

Ribbon categories and (unoriented) CFT: Frobenius algebras, automorphisms, reversions

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ABSTRACT. A Morita class of symmetric special Frobenius algebras A in the modular tensor category of a chiral CFT determines a full CFT on oriented world sheets. For unoriented world sheets, A must in addition possess a reversion, i.e. an isomorphism from A^{opp} to A squaring to the twist. Any two reversions of an algebra A differ by an element of the group $\text{Aut}(A)$ of algebra automorphisms of A . We establish a group homomorphism from $\text{Aut}(A)$ to the Picard group of the bimodule category $\mathcal{C}_{A|A}$, with kernel consisting of the inner automorphisms, and we refine Morita equivalence to an equivalence relation between algebras with reversion.

1. Quantum field theory and categories

A means for getting to the core of a quantum field theory (QFT) is to understand it as a functor from some geometric category \mathcal{X} to an algebraic category \mathcal{A} . Since quantum field theory can be analyzed from diverse points of view, various such functors, between different types of categories, have been studied.

In this paper we consider a specific class of QFT models: two-dimensional conformal field theories, or *CFTs*, for short. Our categorical setup for these can be sketched as follows. The objects X of the geometric category \mathcal{X} are compact two-dimensional manifolds with certain decorations – disjoint labeled (germs of) arcs in the interior of X and/or on the boundary ∂X , and/or labeled curves in the interior of X . The morphisms of \mathcal{X} are mapping classes $\varphi: X \rightarrow X'$ that are compatible with the decorations. The labels for the decorations are taken from data which we collectively denote by \mathcal{O} ; accordingly we write $\mathcal{X} = \mathcal{X}_{\mathcal{O}}$. The algebraic category \mathcal{A} is the category $\text{Vect}_{\mathbb{C}}^{\bullet}$ of pointed finite-dimensional complex vector spaces; its objects are pairs (W, w) consisting of a finite-dimensional \mathbb{C} -vector space and an element $w \in W$, and its morphisms are linear maps. On objects, the CFT functor $\text{cft}_{\mathcal{O}}: \mathcal{X}_{\mathcal{O}} \rightarrow \text{Vect}_{\mathbb{C}}^{\bullet}$ acts by mapping a world sheet X to the pair

$$\text{cft}_{\mathcal{O}}(X) = (H(X), \text{Cor}(X))$$

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consisting of the space of *conformal blocks* on X and of the *correlator* of X . We will refer to objects X of \mathcal{X} as *world sheets* (a terminology borrowed from string theory) and sometimes slightly abuse notation by using the symbol X also for the underlying undecorated manifolds.

Deliberately, several basic aspects of this setup, such as the precise form and physical significance of the decorations and of the data \mathcal{O} , have not been specified above. More explanations will be given in due time, though for lack of space various details will be suppressed. Before doing so it is, however, wise to examine the following simpler situation. Take \mathcal{X} to be the category $\mathcal{Y}_{\mathcal{C}}$ whose objects are compact oriented closed two-manifolds with disjoint labeled arcs and whose morphisms are mapping classes which map arcs to arcs. The labels for the arcs on objects of $\mathcal{Y}_{\mathcal{C}}$ are taken from some data \mathcal{C} , which are required to form a braided monoidal category: an arc is labeled by an object of \mathcal{C} . Further, take \mathcal{A} to be the category $\mathcal{Vect}_{\mathbb{C}}$ of finite-dimensional complex vector spaces. Then we can consider a functor

$$c\text{-}cft_{\mathcal{C}} : \mathcal{Y}_{\mathcal{C}} \rightarrow \mathcal{Vect}_{\mathbb{C}},$$

known as the functor of *chiral conformal field theory*. (To distinguish the functor $cft_{\mathcal{O}}$ above from the one of chiral CFT, we call $cft_{\mathcal{O}}$ the functor of *full conformal field theory*.) If it exists, the functor $c\text{-}cft_{\mathcal{C}}$ is supposed to be determined uniquely by the category \mathcal{C} . When \mathcal{C} is a *modular tensor category* (see section 2), then $c\text{-}cft_{\mathcal{C}}$ exists and is indeed well-known: it is a two-dimensional topological modular functor in the sense of [BK]. To each modular tensor category there is associated such a functor, together with a three-dimensional topological quantum field theory [RT, Tu, BK].

A forgetful functor $F_{\mathcal{X}}: \mathcal{X}_{\mathcal{O}} \rightarrow \mathcal{Y}_{\mathcal{C}}$ is obtained by viewing world sheets X as pairs

$$X = (\widehat{X}, \tau)$$

consisting of their *double* \widehat{X} and an orientation-reversing involution τ of \widehat{X} (see section 3), and then forgetting τ and suitably manipulating the decorations. Together with the obvious forgetful functor $F_{\nu}: \mathcal{Vect}_{\mathbb{C}}^{\bullet} \rightarrow \mathcal{Vect}_{\mathbb{C}}$, $F_{\mathcal{X}}$ fits into a diagram

$$(1) \quad \begin{array}{ccc} \mathcal{X}_{\mathcal{O}} & \xrightarrow{cft_{\mathcal{O}}} & \mathcal{Vect}_{\mathbb{C}}^{\bullet} \\ F_{\mathcal{X}} \downarrow & & \downarrow F_{\nu} \\ \mathcal{Y}_{\mathcal{C}} & \xrightarrow{c\text{-}cft_{\mathcal{C}}} & \mathcal{Vect}_{\mathbb{C}} \end{array}$$

of four categories and four functors (all of which are symmetric monoidal). As we will demonstrate below, the construction of the functor $cft_{\mathcal{O}}$ can be conveniently divided in two steps of which the first amounts to requiring that this diagram is commutative. Thus in particular the spaces of conformal blocks of a full CFT are those of the associated chiral CFT, $H((\widehat{X}, \tau)) = c\text{-}cft_{\mathcal{C}}(\widehat{X})$.

We still need to specify the data \mathcal{O} on which $\mathcal{X}_{\mathcal{O}}$ depends, as well as the corresponding data for the functor $cft_{\mathcal{O}}$. We must in fact consider two different categories $\mathcal{X}_{\mathcal{O}}$, one in which the world sheets are oriented and one in which they are not, and accordingly there are two full CFT functors. Here we restrict our attention to the oriented case; remarks on the unoriented case will be added later. For compatibility with (1), the data \mathcal{O} must fit with those of \mathcal{C} . Now \mathcal{C} is in particular a monoidal category, or what is the same, a 2-category with a single object. \mathcal{O} generalizes this aspect: it is a 2-category with precisely two objects. \mathcal{O} has four 1-cells: the

monoidal category \mathcal{C} , another monoidal category \mathcal{C}^* , and two categories \mathcal{M} and \mathcal{M}' which are right and left module categories [CF] over \mathcal{C} , respectively; \mathcal{O} must have the further properties that \mathcal{C} is braided and that \mathcal{C}^* is equivalent, as a module category over \mathcal{C} , to the category $\text{Func}(\mathcal{M}, \mathcal{M})$ of module endofunctors of \mathcal{M} . Specifying also the functor $cft_{\mathcal{O}}$ requires one additional datum, namely an algebra A in \mathcal{C} such that \mathcal{M} is equivalent to the category $\mathcal{C}_{A|}$ of left A -modules. Such an algebra exists and is determined up to Morita equivalence [Os], and indeed up to equivalence the whole 2-category \mathcal{O} can be reconstructed from \mathcal{C} and A . The algebra A is, however, only auxiliary; from the correlators obtained with a particular choice of A , the correlators obtained with any Morita equivalent algebra A' can be determined uniquely, and indeed the corresponding full CFTs do not differ in any observable quantity. Nevertheless, for the concrete description of the full CFT functor $cft_{\mathcal{O}}$ a choice of A within its Morita class must be made, and accordingly from now on we denote this functor by $cft_{\mathcal{C}, A}$ and write $cft_{\mathcal{C}, A}(X) = (H_{\mathcal{C}}(X), Cor_A(X))$.

The rest of this paper is organized as follows. Sections 2 and 3 provide further information on chiral and full CFT. Some aspects of a ‘TFT construction’ of $Cor_A(X)$ are described in sections 4 to 6, while section 7 gives a brief outlook to related issues that we do not explain in this paper. Finally, sections 8 to 11 settle some questions that arise when discussing full CFT on unoriented world sheets.

Before proceeding, let us briefly mention other possibilities for studying QFT in a categorical framework. In one approach (see e.g. [At, Se]), which has in particular been discussed for CFTs and for topological quantum field theories (TFTs), the relevant geometric category \mathcal{X} is a cobordism category, while \mathcal{A} is a category of vector spaces. The three-dimensional TFT variant of this approach, in a formulation closely following the one of [Tu], will be used as a tool in our analysis of the functor $cft_{\mathcal{C}, A}$. In another, quite distinct, framework which emphasizes the principle of causality [BFV], the geometric category has Lorentzian manifolds (‘space-times’) as objects and isometric embeddings as morphisms, while \mathcal{A} is the category of unital C^* -algebras. Common to all these approaches is that QFT is considered simultaneously on a large class of spaces, or space-times.

