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Algebras in tensor categories and coset conformal field theories

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Abstract: The coset construction is the most important tool to construct rational conformal field theories with known chiral data. For some cosets at small level, so-called maverick cosets, the familiar analysis using selection and identification rules breaks down. Intriguingly, this phenomenon is linked to the existence of exceptional modular invariants. Recent progress in CFT, based on studying algebras in tensor categories, allows for a universal construction of the chiral data of coset theories which in particular also applies to maverick cosets.

1 Coset conformal field theories

The coset construction is among the oldest [1] tools for obtaining rational two-dimensional conformal field theories and has been very successful. It has been used to construct prominent classes of models, such as (super-)Virasoro minimal models and Kazama-Suzuki models. Still, it presents a number of mysteries, even in the case of unitary conformal field theories, to which we will restrict ourselves in this contribution.

The coset construction is based on the following data: A (finite-dimensional, complex, reductive) Lie algebra $\mathfrak{g}$ together with a choice k of levels, i.e. a positive integer for each simple ideal of $\mathfrak{g}$, and a Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$. The embedding of $\mathfrak{g}'$ into $\mathfrak{g}$ determines the levels k' of $\mathfrak{g}'$. The aim of the coset construction is to obtain conformal field theories whose chiral data – like conformal weights, fusion rules, braiding and fusing matrices – are completely known, and moreover can be expressed entirely in terms of the chiral data for $(\mathfrak{g}, k)$ and $(\mathfrak{g}', k')$. This goal can indeed be achieved. However, as will become evident below, the way this aim is reached is quite a bit more subtle than one might anticipate.

At first sight, understanding coset theories proceeds according to the following well-known pattern: The $\mathfrak{g}/\mathfrak{g}'$ coset theory has a description in terms of a gauged WZW sigma model [2] with target space the Lie group G (the compact simply connected covering group associated to $\mathfrak{g}$), in which the action of the subgroup G' of G is gauged. For constructing the space of states, this immediately suggests to start with positive energy representations

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of the (centrally extended) loop algebra $\hat{\mathfrak{g}}$ based on $\mathfrak{g}$ at level k, and require, à la Gupta-Bleuler, that the states of the coset theory are annihilated by the positive modes of the $\mathfrak{g}'$-currents $J^{a'}$ (and some more operators):

$$J^{a'}_n v = 0 \qquad \text{for} \quad n > 0\,.$$

This selects the highest weight spaces of $\hat{\mathfrak{g}}'$; the candidate spaces of states of coset theories are thus the *branching spaces* $\mathcal{H}^{\lambda}_{\lambda'}$ that appear in the decomposition

$$\mathcal{H}^{\mathfrak{g}}_{\lambda} = \textstyle\bigoplus_{\lambda'} \mathcal{H}^{\lambda}_{\lambda'} \otimes \mathcal{H}^{\mathfrak{g}'}_{\lambda'}$$

of irreducible highest weight $\mathfrak{g}$-modules $\mathcal{H}_\lambda$ into $\mathfrak{g}'$-modules $\mathcal{H}_{\lambda'}$.

Looking at simple examples reveals the following properties of these spaces:

- Some branching spaces can be zero.
- Some branching spaces can be isomorphic – not just as graded vector spaces, but even as modules over the coset chiral algebra (which is the commutant of the chiral algebra of $(\mathfrak{g}', k')$ in the chiral algebra of $(\mathfrak{g}, k)$).
- Some branching spaces can be reducible as modules over the coset chiral algebra.

The first two features – selection rules and "field identification" – already arise in the simplest example, the Ising model, which is realized by the coset

$$\frac{\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1}{\mathfrak{su}(2)_2}\,.$$

The branching spaces $\mathcal{H}^{l_1,l_2}_{l}$ of this theory are labeled by three integers $l_{1,2} \in \{0,1\}$ and $l \in \{0,1,2\}$ (twice the respective $\mathfrak{su}(2)$ spins). By the spin coupling rules, branching spaces can be non-zero only if $l_1 + l_2 - l$ is even. Moreover, $\mathcal{H}^{l_1,l_2}_{l}$ and $\mathcal{H}^{1-l_1,1-l_2}_{2-l}$ are isomorphic Virasoro modules, reflecting the familiar symmetry of the Kac table of the Ising model.

The selection rules and field identifications are, at least at first sight, well-understood in the Lagrangian setting: One actually gauges the *adjoint* action of the subgroup G' on the group G [3], so that the common center $Z := Z(G) \cap Z(G')$ acts trivially. As a consequence, the group relevant for gauging is G'/Z, which is non-simply connected. Both the selection rules and the field identifications are implemented by summing over inequivalent G'/Z bundles [4]. In an algebraic formulation, simple currents are the appropriate concept to explain these effects [5]; the selection rules eliminate branching spaces of non-zero monodromy charge, and isomorphic branching spaces form simple current orbits. In this setting also the problem of "fixed point resolution", i.e. of understanding the structure of reducible branching spaces, can be addressed [6].

