Picard groups in rational conformal field theory

Jürg Fröhlich, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert

ABSTRACT. Algebra and representation theory in modular tensor categories can be combined with tools from topological field theory to obtain a deeper understanding of rational conformal field theories in two dimensions: It allows us to establish the existence of sets of consistent correlation functions, to demonstrate some of their properties in a model-independent manner, and to derive explicit expressions for OPE coefficients and coefficients of partition functions in terms of invariants of links in three-manifolds.

We show that a Morita class of (symmetric special) Frobenius algebras A in a modular tensor category $\mathcal{C}$ encodes all data needed to describe the correlators. A Morita-invariant formulation is provided by module categories over $\mathcal{C}$. Together with a bimodule-valued fiber functor, the system (tensor category + module category) can be described by a weak Hopf algebra.

The Picard group of the category $\mathcal{C}$ can be used to construct examples of symmetric special Frobenius algebras. The Picard group of the category of A-bimodules describes the internal symmetries of the theory and allows one to identify generalized Kramers-Wannier dualities.

1. Modular tensor categories and topological field theories

The structure of a modular tensor category appears in a variety of representation theoretic problems in mathematical physics. For the purposes of this contribution, by a modular tensor category we understand an abelian, semi-simple $\mathbb{C}$-linear tensor category $\mathcal{C}$ that comes with a braiding, i.e. a collection of functorial isomorphisms

$$c_{X,Y}: \quad X \otimes Y \xrightarrow{\cong} Y \otimes X \,,$$

for any pair X, Y of objects of $\mathcal{C}$, and a twist, i.e. a collection of functorial isomorphisms

$$\theta_X: \quad X \xrightarrow{\cong} X$$

for any object X of $\mathcal{C}$, such that the following axioms hold: First,

$$\theta_{X \otimes Y} = c_{Y,X} \circ (\theta_Y \otimes \theta_X) \circ c_{X,Y} \,;$$

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second, there is a compatible duality; third, there are only finitely many isomorphism classes of simple objects – a set of representatives of which we denote by $\{U_i\}_{i \in I}$ – and the tensor unit $\mathbf{1} = U_0$ is simple. Finally, the braiding is maximally non-degenerate in the sense that the matrix

$$s_{ij} = \operatorname{tr} c_{U_i, U_j} \circ c_{U_j, U_i}$$

$(i, j \in I)$ is invertible.

Modular tensor categories arise in various contexts. For example, the representation categories of the following algebraic structures can be modular tensor categories: weak quantum groups, conformal nets of von Neumann algebras on the real line, and vertex algebras. In view of the role of the two latter structures in two-dimensional conformal quantum field theory (CFT) (see e.g. [**6**, **21**]), it follows in particular that modular tensor categories constitute the axiomatization of the chiral data – in essence, the monodromy of the conformal blocks – of rational CFTs.

Recently, quite a few results have been obtained that characterize cases when representation categories are modular:

- If H is a connected C^* weak Hopf algebra, then the category of unitary representations of its double is a unitary modular tensor category [**28**].
- Similarly, the representation category of a connected ribbon factorizable weak Hopf algebra over $\mathbb{C}$ (or, more generally, over any algebraically closed field $\mathbb{k}$) with a Haar integral is modular [**28**].
- If a finite-index net of von Neumann algebras on the real line is strongly additive (which for conformal nets is equivalent to Haag duality) and has the split property, its category of local sectors is a modular tensor category [**23**].
- Finally, according to the results of [**22**], if a self-dual vertex algebra that obeys Zhu's C_2 cofiniteness condition and certain conditions on its homogeneous subspaces has a semi-simple representation category, then this category is actually a modular tensor category.

The definition of a modular tensor category was motivated [**36**] by the fact that it allows for the construction of a three-dimensional topological field theory (TFT). Such a TFT furnishes a modular functor, i.e. it assigns finite-dimensional vector spaces to two-manifolds – more precisely, to extended surfaces – and linear maps to cobordisms.

An extended surface (for a given tensor category $\mathcal{C}$) is a closed oriented two-dimensional manifold X with finitely many embedded (germs of) arcs labelled by objects of $\mathcal{C}$, together with the choice of a Lagrangian subspace in $H_1(\mathrm{X}, \mathbb{R})$. There is a natural notion of morphisms of extended surfaces. Given two extended surfaces X and Y, a cobordism M from X to Y is an oriented three-manifold with an embedded ribbon graph such that $\partial \mathrm{M} = \mathrm{X} \sqcup (-\mathrm{Y})$.

The complex vector spaces $\mathcal{H}(\mathrm{X})$ – called the state spaces of the TFT, or the spaces of conformal blocks – assigned to extended surfaces X obey $\mathcal{H}(\emptyset) = \mathbb{C}$ and the multiplicativity property $\mathcal{H}(\mathrm{X} \sqcup \mathrm{Y}) = \mathcal{H}(\mathrm{X}) \otimes_{\mathbb{C}} \mathcal{H}(\mathrm{Y})$. A modular functor associates to each morphism $f \colon \mathrm{X} \to \mathrm{Y}$ of extended surfaces a linear map $f_\sharp \colon \mathcal{H}(\mathrm{X}) \to \mathcal{H}(\mathrm{Y})$, while to a cobordism $(\mathrm{M}, \partial_- \mathrm{M}, \partial_+ \mathrm{M})$ it assigns a linear map

$$Z(\mathrm{M}, \partial_- \mathrm{M}, \partial_+ \mathrm{M}): \quad \mathcal{H}(\partial_- \mathrm{M}) \to \mathcal{H}(\partial_+ \mathrm{M}).$$

In particular, a cobordism of the form $Z(\mathrm{M}, \emptyset, \partial \mathrm{M})$ gives rise to map $\mathbb{C} \to \mathcal{H}(\partial \mathrm{M})$. Put differently, a ribbon graph in a three-manifold M allows one to specify a vector in the space of conformal blocks on the boundary $\partial \mathrm{M}$.

The axioms for the linear maps Z – naturality, multiplicativity, normalization of the cylinder, and functoriality – have two important consequences:

- Each space $\mathcal{H}(\mathrm{X})$ of conformal blocks carries a projective representation of the mapping class group $Map(\mathrm{X})$.
- By gluing two arcs which are labelled by simple objects U_j and $U_j^\vee$ via a tube with embedded U_j-ribbon, one obtains isomorphisms $\bigoplus_{j \in I} \mathcal{H}(\mathrm{X}_j) \xrightarrow{\cong} \mathcal{H}(\mathrm{X}')$, called *factorization rules*.

