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Kramers-Wannier duality from conformal defects

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Abstract
We demonstrate that the fusion algebra of conformal defects of a two-dimensional conformal field theory contains information about the internal symmetries of the theory and allows one to read off generalisations of Kramers-Wannier duality. We illustrate the general mechanism in the examples of the Ising model and the three-states Potts model.

Introduction

Kramers and Wannier found a high/low temperature duality for the Ising model [1] that asserts that a correlator of Ising spins $\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle$ at inverse temperature β is equal to a disorder correlation function $\langle\mu_{x_1}\cdots\mu_{x_n}\rangle$ at the dual inverse temperature $\tilde{\beta}=-\frac{1}{2}\ln\tanh\beta$. In the disorder correlator, the couplings between neighbouring spins dual to the links of $n/2$ lines, with each of the positions x_k at the end of one of the lines, are chosen to be antiferromagnetic (opposite to the standard ferromagnetic nearest-neighbour coupling). This duality has since been considerably generalised, see e.g. [2, 3].

The significance of Kramers-Wannier duality lies in the fact that it relates the high-temperature expansion (weak coupling regime) of a lattice model to its low-temperature expansion (strong coupling regime) and thereby makes the latter accessible to perturbation theory.

Kramers-Wannier-like dualities are also a useful tool in understanding the phase structure of a lattice model. At zero magnetic field, the Ising model has a critical point when $\beta=\tilde{\beta}$. Its universality class is described by a two-dimensional conformal field theory (CFT) with central charge $c=\frac{1}{2}$. Physical quantities like critical exponents can then be determined by a CFT calculation, relating them to scaling dimensions of bulk fields. The critical Ising model is self-dual under Kramers-Wannier duality, so that a correlator involving spin and disorder fields is equal to another correlator in the same CFT, but with spin fields and disorder fields interchanged.

It is clearly desirable to be able to read off the possible high/low temperature dualities leaving a given critical model fixed solely from knowing its universality class, i.e., its CFT description. In this letter, we provide such a method by relating order/disorder dualities of CFT correlators to *conformal defects*. Not every defect can be used to establish a duality, but only what we will call 'duality defects'. Below we present a method that allows us to identify such defects by studying the fusion algebra of all conformal defects. Duality defects relate perturbations of a CFT in different marginal directions, thus allowing one to explore the vicinity of a model in its moduli space, and they also relate different relevant directions, allowing one to extend the order/disorder duality of the CFT to a genuine high/low temperature duality away from the critical point.

Defects in the critical Ising model

Before exhibiting the underlying mechanism in generality, we investigate in some detail the critical Ising model as a first non-trivial example. At central charge $c=\frac{1}{2}$ the Virasoro algebra has three unitary irreducible highest-weight representations, which we denote by $\mathbf{1}$, σ, ε. Their weights are $h_{\mathbf{1}}=0$, $h_\sigma=\frac{1}{16}$ and $h_\varepsilon=\frac{1}{2}$. Correspondingly, there are three primary bulk fields, the identity $\mathbf{1}$, the spin field $\sigma(z)$ and the energy field $\varepsilon(z)$, with chiral/antichiral conformal weights $(0,0)$, $(\frac{1}{16},\frac{1}{16})$ and $(\frac{1}{2},\frac{1}{2})$, respectively.

Next, we introduce conformal defects. One can think of a conformal defect on a surface as being obtained by cutting the surface along the defect line and re-joining the two sides of the cut by an appropriate boundary condition, i.e. a prescription on how bulk fields behave