2 Maverick coset theories

In a *maverick coset theory* the pattern of field identifications and selection rules governed by simple currents breaks down – there are more vanishing branching spaces, and more identifications. The observation that such maverick cosets exist came as a big surprise. The first example was presented in [7]; more examples were found in [8, 6, 9]. A classification is not known to date, but in all known maverick cosets the level is small.

The existence of maverick cosets would not have been that astonishing, though, had one only taken the lesson of *conformal embeddings* seriously. For a conformal embedding of $(\mathfrak{g}', k')$ in $(\mathfrak{g}, k)$ the Virasoro central charges of the respective WZW models coincide,

$c_{(\mathfrak{g}',k')} = c_{(\mathfrak{g},k)}$, so that the corresponding coset theory has central charge zero and hence is trivial as a chiral CFT: For such cosets *all* branching spaces are either zero or the trivial one-dimensional $c = 0$ Virasoro module. Maverick cosets are thus intermediate between 'ordinary' cosets and conformal embeddings.

Let us have a look at the simplest known example [7], $\mathfrak{su}(2)$ embedded via its three-dimensional representation into $\mathfrak{su}(3)$. The coset theory at level 2, $\mathfrak{su}(3)_2/\mathfrak{su}(2)_8$, is maverick. The ordinary selection rules allow all branching spaces $\mathcal{H}_q^{(l_1\, l_2)}$ with q even, and the expected identifications as Virasoro modules are $\mathcal{H}_q^{(l_1\, l_2)} \cong \mathcal{H}_{8-q}^{(l_1\, l_2)}$. However, comparison with the Kac table for the tetracritical Ising model shows that the branching spaces

$$\mathcal{H}_2^{(00)} \cong \mathcal{H}_6^{(00)},\ \mathcal{H}_2^{(20)} \cong \mathcal{H}_6^{(20)},\ \mathcal{H}_2^{(02)} \cong \mathcal{H}_6^{(02)},\ \mathcal{H}_0^{(10)} \cong \mathcal{H}_8^{(10)},\ \mathcal{H}_0^{(01)} \cong \mathcal{H}_8^{(01)},\ \mathcal{H}_0^{(11)} \cong \mathcal{H}_8^{(11)}$$

which a priori are allowed by the selection rules actually vanish as well, and that there are additional identifications

$$\begin{array}{lll}
\mathcal{H}_0^{(00)} \cong \mathcal{H}_8^{(00)} \cong \mathcal{H}_4^{(11)} & \chi(q) = & 1 \qquad + q^2 + 2q^3 + 3q^4 + 4q^5 \cdots \\
\mathcal{H}_2^{(11)} \cong \mathcal{H}_6^{(11)} \cong \mathcal{H}_4^{(00)} & & 1 + 2q + 2q^2 + 4q^3 + 5q^4 + 8q^5 \cdots \\
\mathcal{H}_2^{(10)} \cong \mathcal{H}_6^{(10)} \cong \mathcal{H}_4^{(02)} & & 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 \cdots \\
\mathcal{H}_2^{(01)} \cong \mathcal{H}_6^{(01)} \cong \mathcal{H}_4^{(20)} & & 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 \cdots \\
\mathcal{H}_0^{(20)} \cong \mathcal{H}_8^{(20)} \cong \mathcal{H}_4^{(01)} & & 1 + q + 2q^2 + 2q^3 + 4q^4 + 5q^5 \cdots \\
\mathcal{H}_0^{(02)} \cong \mathcal{H}_8^{(02)} \cong \mathcal{H}_4^{(10)} & & 1 + q + 2q^2 + 2q^3 + 4q^4 + 5q^5 \cdots
\end{array}$$

For all known maverick cosets, it has been observed that there exists a modular invariant torus partition function of extension type for the WZW based on $\mathfrak{g} \oplus \mathfrak{g}'$ in which

- only such pairs (λ, λ') of representations of $\mathfrak{g}$ and $\mathfrak{g}'$ appear that correspond to non-vanishing branching spaces $\mathcal{H}_{\lambda'}^{\lambda}$, and in which
- the way these pairs are combined into irreducible representations of the extended chiral algebra reflects also the additional identifications.