2. Chiral conformal field theory and modular tensor categories

To characterize a specific model of QFT one must in particular have a grasp on its fields and symmetries. The symmetries of a chiral CFT include conformal symmetries. These can be encoded in a Virasoro vertex algebra \mathcal{V}_{Vir} , which furnishes in particular a representation of the Virasoro Lie algebra. But most models have additional symmetries, so that the symmetry structure is a larger conformal vertex algebra $\mathcal{V} \supseteq \mathcal{V}_{\text{Vir}}$ (see e.g. [Hu1]). The spaces of fields of the chiral CFT are then given by \mathcal{V} -modules. The braided monoidal category \mathcal{C} that supplies the data for the geometric category of a chiral CFT is the representation category $\mathcal{R}ep(\mathcal{V})$ of \mathcal{V} , which has \mathcal{V} -modules as objects and intertwiners between \mathcal{V} -modules as morphisms. The monoidal structure on $\mathcal{R}ep(\mathcal{V})$ is given by the tensor product \otimes of \mathcal{V} -modules and of intertwiners, the tensor unit $\mathbf{1}$ being \mathcal{V} itself [HL]. Invoking coherence, we tacitly pass to an equivalent category for which both \otimes and $\mathbf{1}$ are strict.

In the sequel, we will restrict our attention to the case that \mathcal{V} is a *rational* conformal vertex algebra in the sense that it satisfies the conditions of theorem 5.1 of [Hu2]. (That is, \mathcal{V} is C_2 -cofinite and self-dual as a \mathcal{V} -module and satisfies $\mathcal{V}^{(n)} = 0$ for $n < 0$ and $\mathcal{V}^{(0)} = \mathbb{C}\mathbf{1}$, every simple \mathcal{V} -module not isomorphic to \mathcal{V} has positive

conformal weight, and every \mathbb{N} -gradable weak \mathcal{V} -module is fully reducible.) We are then dealing with a (chiral) rational CFT, or *RCFT*, for short. Among the conformal vertex algebras the rational ones are distinguished by the fact¹ [HL, Hu2] that $\mathcal{C} = \mathcal{R}ep(\mathcal{V})$ has a number of peculiar properties:

- (i) The tensor unit is simple.
- (ii) \mathcal{C} is abelian, \mathbb{C} -linear and semisimple.
- (iii) \mathcal{C} is ribbon:² There are families $\{c_{U,V}\}$ of braiding, $\{\theta_U\}$ of twist, and $\{d_U, b_U\}$ of evaluation and coevaluation morphisms satisfying the relevant properties.
- (iv) \mathcal{C} is Artinian (or ‘finite’), i.e. the number of isomorphism classes of simple objects is finite.
- (v) The braiding is maximally non-degenerate: the numerical matrix s with entries $s_{i,j} := (d_{U_j} \otimes \tilde{d}_{U_i}) \circ [id_{U_j^\vee} \otimes (c_{U_i, U_j} \circ c_{U_j, U_i}) \otimes id_{U_i^\vee}] \circ (\tilde{b}_{U_j} \otimes b_{U_i})$ is invertible.

Here we denote by $\{U_i \mid i \in \mathcal{I}\}$ a (finite) set of representatives of isomorphism classes of simple objects; we also take $U_0 := \mathbf{1}$ as the representative for the class of the tensor unit. A monoidal category with the properties listed above is called a *modular tensor category* [Tu].

It is worth mentioning that every ribbon category is *sovereign*, i.e. besides the left duality given by $\{d_U, b_U\}$ there is also a right duality (with evaluation and coevaluation morphisms to be denoted by $\{\tilde{d}_U, \tilde{b}_U\}$), which coincides with the left duality in the sense that ${}^\vee U = U^\vee$ and ${}^\vee f = f^\vee$. Below we will make ample use of Joyal-Street [JS1] type diagrams for morphisms of a strict ribbon category. For instance, the ribbon structure morphisms and the entries of s are depicted by

The diagrams show the following morphisms in a ribbon category:

- $c_{U,V}$: A braiding diagram where two strands cross.
- θ_U : A twist diagram where a strand passes through a loop.
- b_U : A coevaluation diagram where a strand enters a loop from the bottom.
- d_U : An evaluation diagram where a strand exits a loop from the top.
- $s_{i,j}$: A diagram representing the matrix entry, showing two overlapping loops labeled U_i and U_j .

Let us also remark that other approaches to conformal quantum field theory, not based on vertex algebras, exist. In particular one can work with nets of von Neumann algebras instead, see e.g. [Re]. This setting leads again to modular tensor categories, albeit with the extra property that the (quantum) dimensions $\dim(U) := \tilde{d}_U \circ b_U = d_U \circ \tilde{b}_U \in \mathbb{C} id_{\mathbf{1}}$ are real and positive. We will not be concerned with the issue where the category \mathcal{C} under study comes from. Moreover, while in sections 3–7 \mathcal{C} will always stand for a modular tensor category, various results that we will mention actually remain valid in a more general setting. And in sections 8–11, \mathcal{C} will generally not be assumed to be modular, but essentially only to be an additive \mathbb{k} -linear ribbon category.

3. Full conformal field theory

There exist some physical problems, such as the fractional quantum Hall effect (see e.g. [FPSW]), for which chiral CFT is relevant. But more often, e.g. in the study of phase transitions, percolation, impurity problems, or string theory, it is

¹ The conditions in the definition of rationality used here can possibly be relaxed. Various similar notions of rationality that have been discussed in the literature are not sufficiently strong for our purposes, however.

² Besides the qualifier ‘ribbon’ [RT], which emphasizes the similarity with the properties of ribbons in a three-manifold, also the terms ‘tortile’ [JS2] and ‘balanced rigid braided’ are in use.

full rather than chiral CFT that matters. Recall that, unlike in chiral CFT, in full CFT the objects of \mathcal{X} – the world sheets X – can have nonempty boundary. In applications, a boundary can arise e.g. as a cluster boundary in an effectively one-dimensional condensed matter system, or as the world line of an end point of an open string in string theory.³

In physical terminology, an arc on X specifies the location (together with a germ of local coordinates around it) of a ‘field insertion’, while the label of the arc specifies the type of field that is inserted. As mentioned above, in chiral CFT fields correspond to \mathcal{V} -modules, and hence arcs are labeled by objects of \mathcal{C} . In contrast, in full CFT the label of an arc in the interior of X involves a *pair* of objects of \mathcal{C} , say (taking, without loss of generality the objects to be simple) (U_i, U_j) with $i, j \in \mathcal{I}$. Such field insertions are called *bulk fields*. On ∂X one can have another type of fields, the *boundary fields*, and in the presence of defect lines (labeled curves in the interior of X) there are also *defect fields*, which generalize bulk fields. In the sequel we concentrate on the bulk fields.

In terms of the chiral symmetry \mathcal{V} , the prescription to work with a pair of objects of \mathcal{C} means that bulk fields carry two representations of \mathcal{V} ; or put differently: a representation of two copies of \mathcal{V} . The latter formulation is indeed quite suggestive; the two copies are referred to as “left- and right-moving” or “holomorphic and antiholomorphic” world sheet symmetries, respectively, mimicking the common terminology for the two types of solutions to the classical equation of motion of a free boson field in two-dimensional Minkowski and Euclidean space, respectively. (But recall that there are two types of full CFT, with oriented and unoriented (including in particular unorientable) world sheets, respectively. The present terminology is appropriate only in the oriented case.) For oriented full CFT, the total left and right chiral symmetries can actually be different. What we denote by \mathcal{V} is a subalgebra of symmetries that is contained both in the left and right chiral symmetries. Since by assumption \mathcal{V} is a rational conformal vertex algebra, this excludes so-called heterotic RCFTs, in which rationality is only present when different extended symmetries are taken into account for the left- and right-moving part.

When analyzing full RCFT, important tools are supplied by the corresponding chiral CFT that according to the diagram (1) shares the underlying modular tensor category \mathcal{C} . In the sequel we take the attitude that this chiral CFT is sufficiently well under control, so that the interesting part of discussing the full CFT is the particular way it is related to the chiral CFT.

At the level of geometric categories, the relationship is rather simple: The object \widehat{X} of $\mathcal{Y}_{\mathcal{C}}$ to which a world sheet $X \in \text{Obj}(\mathcal{X}_{\mathcal{O}})$ gets mapped by the forgetful functor F_{χ} is the *double* \widehat{X} of X . The world sheet $X = (\widehat{X}, \tau)$ can be obtained from \widehat{X} as the quotient by an orientation-reversing involution τ . Conversely, the double can be recovered from X as the orientation bundle over X modulo identification of the two points in the fiber over each point of ∂X [**AG**, **B(CD)²**, **F³S**]. One may also regard X as a real scheme; then \widehat{X} is its complexification, and the involution τ just implements the action of the generator of the Galois group $\text{Gal}(\mathbb{C}, \mathbb{R})$ [**SF**]. To give some examples: the double of a closed orientable world sheet X is just the

³ Instead of arcs on X , one may also use parametrized circles to implement decorations. Then in addition to the ‘physical’ boundary components of X occurring here, one also has ‘insertion’, or ‘state’, boundary components. For details as well as references see [**RFFS**].

disconnected sum $\widehat{X} = X \sqcup -X$ of two copies of X endowed with opposite orientation; and both the disk and the real projective plane have the two-sphere as their double.

