are idempotents (for $P^l_A(U)$ this has already been shown in lemma 5.2 of [18].)

Definition 3.1:
A special Frobenius algebra A in a ribbon category $\mathcal{C}$ is called *centrally split* iff the idempotents (3.1) are split for every $U\in\mathrm{Obj}(\mathcal{C})$.

Clearly, in a Karoubian ribbon category (and hence in particular in a modular tensor category) every Frobenius algebra is centrally split. Recall, however, that occasionally we want to allow for non-Karoubian categories. Then we need centrally split algebras in order to ensure the existence of the desired endofunctors. Accordingly we make the following

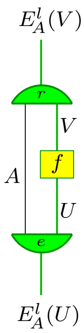
Declaration 3.2:
In the sequel every special Frobenius algebra A will be assumed to be centrally split.

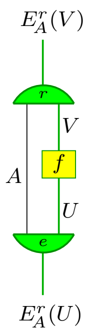
With this agreement in mind, we can now proceed to

Definition 3.3:
For A a symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$, the operations $E^{l/r}_A$ are defined on objects and morphisms of $\mathcal{C}$ as follows.
For $U\in\mathrm{Obj}(\mathcal{C})$, $E^{l/r}_A(U)$ are the retracts

$$E^l_A(U) := \mathrm{Im}\, P^l_A(U) \qquad \text{and} \qquad E^r_A(U) := \mathrm{Im}\, P^r_A(U) \tag{3.2}$$

of the induced module $A\otimes U$, with the idempotents $P^{l/r}_A(U)\in\mathrm{Hom}(A{\otimes}U,A{\otimes}U)$ given by (3.1).
For $f\in\mathrm{Hom}(U,V)$, $E^{l/r}_A(f)\in\mathrm{Hom}(E^{l/r}_A(U),E^{l/r}_A(V))$ are the morphisms

$$E^l_A(f) \;:=\;$$ 

$$E^r_A(f) \;:=\;$$ 

(3.3)

with $e \equiv e_{E_A^{l/r}(U) \prec A\otimes U}$ and $r \equiv r_{A\otimes V \succ E_A^{l/r}(V)}$.

Let us remark that this construction is non-trivial only in a genuinely braided tensor category. For, when $\mathcal{C}$ is a *symmetric* tensor category, the projection just amounts to considering the objects $C \otimes U$, where C is the center of the algebra A. Note that these are precisely the objects that underlie induced C-modules; as we will see later, the objects $E_A^{l/r}(U)$ naturally carry a module structure, too: they are modules over the left and right center of A, respectively.

Proposition 3.4:
The operations $E_A^{l/r}$ are endofunctors of $\mathcal{C}$.

Proof:
Let E stand for one of E_A^l, E_A^r. It follows from the definitions (3.1) and (3.3) that for any $g \in \mathrm{Hom}(U,V)$ we have $E(g) \in \mathrm{Hom}(E(U),E(V))$, i.e. $E(g)$ is in the correct space. It remains to check that for any $g' \in \mathrm{Hom}(V,W)$ one has $E(g'\circ g) = E(g')\circ E(g)$ and that $E(id_U) = id_{E(U)}$. The second property is obvious because $E_A^{l/r}(id_U) = r^{l/r} \circ e^{l/r}$ is indeed nothing but the identity morphism $id_{E(U)}$ on the retract. For the first property we note that, writing out the definitions for $E(g'\circ g)$ and $E(g')\circ E(g)$, these two morphisms only differ by an idempotent (3.1). By functoriality of the braiding we can shift this idempotent past g so that it gets directly composed with the embedding morphism e, and then (2.11) tells us that it can be left out. □

These functors are, however, in general *not* tensor functors.

The following lemma will be used in the proof of Proposition 3.6.

Lemma 3.5:
(i) For every symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ and every $U \in \mathrm{Obj}(\mathcal{C})$, and with right A-actions $\rho_\mathrm{r}^\pm$ defined as in (2.31), we have

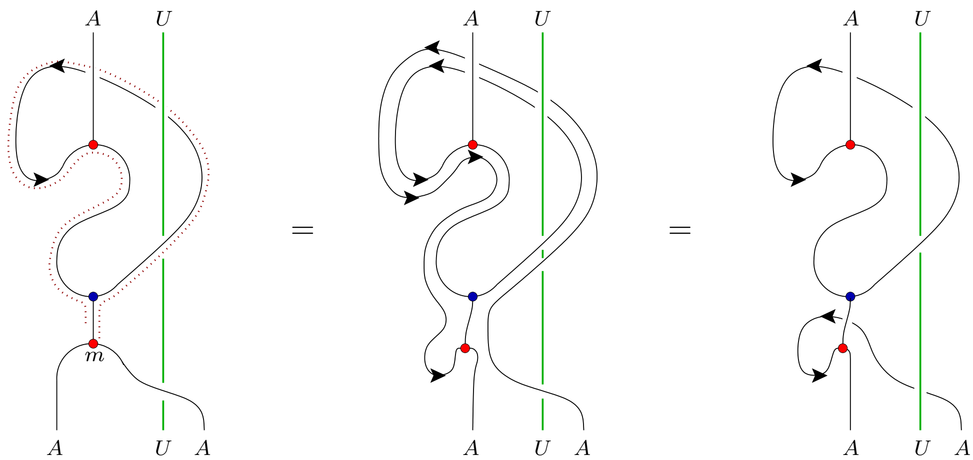
$$\begin{aligned} P_A^l(U)\circ\rho_\mathrm{r}^- &\equiv P_A^l(U)\circ(m\otimes id_U)\circ(id_A\otimes c_{A,U}^{-1}) \\ &= P_A^l(U)\circ\big([m\circ c_{A,A}\circ(id_A\otimes\theta_A)]\otimes id_U\big)\circ(id_A\otimes c_{U,A})\,, \\ P_A^r(U)\circ\rho_\mathrm{r}^+ &\equiv P_A^r(U)\circ(m\otimes id_U)\circ(id_A\otimes c_{U,A}) \\ &= P_A^r(U)\circ\big([m\circ c_{A,A}\circ(id_A\otimes\theta_A^{-1})]\otimes id_U\big)\circ(id_A\otimes c_{A,U}^{-1})\,. \end{aligned} \tag{3.4}$$

(ii) If A is in addition commutative, then

$$P_A(U)\circ\rho_\mathrm{r}^+ = P_A(U)\circ\rho_\mathrm{r}^- \tag{3.5}$$

for $P_A(U) \equiv P_A^{l/r}(U)$.

Proof:
(i) The first of the formulas (3.4) follows by the moves



(3.6)

In the first picture, the dotted line is not part of the morphism, but rather only indicates a path along which the product that is marked explicitly is 'dragged' (using functoriality of the braiding, as well as associativity and the Frobenius property of A) so as to arrive at the first equality. The second equality is obtained by deforming the A-ribbon that results from this dragging.
The second of the formulas (3.4) is seen analogously, with under- and overbraidings interchanged.

(ii) follows immediately form (i) by using that A has trivial twist (Proposition 2.25(i)) and the definition of commutativity. (Also, in the commutative case we actually have $P_A^l(U) = P_A^r(U)$, see the picture (3.23) below.) □

Note that, obviously, the assertions made in the lemma are non-trivial only if the tensor category $\mathcal{C}$ is genuinely braided. The same remark applies to several other statements below, in particular to Theorem 5.20. (Compare also to the considerations at the end of Section 1.3.)

Assume now that there exist right-adjoint functors $(\alpha_A^\pm)^\dagger$ to the $\alpha_A^\pm$-induction functors. The following result shows that in this case the endofunctors $E_A^{l/r}$ can be regarded as the composition of $(\alpha_A^\pm)^\dagger$ with $\alpha_A^\mp$. (The result does not imply that such right-adjoint functors exist. They certainly do exist, though, if $\mathcal{C}$ is semisimple with finite number of non-isomorphic simple objects, in particular if $\mathcal{C}$ is modular.)

Proposition 3.6:
For every symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ and any two objects $U, V \in \mathrm{Obj}(\mathcal{C})$ there are natural bijections

$$\mathrm{Hom}(E_A^l(U), V) \;\cong\; \mathrm{Hom}_{A|A}(\alpha_A^-(U), \alpha_A^+(V)) \;\cong\; \mathrm{Hom}(U, E_A^r(V)) \qquad (3.7)$$

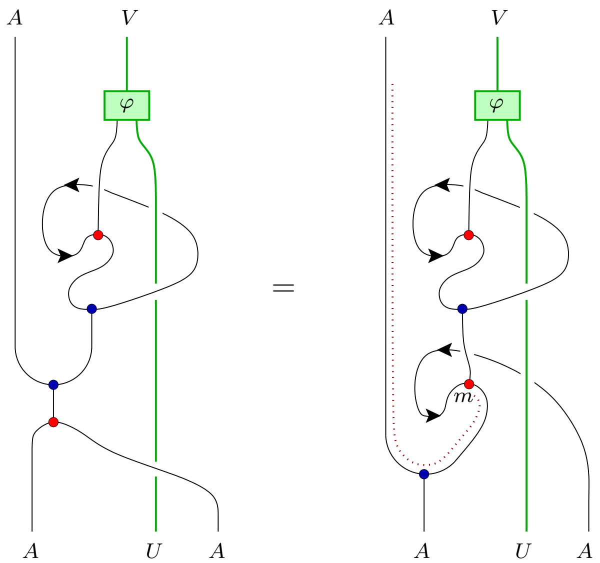
and

$$\mathrm{Hom}(E_A^r(U), V) \;\cong\; \mathrm{Hom}_{A|A}(\alpha_A^+(U), \alpha_A^-(V)) \;\cong\; \mathrm{Hom}(U, E_A^l(V))\,. \qquad (3.8)$$

Proof:
Let us start with the first equivalence in (3.7). Recall that according to the reciprocity relation (2.41) there is a natural bijection Φ: $\mathrm{Hom}(A{\otimes}U,V) \stackrel{\cong}{\to} \mathrm{Hom}_A(\mathrm{Ind}_A(U),\mathrm{Ind}_A(V))$, and note that the target of this bijection contains the middle expression of (3.7) as a natural subspace.
Furthermore, in view of Lemma 2.4(i), by definition of $E_A^l(\cdot)$ we may identify the left hand side of (3.7) with the subspace $\mathrm{Hom}_{(P_A^l(U))}(A{\otimes}U,V)$ of $\mathrm{Hom}(A{\otimes}U,V)$. Thus it is sufficient to show that Φ restricts to a bijection between this subspace and $\mathrm{Hom}_{A|A}(\alpha_A^-(U),\alpha_A^+(V))$. The map Φ and its inverse are defined similarly as in formula (2.69); they act as

$$\varphi \;\mapsto\; (id_A \otimes \varphi) \circ (\Delta \otimes id_U) \qquad \text{and} \qquad \psi \;\mapsto\; (\varepsilon \otimes id_V) \circ \psi \tag{3.9}$$

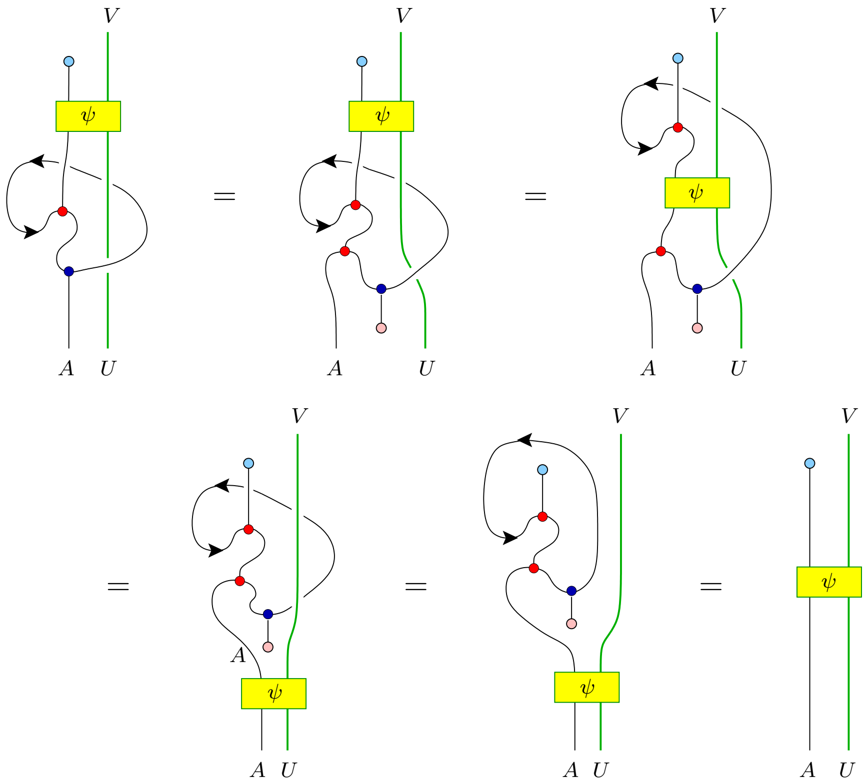
for $\varphi \in \mathrm{Hom}(A{\otimes}U,V)$ and $\psi \in \mathrm{Hom}_A(\mathrm{Ind}_A(U),\mathrm{Ind}_A(V))$, respectively. The following considerations show that Φ and its inverse restrict to linear maps between $\mathrm{Hom}_{(P_A^l(U))}(A{\otimes}U,V)$ and $\mathrm{Hom}_{A|A}(\alpha_A^-(U),\alpha_A^+(V))$.
First, for $\varphi \in \mathrm{Hom}_{(P_A^l(U))}(A{\otimes}U,V)$ the morphism $\Phi(\varphi) \circ \rho_\mathrm{r}^-(U) \equiv \Phi(\varphi \circ P_A^l(U)) \circ \rho_\mathrm{r}^-(U)$ is given by the left hand side of the equality



(3.10)

This equality, in turn, is a straightforward application of lemma 3.5. Further, by dragging the marked product along the path indicated by the dashed line and deforming the resulting ribbon (using functoriality of the braiding) such that the braiding occurs above the morphism φ and omitting again the idempotent $P_A^l(U)$ then yields the graphical description of $\rho_\mathrm{r}^+(U) \circ (\Phi(\varphi){\otimes}id_A)$.
This shows that $\Phi(\varphi)$ is indeed a morphism of α-induced bimodules.
The required property of Φ^{-1} is obtained by the following manipulations, valid for every

$\psi \in \mathrm{Hom}_{A|A}(\alpha_A^-(U), \alpha_A^+(V))$:



(3.11)

The first equality uses that A is Frobenius; the second and third use functoriality of the braiding and the fact that ψ intertwines the A-bimodules $\alpha_A^-(U)$ and $\alpha_A^+(V)$ (more specifically, that ψ is a morphism of left modules for the second, and that it is a morphism of right modules for the third equality); and the fourth is just a deformation of the A-loop. The last equality combines the fact that A is symmetric Frobenius and the identification of the counit with the morphism $\varepsilon_\natural$ (2.39). Thus indeed $\Phi^{-1}(\psi) \circ P_A^l(U) = \Phi^{-1}(\psi)$.

Next consider the second equivalence in (3.8). In this case we can use the natural bijection $\tilde\Phi$: $\mathrm{Hom}(U, A{\otimes}V) \xrightarrow{\cong} \mathrm{Hom}_A(\mathrm{Ind}_A(U), \mathrm{Ind}_A(V))$ as well as the equivalence $\mathrm{Hom}(U, E_A^l(V)) \cong \mathrm{Hom}^{(P_A^l(V))}(U, A{\otimes}V)$, see equation (2.13). Explicitly, the linear map $\tilde\Phi$ and its inverse are given by

$$\varphi \;\mapsto\; (m \otimes id_V) \circ (id_A \otimes \varphi) \qquad \text{and} \qquad \psi \;\mapsto\; \psi \circ (\eta \otimes id_U)\,. \tag{3.12}$$

Similarly to the argument above, one shows that $\tilde\Phi$ and its inverse restrict to linear maps between $\mathrm{Hom}^{(P_A^l(V))}(U, A{\otimes}V)$ and $\mathrm{Hom}_{A|A}(\alpha_A^+(U), \alpha_A^-(V))$. For example, the pictures occurring in the proof of $\tilde\Phi(\varphi) \circ \rho_\mathrm{r}^+(U) = \rho_\mathrm{r}^-(U) \circ (\tilde\Phi(\varphi){\otimes}id_A)$ look like the ones in (3.10) except that they are 'reflected' about a horizontal axis.

The remaining two equivalences are derived analogously. □

Remark 3.7:
When $\mathcal{C}$ is in addition semisimple, then it follows that the objects $E_A^{l/r}(U)$ decompose into simple objects as

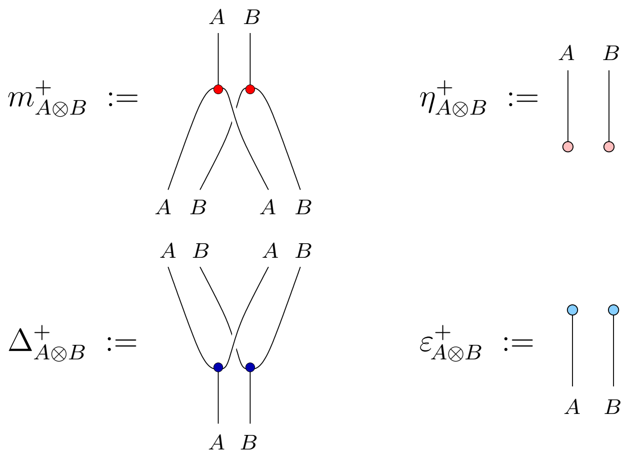
$$E_A^l(U) \cong \bigoplus_{i\in\mathcal{I}} \Big(\sum_{q\in\mathcal{I}} \tilde{Z}(A)_{iq}\, n_q \Big) U_i \quad \text{and} \quad E_A^r(U) \cong \bigoplus_{i\in\mathcal{I}} \Big(\sum_{q\in\mathcal{I}} n_q\, \tilde{Z}(A)_{qi} \Big) U_i \,, \tag{3.13}$$

with the non-negative integers n_q defined by the decomposition $U \cong \bigoplus_q n_q U_q$ of U. Thus when expressing objects as direct sums of the simple objects U_i with $i\in\mathcal{I}$, the action of the functor $E_A^{l/r}(\,\cdot\,)$ on objects amounts to multiplication from the left and right, respectively, with the matrix $\tilde{Z}(A)$.

3.2 Endofunctors on categories of algebras

One datum contained in a Frobenius algebra $(B,m,\Delta,\eta,\epsilon)$ is the object $B\in\mathrm{Obj}(\mathcal{C})$, on which we can consider the action of the endofunctors $E_A^{l/r}$ associated to some symmetric special Frobenius algebra A. We wish to show that the objects $E_A^{l/r}(B)$ carry again the structure of a Frobenius algebra. (This would be obvious if the functors $E_A^{l/r}$ were tensor functors, because then we could simply take $E_A^{l/r}(m)$ as the multiplication morphism. But this is not the case, in general.) This will imply that $E_A^{l/r}$ also provides us with endofunctors on the category of Frobenius algebras in $\mathcal{C}$. Since $E_A^{l/r}(B)$ is a retract of $A\otimes B$, what we first need is the notion of a tensor product of two Frobenius algebras A and B.

For any pair A,B of Frobenius algebras in a ribbon category there are in fact two natural Frobenius algebra structures – to be denoted by $A\otimes^\pm B \equiv (A\otimes B, m^\pm_{A\otimes B}, \eta^\pm_{A\otimes B}, \Delta^\pm_{A\otimes B}, \varepsilon^\pm_{A\otimes B})$ – on the tensor product object $A\otimes B$. For the case of $\otimes^+$, the structural morphisms are



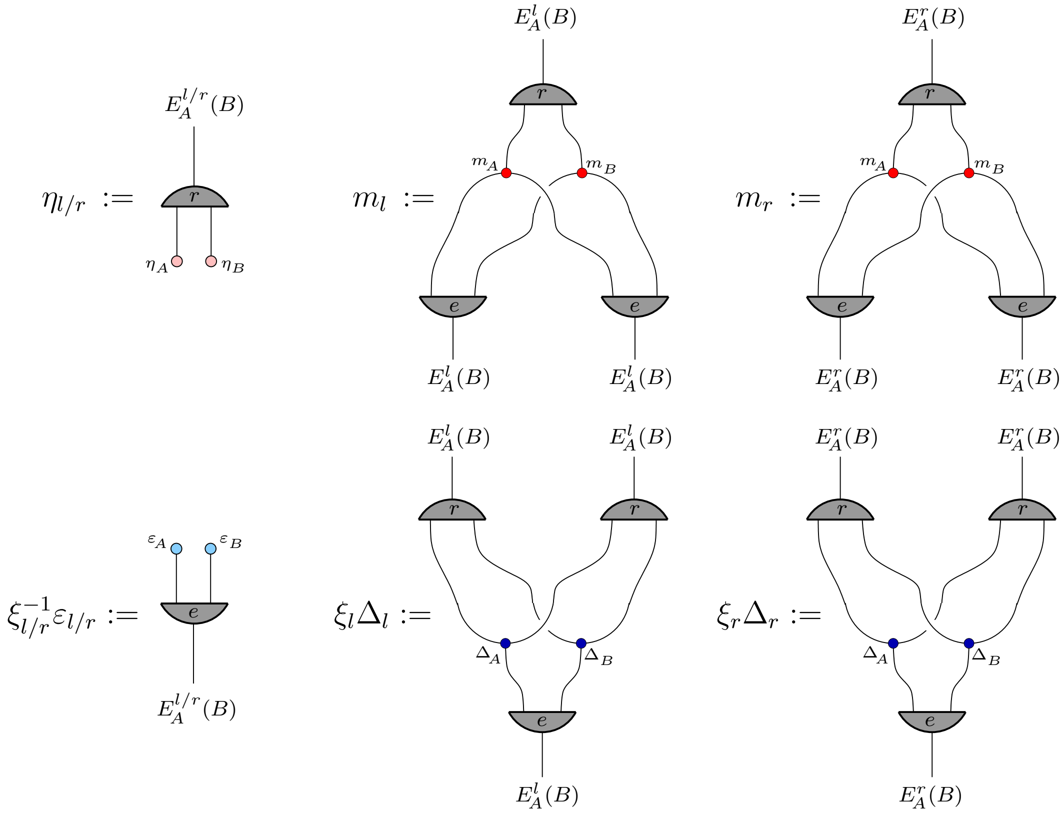
(3.14)

while for $\otimes^-$ over-braiding and under-braiding must be exchanged in the definition of both the product and the coproduct. One verifies by direct substitution that $A\otimes^+B$ is again a Frobenius algebra. Further, if A,B are in addition symmetric and special, then so is $A\otimes^+B$. An analogous statement holds for $A\otimes^-B$.

In the sequel we will work with $\otimes^+$; also, we slightly abuse notation and simply write $A{\otimes}B$ in place of $A{\otimes}^+B$ for the tensor product of two Frobenius algebras.

Note that even when both A and B are commutative, their tensor product $A{\otimes}B$ is not commutative, in general. More precisely, if A and B are commutative, then $A{\otimes}B$ is commutative iff $c_{A,B}\circ c_{B,A} = id_{A\otimes B}$. While this identity holds in a symmetric tensor category, it does not necessarily hold in a genuinely braided tensor category; in this setting it is therefore not advisable to restrict one's attention exclusively to (braided-) commutative algebras.

Proposition 3.8:
Let A be a symmetric special Frobenius algebra and B a Frobenius algebra in a ribbon category. Then the following holds:
(i) $E_A^{l/r}(B) \equiv (E_A^{l/r}(B), m_{l/r}, \eta_{l/r}, \Delta_{l/r}, \varepsilon_{l/r})$, with morphisms given by



(3.15)

with $\xi_{l/r} \in \Bbbk^\times$, is a Frobenius algebra.
(ii) If B is symmetric, then $E_A^l(B)$ and $E_A^r(B)$ are symmetric.
If B is commutative, then $E_A^l(B)$ and $E_A^r(B)$ are commutative.
(iii) If $E_A^l(B)$ is symmetric, B is in addition special, $\dim_\Bbbk \mathrm{Hom}(B, C_r(A)) = 1$, and $\dim(E_A^l(B))$ is non-zero, then $E_A^l(B)$ is in addition haploid and special.
If $E_A^r(B)$ is symmetric, B is in addition special, $\dim_\Bbbk \mathrm{Hom}(B, C_l(A)) = 1$, and $\dim(E_A^r(B))$

is non-zero, then $E_A^r(B)$ is in addition haploid and special.

(iv) If A is commutative, then $E_A^l = E_A^r$ as functors. More precisely, for every $U \in \mathrm{Obj}(\mathcal{C})$ we have the equality $E_A^l(U) = E_A^r(U)$ as objects in $\mathcal{C}$, and for every morphism f of $\mathcal{C}$ we have $E_A^l(f) = E_A^r(f)$.

(v) If A is commutative, then $E_A^l(B) = E_A^r(B)$ as Frobenius algebras.

Remark 3.9:
In [45] the notion of an Azumaya algebra in a braided tensor category has been introduced. The definition in [45] can be seen to be equivalent to the following one: An algebra A in a ribbon category $\mathcal{C}$ is called an *Azumaya algebra* iff the functors α_A^+ and α_A^- from $\mathcal{C}$ to $\mathcal{C}_{A|A}$ are equivalences of tensor categories. If a symmetric special Frobenius algebra A is Azumaya, then $C_l(A) \cong \mathbf{1} \cong C_r(A)$. To see this note that if α_A^+ is an equivalence functor, then it has a left and right adjoint $(\alpha_A^+)^\dagger$, given by $(\alpha_A^+)^{-1}$. In Assertions (i) – (iii) of Proposition 2.36 we have seen that the composition $(\alpha_A^+)^\dagger \circ \alpha_A^+$ corresponds to tensoring with $C_l(A)$. This is an equivalence iff $C_l(A) \cong \mathbf{1}$. A similar argument shows that $C_r(A) \cong \mathbf{1}$. Assertions (i) – (iii) of Proposition 3.8 thus imply in particular that every Azumaya algebra defines two endofunctors of the full subcategory of haploid commutative symmetric special Frobenius algebras in a given ribbon category $\mathcal{C}$. Algebras of the latter type can be used to construct new ribbon categories starting from $\mathcal{C}$, see Proposition 3.21 below.

The proof of Proposition 3.8 will fill the remainder of this section. We need the following three lemmata.

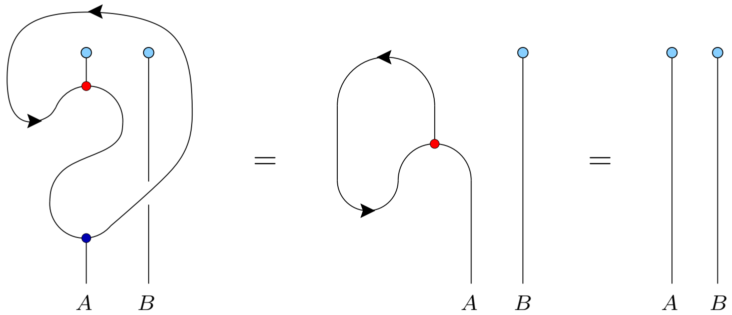
Lemma 3.10:
Let A and B be as in Proposition 3.8, and $\tilde m_l \in \mathrm{Hom}((A{\otimes}B) \otimes (A{\otimes}B), A{\otimes}B)$ denote the morphism obtained from m_l of (3.15) by omitting the embedding and restriction morphisms e, r; define $\tilde m_r$ and $\tilde\Delta_{r/l}$ similarly. Further let $\tilde\eta := \eta_A \otimes \eta_B$ and $\tilde\varepsilon := \varepsilon_A \otimes \varepsilon_B$. The idempotent $P^l \equiv P_A^l(B)$ fulfills

$$\begin{aligned} &P^l \circ \tilde\eta = \tilde\eta\,, \qquad \tilde\varepsilon \circ P^l = \tilde\varepsilon\,, \\ &P^l \circ \tilde m_l \circ (P^l \otimes P^l) = id_{A\otimes B} \circ \tilde m_l \circ (P^l \otimes P^l) \\ &\qquad = P^l \circ \tilde m_l \circ (id_{A\otimes B} \otimes P^l) = P^l \circ \tilde m_l \circ (P^l \otimes id_{A\otimes B})\,. \end{aligned} \tag{3.16}$$

Analogous relations hold for $P^r \equiv P_A^r(B)$ and $\tilde m_r, \tilde\eta, \tilde\varepsilon$, as well as for $P^{l/r}$ and $\tilde\Delta_{l/r}$.
In terms of the graphical calculus, this means that at any product or coproduct vertex for which each of the three attached ribbons carries an idempotent P^l, or each a P^r, any one out of the three idempotents can be omitted.

Proof:
The proof is similar for all relations. As examples we present it for $\tilde\varepsilon \circ P^l = \tilde\varepsilon$ and for

$(P^r \otimes id_{A\otimes B}) \circ \tilde\Delta^r \circ P^r$. The first of these relations is easily seen from

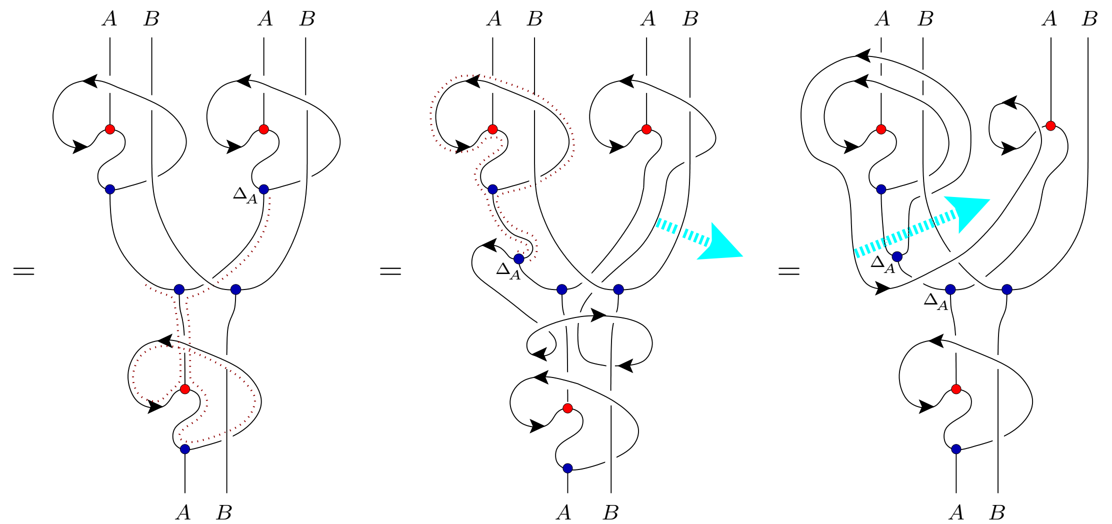


(3.17)

In the first step one substitutes the definition of P^l and deforms the graph slightly; then one uses the Frobenius and counit properties to get rid of the counit of A. The final step re-introduces this counit by using the fact that it obeys $\varepsilon = \varepsilon_\natural$ with $\varepsilon_\natural$ given by (2.39) (and also that A is symmetric Frobenius).

To obtain the second relation one considers the following series of transformations, for which all defining properties of the symmetric special Frobenius algebra A are needed:

$\xi_r\,(P^r \otimes P^r) \circ \tilde\Delta^r \circ P^r$

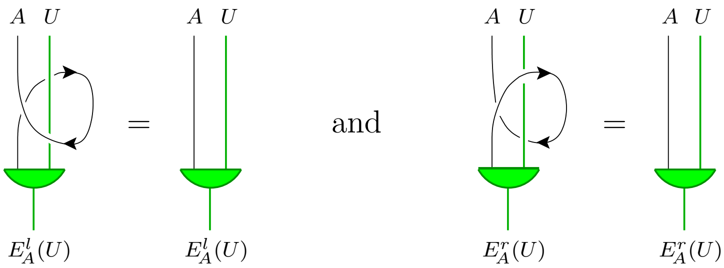


(3.18)

The first equality just consists of writing out the Definition (3.15) of the coproducts and the idempotents. To arrive at the second equality, one drags the coproduct that is marked explicitly in the first graph along the path that is drawn as a dotted line, so that its new location is the one marked in the second graph. The third equality is obtained by first pulling an A-ribbon under the right B-ribbon, which is indicated by the big shaded arrow, and then moving it back in the opposite direction, but this time *over* the B-ribbon (as well as over another A-ribbon). In addition, one continues to drag the coproduct that was already moved during the previous step, now along the dotted path in the second graph; this way it returns to the same location, but is now attached from the opposite side.

Starting from the third graph, one can now pull the left-most A-ribbon in the direction of the shaded arrow, over various A-ribbons as well as over the left B-ribbon; the twists on this ribbon then cancel. Afterwards one can use co-associativity (on the two coproducts that are marked explicitly in the graph) and then the specialness of A so as to arrive at the desired result. □

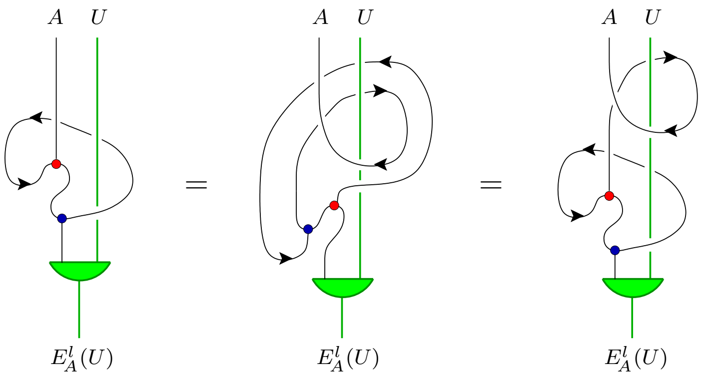
Lemma 3.11:
For every symmetric special Frobenius algebra A we have



(3.19)

as well as the analogous relations for $r_{A\otimes U \succ E^{l/r}_A(U)}$ instead of $e_{E^{l/r}_A(U)\prec A\otimes U}$.

Proof:
We show the moves needed to derive the left equality – the right one and the relations for r follow analogously:



(3.20)

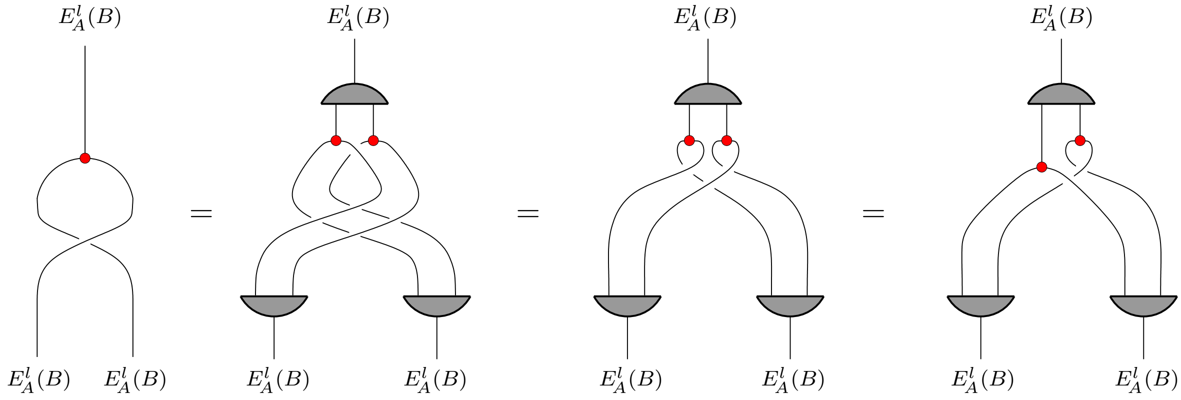
The first expression is the right hand side of the first equality in (3.19), with a redundant idempotent $P^l_A(U)$ inserted. To arrive at the second graph one uses the Frobenius property and suitably drags the resulting coproduct along part of the A-ribbon. A further deformation and application of the Frobenius property results in the graph on the right hand side. In this last expression the idempotent $P^l_A(U)$ is again redundant; removing it yields the left hand side of the first equality in (3.19). □

Lemma 3.12:
Let A be a symmetric special Frobenius algebra and B a Frobenius algebra in a ribbon category. Denote by m_E the multiplication morphism of $E = E^{l/r}_A(B)$ as defined in (3.15). Then

$$m_E \circ c_{E,E} = m_{E'} \tag{3.21}$$

with $E' = E_A^{l/r}(B')$, where $B' = (B, m_B \circ c_{B,B}, \eta_B, \Delta_B \circ c_{B,B}^{-1}, \varepsilon_B)$ (i.e., the opposite algebra of B' is B).

Proof:
We prove the relation for $E_A^l(B)$, the case of $E_A^r(B)$ being analogous. Consider the following moves:



(3.22)

The first step implements the definition (3.15) of the product on $E_A^l(B)$, while in the second step the resulting ribbons are deformed slightly. The third expression in (3.22) is already almost the multiplication of E', except that the braiding $c_{A,A}$ must be removed and the braiding $c_{B,A}$ must be replaced by $c_{A,B}^{-1}$. This is achieved in two steps. First we use the equality $r = r \circ P_A^l(B)$ to insert an idempotent $P_A^l(B)$ before the restriction morphism and then carry out the moves displayed in figure (3.6) backwards. This replaces $m \circ c_{A,A}$ by $m \circ (id_A \otimes \theta_A^{-1})$. After a further slight deformation of ribbons one arrives at a graph for which the right ingoing leg is just given by the leftmost graph in (3.19). Using the first equality in (3.19) we then arrive at the last expression in (3.22), which is precisely the multiplication of E'. □

Proof of Proposition **3.8**:
We restrict our attention to the case of $E_A^l(B)$. For $E_A^r(B)$ the reasoning works in the same way.

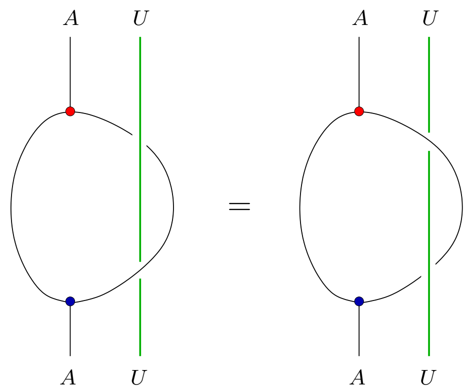
(i) The checks of the (co)associativity, (co)unit and Frobenius properties all work by direct computation: After writing out the definition, one uses Lemma 3.10 to remove the projector on the 'internal' A-ribbon. The (co)associativity, Frobenius and (co)unit relations then follow directly from the corresponding properties of A and B.

(ii) The check that symmetry of B implies symmetry of $E_A^l(B)$ can be performed by the same method as in (i). To see that commutativity of B implies commutativity of $E_A^l(B)$, first note that from Lemma 3.12, $m_E \circ c_{E,E} = m_{E'}$. If B is commutative, then $B' = B$ as a Frobenius algebra, so that (3.21) reduces to $m_E \circ c_{E,E} = m_E$.

(iii) That $E_A^l(B)$ is haploid follows from Proposition 3.6 by specialising to $U = B$ and $V = \mathbf{1}$, together with (2.43) and Lemma 3.13 below, by which we have the bijections

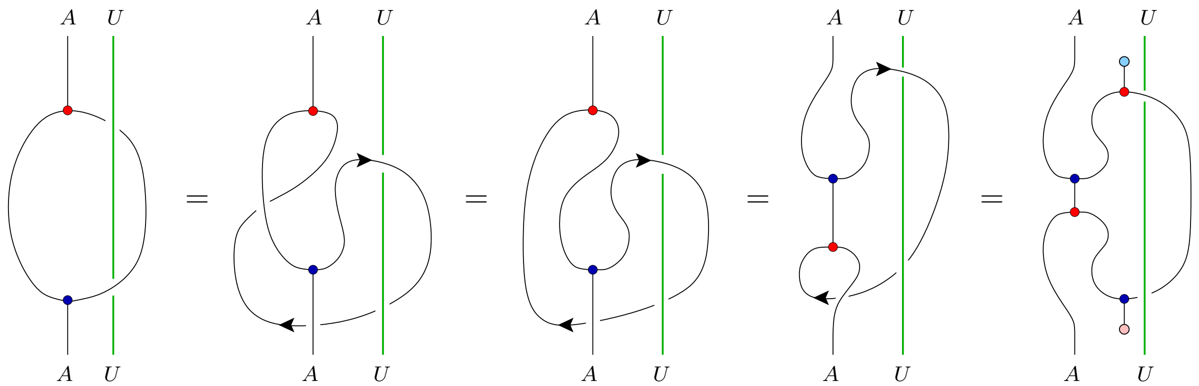
$\mathrm{Hom}(B, C_r(A)) \cong \mathrm{Hom}(B, E_A^r(\mathbf{1})) \cong \mathrm{Hom}(E_A^l(B), \mathbf{1})$. (Note that here the assumption about the non-vanishing of the dimension of $E_A^l(B)$ is not yet needed.)
To see that $E_A^l(B)$ is special, we can use lemma 3.11 of [18], according to which it suffices to show that the counit ε given in (3.15) is a non-zero multiple of $\varepsilon_\natural$ as defined in (2.39) (evaluated for the algebra $E_A^l(B)$). Since $E_A^l(B)$ is haploid, it is guaranteed that $\varepsilon_\natural = \gamma\,\varepsilon$ with $\gamma \in \Bbbk$. The proportionality constant γ can be determined by composing the equality with η; the result is $\gamma = \xi^{-1} \dim(E_A^l(B))/\dim(A)\dim(B)$, which is non-zero by assumption.

(iv) It is sufficient to check that the projectors on $A \otimes U$ are equal, i.e. $P_A^l(U) = P_A^r(U)$. Since A is commutative and symmetric, and thus also has trivial twist, the desired equality can be rewritten as



(3.23)

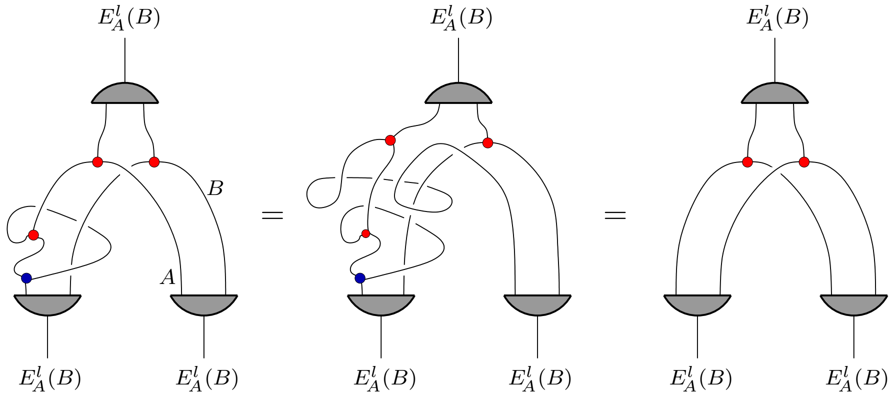
This latter equality, in turn, can be verified as follows. First one deforms the A-loop on the left hand side of (3.23) in such a manner that the order of the braidings $c_{A,U}$ and $c_{U,A}^{-1}$ gets interchanged, and then uses, consecutively, commutativity, the Frobenius property, again commutativity, symmetry to obtain



(3.24)

from which one arrives at the right hand side of (3.23) by another (twofold) use of the Frobenius property.