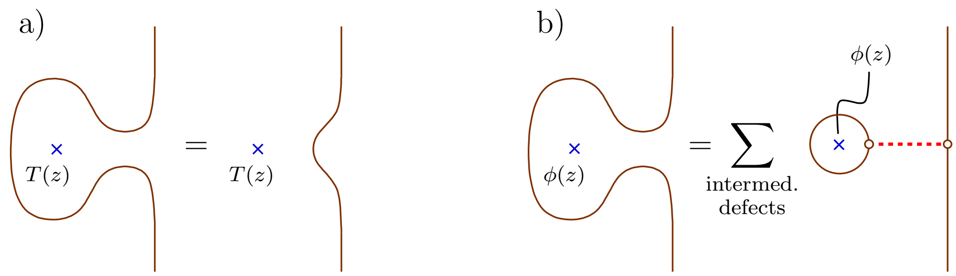

Figure 1: A conformal defect is transparent to the stress tensor (a), while a bulk field ϕ generically becomes a sum of disorder fields (b).

when crossing the cut. This prescription must preserve the conformal symmetry, i.e. both the chiral and antichiral components $T(z)$ and $\bar{T}(\bar{z})$ of the conformal stress tensor must vary continuously across the cut. In contrast, other bulk fields are permitted to exhibit a more complicated behaviour. In fact, dragging a conformal defect across a bulk field other than the stress tensor generally results in disorder fields, as illustrated in figure 1.

Because the defect line commutes with the stress tensor, it can be continuously deformed without changing the value of a correlator. In this sense a conformal defect is tensionless. Defect lines can only start and end on field insertions. Such fields are called *disorder fields*. Since a defect is invisible to T and $\bar{T}$, disorder fields fall into representations of two copies of the Virasoro algebra, just as the bulk fields do.

By an argument similar to one used in the analysis of conformal boundary conditions [4], in the Ising model one finds three conformal defects [5]. They are labelled by the three $c = \frac{1}{2}$ irreps of the Virasoro algebra. The defect of type $\mathbf{1}$ is the trivial defect, in the presence of which all fields are continuous. The ε-defect corresponds to a line of antiferromagnetic couplings in the lattice realisation, while the σ-defect does not have a straightforward lattice interpretation [6] and has long been overlooked. The appearance of the σ-defect illustrates that a systematic analysis of a universality class, using CFT methods, can lead to structural insight not obvious from studying a concrete lattice realisation.

In addition to the bulk fields $\mathbf{1}$, $\sigma(z)$ and $\varepsilon(z)$ we will also consider the disorder field $\mu(z)$. Pairs of disorder fields $\mu(z_1)$ and $\mu(z_2)$ are joined by a defect line of type ε. A disorder field has the same conformal weights as the spin field, i.e. $(\frac{1}{16}, \frac{1}{16})$.

The results reported in this letter are obtained in the approach to CFT [7, 8] that is based on topological field theory (TFT) in three dimensions. A chiral CFT can be described by the boundary degrees of freedom of a three-dimensional topological field theory [9, 10]. The observables of the TFT are (networks of) Wilson lines. Each Wilson line is labelled by a representation of the chiral algebra of the CFT, i.e., by $\mathbf{1}$, σ or ε in the example of the Ising model. The vertices of the network of Wilson lines are labelled by intertwiners of the corresponding representations. In the TFT formalism [7, 8], a CFT correlator on a surface X (oriented, without boundary) with field insertions is described as follows: one first constructs a three-manifold by taking an interval above each point of X, $M = X \times [1, -1]$. The two boundary components $X \times \{1\}$ and $X \times \{-1\}$ support the two chiral degrees of

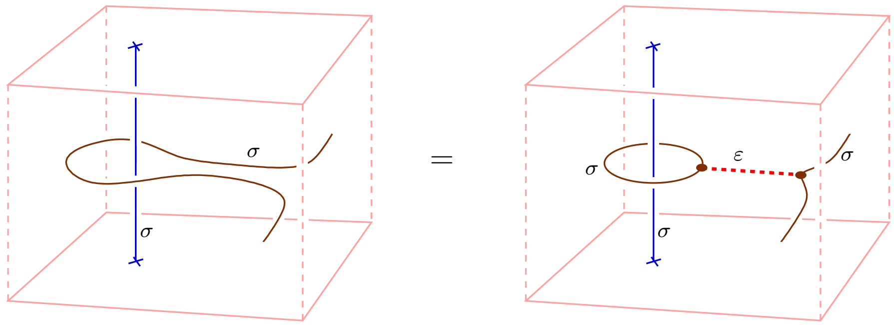

Figure 2: The TFT-representation of the pulling a defect of type σ past a spin field. Collapsing the circular σ-Wilson line on the rhs generates the TFT-representation of the disorder field $\mu(z)$.