At this point we should mention that world sheets X must also be endowed with a conformal structure (and for certain aspects, even with a metric, compare e.g. [RFFS] for details). Analogously, the objects of $\mathcal{Y}_{\mathcal{C}}$ carry a complex structure; the possible choices of complex structure of \widehat{X} are restricted by the requirement that the involution τ is anticonformal. For the relation between chiral and full CFT studied here, the conformal structure on X (and complex structure on \widehat{X}) is inessential.

4. The connecting three-manifold and topological field theory

A world sheet X of a full CFT comes with an arc for each bulk field insertion; each such arc gives rise to two arcs on the double \widehat{X} . (In contrast, for boundary field insertions, which are described by arcs on ∂X , there is just a single arc on \widehat{X} .) To specify the bulk field insertion, the arc on X is labeled by a pair of objects of \mathcal{C} , say (U_i, U_j) as above. A field insertion in chiral CFT, on the other hand, requires a single object as label for its arc. Thus in order to relate full CFT on $X = (\widehat{X}, \tau)$ to chiral CFT on \widehat{X} we must label one of the two arcs on \widehat{X} by U_i and the other one by U_j . In short, at the level of bulk fields, going from full to chiral CFT affords a *geometric separation of left- and right-movers*.

In the chiral CFT the information that the arc labels U_i and U_j arise from one and the same bulk field is ignored. A possibility to retain this information is to regard \widehat{X} as the boundary of a ‘fattened’ world sheet, which we call the *connecting manifold* for X and denote by M_X . M_X can be defined as the interval bundle over X modulo a certain identification over ∂X , or equivalently as

$$M_X := (\widehat{X} \times [-1, 1]) / \sim \quad \text{with} \quad ([x, \text{or}_2], t) \sim ([x, -\text{or}_2], -t).$$

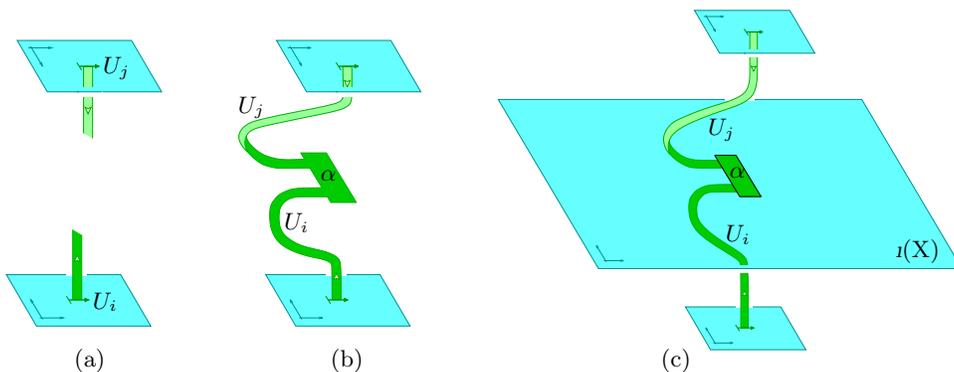
Then $\partial M_X = \widehat{X}$, while X is naturally embedded in M_X via $\iota: X \hookrightarrow X \times \{t=0\} \hookrightarrow M_X$. To relate the theories on X and \widehat{X} , it is desirable that also the connecting three-manifold M_X , just like X and \widehat{X} , comes along with some QFT. Since M_X plays only an auxiliary role, that three-dimensional quantum field theory should require as little structure on M_X as possible – in physics terminology, it should be non-dynamical. This is achieved by demanding it to be a *topological* quantum field theory.

As already mentioned at the end of section 1, a d -dimensional topological quantum field theory, or *TFT*, furnishes a specific version of a QFT functor, in which the geometric category \mathcal{X} is a cobordism category;⁴ the target category \mathcal{A} of a TFT is the category of finite-dimensional complex vector spaces. Details can be found e.g. in [At, Tu, La]. What is relevant for us is actually a variant, called *C-extended* three-dimensional TFT; we denote the corresponding functor by $tft_{\mathcal{C}}$. The objects of the domain category of $tft_{\mathcal{C}}$ are just those of the domain category $\mathcal{Y}_{\mathcal{C}}$ of the relevant chiral CFT, i.e. compact oriented closed two-manifolds E with a chosen Lagrangian subspace of $H_1(E, \mathbb{R})$ and decorated with arcs, which in turn are labeled by objects of \mathcal{C} . The morphisms are decorated as well: they are oriented three-manifolds M with embedded ribbon graphs, i.e. finite collections of disjoint

⁴ Owing to a non-trivial action of the mapping class group on the objects E of the three-dimensional cobordism category, an additional rigidification of the category is required. Following [Tu], here we implement this refinement by assuming a choice of Lagrangian subspace of $H_1(E, \mathbb{R})$ as an auxiliary datum. We refer to [Tu], as well as e.g. to [F³S] or section A.1 of [FjFRS], for details. An alternative treatment is possible in a 2-categorical setting, see [Ti, BCR].

oriented ribbons and coupons. Each ribbon either forms an annulus or connects coupons and/or arcs on ∂M . The pieces of the ribbon graph are labeled by data from \mathcal{C} , too: ribbons by objects of \mathcal{C} , and coupons by morphisms of \mathcal{C} , with domain the tensor product of the objects that label the ribbons entering the coupon and codomain the tensor product for the ribbons leaving the coupon. (Recall that the tensor product \otimes of \mathcal{C} is taken to be strict. Also, by strictness of $\mathbf{1}$, ribbons labeled by $\mathbf{1}$ are irrelevant, and hence will be regarded as invisible.)

An arc on ∂M carries the same label as the ribbon beginning or ending at it. In the situation of our interest, where $M = M_X$, we must in addition account for the arcs on $\iota(X) \subset M_X$ in the interior of M . To fit into the TFT picture, such an arc labeled (U_i, U_j) should result in a coupon with incoming ribbons labeled by U_i and U_j . Let us pretend for the moment that no other ribbons are to be attached to such coupons (as will be explained in the next section, this is only true for a special class of full CFTs). Then we arrive at a TFT description of the bulk fields as



The picture (a) just displays the arcs on \widehat{X} with emanating ribbons; (b) indicates in addition how the ribbons enter the coupon (to be looked at from below) in M_X , while (c) shows how this coupon lies in $\iota(X) \subset M_X$.

5. Full CFT and Frobenius algebras in \mathcal{C}

As shown in [F³S], the description of bulk fields given above can indeed be extended to a consistent scheme for constructing a full CFT from its underlying chiral CFT with the help of three-dimensional TFT. However, only a special class of full CFTs is covered. Indeed, this construction implies in particular that the vector space of bulk fields with given chiral labels $i, j \in \mathcal{I}$ is $\text{Hom}(U_i \otimes U_j, \mathbf{1}) \cong \delta_{i, j^\vee} \mathbb{C}$ ($j^\vee \in \mathcal{I}$ is the unique label such that $U_{j^\vee} \cong U_j^\vee$). This is sometimes referred to as *C-diagonal* CFT or, when also symmetry preserving boundary conditions are included, as the *Cardy case* of full CFT. In contrast, in general full CFTs these spaces can be nonzero for $i \neq j^\vee$, and they can have dimension > 1 .

In [FRS1] the missing piece of data was recognized as a certain module category \mathcal{M} over \mathcal{C} : a full RCFT can be constructed from the corresponding chiral CFT (sharing the relevant modular tensor category \mathcal{C}) together with \mathcal{M} . In short,

$$(2) \quad \text{full CFT} = \text{chiral CFT} + \text{module category } \mathcal{M} \text{ over } \mathcal{C}.$$

A module category \mathcal{M} over \mathcal{C} is a category \mathcal{M} equipped with a bifunctor $\mathcal{M} \times \mathcal{C} \rightarrow \mathcal{M}$ subject to an appropriate associativity constraint. As already mentioned, for any

(semisimple) module category \mathcal{M} there exists [Os] an algebra A in \mathcal{C} , unique up to Morita equivalence, such that \mathcal{M} is equivalent to the category \mathcal{C}_A of left A -modules.