(v) In addition to having $E_A^l(B) \cong E_A^r(B)$ as objects in $\mathcal{C}$, one further verifies that $m_l = m_r$, $\eta_l = \eta_r$, $\Delta_l = \Delta_r$ and $\varepsilon_l = \varepsilon_r$. Let us only show how to check equality of m_l and m_r; for the other morphisms similar arguments apply. One considers the transformations (for better readability we suppress the arrows indicating the duality morphisms)

m_l =  $= m_r$ (3.25)

In the first step the definition of m_l is written out and an idempotent $P^l_A(B)$ is inserted on top of an embedding $E^l_A(B) \prec A \otimes B$. Afterwards the multiplication on A is moved along the idempotent. In the third step the A-ribbon is rearranged and the commutativity of A is used. □

3.3 Centers and endofunctors

From the Definitions 2.31 and 3.3 it is clear that the centers of an algebra can be interpreted as images of the endofunctors $E_A^{l/r}$, i.e. $C_{l/r}(A) \cong E_A^{l/r}(\mathbf{1})$ as objects of $\mathcal{C}$. We will now see that, upon endowing $C_{l/r}(A)$ with the structure of a Frobenius algebra inherited from A, and $E_A^{l/r}(\mathbf{1})$ with the Frobenius structure described in Proposition 3.8, this is even an isomorphism of Frobenius algebras.

Lemma 3.13:
For every symmetric special Frobenius algebra A we have isomorphisms

$$C_l(A) \cong E^l_A(\mathbf{1}) \qquad \text{and} \qquad C_r(A) \cong E^r_A(\mathbf{1}) \tag{3.26}$$

as Frobenius algebras.

Proof:
From Lemma 2.30 we know that $C_{l/r}$ is the image of the split idempotent $P_A^{l/r}$ defined in equation (2.52). Also, comparison with the idempotents (3.1) shows that $P_A^{l/r} = P_A^{l/r}(\mathbf{1})$. Thus $C_{l/r} \cong E_A^{l/r}(\mathbf{1})$ as an object in $\mathcal{C}$. Further, the definition of the algebra structure on $E_A^{l/r}(B)$ in Proposition 3.8(i) reduces to the one of $C_{l/r}$ (as given in equation (2.70)) when $B = \mathbf{1}$. □

This lemma can be used to establish the following more general result for tensor product algebras:

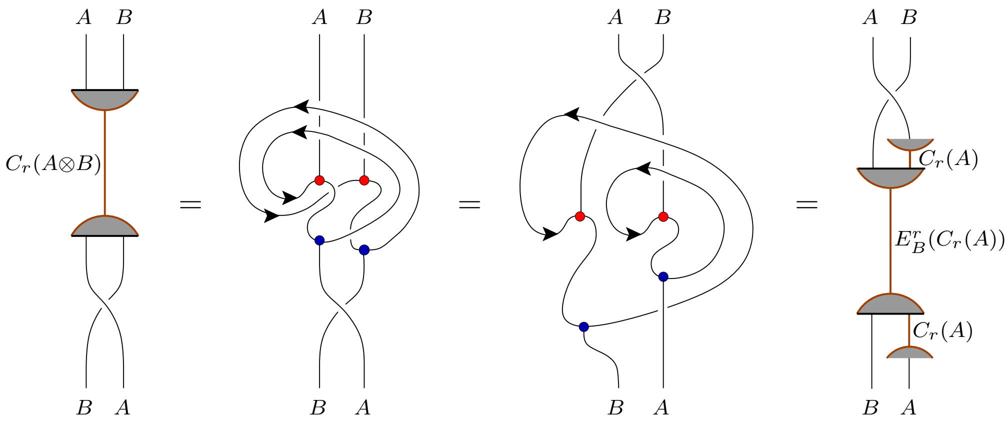
Proposition 3.14:
(i) For any pair A, B of symmetric special Frobenius algebras in a ribbon category $\mathcal{C}$ one has

$$C_l(A{\otimes}B) \;\cong\; E^l_A(C_l(B)) \qquad \text{and} \qquad C_r(A{\otimes}B) \;\cong\; E^r_B(C_r(A)) \tag{3.27}$$

as symmetric Frobenius algebras.

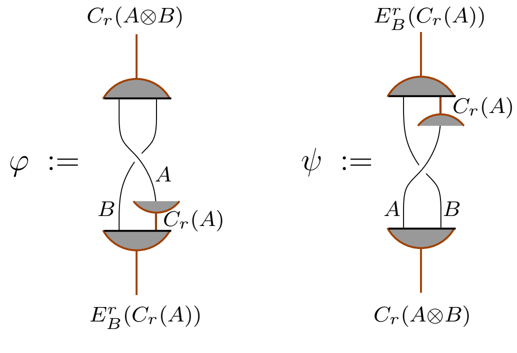
(ii) If in addition $\dim(C_r(A)) \neq 0$, $\dim(C_l(B)) \neq 0$ and $\dim(C_{l/r}(A{\otimes}B)) \neq 0$, as well as $\dim_{\Bbbk} \mathrm{Hom}(C_r(A), C_l(B)) = 1$, then $E^l_A(C_l(B))$ and $E^r_B(C_r(A))$ are haploid and special.

Proof:
(i) Let us start with the second relation in (3.27). The following series of equalities shows that the braiding $(c_{A,B})^{-1}$ relates the idempotents (3.1) for $C_r(A{\otimes}B) \cong E^r_{A\otimes B}(\mathbf{1})$ and for $E^r_B(C_r(A))$:



(3.28)

Define the morphisms $\varphi \in \mathrm{Hom}(E^r_B(C_r(A)), C_r(A{\otimes}B))$ and $\psi \in \mathrm{Hom}(C_r(A{\otimes}B), E^r_B(C_r(A)))$ by



(3.29)

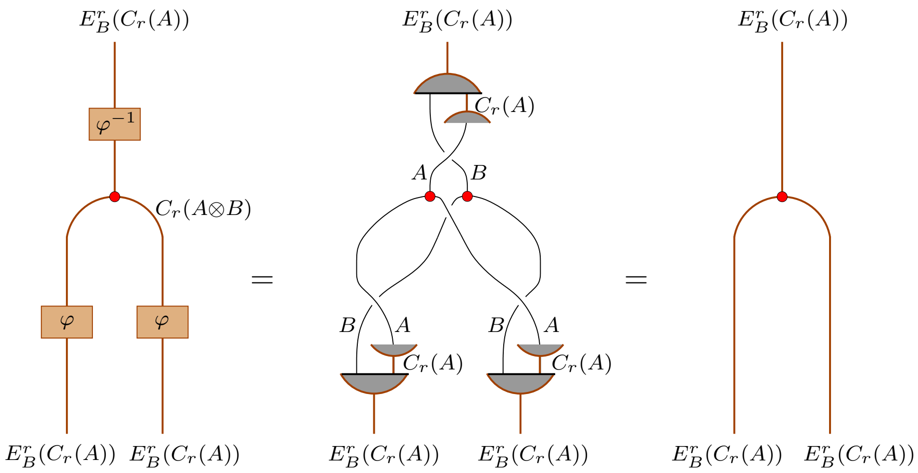
Using (3.28) one can verify that

$$\varphi \circ \psi = id_{C_r(A\otimes B)} \qquad \text{and} \qquad \psi \circ \varphi = id_{E^r_B(C_r(A))} \,. \tag{3.30}$$

Next we would like to see that φ is compatible with the symmetric special Frobenius structure of the two algebras. We need to check that

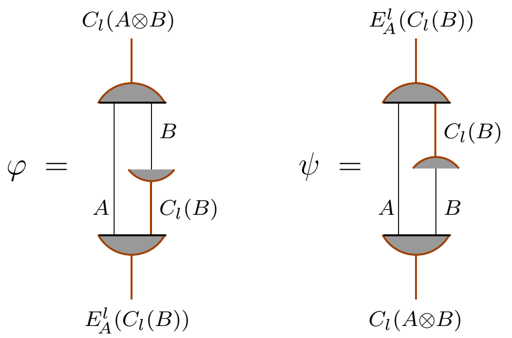
$$\begin{aligned} &\varphi^{-1}{\circ}\, \eta_{C_r(A\otimes B)} = \eta_{E^r_B(C_r(A))} \,, && \varphi^{-1}{\circ}\, m_{C_r(A\otimes B)} \circ (\varphi{\otimes}\varphi) = m_{E^r_B(C_r(A))} \,, \\ &\varepsilon_{C_r(A\otimes B)}{\circ}\, \varphi = \varepsilon_{E^r_B(C_r(A))} \,, && (\varphi^{-1}{\otimes}\varphi^{-1}) \circ \Delta_{C_r(A\otimes B)} \circ \varphi = \Delta_{E^r_B(C_r(A))} \,. \end{aligned} \tag{3.31}$$

The relations for η and ε are immediate when inserting the definitions (3.15) and (2.70). Using again (3.28), for the multiplication we find



(3.32)

The corresponding relation for the comultiplication in (3.31) is demonstrated similarly.
The proof of the first relation in (3.27) works along the same lines, but this time φ and ψ take the easier form



(3.33)

Correspondingly there is no braiding in the analogue of (3.28).
By Propositions 2.37(i) and 3.8(ii), $E_A^l(C_l(B))$ and $E_B^r(C_r(A))$ are symmetric.
(ii) By Proposition 2.37(i), $C_r(A)$ and $C_l(B)$ are commutative symmetric Frobenius algebras. Further, since by Remark 2.23(vi) the tensor unit is a retract of every Frobenius algebra, the condition $\dim_{\Bbbk} \mathrm{Hom}(C_r(A), C_l(B)) = 1$ implies in particular that $C_r(A)$ and $C_l(B)$ are haploid and thus simple. Since their dimensions are non-zero by assumption, Proposition 2.37(iii) then tells us that the two centers are also special. Together with the assumptions $\dim(C_{l/r}(A{\otimes}B)) \neq 0$ and $\dim_{\Bbbk} \mathrm{Hom}(C_r(A), C_l(B)) = 1$, as well as the isomorphisms of part (i), we can finally apply Proposition 3.8(ii) and 3.8(iii) to see that $E_B^r(C_r(A))$ is haploid and special.
Similarly, again by Proposition 3.8(ii) and 3.8(iii), this time together with the bijection (2.43), $E_A^l(C_l(B))$ is haploid and special as well. □

3.4 The ribbon subcategory of local modules

As noticed after (3.3), in symmetric tensor categories the objects $E_A^{l/r}(U)$ are closely related to induced modules over the *center* of the algebra A. We therefore now consider categories of modules over *commutative* symmetric Frobenius algebras. As it turns out, this is still appropriate in the genuinely braided case. Note that according to Proposition 3.8(iv), for commutative A, there is only one endofunctor $E_A(\cdot) \equiv E_A^{l/r}(\cdot)$.

The relevant class of modules is introduced in

Definition 3.15:
A left module $M = (\dot M, \rho_M)$ over a commutative symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ is called *local* iff

$$\rho_M \circ P_A(\dot M) = \rho_M \,, \tag{3.34}$$

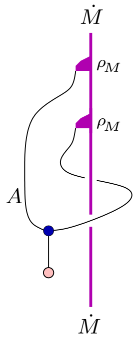
where $P_A(\dot M) \equiv P_A^{l/r}(\dot M)$ is the idempotent defined in (3.1).

As we will see in Proposition 3.17 below, our concept of locality is equivalent to the one of Definition 3.2 of [26]. The latter, which says that M is local iff $\rho_M \circ c_{\dot M,A} \circ c_{A,\dot M} = \rho_M$, had been introduced earlier for modules over an algebra in a general braided tensor category in [40], where such modules were termed *dyslectic*. A main motivation for the introduction of dyslectic modules in [40] was the fact that they form a full subcategory that can be naturally endowed with a tensor structure and a braiding. This property will be crucial in the present context, too. Here we prefer the qualification "local" to the term "dyslectic" because it agrees with the standard use [42] in conformal quantum field theory in the context of so-called simple current extensions (compare Remark 3.19(ii) below).

To show the equivalence between the two characterisations (as well as a third one) we first give the

Lemma 3.16:
For $M = (\dot M, \rho)$ a left module over a commutative symmetric special Frobenius algebra in a ribbon category the morphism

$Q_M :=$



$\in \text{Hom}(\dot M, \dot M)$ (3.35)

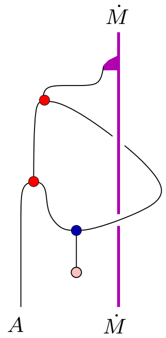
satisfies

$$\text{(i)} \quad \rho_M \circ P_A(\dot M) = \rho_M \circ (id_A \otimes Q_M) \qquad \text{and} \qquad \text{(ii)} \quad Q_M \circ Q_M = Q_M \,. \tag{3.36}$$

(iii) M is local iff $Q_M = id_{\dot M}$.

Proof:
To see (i) we note that

$$\rho_M \circ P_A(\dot M) \;=\; \qquad \qquad = \rho_M \circ (id_A \otimes Q_M)\,. \tag{3.37}$$



The equality on the left uses the fact that A is commutative and has trivial twist (recall the corresponding comment in Section 2.3), as well as the Frobenius property. The equality on the right is obtained by applying the representation property first for the upper and then on the lower of the two products.

To deduce (ii) we combine (i) with the fact that $P_A(\dot M)$ is an idempotent, and insert (i) twice (using also that $id_A \otimes Q_M$ commutes with $P_A(\dot M)$), so as to arrive at

$$\begin{aligned} \rho_M \circ (id_A \otimes Q_M) \;&= \rho_M \circ P_A(\dot M) \\ &= \rho_M \circ P_A(\dot M) \circ P_A(\dot M) = \rho_M \circ (id_A \otimes Q_M) \circ (id_A \otimes Q_M)\,. \end{aligned} \tag{3.38}$$

Property (ii) now follows by composing both sides of (3.38) with $\eta \otimes id_M$.

(iii) For local M (3.36(i)) reads $\rho_M \circ (id_A \otimes Q_M) = \rho_M$, which when composed with $\eta \otimes id_{\dot M}$ yields $Q_M = id_{\dot M}$. Conversely, inserting $Q_M = id_{\dot M}$ into (i) yields the defining property of locality. □

We are now in a position to present

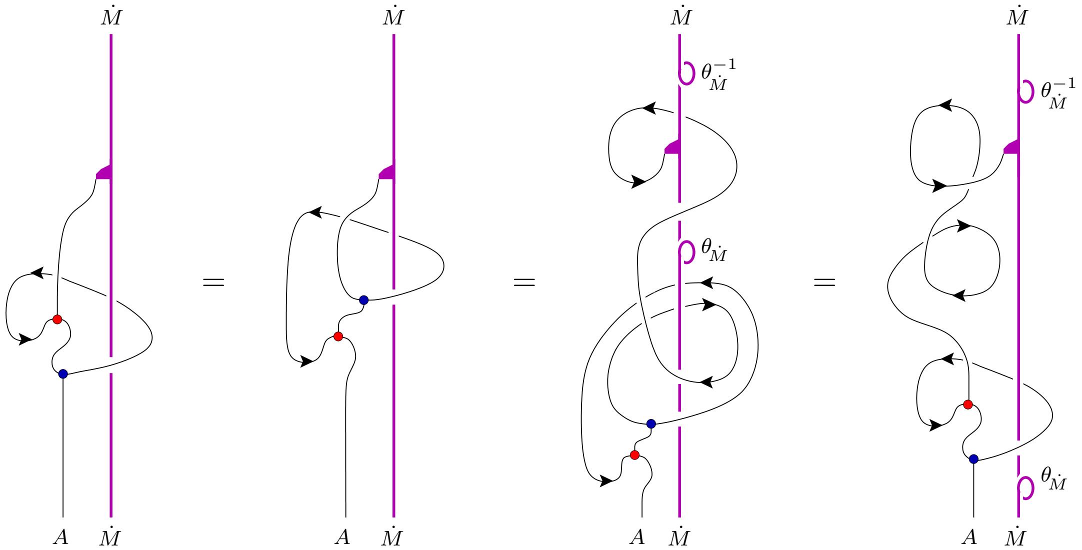
Proposition 3.17:
For a left module $M = (\dot M, \rho)$ over a commutative symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$ the following conditions are equivalent:

(i) M is local.

(ii) $\theta_{\dot M} \in \mathrm{Hom}_A(M, M)$.

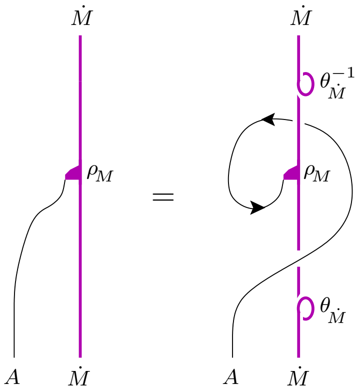
(iii) $\rho \circ c_{\dot M, A} \circ c_{A, \dot M} = \rho$.

Proof:
(i) ⇒ (ii) : We start with the equalities



(3.39)

The first equality uses that A is Frobenius, the second combines the identity



(3.40)

with moves similar to those in figure (3.20), and the third combines a deformation of the upper A-ribbon with an application of the Frobenius property in the lower part of the graph. On the right hand side of (3.39), we can in addition straighten the upper A-ribbon. Afterwards, by composing the left and right hand sides with $\theta_{\dot M}$ from the top and removing the idempotent $P_A(\dot M)$ (as allowed by locality) we arrive at the statement that $\theta_{\dot M} \in \mathrm{End}(\dot M, \dot M)$ is actually in $\mathrm{End}_A(M, M)$.
(ii) ⇒ (i) : By $\theta_A = id_A$ and the compatibility between braiding and twist we have

$$\rho_M \circ c_{\dot M,A} \circ c_{A,\dot M} = \theta_{\dot M} \circ \rho_M \circ (id_A \otimes \theta_{\dot M}^{-1}) \,. \qquad (3.41)$$

To show that Q_M is the identity morphism, we insert this relation into the definition (3.35) of the morphism Q_M. Then by (ii) we can take the lower representation morphism ρ_M past $\theta_{\dot M}$ without introducing any braiding or twist. Using that A is special, the A-ribbon can then be removed, resulting in $Q_M = id_{\dot M}$. By Lemma 3.16(iii) it follows that M is local.

(ii) $\Leftrightarrow$ (iii) follows immediately from relation (3.41).

(For semisimple $\mathcal{C}$ this equivalence is Theorem 3.4.1 of [26].) □

In applications one is often interested in *simple* modules. Therefore we separately state the following result which makes it easy to test if a simple module is local.

Corollary 3.18 :
For a simple module M, with $\dim(M) \neq 0$, over a commutative symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$ the following conditions are equivalent:

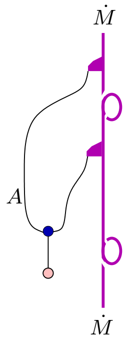
(i) M is local.

(ii) $\operatorname{tr} \theta_{\dot M} \neq 0$.

(iii) $\theta_{\dot M} = \xi_M \, id_{\dot M}$ for some $\xi_M \in \Bbbk^\times$.

Proof:
We first note that when M is simple, then the morphism $Q_M \in \mathrm{Hom}(\dot M, \dot M)$ given by (3.35) satisfies

$$Q_M \; = $$



$$= \frac{\operatorname{tr} \theta_{\dot M}}{\dim(\dot M)} \, \theta_{\dot M}^{-1} \, . \qquad (3.42)$$

To get the first equality, the $\dot M$-ribbon is twisted so as to remove the braidings; because of $\theta_A = id_A$ the resulting twist of the A-ribbon can be left out. The second equality is a consequence of Lemma 4.4 of [18]. By definition (see Definition 4.3 of [18]) of the *A-averaged morphism* $\theta_{\dot M}^{\mathrm{av}}$, which is an element of $\mathrm{Hom}_A(M, M)$, the graph is just $\theta_{\dot M}^{\mathrm{av}} \circ \theta_{\dot M}^{-1}$. Since M is simple, $\mathrm{Hom}_A(M, M)$ is one-dimensional, so that $\theta_{\dot M}^{\mathrm{av}}$ is proportional to $id_{\dot M}$. The constant of proportionality is determined by comparing the traces, resulting in the final expression in (3.42).

(i) $\Rightarrow$ (ii) : By Lemma 3.16(iii), for local M we have $Q_M = id_{\dot M}$. By (3.42) this, in turn, means that

$$\frac{\operatorname{tr} \theta_{\dot M}}{\dim(\dot M)} \, \theta_{\dot M}^{-1} = id_{\dot M} \, . \qquad (3.43)$$

Since $id_{\dot M}$ is invertible, this requires $\operatorname{tr} \theta_{\dot M}$ to be non-zero.

(ii) $\Rightarrow$ (iii) : The equality obtained by inserting (3.42) into the projection property (3.36(ii)) can hold only if $\operatorname{tr} \theta_{\dot M} = 0$ or if

$$\frac{\operatorname{tr} \theta_{\dot M}}{\dim(\dot M)} \, id_{\dot M} = \theta_{\dot M} \, . \qquad (3.44)$$

Since by (ii) the first possibility is excluded, we arrive at (iii) with $\xi_M = \operatorname{tr} \theta_{\dot M} / \dim(\dot M)$ (which is non-zero).

(iii) ⇒ (i) : When combined with (3.42), the statement (iii) implies $Q_M = id_{\dot{M}}$. Together with (3.36(iii)) it then follows that M is local. □

Remark 3.19 :

(i) When $\mathcal{C}$ is semisimple, it follows immediately from the definition that in the decomposition of a local module M into simple modules M_κ all the M_κ are local as well.

(ii) The case when the commutative algebra A is a direct sum of *invertible* simple objects is known in the physics literature as a *simple current extension*. Then the local modules M are those for which the 'monodromy charge' with respect to A vanishes, which means that for all simple subobjects J of A and all simple subobjects U_i of M one has the equality $s_{J,U_i} = s_{\mathbf{1},U_i}$, where s is the s-matrix defined in (2.5). Precisely these modules appear in the chiral conformal field theory obtained by a simple current extension [43].

(iii) For $\mathcal{C} = \mathcal{R}ep_{\text{DHR}}(\mathfrak{C})$ the category of DHR superselection sectors [12, 13] of a local rational quantum field theory $\mathfrak{C}$, there is a bijection between finite index extensions $\mathfrak{C}_{\text{ext}} \supseteq \mathfrak{C}$ and symmetric special Frobenius algebras A in $\mathcal{R}ep_{\text{DHR}}(\mathfrak{C})$, and $\mathfrak{C}_{\text{ext}}$ is again a local quantum field theory iff A is commutative [28].
For the case that $\mathcal{C} = \mathcal{R}ep(\mathfrak{V})$ is the category of modules over a rational vertex algebra $\mathfrak{V}$ with certain nice properties, the fact that $\mathcal{C}^{\ell oc}_{\mathcal{R}ep(\mathfrak{V})}$ is equivalent to $\mathcal{R}ep(\mathfrak{V}_{\text{ext}})$, with $\mathfrak{V}_{\text{ext}}$ the vertex algebra for the extended conformal field theory, has been observed in [26] (Theorem 5.2).

Definition 3.20 :

Let A be a commutative symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$. The *category of local A-modules*, denoted by $\mathcal{C}^{\ell oc}_A$, is the full subcategory of $\mathcal{C}_A$ whose objects are local A-modules.

Under suitable conditions on $\mathcal{C}$ and A, the category $\mathcal{C}^{\ell oc}_A$ inherits various structural properties from $\mathcal{C}$, such as being braided tensor (Theorem 2.5 of [40]) or modular (Theorem 4.5 of [26]). We collect some of these properties in

Proposition 3.21 :

For every commutative symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ the following holds:

(i) $\mathcal{C}^{\ell oc}_A$ is a ribbon category.

(ii) If $\mathcal{C}$ is semisimple, then $\mathcal{C}^{\ell oc}_A$ is semisimple. If $\mathcal{C}$ is closed under direct sums and subobjects, then $\mathcal{C}^{\ell oc}_A$ is closed under direct sums and subobjects.

(iii) If $\mathcal{C}$ is modular and if A is in addition simple, then $\mathcal{C}^{\ell oc}_A$ is modular.

The proof is a straightforward combination of the results contained in the proofs of Theorems 1.10, 1.17 and 4.5 of [26] (which are derived in a semisimple setting and with A

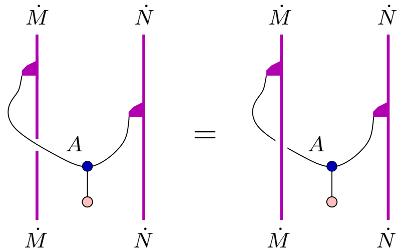
assumed to be haploid, but are easily adapted to the present framework, using in particular the fact that simple commutative algebras are also haploid) and the permanence properties established in Section 5 of [20]. (For the simple current case that was mentioned in Remark 3.19(ii) above, see also [19, 9, 33].)

Let us describe the tensor structure of $\mathcal{C}_A^{\ell oc}$ in some detail. For any algebra A, one defines the tensor product $M \otimes_A N$ of a right A-module M and a left A-module N as the cokernel of the morphism $\rho_M \otimes id_N - id_M \otimes \rho_N$, provided that the cokernel exists. In the present context, i.e. for A a commutative symmetric special Frobenius algebra and M and N two local left A-modules, the tensor product can conveniently be described as the image

$$M \otimes_A N := \mathrm{Im}\, P_{M\otimes N} \tag{3.45}$$

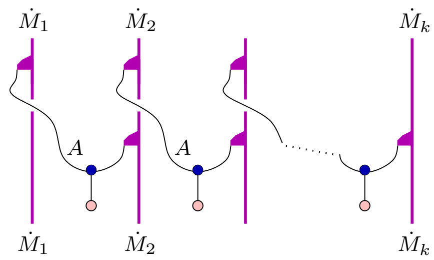
of a suitable idempotent in $\mathrm{End}(\dot M \otimes \dot N)$, provided that this idempotent is split. The idempotent in question is given by (compare lemma 1.21 of [26])

$$P_{M\otimes N} \;=\; \tag{3.46}$$



(Owing to Proposition 3.17(iii), applied to the representation morphism ρ_M for the local module M, the morphisms given by the left and right pictures are equal.) Similarly, multiple tensor products can then be described as images of the idempotents

$$P_{M_1\otimes\cdots\otimes M_k} \;=\; \tag{3.47}$$



Note that this way of defining multiple tensor products is consistent with the iterative application of (3.45). Indeed one easily verifies that the idempotents $P_{(M\otimes_A N)\otimes K}$ and $P_{M\otimes(N\otimes_A K)}$ are both equal to $P_{M\otimes N\otimes K}$.

Finally, denoting by $e_{M_1\otimes\cdots\otimes M_k}$ and $r_{M_1\otimes\cdots\otimes M_k}$ the embedding and restriction morphisms for the idempotent (3.47), the tensor product of morphisms $f_i \in \mathrm{Hom}_A(M_i, N_i)$ $(i = 1, 2, \dots, k)$ takes the form

$$f_1 \otimes_A \cdots \otimes_A f_k = r_{N_1\otimes\cdots\otimes N_k} \circ (f_1 \otimes \cdots \otimes f_k) \circ e_{M_1\otimes\cdots\otimes M_k} \,. \tag{3.48}$$

The definition (3.45) of the tensor product is based on the assumption that the idempotents $P_{M_1\otimes\cdots\otimes M_k}$ are split, for which it is sufficient that $\mathcal{C}$ is Karoubian. If we do not impose Karoubianness of $\mathcal{C}$, it can happen that $P_{M_1\otimes\cdots\otimes M_k}$ is not split; then we must work with the Karoubian envelope of $\mathcal{C}_A^{\ell oc}$ and define

$$M_1 \otimes_A \cdots \otimes_A M_k := (M_1 \otimes \cdots \otimes M_k; P_{M_1\otimes\cdots\otimes M_k}) . \tag{3.49}$$

If $\mathcal{C}$ *is* Karoubian so that we can define the tensor product as an image, we still must select $M{\otimes_A}N$ as a specific object in its isomorphism class (recall that we use the axiom of choice to regard images as objects). We make this choice in a way compatible with (3.49). With this definition of the tensor product the associativity constraints of the category $\mathcal{C}_A^{\ell oc}$ are, just as the ones of $\mathcal{C}$, identities. However, in general $A\otimes_A M$ and M are different objects of $\mathcal{C}_A$ so that the unit constraints are non-trivial. In particular, the module category is in general *not* a strict tensor category.

The ribbon structure of $\mathcal{C}_A^{\ell oc}$ is inherited in a rather obvious manner from $\mathcal{C}$. Concretely, the *braiding* on $\mathcal{C}_A^{\ell oc}$ is given by the family

$$c^A_{M,N} := r \circ c_{\dot M,\dot N} \circ e \;\in \mathrm{Hom}_A(M{\otimes_A}N, N{\otimes_A}M) \tag{3.50}$$

of morphisms, where e is the embedding morphism for the retract $M{\otimes_A}N \prec \dot M{\otimes}\dot N$, $c_{\dot M,\dot N}$ is the braiding in $\mathcal{C}$, and r the restriction morphism for $\dot N{\otimes}\dot M \succ N{\otimes_A}M$. The *twist* on $\mathcal{C}_A^{\ell oc}$ just coincides with the one of $\mathcal{C}$, i.e. $\theta^A_M = \theta_{\dot M}$ (see Proposition 3.17), and the *duality* of $\mathcal{C}_A^{\ell oc}$ is the assignment $M \mapsto M^\vee = (\dot M^\vee, (d_{\dot M}{\otimes}id_{\dot M^\vee}) \circ (id_{\dot M^\vee}{\otimes}\rho_M{\otimes}id_{\dot M^\vee}) \circ (c^{-1}_{\dot M^\vee,A}{\otimes}b_{\dot M}))$ together with the morphisms

$$\begin{aligned} b^A_M &:= r_{M\otimes_A M^\vee} \circ (\rho_M \otimes id_{\dot M^\vee}) \circ (id_A \otimes b_{\dot M}) = \rho_{M\otimes_A M^\vee} \circ [id_A \otimes (r_{M\otimes_A M^\vee} \circ b_{\dot M})] \quad \text{and} \\ d^A_M &:= [id_A \otimes (d_{\dot M} \circ (id_{\dot M^\vee} \otimes \rho_M) \circ (c^{-1}_{\dot M^\vee,A} \otimes id_M)] \circ [(\Delta\circ\eta) \otimes e_{M^\vee\otimes_A M}] \\ &= [id_A \otimes (d_{\dot M} \circ e_{M^\vee\otimes_A M} \circ \rho_{M^\vee\otimes_A M})] \circ [(\Delta\circ\eta) \otimes id_{M^\vee\otimes_A M}] \end{aligned} \tag{3.51}$$

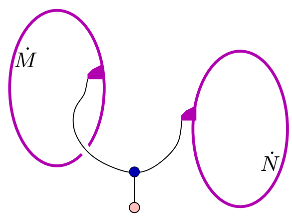
(compare Theorem 1.15 of [26] and section 5.3 of [20]).

Lemma 3.22:
For A a simple commutative special Frobenius algebra in a ribbon category $\mathcal{C}$ and A-modules $M, N \in \mathrm{Obj}(\mathcal{C}_A)$ one has

$$\dim(M \otimes_A N) = \frac{\dim(\dot M)\,\dim(\dot N)}{\dim(A)} . \tag{3.52}$$

Proof:
We have

$$\dim(M{\otimes_A}N) \;=\; \mathrm{tr}(id_{M\otimes_A N}) \;=\; \mathrm{tr}(P_{M\otimes N}) \;= \tag{3.53}$$



Now since A is haploid, for every $\varphi \in \mathrm{Hom}(\mathbf{1}, A)$ we have $\varphi = \beta_{\mathbf{1}}^{-1}(\varepsilon\circ\varphi)\,\eta = (\varepsilon\circ\varphi)\,\eta/\dim(A)$. It follows that removing the A-lines from the graph on the right hand side of (3.53) just amounts to a factor of $1/\dim(A)$; but removing the A-ribbons leaves us just with an $\dot{M}$- and an $\dot{N}$-loop, i.e. with $\dim(\dot{M})\,\dim(\dot{N})$. □

When A is symmetric, this result is also implied by Lemma 2.27, and for the case that in addition $\mathcal{C}$ is semisimple, it has already been established in [26] (corollary to Theorem 1.18).

Remark 3.23:
(i) To a modular tensor category $\mathcal{C}$ one associates a *dimension* $\mathrm{Dim}(\mathcal{C})$ and the (unnormalised) *charges* $p^{\pm}(\mathcal{C})$ by

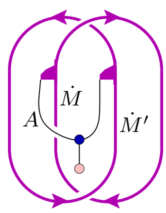
$$\mathrm{Dim}(\mathcal{C}) := \sum_{i\in\mathcal{I}} \dim(U_i)^2 \qquad \text{and} \qquad p^{\pm}(\mathcal{C}) := \sum_{i\in\mathcal{I}} \theta_i^{\pm 1} \dim(U_i)^2\,, \tag{3.54}$$

where $\{U_i\,|\,i\in\mathcal{I}\}$ are representatives of the isomorphism classes of simple objects of $\mathcal{C}$. The numbers $\mathrm{Dim}(\mathcal{C})$ and $p^{\pm}(\mathcal{C})$ are non-zero (see e.g. Corollary 3.1.8 of [1]) and satisfy $p^{+}(\mathcal{C})\,p^{-}(\mathcal{C}) = \mathrm{Dim}(\mathcal{C})$.
Let A be a haploid commutative symmetric special Frobenius algebra in $\mathcal{C}$. Combining Theorem 4.1 of [26] with Theorem 3.1.7 of [1], one sees that the dimension and charge obey

$$\mathrm{Dim}(\mathcal{C}_A^{\ell oc}) = \frac{\mathrm{Dim}(\mathcal{C})}{\big(\dim^{\mathcal{C}}(A)\big)^2} \qquad \text{and} \qquad p^{\pm}(\mathcal{C}_A^{\ell oc}) = \frac{p^{\pm}(\mathcal{C})}{\dim^{\mathcal{C}}(A)}\,. \tag{3.55}$$

Suppose now that $\Bbbk = \mathbb{C}$ and that $\dim(U) \geq 0$ for all U (as is e.g. the case if $\mathcal{C}$ is a *-category [29]). Then one has in fact $\dim(U) \geq 1$ for all non-zero objects, as well as $\mathrm{Dim}(\mathcal{C}) \geq 1$ and $|p^{+}/p^{-}| = 1$. It also follows that either $\dim(A) = 1$ or else $\dim(A) \geq 2$, so that for any non-trivial A the dimension of $\mathcal{C}_A^{\ell oc}$ is at most one quarter of the dimension of $\mathcal{C}$. The relation "being a category of local A-modules" (with A a haploid commutative symmetric special Frobenius algebra in another category) thus induces a partial ordering '>' on modular tensor categories, given by $\mathcal{C} > \mathcal{D}$ iff $\mathcal{D} \cong \mathcal{C}_A^{\ell oc}$ for some $A \not\cong \mathbf{1}$. Also note that owing to $\mathrm{Dim}(\mathcal{C}) \geq 1$ one can repeat the procedure of "going to the category of local modules" only a finite number of times. Conversely, it follows that the dimension of a haploid commutative special Frobenius algebra in a modular tensor category $\mathcal{C}$ is bounded by the square root of the dimension of $\mathcal{C}$.

(ii) In case A is a commutative simple symmetric special Frobenius algebra, the numbers s^A that are the analogs of the numbers (2.5) in the category $\mathcal{C}_A^{\ell oc}$ can be expressed in terms of morphisms of $\mathcal{C}$ as

$$s^A_{M,M'} = \frac{1}{\dim(A)}$$  (3.56)

It follows e.g. that

$$\dim_A(M) \equiv s^A_{M,0} = \dim(\dot{M}) \,/\, \dim(A) \tag{3.57}$$

(see Theorem 1.18 of [26]). Note that the label 0 on s^A refers to the tensor unit of $\mathcal{C}_A^{\ell oc}$, which is the simple local module A itself. In the application to conformal field theory, s^A is also closely related to the modular S-transformation of conformal one-point blocks on a torus with insertion A (see [3] and Section 5.7 of [18]).

Next we study what can be said about Karoubianness of categories of local modules. Recall the statements about A-modules in Remarks 2.17 and 2.19. It follows immediately with the help of the functoriality of the braiding that if the A-module $(\dot{M}, \rho)$ is in addition local, then so are the A-module $(\mathrm{Im}\,(p), r{\circ}\rho{\circ}(id_A{\otimes}e))$ (2.26) in $\mathcal{C}$ and the A-module $((\dot{M};p), p\circ\rho)$ (2.27) in the Karoubian envelope $\mathcal{C}^{\mathrm{K}}$.

According to Remark 2.19, non-split idempotents in $\mathcal{C}$ can be used to build $(A;id_A)$-modules in $\mathcal{C}^{\mathrm{K}}$ which do not come from an A-module in $\mathcal{C}$. Thus in general the category $(\mathcal{C}_A)^{\mathrm{K}}$ is a *proper* subcategory of $(\mathcal{C}^{\mathrm{K}})_A$. On the other hand, we still have the following results, which later on will allow us to establish, in corollary 4.11, equivalence of these two categories if A is not just an algebra but even a special Frobenius algebra.

Lemma 3.24:
(i) If A is a commutative symmetric special Frobenius algebra in a Karoubian ribbon category $\mathcal{C}$, then the category $\mathcal{C}_A^{\ell oc}$ of local A-modules in $\mathcal{C}$ is Karoubian as well.

(ii) For any algebra A in a (not necessarily Karoubian) tensor category $\mathcal{C}$ the category $(\mathcal{C}^{\mathrm{K}})_{(A;id_A)}$ is Karoubian, i.e. one has the equivalence

$$\big((\mathcal{C}^{\mathrm{K}})_{(A;id_A)}\big)^{\mathrm{K}} \cong (\mathcal{C}^{\mathrm{K}})_{(A;id_A)} \tag{3.58}$$

of categories. If $\mathcal{C}$ is ribbon and A is commutative symmetric special Frobenius, then also the category $(\mathcal{C}^{\mathrm{K}})^{\ell oc}_{(A;id_A)}$ is Karoubian, and one has the equivalence

$$\big((\mathcal{C}^{\mathrm{K}})^{\ell oc}_{(A;id_A)}\big)^{\mathrm{K}} \cong (\mathcal{C}^{\mathrm{K}})^{\ell oc}_{(A;id_A)}\,. \tag{3.59}$$

of ribbon categories.

Proof:
(i) Since $\mathcal{C}_A^{\ell oc}$ is a full subcategory of $\mathcal{C}_A$, the assertion follows from immediately from the analogous statement about $\mathcal{C}_A$ in Lemma 2.18.

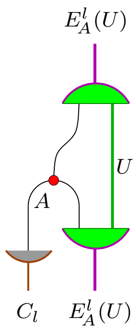
(ii) Since $\mathcal{C}^{\mathrm{K}}$ is Karoubian, the two equivalences are directly implied by Lemma 2.18 and by (i), respectively. That the second equivalence preserves the ribbon structure is easily seen by writing out the equivalence explicitly. □

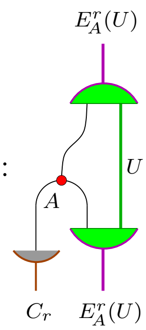
4 Local induction

4.1 The local induction functors

We have already announced above that the endofunctors $E_A^{l/r}$ with respect to a symmetric special Frobenius algebra A are related to *local induction*, i.e. functors from $\mathcal{C}$ to a full subcategory of the category $\mathcal{C}_{C_{l/r}(A)}$ of modules over the left and right center of A, respectively, that share many properties of induction. As shown in Proposition 4.1 below, the objects $E_A^{l/r}(U)$ in the image of these endofunctors possess an additional property: they are *local* $C_{l/r}(A)$-modules. Accordingly, the relevant full subcategories are the categories $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$ of local $C_{l/r}(A)$-modules. The corresponding local induction functors, to be denoted by $\ell\text{-Ind}_A^{l/r}$, from $\mathcal{C}$ to $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$ will be introduced in Definition 4.3 below. In the special case that already A itself is commutative, the centers coincide with A, and accordingly there is only a single local induction procedure, which is a functor from $\mathcal{C}$ to the category $\mathcal{C}^{\ell oc}_A$ of local A-modules.