freedom of the CFT, respectively. At each field insertion on X, a Wilson line with the corresponding label is inserted which runs along the interval $[-1,1]$, thus connecting the two boundary components of M. A defect line on the surface X is described by a Wilson line inserted on $X\times\{0\}\subset M$ and labelled again by σ or ε, depending on the defect type. Consider, for instance, the effect of pulling a σ-defect past a spin field $\sigma(z)$ as in figure 1b. This turns out to generate a disorder field $\mu(z)$ and an ε-defect. In the TFT formalism, this process amounts to the identity in figure 2, which is then easily verified.

A straightforward calculation within the TFT-framework allows one to find the set of rules summarised in figure 3 for taking defects past field insertions. In this figure, the normalisation of the fields is chosen such that $\langle\sigma(z)\,\sigma(w)\rangle = \langle\mu(z)\,\mu(w)\rangle = |z-w|^{-1/4}$ and $\langle\varepsilon(z)\,\varepsilon(w)\rangle = |z-w|^{-2}$. Also, three-valent vertices between two σ-defects and one ε-defect have been labelled with a suitably normalised intertwiner.

We can now obtain the first example for an order/disorder duality, the correlator of four spin fields on the sphere. In this correlator we insert a small circular σ-defect, which changes

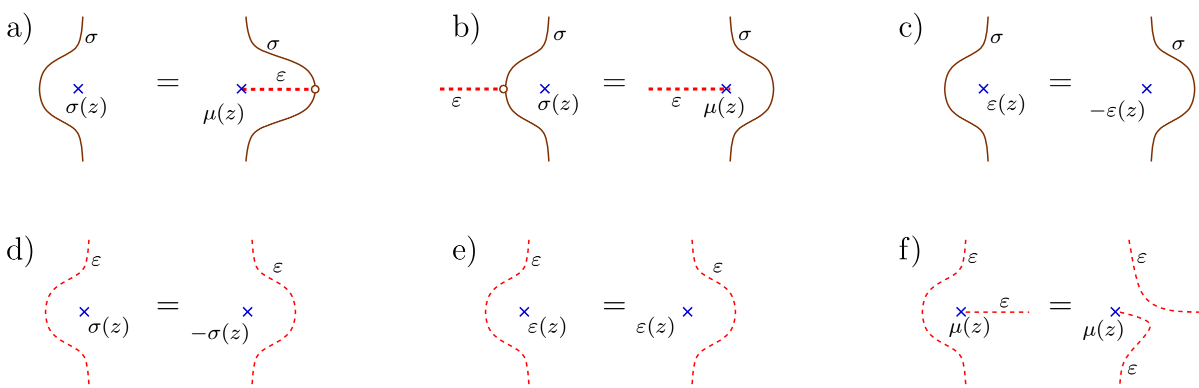

Figure 3: Taking defects of type σ and ε past field insertions. The TFT-representation of a) is given in figure 2.