In our application to RCFT, A must be a *symmetric special Frobenius* algebra, so that the motto (2) may be rephrased in a non-Morita invariant manner as

$$(3) \quad \text{full CFT} = \text{chiral CFT} + \text{sym. sp. Frobenius algebra } A \text{ in } \mathcal{C}.$$

A *Frobenius* algebra in \mathcal{C} is⁵ a quintuple $A = (A, m, \eta, \Delta, \varepsilon)$ such that (A, m, η) is a (unital associative) algebra and (A, Δ, ε) a (counital coassociative) coalgebra, subject to the compatibility condition that Δ is a morphism of A -bimodules, i.e. $(id_A \otimes m) \circ (\Delta \otimes id_A) = \Delta \circ m = (m \otimes id_A) \circ (id_A \otimes \Delta)$; A is called *special* iff $\varepsilon \circ \eta = \gamma id_{\mathbf{1}}$ and $m \circ \Delta = \gamma' id_A$ with nonzero complex numbers γ and γ' (implying in particular that $\dim(A) = \gamma\gamma' \neq 0$); and A is called *symmetric* iff the two isomorphisms

$$(4) \quad \Phi = ((\varepsilon \circ m) \otimes id_{A^\vee}) \circ (id_A \otimes b_A) \quad \text{and} \quad \Phi' = (id_{A^\vee} \otimes (\varepsilon \circ m)) \circ (\tilde{b}_A \otimes id_A)$$

in $\text{Hom}(A, A^\vee)$ are equal.

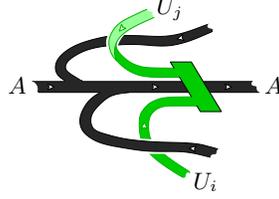
It is worth pointing out that generically \mathcal{C} is not symmetric monoidal, and as a consequence there are theorems of ‘braided algebra’, such as theorem 5.20 of [FFRS1], which do not have any substantial classical analogue.

A constructive method for obtaining the full CFT from \mathcal{C} and A with the help of TFT on the connecting manifold was developed in [FRS1, FRS2, FRS4, FjFRS]. It involves in particular a ribbon graph Γ that is inserted along a triangulation of $\iota(X)$, with all ribbons (along edges of the triangulation) labeled by A and all coupons (at the (trivalent) vertices of the triangulation of $X \setminus \partial X$) labeled by the product or coproduct of A ; for details, see e.g. appendix A of [FjFRS]. As for bulk fields, the construction amounts to extending the prescription of section 4 by introducing additional A -ribbons that enter and leave the coupon in $\iota(X)$ (and are connected to Γ); schematically,

$$(5) \quad \begin{array}{c} \text{Diagram 1: A coupon } \alpha \text{ with two } A \text{-ribbons } U_j \text{ and } U_i \text{ entering from the left and exiting to the right.} \\ \equiv \\ \text{Diagram 2: A coupon } \alpha \text{ with two } A \text{-ribbons } U_j \text{ and } U_i \text{ entering from the left and exiting to the right, with a thick black line } \mathbf{1} \text{ passing through the coupon.} \\ \rightsquigarrow \\ \text{Diagram 3: A coupon } \alpha \text{ with two } A \text{-ribbons } U_j \text{ and } U_i \text{ entering from the left and exiting to the right, with a thick black line } A \text{ passing through the coupon.} \end{array}$$

Thus in the general case a bulk field with chiral labels $i, j \in \mathcal{I}$ is a morphism $\alpha \in \text{Hom}(U_i \otimes A \otimes U_j, A)$. But to ensure that the correlators involving such a bulk field can be nonvanishing, one must in fact further restrict to a certain subspace of A -bimodule morphisms. The relevant bimodule structure on the object $U_i \otimes A \otimes U_j$ must account for the fact that (referring to picture (c) in the description of bulk fields in section 4) all A -ribbons in $\iota(X)$ pass below the U_i -ribbon and above the U_j -ribbon, as illustrated in the following picture:

⁵ In the category $\text{Vect}_{\mathbb{C}}$ of finite-dimensional complex vector spaces, this definition is equivalent to the more conventional one as an algebra with nondegenerate invariant bilinear form [Ab].



Accordingly, the commuting left and right A -actions on $U_i \otimes A \otimes U_j$ are given by $(id_{U_i} \otimes m \otimes id_{U_j}) \circ (c_{U_i, A}^{-1} \otimes id_A \otimes id_{U_j})$ and $(id_{U_i} \otimes m \otimes id_{U_j}) \circ (id_{U_i} \otimes id_A \otimes c_{A, U_j}^{-1})$, respectively. We denote the so defined A -bimodule by $U_i \otimes^+ A \otimes^- U_j$. Thus the space of bulk fields with chiral labels $i, j \in \mathcal{I}$ is given by $\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$.

Before proceeding, we mention two pertinent results and some examples of symmetric special Frobenius algebras. An algebra is called *simple* iff it is simple as a bimodule over itself, i.e. a simple object of the category $\mathcal{C}_{A|A}$ of A -bimodules.

PROPOSITION 1. (i) *An algebra in \mathcal{C} can be endowed with the structure of a symmetric special Frobenius algebra iff the morphism $\Phi_{\mathfrak{h}}$, obtained when replacing ε in formula (4) for Φ by $\xi \varepsilon_{\mathfrak{h}}$ with $\xi \in \mathbb{C}^\times$ and $\varepsilon_{\mathfrak{h}} := d_A \circ (id_A \otimes m) \circ (\tilde{b}_A \otimes id_A)$, is invertible. This structure, if it exists, is unique up to the choice of $\xi \in \mathbb{C}^\times$ and up to isomorphisms of A as a Frobenius algebra.*

(ii) *Every algebra in the Morita class $[A]$ of a simple symmetric special Frobenius algebra in \mathcal{C} is simple symmetric special Frobenius, too.*

PROOF. (i) is shown in lemma 3.12 and theorem 3.6(i) of [FRS1].
(ii) follows immediately from proposition 2.13 of [FRS2]. \square

Concerning examples, first of all the tensor unit $\mathbf{1}$, with all structural morphisms identity morphisms, is trivially a symmetric special Frobenius algebra. Also, for any object U of \mathcal{C} , $(U^\vee \otimes U, id_{U^\vee} \otimes \bar{d}_U \otimes id_U, \bar{b}_u, id_{U^\vee} \otimes b_U \otimes id_U, d_U)$ is a symmetric Frobenius algebra, and (since $\varepsilon \circ \eta = \dim(U) id_{\mathbf{1}}$) it is special iff $\dim(U) \neq 0$. These algebras are Morita equivalent to $\mathbf{1}$, and hence for each of them the TFT construction gives the C -diagonal full CFT.

A large number of nontrivial examples is provided by the following setup [FRS3]. As a monoidal category, the full subcategory $\mathcal{P}ic(\mathcal{C})$ of \mathcal{C} whose simple objects L are the invertible objects of \mathcal{C} is determined uniquely (up to equivalence) by the Picard group $\text{Pic}(\mathcal{C})$ of \mathcal{C} and a class $\psi \in H^3(\text{Pic}(\mathcal{C}), \mathbb{C}^\times)$; ψ specifies the associator of $\mathcal{P}ic(\mathcal{C})$.⁶ Let further $G \leq \text{Pic}(\mathcal{C})$ be a subgroup of $\text{Pic}(\mathcal{C})$ for which there exists a class $\omega \in H^2(G, \mathbb{C}^\times)$ such that $\psi|_G = d\omega$. For each such pair G, ω there is, up to isomorphism, a unique symmetric special Frobenius algebra $A = A_{G; \omega}$ in \mathcal{C} , called a Schellekens algebra, for which the underlying object is $\bigoplus_{g \in G} L_g$, with L_g a simple object of $\mathcal{P}ic(\mathcal{C})$ whose class in $\text{Pic}(\mathcal{C})$ is g . Moreover, for a given subgroup G the algebras associated to distinct classes ω of $H^2(G, \mathbb{C}^\times)$ are nonisomorphic, and every symmetric special Frobenius algebra A in \mathcal{C} all of whose simple subobjects are invertible and for which $\text{Hom}(\mathbf{1}, A) \cong \mathbb{C}$ is of this specific form.

⁶ Indeed, a categorification of a finite group K is, by definition, a monoidal category with Grothendieck ring $\mathbb{Z}K$, and up to equivalence such categories are classified by $H^3(K, \mathbb{C}^\times)$. Similarly, categorifications of a finite abelian group T are naturally defined as braided monoidal categories, and they are classified [JS2, FK] by the abelian group cohomology [EM] $H_{\text{ab}}^3(T, \mathbb{C}^\times)$.

Because of the uniqueness result in part (i) of the proposition, without loss of generality we can (and do) impose the normalization condition $\varepsilon \circ \eta = \dim(A) id_{\mathbf{1}}$ (or equivalently, $m \circ \Delta = id_A$) on symmetric special Frobenius algebras.

6. TFT construction of RCFT correlators

One of the major tasks in the study of any quantum field theory is to obtain the correlation functions, or *correlators*, of the theory. Roughly speaking, a correlator is a global section in a certain vector bundle (involving auxiliary data, such as a metric and choices of local coordinates around insertion points) over the space of configurations of field insertions. A correlator for the empty configuration (no fields at all inserted) is also called a *partition function*. In full CFT the field configuration for the correlator of n bulk fields is specified by the world sheet X and insertion arcs γ_a on X with labels $(U_{i_a}, U_{j_a}, \alpha_a)$, for $a \in \{1, 2, \dots, n\}$, together with further data which do not concern us here, and similarly for correlators involving boundary and/or defect fields. These correlators must satisfy three types of consistency conditions, known as the (chiral) Ward identities, as modular invariance (or locality) and as factorisation constraints, respectively. (For details see e.g. sections 5.2 and 6.1 of [FRS4] and sections 4.2 and 5 of [FRS5], as well as [RFFS].)