Proposition 4.1:
Let A be a symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$. Then for any object U of $\mathcal{C}$, $E_A^l(U)$ is a local $C_l(A)$-module and $E_A^r(U)$ is a local $C_r(A)$-module. The representation morphisms are given by

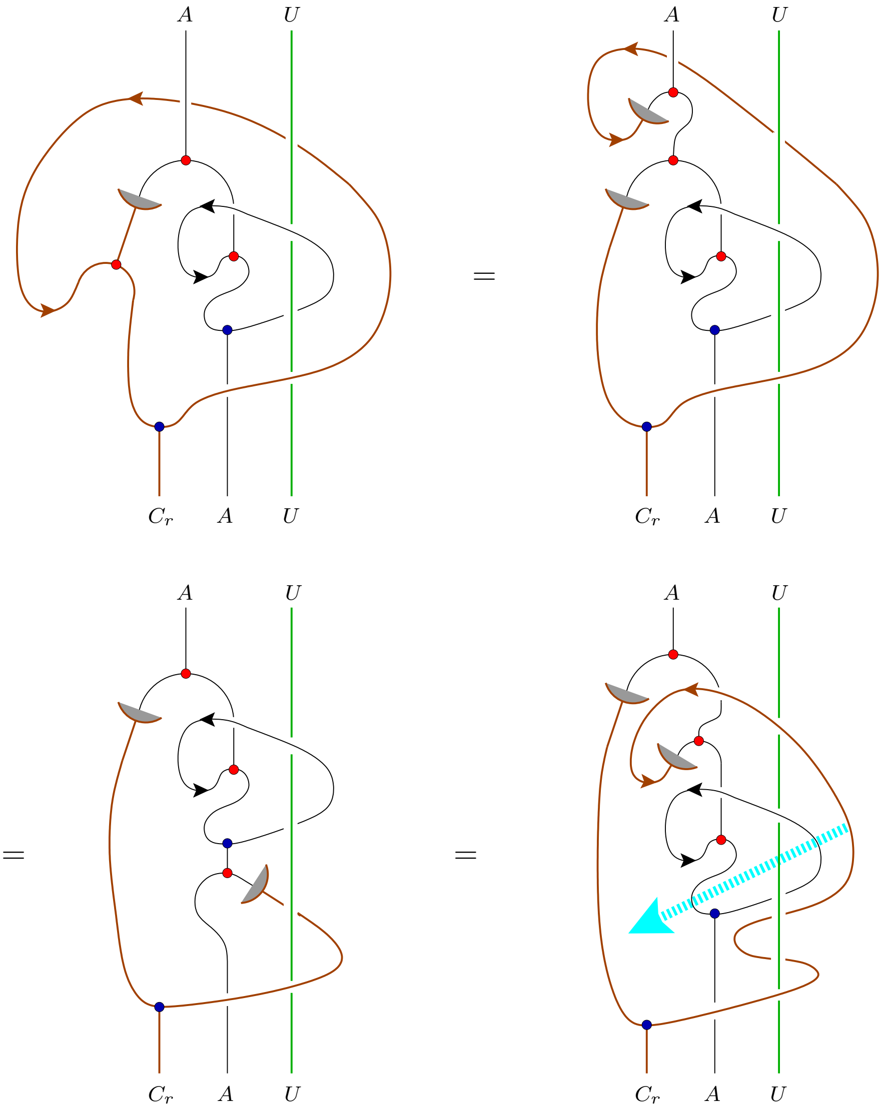
$\rho^{\ell oc}_{C_l(A);U}$ =: 

$\rho^{\ell oc}_{C_r(A);U}$ =: 

(4.1)

Proof:
Using the properties (2.64) it is easily verified that $\rho^{\ell oc}_{C_{l/r}(A);U}$ as defined in (4.1) possess the properties of a representation morphism for $C_l(A)$ and $C_r(A)$, respectively. To establish locality we must check that $\rho^{\ell oc}_{C_{l/r}(A);U} \circ P_{C_{l/r}}(U) = \rho^{\ell oc}_{C_{l/r}(A);U}$. This can be seen by inserting an idempotent $P_A^{l/r}(U)$ in front of the embedding morphism e of $E_A^{l/r}(U)$; afterwards this idempotent can be used to remove $P_{C_{l/r}}(U)$. For the case of $E_A^r(U)$, the corresponding

moves look as follows.

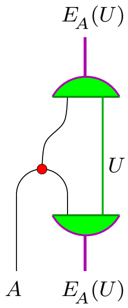


(4.2)

Here the embedding and restriction morphisms for $E_A^r(U) \prec A \otimes U$ are omitted. To establish these equalities one needs in particular (3.16) and the properties (2.64) and (2.65) of C_r. □

Corollary 4.2:
Let A be a commutative symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$ and $U \in \mathrm{Obj}(\mathcal{C})$. Then the object $E_A(U) := E_A^l(U) = E_A^r(U)$ carries a natural structure of local

A-module with representation morphism

$$\rho^{\ell oc}_{A;U} :=$$ 

(4.3)

It follows that given any symmetric special Frobenius algebra A in a ribbon category, by regarding $E_A^{l/r}(U)$ as an object of the category $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$ of local $C_{l/r}$-modules we have a functor from $\mathcal{C}$ to $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$.

Definition 4.3:
The functors $\ell\text{-Ind}_A^{l/r}$, called (left, respectively right) *local induction functors*, from $\mathcal{C}$ to $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$ are defined by

$$\ell\text{-Ind}_A^{l/r}(U) := (E_A^{l/r}(U), \rho^{\ell oc}_{C_{l/r}(A);U}) , \qquad \ell\text{-Ind}_A^{l/r}(f) := E_A^{l/r}(f) . \tag{4.4}$$

When A is commutative, we write $\ell\text{-Ind}_A$ for $\ell\text{-Ind}_A^l = \ell\text{-Ind}_A^r$.

The qualification 'local' used here indicates that the A-module $\ell\text{-Ind}_A(U)$ is local; that we speak of local *induction* is justified by the observation that there exists an embedding of $\ell\text{-Ind}_A(U)$ into the induced module $\text{Ind}_A(U)$. More precisely, we have the following result, which allows us to use reciprocity theorems of ordinary induction when working with local induction.

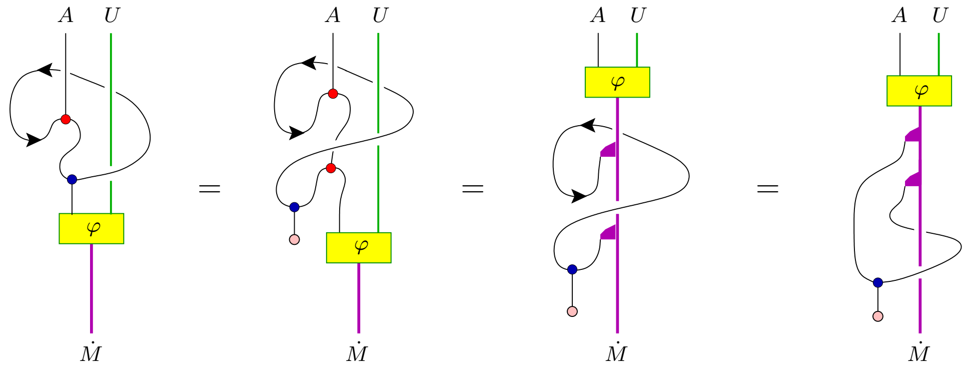
Proposition 4.4:
For A a commutative symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$ and $\ell\text{-Ind}_A(U)$ endowed with the A-module structure given in corollary 4.2, for every local A-module M one has

$$\begin{aligned} \text{Hom}_A(M, \ell\text{-Ind}_A(U)) &\cong \text{Hom}_A(M, \text{Ind}_A(U)) \qquad \text{and} \\ \text{Hom}_A(\ell\text{-Ind}_A(U), M) &\cong \text{Hom}_A(\text{Ind}_A(U), M) . \end{aligned} \tag{4.5}$$

Proof:
Consider the first isomorphism in (4.5). Apply Lemma 2.4 to the objects M and $\text{Ind}_A(U)$ of $\mathcal{C}_A$ to see that there is a natural bijection

$$\text{Hom}_A(M, \ell\text{-Ind}_A(U)) \cong \{\varphi \in \text{Hom}_A(M, \text{Ind}_A(U)) \,|\, P_A(U) \circ \varphi = \varphi\} . \tag{4.6}$$

Further, observe that for every A-module M and every $\varphi \in \mathrm{Hom}_A(M, \mathrm{Ind}_A(U))$ we have



(4.7)

Here the first equality uses that A is commutative and symmetric Frobenius, the second that φ is an A-module morphism, and the third is a rearrangement of the lower A-ribbon that uses that A is commutative and symmetric and that (since it is also Frobenius) it has trivial twist.

When M is local, then by Lemma 3.16(iii) the right hand side of (4.7) equals φ. Further, the left hand side of (4.7) is nothing but $P_A(U) \circ \varphi$. Thus if M is local and φ a morphism in $\mathrm{Hom}_A(M, \mathrm{Ind}_A(U))$, then $P_A(U) \circ \varphi = \varphi$ holds automatically. Together with (4.6) this implies the first bijection in (4.5).

The second of the bijections (4.5) follows analogously by an identity between morphisms that looks like figure (4.7) turned upside down. □

Lemma 4.5:

Let A be an algebra in a (not necessarily Karoubian) tensor category $\mathcal{C}$.

(i) There is an equivalence

$$\big((\mathcal{C}^{\mathrm{K}})^{\mathrm{Ind}}_{(A;id_A)}\big)^{\mathrm{K}} \cong (\mathcal{C}^{\mathrm{Ind}}_A)^{\mathrm{K}} \tag{4.8}$$

between Karoubian envelopes of categories of induced modules.

(ii) If $\mathcal{C}$ is ribbon and A is commutative symmetric special Frobenius, then there is an equivalence

$$\big((\mathcal{C}^{\mathrm{K}})^{\ell\text{-Ind}}_{(A;id_A)}\big)^{\mathrm{K}} \cong (\mathcal{C}^{\ell\text{-Ind}}_A)^{\mathrm{K}} \tag{4.9}$$

between Karoubian envelopes of categories of locally induced modules.

Proof:

(i) We will construct a functor F from $(\mathcal{C}^{\mathrm{Ind}}_A)^{\mathrm{K}}$ to $\mathcal{D} := ((\mathcal{C}^{\mathrm{K}})^{\mathrm{Ind}}_{(A;id_A)})^{\mathrm{K}}$ that satisfies the criterion of Proposition 2.3.

But first we consider the category $\mathcal{D}$ in more detail. Objects of $\mathcal{D}$ are of the form[4] $(\mathrm{Ind}_{(A;id_A)}(U;p);\pi)$ with $U \in \mathrm{Obj}(\mathcal{C})$, and with $p \in \mathrm{End}(U)$ and $\pi \in \mathrm{End}(A \otimes U)$ idempotents satisfying

$$(id_A \otimes p) \circ \pi \circ (id_A \otimes p) = \pi \qquad \text{and} \qquad \pi \circ (m \otimes p) = (m \otimes p) \circ (id_A \otimes \pi)\,. \tag{4.10}$$

[4] We slightly abuse notation by writing just $\mathrm{Ind}_{(A;id_A)}(U;p)$ in place of $\mathrm{Ind}_{(A;id_A)}((U;p))$.

The latter properties imply that

$$\pi \circ (m \otimes id_U) = \pi \circ (m \otimes p) = (m \otimes id_U) \circ (id_A \otimes \pi)\,, \tag{4.11}$$

which in turn allows us to regard π as an idempotent in $\mathrm{End}_{(A;id_A)}(\mathrm{Ind}_{(A;id_A)}(U;id_U))$, i.e. in the space of endomorphisms of an induced $(A;id_A)$-module for which p is replaced by id_U. As a consequence, $(\mathrm{Ind}_{(A;id_A)}(U;id_U);\pi)$ is an object of $\mathcal{D}$, and we have

$$id_{(\mathrm{Ind}_{(A;id_A)}(U;id_U);\pi)} = \pi = id_{(\mathrm{Ind}_{(A;id_A)}(U;p);\pi)}\,. \tag{4.12}$$

(All morphism spaces are regarded as subspaces of the corresponding spaces of morphisms in $\mathcal{C}$.)
Furthermore, again using (4.10), it follows that the morphism spaces of $\mathcal{D}$ of our interest are of the form

$$\begin{aligned}&\mathrm{Hom}^{\mathcal{D}}((\mathrm{Ind}_{(A;id_A)}(U;q);\varpi),(\mathrm{Ind}_{(A;id_A)}(U;q');\varpi'))\\&\quad= \{\, f\in \mathrm{End}(A\otimes U) \,|\, \varpi'\circ f\circ \varpi = f = (id_A\otimes q')\circ f\circ (id_A\otimes q)\\&\qquad\qquad \text{and } f\circ (m\otimes q) = (m\otimes q')\circ(id_A\otimes f)\,\}\,.\end{aligned} \tag{4.13}$$

By similar calculations as in (4.11) one can then check that

$$\begin{aligned}&\pi \in \mathrm{Hom}^{\mathcal{D}}((\mathrm{Ind}_{(A;id_A)}(U;id_U);\pi),(\mathrm{Ind}_{(A;id_A)}(U;p);\pi)) \qquad \text{and}\\&\pi \in \mathrm{Hom}^{\mathcal{D}}((\mathrm{Ind}_{(A;id_A)}(U;p);\pi),(\mathrm{Ind}_{(A;id_A)}(U;id_U);\pi))\,,\end{aligned} \tag{4.14}$$

so that $(\mathrm{Ind}_{(A;id_A)}(U;p);\pi)$ and $(\mathrm{Ind}_{(A;id_A)}(U;id_U);\pi)$ are isomorphic as objects of $\mathcal{D}$,

$$(\mathrm{Ind}_{(A;id_A)}(U;p);\pi) \cong (\mathrm{Ind}_{(A;id_A)}(U;id_U);\pi)\,. \tag{4.15}$$

Finally we observe that objects of $(\mathcal{C}_A^{\mathrm{Ind}})^{\mathrm{K}}$ are of the form $(\mathrm{Ind}_A(U);\pi)$ with $U\in\mathrm{Obj}(\mathcal{C})$ and $\pi\in\mathrm{End}_A(\mathrm{Ind}_A(U))$ an idempotent. Therefore by setting

$$F: \quad (\mathrm{Ind}_A(U);\pi) \;\mapsto\; (\mathrm{Ind}_{(A;id_A)}(U;id_U);\pi) \tag{4.16}$$

for objects and defining F to be the identity map on morphisms provides us with a functor F: $(\mathcal{C}_A^{\mathrm{Ind}})^{\mathrm{K}}\to\mathcal{D}$. Because of (4.15), F is essentially surjective, and it is bijective on morphisms. By Proposition 2.3, F thus furnishes an equivalence of categories.

(ii) The proof works along the same lines as for part (i). First note that objects of the category $\mathcal{D}^{\ell\mathrm{oc}} := \big((\mathcal{C}^{\mathrm{K}})^{\ell\text{-}\mathrm{Ind}}_{(A;id_A)}\big)^{\mathrm{K}}$ are of the form $(\ell\text{-}\mathrm{Ind}_{(A;id_A)}((U;p));\pi)$. On the other hand, by definition we have $\ell\text{-}\mathrm{Ind}_A(U) = (\mathrm{Ind}_A(U);P_A(U))$, so that

$$(\ell\text{-}\mathrm{Ind}_{(A;id_A)}((U;p));\pi) = (\mathrm{Ind}_{(A;id_A)}(U;p);\pi) \tag{4.17}$$

with $P_A(U)\circ\pi\circ P_A(U)=\pi$. The rest of the arguments in (i) go through unmodified, telling us that

$$(\ell\text{-}\mathrm{Ind}_{(A;id_A)}((U;p));\pi) \;\cong\; (\ell\text{-}\mathrm{Ind}_{(A;id_A)}((U;id_U));\pi)\,. \tag{4.18}$$

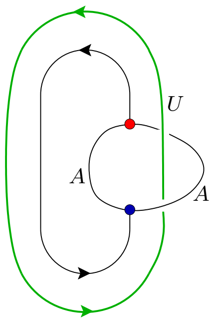
Therefore the functor $F^{\ell oc}$, defined as F in (4.16) with $\ell\text{-Ind}_{(A;id_A)}$ in place of $\text{Ind}_{(A;id_A)}$, is essentially surjective and bijective on morphisms, and hence furnishes an equivalence of categories. □

Remark 4.6 :
For any commutative symmetric special Frobenius algebra A and any object U of $\mathcal{C}$ the dimension of $E_A(U) \in \text{Obj}(\mathcal{C})$ is given by

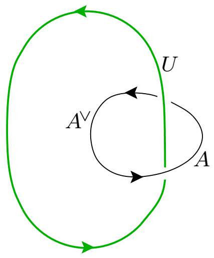
$$\dim(E_A(U)) = s_{U,A} \,. \tag{4.19}$$

(The dimension of $\ell\text{-Ind}_A(U)$ as an object of $\mathcal{C}_A^{\ell oc}$ then follows via (3.57).) The equality (4.19) is easily verified by drawing the corresponding ribbon graphs:

$\dim(E_A(U)) \;=$



$=$

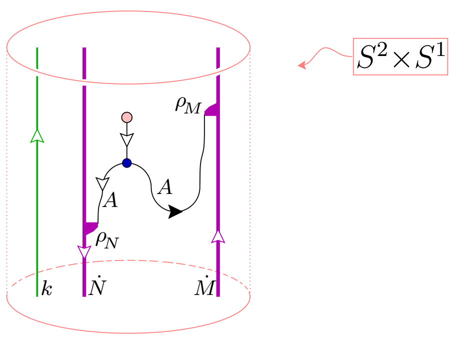


$= \; s_{U,A} \,.$ (4.20)

The first equality uses the fact that for any retract (S,e,r) of U one has $\dim(S) = \text{tr}_S\, id_S = \text{tr}_S\, r\circ e = \text{tr}_U\, e\circ r = \text{tr}_U\, P$, applied to the idempotent $P = P_A$. In the second step the A-loop that does not intersect the U-ribbon is omitted, using in particular the Frobenius property and specialness of A. The resulting graph is equal to $s_{U,A^\vee}$; but $A \cong A^\vee$, since A is Frobenius.

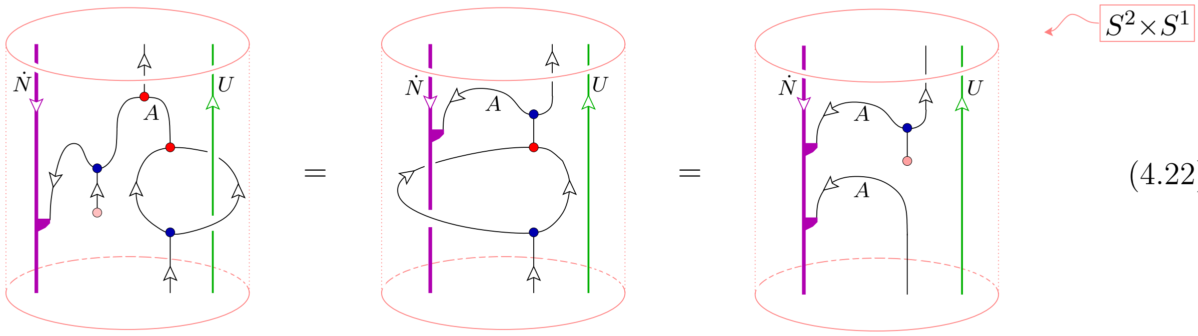
Remark 4.7 :
When $\mathcal{C}$ is modular, one may obtain (4.5) also as follows. Proposition 5.22 of [18] expresses the dimension $\dim \text{Hom}_A(M{\otimes}U_k, N)$ as the invariant of a ribbon graph in $S^2{\times}S^1$:

$\dim\, \text{Hom}_A(M{\otimes}U_k, N) \;=$



(4.21)

Let us consider the case that $U_k = \mathbf{1}$, $M = \ell\text{-Ind}_A(U)$ and N a local module. Inserting the definition (4.3) of $\rho^{E_A(U)}$ and moving the restriction morphism r around the (vertical) S^1-direction so as to combine with the embedding e to a projector, then yields for dim $\text{Hom}_A(\ell\text{-Ind}_A(U), N)$ the graph on the left hand side of



(4.22)

The equalities shown here are obtained as follows. In the first step the A-ribbon of the projector is taken around the (horizontal) S^2-direction until it wraps around the $\dot{N}$-ribbon. This can be transformed into a locality projector for N and thus – as N is local by assumption – be left out. The second step is then completed by using the representation property for N. In the graph on the right hand side one can now move one of the representation morphisms around the S^1-direction, and then use the representation property again; afterwards the A-ribbon can be removed, using that A is special. The invariant of the resulting graph in $S^2 \times S^1$ is dim $\text{Hom}(U, \dot{N})$.

4.2 Local modules from local induction

In the sequel it will be very helpful to express categories of (local) modules in terms of the corresponding categories of (locally) induced modules. A crucial ingredient is the

Lemma 4.8:
Let A be a special Frobenius algebra in a (not necessarily Karoubian) tensor category $\mathcal{C}$.
(i) For every module M over A the object $\dot{M}$ is a retract of $A \otimes \dot{M}$.
(ii) Every module over A is a module retract of an induced A-module.

Proof:
(i) The retract is given by $(\dot{M}, e_M, \rho_M)$ with ρ_M the representation morphism of M and $e_M := (id_A \otimes \rho_M) \circ ((\Delta \circ \eta) \otimes id_{\dot{M}})$. That $\rho_M \circ e_M = id_{\dot{M}}$ is verified by first using the representation property of ρ_M, then specialness of A, and then the unit property of η.
Note that the Frobenius property (2.37) of A is not used in this argument.
(ii) We show that any A-module M is a module retract of $\mathrm{Ind}_A(\dot{M})$. In view of (i), all that needs to be checked is that the morphisms ρ_M and e_M are module morphisms. That $\rho_M \in \mathrm{Hom}_A(\mathrm{Ind}_A(\dot{M}), M)$ follows directly from the representation property of ρ_M, while $e_M \in \mathrm{Hom}_A(M, \mathrm{Ind}_A(\dot{M}))$ is a consequence of the Frobenius property of A. □

This result has already been established in lemma 4.15 of [20]. (There the assumption was made that the category $\mathcal{C}$ of which A is an object is abelian, but the proof does not require this property.)

Proposition 4.9:
Let A be a special Frobenius algebra in a (not necessarily Karoubian) tensor category $\mathcal{C}$. Then, while the module category $\mathcal{C}_A$ is not necessarily Karoubian, still the Karoubian envelopes of $\mathcal{C}_A$ and of its full subcategory $\mathcal{C}_A^{\mathrm{Ind}}$ of induced A-modules coincide:

$$(\mathcal{C}_A)^{\mathrm{K}} \cong (\mathcal{C}_A^{\mathrm{Ind}})^{\mathrm{K}} . \tag{4.23}$$

It follows in particular that in case that $\mathcal{C}$ *is* Karoubian (so that by Lemma 2.18 $\mathcal{C}_A$ is Karoubian, too), then $\mathcal{C}_A \cong (\mathcal{C}_A^{\mathrm{Ind}})^{\mathrm{K}}$.

Proof:
Lemma 4.8 implies in particular that every object of the category $\mathcal{C}_A$ of A-modules in $\mathcal{C}$ is of the form

$$\mathrm{Ind}_A^p(U) := (\mathrm{Im}(p), r \circ (m \otimes id_U) \circ (id_A \otimes e)) \tag{4.24}$$

with a suitable object $U \in \mathrm{Obj}(\mathcal{C})$ and p a split idempotent such that

$$p \in \mathrm{Hom}_A(\mathrm{Ind}_A(U), \mathrm{Ind}_A(U)) , \qquad p \circ p = p , \qquad e \circ r = p , \qquad r \circ e = id_{\mathrm{Im}(p)} . \tag{4.25}$$

This implies the equivalence (4.23). □

Not surprisingly, Lemma 4.8 and Proposition 4.9 have analogues for local modules. Indeed, when combined with the previous result (4.5), they imply:

Corollary 4.10:
Let A be a centrally split commutative symmetric special Frobenius algebra in a (not necessarily Karoubian) ribbon category $\mathcal{C}$. Then every local module over A is a module retract of a locally induced A-module, and we have

$$(\mathcal{C}_A^{\ell\text{oc}})^{\mathrm{K}} \cong (\mathcal{C}_A^{\ell\text{-Ind}})^{\mathrm{K}}. \tag{4.26}$$

The equivalence (4.23) can be combined with previously established equivalences, in particular Lemma 4.5, to establish the following properties of module categories over special Frobenius algebras. They are much stronger than Lemma 4.5, and they do not hold, in general, for algebras that are not special Frobenius.

Corollary 4.11:
(i) For any special Frobenius algebra A in a (not necessarily Karoubian) tensor category $\mathcal{C}$ there is an equivalence

$$(\mathcal{C}^{\mathrm{K}})_{(A;id_A)} \cong (\mathcal{C}_A)^{\mathrm{K}}, \tag{4.27}$$

i.e. the operations of taking the Karoubian envelope and of forming the module category commute.

(ii) For any commutative symmetric special Frobenius algebra A in a (not necessarily Karoubian) ribbon category $\mathcal{C}$ there is an equivalence

$$(\mathcal{C}^{\mathrm{K}})^{\ell\text{oc}}_{(A;id_A)} \cong (\mathcal{C}_A^{\ell\text{oc}})^{\mathrm{K}}, \tag{4.28}$$

i.e. the operations of taking the Karoubian envelope and of forming the category of local modules commute.

Proof:
(i) We have

$$(\mathcal{C}_A)^{\mathrm{K}} \cong (\mathcal{C}_A^{\mathrm{Ind}})^{\mathrm{K}} \cong \big((\mathcal{C}^{\mathrm{K}})^{\mathrm{Ind}}_{(A;id_A)}\big)^{\mathrm{K}} \cong \big((\mathcal{C}^{\mathrm{K}})_{(A;id_A)}\big)^{\mathrm{K}} \cong (\mathcal{C}^{\mathrm{K}})_{(A;id_A)}. \tag{4.29}$$

The last equivalence follows by Lemma 2.18, the second equivalence is the one of Lemma 4.5(i), and the other two equivalences hold by Proposition 4.9.

(ii) Analogously,

$$(\mathcal{C}_A^{\ell\text{oc}})^{\mathrm{K}} \cong (\mathcal{C}_A^{\ell\text{-Ind}})^{\mathrm{K}} \cong \big((\mathcal{C}^{\mathrm{K}})^{\ell\text{-Ind}}_{(A;id_A)}\big)^{\mathrm{K}} \cong \big((\mathcal{C}^{\mathrm{K}})^{\ell\text{oc}}_{(A;id_A)}\big)^{\mathrm{K}} \cong (\mathcal{C}^{\mathrm{K}})^{\ell\text{oc}}_{(A;id_A)}. \tag{4.30}$$

The last equivalence follows by Lemma 3.24(i), the second equivalence is the one of Lemma 4.5(ii) and the other two equivalences hold by corollary 4.10. □

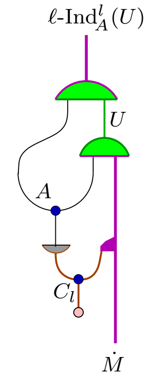
The statements of Proposition 4.4 and the results above about commutative Frobenius algebras that are based on that proposition do not directly generalise to the non-commutative case. However, there is the following substitute:

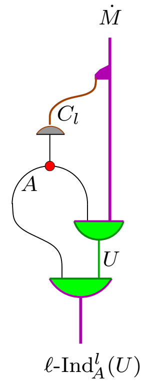
Proposition 4.12:
Let A be a symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$, and assume that the commutative symmetric Frobenius algebra $C_l(A)$ is special.

Then every local $C_l(A)$-module M is a module retract of a locally induced A-module, $M \prec \ell\text{-Ind}_A^l(U)$ with suitable $U \in \mathrm{Obj}(\mathcal{C})$.

Similarly, if $C_r(A)$ is special, then every local $C_r(A)$-module is a module retract of $\ell\text{-Ind}_A^r(U)$ with suitable $U \in \mathrm{Obj}(\mathcal{C})$.

Proof:
We establish the statement for $C_l \equiv C_l(A)$.
Let M be a local C_l-module. Choose $U = \mathrm{Im}\, E_A^r(\dot{M})$ and define morphisms $\tilde{e}$ and $\tilde{r}$ as

$\tilde{e} \;:=$ 

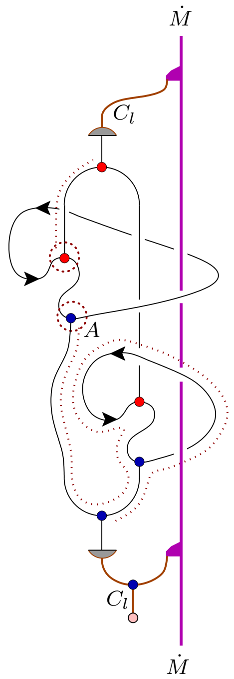
and $\tilde{r} \;:= \dfrac{\dim(A)}{\dim(C_l)}$ 

(4.31)

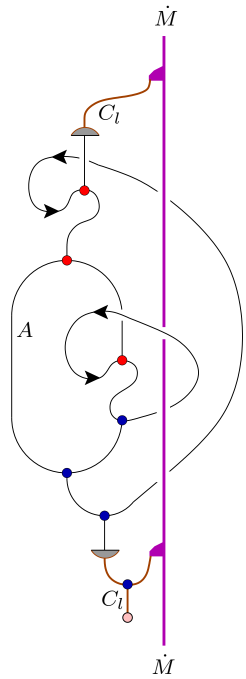
These are C_l-intertwiners, i.e. $\tilde{e} \in \mathrm{Hom}_{C_l}(M, \ell\text{-Ind}_A^l(U))$ and $\tilde{r} \in \mathrm{Hom}_{C_l}(\ell\text{-Ind}_A^l(U), M)$. To establish that $(M, \tilde{e}, \tilde{r})$ is a C_l-module retract of $\ell\text{-Ind}_A^l(U)$ we must show that $\tilde{r} \circ \tilde{e} = id_M$.

This is seen by the following series of moves.

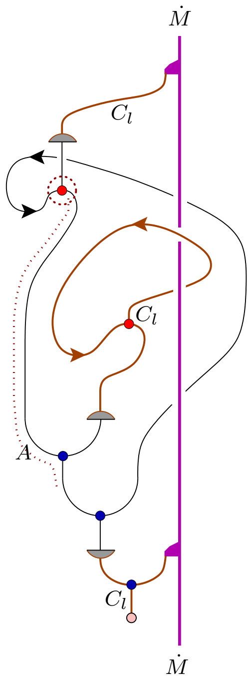
$$\frac{\dim(C_l)}{\dim(A)}\,\tilde{r}\circ\tilde{e} \;=$$



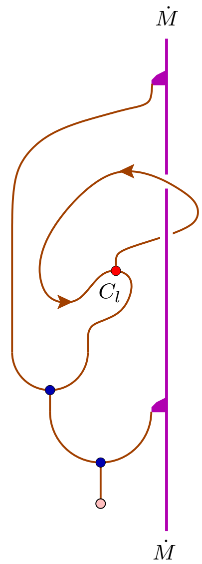
$$=$$



$$=$$



$$= \frac{\dim(C_l)}{\dim(A)}$$



(4.32)

In the first step the idempotents resulting from the composition are drawn explicitly. Then the multiplication and comultiplication are moved along the paths indicated. To the resulting morphism in the second picture one can apply Lemma 2.39 with $U = V = \dot{M}$ and $\Phi = c^{-1}_{A,\dot{M}} \circ c^{-1}_{\dot{M},A}$. This results in the insertion of an idempotent P^l_A. Using Lemma 2.29(iii) and the definition of the multiplication on C_l in (2.70) one arrives at the third morphism. In the final step the marked coproduct is moved along the path indicated, resulting in another idempotent P^l_A, which can be omitted against the embedding morphism e_{C_l}. Inserting the definition of the comultiplication on C_l in (2.70) one finally arrives at the morphism on the

right hand side.
There, the C_l-loop can be rearranged to be equal to $P_{C_l}(M)$, using the fact that C_l is a commutative symmetric Frobenius algebra. Afterwards, by the Definition 3.15 of a local module, the idempotent $P_{C_l}(M)$ can be omitted. The representation property together with specialness of C_l imply that the resulting morphism it is equal to $\dim(C_l)/\dim(A)\, id_M$. Altogether we thus have $\tilde r \circ \tilde e = id_M$, showing that M is indeed a retract of $\ell\text{-Ind}_A^l(U)$. □

Note that specialness of $C_{l/r}(A)$, which is assumed in the proposition, is guaranteed e.g. if A is simple and $\dim(C_{l/r}(A))$ is non-zero, see Proposition 2.37, and also if A is commutative, because then $C_{l/r}(A) = A$ and A is special by assumption.

4.3 Local induction of algebras

Since for any symmetric special Frobenius algebra A the categories $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$ of local modules over the left and right center of A are tensor categories, one can study algebras in these categories and, in particular, ask whether for an algebra B in $\mathcal{C}$ the locally induced module $\ell\text{-Ind}_A^{l/r}(B)$ inherits an algebra structure from B. We shall show that indeed the algebra $E_A^{l/r}(B)$ as defined by Proposition 3.8(i) lifts to an algebra in $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$ and inherits further structural properties. As a consequence, $\ell\text{-Ind}_A^{l/r}$ furnishes a functor from the category of (symmetric special Frobenius) algebras in $\mathcal{C}$ to the category of (symmetric special Frobenius) algebras in $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$.

We start by formulating conditions that allow an algebra B in $\mathcal{C}$ to be 'lifted' to an algebra in $\mathcal{C}^{\ell oc}_A$:

Lemma 4.13:
Let A be a commutative symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$. Let $B \equiv (B, m_B, \eta_B, \Delta_B, \varepsilon_B)$ be a Frobenius algebra. Let (B, ρ_B) carry the structure of a local A-module, and the product m_B on B satisfy

$$m_B \in \mathrm{Hom}_A(B\otimes B, B) \qquad \text{and} \qquad m_B \circ P_{B\otimes B} = m_B\,. \tag{4.33}$$

(i) $\tilde B \equiv (B, \tilde m_B, \tilde\eta_B, \tilde\Delta_B, \tilde\varepsilon_B)$ with

$$\tilde m_B := m_B \circ e_{B\otimes B}\,, \qquad \tilde\eta_B := \rho_B \circ (id_A \otimes \eta_B) \tag{4.34}$$

and

$$\tilde\Delta_B := r_{B\otimes B} \circ \Delta_B\,, \qquad \tilde\varepsilon_B := (id_A \otimes \varepsilon_B) \circ (id_A \otimes \rho_B) \circ ([\Delta_A \circ \eta_A] \otimes id_B) \tag{4.35}$$

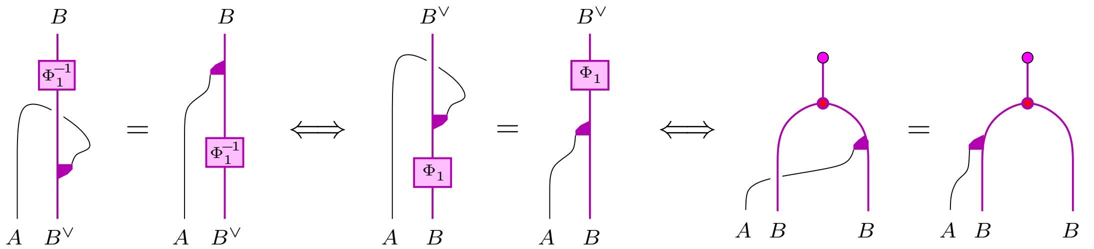
is a Frobenius algebra in $\mathcal{C}^{\ell oc}_A$.

(ii) Let A in addition be simple. If B has in addition any of the properties of being commutative, haploid, simple, special, or symmetric, then so has $\tilde B$.

Proof:
(i) We start by showing that $P_{B\otimes B}\circ\Delta_B=\Delta_B$ is implied by $m_B\circ P_{B\otimes B}=m_B$. The ultimate reason is that the coproduct can be expressed in terms of the product as

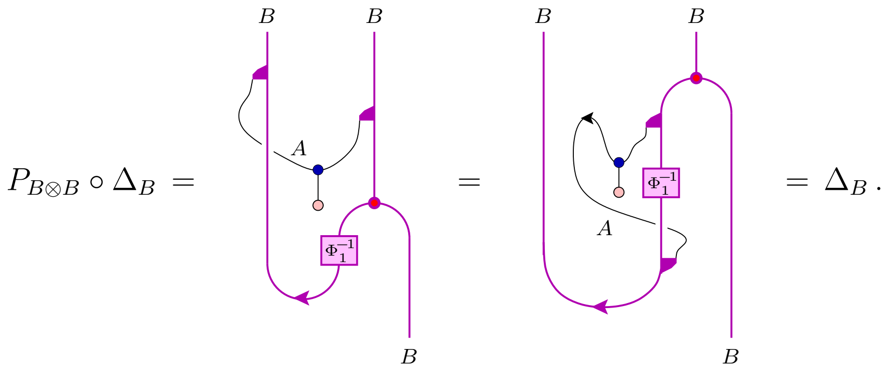
$$\Delta_B = (id_B\otimes m_B)\circ(id_B\otimes\Phi_1^{-1}\otimes id_B)\circ(b_B\otimes id_B) \tag{4.36}$$

with the morphism Φ_1, defined as in (2.35), being invertible because B is a Frobenius algebra (see formula (3.36) of [18] and, for the proof, lemma 3.7 of [18]). Consider the equivalences



(4.37)

The first equivalence follows by composing both sides of the first equality with Φ_1 both from the top and from the bottom. The second equivalence is obtained by composing the middle equality with the duality morphism d_B and writing out the definition (2.35) of Φ_1. Now the last equality in (4.37) indeed holds true. This can be seen by replacing m_B with $m_B\circ P_{B\otimes B}$ and using commutativity and the Frobenius property of A to move the action of A along the resulting A-ribbon from the right B-factor to the left. We can therefore write



(4.38)

The left-most graph is obtained by writing out the definition of $P_{B\otimes B}$ and inserting relation (4.36) for Δ_B. The next step uses in particular that $m_B\in\mathrm{Hom}_A(B\otimes B,B)$. The final step follows from the first equality in (4.37) together with the properties of A to be symmetric and special.

It is easy to check that the morphisms defined in (4.34) are elements of the relevant Hom_A-spaces, i.e. $\tilde m_B\in\mathrm{Hom}_A(B\otimes_A B,B)$ and $\tilde\eta_B\in\mathrm{Hom}_A(A,B)$, and analogously for $\tilde\Delta_B$ and $\tilde\varepsilon_B$. Of the defining properties for $\tilde B$ to be a Frobenius algebra we will verify explicitly only associativity – the other properties are checked analogously.
Associativity is deduced as follows:

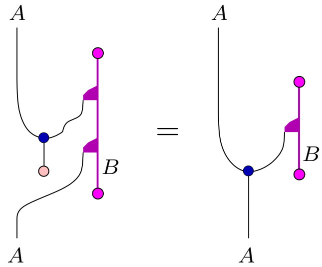
$$\begin{aligned}\tilde m_B\circ(\tilde m_B\otimes_A id_B) &= m_B\circ e_{B\otimes B}\circ r_{B\otimes B}\circ(m_B\otimes id_B)\circ e_{B\otimes B\otimes B}\\ &= m_B\circ(m_B\otimes id_B)\circ e_{B\otimes B\otimes B} = \cdots = \tilde m_B\circ(id_B\otimes_A\tilde m_B)\,.\end{aligned} \tag{4.39}$$

In the first step the definitions (3.48) and (4.34) are inserted; afterwards the idempotent $e_{B\otimes B} \circ r_{B\otimes B} = P_{B\otimes B}$ is omitted, which is allowed by assumption. Afterwards one can apply associativity of B, and then the previous steps are followed in reverse order.

(ii) Note that since A is commutative and simple, by Remark 2.28(i) it is also haploid. Out of the list of properties, let us look at specialness, commutativity and haploidity as examples; the remaining cases are analysed similarly.

Specialness: The first specialness relation for $\tilde{B}$ follows as

$\tilde{\varepsilon}_B \circ \tilde{\eta}_B =$



$$= \frac{\dim(B)}{\dim(A)}\, id_A \,. \qquad (4.40)$$

In the first step the definitions are substituted, while the second step uses the representation property of ρ_B and the Frobenius property of A. The resulting morphism is an element of $\mathrm{Hom}_A(A, A)$. Since A is haploid, this space is one-dimensional, so that the morphism must be proportional to id_A; comparing the traces determines the constant.
The second specialness condition is implied by

$$\tilde{m}_B \circ \tilde{\Delta}_B = m_B \circ e_{B\otimes B} \circ r_{B\otimes B} \circ \Delta_B = m_B \circ \Delta_B = id_B \,. \qquad (4.41)$$

Here in the next to last step we used again that $m_B \circ P_{B\otimes B} = m_B$; the last equality holds because B is special.

Commutativity: When B is commutative it follows directly from the form of the braiding in $\mathcal{C}_A^{\ell oc}$ – i.e. $c^A = r \circ c \circ e$ – and from the definition (4.34) of $\tilde{m}_B$ that $\tilde{B}$ is commutative as well.

Haploidity of $\tilde{B}$ is equivalent to $\dim \mathrm{Hom}_A(A, B) = 1$. Since $A = \mathrm{Ind}_A(\mathbf{1})$, the reciprocity (2.40) implies $\dim \mathrm{Hom}_A(A, B) = \dim \mathrm{Hom}(\mathbf{1}, B)$. If B is haploid, then this equals 1, so that $\tilde{B}$ is haploid as well. □

The following assertion shows that for any simple symmetric special Frobenius algebra A, local induction also supplies us with a functor from the category of Frobenius algebras in $\mathcal{C}$ to the category of Frobenius algebras in $\mathcal{C}_{C_{l/r}(A)}^{\ell oc}$.

Proposition 4.14:
Let A be a symmetric special Frobenius algebra and B a Frobenius algebra in a ribbon category $\mathcal{C}$, and assume that the symmetric Frobenius algebras $C_l(A)$ and $C_r(A)$ are also special.

(i) The local $C_l(A)$-module $\ell\text{-}\mathrm{Ind}_A^l(B) = (E_A^l(B), \rho_{C_l(A);B}^{\ell oc})$ can be endowed with the structure of a Frobenius algebra in the category $\mathcal{C}_{C_l(A)}^{\ell oc}$ of local $C_l(A)$-modules.

(ii) Let A be in addition simple. If the Frobenius algebra $E_A^l(B) \in \mathrm{Obj}(\mathcal{C})$ has any of the properties of being commutative, haploid, simple, symmetric, or special, then so has the Frobenius algebra $\ell\text{-Ind}_A^l(B) \in \mathrm{Obj}(\mathcal{C}_{C_l(A)}^{\ell\mathrm{oc}})$.
Analogous statements apply to $C_r(A)$ and $E_A^r(B)$.

Proof:
We show the claims for $C_l(A)$ and $E_A^l(B)$; the corresponding statements for $C_r(A)$ and $E_A^r(B)$ can be seen similarly. The statements follow by applying Lemma 4.13 to the Frobenius algebra $E_A^l(B)$. Accordingly we must check that the requirements of that lemma are satisfied. Abbreviate $C_l(A)$ by C. Recall the definition (4.1) of $\rho^{\ell\mathrm{oc}}$, which according to Proposition 4.1 gives a local C-module structure on $E_A^l(B)$. Furthermore, we have

$$\rho_{C;B}^{\ell\mathrm{oc}} \circ (id_C \otimes m) = m \circ (\rho_{C;B}^{\ell\mathrm{oc}} \otimes id_{E_A^l(B)})\,, \tag{4.42}$$

i.e. the multiplication m of $E_A^l(B)$ is indeed in $\mathrm{Hom}_C(E_A^l(B) \otimes E_A^l(B), E_A^l(B))$. To see this, we write out the definitions (4.1) and (3.15) for the action of C and the multiplication on $E_A^l(B)$, after which we can replace the resulting combination $e \circ r$ by $P_A^l(B)$; then associativity of A as well as the properties (2.55) and (2.64) of the center C relate the two sides of (4.42).
The equality $m \circ P_{E_A^l(B) \otimes E_A^l(B)} = m$ can be verified in the same way, using in addition that C is special. □

Let us reformulate the statement of Proposition 4.14(i) for later reference:

Corollary 4.15:
Let A a be symmetric special Frobenius algebra such that $C_l(A)$ and $C_r(A)$ are special, and B a Frobenius algebra, in a ribbon category $\mathcal{C}$. Then there is a Frobenius algebra

$$\ell\text{-Ind}_A^{l/r}(B) \;\in \mathrm{Obj}(\mathcal{C}_{C_{l/r}(A)}^{\ell\mathrm{oc}})\,. \tag{4.43}$$

in the category of local $C_{l/r}(A)$-modules. The underlying object of the module $\ell\text{-Ind}_A^{l/r}(B)$ is $E_A^{l/r}(B)$.