Moreover, in all maverick cases this modular invariant for the $\mathfrak{g} \oplus \mathfrak{g}'$ WZW model is not of simple current type, but *exceptional*. Having mentioned the term partition function, it is worth pointing out that here our aim is to understand coset theories as *chiral* conformal field theories.[1] Accordingly, the $\mathfrak{g} \oplus \mathfrak{g}'$ torus partition function in question is nothing but the charge conjugation modular invariant with respect to the extended chiral algebra.

Recently, the study of conformal field theories on surfaces with boundary has given many new insights in the structure of modular invariant partition functions (see [11] and references therein). In particular, novel techniques for exceptional modular invariants have become available. In the rest of this note we present some of these techniques and explain how they allow us to gain a better understanding of maverick cosets.

3 Algebras in tensor categories

The first step is to find a convenient basis-independent way to encode the chiral data of a given rational chiral CFT. This is provided by the representation category $\mathcal{C}$ of the given chiral algebra, which has the structure of a *modular tensor category*. The objects V of $\mathcal{C}$ are representations of the chiral algebra, and the morphisms $f\colon V \to W$ of $\mathcal{C}$ are intertwiners. Fusion of chiral algebra representations is encoded by a tensor product

$$\otimes\colon \quad \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

[1] Indeed, already as chiral theories, they are useful in physical applications [10].

that is associative, with the vacuum representation $\mathbf{1}$ acting as unit, $\mathbf{1}\otimes V = V$. Braid group statistics, obeyed by quantum fields in low dimensions, yields a braiding on $\mathcal{C}$, i.e. for every pair (V,W) of objects an isomorphism $c_{V,W}\colon V\otimes W\to W\otimes V$. Pictorially, the braiding is shown in figure 1. Non-degeneracy of CFT two-point functions gives rise to a notion of dual object (conjugate field), and the fractional part of the conformal weight defines a 'twist' for every object.

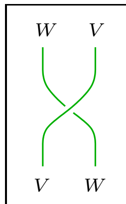

Figure 1

These structures are subject to quite a few axioms, of course. Most of them just amount to the statement that a visualization in terms of ribbon graphs like in figure 1 is possible. In addition, it is required that the matrix with entries $s_{\lambda,\mu} := \mathrm{tr}(c_{\lambda,\mu}\circ c_{\mu,\lambda})$ is non-degenerate. (This trace, depicted in figure 2, is the invariant of the Hopf link in the three-manifold S^3; the V_μ are representatives for the isomorphism classes of simple objects of $\mathcal{C}$.)

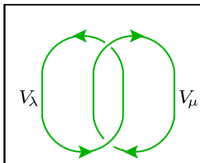

Figure 2

In the case of coset theories we are given two modular tensor categories, $\mathcal{G}$ and $\mathcal{G}'$, for the chiral data of the WZW models based on $(\mathfrak{g},k)$ and $(\mathfrak{g}',k')$. The goal we would like to achieve is then to express the category $\mathcal{Q}$ for the coset theory in terms of $\mathcal{G}$ and $\mathcal{G}'$.

To this end, we use the fact that algebra and representation theory can be developed not just for (real or complex) vector spaces, but also in the much more general context of tensor categories. An algebra (A,m,η) in a tensor category $\mathcal{C}$ consists of an object A of $\mathcal{C}$, a multiplication $m\colon A\otimes A\to A$ that is associative, i.e. fulfils

$$m\circ(id_A\otimes m) = m\circ(m\otimes id_A)\,,$$

and a unit $\eta\colon \mathbf{1}\to A$. The latter should be thought of as the generalization of the map $\mathbb{C}\to A$ that acts as $\xi\mapsto\xi e$ for an algebra A with unit element e in the case that A is a complex vector space. A (left-) representation (M,ρ_M) of A can be defined similarly: It is an object M of $\mathcal{C}$ together with a morphism $\rho_M\colon A\otimes M\to M$ that obeys

$$\rho_M\circ(m\otimes id_M) = \rho_M\circ(id_A\otimes\rho_M) \qquad\text{and}\qquad \rho_M\circ(\eta\otimes id_M) = id_M\,.$$

A particular class of algebras in modular tensor categories, called *symmetric special Frobenius algebras*, is relevant in conformal field theory. All information about a full local CFT based on a given chiral CFT is encoded in such an algebra [11]. For instance, the coefficients of the torus partition function are given by the dimension of the space of intertwiners of certain A-bimodules:

$$Z_{\lambda,\mu} = \dim\mathrm{Hom}_{A|A}(\alpha_A^-(V_\lambda),\alpha_A^+(V_\mu))\,.$$