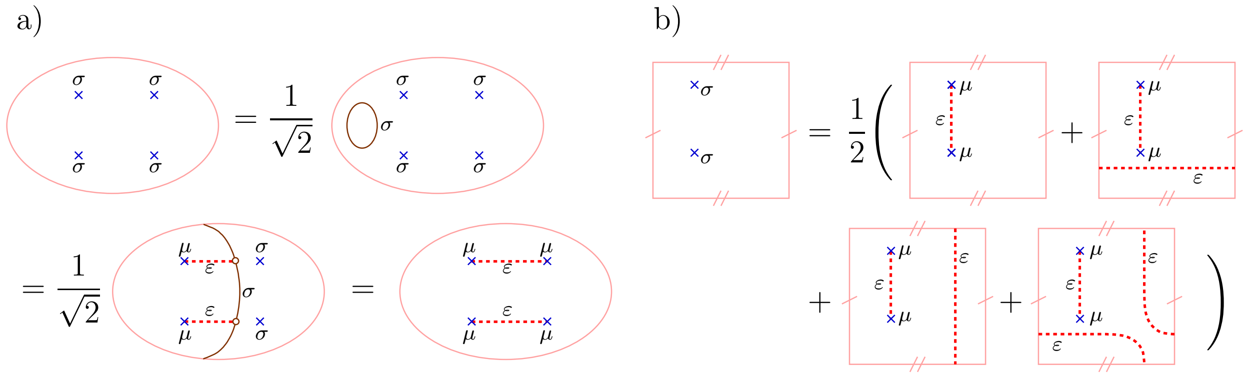

Figure 4: Order/Disorder duality of a correlator of four spin fields on a sphere, and of two spin fields on a torus, as induced by the σ-defect.

the value of the correlator by the quantum dimension $\dim(\sigma) = \sqrt{2}$ of the representation σ. Pulling the defect circle around the sphere and past the field insertions results in a disorder correlator as shown in figure 4a. Using the rules of figure 3, it is easy to verify that repeating this procedure removes the ε-defects and replaces the disorder fields again by spin fields. This is the order/disorder duality of the Ising model on the sphere. The rules in figure 3 immediately imply that when one studies duality on a torus, the non-trivial topology results in a sum over several configurations, as illustrated in figure 4b.

The mechanism can be generalised to surfaces with boundaries. In the Ising model, the boundary conditions are again labelled by the $c = \frac{1}{2}$ irreps [4]: $\mathbf{1}$ and ϵ describe fixed boundary conditions with 'spin up' and 'spin down', respectively, while σ describes the 'free' boundary condition. Owing to the Ising fusion rules $\sigma \star \mathbf{1} = \sigma \star \epsilon = \sigma$ and $\sigma \star \sigma = \mathbf{1} + \varepsilon$, a σ-defect in front of a 'spin up' or a 'spin down' boundary condition can be replaced by a 'free' boundary condition without defect, while a σ-defect in front of a 'free' boundary condition yields the sum of a 'spin up' and a 'spin down' boundary condition. One thus obtains the well-known duality of fixed and free boundary conditions [3].

So far, we have considered the order/disorder duality only at the critical point. However, the rules listed in figure 3 also allow us to establish the duality away from the critical point. For example, note that taking a σ-defect through the energy field $\varepsilon(z)$ results in a change of sign. Perturbing the CFT by $\varepsilon(z)$ amounts to a change of temperature, and applying the duality to each term in a perturbation series leads to the equality

$$\left\langle\, \sigma(x)\, \sigma(x')\, \mathrm{e}^{-\lambda \int \varepsilon(y)\, \mathrm{d}^2 y} \,\right\rangle = \left\langle\, \mu(x)\, \mu(x')\, \mathrm{e}^{\lambda \int \varepsilon(y)\, \mathrm{d}^2 y} \,\right\rangle$$

for the example of a two-point correlator on the sphere.

The general mechanism

We are now in a position to describe a general mechanism that works for all unitary rational conformal field theories. For such models, there is a finite set of primary bulk fields $\phi_a(z)$. One denotes the number of such fields transforming in representations i and j of the chiral and antichiral symmetries, respectively, by Z_{ij}. The matrix Z thus describes the modular invariant torus partition function of the CFT.

We restrict our attention to conformal defects that preserve enough additional symmetry to keep the model rational. We call a defect 'simple', iff it cannot be written as a sum of other defects. The number of simple defects is given by $\mathrm{tr}(ZZ^{\mathrm{t}})$ [5, 8]. Let us denote the set of simple defects by $\{D_\alpha \,|\, \alpha \in \mathcal{K}\}$ for some label set $\mathcal{K}$, with the label for the trivial defect denoted by 'e'. In general, one must assign an orientation to a defect line.