Note that we do not introduce a separate notation to indicate the Frobenius algebra structure of the module (4.43).

For the following statement we take A to be commutative, so that (4.43) is now an algebra in the category of local A-modules, denoted by $\ell\text{-Ind}_A(B)$.

Proposition 4.16:
Let A and B be commutative symmetric special Frobenius algebras in a ribbon category $\mathcal{C}$. Suppose in addition that A is simple and that the Frobenius algebra $E_A(B)$ is special. Then $\ell\text{-Ind}_A(B)$ is special, too, and we have an equivalence

$$(\mathcal{C}_A^{\ell\mathrm{oc}})_{\ell\text{-Ind}_A(B)}^{\ell\mathrm{oc}} \;\cong\; \mathcal{C}_{E_A(B)}^{\ell\mathrm{oc}} \tag{4.44}$$

of ribbon categories.

Proof:
By Proposition 3.8, $E_A(B)$ is a commutative symmetric Frobenius algebra. By assumption it is also special. Since A is simple, by Proposition 4.14(ii) all properties of $E_A(B)$ get transported to $\ell\text{-Ind}_A(B)$. In particular the three algebras A, $\ell\text{-Ind}_A(B)$ and $E_A(B)$ are commutative symmetric special Frobenius algebras, and by Proposition 3.21 all categories of local modules in (4.44) are ribbon categories.
The equivalence (4.44) is established by specifying two functors

$$F:\ \ (\mathcal{C}_A^{\ell oc})_{\ell\text{-Ind}_A(B)}^{\ell oc} \to \mathcal{C}_{E_A(B)}^{\ell oc} \qquad \text{and} \qquad G:\ \ \mathcal{C}_{E_A(B)}^{\ell oc} \to (\mathcal{C}_A^{\ell oc})_{\ell\text{-Ind}_A(B)}^{\ell oc} \tag{4.45}$$

and showing that they are each other's inverse and that they are ribbon.
The functor F: An object M in $(\mathcal{C}_A^{\ell oc})_{\ell\text{-Ind}_A(B)}^{\ell oc}$ can be regarded as a triple $(\dot M, \rho^A, \rho^{\ell\text{-Ind}_A(B)})$ consisting of an object $\dot M$ in $\mathcal{C}$, a representation morphism $\rho^A \equiv \rho_M^A \in \mathrm{Hom}(A{\otimes}\dot M, \dot M)$ that endows $(\dot M, \rho^A)$ with the structure of a local A-module, and a morphism $\rho^{\ell\text{-Ind}_A(B)} \equiv \rho_M^{\ell\text{-Ind}_A(B)} \in \mathrm{Hom}_A(\ell\text{-Ind}_A(B){\otimes}_A M, M)$ such that $(M, \rho^{\ell\text{-Ind}_A(B)})$ is a local $\ell\text{-Ind}_A(B)$-module in $\mathcal{C}_A^{\ell oc}$. To define F on objects we turn M into a local $E_A(B)$-module by providing a morphism $\rho^{E_A(B)} \in \mathrm{Hom}(E_A(B){\otimes}M, M)$ which has the appropriate properties; we set

$$\rho^{E_A(B)} := \rho^{\ell\text{-Ind}_A(B)} \circ r_{E_A(B)\otimes M} \,. \tag{4.46}$$

(Recall from formula (3.48) that $r_{E_A(B)\otimes M}$ is a short hand for $r_{E_A(B)\otimes M \succ E_A(B)\otimes_A M}$; analogous abbreviations are implicit in e_2 and e_3 below.) To check the first representation property in (2.24) one computes – abbreviating $\rho \equiv \rho^{E_A(B)}$, $m \equiv m^{E_A(B)}$, $\tilde\rho \equiv \rho^{\ell\text{-Ind}_A(B)}$, $\tilde m \equiv m^{\ell\text{-Ind}_A(B)}$ as well as $e_2 \equiv e_{E_A(B)\otimes M}$, $e_3 \equiv e_{E_A(B)\otimes E_A(B)\otimes M}$ and similarly for r_2, r_3 – as follows:

$$\begin{aligned} \rho \circ (id_{E_A(B)} \otimes \rho) &\overset{\text{(a)}}{=} \tilde\rho \circ r_2 \circ P_{E_A(B)\otimes M} \circ (id_{E_A(B)} \otimes \tilde\rho) \circ (id_{E_A(B)} \otimes r_2) \\ &\overset{\text{(b)}}{=} \tilde\rho \circ r_2 \circ (id_{E_A(B)} \otimes \tilde\rho) \circ (id_{E_A(B)} \otimes r_2) \circ e_3 \circ r_3 \\ &\overset{\text{(c)}}{=} \tilde\rho \circ (id_{E_A(B)} \otimes_A \tilde\rho) \circ r_3 \overset{\text{(d)}}{=} \tilde\rho \circ (\tilde m \otimes_A id_M) \circ r_3 \\ &\overset{\text{(e)}}{=} \rho \circ (m \otimes id_M) \,. \end{aligned} \tag{4.47}$$

In step (a) definition (4.46) of ρ is substituted and the idempotent $P_{E_A(B)\otimes M} \equiv P_2 = e_2 \circ r_2 \in \mathrm{End}(E_A(B) \otimes M)$ is inserted before the second restriction morphism r_2. Substituting the definition (3.46) for this idempotent, we see that it can be commuted past the first representation and restriction morphisms $\tilde\rho$ and r_2, both these morphisms being in Hom_A, and afterwards due to the presence of $r_2 = r_2 \circ P_2$ it can be replaced by $P_{E_A(B)\otimes E_A(B)\otimes M} \equiv P_3 = e_3 \circ r_3 \in \mathrm{End}(E_A(B) \otimes E_A(B) \otimes M)$; this has been done in (b). In (c) the definition (3.48) of the tensor product over A for morphisms is substituted, while step (d) is the representation property of $\rho^{\ell\text{-Ind}_A(B)}$. Finally in (e) the tensor product over A is replaced by (3.48), the multiplication $\tilde m$ of $\ell\text{-Ind}_A(B)$ expressed through (4.34) and the definition (3.47) substituted for the resulting $e_3 \circ r_3$; then all A-ribbons can be removed, yielding the final expression in

(4.47). The second property in (2.24) can be checked similarly.
Locality of the module $(M,\rho^{E_A(B)})$ is most easily verified with the help of the condition (ii) in Proposition 3.17. Indeed we have $\theta_M\circ\rho=\theta_M\circ\tilde\rho\circ r_2=\tilde\rho\circ(id_{E_A(B)}\otimes_A\theta_M)\circ r_2$, where the second step uses locality of M with respect to $\ell\text{-Ind}_A(B)$. As a consequence, $\theta_M\circ\rho=\tilde\rho\circ r_2\circ(id_{E_A(B)}\otimes\theta_M)\circ P_{E_A(B)\otimes M}=\rho\circ(id_{E_A(B)}\otimes\theta_M)$, where in the first equality the morphism $id_{E_A(B)}\otimes_A\theta_M$ is substituted, giving rise to the appearance of the idempotent $e_2\circ r_2=P_{E_A(B)\otimes M}$, while the second step uses locality of M with respect to A to commute θ_M with the idempotent, which is then omitted against r_2.

A morphism f from M to N in $(\mathcal{C}_A^{\ell oc})^{\ell oc}_{\ell\text{-Ind}_A(B)}$ is an element of $\text{Hom}(\dot M,\dot N)$ that commutes with the two actions ρ^A and $\rho^{\ell\text{-Ind}_A(B)}$. The functor F is defined to act as the identity on morphisms: $F(f):=f$. If f commutes with ρ^A and $\rho^{\ell\text{-Ind}_A(B)}$, then it commutes with $\rho^{E_A(B)}$ as well, because (using abbreviations similar to those in (4.47))

$$\begin{aligned} f\circ\rho_M &= f\circ\tilde\rho_M\circ r_{2,M}=\tilde\rho_N\circ(id_{E_A(B)}\otimes_A f)\circ r_{2,M}\\ &=\tilde\rho_N\circ r_{2,M}\circ(id_{E_A(B)}\otimes f)\circ P_{E_A(B)\otimes M}\\ &=\tilde\rho_N\circ r_{2,M}\circ P_{E_A(B)\otimes M}\circ(id_{E_A(B)}\otimes f)=\rho_N\circ(id_{E_A(B)}\otimes f)\,. \end{aligned}\tag{4.48}$$

In the second step the $\ell\text{-Ind}_A(B)$-intertwiner property of f is used. The fact that f is also in Hom_A allows one to commute it, in the fourth step, with $P_{E_A(B)\otimes M}$.

The functor G: We will be still more sketchy in the definition of G. On morphisms it acts as the identity, $G(f):=f$, just like F. To a local $E_A(B)$-module $(M,\rho^{E_A(B)})$ it assigns the object $G(M,\rho^{E_A(B)}):=(M,\rho^A,\rho^{\ell\text{-Ind}_A(B)})$ of $(\mathcal{C}_A^{\ell oc})^{\ell oc}_{\ell\text{-Ind}_A(B)}$ as follows:

$$\begin{aligned} \rho^A &:= \rho^{E_A(B)}\circ(e_{E_A(B)}\otimes id_M)\circ(id_A\otimes\eta^B\otimes id_M) &&\in\text{Hom}(A\otimes M,M)\,,\\ \rho^{\ell\text{-Ind}_A(B)} &:= \rho^{E_A(B)}\circ e_{\ell\text{-Ind}_A(B)\otimes M} &&\in\text{Hom}(\ell\text{-Ind}_A(B)\otimes_A M,M)\,. \end{aligned}\tag{4.49}$$

To verify the representation property of $\rho^{\ell\text{-Ind}_A(B)}$ one needs the relation

$$\rho^{E_A(B)}\circ P_{E_A(B)\otimes M}=\rho^{E_A(B)}\,,\tag{4.50}$$

which can be seen by combining the definition (3.46) of $P_{E_A(B)\otimes M}$ and of ρ^A in (4.49) with the representation property of $\rho^{E_A(B)}$ and the definition (3.15) of the product on $E_A(B)$. Using the condition of Proposition 3.17(ii) one can further convince oneself that ρ^A and $\rho^{\ell\text{-Ind}_A(B)}$ are local; we omit the calculation.

F and G as inverse functors: F and G are clearly inverse to each other on morphisms. That $F\circ G$ is the identity on objects follows from (4.50). To see $G\circ F=Id$ on objects one must verify that

$$\begin{aligned} \rho^A &= \rho^{\ell\text{-Ind}_A(B)}\circ r_{E_A(B)\otimes M\succ E_A(B)\otimes_A M}\circ(e_{E_A(B)\prec A\otimes B}\otimes id_M)\circ(id_A\otimes\eta^B\otimes id_M)\,,\\ \rho^{\ell\text{-Ind}_A(B)} &= \rho^{\ell\text{-Ind}_A(B)}\circ r_{E_A(B)\otimes M\succ E_A(B)\otimes_A M}\circ e_{E_A(B)\otimes_A M\prec E_A(B)\otimes M}\,. \end{aligned}\tag{4.51}$$

The second equality is obvious. To see the first equality one replaces id_A by $m\circ(id_A\otimes\eta^A)$ and uses the fact that all morphisms are in Hom_A to trade the multiplication first for the representation of A on $E_A(B)$, then on $E_A(B)\otimes_A M$, and finally on M. The morphism $\rho^{\ell\text{-Ind}_A(B)}$ is now applied to the unit of ℓ-$\text{Ind}_A(B)$ and can be left out. The remaining morphism is precisely ρ^A, the action of A on M.

F as tensor functor: Denote by $\otimes_1$ the tensor product in $(\mathcal{C}_A^{\ell\text{oc}})^{\ell\text{oc}}_{\ell\text{-Ind}_A(B)}$ and by $\otimes_2$ the tensor product in $\mathcal{C}^{\ell\text{oc}}_{E_A(B)}$. We need to show that $F(M\otimes_1 N)\cong F(M)\otimes_2 F(N)$; as we will see, the two objects are in fact equal. Since F only changes the representation morphisms of M and N, but not the underlying objects $\dot M$ and $\dot N$ we have (working with the Karoubian envelope, see formula (3.49))

$$F(M\otimes_1 N) = \big((\dot M\otimes\dot N;P_1),\rho_1\big) \qquad\text{and}\qquad F(M)\otimes_2 F(N) = \big((\dot M\otimes\dot N;P_2),\rho_2\big)\,, \quad (4.52)$$

where $\dot M$ and $\dot N$ are objects in $\mathcal{C}$ and $\rho_{1,2}$ are representation morphisms for the algebra $E_A(B)$. Further, P_1 is the idempotent in $\text{End}(\dot M\otimes\dot N)$ whose retract is $M\otimes_1 N$, while P_2 gives the retract $F(M)\otimes_2 F(N)$, i.e.

$$P_1 = e\circ e'\circ r'\circ r \qquad\text{and}\qquad P_2 = e''\circ r''\,, \quad (4.53)$$

where the abbreviations $e=e_{M\otimes_A N\prec M\otimes N}$, $e'=e_{M\otimes_{\ell\text{-Ind}_A(B)}N\prec M\otimes_A N}$, $e''=e_{M\otimes_{E_A(B)}N\prec M\otimes N}$, as well as an analogous notation for r, r', r'' are used. By direct substitution of the definitions one verifies that $P_1=P_2$. It then remains to compare the representation morphisms ρ_1 and ρ_2. Again by substituting the definitions one finds that they are

$$\rho_1 = \rho_2 = (\rho_M^{\ell\text{-Ind}_A(B)}\circ r_{E_A(B)\otimes M})\otimes id_N \quad \in \text{Hom}(E_A(B)\otimes M\otimes N, M\otimes N)\,. \quad (4.54)$$

F as a ribbon functor: The duality and braiding are defined as in (2.16) and (2.17), with the idempotents given by the idempotents (3.47) for the corresponding tensor products. But since the idempotents $P_{1,2}$ which define the retracts $M\otimes_1 N\prec\dot M\otimes\dot N$ and $F(M)\otimes_2 F(N)$ $\prec\dot M\otimes\dot N$ are equal and F acts as the identity on morphisms, duality and braiding of $(\mathcal{C}_A^{\ell\text{oc}})^{\ell\text{oc}}_{\ell\text{-Ind}_A(B)}$ get mapped to duality and braiding of $\mathcal{C}^{\ell\text{oc}}_{E_A(B)}$. □

5 Local modules and a subcategory of bimodules

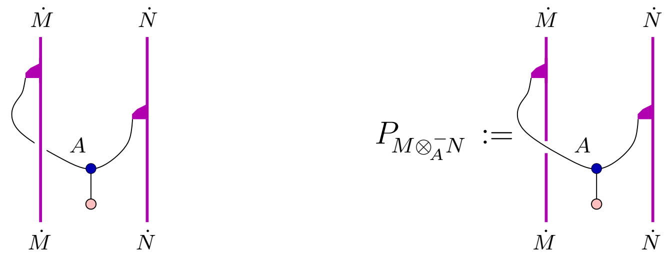
The aim of this section is to establish – in Theorem 5.20 – an equivalence between the three ribbon categories $\mathcal{C}^{\ell oc}_{C_l(A)}$, $\mathcal{C}^{\ell oc}_{C_r(A)}$ and $\mathcal{C}^0_{A|A}$. Here $\mathcal{C}^0_{A|A}$ denotes the full subcategory of $\mathcal{C}_{A|A}$ whose objects are those A-bimodules which are at the same time a sub-bimodule of an α^+_A-induced and of an α^-_A-induced bimodule.

To obtain this equivalence we introduce families of morphisms in the category of left modules and in the category of bimodules over a symmetric special Frobenius algebra. These families will be called pre-braidings. The terminology derives from the fact that for left modules the pre-braiding restricts to the braiding defined in (3.50) if the algebra is commutative and the modules are local, while for bimodules it gives rise to a braiding when restricted to $\mathcal{C}^0_{A|A}$ (Propositions 5.5 and 5.12).

After discussing these preparatory concepts, a tensor functor from $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$ to $\mathcal{C}^0_{A|A}$ is constructed. Then it is first shown that this functor respects the braiding, and finally that it provides an equivalence, thus establishing the theorem.

5.1 Braiding and left modules

Let A be a symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$. If A is in addition *commutative*, then one can define two tensor products $\otimes^{\pm}_A$ on the category $\mathcal{C}_A$ of left A-modules, by extending the tensor product on its full subcategory $\mathcal{C}^{\ell oc}_A$ of local A-modules (see Section 3.4) in two different ways. The basic ingredients are the idempotents introduced in (3.46), i.e.

$P_{M\otimes^+_A N}$:=  $P_{M\otimes^-_A N}$:= (5.1)

for any pair M, N of A-modules.

If M is local, then $P_{M\otimes^+_A N} = P_{M\otimes^-_A N} = P_{M\otimes_A N}$ as defined in (3.46), and one deals with tensor product $\otimes_A$ on $\mathcal{C}^{\ell oc}_A$ described in Section 3.4. In contrast, for general A-modules we get two distinct tensor products $\otimes^{\pm}_A$. If, for $\nu \in \{\pm\}$, the idempotent $P_{M\otimes^\nu_A N}$ is split, we denote the associated retract by $(\mathrm{Im}\, P_{M\otimes^\nu_A N}, e^\nu_{M\otimes N}, r^\nu_{M\otimes N})$, and thus the tensor product $\otimes^\nu_A$ is given by

$$M \otimes^\nu_A N = \mathrm{Im}\, P_{M\otimes^\nu_A N} \qquad \text{and} \qquad f \otimes^\nu_A g = r^\nu_{M'\otimes N'} \circ (f \otimes g) \circ e^\nu_{M\otimes N} \tag{5.2}$$

for $M, M', N, N' \in \mathrm{Obj}(\mathcal{C}_A)$ and $f \in \mathrm{Hom}_A(M, M')$, $g \in \mathrm{Hom}_A(N, N')$. If $P_{M\otimes^\nu_A N}$ is not split, we must instead work with the Karoubian envelope; then the same comments apply as in the case of $\mathcal{C}^{\ell oc}_A$ that was discussed in Section 3.4.

When the symmetric special Frobenius algebra A is *not* commutative, $\mathcal{C}_A$ is, in general, not a tensor category. However, we can still perform an operation that has some similarity with a tensor product. This then allows us in particular to introduce a 'pre-braiding' on $\mathcal{C}_A$ that shares some properties of a genuine braiding. To this end we restrict, for the moment, our attention to induced modules. For any pair U, V of objects of $\mathcal{C}$ we introduce the endomorphisms

$$\begin{aligned} P_{\hat\otimes_A^+}(U,V) &:= \big[(m \otimes id_U \otimes id_A) \circ (id_A \otimes c_{U,A} \otimes id_A) \circ (id_A \otimes id_U \otimes \Delta)\big] \otimes id_V \qquad \text{and} \\ P_{\hat\otimes_A^-}(U,V) &:= \big[(m \otimes id_U \otimes id_A) \circ (id_A \otimes c_{A,U}^{-1} \otimes id_A) \circ (id_A \otimes id_U \otimes \Delta)\big] \otimes id_V \end{aligned} \tag{5.3}$$

in $\mathrm{End}_A(\mathrm{Ind}_A(U{\otimes}A{\otimes}V))$, with c the braiding on $\mathcal{C}$.

Lemma 5.1 :
The morphisms $P_{\hat\otimes_A^\pm}(U,V)$ are split idempotents, with image $\mathrm{Ind}_A(U{\otimes}V)$.

Proof:
That $P_{\hat\otimes_A^\pm}(U,V)$ are idempotents follows easily by using (co)associativity and specialness of A. To show that they are split, we just give explicitly the corresponding embedding and restriction morphisms $e_{UV}^\pm = e_{\hat\otimes_A^\pm}(U,V) \in \mathrm{Hom}_A(\mathrm{Ind}_A(U{\otimes}V), \mathrm{Ind}_A(U{\otimes}A{\otimes}V))$ and $r_{UV}^\pm = r_{\hat\otimes_A^\pm}(U,V) \in \mathrm{Hom}_A(\mathrm{Ind}_A(U{\otimes}A{\otimes}V), \mathrm{Ind}_A(U{\otimes}V))$:

$$\begin{aligned} e_{UV}^+ &= \big[(id_A{\otimes}c_{U,A}^{-1}) \circ (\Delta{\otimes}id_U)\big] \otimes id_V\,, \qquad & r_{UV}^+ &= \big[(m{\otimes}id_U) \circ (id_A{\otimes}c_{U,A})\big] \otimes id_V\,, \\ e_{UV}^- &= \big[(id_A{\otimes}c_{A,U}) \circ (\Delta{\otimes}id_U)\big] \otimes id_V\,, \qquad & r_{UV}^- &= \big[(m{\otimes}id_U) \circ (id_A{\otimes}c_{A,U}^{-1})\big] \otimes id_V\,. \end{aligned} \tag{5.4}$$

That $e_{UV}^\nu \circ r_{UV}^\nu = P_{\hat\otimes_A^\nu}(U,V)$ is an immediate consequence of the Frobenius property of A. Further, as a result of specialness of A the composition $r_{UV}^\nu \circ e_{UV}^\nu$ is equal to $id_A{\otimes}id_U{\otimes}id_V$, hence the statement about the image. □

The module retracts associated to the idempotents $P_{\hat\otimes_A^\pm}(U,V)$ are used in

Definition 5.2 :
The operations $\hat\otimes_A^\nu$: $\mathcal{C}_A^{\mathrm{Ind}} {\times} \mathcal{C}_A^{\mathrm{Ind}} \to \mathcal{C}_A^{\mathrm{Ind}}$ $(\nu \in \{\pm\})$ are given by

$$\mathrm{Ind}_A(U) \,\hat\otimes_A^\nu\, \mathrm{Ind}_A(V) := \mathrm{Im}\, P_{\hat\otimes_A^\nu}(U,V) = (\mathrm{Ind}_A(U{\otimes}V), e_{UV}^\nu, r_{UV}^\nu) \tag{5.5}$$

and

$$f \,\hat\otimes_A^\nu\, g := r_{U'V'}^\nu \circ (f \otimes g) \circ e_{UV}^\nu \tag{5.6}$$

for $U, V, U', V' \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_A(\mathrm{Ind}_A(U), \mathrm{Ind}_A(U'))$, $g \in \mathrm{Hom}_A(\mathrm{Ind}_A(V), \mathrm{Ind}_A(V'))$.

In general, $(f_1 \hat\otimes_A^\nu g_1) \circ (f_2 \hat\otimes_A^\nu g_2)$ is not equal to $(f_1 {\circ} f_2) \,\hat\otimes_A^\nu\, (g_1 {\circ} g_2)$, so that $\hat\otimes_A^\nu$ is not a functor from $\mathcal{C}_A^{\mathrm{Ind}} {\times} \mathcal{C}_A^{\mathrm{Ind}}$ to $\mathcal{C}_A^{\mathrm{Ind}}$, and hence in particular it is not a tensor product. However, for *commutative* algebras $\hat\otimes_A^\nu$ does constitute a tensor product on $\mathcal{C}_A^{\mathrm{Ind}}$. Indeed, the following statement can be verified by direct substitution of the respective definitions:

Lemma 5.3:
For every commutative symmetric special Frobenius algebra A the operations $\hat\otimes_A^\nu$ and $\otimes_A^\nu$ coincide on $\mathcal{C}_A^{\rm Ind}\times\mathcal{C}_A^{\rm Ind}$, i.e. $\mathrm{Ind}_A(U)\,\hat\otimes_A^\nu\,\mathrm{Ind}_A(V) = \mathrm{Ind}_A(U)\otimes_A^\nu\mathrm{Ind}_A(V)$ and $f\,\hat\otimes_A^\nu\,g = f\otimes_A^\nu g$ for all $U,V,U',V'\in\mathrm{Obj}(\mathcal{C})$ and all $f\in\mathrm{Hom}_A(\mathrm{Ind}_A(U),\mathrm{Ind}_A(U'))$, $g\in\mathrm{Hom}_A(\mathrm{Ind}_A(V),\mathrm{Ind}_A(V'))$.

Definition 5.4:
Let A be a (not necessarily commutative) symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$. For $\mu,\nu\in\{\pm\}$, we denote by $\gamma^{A\,\mu\nu}$ the family of morphisms

$$\gamma_{UV}^{A\,\mu\nu} := id_A\otimes c_{U,V} \qquad \text{for} \quad U,V\in\mathrm{Obj}(\mathcal{C}) \tag{5.7}$$

in $\mathrm{Hom}_A(\mathrm{Ind}_A(U)\hat\otimes_A^\mu\mathrm{Ind}_A(V),\mathrm{Ind}_A(V)\hat\otimes_A^\nu\mathrm{Ind}_A(U))$.

We will refer to the family $\gamma^{A\,\mu\nu}$, and likewise to similar structures occurring below, as a *pre-braiding* on $\mathcal{C}_A^{\rm Ind}$. While $\gamma^{A\,\mu\nu}$ is itself not a braiding, it will give rise to one when restricted to a suitable subcategory.

For the rest of this subsection we suppose that the symmetric special Frobenius algebra A is commutative. Then $\gamma^{A\,\mu\nu}$ can indeed be used to obtain a braiding on the category $\mathcal{C}_A^{\ell oc}$ of local A-modules, and this braiding coincides with the one already described in (3.50). To obtain a statement about $\mathcal{C}_A^{\ell oc}$ we must, however, get rid of the restriction to induced modules. To this end we recall from Lemma 4.8(ii) that every A-module, and hence in particular every local A-module, is a module retract of an induced module. Accordingly for each local A-module M we select an object $U_M\in\mathrm{Obj}(\mathcal{C})$ such that (M,e_M,r_M) is a module retract of $\mathrm{Ind}_A(U_M)$. Then for $\mu,\nu\in\{\pm\}$ we define a family $\Gamma_{MN}^{A\,\mu\nu}$ of morphisms of $\mathcal{C}_A^{\ell oc}$ by

$$\Gamma_{MN}^{A\,\mu\nu} := (r_N\otimes_A^\nu r_M)\circ\gamma_{U_M U_N}^{A\,\mu\nu}\circ(e_M\otimes_A^\mu e_N) \tag{5.8}$$

for $M,N\in\mathrm{Obj}(\mathcal{C}_A^{\ell oc})$. Note that even though $\otimes_A^\pm=\otimes_A$ for local modules, here we still must use the operation $\otimes_A^\pm$, because the induced module $\mathrm{Ind}_A(U_M)$ is not necessarily local, so that e.g. the morphism $e_M\in\mathrm{Hom}_A(M,\mathrm{Ind}_A(U_M))$ is, in general, only a morphism in $\mathcal{C}_A$, but not in $\mathcal{C}_A^{\ell oc}$.

The following result implies that $\Gamma^{A\,\mu\nu}$ does not depend on the particular choice of the triple (U_M,e_M,r_M). It also establishes that $\Gamma^{A\,\mu\nu}$ is actually independent of μ and ν, that it furnishes a braiding on $\mathcal{C}_A^{\ell oc}$, and that this braiding coincides with the braiding c^A defined in (3.50).

Proposition 5.5:
Let A be a commutative symmetric special Frobenius algebra and M,N be local A-modules. Then

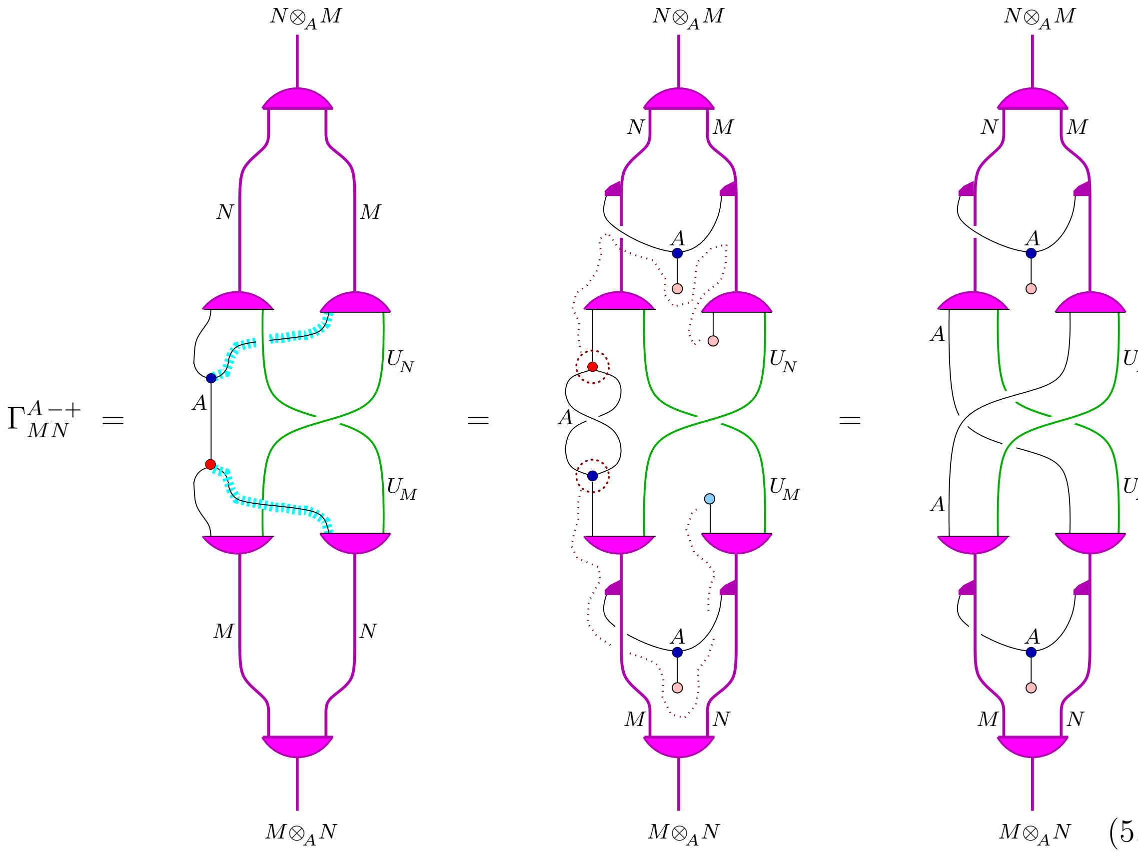
$$\Gamma_{MN}^{A\,\mu\nu} = c_{MN}^A \tag{5.9}$$

for $\mu,\nu\in\{\pm\}$.

Proof:
Writing out the definition of $\Gamma^{A\,\mu\nu}_{MN}$ gives

$$\Gamma^{A\,\mu\nu}_{MN} = r_{N\otimes M} \circ (r_N \otimes r_M) \circ e_{\hat\otimes^\nu_A} \circ (id_A \otimes c_{U_M,U_N}) \circ r_{\hat\otimes^\mu_A} \circ (e_M \otimes e_N) \circ e_{M\otimes N}\,. \tag{5.10}$$

In the sequel we consider the case $\mu = -$, $\nu = +$ as an example. (The other cases are verified similarly.) In pictorial notation, formula (5.10) is the first equality in the following series of transformations:



(5.11)

The second step of these manipulations involves a rewriting of the marked A-ribbons as idempotents $P_{M\otimes^\pm_A N}$, which uses in particular that A is commutative and that M and N are local. Furthermore, the identity $id_A = m \circ c^{-1}_{A,A} \circ \Delta$, which holds because A is special and commutative, is inserted. In the last step, the marked multiplication and comultiplication morphisms are dragged along the paths indicated (becoming representation morphisms for part of the way); this relies again on A being commutative.
In the final picture, the idempotents $P_{M\otimes^\pm_A N}$ can be removed, while the morphisms $e_{M/N}$ and $r_{M/N}$ combine to the identity morphism on M and N, respectively. Comparison with (3.50) then shows that $\Gamma^{A\,-+}_{MN} = c^A_{MN}$, as claimed. $\square$

5.2 Braiding and bimodules

From now on A is again a general symmetric special Frobenius algebra, not necessarily commutative.

The category $\mathcal{C}_{A|A}$ of A-bimodules contains interesting full subcategories which were studied in [7] and [37].

Definition 5.6:
The full subcategories of $\mathcal{C}_{A|A}$ whose objects are the α_A^+-induced and the α_A^--induced bimodules, respectively, are denoted by $\mathcal{C}_{A|A}^{\alpha^+\text{-Ind}}$ and $\mathcal{C}_{A|A}^{\alpha^-\text{-Ind}}$, and their Karoubian envelopes by

$$\mathcal{C}_{A|A}^{\pm} := \big(\mathcal{C}_{A|A}^{\alpha^\pm\text{-Ind}}\big)^{\mathrm{K}} . \tag{5.12}$$

The category $\mathcal{C}_{A|A}^{0}$ of *ambichiral* A-bimodules is the full subcategory of $\mathcal{C}_{A|A}$ whose objects are both in $\mathcal{C}_{A|A}^{+}$ and in $\mathcal{C}_{A|A}^{-}$, i.e.

$$\mathcal{C}_{A|A}^{0} := \mathcal{C}_{A|A}^{+} \cap \mathcal{C}_{A|A}^{-} . \tag{5.13}$$

One can wonder whether the pre-braiding $\gamma^{A\,\mu\nu}$ on $\mathcal{C}_A^{\text{Ind}}$ can be lifted to the bimodule category $\mathcal{C}_{A|A}$. We will see that this is indeed possible, by constructing families $\tilde\gamma^{A\,\mu\nu}$ of morphisms satisfying $R_A(\tilde\gamma_{UV}^{A\,\mu\nu}) = \gamma_{UV}^{A\,\mu\nu}$, where

$$R_A: \quad \mathcal{C}_{A|A} \to \mathcal{C}_A \tag{5.14}$$

is the restriction functor whose action on objects consists in forgetting the right-action of A on a bimodule. To do so first note that, as follows again by a straightforward application of the definitions, we have

$$\alpha^\mu(U) \otimes_A \alpha^\nu(V) = (\alpha^\nu(U{\otimes}V), e_{UV}^\mu, r_{UV}^\mu) , \tag{5.15}$$

with $e_{UV}^\pm$ and $r_{UV}^\pm$ defined as in (5.4), as a bimodule retract of $\alpha^\mu(U) \otimes \alpha^\nu(V)$. To proceed we set

$$\tilde\gamma_{UV}^{A\,\mu\nu} := id_A \otimes c_{U,V} \tag{5.16}$$

for $\mu,\nu \in \{\pm\}$ and $U,V \in \mathrm{Obj}(\mathcal{C})$ as in formula (5.7), but now regarded as morphisms from $\alpha_A^\mu(U){\otimes_A}\alpha_A^\nu(V)$ to $\alpha_A^\nu(V){\otimes_A}\alpha_A^\mu(U)$. These families will again be called pre-braidings.

Lemma 5.7:
The pre-braidings $\tilde\gamma_{UV}^{A\,\mu\nu}$ defined by (5.16) have the following properties.
(i) For $(\mu\nu) \in \{(++),(+-),(--)\}$ they are bimodule morphisms, i.e.

$$\tilde\gamma_{UV}^{A\,\mu\nu} \in \mathrm{Hom}_{A|A}(\alpha_A^\mu(U){\otimes_A}\alpha_A^\nu(V), \alpha_A^\nu(V){\otimes_A}\alpha_A^\mu(U)) . \tag{5.17}$$

(ii) They fulfill

$$R_A(\tilde\gamma^{A\,\mu\nu}_{UV}) = \gamma^{A\,\mu\nu}_{UV}\,, \tag{5.18}$$

with R_A the restriction functor (5.14).

Proof:
(i) Compatibility of $\tilde\gamma^{A\,\mu\nu}$ with the left action of A is clear. In the case of the right action $\rho_{\rm r}^{\pm}$, given for α-induced bimodules in (2.31), we must show that

$$\tilde\gamma^{A\,\mu\nu}_{UV} \circ (id_{\alpha^\mu_A(U)} \otimes_A \rho^\nu_{\rm r}(V)) = (id_{\alpha^\nu_A(V)} \otimes_A \rho^\mu_{\rm r}(U)) \circ (\tilde\gamma^{A\,\mu\nu}_{UV} \otimes id_A)\,. \tag{5.19}$$

Writing out the definitions, this amounts to

$$\begin{aligned}&(id_A \otimes c_{U,V}) \circ r^\mu_{UV} \circ (id_A \otimes id_U \otimes \rho^\nu_{\rm r}(V)) \circ (e^\mu_{UV} \otimes id_A)\\ &\quad = r^\nu_{VU} \circ (id_A \otimes id_V \otimes \rho^\mu_{\rm r}(U)) \circ (e^\nu_{VU} \otimes id_A) \circ (id_A \otimes c_{U,V} \otimes id_A)\,.\end{aligned} \tag{5.20}$$

Inserting also the definitions of $\rho_{\rm r}^{\pm}$, e and r one verifies, separately for each choice of $(\mu\nu) \in \{(++),(+-),(--)\}$, that this equality follows from the properties of A and of the braiding in $\mathcal{C}$.
(ii) For α-induced bimodules we have $R_A(\alpha^\pm(U)) = {\rm Ind}_A(U)$, so that

$$R_A(\alpha^\mu_A(U) {\otimes_A} \alpha^\nu_A(V)) = R_A(\alpha^\nu_A(U{\otimes}V)) = {\rm Ind}_A(U{\otimes}V)\,. \tag{5.21}$$

Thus R_A maps the source and target objects of $\tilde\gamma^{A\,\mu\nu}_{UV}$ to those of $\gamma^{A\,\mu\nu}_{UV}$. As a consequence, the equality

$$R_A(\tilde\gamma^{A\,\mu\nu}_{UV}) = R_A(id_A {\otimes} c_{U,V}) = id_A \otimes c_{U,V} = \gamma^{A\,\mu\nu}_{UV} \tag{5.22}$$

follows immediately. □

The morphisms $\tilde\gamma^{A\,\mu\nu}_{UV}$ are not all functorial, as would be required for a braiding. But still we have the following properties.

Lemma 5.8:
For any $U, V, R, S \in {\rm Obj}(\mathcal{C})$ the following identities hold in $\mathcal{C}_{A|A}$.

(i) $\tilde\gamma^{A++}_{UV} \circ (id{\otimes_A}g) = (g{\otimes_A}id) \circ \tilde\gamma^{A++}_{US}$ for $g \in {\rm Hom}_{A|A}(\alpha^+_A(S), \alpha^+_A(V))\,.$

(ii) $\tilde\gamma^{A--}_{UV} \circ (f{\otimes_A}id) = (id{\otimes_A}f) \circ \tilde\gamma^{A--}_{RV}$ for $f \in {\rm Hom}_{A|A}(\alpha^-_A(R), \alpha^-_A(U))\,.$

(iii) $\tilde\gamma^{A+-}_{UV} \circ (id{\otimes_A}g) = (g{\otimes_A}id) \circ \tilde\gamma^{A++}_{US}$ for $g \in {\rm Hom}_{A|A}(\alpha^+_A(S), \alpha^-_A(V))\,.$

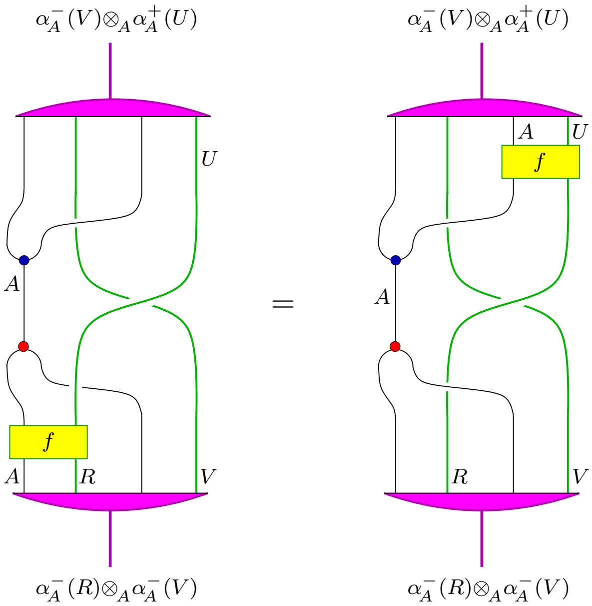
(iv) $\tilde\gamma^{A+-}_{UV} \circ (f{\otimes_A}id) = (id{\otimes_A}f) \circ \tilde\gamma^{A--}_{RV}$ for $f \in {\rm Hom}_{A|A}(\alpha^-_A(R), \alpha^+_A(U))\,.$

(v) $\tilde\gamma^{A+-}_{UV} \circ (f{\otimes_A}g) = (g{\otimes_A}f) \circ \tilde\gamma^{A+-}_{RS}$ for $f \in {\rm Hom}_{A|A}(\alpha^+_A(R), \alpha^+_A(U))$ and $g \in {\rm Hom}_{A|A}(\alpha^-_A(S), \alpha^-_A(V))\,.$

Proof:
The statements are all verified in a similar manner; we present the proof of (iv) as an

example. Substituting the definitions we find

$$\tilde{\gamma}_{UV}^{A+-} \circ (f \otimes_A id) = \quad \text{[diagram]} \quad = \quad \text{[diagram]} \tag{5.23}$$



In the first step the definition of $\tilde{\gamma}^{A+-}$ and of the tensor product of morphisms is inserted. The second step uses first that the morphism f intertwines the right action of $\alpha_A^-(R)$ and $\alpha_A^+(U)$ so as to take it past the multiplication, and next that it intertwines the left action (and hence, by the Frobenius property, the left co-action as well) to commute it past the comultiplication. The resulting morphism on the right hand side is equal to $(id \otimes_A f) \circ \tilde{\gamma}_{RV}^{A--}$. □

So far we have a pre-braiding on the categories $\mathcal{C}_{A|A}^{\alpha^\pm\text{-Ind}}$ of α-induced bimodules. We proceed to construct pre-braidings $\tilde{\Gamma}^{A\,\mu\nu}$ for $\mathcal{C}_{A|A}^{\pm}$.