To formulate a TFT one employs a cobordism category of topological manifolds. The use of the term *conformal* block therefore needs to be justified. Given a conformal vertex algebra $\mathfrak{V}$ and an m-tuple $(\mathcal{H}_{\lambda_1}, \mathcal{H}_{\lambda_2}, \dots, \mathcal{H}_{\lambda_m})$ of $\mathfrak{V}$-modules, conformal blocks are constructed [8] as vector bundles $\mathcal{B}(\lambda_1, \lambda_2, \dots, \lambda_m)$ with a projectively flat connection over the moduli space $\mathcal{M}_{g,m}$ of complex curves of genus g with m marked points. The monodromy data of conformal blocks on $\mathbb{CP}^1$ can then be used to equip the representation category $\mathcal{R}ep(\mathfrak{V})$ with additional structure like a tensor product (fusion) and a braiding. In certain cases, e.g. the ones described in [22], this endows $\mathcal{R}ep(\mathfrak{V})$ with the structure of a modular tensor category. From this modular tensor category, one obtains representations of the mapping class groups. It is an important open conjecture that these representations are isomorphic to the ones provided by the monodromies of the vector bundles $\mathcal{B}$. We will assume that this conjecture holds true; this allows us to pass tacitly between topological categories of topological two-manifolds and categories of conformal or complex manifolds.

The vector bundles of conformal blocks are – up to the choice of local coordinates, a subtlety we ignore for the purposes of this contribution – constructed as subbundles of the trivial bundle

$$\mathcal{M}_{g,m} \times (\mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_2} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m})^*.$$

Applying a flat section of the subbundle to an m-tuple of vectors $v_i \in \mathcal{H}_{\lambda_i}$ therefore yields a multivalued function on $\mathcal{M}_{g,m}$. A central question, to be addressed in the next section, is how these multivalued functions are related to physical correlation functions of the conformal field theory.

2. Geometry for correlators

To describe the correlators of a local conformal field theory on a surface X – which may have a non-empty boundary and, for the moment, is supposed to be oriented – we associate to X its (oriented) double $\hat{\mathrm{X}}$. The double $\hat{\mathrm{X}}$ comes with an orientation reversing involution σ such that $\mathrm{X} = \hat{\mathrm{X}}/\{1, \sigma\}$. If we work in a geometric category, i.e. if X is supposed to be a conformal manifold, then the double has a complex structure and σ is anti-conformal.

Correlators on X are specific vectors $C(\mathrm{X})$ in the spaces $\mathcal{H}(\hat{\mathrm{X}})$ (this is known as the 'principle of holomorphic factorization'; it has been derived [37] for important classes of rational CFTs like (gauged) WZW models by introducing a background gauge field on the world sheet). These particular elements of $\mathcal{H}(\hat{\mathrm{X}})$ satify two types of conditions:

- The vector $C(\mathrm{X}) \in \mathcal{H}(\hat{\mathrm{X}})$ is required to be invariant under the mapping class group $Map(\mathrm{X})$, which can be identified with the subgroup of $Map(\hat{\mathrm{X}})$ that commutes with σ. (Recall that $Map(\hat{\mathrm{X}})$ acts projectively on $\mathcal{H}(\hat{\mathrm{X}})$.)

- The correlators must obey factorization constraints. We do not write down these explicitly here; in short, the image of a correlator under an isomorphism that is given by the factorization rules for the conformal blocks on the double is again a correlator. (See [**17**] for precise statements and proofs.)

In view of the properties of TFT discussed above, it is natural to look for an oriented three-manifold M_X with boundary $\partial M_X = \hat{X}$ such that $C(X) := Z(M_X, \emptyset, \hat{X})(1)$ is the correlator. The following ansatz turns out to be successful: take the product of the double $\hat{X}$ with the interval $[-1, 1]$ and mod out by the orientation preserving $\mathbb{Z}_2$ action for which the non-trivial element acts like σ on $\hat{X}$ and like $t \mapsto -t$ on the interval:

$$M_X = \left(\hat{X} \times [-1, 1] \right) / (\sigma, t \mapsto -t).$$

This so-called connecting manifold [**7**] has boundary $\partial M_X = \hat{X}$; there is a natural embedding $\iota \colon X \to M_X$ acting as $x \mapsto [x, 0]$, which shows that X is a retract of M_X. For example, when X is a disk, M_X is a full ball.

To specify a correlator $C(X) \in \mathcal{H}(\hat{X})$, the three-manifold M_X must be endowed with a ribbon graph. Points in the interior of X have two pre-images in $\hat{X}$, points on ∂X only one. For each field insertion point p we place a ribbon along the distinguished interval in M_X that joins $\iota(p)$ to the pre-image(s) of p; we also put ribbons along the boundary of $\iota(X)$. All ribbons are labelled by objects of $\mathcal{C}$, in a manner to be described in the next section. Finally, we triangulate $\iota(X)$, in such a manner that only trivalent vertices occur. (Faces can have an arbitrary number of edges; thus, properly speaking, we are dealing with the dual of a triangulation.) Each marked point $\iota(p)$ for p in the interior of X must lie on an edge of the triangulation T_X. We place ribbons along the edges of T_X. This gives in particular rise to trivalent and quadrivalent intersections, at which we put coupons on which the ribbons end – trivalent for vertices of the triangulation and for the points $\iota(p)$ with $p \in \partial X$, and quadrivalent for the points $\iota(p)$ with p in the interior of X.

For each of the coupons an element in the appropriate morphism space must be chosen. For the part of the triangulation that lies in the interior of $\iota(X)$ this is done as follows. Choose an object A of $\mathcal{C}$. On each edge we place two ribbons labeled by A which start at one of the vertices and have orientation pointing away from that vertex, and join them with a suitable morphism Φ in $\mathrm{Hom}(A, A^\vee)$, to be specified below. At each vertex we need an element in $\mathrm{Hom}(\mathbf{1}, A \otimes A \otimes A)$. A comparison of this situation with the analysis of so-called lattice TFTs [**20**, **1**], where $\mathcal{C}$ is the category of finite-dimensional $\mathbb{C}$-vector spaces, leads us to expect that A is a generalization of a Frobenius algebra to more general tensor categories $\mathcal{C}$. As we will see, this allows us to give a model-independent approach to correlators of rational CFTs that is based on a combination of TFT in three dimensions and of non-commutative algebra in tensor categories.