Consider two simple defects running parallel to each other and with the same orientation. In the limit of vanishing distance they fuse to a single defect which is, in general, a superposition of simple defects. This gives rise to a (not necessarily commutative) fusion algebra of defects [5, 11], written schematically as

$$D_\alpha \otimes D_\beta = \sum_{\gamma \in \mathcal{K}} \hat{N}_{\alpha\beta}^{\ \ \gamma} \, D_\gamma \,.$$

In the TFT formalism, the general class of models we are studying now is described by an algebra A in the category of representations of the chiral algebra of the CFT. Defects are then described as bimodules of A, and the defect fusion rules above amount to decomposing the tensor product over A of two bimodules into a direct sum of simple bimodules, which can be performed explicitly. The bimodule describing the trivial defect D_e turns out to be A itself. If the two parallel defects have opposite direction we write $D_\alpha \otimes D_\beta^\vee$.

Two subsets of defects turn out to be of particular interest. The first one is the set $\mathcal{G}$ of *group-like* defects. A defect X is called group-like, iff $X \otimes X^\vee = D_e$. One can show that group-like defects are simple, so that $\mathcal{G} \subseteq \mathcal{K}$. Further, for two group-like defects D and D', their fusion $D \otimes D'$ is again group-like. This turns $\mathcal{G}$ into a (in general nonabelian) group with unit D_e, via $D_g \otimes D_h = D_{gh}$ and $D_{g^{-1}} = D_g^\vee$. From figure 1b we see that taking any group-like defect past a bulk field results in a sum of bulk fields, since the only intermediate defect that does occur is the trivial one, $D_g \otimes D_g^\vee = D_e$. Commuting a group-like defect past all bulk fields in a correlator results in a correlator of different bulk fields, but having the same value. Thus, *group-like defects produce an internal symmetry of the CFT*. For the Ising model one has $\mathcal{G} = \{\mathbf{1}, \varepsilon\}$, a $\mathbb{Z}_2$ group, and from figure 3d we see that the defect ε indeed acts by reversing the sign of the spin field.

The second and larger subset is formed by the *duality defects*. A defect X is a duality defect, iff there exists another defect Y such that taking first X and then Y past a bulk field results only in a sum of bulk fields, with no disorder fields present. In other words, commuting X past all fields in an order correlator in general gives a disorder correlator. However, subsequently commuting Y past all fields in this disorder correlator gives back an order correlator. Thus, *duality defects produce order-disorder dualities of the CFT.* Using the TFT formalism, one can establish the following simple characterisation of duality defects: X is a duality defect, if and only if every simple defect in $X \otimes X^\vee$ is a group-like defect. A detailed proof will be presented elsewhere. Clearly, the set $\mathcal{D}$ of simple duality defects satisfies $\mathcal{G} \subseteq \mathcal{D} \subseteq \mathcal{K}$.

Note that in order to determine $\mathcal{G}$ and $\mathcal{D}$ in a given model, it suffices to know the fusion algebra of defects. In the Ising model one finds $\mathcal{D} = \{\mathbf{1}, \sigma, \varepsilon\} = \mathcal{K}$. The duality defect σ generates the original Kramers-Wannier duality.

The above discussion is limited to the critical point. However, suppose that for a given duality defect D_α we can find a bulk field $\phi(z)$ such that taking D_α past $\phi(z)$ results in another bulk field $\tilde{\phi}(z)$, rather than in a sum of bulk fields and disorder fields. (In the Ising model, the field $\varepsilon(z)$ has this property with respect to the defect labelled by σ, see figure 3c.) Then the duality induced by D_α provides an equality between a correlator of the CFT perturbed by $\int \phi(z)\, \mathrm{d}^2 z$ and the dual correlator perturbed by $\int \tilde{\phi}(z)\, \mathrm{d}^2 z$.