Definition 5.9 :
Select, for each bimodule $X \in \text{Obj}(\mathcal{C}_{A|A}^{\mu})$ and $\mu \in \{\pm\}$, an object $U_X^\mu \in \text{Obj}(\mathcal{C})$ and morphisms e_X^μ, r_X^μ such that (X, e_X^μ, r_X^μ) is a bimodule retract of $\alpha_A^\mu(U_X^\mu)$. Then for $X \in \text{Obj}(\mathcal{C}_{A|A}^{\mu})$, $Y \in \text{Obj}(\mathcal{C}_{A|A}^{\nu})$ and $(\mu\nu) \in \{(++), (+-), (--)\}$ the morphism $\tilde{\Gamma}_{XY}^{A\,\mu\nu}$ is defined as

$$\tilde{\Gamma}_{XY}^{A\,\mu\nu} := (r_Y^\nu \otimes_A r_X^\mu) \circ \tilde{\gamma}_{U_X^\mu U_Y^\nu}^{A\,\mu\nu} \circ (e_X^\mu \otimes_A e_Y^\nu) \,. \tag{5.24}$$

We will now show that the families $\tilde{\Gamma}^{A\,\mu\nu}$ of morphisms have similar properties as those of the pre-braidings $\tilde{\gamma}^{A\,\mu\nu}$ that were listed in lemma 5.8. In particular, the morphisms $\tilde{\Gamma}_{XY}^{A+-}$ turn out to be functorial and thus furnish a *relative braiding* between $\mathcal{C}_{A|A}^{+}$ and $\mathcal{C}_{A|A}^{-}$, which coincides with the relative braiding introduced in Proposition 4 of [37]. Indeed we have

Lemma 5.10:
For any $X^\mu, Y^\mu, R^\mu, S^\mu \in \mathrm{Obj}(\mathcal{C}^\mu_{A|A})$ $(\mu \in \{\pm\})$ the following identities hold in $\mathcal{C}_{A|A}$.

(i) $\tilde\Gamma^{A\,++}_{XY} \circ (id \otimes_A g) = (g \otimes_A id) \circ \tilde\Gamma^{A\,++}_{XS}$ for $g \in \mathrm{Hom}_{A|A}(S^+, Y^+)$.

(ii) $\tilde\Gamma^{A\,--}_{XY} \circ (f \otimes_A id) = (id \otimes_A f) \circ \tilde\Gamma^{A\,--}_{RY}$ for $f \in \mathrm{Hom}_{A|A}(R^-, X^-)$.

(iii) $\tilde\Gamma^{A\,+-}_{XY} \circ (id \otimes_A g) = (g \otimes_A id) \circ \tilde\Gamma^{A\,++}_{XS}$ for $g \in \mathrm{Hom}_{A|A}(X^+, Y^-)$.

(iv) $\tilde\Gamma^{A\,+-}_{XY} \circ (f \otimes_A id) = (id \otimes_A f) \circ \tilde\Gamma^{A\,--}_{RY}$ for $f \in \mathrm{Hom}_{A|A}(R^-, X^+)$.

(v) $\tilde\Gamma^{A\,+-}_{XY} \circ (f \otimes_A g) = (g \otimes_A f) \circ \tilde\Gamma^{A\,+-}_{RS}$ for $f \in \mathrm{Hom}_{A|A}(R^+, X^+)$ and $g \in \mathrm{Hom}_{A|A}(S^-, Y^-)$.

Here the abbreviations $\tilde\Gamma^{A\,++}_{XY} = \tilde\Gamma^{A\,++}_{X^+Y^+}$ etc. are used.

Proof:
These properties of $\tilde\Gamma^{A\,\mu\nu}$ are easily reduced to the corresponding properties of $\tilde\gamma^{A\,\mu\nu}$ in Lemma 5.8. Let us treat (i) as an example. Writing out the definition of $\tilde\Gamma^{A\,++}$ on the left hand side of (i) gives (abbreviating also $r^+_X = r^+_{X^+}$ etc.)

$$\tilde\Gamma^{A\,++}_{XY} \circ (id_{X^+} \otimes_A g) = (r^+_Y \otimes_A r^+_X) \circ \tilde\gamma^{A\,++}_{U_X U_Y} \circ (e^+_X \otimes_A (e^-_Y \circ g)) \,, \tag{5.25}$$

while for the right hand side we have

$$(g \otimes_A id_{X^+}) \circ \tilde\Gamma^{A\,++}_{XS} = ((g \circ r^+_S) \otimes_A r^+_X) \circ \tilde\gamma^{A\,++}_{U_X\, U_S} \circ (e^+_X \otimes_A e^+_S) \,. \tag{5.26}$$

Since $e^+_Y \otimes_A e^+_X$ is monic and $r^+_X \otimes_A r^+_S$ is epi, it is sufficient to show equality after composing the two expressions (5.25) and (5.26) with $e^+_Y \otimes_A e^+_X$ from the left and with $r^+_X \otimes_A r^+_S$ from the right. The resulting expressions are indeed equal, as is seen by using Lemma 5.8(i) with $id_{X^+} \otimes_A (e^-_Y \circ g \circ r^+_S)$ in place of $id \otimes_A g$. □

The pre-braiding $\tilde\Gamma^{A\,\mu\nu}$ gives rise to a braiding on $\mathcal{C}^0_{A|A}$. The following observations will be instrumental to establish this result.

Lemma 5.11:
(i) The morphisms $\tilde\Gamma^{A\,++}$ satisfy

$$\tilde\Gamma^{A\,++}_{XY} \circ (f \otimes_A g) = (g \otimes_A f) \circ \tilde\Gamma^{A\,++}_{RS} \tag{5.27}$$

for $X, R, S \in \mathrm{Obj}(\mathcal{C}^+_{A|A})$, $Y \in \mathrm{Obj}(\mathcal{C}^0_{A|A})$, and $f \in \mathrm{Hom}_{A|A}(R, X)$, $g \in \mathrm{Hom}_{A|A}(S, Y)$.
(ii) The morphisms $\tilde\Gamma^{A\,--}$ satisfy

$$\tilde\Gamma^{A\,--}_{XY} \circ (f \otimes_A g) = (g \otimes_A f) \circ \tilde\Gamma^{A\,--}_{RS} \tag{5.28}$$

for $X \in \mathrm{Obj}(\mathcal{C}^0_{A|A})$, $Y, R, S \in \mathrm{Obj}(\mathcal{C}^-_{A|A})$, and $f \in \mathrm{Hom}_{A|A}(R, X)$, $g \in \mathrm{Hom}_{A|A}(S, Y)$.

(iii) When restricted to $\mathcal{C}^+_{A|A} \times \mathcal{C}^0_{A|A}$, the morphisms $\tilde\Gamma^{A\,++}$ are functorial; when restricted to $\mathcal{C}^-_{A|A} \times \mathcal{C}^0_{A|A}$, the morphisms $\tilde\Gamma^{A\,--}$ are functorial.

Proof:
We establish (i); the proof of (ii) works analogously, while (iii) is an immediate consequence of (i) and (ii).
By assumption on Y there are bimodule retracts (Y, e_Y^+, r_Y^+) of $\alpha_A^+(U_Y^+)$ and (Y, e_Y^-, r_Y^-) of $\alpha_A^-(U_Y^-)$. Since $e_Y^- \in \mathrm{Hom}_{A|A}(Y, \alpha_A^-(U_Y^-))$ is a monic, it is sufficient to verify that

$$(e_Y^- \otimes_A id_X) \circ \tilde{\Gamma}^{A++}_{XY} \circ (f \otimes_A g) = (e_Y^- \otimes_A id_X) \circ (g \otimes_A f) \circ \tilde{\Gamma}^{A++}_{RS} . \tag{5.29}$$

That this equality holds can be seen by using the properties of $\tilde{\Gamma}^{A\,\mu\nu}$ established in Lemma 5.10:

$$\begin{aligned} (e_Y^- \otimes_A id_X) \circ \tilde{\Gamma}^{A++}_{XY} \circ (f \otimes_A g) &\overset{\text{(iii)}}{=} \tilde{\Gamma}^{A+-}_{X\,\alpha^-(U_Y^-)} \circ (id_X \otimes_A e_Y^-) \circ (f \otimes_A g) \\ &\overset{\text{(v)}}{=} (id_{\alpha^-(U_Y^-)} \otimes_A f) \circ \tilde{\Gamma}^{A+-}_{R\,\alpha^-(U_Y^-)} \circ (id_R \otimes_A e_Y^-) \circ (id_R \otimes_A g) \\ &\overset{\text{(iii)}}{=} (id_{\alpha^-(U_Y^-)} \otimes_A f) \circ (e_Y^- \otimes_A id_X) \circ \tilde{\Gamma}^{A++}_{RY} \circ (id_R \otimes_A g) \\ &\overset{\text{(i)}}{=} (e_Y^- \otimes_A id_X) \circ (g \otimes_A f) \circ \tilde{\Gamma}^{A++}_{RS} \end{aligned} \tag{5.30}$$

(above the equality signs it is indicated which part of Lemma 5.10 is used). □

Proposition 5.12:
When restricting $\tilde{\Gamma}^{A\,\mu\nu}$ with $(\mu\nu) \in \{(++), (+-), (--)\}$ to $\mathcal{C}^0_{A|A} \times \mathcal{C}^0_{A|A}$, we have:
(i) The three families $\tilde{\Gamma}^{A\,\mu\nu}$ coincide. Thus we can set

$$\tilde{\Gamma}^{A}_{XY} := \tilde{\Gamma}^{A++}_{XY} = \tilde{\Gamma}^{A+-}_{XY} = \tilde{\Gamma}^{A--}_{XY} \tag{5.31}$$

for all $X, Y \in \mathrm{Obj}(\mathcal{C}^0_{A|A})$.
(ii) The morphism $\tilde{\Gamma}^{A}_{XY}$ is independent of the choices $e^{\pm}_{X,Y}$, $r^{\pm}_{X,Y}$ and $U^{\pm}_{X,Y}$ that are used in its definition.
(iii) The family $\tilde{\Gamma}^{A}$ of morphisms furnishes a braiding on $\mathcal{C}^0_{A|A}$.
Proof:
(i) We demonstrate explicitly only the case $\tilde{\Gamma}^{A++}_{XY} = \tilde{\Gamma}^{A+-}_{XY}$; the case $\tilde{\Gamma}^{A--}_{XY} = \tilde{\Gamma}^{A+-}_{XY}$ can be shown in the same way.
We have $X, Y \in \mathrm{Obj}(\mathcal{C}^+_{A|A})$, so there are bimodule retracts (X, e_X^+, r_X^+) of $\alpha_A^+(U_Y^+)$ and (Y, e_Y^+, r_Y^+) of $\alpha_A^+(U_Y^+)$. Furthermore $r_X^+ \otimes_A r_Y^+$ is epi, so that it is sufficient to establish that

$$\tilde{\Gamma}^{A++}_{XY} \circ (r_X^+ \otimes_A r_Y^+) = \tilde{\Gamma}^{A+-}_{XY} \circ (r_X^+ \otimes_A r_Y^+) . \tag{5.32}$$

Because of $Y \in \mathrm{Obj}(\mathcal{C}^0_{A|A})$ we can apply lemma 5.11(i) to the left hand side, yielding

$$\tilde{\Gamma}^{A++}_{XY} \circ (r_X^+ \otimes_A r_Y^+) = (r_Y^+ \otimes_A r_X^+) \circ \tilde{\Gamma}^{A++}_{\alpha_A^+(U_X^+)\,\alpha_A^+(U_Y^+)} . \tag{5.33}$$

For the right hand side of (5.32) we get

$$\begin{aligned}\tilde\Gamma^{A\,+-}_{XY} \circ (r_X^+ \otimes_A r_Y^+) &= (id_Y \otimes_A r_X^+) \circ \tilde\Gamma^{A\,+-}_{\alpha_A^+(U_X^+)\,Y} \circ (id_X \otimes_A r_Y^+) \\ &= (r_Y^+ \otimes_A r_X^+) \circ \tilde\Gamma^{A\,++}_{\alpha_A^+(U_X^+)\,\alpha_A^+(U_Y^+)}\,,\end{aligned} \tag{5.34}$$

where the first step amounts to Lemma 5.10(v), while in the second step Lemma 5.10(iii) is used, which is allowed because the source of the morphism $r_Y^+ \in \mathrm{Hom}_{A|A}(\alpha_A^+(U_Y^+),Y)$ is in $\mathcal{C}^+_{A|A}$ and its target is in $\mathcal{C}^0_{A|A}$ and thus in particular in $\mathcal{C}^-_{A|A}$.
Comparing (5.33) and (5.34) we see that (5.32) indeed holds true.

(ii) is implied by (i). Indeed, $\tilde\Gamma^{A\,++}_{X,Y}$ cannot depend on the choices of $e^+_{X/Y}$, $r^+_{X/Y}$ or $U^+_{X/Y}$, because $\tilde\Gamma^{A\,--}_{X,Y}$ manifestly does not. Conversely, $\tilde\Gamma^{A\,--}_{XY}$ must be independent of $e^-_{X/Y}$, $r^-_{X/Y}$ and $U^-_{X/Y}$. Likewise, since $\tilde\Gamma^{A\,+-}_{XY}$ equals $\tilde\Gamma^{A\,++}_{XY}$, it is independent of the choices for e^+_X, r^+_X and U^+_X, and since it equals $\tilde\Gamma^{A\,--}_{XY}$, it is independent of the choices for e^-_Y, r^-_Y and U^-_Y.

For the proof of (iii) the tensoriality of the braiding – the second line of formula (2.2) – must be verified. This can be done by direct computation. We do not present this calculation, but rather prefer to use a different argument later on, as part of the proof of Theorem 5.20 in Section 5.3. □

5.3 A ribbon equivalence between local modules and ambichiral bimodules

Given a symmetric special Frobenius algebra A and any pair U, V of objects of a ribbon category $\mathcal{C}$, define the linear maps $\Phi^{l/r}_{A;UV}$ by

$$\begin{aligned}\Phi^{l/r}_{A;UV}:\quad \mathrm{Hom}_A(\mathrm{Ind}_A(U),\mathrm{Ind}_A(V)) &\to \mathrm{Hom}(C_{l/r}{\otimes}U, C_{l/r}{\otimes}V) \\ f &\mapsto (r_{C_{l/r}} \otimes id_V) \circ f \circ (e_{C_{l/r}} \otimes id_U)\,,\end{aligned} \tag{5.35}$$

where $C_{l/r}$ stands for $C_l(A)$ and $C_r(A)$, respectively, and $r_{C_{l/r}}$ and $e_{C_{l/r}}$ are the restriction and embedding morphisms for the retract $C_{l/r} \prec A$. One checks that $\Phi^{l/r}_{A;UV}(f)$ commutes with the action of $C_{l/r}$, i.e. we have

$$\Phi^{l/r}_{A;UV}:\quad \mathrm{Hom}_A(\mathrm{Ind}_A(U),\mathrm{Ind}_A(V)) \longrightarrow \mathrm{Hom}_{C_{l/r}}(\mathrm{Ind}_{C_{l/r}}(U),\mathrm{Ind}_{C_{l/r}}(V))\,. \tag{5.36}$$

Definition 5.13:
For $x \in \{l,r\}$, the operations

$$\Phi^x_A:\quad \mathcal{C}^{\mathrm{Ind}}_A \to \mathcal{C}^{\mathrm{Ind}}_{C_x} \tag{5.37}$$

are defined on objects as

$$\Phi^x_A(\mathrm{Ind}_A(U)) := \mathrm{Ind}_{C_x}(U) \tag{5.38}$$

for $U \in \mathrm{Obj}(\mathcal{C})$, and on morphisms as

$$\Phi_A^x(f) := \Phi_{A;UV}^x(f)\,, \tag{5.39}$$

with $\Phi_{A;UV}^x$ defined by (5.35), for $f \in \mathrm{Hom}_A(\mathrm{Ind}_A(U), \mathrm{Ind}_A(V))$.

The following properties of the maps $\Phi_{A;UV}^{l/r}$ are immediate consequences of the definitions.

Lemma 5.14:
The maps $\Phi_{A;UV}^x$ defined in (5.35) fulfill

$$\Phi_{A;UU}^x(id_A \otimes id_U) = id_{C_x} \otimes id_U \tag{5.40}$$

as well as

$$\Phi_{A;VW}^x(g) \circ \Phi_{A;UV}^x(f) = \Phi_{A;UW}^x(g \circ (P_A^x \otimes id_V) \circ f) \tag{5.41}$$

for $f \in \mathrm{Hom}_A(\mathrm{Ind}_A(U), \mathrm{Ind}_A(V))$ and $g \in \mathrm{Hom}_A(\mathrm{Ind}_A(V), \mathrm{Ind}_A(W))$.

As indicated by the appearance of the idempotent $P_A^{l/r}$ on the right hand side of (5.41), the operation $\Phi_A^{l/r}$ is not a functor. However, as will be seen below, $\Phi_A^{l/r}$ can be used to define a functor from $\mathcal{C}_{A|A}^{\pm}$ to $\mathcal{C}_{C_{l/r}}$.

Lemma 5.15:
The operations $\Phi_A^{l/r}$ are compatible with the pre-braiding γ^A in the sense that

$$\Phi_A^l(\gamma_{UV}^{A\,++}) = \gamma_{UV}^{C_l\,++} \qquad \text{and} \qquad \Phi_A^r(\gamma_{UV}^{A\,--}) = \gamma_{UV}^{C_r\,--} \tag{5.42}$$

for all $U, V \in \mathrm{Obj}(\mathcal{C})$.
Proof:
As a straightforward application of the definitions, we have

$$\Phi_A^l(\gamma_{UV}^{A\,++}) = \Phi_{A;U\otimes V,V\otimes U}^l(id_A \otimes c_{U,V}) = (r_{C_l} \circ id_A \circ e_{C_l}) \otimes c_{U,V} = \gamma_{UV}^{C_l\,++}\,. \tag{5.43}$$

and similarly for $\Phi_A^r(\gamma_{UV}^{A\,--})$. □

Lemma 5.16:
The map $\Phi_{A;UV}^{l/r}$ restricts as follows to bijections between spaces of bimodule morphisms of α-induced A-bimodules and module morphisms of (locally) induced $C_{l/r}$-modules:

$$\begin{aligned} \Phi_{A;UV}^{l\,++}: &\quad \mathrm{Hom}_{A|A}(\alpha_A^+(U), \alpha_A^+(V)) \xrightarrow{\cong} \mathrm{Hom}_{C_l}(\mathrm{Ind}_{C_l}(U), \mathrm{Ind}_{C_l}(V))\,, \\ \Phi_{A;UV}^{r\,--}: &\quad \mathrm{Hom}_{A|A}(\alpha_A^-(U), \alpha_A^-(V)) \xrightarrow{\cong} \mathrm{Hom}_{C_r}(\mathrm{Ind}_{C_r}(U), \mathrm{Ind}_{C_r}(V))\,. \end{aligned} \tag{5.44}$$

Proof:
This is a consequence of the Proposition 2.36 together with the reciprocity relations (see Remark 2.23(iii))

$$\mathrm{Hom}(C_{l/r}\otimes U,V) \;\cong\; \mathrm{Hom}_{C_{l/r}}(\mathrm{Ind}_{C_{l/r}}(U),\mathrm{Ind}_{C_{l/r}}(V))\,. \tag{5.45}$$

Using the explicit form (2.42) and (2.69) of these isomorphisms, one can check that they are indeed given by restrictions of the maps $\Phi^{l/r}_{A;UV}$. □

We now compose the operations $\Phi^{l/r}_A$ with the restriction functor R_A (5.14).

Definition 5.17:
For A a symmetric special Frobenius algebra in a ribbon category, the operations

$$G^{\mathrm{Ind}}_+ :\quad \mathcal{C}^{\alpha^+\text{-Ind}}_{A|A} \to \mathcal{C}^{\mathrm{Ind}}_{C_l} \qquad \text{and} \qquad G^{\mathrm{Ind}}_- :\quad \mathcal{C}^{\alpha^-\text{-Ind}}_{A|A} \to \mathcal{C}^{\mathrm{Ind}}_{C_r} \tag{5.46}$$

are defined as the compositions $G^{\mathrm{Ind}}_+ := \Phi^l_A \circ R_A$ and $G^{\mathrm{Ind}}_- := \Phi^r_A \circ R_A$ of the operations (5.37) with the restriction functor (5.14).

Lemma 5.18:
Let A be a symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$ such that the symmetric Frobenius algebras $C_l(A)$ and $C_r(A)$ are special. Then we have:
(i) The operations $G^{\mathrm{Ind}}_\pm$ are functors.
(ii) They constitute tensor equivalences between the categories $\mathcal{C}^{\alpha^\pm\text{-Ind}}_{A|A}$ and $\mathcal{C}^{\mathrm{Ind}}_{C_{l/r}}$.
(iii) They satisfy

$$G^{\mathrm{Ind}}_+(\gamma^{A\,++}_{UV}) = \gamma^{C_l\,++}_{UV} \qquad \text{and} \qquad G^{\mathrm{Ind}}_-(\gamma^{A\,--}_{UV}) = \gamma^{C_r\,--}_{UV}\,. \tag{5.47}$$

Proof:
We establish the properties for G^{Ind}_+; the proofs for G^{Ind}_- work analogously.
(i) G^{Ind}_+ *is a functor*: Recall from the comment before Lemma 5.15 that G^{Ind}_+ is not a priori a functor, since Φ^l_A is not. However. after composition with R_A we have, for $f\in\mathrm{Hom}_{A|A}(\alpha^+_A(U),\alpha^+_A(V)$ and $g\in\mathrm{Hom}_{A|A}(\alpha^+_A(V),\alpha^+_A(W)$,

$$\Phi^l_A \circ R_A(g{\circ}f) = \Phi^l_A(g{\circ}f) = \Phi^l_{A;UW}(g{\circ}f) \tag{5.48}$$

as well as

$$\begin{aligned}\Phi^l_A(R_A(g))\circ\Phi^l_A(R_A(f)) &= \Phi^l_A(g)\circ\Phi^l_A(f) = \Phi^l_{A;VW}(g)\circ\Phi^l_{A;UV}(f)\\ &= \Phi^l_{A;UW}(g\circ(P^l_A\otimes id_V)\circ f) = \Phi^l_{A;UW}((P^l_A\otimes id_W)\circ g\circ f)\,.\end{aligned} \tag{5.49}$$

Here in the third step Lemma 5.14 is used, and in the last step the idempotent P^l_A is moved past g, which is allowed by Lemma 2.35. Finally, when inserting (5.35) for $\Phi^l_{A;UW}(\cdot)$, the

idempotent P^l_A can be left out because of the presence of r_{C_l}, thereby yielding the right hand side of (5.48). Thus $G^{\rm Ind}_+(g\circ f) = G^{\rm Ind}_+(g)\circ G^{\rm Ind}_+(f)$.
That $G^{\rm Ind}_+(id) = id$ follows again from Lemma 5.14.

(ii) $G^{\rm Ind}_+$ *is an equivalence functor*: Clearly $G^{\rm Ind}_+$ is essentially surjective on objects. Further, by the first equivalence in Lemma 5.16, $G^{\rm Ind}_+$ is an isomorphism on morphisms,

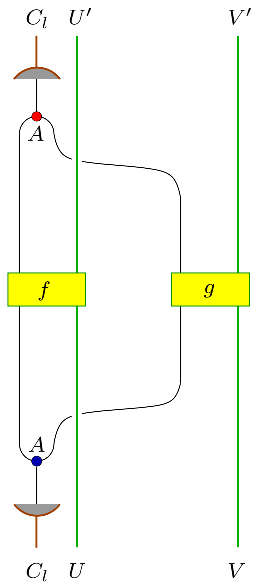
$$G^{\rm Ind}_+ : \quad {\rm Hom}_{A|A}(\alpha^+_A(U),\alpha^+_A(V)) \xrightarrow{\cong} {\rm Hom}_{C_l}({\rm Ind}_{C_l}(U),{\rm Ind}_{C_l}(V))\,. \tag{5.50}$$

Thus by the criterion of Proposition 2.3, $G^{\rm Ind}_+$ is an equivalence functor.

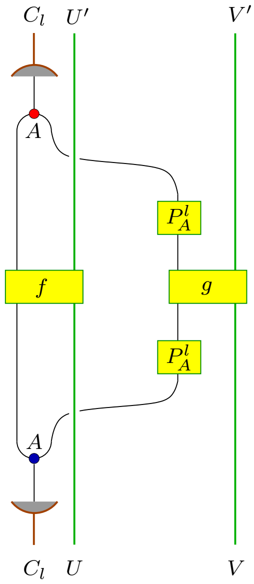
$G^{\rm Ind}_+$ *is a tensor functor*: Using equation (5.20) we have

$$\begin{aligned} G^{\rm Ind}_+(\alpha^+_A(U)\otimes_A \alpha^+_A(V)) &= \Phi^l_A({\rm Ind}_A(U{\otimes}V)) \\ &= {\rm Ind}_{C_l}(U{\otimes}V) = {\rm Ind}_{C_l}(U)\otimes_{C_l}{\rm Ind}_{C_l}(V)\,. \end{aligned} \tag{5.51}$$

The right hand side of (5.51) is equal to $G^{\rm Ind}_+(\alpha^+_A(U))\otimes_{C_l} G^{\rm Ind}_+(\alpha^+_A(V))$. Thus $G^{\rm Ind}_+$ is tensorial on objects.
For $f\in{\rm Hom}_{A|A}(\alpha^+_A(U),\alpha^+_A(U'))$ and $g\in{\rm Hom}_{A|A}(\alpha^+_A(V),\alpha^+_A(V'))$ the morphisms $G^{\rm Ind}_+(f{\otimes_A}g)$ and $G^{\rm Ind}_+(f)\otimes_{C_l} G^{\rm Ind}_+(g)$ read

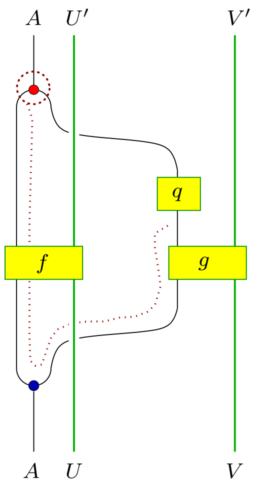
$G^{\rm Ind}_+(f\otimes_A g)\;=$  (5.52) and

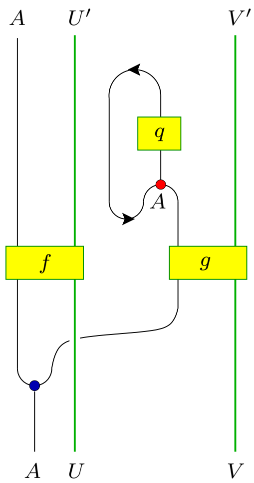
$$G_+^{\rm Ind} \otimes_{C_l} G_+^{\rm Ind}(g) \;=\; \frac{\dim(A)}{\dim(C_l)} \qquad (5.53)$$



In (5.53) the definition of the (co)multiplication on C_l has been substituted and Lemma 2.29(iii) has been used to omit one of the two resulting idempotents P_A^l at m and Δ.
To see that (5.52) and (5.53) are equal we consider the following identity, which can be obtained by dragging the marked multiplication along the path indicated. In order to do so one first uses that $f \in {\rm Hom}_{A|A}(\alpha_A^+(U), \alpha_A^+(U'))$ (applied to the right action of A) and next that $g \in {\rm Hom}_{A|A}(\alpha_A^+(V), \alpha_A^+(V'))$ (applied to the left action of A).

$$h(q) \;:= \qquad = \qquad (5.54)$$





For $q = id_A$, the A-loop on the right hand side is equal to the counit ε_A. On the other hand, for $q = P_A^l$, Lemma 2.29(iii) allows us to replace the A-loop by a C_l-loop, which by specialness of C_l is equal to the counit of C_l and a restriction to C_l, i.e. to replace the A-loop by $\varepsilon_{C_l} \circ r_{C_l}$. The latter, in turn, is equal to $\dim(C_l)/\dim(A)\,\varepsilon_A$. Thus

$$h(id_A) = \frac{\dim(C_l)}{\dim(A)}\, h(P_A^l)\,. \qquad (5.55)$$

Now the right hand side of (5.52) is equal to $(r_{C_l} \otimes id_{U'} \otimes id_{V'}) \circ h(id_A) \circ (e_{C_l} \otimes id_U \otimes id_V)$, while the right hand side of (5.53) equals – after eliminating one of the two idempotents with the help of Lemma 2.35 – $\dim(A)/\dim(C_l)\, (r_{C_l} \otimes id_{U'} \otimes id_{V'}) \circ h(P_A^l) \circ (e_{C_l} \otimes id_U \otimes id_V)$. Hence the equality (5.55) implies that $G_+^{\text{Ind}}(f \otimes_A g) = G_+^{\text{Ind}}(f) \otimes_{C_l} G_+^{\text{Ind}}(g)$.

(iii) G_+^{Ind} *is compatible with* γ^A: The equality

$$G_+^{\text{Ind}}(\tilde{\gamma}_{UV}^{A\,\mu\nu}) = \Phi_A^l(R_A(\tilde{\gamma}_{UV}^{A\,\mu\nu})) = \Phi_A^l(\gamma_{UV}^{A\,\mu\nu}) = \gamma_{UV}^{A\,\mu\nu} \tag{5.56}$$

follows by just combining Lemma 5.7(ii) and Lemma 5.15. □

Via Karoubification the functors $G_\pm^{\text{Ind}}$ induce functors

$$G_+ : \quad \mathcal{C}_{A|A}^+ \to \mathcal{C}_{C_l} \qquad \text{and} \qquad G_- : \quad \mathcal{C}_{A|A}^- \to \mathcal{C}_{C_r} \,. \tag{5.57}$$

Proposition 5.19:
The functors $G_\pm$ are tensor equivalences and satisfy

$$\begin{aligned} G_+(\tilde{\Gamma}_{XY}^{A++}) &= \Gamma_{G_+(X)\,G_+(Y)}^{C_l\,++} \quad \text{for } X, Y \in \text{Obj}(\mathcal{C}_{A|A}^+) \qquad \text{and} \\ G_-(\tilde{\Gamma}_{XY}^{A--}) &= \Gamma_{G_-(X)\,G_-(Y)}^{C_r\,--} \quad \text{for } X, Y \in \text{Obj}(\mathcal{C}_{A|A}^-) \,. \end{aligned} \tag{5.58}$$

Proof:
By Proposition 4.9 we have $\mathcal{C}_{C_l} \cong (\mathcal{C}_{C_l}^{\text{Ind}})^{\text{K}}$. That $G_\pm$ is a tensor equivalence then follows from the corresponding property of $G_\pm^{\text{Ind}}$ established in Lemma 5.18 by invoking Lemma 2.9.
The proof of the property (5.58) will be given for G_+ only, the one for G_- being analogous. Using the realisation of the Karoubian envelope via idempotents, let $X = (\alpha_A^+(U_X^+); p_X^+)$ and $Y = (\alpha_A^+(U_Y^+); p_Y^+)$. Then

$$\tilde{\Gamma}_{XY}^{A++} = (p_Y^+ \otimes_A p_X^+) \circ \gamma_{U_X^+ U_Y^+}^{A\,++} \circ (p_X^+ \otimes_A p_Y^+) \,. \tag{5.59}$$

Also, if $M = (\text{Ind}_{C_l}(U); p)$ and $N = (\text{Ind}_{C_l}(V); q)$ are objects in $(\mathcal{C}_{C_l}^{\text{Ind}})^{\text{K}}$, then

$$\Gamma_{MN}^{A\,++} = (q \otimes_{C_l} p) \circ \gamma_{UV}^{A\,++} \circ (p \otimes_{C_l} q) \,. \tag{5.60}$$

By definition, $G_+(X) = (G_+^{\text{Ind}}(\alpha_A^+(U_X^+)); G_+^{\text{Ind}}(p_X^+))$; the desired property of G_+ thus follows from the equalities

$$\begin{aligned} G_+(\tilde{\Gamma}_{XY}^{A++}) &= G_+^{\text{Ind}}((p_Y^+ \otimes_A p_X^+) \circ \gamma_{U_X^+ U_Y^+}^{A\,++} \circ (p_X^+ \otimes_A p_Y^+)) \\ &= [G_+^{\text{Ind}}(p_Y^+) \otimes_{C_l} G_+^{\text{Ind}}(p_X^+)] \circ \Gamma_{U_X^+ U_Y^+}^{C_l\,++} \circ [G_+^{\text{Ind}}(p_X^+) \otimes_{C_l} G_+^{\text{Ind}}(p_Y^+)] \\ &= \Gamma_{G_+(X)\,G_+(Y)}^{C_l\,++} \,, \end{aligned} \tag{5.61}$$

where we also used the compatibility of γ^A with G_+^{Ind} from Lemma 5.18(iii). □

We are now in a position to present our first main result, the ribbon equivalences between local $C_{l/r}(A)$-modules and ambichiral A-bimodules; based on results of [7], these equivalences have been conjectured in 'claim 5' of [37].

Theorem 5.20:
Let A be a symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$ such that the symmetric Frobenius algebras $C_{l/r}(A)$ are special as well. Then there are equivalences

$$\mathcal{C}^{\ell\mathrm{oc}}_{C_l(A)} \;\cong\; \mathcal{C}^{\,0}_{A|A} \;\cong\; \mathcal{C}^{\ell\mathrm{oc}}_{C_r(A)} \tag{5.62}$$

of ribbon categories.

We will only present the proof of the equivalence $\mathcal{C}^{\ell\mathrm{oc}}_{C_l(A)} \cong \mathcal{C}^{\,0}_{A|A}$ explicitly; the second equivalence can be shown by similar means. [5] As a preparation we need the following two lemmata.

Lemma 5.21:
We have the following bijections between spaces of bimodule morphisms of α-induced A-bimodules and module morphisms of (locally) induced C_l-modules:

$$\begin{aligned} \Psi^{l\,+-}_{A;UV}: &\quad \mathrm{Hom}_{A|A}(\alpha_A^+(U), \alpha_A^-(V)) \;\xrightarrow{\;\cong\;}\; \mathrm{Hom}_{C_l}(\mathrm{Ind}_{C_l}(U), \ell\text{-}\mathrm{Ind}^l_A(V)) \\ \Psi^{l\,-+}_{A;UV}: &\quad \mathrm{Hom}_{A|A}(\alpha_A^-(U), \alpha_A^+(V)) \;\xrightarrow{\;\cong\;}\; \mathrm{Hom}_{C_l}(\ell\text{-}\mathrm{Ind}^l_A(U), \mathrm{Ind}_{C_l}(V))\,. \end{aligned} \tag{5.63}$$

The maps $\Psi^{l\,+-}_{A;UV}$ and $\Psi^{l\,-+}_{A;UV}$ are given by

$$\Psi^{l\,+-}_{A;UV}(f) = f\circ(e_{C_l}\otimes id_U) \qquad \text{and} \qquad \Psi^{l\,-+}_{A;UV}(g) = (r_{C_l}\otimes id_V)\circ g\,. \tag{5.64}$$

In the definition of $\Psi^{l\,+-}_{A;UV}$, the realisation of $\ell\text{-}\mathrm{Ind}^l_A(V)$ as $(\mathrm{Ind}_A(V); P^l_A(V))$ is implied for obtaining the relevant subspace of $\mathrm{Hom}_{C_l}(\mathrm{Ind}_A(U), \mathrm{Ind}_A(V))$, and similar implications hold for the definition of $\Psi^{l\,-+}_{A;UV}$. The bijections (5.63) satisfy

$$\Psi^{l\,-+}_{A;VU}(g)\circ\Psi^{l\,+-}_{A;UV}(f) = \Phi^{l\,++}_{A;UU}(g\circ f)\,. \tag{5.65}$$

Proof:
This is a consequence of Proposition 3.6 together with the reciprocity relations (see Remark 2.23(iii))

$$\begin{aligned} &\mathrm{Hom}(U, E^l_A(V)) \cong \mathrm{Hom}_{C_l}(\mathrm{Ind}_{C_l}(U), \ell\text{-}\mathrm{Ind}^l_A(V)) \qquad \text{and} \\ &\mathrm{Hom}(E^l_A(U), V) \cong \mathrm{Hom}_{C_l}(\ell\text{-}\mathrm{Ind}^l_A(U), \mathrm{Ind}_{C_l}(V))\,. \end{aligned} \tag{5.66}$$

Using the explicit form (2.42) and (3.9) of these bijections, one checks that $\Psi^{l\,+-}_{A;UV}$ and $\Psi^{l\,-+}_{A;UV}$ are indeed given by the maps (5.64). Furthermore, substituting (5.64) and the definition (5.35), it is immediate that (5.65) holds true. □

[5] Recall also declarations 2.10 and 3.2.

Lemma 5.22:
The following two statements are equivalent:
(i) $((\alpha_A^+(U);p),e,r)$ is a A-bimodule retract of $\alpha_A^-(V)$,
(ii) $((\mathrm{Ind}_{C_l}(U);\Phi^l_{A;UU}(p)),\Psi^{l\,+-}_{A;UV}(e),\Psi^{l\,-+}_{A;VU}(r))$ is a C_l-module retract of ℓ-$\mathrm{Ind}^l_A(V)$.

Proof:
We will need two series of identities, both of which hold for any choice of morphisms $p\in\mathrm{Hom}_{A|A}(\alpha_A^+(U),\alpha_A^+(U))$, $e\in\mathrm{Hom}_{A|A}(\alpha_A^+(U),\alpha_A^-(V))$ and $r\in\mathrm{Hom}_{A|A}(\alpha_A^-(V),\alpha_A^+(U))$. The first series of identities is

$$\begin{aligned}\Phi^l_{A;UU}(p)\circ\Phi^l_{A;UU}(p) &= \Phi^l_{A;UU}(p\circ(P^l_A\circ\mathrm{id}_U)\circ p)\\ &= \Phi^l_{A;UU}((P^l_A\circ\mathrm{id}_U)\circ p\circ p) = \Phi^l_{A;UU}(p\circ p)\,.\end{aligned}\tag{5.67}$$

Here the first step holds by Lemma 5.14, in the second step Lemma 2.35 is used, and in the third step the idempotent P^l_A is omitted against the restriction morphism r_{C_l} contained, by definition, in $\Phi^l_{A;UU}$. The second series of identities is

$$\Psi^{l\,-+}_{A;VU}(r)\circ\Psi^{l\,+-}_{A;UV}(e) = \Phi^{l\,++}_{A;UU}(r\circ e) = \Phi^l_{A;UU}(r\circ e)\,,\tag{5.68}$$

where the first equality uses (5.65) and in the second equality holds because $\Phi^{l\,++}_{A;UU}$ is just a restriction of $\Phi^l_{A;UU}$ to a subspace.

(i) $\Rightarrow$ (ii): By assumption (i), p is an idempotent and we have $r\circ e=p$. By (5.67) this implies that $\Phi^l_{A;UU}(p)$ is an idempotent, too. Furthermore, by the equality (5.68) we have $\Psi^{l\,-+}_{A;VU}(r)\circ\Psi^{l\,+-}_{A;UV}(e)=\Phi^l_{A;UU}(p)$, which is equal to the identity morphism of $(\mathrm{Ind}_{C_l}(U);\Phi^l_{A;UU}(p))$, thus establishing that we are indeed dealing with a C_l-module retract.

(ii) $\Rightarrow$ (i): Conversely, suppose that $\Psi^{l\,-+}_{A;VU}(r)\circ\Psi^{l\,+-}_{A;UV}(e)=\Phi^l_{A;UU}(p)$ and that $\Phi^l_{A;UU}(p)$ is an idempotent. Then equations (5.67) and (5.68) tell us that also $\Phi^l_{A;UU}(p\circ p)=\Phi^l_{A;UU}(p)$ and $\Phi^l_{A;UU}(r\circ e)=\Phi^l_{A;UU}(p)$. Since, by the first isomorphism in Lemma 5.16, $\Phi^l_{A;UU}$ is injective on $\mathrm{End}_{A|A}(\alpha_A^+(U))$, it follows that p is an idempotent, and that $e\circ r=p$, which is the identity morphism in $\mathrm{End}_{A|A}((\alpha_A^+(U);p))$. □

Proof of Theorem **5.20**:
Denote by G: $\mathcal{C}^0_{A|A}\to\mathcal{C}_{C_l}$ the restriction of G_+ to $\mathcal{C}^0_{A|A}$. We will show that G is a ribbon equivalence between $\mathcal{C}^0_{A|A}$ and $\mathcal{C}^{\ell oc}_{C_l(A)}$.

(i) *The image of G consists of local modules*: Objects in $\mathcal{C}^+_{A|A}$ are of the form $B=(\alpha_A^+(U);p)$. If B is also in $\mathcal{C}^-_{A|A}$, then there exist morphisms e,r such that $((\alpha_A^+(U);p),e,r)$ is a bimodule retract of $\alpha_A^-(V)$ for some $V\in\mathrm{Obj}(\mathcal{C})$. By Lemma 5.22 it follows that $((\mathrm{Ind}_{C_l}(U);\Phi^l_{A;UU}(p))$, $\Psi^{l\,+-}_{A;UV}(e),\Psi^{l\,-+}_{A;VU}(r))$ is a C_l-module retract of the local module ℓ-$\mathrm{Ind}^l_A(V)$.
Thus $G(B)=(\mathrm{Ind}_{C_l}(U);\Phi^l_{A;UU}(p))$ is a retract of a local module, and hence local itself.