3. Frobenius algebras in modular tensor categories

It is in fact not hard to generalize many notions of algebra and representation theory from vector spaces (or modules over commutative rings) to more general tensor categories. A (unital, associative) algebra $A \equiv (A, m, \eta)$ in a tensor category $\mathcal{C}$, for example, is an object A of $\mathcal{C}$ together with a multiplication morphism $m \in \mathrm{Hom}(A \otimes A, A)$ that obeys

$$m \circ (id_A \otimes m) = m \circ (m \otimes id_A)$$

and together with a unit morphism $\eta \in \mathrm{Hom}(\mathbf{1}, A)$ such that

$$m \circ (\eta \otimes id_A) = id_A = m \circ (id_A \otimes \eta).$$

A (coassociative, counital) coalgebra (A, Δ, ε) is defined analogously. A *Frobenius algebra* in $\mathcal{C}$ is an algebra that is also a coalgebra, with the additional property that the coproduct $\Delta \in \mathrm{Hom}(A, A \otimes A)$ is a morphism of A-bimodules.

The Frobenius algebras of interest to us possess two more properties. First, as in most applications in representation theory, they are *symmetric*: denote by $d_A \in \mathrm{Hom}(\mathbf{1}, A \otimes A^\vee)$ and $\tilde{d}_A \in \mathrm{Hom}(\mathbf{1}, A^\vee \otimes A)$ the two coevaluations of the category $\mathcal{C}$ (which we assume to be sovereign). There are two natural isomorphisms – in fact isomorphisms of A-bimodules –

$$\Phi_1 := ((\varepsilon \circ m) \otimes id_{A^\vee}) \circ (id_A \otimes d_A), \qquad \Phi_2 := (id_{A^\vee} \otimes (\varepsilon \circ m)) \circ (\tilde{d}_A \otimes id_A)$$

in $\mathrm{Hom}(A, A^\vee)$; in a symmetric Frobenius algebra, these two isomorphisms coincide. It is the morphism $\Phi \equiv \Phi_1 = \Phi_2$ that we use along the edges of the triangulation of $\iota(\mathrm{X})$. Second, our Frobenius algebras are *special*, which means that Δ is a right-inverse of the multiplication – this generalizes the notion of a separable algebra over a field – and that $\varepsilon \circ \eta = \dim(A) \, id_\mathbf{1}$.

It can be shown [**13**] that in a rational CFT the operator product (OPE) for boundary fields that preserve a given boundary condition leads to a symmetric special Frobenius algebra in the modular tensor category $\mathcal{C}$ that describes the chiral data of the CFT. The main ingredients are the associativity of the OPE and the non-degeneracy of the two-point correlation function of two boundary fields on a disk, which furnishes the non-degenerate invariant bilinear form. It should also be appreciated that a Frobenius algebra obtained this way from boundary fields in CFT is not necessarily (braided-)commutative, and that the underlying boundary condition is not required to be 'elementary'.

With this in mind, the main insight of our construction can be summarized as follows [**12**]. For given chiral data $\mathcal{C}$, a full local CFT – which we denote as $\mathrm{CFT}(A)$ – can be constructed from a symmetric special Frobenius algebra A in $\mathcal{C}$. This Frobenius algebra is the algebra of boundary fields (or, in string theory terminology, of open string states) for a single boundary condition.

As for other boundary conditions than the one used to define A, the analysis of boundary OPEs involving also boundary condition changing operators shows [**13**] that they correspond to modules over the Frobenius algebra A. Here modules are defined in the obvious way: a (left) A-module is a pair $M \equiv (\dot{M}, \rho_M)$ consisting of an object $\dot{M}$ of $\mathcal{C}$ and a morphism $\rho_M \in \mathrm{Hom}(A \otimes \dot{M}, \dot{M})$ such that $\rho_M \circ (m \otimes id_{\dot{M}}) = \rho_M \circ (id_A \otimes \rho_M)$ and $\rho_M \circ (\eta \otimes id_{\dot{M}}) = id_{\dot{M}}$. Many aspects of representation theory can be generalized, see e.g. [**30, 24, 19**] (in fact, some peculiar aspects can only be seen in a braided setting, compare [**9**]). For instance, there is a notion of induced module, reciprocity theorems hold, every simple module appears in the decomposition of an induced module, and one can show that the module category of a special Frobenius algebra in a semi-simple tensor category is again semi-simple.

These observations supply us with the first few entries in a dictionary relating physical concepts to algebraic notions: boundary conditions are A-modules, 'elementary' boundary conditions are simple A-modules; a direct sum of simple

A-modules amounts to introducing boundary conditions with Chan-Paton multiplicities. For our construction of a ribbon graph in the connecting manifold M_X, we conclude that ribbons labelled with the object underlying a boundary condition are to be placed along the boundary segments of $\iota(X)$.

This dictionary can be extended, and this extension at the same time completes our labeling of the ribbons in the connecting manifold M_X. A boundary field Ψ_U^{MN} that changes the boundary condition from M to N has a single chiral insertion U and a trivalent vertex at the boundary of $\iota(X)$ that must be labeled by an element of $\mathrm{Hom}_A(M \otimes U, N)$, where $M \otimes U$ carries the obvious structure of a left A-module and Hom_A denotes morphisms of left A-modules.

Field insertions p in the interior of X have two pre-images. Bulk fields are thus labeled Φ_{UV}: the two ribbons in M_X that originate from the pre-images of p are inward-pointing and are labeled by U and V, respectively. Further, these ribbons hit the A-ribbon that is placed on the edge of the triangulation T_X passing through $\iota(p)$, and the corresponding quadri-valent vertex is labeled by an element of $\mathrm{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A)$, a morphism of A-bimodules. Here the superscripts $\pm$ indicate that the object $U \otimes A \otimes V$ is given the following structure of an A-bimodule: the left action is $(id_U \otimes m \otimes id_V) \circ (c_{U,A}^{-1} \otimes id_{A \otimes V})$, while the right A-action is $(id_U \otimes m \otimes id_V) \circ (id_{U \otimes A} \otimes c_{A,V}^{-1})$.

It is natural to not only consider the special A-bimodule A itself, but allow for arbitrary A-bimodules B as well. They correspond to (tensionless) conformal defect lines which can be added to the triangulation. Defect fields can change such defects; the corresponding quadri-valent vertices for a change of defect from B_1 to B_2 are labeled by A-bimodule morphisms in $\mathrm{Hom}_{A|A}(U \otimes^+ B_1 \otimes^- V, B_2)$.