The critical three-states Potts model

The critical three-states Potts model has central charge $c = 4/5$ and corresponds to a D-type model in the classification of Virasoro-minimal models. It has first been considered in [12]. The number of simple conformal defects in this model is $\mathrm{tr}(ZZ^{\mathrm{t}}) = 16$ (and there are 8 conformal boundary conditions). The defect fusion rules can be computed using Ocneanu quantum algebras [5, 11], or weak Hopf algebras, or by TFT methods. The result can be summarised as follows. The set of defect labels can be written as $\mathcal{K} = \mathcal{K}_x \times \mathcal{K}_y$ with $\mathcal{K}_x = S_3 \cup \{u_+, u_-\}$ and $\mathcal{K}_y = \{\mathbf{1}, \varphi\}$, where S_3 denotes the permutation group of three symbols. The fusion product $D_{x,y} \otimes D_{x',y'} = \sum_{r \in x \cdot x'} \sum_{s \in y \cdot y'} D_{r,s}$ is obtained by the following rules. The product in $\mathcal{K}_y$ is given by Lee-Yang fusion rules $\varphi \cdot \varphi = \mathbf{1} + \varphi$, while the product in $\mathcal{K}_x$ is described as follows. For $p, p' \in S_3$, $p \cdot p'$ is given by the product in S_3, and $p \cdot u_\varepsilon = u_{\varepsilon'}$ with $\varepsilon \in \{\pm 1\}$ and $\varepsilon' = \varepsilon\, \mathrm{sgn}(p)$; finally, denoting the elements of S_3 by e (identity), p_{12}, p_{13}, p_{23} (transpositions), and p_{123}, p_{132} (cyclic permutations), we have $u_+ \cdot u_+ = u_- \cdot u_- = e + p_{123} + p_{312}$ and $u_+ \cdot u_- = u_- \cdot u_+ = p_{12} + p_{13} + p_{23}$. Owing to the presence of S_3, the fusion algebra of defects is non-commutative in this model. One can convince oneself that the group-like defects are $\mathcal{G} = \{(p, \mathbf{1}) \,|\, p \in S_3\}$ and the duality-defects are $\mathcal{D} = \{(x, \mathbf{1}) \,|\, x \in \mathcal{K}_x\}$.

The S_3-structure of the group-like defects could again have been expected from the lattice model realisation of the three-states Potts model; it amounts to a permutation of the three possible values of the spin.

The critical three-states Potts model contains 12 primary bulk fields and 208 primary disorder fields. Of these, we consider the energy operator $E(z)$ of left/right conformal weight $(\frac{2}{5}, \frac{2}{5})$, the two spin fields $S_\pm(z)$ of weight $(\frac{1}{15}, \frac{1}{15})$ and the two disorder fields $Z_\pm(z)$ of the same weight, where Z_+ generates a defect of type $(p_{123}, \mathbf{1})$ and Z_- one of type $(p_{132}, \mathbf{1})$. We find that taking a duality defect of type $(u_\varepsilon, \mathbf{1})$ through a spin field $S_\nu(z)$, for $\varepsilon, \nu \in \{\pm 1\}$, generates a disorder field $Z_{\varepsilon\nu}(z)$, and vice versa. Furthermore, taking $D_{u_\pm, \mathbf{1}}$ past the energy field $E(z)$ gives $-E(z)$, so that the the order/disorder duality at the critical point extends to a high/low temperature duality off the critical point.

Conclusions

We have demonstrated that the fusion algebra of defects in a CFT contains a lot of physical information: Internal symmetries correspond to group-like defects, and the order/disorder dualities to duality defects. The analysis is carried out within CFT, it allows one to study symmetry properties of universality classes of critical behaviour without reference to a particular lattice realisation. To compute the dual correlator one must simply commute a given duality defect past all field insertions. This procedure can be applied to correlators on surfaces of arbitrary genus and even with boundary. Via conformal perturbation theory, one can also identify high/low temperature dualities in the vicinity of the critical point.

To conclude, we mention that these considerations can also be applied to the free boson. One then finds that T-duality is induced by duality defects, too. The defect line in this example is labelled by the $\mathbb{Z}_2$-twisted representation of the $U(1)$-current algebra.

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