(ii) *G is essentially surjective on the category of local modules*: From (i) we know that G is a functor from $\mathcal{C}^0_{A|A}$ to $\mathcal{C}^{\ell oc}_{C_l(A)}$. By Proposition 4.12 every local module M is isomorphic to a retract of a locally induced module ℓ-$\mathrm{Ind}^l_A(V)$ for some $V\in\mathrm{Obj}(\mathcal{C})$. We can write

$M\cong(\ell\text{-Ind}_A^l(V);q)$ for some idempotent $q\in\text{End}_{C_l}(\ell\text{-Ind}_A^l(V))$. However, we want to make a statement involving C_l-modules rather than locally induced A-modules. To this end we introduce the morphisms

$$e' := q\circ r_{A\otimes V\succ\ell\text{-Ind}_l^A(V)}\circ(m\otimes id_V)\circ(e_{C_l}\otimes id_A\otimes id_V)\ \in \text{Hom}(C_l{\otimes}A{\otimes}V,\ell\text{-Ind}_A^l(V)) \quad \text{and}$$

$$r' := (r_{C_l}\otimes id_A\otimes id_V)\circ(\Delta\otimes id_V)\circ e_{\ell\text{-Ind}_l^A(V)\prec A\otimes V}\circ q\ \in \text{Hom}(\ell\text{-Ind}_A^l(V),C_l{\otimes}A{\otimes}V)\,. \tag{5.69}$$

These morphisms fulfill $e'\circ r'=q$, as can be seen as follows. First note that Lemma 2.39, specialised to $U=V=\mathbf{1}$ and $\Phi=id_A$, together with Lemma 2.29(iii) and 2.29(ii) as well as specialness of C_l, implies that $m\circ(id_A\circ P_A^r)\circ\Delta=id_A$. It is then easy to convince oneself that an appropriately modified version of Lemma 2.39 gives rise to the analogous identity $m\circ(P_A^l\circ id_A)\circ\Delta=id_A$. This, in turn, implies $e'\circ r'=q$.
Next define $p':=r'\circ e'$. Because of $q\circ e'=e'$, p' is an idempotent. Thus by construction we have an isomorphism

$$((\text{Ind}_{C_l}(A\otimes V);p'),e',r')\cong(\ell\text{-Ind}_A^l(V);q) \tag{5.70}$$

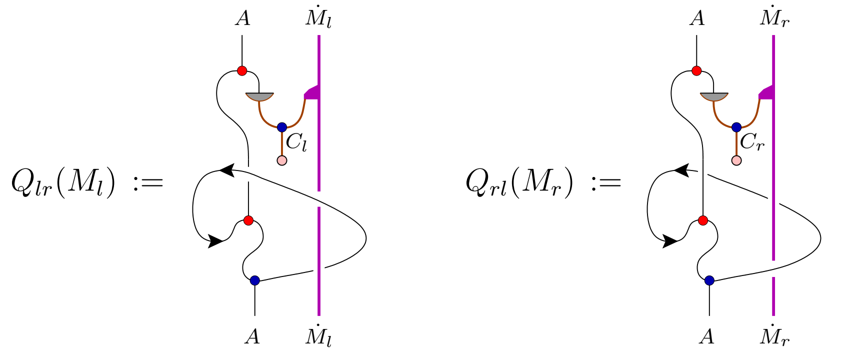
of C_l-modules. Thus $((\text{Ind}_{C_l}(U);p'),e',r')$ with $U:=A\otimes V$ is a module retract of $\ell\text{-Ind}_A^l(V)$. By the Lemmata 5.16 and 5.21 we can now find morphisms $p\in\text{End}_{A|A}(\alpha_A^+(U))$, $e\in\text{Hom}_{A|A}(\alpha_A^+(U),\alpha_A^-(U))$ and $r\in\text{Hom}_{A|A}(\alpha_A^-(U),\alpha_A^+(U))$ such that $\Phi^{l\,++}_{A;UU}(p)=p'$, $\Psi^{l\,+-}_{A;UU}(e)=e'$ and $\Psi^{l\,-+}_{A;UU}(r)=r'$. Then we can use lemma 5.22 to conclude that $((\alpha_A^+(U);p),e,r)$ is an A-bimodule retract of $\alpha_A^-(V)$. Thus we have found an object $B=(\alpha_A^+(U);p)$ in $\mathcal{C}^0_{A|A}$ such that $G(B)\cong M$.
(iii) *G is an equivalence of ribbon categories*: Note that G: $\mathcal{C}^0_{A|A}\to\mathcal{C}^{\ell oc}_{C_l}$ is an equivalence functor because first, it is essentially surjective on objects, and second, it is a restriction of G_+, which is bijective on morphisms. Since G_+ is a tensor functor, so is G. Furthermore, for the family $\tilde\Gamma^A_{XY}$ of morphisms we have

$$G(\tilde\Gamma^A_{XY})=G_+(\tilde\Gamma^{A\,++}_{XY})=\Gamma^{C_l\,++}_{G_+(X)\,G_+(Y)}=c^{C_l}_{G(X)\,G(Y)}\,, \tag{5.71}$$

where we first used Proposition 5.12 (ii), then Proposition 5.19 and finally Proposition 5.5. Since $\tilde\Gamma^A_{XY}$ is mapped to the braiding c^{C_l} on $\mathcal{C}^{\ell oc}_{C_l}$ by an equivalence functor, it follows that $\tilde\Gamma^A_{XY}$ defines a braiding on $\mathcal{C}^0_{A|A}$. – This completes the proof of Proposition 5.12 by also establishing part (iii) of the proposition.
Hence the tensor equivalence G is compatible with the braiding. Thus G is an equivalence of braided tensor categories, and thereby also of ribbon categories. □

Remark 5.23:
Denote by G_{lr}: $\mathcal{C}^{\ell oc}_{C_l(A)}\to\mathcal{C}^{\ell oc}_{C_r(A)}$ and G_{rl}: $\mathcal{C}^{\ell oc}_{C_r(A)}\to\mathcal{C}^{\ell oc}_{C_l(A)}$ the functorial equivalences of the ribbon categories $\mathcal{C}^{\ell oc}_{C_l(A)}$ and $\mathcal{C}^{\ell oc}_{C_r(A)}$ constructed in Theorem 5.20.

One can give an explicit representation of $G_{l/r}$ using retracts. Consider the two morphisms $Q_{lr}(M_l) \in \mathrm{Hom}(A \otimes \dot{M}_l, A \otimes \dot{M}_l)$ and $Q_{rl}(M_r) \in \mathrm{Hom}(A \otimes \dot{M}_r, A \otimes \dot{M}_r)$ given by

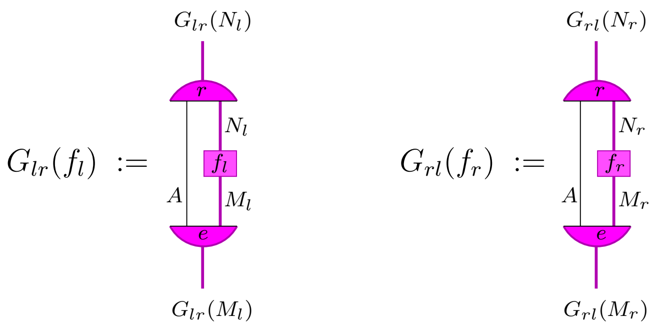


(5.72)

By combining several previous results one sees that $Q_{lr/rl}(M_{l/r})$ are idempotents: the morphisms $P_A^{l/r}(M_{r/l})$ from (3.1) are idempotents, (2.65) can be used to commute the ribbon connecting A to $M_{r/l}$ past the A-loop, and finally one can use (3.16) together with specialness of $C_{l/r}$, which holds by the assumptions in Theorem 5.20.
For local $C_{l/r}$-modules $M_{l/r}$ one has

$$G_{lr}(M_l) = \mathrm{Im}\, Q_{lr}(M_l) \qquad \text{and} \qquad G_{rl}(M_r) = \mathrm{Im}\, Q_{rl}(M_r) \tag{5.73}$$

(recall that we work with Karoubian categories, so that all idempotents are split), and the action of the functors G_{lr} and G_{rl} on morphisms reads



(5.74)

for $f_r \in \mathrm{Hom}_{C_r}(M_r, N_r)$ and $f_l \in \mathrm{Hom}_{C_l}(M_l, N_l)$.

Remark 5.24:
(i) The equivalence of the categories of local modules over the left and right centers given in Theorem 5.20 is a category theoretic analogue of Theorem 5.5 of [5], which was obtained in the study of relations between nets of braided subfactors and modular invariants. In the context of module categories, the equivalence, including the relation to the category of ambichiral bimodules, has been formulated, as a conjecture, in Section 5.4 of [37].

(ii) It is known [32] that in conformal quantum field theory, every modular invariant torus partition function can be described in terms of extensions of the chiral algebras for left movers and right movers. The two extensions need not be the same, but they should lead to extended theories with isomorphic fusion rules. The additional information in a modular invariant partition function is the choice of an isomorphism of these fusion rules. This structure is sometimes summarised by saying that the torus partition function of every full conformal field theory has the form of 'a fusion rule isomorphism on top of (maximal) extensions of the chiral algebras'.

This statement has been obtained in [32] using the action of the (cover of the) modular group $SL(2,\mathbb{Z})$ on the characters of a chiral CFT, the invariance of the torus partition function under this action, and the non-negativity of its coefficients.

The connection between this description of partition functions and our study of algebras in tensor categories is supplied by the insight [17, 18] that, given a chiral rational conformal field theory, a full rational CFT, including in particular its torus and annulus partition functions, can be constructed from a symmetric special Frobenius algebra A in the modular tensor category $\mathcal{C}$ that describes the chiral data of the CFT. (But not every modular invariant bilinear combination of characters of the chiral CFT is the torus partition functions of some full CFT.) The structure of partition functions described above can be obtained from Theorem 5.20 as follows. The procedure of 'extending the chiral algebra for left movers and right movers' corresponds to passing to the modular tensor categories $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$ of local modules of the left center and the right center, respectively, of A. By Theorem 5.20 these two categories are equivalent, so that in particular they have isomorphic fusion rules,

$$K_0\big(\mathcal{C}^{\ell oc}_{C_l(A)}\big) \cong K_0\big(\mathcal{C}^{\ell oc}_{C_r(A)}\big)\,. \tag{5.75}$$

We may lift the algebra A to algebras in $\mathcal{C}^{\ell oc}_{C_{l/r}(A)}$ via lemma 4.13 to obtain algebras with trivial center. In this sense, the two extensions are 'maximal' and the isomorphism of the fusion rules is encoded in the 'non-commutative part' of the algebra A.

6 Product categories and trivialisability

In many respects the simplest tensor categories are the categories of finite-dimensional vector spaces over some field $\Bbbk$; we denote the latter category by $\mathcal{V}ect_{\Bbbk}$. It is therefore interesting to find commutative symmetric special Frobenius algebras A in ribbon categories which are 'trivialising' in the sense that the category $\mathcal{C}^{\ell oc}_A$ of local A-modules is equivalent to $\mathcal{V}ect_{\Bbbk}$. For a generic ribbon category $\mathcal{C}$ such a trivialising algebra need not exist. A class of categories for which a trivialising algebra does exist is provided by the representation categories for so-called holomorphic orbifolds [11, 2]: for these, the trivialising algebra affords the extension of the corresponding orbifold conformal field theory to the underlying un-orbifolded theory.

We may, however, relax the requirement and instead look, for given $\mathcal{C}$, for some 'compensating' ribbon category $\mathcal{C}'$ and a trivialising algebra T in the (suitably defined) product of $\mathcal{C}$ with $\mathcal{C}'$ – for the precise formulation of this concept of trivialisability, see Definition 6.4 below. The main purpose of this section is to establish that such a category $\mathcal{C}'$ and algebra T always exist when $\mathcal{C}$ is a *modular* tensor category. In that case, for $\mathcal{C}'$ we can take the category dual to $\mathcal{C}$, a concept that will be discussed in Section 6.2.

6.1 Product categories and the notion of trivialisability

But first we must introduce a suitable concept of product which to any pair of $\mathbb{k}$-linear categories $\mathcal{C}$ and $\mathcal{D}$ associates a product category that shares with $\mathcal{C}$ and $\mathcal{D}$ all the relevant properties, such as the basic properties listed in the declaration 2.10. This is done in

Definition 6.1:
Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbb{k}$-linear categories.
(i) The category $\mathcal{C}\otimes_{\mathbb{k}}\mathcal{D}$ is the category whose objects are pairs $U \times X$ with $U \in \mathrm{Obj}(\mathcal{C})$ and $X \in \mathrm{Obj}(\mathcal{D})$ and whose morphism spaces are tensor products (over $\mathbb{k}$)

$$\mathrm{Hom}^{\mathcal{C}\otimes_{\mathbb{k}}\mathcal{D}}(U\times X, V\times Y) := \mathrm{Hom}^{\mathcal{C}}(U,V) \otimes_{\mathbb{k}} \mathrm{Hom}^{\mathcal{D}}(X,Y) \tag{6.1}$$

of those of $\mathcal{C}$ and $\mathcal{D}$.
(ii) The *Karoubian product* $\mathcal{C}\boxtimes\mathcal{D}$ is the Karoubian envelope of $\mathcal{C}\otimes_{\mathbb{k}}\mathcal{D}$,

$$\mathcal{C}\boxtimes\mathcal{D} := (\mathcal{C}\otimes_{\mathbb{k}}\mathcal{D})^{\mathrm{K}} . \tag{6.2}$$

Remark 6.2:
(i) Taking the tensor product over $\mathbb{k}$ rather than the Kronecker product of the morphism sets accounts for the fact that the categories of our interest are enriched over $\mathcal{V}ect_{\mathbb{k}}$. The price to pay is that $\mathcal{C}\otimes_{\mathbb{k}}\mathcal{D}$ has idempotents that are not tensor products of idempotents in $\mathcal{C}$ and $\mathcal{D}$, so that even when $\mathcal{C}$ and $\mathcal{D}$ are Karoubian we get, in general, a Karoubian product category only after taking the Karoubian envelope.

(ii) In accordance with Remark 2.8(iii) we regard the category $\mathcal{C}\otimes_{\mathbb{k}}\mathcal{D}$ as a full subcategory of $\mathcal{C}\boxtimes\mathcal{D}$, i.e. in particular identify $U\times X \in \mathrm{Obj}(\mathcal{C}\otimes_{\mathbb{k}}\mathcal{D})$ with $(U\times X, id_U\otimes_{\mathbb{k}} id_X) \in \mathrm{Obj}(\mathcal{C}\boxtimes\mathcal{D})$.

(iii) When $\mathcal{C}$ and $\mathcal{D}$ are small categories, then so are $\mathcal{C}\otimes_{\mathbb{k}}\mathcal{D}$ and $\mathcal{C}\boxtimes\mathcal{D}$. When $\mathcal{C}$ and $\mathcal{D}$ are additive, then so is $\mathcal{C}\boxtimes\mathcal{D}$. When $\mathcal{C}$ and $\mathcal{D}$ are semisimple, then so is $\mathcal{C}\boxtimes\mathcal{D}$.

(iv) When $\mathcal{C}$ and $\mathcal{D}$ are modular tensor categories, then so is their Karoubian product $\mathcal{C}\boxtimes\mathcal{D}$, see Proposition 6.3(iii) below. It is easy to verify that the dimension and charge of modular tensor categories, as defined in (3.54), are multiplicative, i.e.

$$\mathrm{Dim}(\mathcal{C}\boxtimes\mathcal{D}) = \mathrm{Dim}(\mathcal{C})\,\mathrm{Dim}(\mathcal{D}) \qquad \text{and} \qquad p^{\pm}(\mathcal{C}\boxtimes\mathcal{D}) = p^{\pm}(\mathcal{C})\,p^{\pm}(\mathcal{D}) . \tag{6.3}$$

Proposition 6.3:
(i) When $\mathcal{C}$ and $\mathcal{D}$ are tensor categories, then $\mathcal{C}\otimes_{\Bbbk}\mathcal{D}$ can be naturally equipped with the structure of a tensor category, by setting

$$(U\times X)\otimes^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}(V\times Y) := (U\otimes^{\mathcal{C}}V)\times(X\otimes^{\mathcal{D}}Y)\,, \qquad \mathbf{1}^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}} := \mathbf{1}^{\mathcal{C}}\times\mathbf{1}^{\mathcal{D}} \qquad \text{and}$$
$$(f\otimes_{\Bbbk} g)\otimes^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}(f'\otimes_{\Bbbk} g') := (f\otimes^{\mathcal{C}}f')\otimes_{\Bbbk}(g\otimes^{\mathcal{D}}g')\,. \tag{6.4}$$

(ii) Similarly, $\mathcal{C}\otimes_{\Bbbk}\mathcal{D}$ inherits from $\mathcal{C}$ and $\mathcal{D}$ the properties of having a (left or right) duality, a braiding, and a twist, by setting

$$\begin{aligned} d^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}_{U\times X} &:= d^{\mathcal{C}}_{U}\otimes_{\Bbbk} d^{\mathcal{D}}_{V} \qquad \text{etc.}\,,\\ c^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}_{U\times X,V\times Y} &:= c^{\mathcal{C}}_{U,V}\otimes_{\Bbbk} c^{\mathcal{D}}_{X,Y}\,,\\ \theta^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}_{U\times X} &:= \theta^{\mathcal{C}}_{U}\otimes_{\Bbbk}\theta^{\mathcal{D}}_{V}\,. \end{aligned} \tag{6.5}$$

In particular, when $\mathcal{C}$ and $\mathcal{D}$ are ribbon categories, then $\mathcal{C}\otimes_{\Bbbk}\mathcal{D}$ is naturally equipped with the structure of a ribbon category. Moreover,

$$s^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}_{U\times X,V\times Y} = s^{\mathcal{C}}_{U,V}\, s^{\mathcal{D}}_{X,Y}\,; \tag{6.6}$$

in particular, the dimensions in $\mathcal{C}\otimes_{\Bbbk}\mathcal{D}$ are given by

$$\dim^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}(U\times X) = \dim^{\mathcal{C}}(U)\,\dim^{\mathcal{D}}(X)\,. \tag{6.7}$$

(iii) Analogous statements as in (i) and (ii) apply to the Karoubian product $\mathcal{C}\boxtimes\mathcal{D}$. In addition, if $\mathcal{C}$ and $\mathcal{D}$ are modular tensor categories, then the category $\mathcal{C}\boxtimes\mathcal{D}$ has a natural structure of modular tensor category.

Proof:
(i), (ii) Using the relevant properties of $\mathcal{C}$ and $\mathcal{D}$, it is straightforward to check that with the definitions (6.4) and (6.5), all required relations for morphisms in $\mathcal{C}\otimes_{\Bbbk}\mathcal{D}$ are satisfied.
(iii) then holds by combining these results with the properties of the Karoubian envelope listed in Remark 2.8(iv). For modular $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{C}\boxtimes\mathcal{D}$ is additive and semisimple by Remark 6.2(iii), and the s-matrix (6.6) is non-degenerate because those of $\mathcal{C}$ and $\mathcal{D}$ are. Thus $\mathcal{C}\boxtimes\mathcal{D}$ is indeed modular. □

We are now in a position to introduce the concept of *trivialisability* of $\mathcal{C}$:

Definition 6.4:
A ribbon category $\mathcal{C}$ is called *trivialisable* iff there exist a ribbon category $\mathcal{C}'$ and a commutative symmetric special Frobenius algebra T in $\mathcal{C}\boxtimes\mathcal{C}'$ such that the category of local T-modules is equivalent to the category of finite-dimensional vector spaces over $\Bbbk$,

$$(\mathcal{C}\boxtimes\mathcal{C}')^{\ell oc}_{T} \cong \mathcal{V}ect_{\Bbbk}\,. \tag{6.8}$$

The data $\mathcal{C}'$ and T are then called a *trivialisation* of $\mathcal{C}$.

The rest of this subsection is devoted to the study of the Karoubian product of tensor categories and its behaviour in the context of module categories.

Lemma 6.5:
The Karoubian product of two categories is equivalent to the Karoubian product of their Karoubian envelopes,

$$\mathcal{C} \boxtimes \mathcal{D} \cong \mathcal{C}^{\mathrm{K}} \boxtimes \mathcal{D}^{\mathrm{K}}. \tag{6.9}$$

If $\mathcal{C}$ and $\mathcal{D}$ are ribbon, then this is an equivalence of ribbon categories.

Proof:
According to Proposition 2.3 to show the equivalence it is sufficient to construct a functor $F: \mathcal{C}^{\mathrm{K}} \boxtimes \mathcal{D}^{\mathrm{K}} \to \mathcal{C} \boxtimes \mathcal{D}$ that is essentially surjective on objects and bijective on morphisms. The objects of $\mathcal{C} \boxtimes \mathcal{D} \equiv (\mathcal{C} \otimes_{\Bbbk} \mathcal{D})^{\mathrm{K}}$ are triples $(U \times X; \pi)$ with π an idempotent in $\mathrm{End}(U \times X) \cong \mathrm{End}(U) \otimes_{\Bbbk} \mathrm{End}(X)$, while the objects of $\mathcal{C}^{\mathrm{K}} \boxtimes \mathcal{D}^{\mathrm{K}}$ are quintuples $((U;p) \times (X;q); \hat\pi)$, where $U \in \mathrm{Obj}(\mathcal{C})$, $X \in \mathrm{Obj}(\mathcal{D})$, $p \in \mathrm{End}(U)$ and $q \in \mathrm{End}(X)$ are idempotents in $\mathcal{C}$ and $\mathcal{D}$, respectively, and $\hat\pi \in \mathrm{End}(U \times X)$ is an idempotent obeying the Karoubi condition

$$(p \otimes_{\Bbbk} q) \circ \hat\pi = \hat\pi = \hat\pi \circ (p \otimes_{\Bbbk} q)\,. \tag{6.10}$$

We define the functor F on objects as

$$F\big(((U;p) \times (X;q); \pi)\big) := (U \times X; \pi)\,. \tag{6.11}$$

It then follows that we get every object $(U \times X; \pi)$ of $\mathcal{C} \boxtimes \mathcal{D}$ as the image under F of the object $((U; id_U) \times (X; id_X); \pi)$. Hence F is surjective on objects.
To define F on morphisms, we first introduce, for any two objects $((U;p) \times (X;q); \pi)$ and $((V;p') \times (Y;q'); \pi')$ of $\mathcal{C}^{\mathrm{K}} \boxtimes \mathcal{D}^{\mathrm{K}}$, certain endomorphisms P, Q and Π of vector spaces:

$$\begin{array}{rrcl} P: & \mathrm{Hom}^{\mathcal{C}}(U,V) & \to & \mathrm{Hom}^{\mathcal{C}}(U,V) \\ & f & \mapsto & p' \circ f \circ p \end{array} \quad \text{and} \quad \begin{array}{rrcl} Q: & \mathrm{Hom}^{\mathcal{D}}(X,Y) & \to & \mathrm{Hom}^{\mathcal{D}}(X,Y) \\ & g & \mapsto & q' \circ g \circ q \end{array} \tag{6.12}$$

as well as

$$\begin{array}{rrcl} \Pi: & \mathrm{Hom}^{\mathcal{C} \otimes_{\Bbbk} \mathcal{D}}(U \times X, V \times Y) & \to & \mathrm{Hom}^{\mathcal{C} \otimes_{\Bbbk} \mathcal{D}}(U \times X, V \times Y) \\ & \psi & \mapsto & \pi' \circ \psi \circ \pi\,; \end{array} \tag{6.13}$$

P, Π and Q are idempotents of vector spaces. One checks that, by definition of the Karoubian envelope,

$$\mathrm{Hom}^{\mathcal{C}^{\mathrm{K}} \boxtimes \mathcal{D}^{\mathrm{K}}}\big(((U;p) \times (X;q); \pi), ((V;p') \times (Y;q'); \pi')\big) \cong \mathrm{Im}(P) \otimes_{\Bbbk} \mathrm{Im}(Q) \cap \mathrm{Im}(\Pi)\,, \tag{6.14}$$

while

$$\mathrm{Hom}^{\mathcal{C} \boxtimes \mathcal{D}}((U \times X; \pi), (V \times Y; \pi')) \cong \mathrm{Im}(\Pi)\,. \tag{6.15}$$

In addition, from (6.10) it follows that $(P\otimes_{\Bbbk} Q)\circ\Pi=\Pi=\Pi\circ(P\otimes_{\Bbbk} Q)$, which in turn implies that

$$\mathrm{Im}(\Pi) \subseteq \mathrm{Im}(P\otimes_{\Bbbk} Q) = \mathrm{Im}(P)\otimes_{\Bbbk}\mathrm{Im}(Q)\,. \tag{6.16}$$

We can thus conclude that the morphism spaces (6.14) and (6.15) are actually identical subspaces of $\mathrm{Hom}^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}(U{\times}X,V{\times}Y)=\mathrm{Hom}^{\mathcal{C}}(U,V)\otimes_{\Bbbk}\mathrm{Hom}^{\mathcal{D}}(X,Y)$.
We now simply define F to be the identity map on morphisms, so that F is in particular bijective on morphisms. It is easy to check that together with (6.11) this yields a functor from $\mathcal{C}^{\mathrm{K}}\boxtimes\mathcal{D}^{\mathrm{K}}$ to $\mathcal{C}\boxtimes\mathcal{D}$.

Thus F is an equivalence functor from $\mathcal{C}^{\mathrm{K}}\boxtimes\mathcal{D}^{\mathrm{K}}$ to $\mathcal{C}\boxtimes\mathcal{D}$. Suppose now that $\mathcal{C}$ and $\mathcal{D}$ are ribbon. Instead of directly verifying that F is a ribbon equivalence, it is slightly more convenient to work with its functorial inverse, to be denoted by G. On objects $R=(U{\times}X;\pi)$ of $\mathcal{C}\boxtimes\mathcal{D}$ we have $G(R)=((U;id_U)\times(X;id_X);\pi)$, while on morphisms G acts as the identity map. Using the definition of the ribbon structure on the Karoubian envelope of a category and on the Karoubian product of categories, as given in Remark 2.8(iv) and in Proposition 6.3, respectively, one verifies by direct substitution that G is an equivalence of ribbon categories. We present details of the calculation only for the tensor product and for the braiding.
Let $R=(U{\times}X;\pi)$ and $S=(V{\times}Y;\varpi)$ be objects of $\mathcal{C}\boxtimes\mathcal{D}$. Using (2.15) and (6.4) we get

$$\begin{aligned} G(R\otimes^{\mathcal{C}\boxtimes\mathcal{D}}S) &= G\big(((U\otimes^{\mathcal{C}}V)\times(X\otimes^{\mathcal{D}}Y);\pi\otimes^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}\varpi)\big) \\ &= ((U\otimes^{\mathcal{C}}V;id_{U\otimes^{\mathcal{C}}V})\times(X\otimes^{\mathcal{D}}Y;id_{X\otimes^{\mathcal{D}}Y});\pi\otimes^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}\varpi) \end{aligned} \tag{6.17}$$

as well as

$$G(R)\otimes^{\mathcal{C}^{\mathrm{K}}\boxtimes\mathcal{D}^{\mathrm{K}}}G(S) = ((U;id_U)\times(X;id_X);\pi)\otimes^{\mathcal{C}^{\mathrm{K}}\otimes_{\Bbbk}\mathcal{D}^{\mathrm{K}}}((V;id_V)\times(Y;id_Y);\varpi)\,, \tag{6.18}$$

so that indeed $G(R\otimes^{\mathcal{C}\boxtimes\mathcal{D}}S)=G(R)\otimes^{\mathcal{C}^{\mathrm{K}}\boxtimes\mathcal{D}^{\mathrm{K}}}G(S)$. For morphisms, equality of $G(f\otimes^{\mathcal{C}\boxtimes\mathcal{D}}g)$ and $G(f)\otimes^{\mathcal{C}^{\mathrm{K}}\boxtimes\mathcal{D}^{\mathrm{K}}}G(g)$ is immediate because G is the identity on morphisms.
Concerning the braiding note that, using (2.16) and (6.5),

$$G(c_{R,S}) = G(c_{(U\times X;\pi),(V\times Y;\varpi)}) = G((\varpi\otimes^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}\pi)\circ(c_{U,V}\otimes_{\Bbbk}c_{X,Y})) \tag{6.19}$$

and

$$c_{G(R),G(S)} = c_{((U;id_U)\times(X;id_X);\pi),((V;id_V)\times(Y;id_Y);\varpi)} = (\varpi\otimes^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}\pi)\circ(c_{U,V}\otimes_{\Bbbk}c_{X,Y}))\,. \tag{6.20}$$

Since G is the identity on morphisms, this implies that $G(c_{R,S})=c_{G(R),G(S)}$. □

Remark 6.6:
The product $\otimes_{\Bbbk}$ of categories is associative. Together with lemma 6.5, this implies in particular that the Karoubian product of categories is associative as well, i.e. we have

$$(\mathcal{C}\boxtimes\mathcal{D})\boxtimes\mathcal{E} \cong (\mathcal{C}\otimes_{\Bbbk}\mathcal{D})\boxtimes\mathcal{E} \cong (\mathcal{C}\otimes_{\Bbbk}\mathcal{D}\otimes_{\Bbbk}\mathcal{E})^{\mathrm{K}} \cong \mathcal{C}\boxtimes(\mathcal{D}\otimes_{\Bbbk}\mathcal{E}) \cong \mathcal{C}\boxtimes(\mathcal{D}\boxtimes\mathcal{E}) \tag{6.21}$$

for any triple $\mathcal{C},\ \mathcal{D},\ \mathcal{E}$ of categories. If $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are ribbon, then these are equivalences of ribbon categories.

Lemma 6.7:
For any (additive, $\Bbbk$-linear) category $\mathcal{C}$, taking the product, in the sense of (6.1), with the category $\mathcal{V}ect_\Bbbk$ of finite-dimensional vector spaces yields a category equivalent to $\mathcal{C}$,

$$\mathcal{C} \otimes_\Bbbk \mathcal{V}ect_\Bbbk \;\cong\; \mathcal{C}\,, \tag{6.22}$$

while taking the Karoubian product with $\mathcal{V}ect_\Bbbk$ yields the Karoubian envelope of $\mathcal{C}$,

$$\mathcal{C} \boxtimes \mathcal{V}ect_\Bbbk \;\cong\; \mathcal{C}^{\rm K}\,. \tag{6.23}$$

If $\mathcal{C}$ is ribbon, then these are equivalences of ribbon categories.

Proof:
Consider the functor $F\colon \mathcal{C} \to \mathcal{C}\otimes_\Bbbk \mathcal{V}ect_\Bbbk$ defined by $F(U) := U \times \Bbbk$ on objects and by $F(f) := f \otimes_\Bbbk id_\Bbbk$ on morphisms. Clearly, F is bijective on morphisms. Next, note that every object $X \in {\rm Obj}(\mathcal{V}ect_\Bbbk)$ is isomorphic to a direct sum $X \cong \Bbbk \oplus \cdots \oplus \Bbbk$. Furthermore we have an isomorphism $(U \oplus \cdots \oplus U) \times \Bbbk \cong U \times (\Bbbk \oplus \cdots \oplus \Bbbk)$. Thus every object $U \times X$ of $\mathcal{C}\otimes_\Bbbk \mathcal{V}ect_\Bbbk$ is isomorphic to an object of the form $U' \times \Bbbk$, implying in particular that F is essentially surjective, and hence provides an equivalence of categories by Proposition 2.3. This establishes (6.22).
Suppose now that $\mathcal{C}$ is ribbon. Using the definition of the ribbon structure on $\mathcal{C} \boxtimes \mathcal{V}ect_\Bbbk$ as given in Proposition 6.3, one immediately verifies that in this case F is a ribbon functor.
The equivalence (6.23) is obtained from (6.22) by taking the Karoubian envelope on both sides, using Lemma 2.9. □

Lemma 6.8:
(i) When A and B are algebras in tensor categories $\mathcal{C}$ and $\mathcal{D}$, respectively, then setting

$$m_{A\times B}^{\mathcal{C}\otimes_\Bbbk\mathcal{D}} := m_A^{\mathcal{C}} \otimes_\Bbbk m_B^{\mathcal{D}} \qquad \text{and} \qquad \eta_{A\times B}^{\mathcal{C}\otimes_\Bbbk\mathcal{D}} := \eta_A^{\mathcal{C}} \otimes_\Bbbk \eta_B^{\mathcal{D}} \tag{6.24}$$

endows $A \times B \in {\rm Obj}(\mathcal{C}\otimes_\Bbbk\mathcal{D})$ with the structure of an algebra in $\mathcal{C}\otimes_\Bbbk\mathcal{D}$.
(ii) An analogous statement holds for coalgebras, with

$$\Delta_{A\times B}^{\mathcal{C}\otimes_\Bbbk\mathcal{D}} := \Delta_A^{\mathcal{C}} \otimes_\Bbbk \Delta_B^{\mathcal{D}} \qquad \text{and} \qquad \varepsilon_{A\times B}^{\mathcal{C}\otimes_\Bbbk\mathcal{D}} := \varepsilon_A^{\mathcal{C}} \otimes_\Bbbk \varepsilon_B^{\mathcal{D}}\,. \tag{6.25}$$

(iii) If A and B are haploid, then so is $A \times B$.

(iv) If in addition $\mathcal{C}$ and $\mathcal{D}$ are braided and A and B are (co-)commutative, then $A \times B$ is (co-)commutative as well.

(v) When A and B are Frobenius algebras in ribbon categories $\mathcal{C}$ and $\mathcal{D}$, respectively, then (6.24) and (6.25) equip $A \times B \in {\rm Obj}(\mathcal{C}\otimes_\Bbbk\mathcal{D})$ with the structure of a Frobenius algebra in $\mathcal{C}\otimes_\Bbbk\mathcal{D}$. If in addition both A and B are symmetric and/or special, then so is $A \times B$.

Proof:
All required relations of the structural morphisms $m_{A\times B}^{\mathcal{C}\otimes_\Bbbk\mathcal{D}}$, $\eta_{A\times B}^{\mathcal{C}\otimes_\Bbbk\mathcal{D}}$ etc. easily follow from the corresponding ones of A and B. □

Just like in many other respects, special Frobenius algebras are especially well-behaved also with respect to taking product categories. In particular, we have

Lemma 6.9:
For A and B special Frobenius algebras in (not necessarily Karoubian) ribbon categories $\mathcal{C}$ and $\mathcal{D}$, respectively, there is an equivalence

$$(\mathcal{C}\boxtimes\mathcal{D})_{(A\times B;id_A\otimes_{\Bbbk}id_B)} \cong \big((\mathcal{C}\otimes_{\Bbbk}\mathcal{D})_{A\times B}\big)^{\mathrm{K}} . \tag{6.26}$$

If A and B are in addition symmetric and commutative, then there is also an equivalence

$$(\mathcal{C}\boxtimes\mathcal{D})^{\ell\mathrm{oc}}_{(A\times B;id_A\otimes_{\Bbbk}id_B)} \cong \big((\mathcal{C}\otimes_{\Bbbk}\mathcal{D})^{\ell\mathrm{oc}}_{A\times B}\big)^{\mathrm{K}} . \tag{6.27}$$

involving categories of local modules.

Proof:
The assertions follow immediately by applying corollary 4.11(i) and (ii), respectively, to the special Frobenius algebra $A\times B$ in the ribbon category $\mathcal{C}\otimes_{\Bbbk}\mathcal{D}$. □

In the sequel we will often identify $\mathrm{Obj}(\mathcal{C}\otimes_{\Bbbk}\mathcal{D})$ with the corresponding full subcategory of $\mathrm{Obj}(\mathcal{C}\boxtimes\mathcal{D})$, and accordingly identify the algebra $(A\times B;id_A\otimes_{\Bbbk}id_B)$ with the algebra $A\times B\in\mathrm{Obj}(\mathcal{C}\otimes_{\Bbbk}\mathcal{D})\subseteq\mathrm{Obj}(\mathcal{C}\boxtimes\mathcal{D})$.

A natural question is to which extent the modules over $A\times B$ can be understood in terms of A- and B-modules. We first note

Lemma 6.10:
(i) For A and B algebras in tensor categories $\mathcal{C}$ and $\mathcal{D}$, and $A\times B\in\mathrm{Obj}(\mathcal{C}\otimes_{\Bbbk}\mathcal{D})$ endowed with the algebra structure (6.24), we have the equivalence

$$\mathcal{C}_A^{\mathrm{Ind}}\otimes_{\Bbbk}\mathcal{D}_B^{\mathrm{Ind}} \;\cong\; (\mathcal{C}\otimes_{\Bbbk}\mathcal{D})^{\mathrm{Ind}}_{A\times B} \tag{6.28}$$

of categories of induced modules.

(ii) If in addition $\mathcal{C}$ and $\mathcal{D}$ are (not necessarily Karoubian) ribbon categories and A and B are centrally split commutative symmetric special Frobenius algebras then we have the equivalence

$$\mathcal{C}_A^{\ell\text{-}\mathrm{Ind}}\otimes_{\Bbbk}\mathcal{D}_B^{\ell\text{-}\mathrm{Ind}} \;\cong\; (\mathcal{C}\otimes_{\Bbbk}\mathcal{D})^{\ell\text{-}\mathrm{Ind}}_{A\times B} \tag{6.29}$$

of categories of locally induced modules.

Proof:
(i) The induced $A\times B$-modules in $\mathcal{C}\otimes_{\Bbbk}\mathcal{D}$ are pairs consisting of objects $(A\otimes U)\times(B\otimes X)$ and the $A\times B$-action $(m_A\otimes id_U)\otimes_{\Bbbk}(m_B\otimes id_X)$. They are thus in natural bijection with the objects $(A\otimes U, m_A\otimes id_U)\times(B\otimes X, m_B\otimes id_X)$ of $\mathcal{C}_A^{\mathrm{Ind}}\otimes_{\Bbbk}\mathcal{D}_B^{\mathrm{Ind}}$. Analogously there are natural isomorphisms between the respective morphism spaces.

(ii) follows from (i) because also the idempotents (3.1) in the two categories that define the locally induced modules coincide. □

The following is yet another result for which it is essential that the algebras are special Frobenius:

Proposition 6.11:
(i) For A and B special Frobenius algebras in (not necessarily Karoubian) ribbon categories $\mathcal{C}$ and $\mathcal{D}$, there is an equivalence

$$\mathcal{C}_A \boxtimes \mathcal{D}_B \;\cong\; (\mathcal{C}\boxtimes\mathcal{D})_{A\times B} \tag{6.30}$$

of categories.
(ii) If in addition A and B are centrally split, symmetric and commutative, then there is an equivalence

$$\mathcal{C}_A^{\ell oc} \boxtimes \mathcal{D}_B^{\ell oc} \;\cong\; (\mathcal{C}\boxtimes\mathcal{D})_{A\times B}^{\ell oc} \tag{6.31}$$

of ribbon categories.

Proof:
We combine the Lemmata 6.5, 6.9 and 6.10, Proposition 4.9 and corollary 4.10.
(i) We have

$$\begin{aligned}\mathcal{C}_A \boxtimes \mathcal{D}_B \;&\cong\; (\mathcal{C}_A)^{\mathrm{K}} \boxtimes (\mathcal{D}_B)^{\mathrm{K}} \;\cong\; (\mathcal{C}_A^{\mathrm{Ind}})^{\mathrm{K}} \boxtimes (\mathcal{D}_B^{\mathrm{Ind}})^{\mathrm{K}} \\ &\cong\; \mathcal{C}_A^{\mathrm{Ind}} \boxtimes \mathcal{D}_B^{\mathrm{Ind}} \;\equiv\; (\mathcal{C}_A^{\mathrm{Ind}} \otimes_{\Bbbk} \mathcal{D}_B^{\mathrm{Ind}})^{\mathrm{K}} \\ &\cong\; ((\mathcal{C}\otimes_{\Bbbk}\mathcal{D})_{A\times B}^{\mathrm{Ind}})^{\mathrm{K}} \;\cong\; ((\mathcal{C}\otimes_{\Bbbk}\mathcal{D})_{A\times B})^{\mathrm{K}} \;\cong\; (\mathcal{C}\boxtimes\mathcal{D})_{(A\times B;\, id_A \otimes_{\Bbbk} id_B)} \,,\end{aligned} \tag{6.32}$$

where in the first line we use first (6.9) and then (4.23), in the second line again (6.9), and in the last line (6.28), (4.23) and finally (6.26).
(ii) Analogously,

$$\begin{aligned}\mathcal{C}_A^{\ell oc} \boxtimes \mathcal{D}_B^{\ell oc} \;&\cong\; (\mathcal{C}_A^{\ell oc})^{\mathrm{K}} \boxtimes (\mathcal{D}_B^{\ell oc})^{\mathrm{K}} \;\cong\; (\mathcal{C}_A^{\ell\text{-Ind}})^{\mathrm{K}} \boxtimes (\mathcal{D}_B^{\ell\text{-Ind}})^{\mathrm{K}} \\ &\cong\; \mathcal{C}_A^{\ell\text{-Ind}} \boxtimes \mathcal{D}_B^{\ell\text{-Ind}} \;\equiv\; (\mathcal{C}_A^{\ell\text{-Ind}} \otimes_{\Bbbk} \mathcal{D}_B^{\ell\text{-Ind}})^{\mathrm{K}} \\ &\cong\; ((\mathcal{C}\otimes_{\Bbbk}\mathcal{D})_{A\times B}^{\ell\text{-Ind}})^{\mathrm{K}} \;\cong\; ((\mathcal{C}\otimes_{\Bbbk}\mathcal{D})_{A\times B}^{\ell oc})^{\mathrm{K}} \;\cong\; (\mathcal{C}\boxtimes\mathcal{D})_{A\times B}^{\ell oc} \,,\end{aligned} \tag{6.33}$$

where in the first line we use first (6.9) and then (4.26), in the second line again (6.9), and in the last line (6.29), (4.26) and finally (6.27).
Next we note that, by corollary 4.10, objects of $\mathcal{C}_A^{\ell oc} \boxtimes \mathcal{D}_B^{\ell oc}$ can be written as $((\ell\text{-Ind}_A(U);p)\times(\ell\text{-Ind}_B(X);q);\pi)$ with $U\in\mathrm{Obj}(\mathcal{C})$, $X\in\mathrm{Obj}(\mathcal{D})$, p and q the respective idempotents that describe a local module as module retract of a locally induced module, and π the idempotent that arises in taking the Karoubian envelope of $\mathcal{C}_A^{\ell oc}\otimes_{\Bbbk}\mathcal{D}_B^{\ell oc}$. Similarly, objects of $(\mathcal{C}\boxtimes\mathcal{D})_{A\times B}^{\ell oc}$ can be written as $(\ell\text{-Ind}_{A\times B}((V\times Y;\varpi));\hat\pi)$ with $V\in\mathrm{Obj}(\mathcal{C})$, $Y\in\mathrm{Obj}(\mathcal{D})$, ϖ the idempotent arising in taking the Karoubian envelope of $\mathcal{C}\otimes_{\Bbbk}\mathcal{D}$, and $\hat\pi$ the idempotent describing a local $A\times B$-module as module retract of a locally induced $A\times B$-module.
With this description of the objects, the functor $F\colon \mathcal{C}_A^{\ell oc}\boxtimes\mathcal{D}_B^{\ell oc} \xrightarrow{\cong} (\mathcal{C}\boxtimes\mathcal{D})_{A\times B}^{\ell oc}$ that maps the left hand side of (6.33) to the right hand side is given by

$$F:\quad ((\ell\text{-Ind}_A(U);p)\times(\ell\text{-Ind}_B(X);q);\pi) \;\mapsto\; (\ell\text{-Ind}_{A\times B}((U\times X;id_{U\times X}));\pi) \tag{6.34}$$

on objects, and is the identity map on morphisms, with the latter regarded as elements in (a subspace of) $\mathrm{Hom}^{\mathcal{C}\otimes_{\Bbbk}\mathcal{D}}((A{\otimes}U)\times(B{\otimes}X),(A{\otimes}V)\times(B{\otimes}Y))$. (That the idempotents p and q do not appear on the right hand side of (6.34) is seen by the same reasoning as in the proof of Lemma 6.5.)
Now one checks by inserting the relevant definitions – formula (6.4) for the tensor product on products of categories, formula (2.15) for the tensor product on the Karoubian envelope of a category, as well as formula (3.49) for the tensor product of local modules – that the prescription (6.34) respects the tensor product, i.e. $R\otimes^{\mathcal{C}_A^{\ell\mathrm{oc}}\boxtimes\mathcal{D}_B^{\ell\mathrm{oc}}}S\stackrel{F}{\mapsto}F(R)\otimes^{(\mathcal{C}\boxtimes\mathcal{D})_{A\times B}^{\ell\mathrm{oc}}}F(S)$ (together with an analogous equality for the tensor product of morphisms, which follows trivially). Thus F is a tensor functor.
Similarly, using the formulas (6.5) for the braiding on products of categories, (2.16) for the braiding on the Karoubian envelope, and (3.50) for the braiding of local modules, one verifies that the braidings on $\mathcal{C}_A^{\ell\mathrm{oc}}\boxtimes\mathcal{D}_B^{\ell\mathrm{oc}}$ and on $(\mathcal{C}\boxtimes\mathcal{D})_{A\times B}^{\ell\mathrm{oc}}$ are compatible in the sense that $c_{R,S}^{\mathcal{C}_A^{\ell\mathrm{oc}}\boxtimes\mathcal{D}_B^{\ell\mathrm{oc}}}=c_{F(R),F(S)}^{(\mathcal{C}\boxtimes\mathcal{D})_{A\times B}^{\ell\mathrm{oc}}}$. Since F is the identity on morphisms, this means that F is braided, and hence that F is a ribbon functor. □

Corollary 6.12:
If $\mathcal{C}$ and $\mathcal{D}$ are (not necessarily Karoubian) ribbon categories and A is a centrally split commutative symmetric special Frobenius algebra in $\mathcal{C}$, then there are equivalences

$$(\mathcal{C}\otimes_{\Bbbk}\mathcal{D})_{A\times\mathbf{1}_{\mathcal{D}}}^{\ell\text{-Ind}}\cong\mathcal{C}_A^{\ell\text{-Ind}}\otimes_{\Bbbk}\mathcal{D}\qquad\text{and}\qquad(\mathcal{C}\boxtimes\mathcal{D})_{A\times\mathbf{1}_{\mathcal{D}}}^{\ell\mathrm{oc}}\cong\mathcal{C}_A^{\ell\mathrm{oc}}\boxtimes\mathcal{D}\,. \tag{6.35}$$

The first is an equivalence of categories, the second an equivalence of ribbon categories.