In the construction of [FRS1, FRS2, FRS4, FjFRS], for brevity to be referred to as the *TFT construction*, any RCFT correlator is obtained as the invariant of a three-manifold with embedded ribbon graph that is built from the data just mentioned. In more detail, the correlator $Cor_A(X)$ of the full CFT with modular tensor category \mathcal{C} and symmetric special Frobenius algebra A for a world sheet $X = (\widehat{X}, \tau)$ (with field insertions, which we suppress in the notation) is the vector

$$(6) \quad Cor_A(X) := tft_{\mathcal{C}}(M_{\emptyset, \widehat{X}}) 1 \in tft_{\mathcal{C}}(\widehat{X})$$

in the TFT state space of the double \widehat{X} of the world sheet. Here $1 \in tft_{\mathcal{C}}(\emptyset) = \mathbb{C}$, and $M_{\emptyset, \widehat{X}} = M_X$ is the connecting manifold, regarded as a cobordism from the empty set to \widehat{X} , with an embedded ribbon graph. This ribbon graph, in turn, is obtained by assembling various building blocks, such as a fragment of the form displayed in the last graph of (5) for each bulk field insertion and the A -colored ribbon graph Γ along a triangulation of $\imath(X)$. For the entire prescription, see appendix A of [FjFRS]. That $Cor_A(X)$ obeys the Ward identities is already guaranteed by the fact that it is an element of $tft_{\mathcal{C}}(\widehat{X})$; for the particular vector (6) in $tft_{\mathcal{C}}(\widehat{X})$, the modular invariance and factorisation constraints are satisfied as well.

When X is oriented and its boundary ∂X is empty, so that $\widehat{X} = X \sqcup -X$, one has $tft_{\mathcal{C}}(\widehat{X}) = tft_{\mathcal{C}}(X) \otimes_{\mathbb{C}} tft_{\mathcal{C}}(X)^*$. According to (6) the correlator is then an element in the tensor product of a space for ‘left-movers’ and one for ‘right-movers’. This property is known as the *holomorphic factorisation* of correlators [Wi].

The description of correlators as global sections requires that $Cor_A(X)$ is a single-valued function of the field insertion points and of the moduli of X . That we specify here $Cor_A(X)$ instead as an element of some vector space complies with this requirement upon realizing that vector space as a space of (multivalued) functions, the space of *conformal blocks* (also known as chiral blocks).⁷ That the TFT

⁷ In *chiral* CFT one actually does not have correlators in the sense used here. Rather, there are only the vector spaces of conformal blocks, which constitute the fibers of (generically) nontrivial bundles over the relevant space of field configurations.

state spaces can indeed be identified with the spaces of conformal blocks of the corresponding CFT is still an assumption for general RCFT models, but has been established for several important classes such as the Wess–Zumino–Witten models.

7. Dictionary

Among the results established through the TFT construction, we mention here only that factorisation and modular invariance properties of the construction were proven in **[FjFRS]**; that various properties of the most interesting partition functions were established in section 5 of **[FRS1]** and section 3 of **[FRS2]**; that the case of Schellekens algebras is treated in **[FRS3]**; and that when expressing $Cor_A(X)$ for a few particular world sheets X (the ‘fundamental correlators’) in certain standard bases of conformal blocks, one obtains the structure constants of various ‘operator product expansions’, see section 4 of **[FRS4]**.

Another important fact is that, in accordance with (2), for oriented X the assignment of a suitably normalized correlator $Cor_A(X)$ to X actually depends only on the Morita class of the symmetric special Frobenius algebra A ; this has been discussed e.g. in section 4.1 of **[FRS1]** and section 1.4 of **[FRS2]**, and will be proven in **[FFRS4]**. Also, as already pointed out in section 1, a Morita invariant formulation leads naturally to a setup in terms of a 2-category \mathcal{O} that has precisely two objects. The 1-cells of \mathcal{O} are \mathcal{C} , $\mathcal{M} \cong \mathcal{C}_{A|}$, $\mathcal{M}' \cong \mathcal{C}_{|A}$ and $\mathcal{C}^* = \text{Func}(\mathcal{M}, \mathcal{M}) \cong \mathcal{C}_{A|A}$; see **[Mü]** and section 4 of **[FFRS3]**.

Instead of providing any further details, we content ourselves to illustrate some of these aspects in a brief dictionary below.

DICTIONARY between physical concepts and mathematical structures	
chiral label	object $U \in \text{Obj}(\mathcal{C})$
full CFT on oriented X	Morita class $[A]$ of symmetric special Frobenius algebras in \mathcal{C}
full CFT on unoriented X	Jandl algebra (A, σ) in \mathcal{C}
space $\{\Phi_{ij}\}$ of bulk fields	vector space $\text{Hom}_{A A}(U_i \otimes^+ A \otimes^- U_j, A)$ of bimodule morphisms
boundary condition	A -module $M \in \text{Obj}(\mathcal{C}_{A })$
space $\{\Psi_i^{MM'}\}$ of boundary fields	vector space $\text{Hom}_A(M \otimes U_i, M')$ of module morphisms
defect line	A - B -bimodule $Y \in \text{Obj}(\mathcal{C}_{A B})$
space $\{\Theta_{ij}^{YY'}\}$ of defect fields	vector space $\text{Hom}_{A B}(U_i \otimes^+ Y \otimes^- U_j, Y')$ of bimodule morphisms
simple current model	Schellekens algebra $A \cong \bigoplus_{g \in G} L_g$, $G \leq \text{Pic}^\circ(\mathcal{C})$
internal symmetries	Picard group $\text{Pic}(\mathcal{C}_{A A})$
Kramers–Wannier like dualities	duality bimodules $Y \in \text{Obj}(\mathcal{C}_{A A})$: bimodules obeying $Y^\vee \otimes_A Y \in \text{Pic}(\mathcal{C}_{A A})$

For the notions that occur in this list without having been introduced before, we mainly refer to the cited literature, and only explain the one of a Jandl algebra.

DEFINITION 2. A *Jandl algebra* (A, σ) in \mathcal{C} is a symmetric special Frobenius algebra A in \mathcal{C} together with a morphism $\sigma \in \text{Hom}(A, A)$ satisfying

$$(7) \quad \sigma \circ \eta = \eta, \quad \sigma \circ m = m \circ c_{A,A} \circ (\sigma \otimes \sigma), \quad \sigma \circ \sigma = \theta_A.$$

In words, σ is an algebra isomorphism from the opposite algebra A^{opp} to A that squares to the twist; we call it a *reversion* on A .

8. Automorphisms of Frobenius algebras

There is obviously a flaw in the above dictionary: Just like for full CFT on oriented world sheets a Morita class rather than a single algebra is relevant, also in the non-oriented case the relevant datum should be a suitable class of Jandl algebras, not an individual one. The rest of this paper is devoted to structures which are needed to remedy this flaw. Since some of them are of interest in their own right, independently of the application to CFT, from this point on \mathcal{C} will no longer be required to be a modular tensor category. Rather, \mathcal{C} is only assumed to be a small additive idempotent complete strict ribbon category enriched over $\text{Vect}_{\mathbb{k}}$, with \mathbb{k} a field, and with the tensor unit $\mathbf{1}$ being simple and absolutely simple, i.e. $\text{End}(\mathbf{1}) = \mathbb{k} \text{id}_{\mathbf{1}}$. A will again be a symmetric special Frobenius algebra in \mathcal{C} .

We first study automorphisms of A .

DEFINITION 3. A (unital) *algebra automorphism* of an algebra (A, m, η) is an isomorphism $\varphi \in \text{End}(A)$ such that $m \circ (\varphi \otimes \varphi) = \varphi \circ m$ and $\varphi \circ \eta = \eta$.

By composition, the algebra automorphisms of A form a group, denoted by $\text{Aut}(A)$. One may analogously define a (co-unital) co-algebra automorphism of a co-algebra (A, Δ, ε) as an isomorphism $\varphi \in \text{End}(A)$ satisfying $(\varphi \otimes \varphi) \circ \Delta = \Delta \circ \varphi$ and $\varepsilon \circ \varphi = \varepsilon$, and a Frobenius automorphism of a Frobenius algebra as one that is both an algebra and a co-algebra automorphism. But when A is symmetric special Frobenius, there is no need to distinguish between these concepts:

LEMMA 4. *Let A be a symmetric special Frobenius algebra. A morphism in $\text{End}(A)$ is an algebra automorphism of A iff it is a co-algebra automorphism of A .*

PROOF. We show that any algebra automorphism φ of a symmetric special Frobenius algebra A is also a co-algebra automorphism. The opposite implication follows by analogous arguments, in which the role of the algebra and co-algebra structures are interchanged.