Using the ansatz for obtaining the correlators in terms of three-manifolds with embedded ribbon graphs described above, factorization and invariance under the action of the mapping class group $Map(X)$ can be proven [**17**]. Also, other mathematical objects defined previously in the discussion of rational conformal field theory, like the so-called classifying algebra or NIMreps, are recovered naturally and their properties can be established rigorously [**13**]. It is worth emphasizing that our formalism provides a unified treatment of all modular invariant torus partition functions – both those of simple current type and exceptional modular invariants.

Note that we have started from a single arbitrary boundary condition to construct the (symmetric special) Frobenius algebra A. A different boundary condition will, in general, give us a different Frobenius algebra A'. But as it turns out, for a given CFT any two such Frobenius algebras are Morita equivalent and, moreover, Morita equivalent Frobenius algebras give equivalent correlators; we express this as $\mathrm{CFT}(A') \cong \mathrm{CFT}(A)$.

Finally, we mention that our construction can be extended [**14**] to the situation that X is not oriented, and possibly not even orientable. In that case, the Frobenius algebra A must be equipped with the additional structure of a *Jandl algebra*. For a Jandl algebra A there is an isomorphism of algebras $\sigma \colon A^{\mathrm{opp}} \to A$ which squares to the twist, $\sigma^2 = \theta_A$. In the special case that $\mathcal{C}$ is the category of vector spaces, the structure of a Jandl algebra reduces to a (symmetric special Frobenius) algebra with an involution.

4. Relation to other structures

Next we wish to describe the relation of the approach to rational CFT based on symmetric special Frobenius algebras to other algebraic structures whose relevance to the problem has been suggested.

As a first step, we notice that the category $\mathcal{C}_A$ of left modules over an algebra A in a tensor category $\mathcal{C}$ carries the structure of a so-called module category [30, 4] over $\mathcal{C}$: there is a "mixed" tensor functor

$$\otimes : \quad \mathcal{C}_A \times \mathcal{C} \to \mathcal{C}_A$$

with an associativity constraint that satisfies generalized triangle and pentagon axioms. Morita equivalent algebras can be characterized [29] by the fact that they yield equivalent module categories.

For a symmetric special Frobenius algebra in an abelian semi-simple tensor category, the module category $\mathcal{M} = \mathcal{C}_A$ is abelian and semi-simple. We can therefore find a complex semi-simple algebra R whose representation category $\mathcal{M}od(R)$ is equivalent, as an abelian category, to $\mathcal{M}$. Obviously, R is a direct sum of full matrix algebras whose number of minimal ideals equals the number of simple objects in $\mathcal{M}$. Since only the number of minimal ideals, but not their dimension, matters, R is a non-canonical object. In any case, this equivalence endows $\mathcal{M}od(R)$ with the structure of a module category over $\mathcal{C}$.

Next, recall the elementary fact that an abelian group M is a module over a ring S iff there is a morphism of rings from S into $\mathrm{End}(M)$. An analogous theorem is valid for categories [29]; the ring S is replaced by the tensor category $\mathcal{C}$, the module M by the module category $\mathcal{M}$, and morphisms of rings by fiber functors, i.e., in the setting we are interested in, by monoidal functors. If $\mathcal{M} \cong \mathcal{M}od(R)$ as an abelian category, then the bimodules $\mathcal{B}imod(R)$ play the role of $\mathrm{End}(M)$, and we have a natural bijection between fiber functors from $\mathcal{C}$ into $\mathcal{B}imod(R)$ and structures of a module category over $\mathcal{C}$ on $\mathcal{M}od(R)$.

Thus in the situation of interest to us we obtain a bimodule-valued fiber functor

$$\omega_R : \quad \mathcal{C} \to \mathcal{B}imod(R) \,.$$

One would now like to apply familiar arguments from Tannaka theory to the algebra $H_R := \mathrm{End}(\omega_R)$ of endomorphisms of the fiber functor to endow it with some structure that generalizes Hopf algebras. This can indeed be done [35], provided that separability data for R are chosen, i.e. a right-inverse of the multiplication and a left-inverse of the unit. (Except for the case that all minimal ideals of R are one-dimensional, there is no canonical separability structure.) It turns out that H_R can then be endowed with the structure of a weak Hopf algebra and that $\mathcal{M}od(H_R)$ is equivalent, as a tensor category, to $\mathcal{C}$.

This construction has a converse: any weak Hopf algebra H gives rise to the tensor category $\mathcal{C} := \mathcal{M}od(H)$ of left H-modules and a module category $\mathcal{M}$ over $\mathcal{C}$: A weak Hopf algebra contains two commuting associative unital subalgebras H_s and H_t which are related by the antipode. Since the antipode is an anti-morphism of algebras, H_t can be identified with the opposite algebra of H_s. By restriction to H_s and H_t, any left H-module can be seen to be an H_t-bimodule, and hence we have found a tensor functor from $\mathcal{C}$ to $\mathcal{B}imod(H_t)$. The general result mentioned above now implies that the category $\mathcal{M}od(H_t)$ is a module category over $\mathcal{C}$.

Now indeed, as has been argued in [**3, 32**], the structure of a weak Hopf algebra (called Ocneanu double triangle algebra) can be abstracted from a rational CFT and its boundary conditions. The discussion above shows that this algebra is not canonical; there are infinitely many non-isomorphic weak Hopf algebras which lead to equivalent module categories and hence to one and the same CFT. It is also worth mentioning that in this description the braiding on $\mathcal{C}$ must be expressed in terms of an R-matrix for the weak Hopf algebra, a structure that is not as well amenable to explicit computations as our pictorial calculus using ribbon graphs. Still, this approach can provide non-trivial insight; for instance, using the fact that Davydov–Yetter cohomology of the pair $\mathcal{M}, \mathcal{C}$ can be expressed in terms of Hochschild cohomology of any of the Hopf algebras H_R, it was shown in [**5**] that rational conformal field theories cannot be deformed within the class of rational conformal field theories.