Proof:
These equivalences follow by setting $B=\mathbf{1}_{\mathcal{D}}$ in the equivalences (6.29) and (6.31), respectively. □

Before we specialise to a special situation of particular interest – $\mathcal{C}$ a modular tensor category and $\mathcal{C}'$ being dual to $\mathcal{C}$ – let us mention that another large class of trivialisable pairs $\mathcal{C}$ and $\mathcal{C}'$ is provided by conformal embeddings similar to those listed in (1.16).

6.2 The dual of a tensor category

As already mentioned above, an important class of trivialisable categories is given by modular tensor categories, and for these $\mathcal{C}'$ is the dual of $\mathcal{C}$. We therefore turn to the discussion of the concept of dual tensor category.

Definition 6.13:
The *dual category* $\overline{\mathcal{C}}$ of a tensor category $(\mathcal{C},\otimes)$ is the tensor category $(\mathcal{C}^{\mathrm{opp}},\otimes)$.

More concretely, when marking quantities in $\overline{\mathcal{C}}$ by an overline, we have

$$\begin{array}{ll} \text{Objects :} & \mathrm{Obj}(\overline{\mathcal{C}}) = \mathrm{Obj}(\mathcal{C})\,, \text{ i.e. } \overline{U} \in \mathrm{Obj}(\overline{\mathcal{C}}) \text{ iff } U \in \mathrm{Obj}(\mathcal{C})\,, \\ \text{Morphisms :} & \overline{\mathrm{Hom}}(\overline{U},\overline{V}) = \mathrm{Hom}(V,U)\,, \\ \text{Composition :} & \overline{f}\,\overline{\circ}\,\overline{g} = \overline{g\circ f}\,, \\ \text{Tensor product :} & \overline{U}\,\overline{\otimes}\,\overline{V} = \overline{U\otimes V}\,, \quad \overline{f}\,\overline{\otimes}\,\overline{g} = \overline{f\otimes g}\,, \\ \text{Tensor unit :} & \overline{\mathbf{1}} = \mathbf{1}\,. \end{array} \tag{6.36}$$

Remark 6.14:
(i) Since $\mathcal{C}$ is strict, $\overline{\mathcal{C}}$ is indeed again a (strict) tensor category. If the tensor category $\mathcal{C}$ is small, then so is $\overline{\mathcal{C}}$. If $\mathcal{C}$ is additive, then so is $\overline{\mathcal{C}}$. If $\mathcal{C}$ is semisimple, then so is $\overline{\mathcal{C}}$.
(ii) If the tensor category $\mathcal{C}$ is Karoubian, then so is $\overline{\mathcal{C}}$. More generally, since the idempotents in $\mathcal{C}$ coincide with the idempotents in $\overline{\mathcal{C}}$, for any tensor category $\mathcal{C}$ the Karoubian envelope of $\overline{\mathcal{C}}$ is the dual category of the Karoubian envelope of $\mathcal{C}$, i.e. $\overline{\mathcal{C}}^{\mathrm{K}} = \overline{\mathcal{C}^{\mathrm{K}}}$.

The following result is analogous to lemma 2.9 of [34]:

Lemma 6.15:
(i) If the tensor category $\mathcal{C}$ has a left (right) duality, then its dual category $\overline{\mathcal{C}}$ has a right (left) duality. If $\mathcal{C}$ has a braiding, then so has $\overline{\mathcal{C}}$, and if $\mathcal{C}$ has a twist, then so has $\overline{\mathcal{C}}$.
In particular, the dual $\overline{\mathcal{C}}$ of a ribbon category $\mathcal{C}$ is naturally a ribbon category, too.
The values of s for $\mathcal{C}$ and $\overline{\mathcal{C}}$ are related via

$$\overline{s}_{\overline{U},\overline{V}} = s_{U,V^\vee} \quad (= s_{U^\vee,V})\,, \tag{6.37}$$

so that in particular

$$\overline{\dim}(\overline{U}) = \dim(U)\,. \tag{6.38}$$

(ii) The dual category $\overline{\mathcal{C}}$ of a modular tensor category $\mathcal{C}$ carries a natural structure of a modular tensor category.

Proof:
(i) We set

$$\overline{U}^\vee := \overline{{}^\vee U}\,, \qquad {}^\vee\overline{U} := \overline{U^\vee} \tag{6.39}$$

and

$$\begin{array}{ll} \text{Dualities :} & \overline{b}_{\overline{U}} := \overline{(\tilde{d}_U)} \in \overline{\mathrm{Hom}}(\overline{\mathbf{1}},\overline{U}\,\overline{\otimes}\,\overline{U}^\vee)\,, \quad \overline{d}_{\overline{U}} := \overline{(\tilde{b}_U)} \in \overline{\mathrm{Hom}}(\overline{U}^\vee\,\overline{\otimes}\,\overline{U},\overline{\mathbf{1}})\,, \\ & \overline{\tilde{b}}_{\overline{U}} := \overline{(d_U)} \in \overline{\mathrm{Hom}}(\overline{\mathbf{1}},{}^\vee\overline{U}\,\overline{\otimes}\,\overline{U})\,, \quad \overline{\tilde{d}}_{\overline{U}} := \overline{(b_U)} \in \overline{\mathrm{Hom}}(\overline{U}\,\overline{\otimes}\,{}^\vee\overline{U},\overline{\mathbf{1}})\,, \\ \text{Braiding :} & \overline{c}_{\overline{U},\overline{V}} := \overline{(c_{U,V})^{-1}} \in \overline{\mathrm{Hom}}(\overline{U}\,\overline{\otimes}\,\overline{V},\overline{V}\,\overline{\otimes}\,\overline{U})\,, \\ \text{Twist :} & \overline{\theta}_{\overline{U}} := \overline{(\theta_U^{-1})} \in \overline{\mathrm{Hom}}(\overline{U},\overline{U})\,. \end{array} \tag{6.40}$$

By direct substitution one verifies that these morphisms satisfy all properties of dualities, braiding and twist.
For s as defined by (2.5) one computes

$$\begin{aligned} \overline{s}_{\overline{U},\overline{V}} &= (\overline{d}_{\overline{V}} \,\overline{\otimes}\, \overline{\tilde{d}}_{\overline{U}}) \,\overline{\circ}\, [\, id_{\overline{V}^\vee} \,\overline{\otimes}\, (\overline{c}_{\overline{U},\overline{V}} \,\overline{\circ}\, \overline{c}_{\overline{V},\overline{U}}) \,\overline{\otimes}\, id_{\overline{U}^\vee} \,] \,\overline{\circ}\, (\overline{\tilde{b}}_{\overline{V}} \,\overline{\otimes}\, \overline{b}_{\overline{U}}) \\ &= (\overline{(\tilde{b}_V)} \,\overline{\otimes}\, \overline{(b_U)}) \,\overline{\circ}\, [\, id_{\overline{V^\vee}} \,\overline{\otimes}\, (\overline{(c_{U,V})^{-1}} \,\overline{\circ}\, \overline{(c_{V,U})^{-1}}) \,\overline{\otimes}\, id_{\overline{U^\vee}}] \,\overline{\circ}\, (\overline{(d_V)} \,\overline{\otimes}\, \overline{(\tilde{d}_U)}) \\ &= \overline{(d_V \otimes \tilde{d}_U) \circ [\, id_{U^\vee} \otimes ((c_{V,U})^{-1} \circ (c_{U,V})^{-1}) \otimes id_{V^\vee}] \circ (\tilde{b}_V \otimes b_U)} \\ &= s_{U,V^\vee} = s_{U^\vee,V}\,. \end{aligned} \tag{6.41}$$

The manipulations leading to the last two equalities may be summarised in the language of ribbon graphs, analogously as in (2.5): The second-to-last corresponds to a 180° rotation of the V-ribbon, and the last to a 180° rotation of the U-ribbon.

(ii) The simple objects of $\overline{\mathcal{C}}$ are $\overline{V}$ with V a simple object of $\mathcal{C}$; in particular, $\overline{\mathcal{C}}$ has as many isomorphism classes of simple objects as $\mathcal{C}$ has. Finally, owing to (6.37) invertibility of the matrix $\overline{s} \equiv (\overline{s}_{i,j})$ follows immediately from invertibility of s. □

Remark 6.16:
As in Remarks 3.23(i) and 6.2(iv) we may consider the behaviour of the dimension and charge of a modular tensor category. One verifies that under taking duals one has

$$\mathrm{Dim}(\overline{\mathcal{C}}) = \mathrm{Dim}(\mathcal{C}) \qquad \text{and} \qquad p^{\pm}(\overline{\mathcal{C}}) = p^{\mp}(\mathcal{C})\,. \tag{6.42}$$

Lemma 6.17:
(i) If (A, m, η) is an algebra in a tensor category $\mathcal{C}$, then $(\overline{A}, \overline{m}, \overline{\eta})$ is a coalgebra in $\overline{\mathcal{C}}$, and if (A, Δ, ε) is a coalgebra in $\mathcal{C}$, then $(\overline{A}, \overline{\Delta}, \overline{\varepsilon})$ is an algebra in $\overline{\mathcal{C}}$.

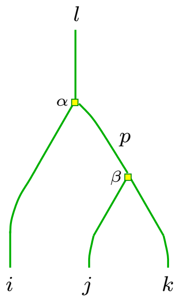
(ii) If $(A, m, \eta, \Delta, \varepsilon)$ is a (commutative) symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$, then $(\overline{A}, \overline{\Delta}, \overline{\varepsilon}, \overline{m}, \overline{\eta})$ is a (commutative) symmetric special Frobenius algebra in $\overline{\mathcal{C}}$.

Proof:
The relevant properties in the dual category are nothing but the corresponding properties of the dual morphisms. □

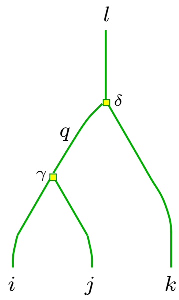
For the rest of this subsection we assume that $\mathcal{C}$ is a tensor category with a finite number of isomorphism classes of simple objects, i.e. that the index set $\mathcal{I}$ (see Section 2.1) is finite. Then for every triple of simple objects U_i, U_j, U_k with $i, j, k \in \mathcal{I}$ we fix once and for all a basis $\{\alpha\} \subset \mathrm{Hom}(U_i \otimes U_j, U_k)$ and a dual basis $\{\alpha\} \subset \mathrm{Hom}(U_k, U_i \otimes U_j)$. [6] Then the

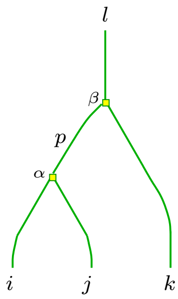
[6] See Section 2.2 of [18] for more details. There the notation $\bar{\alpha}$ was used for the second type of basis elements; here the overbar is suppressed to avoid confusion with quantities referring to the dual category $\overline{\mathcal{C}}$.

6j-symbols, or fusing matrices, F, of $\mathcal{C}$ and their inverses G are defined by (in the figures we abbreviate the simple objects U_i by their labels i)

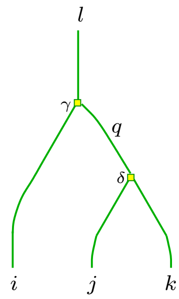


$$= \sum_{q\in\mathcal{I}} \sum_{\gamma,\delta} \mathsf{F}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta} \tag{6.43}$$

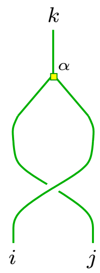




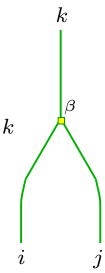
$$= \sum_{q,\gamma,\delta} \mathsf{G}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta} \tag{6.44}$$



Furthermore, when $\mathcal{C}$ is braided, then the braiding matrices R of $\mathcal{C}$ are defined by



$$=: \sum_{\beta} \mathsf{R}^{(i\,j)k}_{\alpha\,\beta} \tag{6.45}$$



$\mathsf{R}^{(i\,j)k}$ is a square matrix with rows and columns labelled by the basis $\{\alpha\}$ of $\mathrm{Hom}(U_i{\otimes}U_j, U_k)$; its inverse with respect to this matrix structure is $\mathsf{R}^{-\,(j\,i)k}$, which is defined analogously as $\mathsf{R}^{(j\,i)k}$, but with an under-braiding instead of an over-braiding.

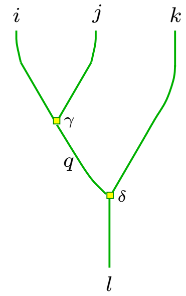
The choice of bases in the spaces $\mathrm{Hom}(U_i{\otimes}U_j, U_k)$ and $\mathrm{Hom}(U_k, U_i{\otimes}U_j)$ of $\mathcal{C}$ allow us to choose a correlated basis in $\overline{\mathcal{C}}$. For example to pick a basis $\{\overline{\alpha}\} \subset \overline{\mathrm{Hom}}(\overline{U}_i \,\overline{\otimes}\, \overline{U}_j, \overline{U}_k)$ we use that by definition $\overline{\mathrm{Hom}}(\overline{U}_i \,\overline{\otimes}\, \overline{U}_j, \overline{U}_k) = \mathrm{Hom}(U_k, U_i{\otimes}U_j)$ and take the basis we have already chosen in the latter.

To simplify notation, in the remainder of the paper we will omit the overlines on quantities of the dual category $\overline{\mathcal{C}}$ whenever from the context it is so obvious that $\overline{\mathcal{C}}$-quantities are meant that no confusion can arise. For instance, we write the fusing matrices of $\overline{\mathcal{C}}$ as $\overline{\mathsf{F}}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta}$ instead of $\overline{\mathsf{F}}^{(\overline{i}\,\overline{j}\,\overline{k})\,\overline{l}}_{\overline{\alpha}\overline{p}\overline{\beta},\overline{\gamma}\overline{q}\overline{\delta}}$.

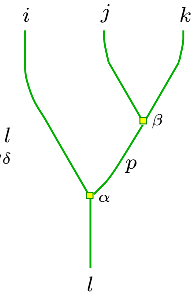
Lemma 6.18:
The fusing and braiding matrices of the dual $\overline{\mathcal{C}}$ of a braided tensor category $\mathcal{C}$ with finite index set $\mathcal{I}$ are given by

$$\overline{\mathsf{F}}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta} = \mathsf{G}^{(i\,j\,k)\,l}_{\gamma q\delta,\alpha p\beta}\,, \quad \overline{\mathsf{G}}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta} = \mathsf{F}^{(i\,j\,k)\,l}_{\gamma q\delta,\alpha p\beta}\,, \quad \overline{\mathsf{R}}^{(i\,j)k}_{\alpha\,\beta} = \mathsf{R}^{-\,(j\,i)k}_{\beta\,\alpha}\,, \quad \overline{\mathsf{R}}^{-\,(i\,j)k}_{\alpha\,\beta} = \mathsf{R}^{(j\,i)k}_{\beta\,\alpha}\,. \qquad (6.46)$$

Proof:
It follows from the definition of dual bases that the fusing matrices also appear in the relation



$$= \sum_{p\in\mathcal{I}} \sum_{\alpha,\beta} \mathsf{F}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta} \qquad (6.47)$$



Combining this result for the category $\mathcal{C}$ with the definition of the morphisms $\overline{\mathrm{Hom}}$ and their composition $\overline{\circ}$ in $\overline{\mathcal{C}}$ one arrives at the first equality. The other relations follow by an analogous reasoning. □

6.3 The trivialising algebra $T_\mathcal{G}$

Recall that we denote by $\mathcal{I}$ the index set such that $\{U_i \,|\, i \in \mathcal{I}\}$ is a collection of representatives for the equivalence classes of simple objects in a category. In this subsection we consider ribbon categories $\mathcal{G}$ which are semisimple and have finite index set $\mathcal{I}_\mathcal{G}$.

We start by introducing an interesting algebra $T \equiv T_\mathcal{G}$ in the Karoubian product $\mathcal{G}\boxtimes\overline{\mathcal{G}}$ of $\mathcal{G}$ with its dual. This is done in the following lemma, which is essentially Proposition 4.1 of [35]:

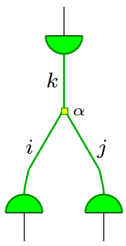
Lemma 6.19:
Let $\mathcal{G}$ be a semisimple ribbon category with a finite number of equivalence classes of simple objects.

(i) The triple $T_\mathcal{G} \equiv (T_\mathcal{G}, m, \eta)$ with

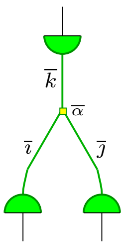
$$T_\mathcal{G} := \bigoplus_{k\in\mathcal{I}_\mathcal{G}} U_k \times \overline{U_k} \;\in \mathrm{Obj}(\mathcal{G}\boxtimes\overline{\mathcal{G}})\,,$$

$$\eta := e_{\mathbf{1}\times\overline{\mathbf{1}}\prec T_\mathcal{G}} \;\in \mathrm{Hom}^{\mathcal{G}\boxtimes\overline{\mathcal{G}}}(\mathbf{1}\times\overline{\mathbf{1}}, T_\mathcal{G})\,,$$

$$m := \sum_{i,j,k\in\mathcal{I}_\mathcal{G}} \sum_{\alpha} \qquad (6.48)$$



$\otimes_\Bbbk$



$\in \mathrm{Hom}^{\mathcal{G}\boxtimes\overline{\mathcal{G}}}(T_\mathcal{G}\otimes T_\mathcal{G}, T_\mathcal{G})$

is an algebra in $\mathcal{G}\boxtimes\overline{\mathcal{G}}$.

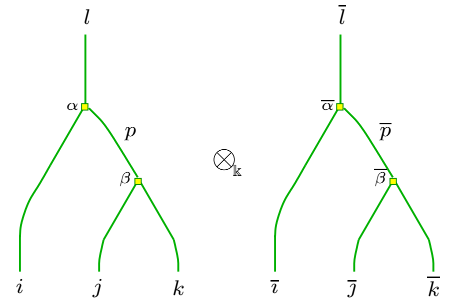
(ii) The algebra $(T_{\mathcal{G}}, m, \eta)$ extends to a haploid commutative symmetric special Frobenius algebra in $\mathcal{G}\boxtimes\overline{\mathcal{G}}$.

Proof:

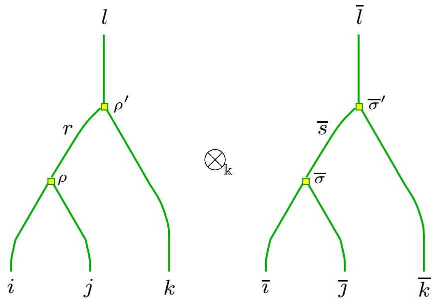
(i) The unit property of the multiplication m follows from the normalisation of the morphisms that was chosen in (2.33) of [18], which states that the basis vector chosen in $\mathrm{Hom}(U_i\otimes\mathbf{1}, U_i)$ and $\mathrm{Hom}(\mathbf{1}\otimes U_i, U_i)$ is id_{U_i}.

To see associativity one notes that

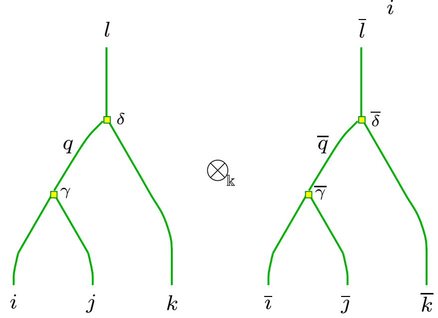
$$\sum_{p,\alpha,\beta}$$



$$= \sum_{r,\rho,\rho'} \sum_{s,\sigma,\sigma'} \sum_{p,\alpha,\beta} \mathsf{F}^{(i\,j\,k)\,l}_{\alpha p\beta,\rho r\rho'} \, \overline{\mathsf{F}}^{(i\,j\,k)\,l}_{\alpha p\beta,\sigma s\sigma'} \qquad (6.49)$$



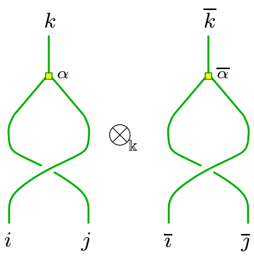
$$= \sum_{q,\gamma,\delta}$$



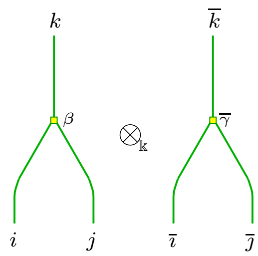
The second step uses Lemma 6.18 to relate $\overline{\mathsf{F}}$ to the inverse of F.

(ii) Thus $T_{\mathcal{G}}$ is an algebra. It is clearly haploid. Commutativity follows from

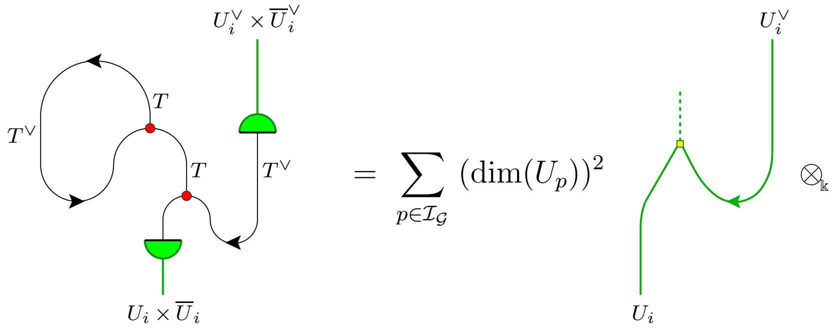
$$\sum_{\alpha}$$



$$= \sum_{\beta,\gamma} \sum_{\alpha} \mathsf{R}^{(i\,j)k}_{\alpha\,\beta} \, \overline{\mathsf{R}}^{(i\,j)k}_{\alpha\,\gamma} \qquad (6.50)$$



together with Lemma 6.18.
To show that $T_\mathcal{G}$ extends to a symmetric special Frobenius algebra, by Remark 2.23(iv) it is sufficient to verify that the morphism $\Phi_{1,\natural}$, which was defined after (2.39), is invertible. Now for every $i \in \mathcal{I}_\mathcal{G}$ we have



(6.51)

because only the tensor unit of $\mathcal{G}\boxtimes\overline{\mathcal{G}}$ contributes in the $T_\mathcal{G}$-ribbon that is connected to the $T_\mathcal{G}$-loop and the resulting isolated $T_\mathcal{G}$-loop amounts to a factor $\dim(T_\mathcal{G})$. Substituting the definition of m then gives the right hand side of (6.51). Since the morphism on the right hand side is invertible for every $i \in \mathcal{I}_\mathcal{G}$, so is $\Phi_{1,\natural}$. $\square$

Lemma 6.20 :
With $T_\mathcal{G}$ defined by (6.48), we have:
(i) The induced $T_\mathcal{G}$-modules

$$M_k := \mathrm{Ind}_{T_\mathcal{G}}(\mathbf{1} \times \overline{U}_k) \tag{6.52}$$

$(k \in \mathcal{I}_\mathcal{G})$ are mutually distinct and simple.
(ii) The induced modules $\mathrm{Ind}_{T_\mathcal{G}}(U_k \times \overline{U}_l)$ decompose into a direct sum of simple $T_\mathcal{G}$-modules according to

$$\mathrm{Ind}_{T_\mathcal{G}}(U_k \times \overline{U}_l) \;\cong\; \bigoplus_{r\in\mathcal{I}_\mathcal{G}} N_{kr}^{\;\;l}\, M_r \,, \tag{6.53}$$

with $N_{ij}^{\;\;k}$ the dimension of $\mathrm{Hom}(U_i \otimes U_j, U_k)$, as introduced in (2.7).
Proof:
(i) Since $\mathcal{G}$ is semisimple, $\mathcal{G}\boxtimes\overline{\mathcal{G}}$ is semisimple as well, and hence the object $\dot{M}_k$ underlying induced module M_k is a direct sum of simple objects of $\mathcal{G}\boxtimes\overline{\mathcal{G}}$. The decomposition into simple objects reads

$$\dot{M}_k = T_\mathcal{G} \otimes (\mathbf{1} \times \overline{U}_k) \;\cong\; \bigoplus_{r,s\in\mathcal{I}_\mathcal{G}} N_{rk}^{\;\;s}\, U_r \times \overline{U}_s \,, \tag{6.54}$$

with $N_{ij}^{\;\;k} = \dim \mathrm{Hom}(U_i \otimes U_j, U_k)$. When combined with the reciprocity relation (2.40), this implies

$$\begin{aligned} \mathrm{Hom}_{T_\mathcal{G}}(M_k, M_l) \;&\cong \textstyle\bigoplus_{r,s\in\mathcal{I}_\mathcal{G}} \mathrm{Hom}^\mathcal{G}(U_r \otimes U_l, U_s) \otimes \mathrm{Hom}^{\mathcal{G}\boxtimes\overline{\mathcal{G}}}(\mathbf{1} \times \overline{U}_k, U_r \times \overline{U}_s) \\ &\cong \mathrm{Hom}^\mathcal{G}(U_l, U_k) \;\cong\; \delta_{k,l}\, \Bbbk \,, \end{aligned} \tag{6.55}$$

which proves the claim.

(ii) We first check that the simple modules M_r appear in $\mathrm{Ind}_{T_\mathcal{G}}(U_k{\times}\overline{U}_l)$ with multiplicity ${N_{kr}}^l$. To this end we we use again reciprocity:

$$\mathrm{Hom}_{T_\mathcal{G}}^{\mathcal{G}\boxtimes\overline{\mathcal{G}}}(M_r,\mathrm{Ind}_{T_\mathcal{G}}(U_r{\times}\overline{U}_l)) \;\cong\; \mathrm{Hom}^{\mathcal{G}\boxtimes\overline{\mathcal{G}}}(\dot{M}_r,U_k{\times}\overline{U}_l) \;\cong\; \Bbbk^{{N_{kr}}^l} . \tag{6.56}$$

The last equality follows from the decomposition of $\dot{M}_r$ into simple objects given in (6.54). We now know that the right hand side of (6.53) is a submodule of $\mathrm{Ind}_{T_\mathcal{G}}(U_k{\times}\overline{U}_l)$. Next we check that $\mathrm{Ind}_{T_\mathcal{G}}(U_k{\times}\overline{U}_l)$ does not contain any further submodules. It is sufficient to verify that (6.53) is correct as a relation for objects in $\mathcal{G}\boxtimes\overline{\mathcal{G}}$. For the two sides of (6.53) we find

$$\begin{aligned} \mathrm{Ind}_{T_\mathcal{G}}(U_k{\times}\overline{U}_l) \;&\cong\; \bigoplus_{r,u,v\in\mathcal{I}_\mathcal{G}} {N_{rk}}^u {N_{rl}}^v\, U_u\times\overline{U}_v \qquad \text{and} \\ \bigoplus_{r\in\mathcal{I}_\mathcal{G}} {N_{kr}}^l M_r \;&\cong\; \bigoplus_{r,u,v\in\mathcal{I}_\mathcal{G}} {N_{kr}}^l {N_{ur}}^v\, U_u\times\overline{U}_v\,, \end{aligned} \tag{6.57}$$

respectively. Using the identities ${N_{rk}}^u = {N_{u\bar{k}}}^r$ and ${N_{kr}}^l = {N_{\bar{k}l}}^r$, we see that the two expressions coincide owing to associativity of the tensor product. □

6.4 Modularity implies trivialisability

We will now apply some of the results above in the particular case that the tensor category under consideration is even modular. We are going to show that such categories are trivialisable, with the compensating category given by the dual and the trivialising algebra of the form given in Lemma 6.19.

In this subsection $\mathcal{G}$ always denotes a modular tensor category. As a preparation we need

Lemma 6.21:
(i) Let U_k be a simple object in a modular tensor category $\mathcal{C}$. If the relation $\theta_s/(\theta_k\theta_r)=1$ holds for all simple objects U_r, U_s $(r,s\in\mathcal{I})$ such that ${N_{rk}}^s\neq 0$, then $U_k=\mathbf{1}$.

(ii) Conversely, let $\mathcal{C}$ be a semisimple additive ribbon category with ground field $\Bbbk$ and with finite index set $\mathcal{I}$. If the equality $\theta_s/\theta_k\theta_r=1$ for all $r,s\in\mathcal{I}$ such that ${N_{rk}}^s\neq 0$ implies that $k=0$, then $\mathcal{C}$ is modular.

Proof:
(i) Fix a basis $\{\lambda^s_{kr,\alpha}\}\subset\mathrm{Hom}(U_k{\otimes}U_r,U_s)$. Then one has

$$\lambda^s_{kr,\alpha}\circ c_{r,k}\circ c_{k,r} = \frac{\theta_s}{\theta_k\theta_r}\,\lambda^s_{kr,\alpha} \tag{6.58}$$

(see e.g. Section 2.2 of [18] for more details). By assumption, all the factors $\theta_s/(\theta_k\theta_r)$ in this expression are equal to one. Since s and α run over a basis, this implies that

$$c_{r,k}\circ c_{k,r} = id_{U_k\otimes U_r} \tag{6.59}$$

for all $r \in \mathcal{I}$. Taking the trace of this formula yields $s_{r,k} = s_{k,0} s_{r,0}$. Thus the kth column of the s-matrix (2.8) is proportional to the **1**-column, with a factor of proportionality equal to $s_{k,0}$. Since the s-matrix is invertible, this is only possible if $k = 0$.

(ii) The same calculations show that the conditions are equivalent to the statement that the equality $c_{U_r,U_k} c_{U_k,U_r} = id_{U_k \otimes U_r}$ for all $r \in \mathcal{I}$ implies that $k = 0$. Taking the trace, we learn that $k = 0$ is the only element of $\mathcal{I}$ such that $s_{U_r,U_k} = \dim(U_k)\dim(U_r)$ for all $r \in \mathcal{I}$. According to Proposition 1.1 of [9], this property in turn implies that the ribbon category $\mathcal{C}$ is modular. □

Lemma 6.22:

For $\mathcal{G}$ a modular tensor category and $T_\mathcal{G}$ as defined in lemma 6.19, up to isomorphism the only local simple $T_\mathcal{G}$-module is $M_\mathbf{1} = T_\mathcal{G}$ itself.

Proof:

By corollary 3.18 it is enough to compute the twist on the simple modules M_k and check whether it is of the form $\xi_k id_{M_k}$ for some $\xi_k \in \Bbbk$. Since $\mathbf{1} \times \overline{U}_k$ is always a subobject of M_k, if it exists ξ_k must be equal to θ_k^{-1}. Evaluating the twist for all other subobjects of M_k we find the following condition: M_k is local iff $\theta_r \theta_s^{-1} = \theta_k^{-1}$ for all r, s such that ${N_{rk}}^s \neq 0$. By Lemma 6.21 this implies that $k = 0$. □

Proposition 6.23:

For $\mathcal{G}$ a modular tensor category and $T_\mathcal{G}$ as defined in lemma 6.19, there is an equivalence

$$(\mathcal{G} \boxtimes \overline{\mathcal{G}})_{T_\mathcal{G}}^{\ell oc} \cong \mathcal{V}ect_\Bbbk \tag{6.60}$$

of modular tensor categories.

Proof:

Combining the Lemmata 6.20 − 6.22 above, we conclude that $(\mathcal{G} \boxtimes \overline{\mathcal{G}})_{T_\mathcal{G}}^{\ell oc}$ is a modular tensor category that, up to isomorphism, has the tensor unit **1** as its single simple object. Any such category is equivalent to $\mathcal{V}ect_\Bbbk$.

7 Correspondences of tensor categories

7.1 Ribbon categories

We are now finally in a position to establish correspondences between certain ribbon categories $\mathcal{Q}$ and $\mathcal{G}$. They make use of another ribbon category $\mathcal{H}$, which must be trivialisable. The strongest result, to be derived in Section 7.2, is obtained when $\mathcal{H}$ is even a modular tensor category. In the present subsection, this special property of $\mathcal{H}$ is not required. Also, $\mathcal{Q}$ and $\mathcal{H}$ are not assumed to be Karoubian. Given $\mathcal{Q}$ and $\mathcal{H}$, we consider a ribbon category $\mathcal{G}$ that is obtained as the category of local modules over a suitable algebra L in the Karoubian product of $\mathcal{Q}$ and $\mathcal{H}$.

Proposition 7.1:
Let $\mathcal{Q}$ be a ribbon category, $\mathcal{H}$ a trivialisable ribbon category, with trivialisation data $\mathcal{H}'$ and T, and let L be a haploid commutative symmetric special Frobenius algebra in the category $\mathcal{Q}\boxtimes\mathcal{H}$ satisfying $\dim_{\Bbbk} \mathrm{Hom}(\mathbf{1}_{\mathcal{Q}}{\times}T, L{\times}\mathbf{1}_{\mathcal{H}'}) = 1$. Denote by $\mathcal{G}$ the ribbon category of local L-modules,

$$\mathcal{G} := (\mathcal{Q}\boxtimes\mathcal{H})_L^{\ell oc} . \tag{7.1}$$

Further, let Υ be the object

$$\Upsilon := \ell\text{-Ind}_{L\times\mathbf{1}_{\mathcal{H}'}}(\mathbf{1}_{\mathcal{Q}}{\times}T) \tag{7.2}$$

in $\mathcal{G}\boxtimes\mathcal{H}'$, endowed with the structure of Frobenius algebra in $\mathcal{G}\boxtimes\mathcal{H}'$ via the prescription given in the proof of Proposition 4.14; similarly, let Γ be the Frobenius algebra

$$\Gamma := \ell\text{-Ind}_{\mathbf{1}_{\mathcal{Q}}\times T}(L{\times}\mathbf{1}_{\mathcal{H}'}) \tag{7.3}$$

in $(\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\mathcal{H}')^{\ell oc}_{\mathbf{1}_{\mathcal{Q}}\times T}$. We have

$$\mathcal{Q}^{\mathrm{K}} \cong (\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\mathcal{H}')^{\ell oc}_{\mathbf{1}_{\mathcal{Q}}\times T} . \tag{7.4}$$

Furthermore, if Υ and Γ have non-zero dimension, then they are haploid commutative symmetric special Frobenius algebras, and there is an equivalence

$$(\mathcal{Q}^{\mathrm{K}})^{\ell oc}_{\Gamma} \;\cong\; (\mathcal{G}\boxtimes\mathcal{H}')^{\ell oc}_{\Upsilon} \tag{7.5}$$

of (Karoubian) ribbon categories.

Proof:
(i) To verify the equivalence (7.4), we first apply Lemma 6.7, then the fact that, by assumption, $\mathcal{H}'$ and T provide a trivialisation for $\mathcal{H}$, and then corollary 6.12:

$$\mathcal{Q}^{\mathrm{K}} \;\cong\; \mathcal{Q}\boxtimes\mathcal{V}ect_{\Bbbk} \;\cong\; \mathcal{Q}\boxtimes(\mathcal{H}\boxtimes\mathcal{H}')^{\ell oc}_T \;\cong\; (\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\mathcal{H}')^{\ell oc}_{\mathbf{1}_{\mathcal{Q}}\times T} . \tag{7.6}$$

(ii) That Υ and Γ are haploid commutative symmetric special Frobenius algebras can be seen by combining Proposition 3.8 and corollary 4.15 as well as Proposition 4.14(ii). Note in particular that we can apply Proposition 3.8(iii), because both L and T are symmetric and special, the dimensions of Υ and Γ are non-vanishing, and the condition on the centers is implied by $\dim_{\Bbbk} \mathrm{Hom}(\mathbf{1}_{\mathcal{Q}}{\times}T, L{\times}\mathbf{1}_{\mathcal{H}'}) = 1$ together with the commutativity of L and T.

(iii) For the next two preparatory calculations, we invoke successively Proposition 4.16, corollary 6.12 and the definition (7.1) of $\mathcal{G}$ (as well as the associativity of the Karoubian product $\boxtimes$ from Remark 6.6) to write

$$\begin{aligned} (\mathcal{Q}{\boxtimes}\mathcal{H}{\boxtimes}\mathcal{H}')^{\ell oc}_{E_{L\times\mathbf{1}_{\mathcal{H}'}}(\mathbf{1}_{\mathcal{Q}}\times T)} &\cong \big(((\mathcal{Q}{\boxtimes}\mathcal{H}){\boxtimes}\mathcal{H}')^{\ell oc}_{L\times\mathbf{1}_{\mathcal{H}'}}\big)^{\ell oc}_{\ell\text{-Ind}_{L\times\mathbf{1}_{\mathcal{H}'}}(\mathbf{1}_{\mathcal{Q}}\times T)} \\ &\cong \big((\mathcal{Q}{\boxtimes}\mathcal{H})^{\ell oc}_{L} \boxtimes \mathcal{H}'\big)^{\ell oc}_{\ell\text{-Ind}_{L\times\mathbf{1}_{\mathcal{H}'}}(\mathbf{1}_{\mathcal{Q}}\times T)} \\ &\cong (\mathcal{G}{\boxtimes}\mathcal{H}')^{\ell oc}_{\ell\text{-Ind}_{L\times\mathbf{1}_{\mathcal{H}'}}(\mathbf{1}_{\mathcal{Q}}\times T)} \end{aligned} \tag{7.7}$$

and similarly, using (7.6) in the second step,

$$\begin{aligned} (\mathcal{Q}{\boxtimes}\mathcal{H}{\boxtimes}\mathcal{H}')^{\ell oc}_{E_{\mathbf{1}_{\mathcal{Q}}\times T}(L\times\mathbf{1}_{\mathcal{H}'})} &\cong \big((\mathcal{Q}{\boxtimes}(\mathcal{H}{\boxtimes}\mathcal{H}'))^{\ell oc}_{\mathbf{1}_{\mathcal{Q}}\times T}\big)^{\ell oc}_{\ell\text{-Ind}_{\mathbf{1}_{\mathcal{Q}}\times T}(L\times\mathbf{1}_{\mathcal{H}'})} \\ &\cong (\mathcal{Q}^{\mathrm{K}})^{\ell oc}_{\ell\text{-Ind}_{\mathbf{1}_{\mathcal{Q}}\times T}(L\times\mathbf{1}_{\mathcal{H}'})} \,. \end{aligned} \tag{7.8}$$

(Recall from Lemma 3.24(i) that the category of local modules over any commutative symmetric special Frobenius algebra in a Karoubian ribbon category is again Karoubian. Thus all the module categories appearing here are Karoubian.)

(iv) Consider now the tensor product algebra

$$F := (\mathbf{1}_{\mathcal{Q}}{\times}T) \otimes (L{\times}\mathbf{1}_{\mathcal{H}'}) \tag{7.9}$$

in $\mathcal{Q}{\boxtimes}\mathcal{H}{\boxtimes}\mathcal{H}'$. Recall that in a braided setting the tensor product of two commutative algebras is not commutative, in general. Concretely, applying Proposition 3.14 we learn that the left and right centers of F are

$$C_l(F) \cong E_{\mathbf{1}_{\mathcal{Q}}\times T}(L{\times}\mathbf{1}_{\mathcal{H}'}) \qquad \text{and} \qquad C_r(F) \cong E_{L\times\mathbf{1}_{\mathcal{H}'}}(\mathbf{1}_{\mathcal{Q}}{\times}T)\,, \tag{7.10}$$

respectively. Further, by Theorem 5.20 the categories of local $C_l(F)$- and local $C_r(F)$-modules are equivalent,

$$(\mathcal{Q}{\boxtimes}\mathcal{H}{\boxtimes}\mathcal{H}')^{\ell oc}_{C_l(F)} \cong (\mathcal{Q}{\boxtimes}\mathcal{H}{\boxtimes}\mathcal{H}')^{\ell oc}_{C_r(F)} \,. \tag{7.11}$$

Combining this information with the results in step (iii) and (7.10), we finally obtain

$$(\mathcal{G}{\boxtimes}\mathcal{H}')^{\ell oc}_{\ell\text{-Ind}_{L\times\mathbf{1}_{\mathcal{H}'}}(\mathbf{1}_{\mathcal{Q}}\times T)} \cong (\mathcal{Q}^{\mathrm{K}})^{\ell oc}_{\ell\text{-Ind}_{\mathbf{1}_{\mathcal{Q}}\times T}(L\times\mathbf{1}_{\mathcal{H}'})} \,, \tag{7.12}$$

thus establishing the equivalence (7.5). This is a ribbon equivalence because all the intermediate equivalences we used are ribbon. □

7.2 Modular tensor categories

It is desirable to find also a description of the category $\mathcal{Q}^{\rm K}$ itself, not just of some module category over $\mathcal{Q}^{\rm K}$, in terms of $\mathcal{G}$ and $\mathcal{H}'$. As it turns out, this can be achieved if we assume that $\mathcal{H}$ is *modular* such that it has a trivialisation of the form described in Proposition 6.23, i.e.

$$\mathcal{H}' = \overline{\mathcal{H}} \qquad \text{and} \qquad T = T_{\mathcal{H}} \tag{7.13}$$

with $T_{\mathcal{H}}$ as given in Lemma 6.19. In addition, also one further condition on the algebra L and one further condition on the category $\mathcal{Q}$ must be imposed; these properties are the following.