First, by the defining property of φ and sovereignty of \mathcal{C} it follows that the morphism $\varepsilon_{\mathfrak{h}} = d_A \circ (\text{id}_A \otimes m) \circ (\tilde{b}_A \otimes \text{id}_A) \in \text{Hom}(A, \mathbf{1})$ obeys $\varepsilon_{\mathfrak{h}} \circ \varphi = \varepsilon_{\mathfrak{h}}$. Invoking proposition 1 we can set $\varepsilon = \xi \varepsilon_{\mathfrak{h}}$; hence $\varepsilon \circ \varphi = \varepsilon$. Further, using this result together with the defining property of φ , one shows that the isomorphism $\Phi \in \text{Hom}(A, A^\vee)$ introduced in (4) satisfies $\Phi \circ \varphi = (\varphi^{-1})^\vee \circ \Phi$. Now the coproduct of A can be expressed through m and Φ^{-1} , see formula (3.40) of [FRS1]; when combined with that expression, the equality just obtained is easily seen to imply that $(\varphi \otimes \varphi) \circ \Delta = \Delta \circ \varphi$. \square

Next we introduce inner automorphisms; to this end we need the \mathbb{k} -vector space

$$A_{\circ} := \text{Hom}(\mathbf{1}, A).$$

A_\circ is a \mathbb{k} -algebra with unit element η and multiplication $\alpha * \beta := m \circ (\alpha \otimes \beta)$ for $\alpha, \beta \in A_\circ$. If α is invertible with respect to this product, we write α^{-1} for its inverse, so that $\alpha * \alpha^{-1} = \eta = \alpha^{-1} * \alpha$, and we denote by A_\circ^\times the group of invertible elements of A_\circ . For $\alpha \in A_\circ^\times$ we set

$$(8) \quad \varpi_\alpha := m \circ (m \otimes \alpha^{-1}) \circ (\alpha \otimes id_A).$$

We also introduce the endomorphisms

$$P_l := \begin{array}{c} A \\ \downarrow \\ \text{---} \text{---} \text{---} \\ \uparrow \\ A \end{array} \quad \text{and} \quad P_r := \begin{array}{c} A \\ \downarrow \\ \text{---} \text{---} \text{---} \\ \uparrow \\ A \end{array}$$

of A , where $\begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \end{array} \in \text{Hom}(A \otimes A, A)$ is the product and $\begin{array}{c} \downarrow \\ \text{---} \text{---} \text{---} \end{array} \in \text{Hom}(A, A \otimes A)$ is the coproduct.

One checks that $P_l \circ \gamma = P_r \circ \gamma$ for $\gamma \in A_\circ$. (Also, P_l and P_r are idempotents; their images $C_l(A) := \text{Im } P_l$ and $C_r(A) := \text{Im } P_r$ are the left and right *centers* of A , see definition 2.31 of [FFRS1]. A few further properties of $P_{l/r}$ are given in lemma 2.29 of [FFRS1].) We write

$$(9) \quad C_\circ := \{\gamma \in A_\circ \mid P_l \circ \gamma = \gamma\}$$

(by lemma 2.29(iii) of [FFRS1], C_\circ is a \mathbb{k} -subalgebra of A_\circ) and denote by C_\circ^\times the group of invertible elements of C_\circ .

LEMMA 5. *The mapping $\alpha \mapsto \varpi_\alpha$ is a group homomorphism from A_\circ^\times to $\text{Aut}(A)$, with kernel C_\circ^\times . Thus in particular C_\circ^\times is a normal subgroup of A_\circ^\times .*

PROOF. By associativity of A one has $(\alpha * \beta)^{-1} = \beta^{-1} * \alpha^{-1}$, which implies that $\varpi_\alpha \circ \varpi_\beta = \varpi_{\alpha * \beta}$. In particular, $\varpi_\eta = id_A$, and $\varpi_{\alpha^{-1}}$ is inverse to ϖ_α . That for any $\alpha \in A_\circ^\times$ the morphism ϖ_α is an algebra automorphism of A then follows by just using associativity of A and the unit property of η . Finally, again by associativity and the unit property, the equality $\varpi_\gamma = id_A$ implies that $m \circ (id_A \otimes \gamma) = m \circ (\gamma \otimes id_A)$, which in turn is equivalent to $\gamma \in C_\circ^\times$ (recall that $C_l(A) := \text{Im } P_l$). \square

A slight extension of the consideration in the last part of the proof also shows that C_\circ is contained in the center of A_\circ . Furthermore, by definition of C_\circ and of the left and right centers $C_{l/r}(A)$ one has $\dim_{\mathbb{k}} C_\circ = \dim_{\mathbb{k}} \text{Hom}(\mathbf{1}, C_{l/r}(A))$.

We call the algebra automorphisms of A that are of the form (8) the *inner automorphisms* of A , and denote by

$$(10) \quad \text{Inn}(A) := \{\varpi_\alpha \mid \alpha \in A_\circ^\times\}$$

the group of inner automorphisms. Note that, by lemma 5,

$$\text{Inn}(A) \cong A_\circ^\times / C_\circ^\times.$$

9. Automorphisms and the Picard group of $\mathcal{C}_{A|A}$

The inner automorphisms play a special role when one twists the action of A on itself. Owing to associativity and the defining property of an algebra automorphism,

twisting the action of A on a left- or right-module by elements $\varphi, \psi \in \text{Aut}(A)$ yields another left- or right-module structure on the same object. In particular,

$$(11) \quad {}_{\varphi}A_{\psi} := (A, m \circ (\varphi \otimes id_A), m \circ (id_A \otimes \psi))$$

defines an A -bimodule structure on the object A . Not all of these bimodule structures are new, though (compare [VZ, p. 112]):

LEMMA 6. (i) For all $\varphi, \psi, \gamma \in \text{Aut}(A)$ we have

$${}_{\varphi}A_{\psi} \cong {}_{\gamma\varphi}A_{\gamma\psi} \quad \text{as } A\text{-bimodules.}$$

(ii) For all $\varphi, \psi, \varphi', \psi' \in \text{Aut}(A)$ we have, as A -bimodules,

$${}_{\varphi}A_{\psi} \otimes_A {}_{\varphi'}A_{\psi'} \cong {}_{\psi^{-1}\varphi}A_{\varphi'^{-1}\psi'}.$$

(iii) For any $\varphi, \psi \in \text{Aut}(A)$, ${}_{\varphi}A_{\psi}$ is an invertible object of $\mathcal{C}_{A|A}$.

(iv) We have $idA_{\psi} \cong idA_{id} \iff \psi \in \text{Inn}(A)$.

PROOF. (i) Using that γ is an algebra automorphism, one easily sees that γ intertwines both the left and the right action of A on ${}_{\varphi}A_{\psi}$ and on ${}_{\gamma\varphi}A_{\gamma\psi}$. Since γ is invertible, this establishes the claim.

(ii) The statement follows from the fact that ${}_{\varphi}A_{\psi} \otimes_A {}_{\varphi'}A_{\psi'} \cong {}_{\psi^{-1}\varphi}A_{id} \otimes_A idA_{\varphi'^{-1}\psi'}$, which in turn is a direct consequence of (i).

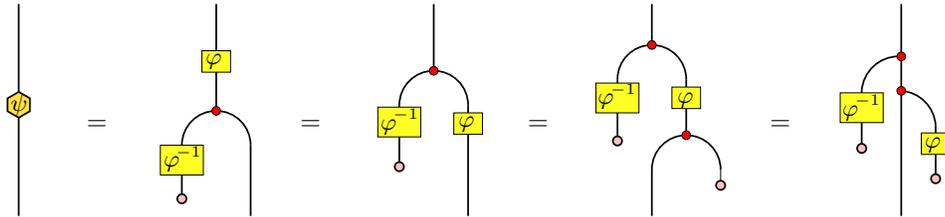
(iii) By (i) and (ii) we have in particular ${}_{\varphi}A_{\psi} \otimes_A {}_{\psi}A_{\varphi} \cong {}_{\psi^{-1}\varphi}A_{\psi^{-1}\varphi} \cong idA_{id} = A$.

(iv) Let $\psi = \varpi_{\alpha^{-1}} \in \text{Inn}(A)$. Then $m \circ (id_A \otimes \alpha) \in \text{End}(A)$ (which is invertible, the inverse being $m \circ (id_A \otimes \alpha^{-1})$) intertwines the right A -action on the bimodules idA_{ψ} and idA_{id} (and, trivially, the left A -action as well). Thus idA_{ψ} and idA_{id} are isomorphic as A -bimodules.

Conversely, let $\varphi \in \text{End}(A)$ be an intertwiner between idA_{ψ} and idA_{id} , i.e. satisfy



Composing the second of these equalities with $(\varphi^{-1} \circ \eta) \otimes id_A$ and using the first of the equalities twice, one shows that



Thus $\psi = \varpi_{\varphi^{-1} \circ \eta}$; in particular ψ is inner. \square

Because of part (i) of the lemma we may restrict our attention to the particular bimodules idA_{ψ} . This yields the following result, which is a variant of proposition 3.14 of [VZ] (the latter is restated as corollary 9 below):

PROPOSITION 7. *There is an exact sequence*

$$(12) \quad 1 \rightarrow \text{Inn}(A) \rightarrow \text{Aut}(A) \rightarrow \text{Pic}(\mathcal{C}_{A|A})$$

of groups.