Given a module category $\mathcal{M} \cong \mathcal{C}_A$ over a tensor category $\mathcal{C}$, immediately a third category enters the game the tensor category $\mathcal{C}^* \cong {}_A\mathcal{C}_A$ of A-bimodules. In our situation the latter category is actually equivalent to the category of module functors $\mathcal{M} \to \mathcal{M}$ and thus does not depend on the choice of A in a Morita class. In contrast to $\mathcal{M}$ the category $\mathcal{C}^*$ is a tensor category, with tensor unit A. While in our case $\mathcal{C}$ is braided, $\mathcal{C}^*$ is not braided, in general. Indeed, the objects of $\mathcal{C}^*$ have the physical interpretation of tensionless defect lines whose fusion [**31, 32**] is described by the tensor product on $\mathcal{C}^*$, and there is no reason for the fusion of defects to be commutative.
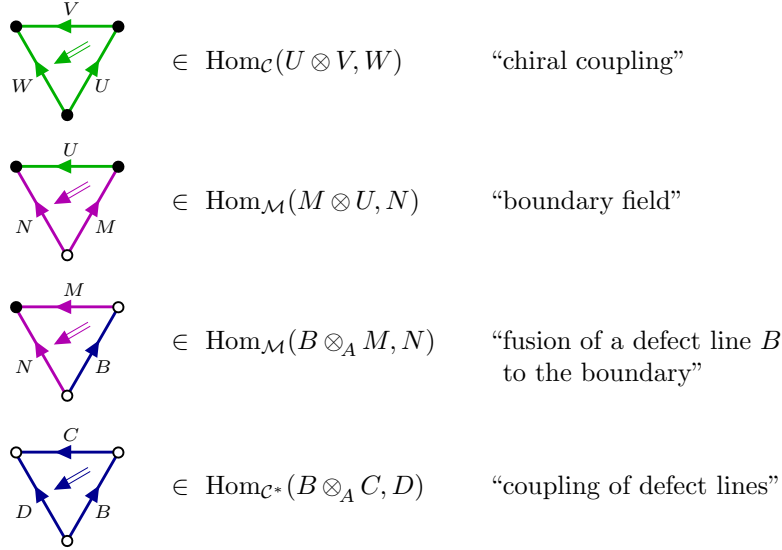
In the present situation we then deal with four bifunctors [**29**]: the tensor products of $\mathcal{C}$ and $\mathcal{C}^*$, the defining operation $\mathcal{M} \times \mathcal{C} \to \mathcal{M}$ of the module category $\mathcal{M}$, and a functor

$$\mathcal{C}^* \times \mathcal{M} \to \mathcal{M}$$

acting on objects as $(B, M) \mapsto B \otimes_A M$. There are five associativity constraints: one for each of the tensor categories $\mathcal{C}$ and $\mathcal{C}^*$, and three mixed constraints for the threefold products $\mathcal{M} \times \mathcal{C} \times \mathcal{C}$, $\mathcal{C}^* \times \mathcal{C}^* \times \mathcal{M}$ (stating that $\mathcal{M}$ is a module category over both $\mathcal{C}$ and $\mathcal{C}^*$) and $\mathcal{C}^* \times \mathcal{M} \times \mathcal{C}$. Associativity of higher products is ensured by six axioms of pentagon type [**29**, Section 4.3]. It should be emphasized that the braiding on $\mathcal{C}$ – which is a crucial structure in the application to CFT – is not accounted for in this setup.

Other notions that have been discussed in this context are graphs and cells [**32**]. The term 'cell' is motivated by the following visualization of the structure described above in terms of oriented simplices in three dimensions. Vertices can be coloured "black" or "white". This gives three types of edges: those joining two black vertices are to be labelled by simple objects of $\mathcal{C}$, those joining two white vertices by simple objects of $\mathcal{C}^*$. An edge joining a white vertex to a black one is labelled by a simple object of $\mathcal{M}$. (In principle, one can also admit edges joining a black vertex to a white vertex. If the module category $\mathcal{M}$ is realized as a category of left A-modules, then one should label these edges by right A-modules.)

These edges can form triangles; one only needs to consider triangles for which the edges are oriented in such a way that the boundary does not form a closed oriented path. The corresponding composition of two objects to a third object is naturally interpreted as a (possibly mixed) tensor product. Concretely, we have the following possibilities:

 $\in$ $\mathrm{Hom}_{\mathcal{C}}(U \otimes V, W)$ "chiral coupling"

 $\in$ $\mathrm{Hom}_{\mathcal{M}}(M \otimes U, N)$ "boundary field"

 $\in$ $\mathrm{Hom}_{\mathcal{M}}(B \otimes_A M, N)$ "fusion of a defect line B to the boundary"

 $\in$ $\mathrm{Hom}_{\mathcal{C}^*}(B \otimes_A C, D)$ "coupling of defect lines"

The triangles are to be labelled with elements of a basis of the morphism space that is given in the table. The interpretation of these morphism spaces in conformal field theory is also indicated in the table. As is familiar from Ponzano–Regge calculus, $6j$-symbols with respect to these bases are described in terms of scalars associated to tetrahedra whose faces are triangles of the types shown above. According to the labelling of the vertices, there are then five types of tetrahedra. In this description, all the six pentagon axioms are interpreted as follows: Glue two tetrahedra along a common face, and cut them again along an additional edge that connects the two vertices not belonging to that face, which results in three tetrahedra.

These observations are conceptually clarified when using the language of two-categories (see e.g. section 4 of [**26**] for an introduction). The relevant two-category has two objects $\bullet$ and $\circ$, corresponding to the black and white vertex. The morphism sets of a two-category are categories; the endomorphism sets, in particular, are tensor categories. Thus the tensor category $\mathcal{C}$ can be identified with the endomorphism set of $\bullet$, and the tensor category $\mathcal{C}^*$ with the endomorphism set of $\circ$. For $x, y \in \{\bullet, \circ\}$, the category $\mathrm{Hom}(x, y)$ is naturally a left-module category over the tensor category $\mathrm{End}(x)$ and a right-module category over $\mathrm{End}(y)$. Moreover, the category $\mathrm{Hom}(\circ, \bullet)$ is just $\mathcal{M}$, while $\mathrm{Hom}(\bullet, \circ)$ is the category of right A-modules.

The whole situation can be understood as a category-theoretic analogue of a Morita context (compare [**27**]). In fact, applying the K_0 functor gives us a Morita context of complex algebras: the Grothendieck groups of the tensor categories $\mathcal{C}$ and $\mathcal{C}^*$ are even rings, $K_0(\mathcal{C})$ being a commutative ring. $K_0(\mathcal{M})$ is a right module over the ring $K_0(\mathcal{C})$ and a left module over the ring $K_0(\mathcal{C}^*)$. Thus it plays the role of an interpolating bimodule.