(It follows from general results that this is always modular invariant.) This partition function is of extension type if A is commutative in the sense that $m\circ c_{A,A} = m$. In that case, the algebra describes just the vacuum sector of the extension, while the other sectors of the extended theory correspond to so-called *local* A-modules. Locality of an A-module means that the relation displayed in figure 3 is satisfied. The category $\mathcal{C}_A^{\ell oc}$ of local modules over a symmetric special Frobenius algebra A in a modular tensor category $\mathcal{C}$ is again modular. This gives a very concise handle on the chiral data of the extended theory:

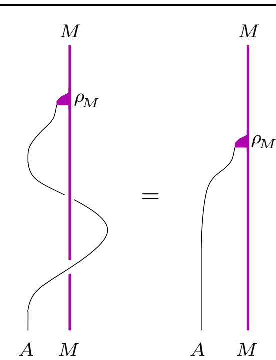

Figure 3

$$\mathcal{C}_{\rm ext} = \mathcal{C}_A^{\ell oc}\,.$$

The vacuum sector $\mathcal{H}_\Omega^{\mathfrak{g}}$ of the $\mathfrak{g}$-theory decomposes in terms of sectors of the $\mathfrak{g}'$-theory and of the coset theory according to

$$\mathcal{H}_\Omega^{\mathfrak{g}} = \bigoplus\nolimits_{\lambda'} \mathcal{H}_{\lambda'}^{\Omega} \otimes \mathcal{H}_{\lambda'}^{\mathfrak{g}'} .$$

It follows that the modular tensor category $\mathcal{G}$ of the $\mathfrak{g}$-theory can be expressed as

$$\mathcal{G} = (\mathcal{Q} \otimes \mathcal{G}')_A^{\ell oc} \qquad (1)$$

through the categories $\mathcal{Q}$ and $\mathcal{G}'$ and a suitable commutative symmetric special Frobenius algebra A in $\mathcal{Q} \otimes \mathcal{G}'$ that encodes the decomposition of the $\mathfrak{g}$-vacuum $\mathcal{H}_\Omega^{\mathfrak{g}}$ given above.

For the understanding of the coset category $\mathcal{Q}$ the following result is crucial:

Theorem [12]: There exists a (braided-) commutative symmetric special Frobenius algebra B in the modular tensor category $\mathcal{G} \otimes \overline{\mathcal{G}'}$ such that

$$\mathcal{Q} = (\mathcal{G} \otimes \overline{\mathcal{G}'})_B^{\ell oc} . \qquad (2)$$

This algebra B corresponds to a modular invariant of extension type for the $\mathfrak{g} \oplus \mathfrak{g}'$ theory.

This is the desired expression of the chiral data $\mathcal{Q}$ of the coset theory in terms of the chiral data $\mathcal{G}$ and $\mathcal{G}'$ of the parent WZW models. The category $\overline{\mathcal{G}'}$ in the theorem is obtained in a straightforward manner from $\mathcal{G}'$. Basically, one applies complex conjugation to all chiral data. For details we refer to [12]. Also, B can be constructed explicitly from the embedding $\mathfrak{g}' \hookrightarrow \mathfrak{g}$, and modularity of the tensor category $\mathcal{Q}$ is *derived* from the modularity of $\mathcal{G}$ and $\mathcal{G}'$ (and from some much weaker assumptions on $\mathcal{Q}$).

A detailed discussion of the proof of the theorem is beyond the scope of this note. Let us, however, mention three crucial aspects.

- In [13, 12], commutative algebra in a braided setting is developed. This theory turns out to be much richer than ordinary commutative algebra for vector spaces (which is already a rich theory). New notions, like local modules, two different centers of an algebra, and new types of induction functors, play an essential role in the proof.
- Another important ingredient is the unitarity of the modular matrix S of rational CFTs. This property, which means that the braiding in two-dimensional CFTs is, in a sense, maximally non-degenerate, allows to "solve" the equality (1) for $\mathcal{Q}$ in the form (2).
- All sectors appearing in the algebra B contain fields of coset conformal weight 0. The transition to local B-modules can therefore be thought of as a means for removing "redundant" vacua in the collection of branching spaces.

4 Conclusions

Despite the progress reported here, the coset construction still presents open problems:

- The classification of coset theories $(\mathfrak{g}, \mathfrak{g}', k)$ that are maverick remains open.
- Mavericks seem to be a low level phenomenon. For the group manifold as a sigma model target space, low level means large curvature. An understanding of the additional selection rules and identifications in maverick cosets as large curvature effects in a Lagrangian setting seems to be far beyond the reach of today's methods.

Finally we point out that the results presented here arose within a larger research program [11, 14], which aims at constructing full local CFTs on world sheets of arbitrary topology from chiral CFTs using algebras in tensor categories. It is gratifying that the tools developed in this approach also shed new light on old and mysterious problems like the existence of exceptional modular invariants associated to maverick coset theories.

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