PROOF. From (i) and (ii) of lemma 6 we deduce that ${}_{id}A_\psi \otimes_A {}_{id}A_{\psi'} \cong {}_{id}A_{\psi\psi'}$. Thus the mapping $\psi \mapsto [{}_{id}A_\psi]$ from $\text{Aut}(A)$ to the Picard group $\text{Pic}(\mathcal{C}_{A|A})$ of the category of A -bimodules is a group homomorphism. According to part (iv) of the lemma, the kernel of this homomorphism is $\text{Inn}(A)$. \square

Note that, up to isomorphism, $\text{Pic}(\mathcal{C}_{A|A})$ depends only on the Morita class of A , whereas different representatives of a Morita class can have rather different automorphism groups. Also, the last morphism in (12) is not, in general, an epimorphism. In the CFT context this means that in general not every internal symmetry can be detected as an automorphism of the algebra A used in the TFT construction; an example of this phenomenon has been presented in [FFRS2].

10. Azumaya algebras

For A an algebra in a braided monoidal category \mathcal{C} , the monoidal functors

$$\alpha_A^\pm : \mathcal{C} \rightarrow \mathcal{C}_{A|A}$$

of α -induction are defined by $\alpha_A^\pm(f) := id_A \otimes f \in \text{Hom}(A \otimes U, A \otimes V)$ for morphisms $f \in \text{Hom}(U, V)$ and $\alpha_A^\pm(V) := (A \otimes V, m \otimes id_V, \rho^\pm)$ for objects $V \in \text{Obj}(\mathcal{C})$. Here the representation morphisms $\rho^\pm \equiv \rho_V^\pm \in \text{Hom}(A \otimes V \otimes A, A \otimes V)$ are given by

$$\rho^+ := (m \otimes id_V) \circ (id_A \otimes c_{V,A}) \quad \text{and} \quad \rho^- := (m \otimes id_V) \circ (id_A \otimes (c_{A,V})^{-1}),$$

respectively. These functors first appeared in the context of subfactors in [LR] and for symmetric monoidal categories in [Pa1] (see also e.g. [BEK, Os, FFRS1]). They are used for introducing the following concept [Pa1, VZ].

DEFINITION 8. A is called an *Azumaya algebra* in \mathcal{C} iff the functors α_A^\pm are monoidal equivalences.

By remark 3.9 of [FFRS1], the left and right centers of an Azumaya algebra are isomorphic to $\mathbf{1}$. Also, one has $\alpha_A^\pm(\mathbf{1}) = A$, implying that every Azumaya algebra is simple. Further, for Azumaya algebras the functors α_A^\pm induce a group isomorphism (in fact, in general even two different isomorphisms) $\text{Pic}(\mathcal{C}) \xrightarrow{\cong} \text{Pic}(\mathcal{C}_{A|A})$, so that from proposition 7 we obtain the Rosenberg-Zelinsky type sequence that was established in proposition 3.14 of [VZ]:

COROLLARY 9. *If A is Azumaya, then there is an exact sequence*

$$1 \rightarrow \text{Inn}(A) \rightarrow \text{Aut}(A) \rightarrow \text{Pic}(\mathcal{C}).$$

Note that for $\mathcal{C} = \text{Vect}_{\mathbb{k}}$, this reduces to Noether's classical result that all automorphisms of a central simple \mathbb{k} -algebra are inner. For non-Azumaya algebras, α_A^\pm still induce group homomorphisms from $\text{Pic}(\mathcal{C})$ to $\text{Pic}(\mathcal{C}_{A|A})$, but in general they are neither injective nor surjective (counter examples from RCFT are given by the modular tensor categories of the D_{even} series of $\mathfrak{sl}(2)$ Wess-Zumino-Witten models).

For algebras in a braided monoidal category one defines the *opposite* algebra of A as $(A, m \circ c_{A,A}^{-1}, \eta)$, and the *product* of A and B as algebras by $(A \otimes B, (m_A \otimes m_B) \circ$

$(id_A \otimes c_{A,B}^{-1} \otimes id_B), \eta_A \otimes \eta_B$). Analogously, when both $A = (A, m_A, \eta_A, \Delta_A, \varepsilon_A)$ and $B = (B, m_B, \eta_B, \Delta_B, \varepsilon_B)$ are symmetric special Frobenius algebras, one sets

$$(13) \quad \begin{aligned} A^{\text{opp}} &:= (A, m \circ c_{A,A}^{-1}, \eta, c_{A,A} \circ \Delta, \varepsilon) \quad \text{and} \\ A \# B &:= (A \otimes B, (m_A \otimes m_B) \circ (id_A \otimes c_{A,B}^{-1} \otimes id_B), \eta_A \otimes \eta_B, \\ &\quad (id_A \otimes c_{A,B} \otimes id_B) \circ (\Delta_A \otimes \Delta_B), \varepsilon_A \otimes \varepsilon_B) \end{aligned}$$

and shows (propositions 3.18(ii) and 3.22(ii) of **[FRS1]**) that the opposite algebra A^{opp} and the product algebra $A \# B$ are again symmetric special Frobenius.

If A and B are Azumaya, then so are A^{opp} and $A \# B$, and forming the opposite algebra and the product is compatible with Morita equivalence of algebras in \mathcal{C} , see theorems 3.3 and 3.4 of **[VZ]**. Thus the set of Morita classes of Azumaya algebras in \mathcal{C} carries a natural group structure, with product induced by $\#$ and inverse induced by taking the opposite algebra; this group is called the Brauer group of \mathcal{C} .⁸

For the rest of this section, \mathcal{C} is again a modular tensor category. Recall from proposition 1 that then the property of an algebra in \mathcal{C} of being simple symmetric special Frobenius is Morita stable. It thus makes sense to study the group of Morita classes of symmetric special Frobenius Azumaya algebras in \mathcal{C} ; we call this group the *Frobenius–Brauer group* of \mathcal{C} and denote it by $\text{FBr}(\mathcal{C})$. In the braided setting, $\text{FBr}(\mathcal{C})$ is not necessarily abelian.

The formulas (13) still make sense and are compatible with Morita equivalence when A (and/or B) is not Azumaya. But $A \otimes A^{\text{opp}}$ is no longer Morita equivalent to $\mathbf{1}$ if A is not Azumaya. Thus the set of Morita classes of arbitrary symmetric special Frobenius algebras in \mathcal{C} is no longer a group, though still a monoid, see **[FRS1, remark 5.4]** and **[EP]**.

Using notation from section 5, we denote by $Z(A)$ the square matrix with entries

$$Z(A)_{ij} := \dim_{\mathbb{C}} \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A) = \dim_{\mathbb{C}} \text{Hom}_{A|A}(\alpha_A^+(U_i), \alpha_A^-(U_j^\vee))$$

for $i, j \in \mathcal{I}$. Here the second equality follows by comparing the definition of \otimes^\pm with that of α -induction. One has $Z(\mathbf{1})_{ij} = \delta_{i,j^\vee}$ with $U_{j^\vee} \cong U_j^\vee$ as well as (by proposition 5.3 of **[FRS1]**) $Z(A^{\text{opp}}) = Z(A)^t$ and $Z(A \# B) = Z(A)Z(\mathbf{1})Z(B)$.

PROPOSITION 10. *A symmetric special Frobenius algebra A is Azumaya iff $Z(A)$ is a permutation matrix.*

PROOF. According to proposition 2.36 of **[FFRS1]** one has

$$\begin{aligned} \text{Hom}_{A|A}(\alpha_A^+(U_i), \alpha_A^+(U_j)) &= \text{Hom}(U_i, C_l(A) \otimes U_j) \cong \mathbb{C}^{\sum_k N_{jk}^i Z(A)_{k0}} \quad \text{and} \\ \text{Hom}_{A|A}(\alpha_A^-(U_i), \alpha_A^-(U_j)) &= \text{Hom}(U_i, C_r(A) \otimes U_j) \cong \mathbb{C}^{\sum_k N_{jk}^i Z(A)_{0k}}, \end{aligned}$$

where $N_{jk}^i = \dim_{\mathbb{C}} \text{Hom}(U_j \otimes U_k, U_i)$ and where the equality sign indicates a canonical isomorphism (given in (2.69) of **[FFRS1]**), while the second isomorphisms follow from remark 3.7 and lemma 3.13 of **[FFRS1]**. Now if $Z(A)$ is a permutation matrix, then the number of isomorphism classes of simple A -bimodules equals the number $|\mathcal{I}|$ of isomorphism classes of simple objects of \mathcal{C} (see remark 5.19(ii) of **[FRS1]**). Moreover, because of $\alpha_A^\pm(\mathbf{1}) = A$ we have $Z(A)_{i0} = \delta_{i,0}$, from which it follows that $\text{Hom}_{A|A}(\alpha_A^\pm(U_i), \alpha_A^\pm(U_j)) \cong \delta_{i,j} \mathbb{C}$. Thus the functors α_A^\pm are essentially surjective and fully faithful, and hence are equivalences.