The problem of finding an "inverse" of the K_0 functor for a given algebraic structure is known as *categorification*. In the case at hand, it is equivalent to the problem of finding consistent values for all tetrahedra. In many respects, categorification seems to behave like a cohomology theory. Indeed, as described in the next section, on the Picard subcategory of $\mathcal{C}$ it reduces to questions about the cohomology of abelian groups.

5. Picard groups

One might worry at this point whether there are interesting examples of symmetric special Frobenius algebras. In every modular tensor category, the tensor unit $\mathbf{1}$ provides such an example; the corresponding full CFT is also known as the "Cardy case". A larger class of examples is provided by the following general construction.

In a tensor category with duality, the isomorphism classes of invertible objects, i.e. of objects such that $V \otimes V^{\vee} \cong \mathbf{1}$, form a group, the Picard group $\mathrm{Pic}(\mathcal{C})$. We denote a set of representatives of these isomorphism classes by $\{L_g \,|\, g \in \mathrm{Pic}(\mathcal{C})\}$, with $L_e = \mathbf{1}$. If the category $\mathcal{C}$ is braided, then the Picard group is abelian. In the physics literature, the invertible objects are known as *simple currents* [**33**].

Technically, it is convenient to consider the full subcategory $\mathcal{P}ic(\mathcal{C})$ of direct sums of invertible objects in $\mathcal{C}$. The Grothendieck group of this Picard subcategory is just the group ring of the Picard group,

$$K_0(\mathcal{P}ic(\mathcal{C})) \cong \mathbb{Z}\mathrm{Pic}(\mathcal{C}) .$$

In this situation, categorification amounts to the following task: given a group G, find a category $\mathcal{C}$, such that $K_0(\mathcal{C}) = \mathbb{Z}G$. This problem has a close cousin that is of independent interest: given an *abelian* group G, find a *braided* tensor category such that $K_0(\mathcal{C}) = \mathbb{Z}G$. As it turns out, both problems possess nice answers in terms of suitable cohomology theories. Inequivalent categorifications of a group G correspond to elements of $H^3(G, \mathbb{C}^{\times})$ in group cohomology, while inequivalent braided categorifications of an abelian group G are described by Eilenberg and Mac Lane's abelian group cohomology $H^3_{\mathrm{ab}}(G, \mathbb{C}^{\times})$ (see [**15**] for more explanation and references).

It is an important result that elements of $H^3_{\mathrm{ab}}(G, \mathbb{C}^{\times})$ are in natural bijection to quadratic forms on G with values in $\mathbb{C}^{\times}$. In fact, the braided tensor structure of the Picard category $\mathcal{P}ic(\mathcal{C})$ is characterized by the twist of $\mathcal{C}$, which gives a quadratic form $g \mapsto \theta_g \equiv \theta_{L_g}$ on $\mathrm{Pic}(\mathcal{C})$. (The value $\theta_{gh}/(\theta_g \theta_h)$ of the associated bilinear form on $\mathrm{Pic}(\mathcal{C})$ is called the (exponentiated) *monodromy charge* of L_g with respect to L_h.) As a consequence, for Picard categories the chiral data are particularly well accessible; this is one of the sources of the power of simple current methods in CFT (see e.g. [**33, 34, 18, 2**]).

We are now in a position to construct non-trivial examples of symmetric special Frobenius algebras. We call such an algebra *simple* iff it is simple as a bimodule over itself. (For the associated full CFT, simplicity amounts to the property that the CFT has a unique vacuum.) A stronger condition is that such an algebra is simple as a left module over itself; in that case we call the algebra *haploid*. Now from Frobenius-Perron theory, one can derive the estimate $\dim_{\mathbb{C}} \mathrm{Hom}(U, A) \le \dim(U)$. for haploid Frobenius algebras. This estimate is particularly stringent for those algebras for which any simple subobject U is invertible and hence has $\dim(U) = 1$: in these algebras the multiplicity of any simple subobject is either 0 or 1. We call such a haploid symmetric special Frobenius algebra a *Schellekens algebra*.

The associativity constraint of $\mathcal{C}$ gives a three-cochain ψ on $\mathrm{Pic}(\mathcal{C})$. All Schellekens algebras in a modular tensor category $\mathcal{C}$ can be constructed by finding a subgroup H of the Picard group $\mathrm{Pic}(\mathcal{C})$ and a two-cochain $\omega \colon H \times H \to \mathbb{C}^{\times}$ with the property that $\mathrm{d}\omega = \psi_{|H}$. It turns out that this can be done if and only if for every $h \in H$ the twist obeys $(\theta_h)^{N_h} = 1$, where N_h is the order of h.

Since the two-cochain ω depends on various gauge choices, it is somewhat awkward to work with ω. It is therefore helpful to remember that for an *abelian* group G the second cohomology group $H^2(G, \mathbb{C}^\times)$ – which classifies twisted group algebras – is canonically isomorphic to the group $AB(G, \mathbb{C}^\times)$ of alternating bihomomorphisms on G with values in $\mathbb{C}^\times$. The isomorphism sends a representative ω of a cohomology class to its commutator two-cocycle ξ, which is defined by $\xi(g, h) := \omega(g, h)/\omega(h, g)$.

In the braided setting we are interested in, the notion of alternating bihomomorphism must be generalized; the relevant generalization is the notion of a *Kreuzer–Schellekens bihomomorphism* (KSB) Ξ, obeying

$$\Xi(g, h)\, \Xi(h, g) \,=\, \frac{\theta_g\, \theta_h}{\theta_{gh}}\,.$$

Here the right hand side is not equal to 1 any more, but rather is expressed in terms of the twist, i.e. of the quadratic form that characterizes the Picard category.

The crucial observation is now that the multiplication on a Schellekens algebra A supplies us with a KSB Ξ_A on the support of A, i.e. on the subgroup $H(A) := \{g \in G \mid \dim_{\mathbb{C}} \mathrm{Hom}(L_g, A) > 0\}$ of $\mathrm{Pic}(\mathcal{C})$, via the following relation which we display graphically:

$$=: \Xi_A(h, g)$$
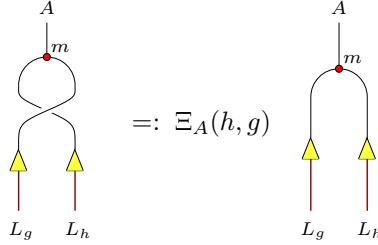
Here m is the multiplication morphism of A, and the triangles indicate non-zero morphisms from simple objects into A. The two graphs are thus morphisms in the one-dimensional space $\mathrm{Hom}(L_g \otimes L_h, A)$ and hence proportional.