Definition 7.2:
An algebra A in the Karoubian product $\mathcal{C}\boxtimes\mathcal{D}$ of two tensor categories $\mathcal{C}$ and $\mathcal{D}$ is called $\mathcal{C}$*-haploid* iff

$$\mathrm{Obj}(\mathcal{C}\boxtimes\mathcal{D}) \ni U \times \mathbf{1}_{\mathcal{D}} \prec A \ \Rightarrow \ U \cong \mathbf{1}_{\mathcal{C}}\,, \tag{7.14}$$

i.e. iff up to isomorphism the only retract of A of the form $U{\times}\mathbf{1}_{\mathcal{D}}$ is $\mathbf{1}_{\mathcal{C}}{\times}\mathbf{1}_{\mathcal{D}}$.

Definition 7.3:
A sovereign tensor category $\mathcal{C}$ is called *separable* if every idempotent p with $\mathrm{tr}(p)=0$ is the zero morphism.

Remark 7.4:
(i) It follows from Remark 2.23(vi) that if $\dim_{\Bbbk}\mathrm{Hom}(\mathbf{1},A)=d$ for a Frobenius algebra A in $\mathcal{C}\boxtimes\mathcal{D}$, then $I_{\mathcal{C}}^{(d)}\times\mathbf{1}_{\mathcal{D}}$ with $I_{\mathcal{C}}^{(d)}=\mathbf{1}_{\mathcal{C}}{\oplus}\mathbf{1}_{\mathcal{C}}{\oplus}\cdots{\oplus}\mathbf{1}_{\mathcal{C}}$ (d summands) is a retract of A, and hence in particular A is not $\mathcal{C}$-haploid. Conversely, if A is $\mathcal{C}$-haploid, then it is in particular haploid.

Also, when $\mathcal{C}\cong\mathcal{V}ect_{\Bbbk}$, for Frobenius algebras the notions of haploidity in $\mathcal{D}$ and of $\mathcal{C}$-haploidity coincide upon identifying $\mathcal{C}\boxtimes\mathcal{D}$ with $\mathcal{D}$. This is the reason for the choice of terminology.

(ii) Since every idempotent in the Karoubian envelope $\mathcal{C}^{\rm K}$ of a sovereign tensor category $\mathcal{C}$ is also an idempotent in $\mathcal{C}$, separability of $\mathcal{C}$ implies separability of $\mathcal{C}^{\rm K}$; owing to the functorial embedding $\mathcal{C}\to\mathcal{C}^{\rm K}$, the converse holds true, too. Also, if $\mathcal{C}$ is separable, then so is its dual $\overline{\mathcal{C}}$.
If $\mathcal{C}$ and $\mathcal{D}$ are sovereign tensor categories such that their product $\mathcal{C}\otimes_{\Bbbk}\mathcal{D}$ (or $\mathcal{C}\boxtimes\mathcal{D}$) is separable, then already $\mathcal{C}$ and $\mathcal{D}$ are separable.
Furthermore, since, for A an algebra in a sovereign tensor category $\mathcal{C}$, every idempotent in $\mathcal{C}_A$ is also an idempotent in $\mathcal{C}$, separability of $\mathcal{C}$ implies separability of $\mathcal{C}_A$. By the same argument, the category $\mathcal{C}_A^{\ell oc}$ of local modules over a commutative symmetric special Frobenius algebra A in a separable ribbon category $\mathcal{C}$ is separable.
Modular categories are in particular separable.

The proof of the stronger result involving modular tensor categories relies also on the following

Lemma 7.5:
Let S, S' be two retracts of an object U in a (not necessarily Karoubian) separable sovereign tensor category $\mathcal{C}$. Suppose that the corresponding split idempotents satisfy $P_S P_{S'} = P_{S'} P_S$ and $\mathrm{tr}_U(P_S) = \mathrm{tr}_U(P_S P_{S'}) = \mathrm{tr}_U(P_{S'})$. Then $P_S = P_{S'}$ and $S \cong S'$ as retracts.

Proof:
We write $S = (S, e, r)$ and $S' = (S', e', r')$, and consider the morphisms $f \in \mathrm{Hom}(S, S')$ and $g \in \mathrm{Hom}(S', S)$ given by $f := r' \circ e$ and $g := r \circ e'$. Using the assumptions we see that $p := g \circ f$ satisfies $p \circ p = r \circ P_{S'} \circ P_S \circ P_{S'} \circ e = r \circ P_{S'} \circ e = p$, i.e. p is an idempotent. Further we have

$$\mathrm{tr}_S\, p = \mathrm{tr}_U(P_S P_{S'}) = \mathrm{tr}_U P_S = \dim(S)\,. \tag{7.15}$$

It follows that $\mathrm{tr}_S(id_S - p) = 0$. By separability this implies that $id_S - p = 0$ so that $p = id_S$. In the same way one shows that $f \circ g = id_{S'}$. Thus S and S' are isomorphic as objects. From $id_S = g \circ f = r \circ P_{S'} \circ e$ we deduce (composing with e from the left) that $e = P_{S'} \circ e = e' \circ f$ and (composing with r from the right) that $r = r \circ P_{S'} = g \circ r'$. The relation $e = e' \circ f$ implies that S and S' are isomorphic as subobjects, and $P_S = e \circ r = e' \circ f \circ g \circ r' = P_{S'}$ shows that they are isomorphic even as retracts. □

Having these ingredients at hand,[7] we can formulate a much stronger result than the one of Proposition 7.1:

Theorem 7.6:
Let $\mathcal{Q}$ be a (not necessarily Karoubian) ribbon category and $\mathcal{H}$ a modular tensor category (with trivialisation data $\overline{\mathcal{H}}$, $T \equiv T_{\mathcal{H}}$) such that the product $\mathcal{Q} \boxtimes \mathcal{H} \boxtimes \overline{\mathcal{H}}$ is separable, and let L be a $\mathcal{Q}$-haploid commutative symmetric special Frobenius algebra in the Karoubian product $\mathcal{Q} \boxtimes \mathcal{H}$.
(i) The Frobenius algebra

$$L' := \ell\text{-Ind}_{L \times \mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_{\mathcal{Q}} \times T) \tag{7.16}$$

is haploid, commutative, symmetric and special, and there is an equivalence

$$\mathcal{Q}^{\mathrm{K}} \cong (\mathcal{G} \boxtimes \overline{\mathcal{H}})_{L'}^{\ell oc} \tag{7.17}$$

of ribbon categories, with $\mathcal{G} = (\mathcal{Q} \boxtimes \mathcal{H})_L^{\ell oc}$.
(ii) The Frobenius algebra L' in $\mathcal{G} \boxtimes \overline{\mathcal{H}}$ is even $\mathcal{G}$-haploid.

Proof of (i):
1) We start by checking that the conditions of Proposition 7.1 are fulfilled. Note that

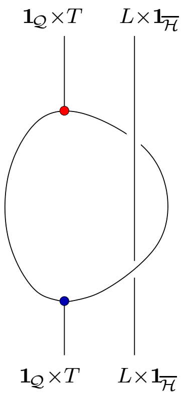
$$\begin{aligned} \dim_{\Bbbk} \mathrm{Hom}(\mathbf{1}_{\mathcal{Q}} \times T_{\mathcal{H}}, L \times \mathbf{1}_{\overline{\mathcal{H}}}) &= \sum_{k \in \mathcal{I}_{\mathcal{H}}} \dim_{\Bbbk} \mathrm{Hom}(\mathbf{1}_{\mathcal{Q}} \times U_k \times \overline{U_k}, L \times \mathbf{1}_{\overline{\mathcal{H}}}) \\ &= \dim_{\Bbbk} \mathrm{Hom}(\mathbf{1}_{\mathcal{Q}} \times \mathbf{1}_{\mathcal{H}}, L) = 1\,, \end{aligned} \tag{7.18}$$

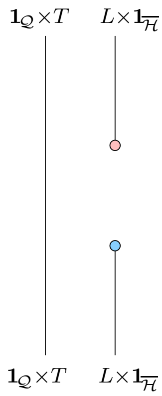
[7] Recall also declarations 2.10 and 3.2.

since L is in particular haploid, by Remark 7.4(i). Next we need to show that the algebras $\varUpsilon \in \mathrm{Obj}(\mathcal{G}\boxtimes\overline{\mathcal{H}})$ and $\varGamma \in \mathrm{Obj}(\mathcal{Q}^{\mathrm{K}})$ appearing in Proposition 7.1 have non-zero dimension. To see this, note that according to Remark 7.4(ii) the categories $\mathcal{Q}^{\mathrm{K}}$ and $\mathcal{G}\boxtimes\overline{\mathcal{H}}$ are separable. Hence for any object U in one of these categories, the vanishing of $\dim(U)$ implies that $\mathrm{tr}(id_U) = 0$ and thus $id_U = 0$, so that U is a zero object. On the other hand, by Remark 2.23(vi), any Frobenius algebra has the tensor unit as a retract, and hence cannot be a zero object.

We can therefore apply Proposition 7.1; in particular $L' = \varUpsilon$ is haploid, commutative, symmetric and special. To establish (7.17), it remains to be shown that $\varGamma = \ell\text{-}\mathrm{Ind}_{\mathbf{1}_\mathcal{Q}\times T}(L\times\mathbf{1}_{\mathcal{H}'})$ is trivial, $\varGamma \cong \mathbf{1}_\mathcal{Q}$.

2) We regard $\mathcal{Q}\otimes_\Bbbk\mathcal{H}$ as a subcategory of $\mathcal{Q}\boxtimes\mathcal{H} = (\mathcal{Q}\otimes_\Bbbk\mathcal{H})^{\mathrm{K}}$ in the usual manner, and likewise for $\mathcal{G}\otimes_\Bbbk\overline{\mathcal{H}}$. We start by noticing that the two algebras $E_{\mathbf{1}_\mathcal{Q}\times T}(L\times\mathbf{1}_{\overline{\mathcal{H}}}) \cong C_l(F)$ and $\mathbf{1}_\mathcal{Q}\times T$ are both retracts of $F := (\mathbf{1}_\mathcal{Q}\times T)\otimes(L\times\mathbf{1}_{\overline{\mathcal{H}}})$. The associated idempotents are

$$P_{C_l(F)} \;=\;$$ 

and $$P_{\mathbf{1}_\mathcal{Q}\times T} \;=\; \frac{1}{\dim(L)}$$ 

(7.19)

respectively. The idempotent $P_{C_l(F)}$ is split by declaration 3.2. To see that $P_{\mathbf{1}_\mathcal{Q}\times T}$ is split as well, consider $\mathbf{1}_\mathcal{Q}\times T$ as a retract of F, with embedding and restriction morphisms $e = id_{\mathbf{1}_\mathcal{Q}\times T}\otimes\eta_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}$ and $r = id_{\mathbf{1}_\mathcal{Q}\times T}\otimes\varepsilon_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}/\dim(L)$, where in the definition of e and r the isomorphism $\mathbf{1}_\mathcal{Q}\times T \cong (\mathbf{1}_\mathcal{Q}\times T)\otimes(\mathbf{1}_\mathcal{Q}\times\mathbf{1}_\mathcal{H}\times\mathbf{1}_{\overline{\mathcal{H}}})$ is implicit; clearly, $e\circ r = id_{\mathbf{1}_\mathcal{Q}\times T}$ and $r\circ e = P_{\mathbf{1}_\mathcal{Q}\times T}$.

Using the specialness of the algebra T, one easily verifies that the idempotents (7.19) satisfy

$$P_{C_l(F)}\circ P_{\mathbf{1}_\mathcal{Q}\times T} = P_{\mathbf{1}_\mathcal{Q}\times T} = P_{\mathbf{1}_\mathcal{Q}\times T}\circ P_{C_l(F)}\,. \tag{7.20}$$

Their traces are computed as $\mathrm{tr}(P_{\mathbf{1}_\mathcal{Q}\times T}) = \dim(T)$ and as

$$\mathrm{tr}(P_{C_l(F)}) = s^{\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\overline{\mathcal{H}}}_{\mathbf{1}_\mathcal{Q}\times T,\,L\times\mathbf{1}_{\overline{\mathcal{H}}}} = \sum_{k\in\mathcal{I}_\mathcal{H}} s^{\mathcal{Q}\boxtimes\mathcal{H}}_{\mathbf{1}\times U_k,L}\, s^{\overline{\mathcal{H}}}_{\overline{U}_k,\mathbf{1}}\,, \tag{7.21}$$

respectively, where the first equality holds by Remark 4.6, while in the second equality the explicit form (6.48) of T is inserted.

3) Next we use the fact that $\mathcal{H}$ is modular and thus in particular semisimple. Hence writing $L\in\mathrm{Obj}(\mathcal{Q}\boxtimes\mathcal{H})$ as $L = (L_\mathcal{Q}\times L_\mathcal{H};\pi)$ with suitable objects $L_\mathcal{Q}$ of $\mathcal{Q}$ and $L_\mathcal{H}$ of $\mathcal{H}$

and an idempotent $\pi \in \mathrm{End}(L_\mathcal{Q} \times L_\mathcal{H})$, we know that $L_\mathcal{H}$ is a direct sum of simple objects U_j of $\mathcal{H}$, with j in the finite index set $\mathcal{I}_\mathcal{H}$, and as a consequence

$$L \cong \bigoplus_{j\in\mathcal{I}_\mathcal{H}} L_j \times U_j \tag{7.22}$$

with suitable objects L_j of $\mathcal{Q}$. Inserting this decomposition into formula (7.21) we obtain

$$\mathrm{tr}(P_{C_l(F)}) = \sum_{j,k\in\mathcal{I}_\mathcal{H}} s^{\mathcal{Q}}_{\mathbf{1},L_j}\, s^{\mathcal{H}}_{U_k,U_j}\, s^{\overline{\mathcal{H}}}_{\overline{U}_k,\mathbf{1}} = \sum_{j\in\mathcal{I}_\mathcal{H}} s^{\mathcal{Q}}_{\mathbf{1},L_j} \sum_{k\in\mathcal{I}_\mathcal{H}} s^{\mathcal{H}}_{U_k,U_j}\, s^{\mathcal{H}}_{U_k,\mathbf{1}}\,. \tag{7.23}$$

By the identity (2.9), modularity of $\mathcal{H}$ also implies that the k-summation in the expression on the right hand side can be carried out, yielding $\delta_{j,0} \sum_{k\in\mathcal{I}_\mathcal{H}} (s^{\mathcal{H}}_{U_k,\mathbf{1}})^2 = \delta_{j,0}\dim(T)$, and hence $\mathrm{tr}(P_{C_l(F)}) = \dim(T)\, s^{\mathcal{Q}}_{\mathbf{1},L_0}$. Further, the hypothesis that L is $\mathcal{Q}$-haploid means that $L_0 \cong \mathbf{1}_\mathcal{Q}$; thus we finally get

$$\mathrm{tr}(P_{C_l(F)}) = \dim(T)\, s^{\mathcal{Q}}_{\mathbf{1},\mathbf{1}} = \dim(T)\,. \tag{7.24}$$

It follows that $\mathrm{tr}(P_{C_l(F)}) = \mathrm{tr}(P_{C_l(F)} \circ P_{\mathbf{1}_\mathcal{Q}\times T}) = \mathrm{tr}(P_{\mathbf{1}_\mathcal{Q}\times T})$. By Lemma 7.5 this implies, in turn, that the two idempotents (7.19) coincide, $P_{C_l(F)} = P_{\mathbf{1}_\mathcal{Q}\times T}$. We conclude that

$$E_{\mathbf{1}_\mathcal{Q}\times T}(L\times \mathbf{1}_{\overline{\mathcal{H}}}) \cong \mathbf{1}_\mathcal{Q} \times T \tag{7.25}$$

as retracts of F.

It is also not difficult to check that the multiplication induced on $\mathbf{1}_\mathcal{Q}\times T$ via its embedding in the algebra F agrees with the one defined in Lemma 6.19. The same holds for $E_{\mathbf{1}_\mathcal{Q}\times T}(L\times \mathbf{1}_{\overline{\mathcal{H}}}) \cong C_l(F)$, as follows from Proposition 3.14. The isomorphism (7.25) therefore also holds as an isomorphism of algebras, and in fact even as an isomorphism of symmetric special Frobenius algebras.

But the object $\mathbf{1}_\mathcal{Q}\times T$ is the tensor unit in the category $(\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\overline{\mathcal{H}})^{\ell oc}_{\mathbf{1}_\mathcal{Q}\times T} \cong \mathcal{Q}^{\mathrm{K}}$, implying that $\ell\text{-Ind}_{\mathbf{1}_\mathcal{Q}\times T}(L\times \mathbf{1}_{\mathcal{H}'}) \cong \mathbf{1}$ as an object in $\mathcal{Q}^{\mathrm{K}}$. The relation (7.17) now follows from (7.5) with $L' = \Upsilon = \ell\text{-Ind}_{L\times \mathbf{1}_{\mathcal{H}'}}(\mathbf{1}_\mathcal{Q}\times T)$.

Proof of (ii):

It remains to be shown that the algebra L' in $\mathcal{G}\boxtimes\overline{\mathcal{H}}$ is $\mathcal{G}$-haploid. We will establish that any object M of $\mathcal{G}$ with the property that $M\times \mathbf{1}_{\overline{\mathcal{H}}}$ is a retract of L', is itself a retract of $\mathbf{1}_\mathcal{G}$. Since $\mathbf{1}_\mathcal{G}$ is simple, this implies that $M \cong \mathbf{1}_\mathcal{G}$, and hence (ii).

Let us formulate these statements in terms of the category $\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\overline{\mathcal{H}}$. L' is the algebra $E_{L\times \mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_\mathcal{Q}\times T)$, while M is a local L-module in $\mathcal{Q}\boxtimes\mathcal{H}$. That $(M\times \mathbf{1}_{\overline{\mathcal{H}}}, e, r)$ is a retract of L' in $\mathcal{G}\boxtimes\overline{\mathcal{H}}$ thus means that

$$\begin{aligned} &e \in \mathrm{Hom}_{L\times \mathbf{1}_{\overline{\mathcal{H}}}}(M\times \mathbf{1}_{\overline{\mathcal{H}}}, \ell\text{-Ind}_{L\times \mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_\mathcal{Q}\times T)) \qquad \text{and} \\ &r \in \mathrm{Hom}_{L\times \mathbf{1}_{\overline{\mathcal{H}}}}(\ell\text{-Ind}_{L\times \mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_\mathcal{Q}\times T), M\times \mathbf{1}_{\overline{\mathcal{H}}}) \end{aligned} \tag{7.26}$$

as morphisms of $\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\overline{\mathcal{H}}$. Now by the isomorphisms of Proposition 4.4 and the reciprocity relation (2.41), we have

$$\mathrm{Hom}_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}(M\times\mathbf{1}_{\overline{\mathcal{H}}}, \ell\text{-}\mathrm{Ind}_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_{\mathcal{Q}}\times T)) \cong \mathrm{Hom}(\dot{M}\times\mathbf{1}_{\overline{\mathcal{H}}}, \mathbf{1}_{\mathcal{Q}}\times T)\,. \tag{7.27}$$

Using the explicit form of T from formula (6.48), this morphism space in $\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\overline{\mathcal{H}}$ is, in turn, isomorphic to the space $\mathrm{Hom}(\dot{M}, \mathbf{1}_{\mathcal{Q}}\times\mathbf{1}_{\mathcal{H}})$ of morphisms in $\mathcal{Q}\boxtimes\mathcal{H}$, and hence to $\mathrm{Hom}_L(M, L)$. Together with a similar argument for the second morphism space in (7.26) we can conclude that there are bijections

$$\begin{aligned} f:\quad & \mathrm{Hom}_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}(M\times\mathbf{1}_{\overline{\mathcal{H}}}, \ell\text{-}\mathrm{Ind}_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_{\mathcal{Q}}\times T)) \xrightarrow{\cong} \mathrm{Hom}_L(M, L) \qquad \text{and} \\ g:\quad & \mathrm{Hom}_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}(\ell\text{-}\mathrm{Ind}_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_{\mathcal{Q}}\times T), M\times\mathbf{1}_{\overline{\mathcal{H}}}) \xrightarrow{\cong} \mathrm{Hom}_L(L, M)\,. \end{aligned} \tag{7.28}$$

Substituting the explicit form of these isomorphisms one can verify that for the morphisms e and r of (7.26) we have $g(r)\circ f(e) = id_M$. It follows that $(M, f(e), g(r))$ is a retract of L. Moreover, since $f(e)$ and $g(r)$ are morphisms of L-modules and L is the tensor unit of the category $\mathcal{G}$, this implies that M is a retract of $\mathbf{1}_{\mathcal{G}}$ in $\mathcal{G}$. □

Combining Theorem 7.6 with Proposition 3.21 we arrive at the following statements about the category $\mathcal{Q}^{\mathrm{K}}$:

Corollary 7.7:
For $\mathcal{Q}$ a (not necessarily Karoubian) ribbon category and $\mathcal{H}$ a modular tensor category such that the product $\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\overline{\mathcal{H}}$ is separable, and L a $\mathcal{Q}$-haploid commutative symmetric special Frobenius algebra in $\mathcal{Q}\boxtimes\mathcal{H}$, we have:

(i) If $(\mathcal{Q}\boxtimes\mathcal{H})_L^{\ell oc}$ is semisimple, then so is $\mathcal{Q}^{\mathrm{K}}$.

(ii) If $(\mathcal{Q}\boxtimes\mathcal{H})_L^{\ell oc}$ is a modular tensor category, then so is $\mathcal{Q}^{\mathrm{K}}$.

Theorem 7.6 allows us to construct the tensor category $\mathcal{Q}$ from the knowledge of the categories $\mathcal{G}$ and $\mathcal{H}$ and of the algebra $L' = \ell\text{-}\mathrm{Ind}_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_{\mathcal{Q}}\times T)$ in $\mathcal{G}\boxtimes\overline{\mathcal{H}}$. For applications, e.g. in conformal quantum field theory, it turns out to be important to gain information about L' by using as little information about the category $\mathcal{Q}$ as possible. The following result helps to determine L' as an object of $\mathcal{G}\boxtimes\overline{\mathcal{H}}$ in case that $\mathcal{G}$ is a *modular* tensor category (and hence, by corollary 7.7(ii), $\mathcal{Q}^{\mathrm{K}}$ is a modular tensor category, too), so that in particular the set $\{M_\kappa \,|\, \kappa\in\mathcal{I}_{\mathcal{G}}\}$ of isomorphism classes of simple objects in $\mathcal{G}$ (i.e. of simple local L-modules in $\mathcal{Q}\boxtimes\mathcal{H}$) is finite.

Lemma 7.8:
Let $\mathcal{Q}$, $\mathcal{H}$ and L be as in Theorem 7.6, and assume that $\mathcal{G} := (\mathcal{Q}\boxtimes\mathcal{H})_L^{\ell oc}$ is modular. Then as an object in $\mathcal{G}\boxtimes\overline{\mathcal{H}}$ the algebra $L' := \ell\text{-}\mathrm{Ind}_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_{\mathcal{Q}}\times T)$ decomposes as

$$L' \cong \bigoplus_{\kappa\in\mathcal{I}_{\mathcal{G}}}\bigoplus_{l\in\mathcal{I}_{\mathcal{H}}} \dim\left[\mathrm{Hom}^{\mathcal{Q}\boxtimes\mathcal{H}}(\dot{M}_\kappa, \mathbf{1}_{\mathcal{Q}}\times U_l)\right] M_\kappa\times\overline{U_l}\,. \tag{7.29}$$

Proof:
By Theorem 7.6, L' is a lift to $\mathcal{G}\boxtimes\overline{\mathcal{H}} \cong (\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\overline{\mathcal{H}})^{\ell oc}_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}$ of the algebra $E_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_{\mathcal{Q}}\times T)$, which is a local $L\times\mathbf{1}_{\overline{\mathcal{H}}}$-module. Now owing to relation (6.35) every simple local $L\times\mathbf{1}_{\overline{\mathcal{H}}}$-module is of the form $M\times\overline{U_l}$, with M a simple local L-module and $\overline{U_l}$ a simple object of $\overline{\mathcal{H}}$. Invoking Proposition 4.4 and the reciprocity relation (2.41), it follows that the algebra L' decomposes according to

$$E_{L\times\mathbf{1}_{\overline{\mathcal{H}}}}(\mathbf{1}_{\mathcal{Q}}\times T) \;\cong\; \bigoplus_{\kappa\in\mathcal{I}_{\mathcal{G}}}\bigoplus_{l\in\mathcal{I}_{\mathcal{H}}} \dim\left[\mathrm{Hom}^{\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\overline{\mathcal{H}}}(\dot{M}_\kappa\times\overline{U_l},\mathbf{1}_{\mathcal{Q}}\times T)\right] M_\kappa\times\overline{U_l} \tag{7.30}$$

into simple local $L\times\mathbf{1}_{\overline{\mathcal{H}}}$-modules. Moreover, the morphism spaces appearing here obey

$$\begin{aligned}\mathrm{Hom}^{\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\overline{\mathcal{H}}}(\dot{M}\times\overline{U_l},\mathbf{1}_{\mathcal{Q}}\times T) &\cong \mathrm{Hom}^{\mathcal{Q}\boxtimes\mathcal{H}\boxtimes\overline{\mathcal{H}}}(\dot{M}\times\overline{U_l},\mathbf{1}_{\mathcal{Q}}\times U_l\times\overline{U_l})\\ &\cong \mathrm{Hom}^{\mathcal{Q}\boxtimes\mathcal{H}}(\dot{M},\mathbf{1}_{\mathcal{Q}}\times U_l)\,,\end{aligned} \tag{7.31}$$

where the first isomorphism follows by inserting the explicit form of T from (6.48) and observing that only the component $U_l\times\overline{U_l}$ contributes. □

Remark 7.9:
If $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{H}$ are modular, then from the observations in Remarks 3.23(i), 6.2(iv) and 6.16 one can easily determine the dimension of the algebra L'. Indeed, because of $\mathcal{G}\cong(\mathcal{Q}\boxtimes\mathcal{H})^{\ell oc}_L$ and $\mathcal{Q}\cong(\mathcal{G}\boxtimes\overline{\mathcal{H}})^{\ell oc}_{L'}$ we have

$$p^+(\mathcal{G}) = \frac{p^+(\mathcal{Q})\,p^+(\mathcal{H})}{\dim^{\mathcal{Q}\boxtimes\mathcal{H}}(L)} \qquad\text{and}\qquad p^+(\mathcal{Q}) = \frac{p^+(\mathcal{G})\,p^-(\mathcal{H})}{\dim^{\mathcal{G}\boxtimes\overline{\mathcal{H}}}(L')}\,. \tag{7.32}$$

As a consequence,

$$\dim^{\mathcal{Q}\boxtimes\mathcal{H}}(L)\,\dim^{\mathcal{G}\boxtimes\overline{\mathcal{H}}}(L') = \mathrm{Dim}(\mathcal{H})\,. \tag{7.33}$$

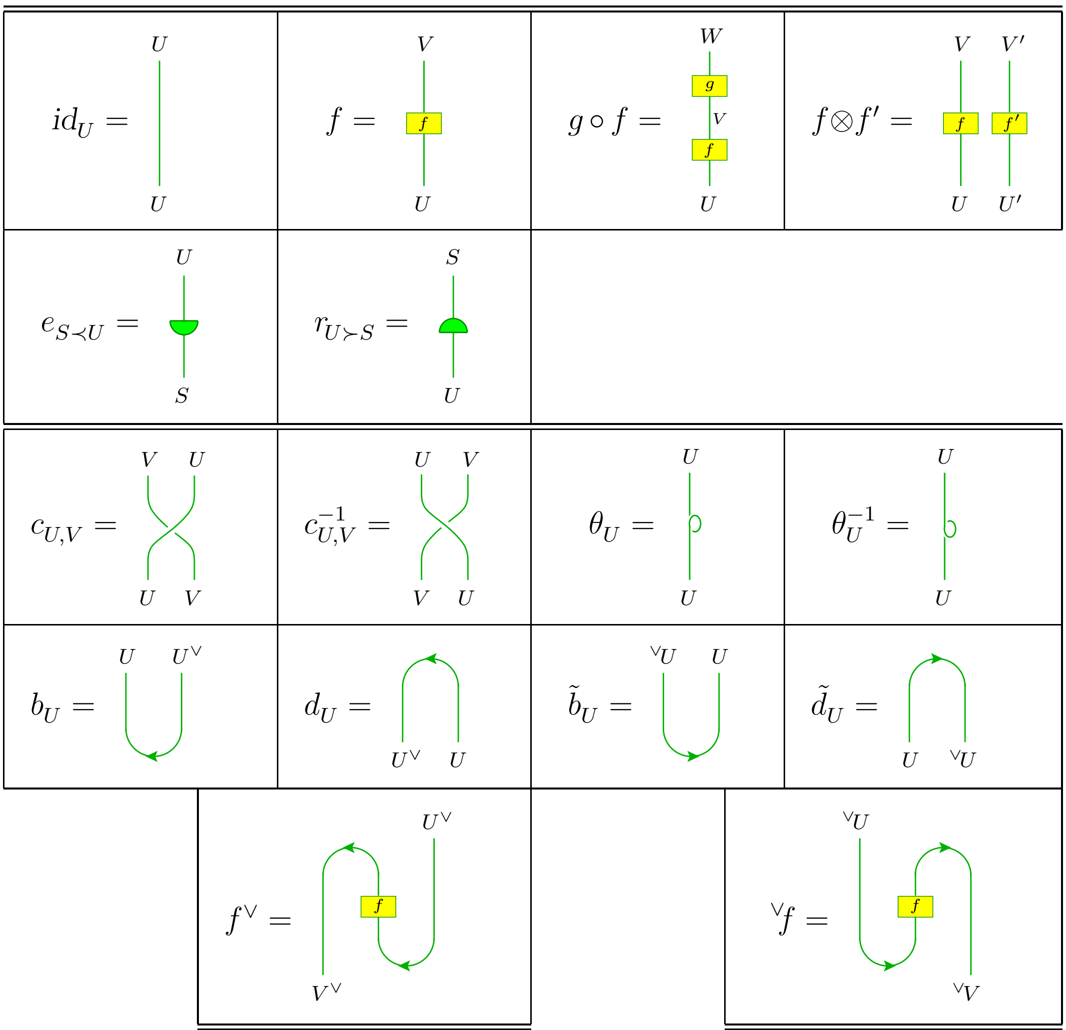
This expresses the dimension of L' in terms of those of L and $\mathcal{H}$.

A Graphical calculus

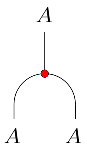
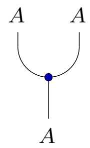
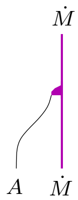
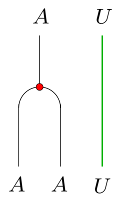
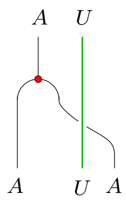
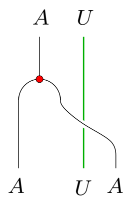
The computations in this paper are often presented in terms of a graphical calculus for ribbon categories, which was first advocated in [21]. To make these manipulations more easily accessible, we summarise in this appendix our conventions, and in particular recall the definition of various specific morphisms that are used in the main text.

A.1 Morphisms

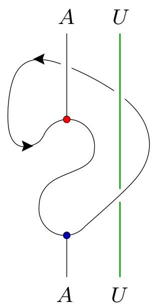
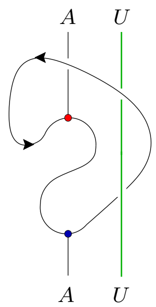
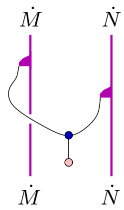
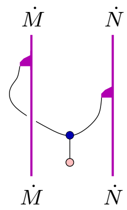
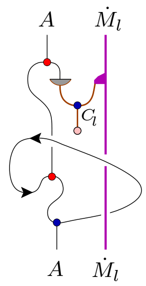
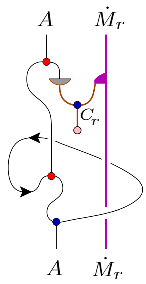
In the following table we present the graphical notation for general morphisms of a tensor category, their composition and tensor product, and for the embedding and restriction morphisms (see equation (2.10)) of retracts. Also shown are the structural morphisms of a ribbon category: the braiding, twist, and left and right dualities (see Definition 2.1), as well as the definition of the (left and right) dual of a general morphism:



The next table lists the structural morphisms of a (co)algebra: the product, unit, coproduct, and counit (see equations (2.22) and (2.23)); the representation morphism for a general left-module (see equation (2.24)); the representation morphism for an induced left-module as well as the right-representation morphisms for α-induced modules (see (2.31)):

$m =$ 	$\eta =$ 	$\Delta =$ 	$\varepsilon =$ 
$\rho_M =$ 	$\rho^{\text{left}}_{\alpha^+_A(U)} = \rho^{\text{left}}_{\alpha^-_A(U)} = \rho_{\text{Ind}_A(U)} =$ 		
$\rho^{\text{right}}_{\alpha^+_A(U)} =$ 		$\rho^{\text{right}}_{\alpha^-_A(U)} =$ 	

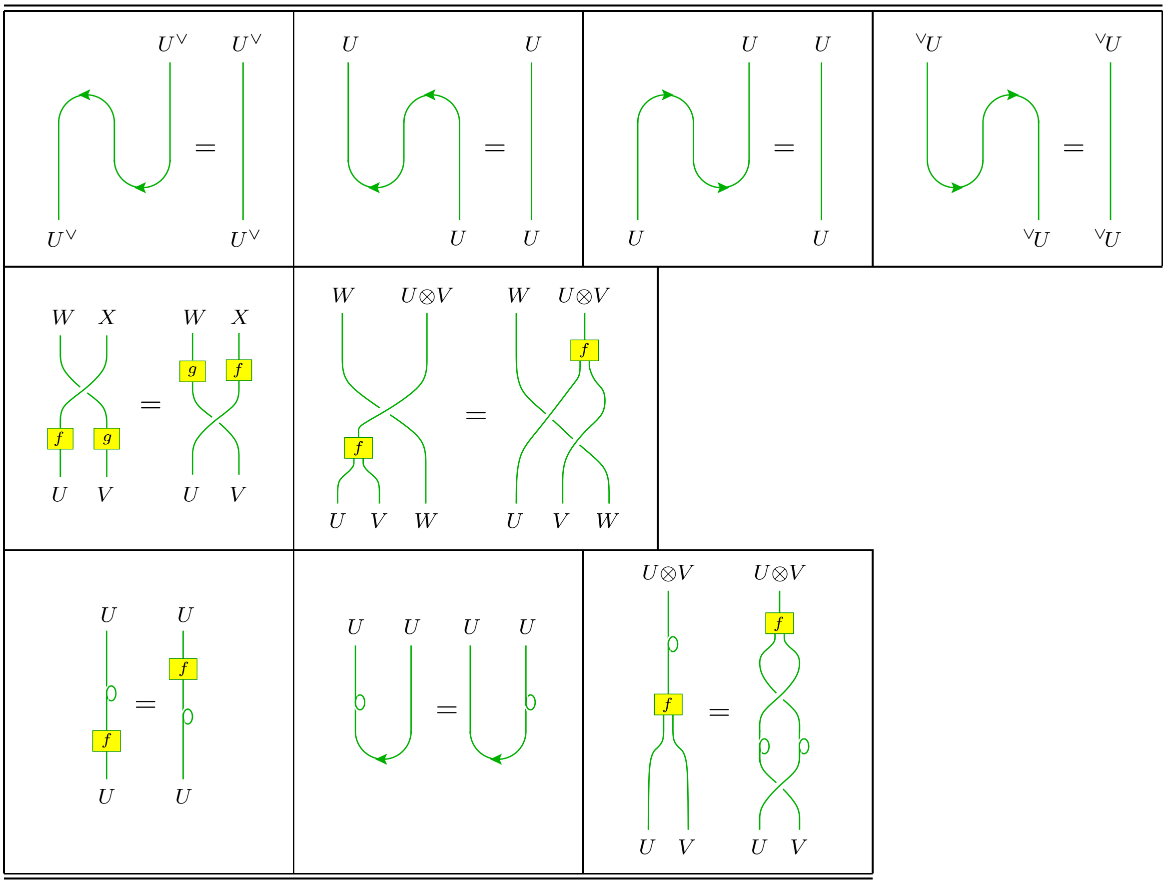
In the following table we list some specific idempotents: the idempotents $P_A^{l/r}(U)$ (see equation (3.1)) on which the left and right local induction are based and which appear in the Definition 3.1 of a centrally split Frobenius algebra; those appearing in the definition of the tensor product of local modules ($P_{M\otimes N}$, see formula (3.46)); and also the idempotents $Q_{r/l}(M_{l/r})$ defined in (5.72), which appear in the functorial equivalences between $\mathcal{C}^{\ell oc}_{C_l(A)}$ and $\mathcal{C}^{\ell oc}_{C_r(A)}$.

$P_A^l(U)$:= 	$P_A^r(U)$:= 
$P_{M\otimes N}$:=  = 	
$Q_r(M_l)$:= 	$Q_l(M_r)$:= 

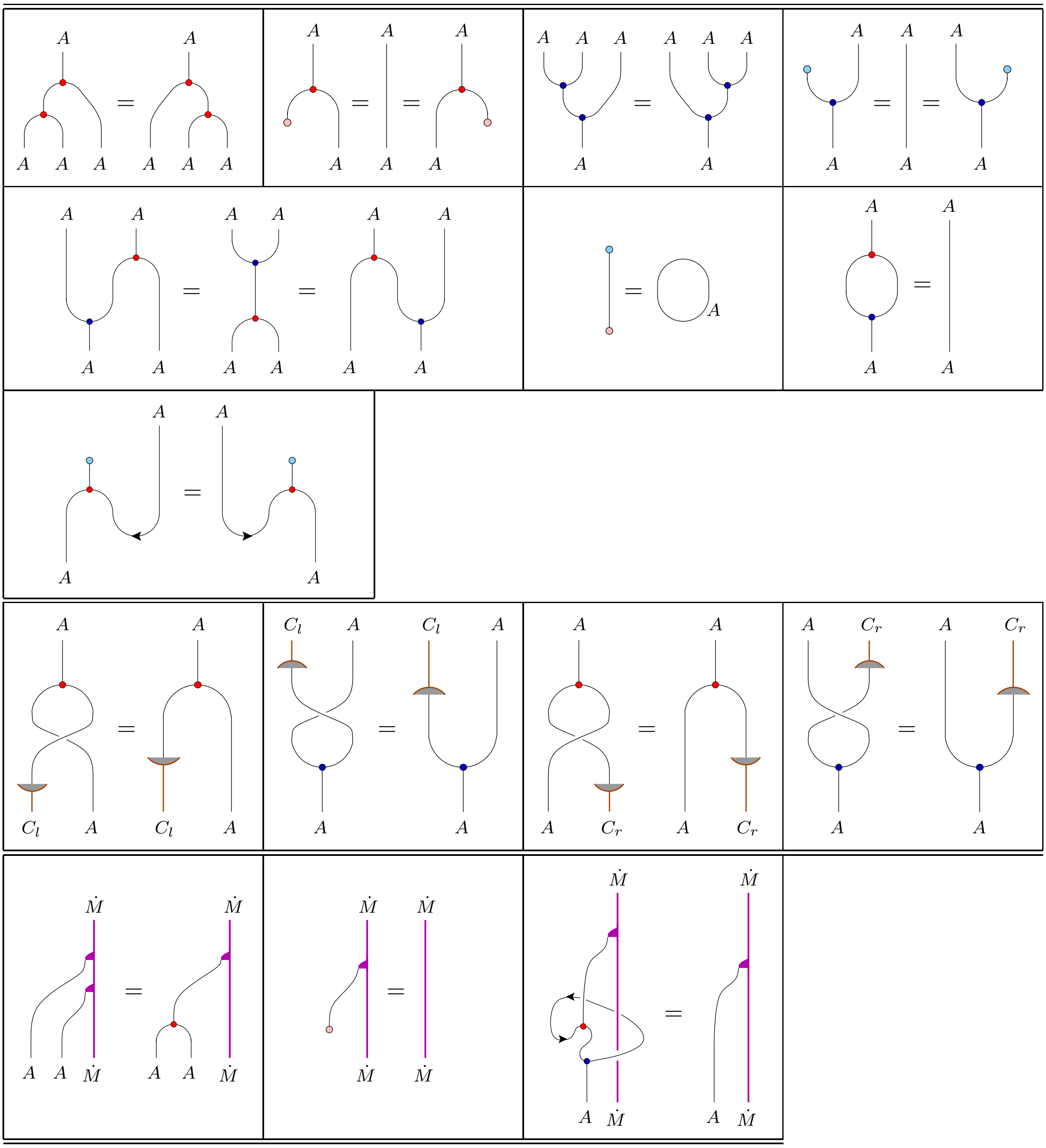
A.2 Defining properties

We now present the defining properties of some of the morphisms displayed in Section A.1.

We start with the axioms of a ribbon category: the defining properties of dualities; the functoriality and tensoriality of the braiding; the functoriality of the twist, and the compatibility of the twist with duality and with braiding, see equation (2.2):



Next we display the axioms of a symmetric special Frobenius algebra A: associativity of the product, the unit property, coassociativity of the coproduct, and the counit property, see equations (2.22) and (2.23); the Frobenius property, the two specialness properties (with the normalisation $\beta_A = 1$) and the symmetry property, see Definition 2.22. Finally we show the defining properties of the left and right centers $C_{l/r} = C_{l/r}(A)$ (see equation (2.64)) as well as the two defining properties of a (left) representation, and the defining property of a local (left) representation, see equations (2.24) and (3.34).



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