Conversely, one can show [15] that a Schellekens algebra is uniquely characterized by its support H and a KSB on the group H. Hence, Schellekens algebras are a generalization of twisted group algebras to the braided setting.

It is now a central goal to express as many quantities of a local CFT built from a Schellekens algebra as possible in terms of the KSB and other computable quantities. This way, one obtains proofs for various simple current formulae that had been conjectured in the literature.

One example is the Kreuzer–Schellekens formula [25] for the torus partition function, which reads

$$Z_{ij}(A) = \frac{1}{|H(A)|} \sum_{g, h \in H(A)} \chi_{U_i}(h) \cdot \Xi_A(h, g) \cdot \delta_{\bar{\jmath}, gi}\,,$$

where for each simple object U of $\mathcal{C}$, χ_U – also called the monodromy charge of U – is the character of $H(A)$ given by $\chi_U(g) := \theta_{L_g \otimes U}/(\theta_g \theta_U)$. (Note that for any simple U and any $g \in \mathrm{Pic}(\mathcal{C})$, $L_g \otimes U$ is again a simple object. Thus the Picard group $\mathrm{Pic}(\mathcal{C})$ acts on the set of isomorphism classes of simple objects of $\mathcal{C}$.)

Elementary boundary conditions are simple A-modules, which are obtained from the decomposition of induced A-modules. One finds that a simple A-module $M \equiv M_{O_U, \psi}$ is described by an orbit O_U of the action of $\mathrm{Pic}(\mathcal{C})$ on the isomorphism classes of simple objects and an irreducible representation ψ of the twisted group

algebra $\mathbb{C}_{\epsilon_U}\mathcal{S}_U$. Here $\mathcal{S}_U$ is the stabilizer of the action of $\mathrm{Pic}(\mathcal{C})$ on the simple object U, and the twist of the group algebra is characterized by the alternating bihomomorphism $\epsilon_U(g,h) := \Phi_U(g,h) \cdot \Xi_A(h,g)$, where Φ_U is a certain gauge independent $6j$-symbol. This way, one reproduces the results of [**11**] for the labelling of boundary conditions. Similar formulae can be derived [**15**] for defect lines, boundary states and other quantities in the theory.

It is natural to consider the Picard group $\mathrm{Pic}(_A\mathcal{C}_A)$ of invertible bimodules as well. This group has a nice physical interpretation [**10**]: the corresponding defects B act by internal symmetries on the correlators of the theory; explicitly:

(i)   A boundary condition described by a left module M is mapped to the one described by the left module

$$B \otimes_A M =: {}^B M.$$

(ii) A boundary field $\Psi_U^{M_1,M_2}$ is mapped to a boundary field $\Psi_U^{{}^B M_1, {}^B M_2}$, where the degeneracy spaces are related by the obvious maps $\mathrm{Hom}(M_1 \otimes U, M_2) \to \mathrm{Hom}((B \otimes_A M_1) \otimes U, B \otimes_A M_2)$.

(iii) The action on bulk fields is given by
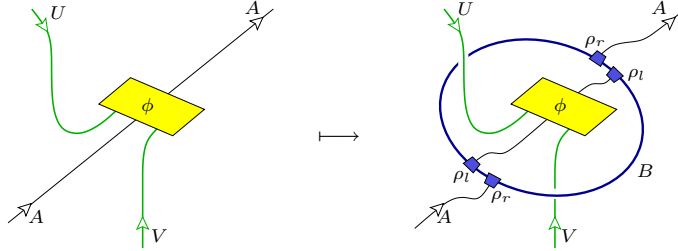
which defines an endomorphism of the vector space $\mathrm{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A)$. Here the morphisms $\rho_{l/r}$ denote the left and right action of A on B, respectively.

With the ansatz for CFT correlators described in sections 2 and 3, it is easy to check that this action preserves the correlation functions: without changing the value of the correlator, one can insert in $\iota(\mathrm{X})$ an unknot ribbon labeled by B. Since $B^\vee \otimes_A B \cong A$, one can use contour deformation arguments familiar from complex analysis. This way, B-loops will run parallel to each boundary component and encircle bulk insertions. Everything is still connected by a network of A-ribbons, so that all tensor products are to be taken over A. This gives precisely the transformation rules presented above.

It is therefore appropriate to identify elements of $\mathrm{Pic}(_A\mathcal{C}_A)$ with internal symmetries of the theory. This is confirmed by the computation of this Picard group for concrete models: for the critical Ising model one obtains $\mathbb{Z}_2$, with the non-trivial element corresponding to a global flip of the Ising spin, while for the critical 3-state Potts model one obtains the symmetric group S_3 that permutes the three possible values of the Potts spin.

6. Order-disorder duality from bimodules

The discussion above shows that any bimodule in $\text{Pic}(_A\mathcal{C}_A)$ describes a symmetry of $\text{CFT}(A)$ that takes the form of equalities between different correlators of $\text{CFT}(A)$. Let us see how this construction gets modified if instead we take a defect labelled by an arbitrary bimodule $B \in \mathcal{O}bj(_A\mathcal{C}_A)$. As it turns out, even in this much more general situation we get equalities between (sums of) correlators.

To see this, we start from the correlator for a given world sheet X and insert in $\iota(\text{X})$ a small annular ribbon labelled by B. This changes the correlator by a factor $\dim^\mathcal{C}(\dot{B})/\dim^\mathcal{C}(A)$, which is the dimension of B as an object of $_A\mathcal{C}_A$. The B-loop divides X in regions "outside" and "inside" the loop. Now deform the loop until the "outside" area has shrunk to zero (here we assume X to be connected). It is not difficult to check that by this procedure of "applying the defect B to the world sheet X" one recovers the action on boundary conditions and boundary fields described in (i) and (ii) above.

However, in addition there are two new effects. First, if X has a non-contractible cycle, additional defect lines labelled by $B \otimes_A B^\vee \equiv {}^B(B^\vee)$ appear. For instance, for X an annulus with boundary conditions M and N one gets
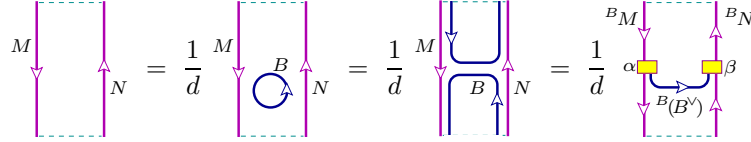

with $d = \dim^\mathcal{C}(\dot{B})/\dim^\mathcal{C}(A)$ and suitable module morphisms $\alpha \in \text{Hom}_A({}^BM, {}^B(B^\vee) \otimes_A {}^BM)$ and $\beta \in \text{Hom}_A({}^B(B^\vee) \otimes_A {}^BN, {}^BN)$. Second, when taking the defect past a bulk field Φ, in general one turns the bulk field into a *disorder field* Θ, according to
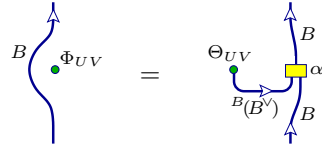
with $\Theta_{UV} \in \text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, B)$ a morphism of A-bimodules describing a disorder field, and $\alpha \in \text{Hom}_{A|A}({}^B(B^\vee) \otimes_A B, B)$ a coupling of defect lines. Applying a generic defect B to a world sheet thus yields an equality between a correlator of bulk fields and a correlator of disorder fields.

For this relationship to constitute an actual order-disorder duality symmetry, we must also be able to turn the disorder correlator back into a correlator of bulk fields. In other words, we need the existence of another defect B' such that first taking B past some bulk field Φ and afterwards taking B' past the resulting disorder field Θ results in a sum of bulk field Φ^β; pictorially,
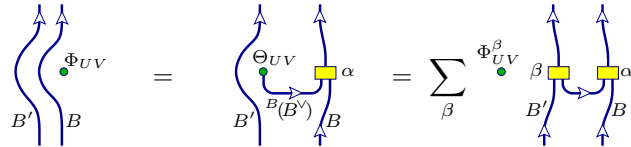
One can prove that such a bimodule B' exists iff

$$B^\vee \otimes_A B \in \mathcal{O}bj(\mathcal{P}ic(_A\mathcal{C}_A)),$$

in which case one can take $B' = B^\vee$. This generalizes the condition $B^\vee \otimes_A B \cong A$ that must hold for the symmetry generating bimodules discussed in the previous section. We call a defect line labelled by a bimodule B obeying $B^\vee \otimes_A B \in \mathcal{O}bj(\mathcal{P}ic(_A\mathcal{C}_A))$ a *duality defect*. Applying a duality defect to a world sheet results in a Kramers–Wannier like duality, and indeed one reproduces known dualities in this way [**10**].

The concept of a duality defect can still be generalized further. Consider two simple symmetric special Frobenius algebras A_1 and A_2. Objects B of the category $_{A_1}\mathcal{C}_{A_2}$ of A_1-A_2-bimodules label tensionless conformal interfaces between $\mathrm{CFT}(A_1)$ and $\mathrm{CFT}(A_2)$ [**13**]. Furthermore, one checks that if $B^\vee \otimes_{A_1} B \in \mathcal{O}bj(\mathcal{P}ic(_{A_2}\mathcal{C}_{A_2}))$ and $B \otimes_{A_2} B^\vee \in \mathcal{O}bj(\mathcal{P}ic(_{A_1}\mathcal{C}_{A_1}))$, then B gives rise to an order-disorder duality symmetry as above, equating this time an order correlator (i.e. involving no defect fields) of $\mathrm{CFT}(A_1)$ to a disorder correlator of $\mathrm{CFT}(A_2)$ and vice versa.

As an illustration, take $A_1 = \mathbf{1}$ (so that $\mathrm{CFT}(A_1)$ is the Cardy case) and let A_2 be a Schellekens algebra. Set $B = A_2$, which becomes an A_1-A_2-bimodule by taking the trivial action of A_1 for the left action and the product of A_2 for the right action. Then $B \otimes_{A_2} B^\vee \cong A_2$ is in $\mathcal{P}ic(_{A_1}\mathcal{C}_{A_1}) \equiv \mathcal{P}ic(\mathcal{C})$ by definition, and one can also show that $B^\vee \otimes_{A_1} B = B^\vee \otimes B$ is in $\mathcal{P}ic(_{A_2}\mathcal{C}_{A_2})$. Thus we can conclude that the correlators of any simple current CFT are related to the correlators in the corresponding Cardy case by an order-disorder duality.

7. Conclusions

We have developed a rigorous algebraic approach to correlation functions in rational conformal field theory. One aspect of this approach we have not emphasized in this contribution is its computational power. Indeed, in the construction of a rational conformal field theory for known chiral data, there is only a single non-linear constraint to be solved: associativity of the multiplication of the Frobenius algebra. The rest of the algorithm is linear and allows, in the end, to find model-independent expressions for interesting CFT quantities like OPE coefficients [**16**] or coefficients of partition functions [**13, 14**] in terms of invariants of knots and links in three-manifolds.

The algebraic approach we have presented in this contribution allows to translate old physical problems to standard problems in algebra and representation theory. We conclude by summarizing this in the following table:

Physical problem	$\longrightarrow$	Algebraic problem
Classification of CFTs based on chiral data $\mathcal{C}$	$\longrightarrow$	Morita classes of algebras in $\mathcal{C}$ (Many examples from Picard group of $\mathcal{C}$)
Classification of $\left.\begin{array}{l}\text{boundary conditions}\\\text{defects}\end{array}\right\}$	$\longrightarrow$	Classification of $\left\{\begin{array}{l}\text{left modules}\\\text{bimodules}\end{array}\right.$
Internal symmetries	$\longrightarrow$	Picard group of $_A\mathcal{C}_A$
Dualities	$\longrightarrow$	Duality defects
Deformation of CFTs	$\longrightarrow$	Deformation of algebras (and of categories)

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INSTITUT FÜR THEORETISCHE PHYSIK, ETH ZÜRICH, CH–8093 ZÜRICH
E-mail address: juerg@itp.phys.ethz.ch

INSTITUTIONEN FÖR FYSIK, KARLSTADS UNIVERSITET, UNIVERSITETSG. 5, S–65188 KARLSTAD
E-mail address: jfuchs@fuchs.tekn.kau.se

MAX-PLANCK-INSTITUT FÜR GRAVITATIONSPHYSIK, AM MÜHLENBERG 1, D–14476 GOLM
E-mail address: ingo@aei.mpg.de

FACHBEREICH MATHEMATIK, UNIVERSITÄT HAMBURG, BUNDESTRASSE 55, D–20146 HAMBURG
E-mail address: schweigert@math.uni-hamburg.de