⁸ In **[Pa1, VZ]** two different Brauer groups are considered. They coincide when the tensor unit is projective, and hence in particular when \mathcal{C} is semisimple.

Conversely, if α_A^\pm are monoidal equivalences, then the bimodules $\alpha_A^+(U_i)$ and $\alpha_A^-(U_i)$ are simple for all $i \in \mathcal{I}$, and moreover, each simple A -bimodule is isomorphic to one of the $\alpha_A^+(U_i)$ and to one of the $\alpha_A^-(U_j)$. Thus $Z(A)$ is a permutation matrix. \square

PROPOSITION 11. *The Frobenius–Brauer group $\text{FBr}(\mathcal{C})$ of a modular tensor category \mathcal{C} is finite.*

PROOF. Every simple algebra in a modular tensor category \mathcal{C} is Morita equivalent to an algebra A satisfying $\text{Hom}(\mathbf{1}, A) = \mathbb{C}\eta$ [Os]. And the number of symmetric special Frobenius algebras in \mathcal{C} having this property is finite (proposition 3.6 of [FRS3]). \square

REMARK 12. In RCFT the integers $Z(A)_{ij}$ possess the interpretation of the coefficients of the torus partition function in a standard basis, called the basis of characters, of one-point conformal blocks on the double of the torus. $Z(A)$ can thus be described in terms of maximal extensions of the chiral symmetries for left and right movers. As an additional information, $Z(A)$ encodes an isomorphism of the fusion rules of the two extensions [MS]. The procedure of ‘extending the chiral symmetries’ corresponds to passing to the modular tensor categories $\mathcal{C}_{C_{l/r}(A)}^{\text{loc}}$ of local modules [Pa2, FFRS1] of the left and right centers $C_{l/r}(A)$, respectively. These two categories are monoidally equivalent (theorem 5.20 of [FFRS1]), so that in particular they have indeed isomorphic Grothendieck rings, $K_0(\mathcal{C}_{C_l(A)}^{\text{loc}}) \cong K_0(\mathcal{C}_{C_r(A)}^{\text{loc}})$. Moreover, one can lift the algebra A to Azumaya algebras in $\mathcal{C}_{C_{l/r}(A)}^{\text{loc}}$ (see proposition 4.14 of [FFRS1]). In this sense, the two extensions are maximal, and the isomorphism of the fusion rules is encoded in the ‘Azumaya part’ of A . In particular, when A is Azumaya, then $[A] \mapsto Z(A)Z(\mathbf{1})$ is a group homomorphism from $\text{FBr}(\mathcal{C})$ to $\text{Aut}(K_0(\mathcal{C}))$.

11. Reversions

Let us now return to the question of finding a suitable equivalence relation for Jandl algebras, such that different algebras in the same class yield, via the TFT construction, one and the same full CFT, i.e. full CFTs that, up to possibly certain scalar factors (for details see [FFRS4]) have the same correlators.

We recall from definition 2 that a Jandl algebra $A \equiv (A, \sigma)$ in a ribbon category \mathcal{C} consists of a symmetric special Frobenius algebra A in \mathcal{C} and a *reversion* $\sigma \in \text{Hom}(A, A)$, obeying the relations (7). Below we will also need the fact that in a ribbon category the bidual functor $?^{\vee\vee}$ is naturally equivalent to $\text{id}_{\mathcal{C}}$, via the isomorphisms $\delta_U = (\text{id}_{U^{\vee\vee}} \otimes d_U) \circ [(c_{U^{\vee}, U^{\vee\vee}} \circ b_{U^{\vee}}) \otimes \theta_U] \in \text{Hom}(U, U^{\vee\vee})$.

On the set of isomorphism classes of Jandl algebras in \mathcal{C} an equivalence relation can be obtained as follows. Let (A, σ) be a simple Jandl algebra, $M = (\dot{M}, \rho)$ a left A -module and $B \in [A]$ the algebra $B = M^\vee \otimes_A M$. Here M^\vee is the right A -module $M^\vee = (\dot{M}^\vee, (\text{id}_{\dot{M}^\vee} \otimes d_A) \circ (\rho^\vee \otimes \text{id}_A))$. Via the reversion one can endow the object \dot{M}^\vee also with a *left* A -module structure, denoted by M^σ (definition 2.6 of [FRS2]): $M^\sigma = (\dot{M}^\vee, (\tilde{d}_A \otimes \text{id}_{\dot{M}^\vee}) \circ (\sigma \otimes^\vee (\rho \circ c_{\dot{M}, A})))$. Now suppose that there is an isomorphism $g \in \text{Hom}_A(M, M^\sigma)$ such that

$$(14) \quad g = \nu g^\vee \circ \delta_{\dot{M}}$$

for some $\nu \in \mathbb{C}$. Then as shown in proposition 2.16 (combined with theorem 2.14) of [FRS2], one can use g to construct a reversion

$$(15) \quad \sigma_g := r \circ (g \otimes g^{-1}) \circ (\theta_{\dot{M}} \otimes id_{\dot{M}^\vee}) \circ c_{\dot{M}^\vee, \dot{M}} \circ e$$

of the algebra B (here $e \in \text{Hom}(B, \dot{M}^\vee \otimes \dot{M})$ and $r \in \text{Hom}(\dot{M}^\vee \otimes \dot{M}, B)$ are the embedding and restriction morphisms for B as a retract of $\dot{M}^\vee \otimes \dot{M}$; see formula (2.60) of [FRS2]).

DEFINITION 13. (i) Two Jandl algebras (A, σ) and (B, τ) are called isomorphic, denoted by $(A, \sigma) \cong (B, \tau)$, iff there exists an isomorphism $\varphi: A \rightarrow B$ of A and B as symmetric special Frobenius algebras such that $\varphi \circ \sigma = \tau \circ \varphi$.

(ii) Two Jandl algebras (A, σ) and (B, τ) are called equivalent, denoted by

$$(16) \quad (A, \sigma) \smile (B, \tau),$$

iff (B, τ) is isomorphic to $(M^\vee \otimes_A M, \sigma_g)$ with M and σ_g as above.

LEMMA 14. *The relation ‘ \smile ’ (16) is an equivalence relation.*

PROOF. That (16) is symmetric is shown by proposition 2.17 [FRS2]. Reflexivity is seen by taking $M = A$ and $g = \Phi \circ \sigma^{-1}$ with Φ the isomorphism given in (4). Then e.g. the equality (14) (with $\nu = 1$) follows with the help of $\Phi = \Phi'$ (which holds because A is symmetric) and of $\varepsilon \circ m \circ (id_A \otimes \sigma) = \varepsilon \circ m \circ (\sigma \otimes id_A)$. Transitivity is shown by verifying (which is a bit lengthy, but straightforward) that when $(A, \sigma) \smile (B, \tau)$ via an A - B -bimodule M and isomorphism g , and $(B, \tau) \smile (C, \varrho)$ via a B - C -bimodule N and isomorphism h , then $(A, \sigma) \smile (C, \varrho)$ via the A - C -bimodule $M \otimes_B N$ and the morphism $c_{\dot{M}^\vee, \dot{N}^\vee} \circ (g \otimes h)$, which is an isomorphism between $M \otimes_B N$ and $(M \otimes_B N)^\sigma$. \square

It can be checked that the equivalence relation (16) indeed refines Morita equivalence. In particular, for the allowed interpolating bimodules M the functor $F_M = M \otimes_B - : \mathcal{C}_{B|} \rightarrow \mathcal{C}_{A|}$ has the property that $F_M \circ F_\tau$ and $F_\sigma \circ F_M$ are naturally isomorphic, where F_σ is the endofunctor of $\mathcal{C}_{A|}$ that acts on objects as $X \mapsto X^\sigma$ and on morphisms (when regarded as morphisms of \mathcal{C}) as $f \mapsto f^\vee$, and analogously for the endofunctor F_τ of $\mathcal{C}_{B|}$.

Let us also remark that the class of objects \dot{M} for which there can exist module isomorphisms satisfying (14) is quite restricted. As shown in [FRS2], the number ν appearing in (14) can only take the values ± 1 . Moreover, by comparison with formula (3.29) of [FS] one learns that every simple subobject of \dot{M} has ν as the value of its Frobenius-Schur indicator, i.e. this value must be the same for all subobjects (also, ν does not depend on g).

Next we note that the existence of reversions does not behave nicely with respect to forming the product of algebras. It is easy to check that for given Jandl algebras (A, σ) and (B, σ') , the morphism $\sigma \otimes \sigma'$ is not, in general, a reversion of the product algebra $A \# B$. In fact, $A \# B$ may not possess any reversion at all: owing to $Z(A^{\text{opp}}) = Z(A)^t$, existence of a reversion implies that the matrix $Z(A)$ is symmetric, and this property is not preserved under $\#$. Consider for instance the case that A and B are Azumaya and Jandl. Then $Z(A)_{ij} = \delta_{i^\vee, \pi_A(j)}$ with π_A a permutation of order 2, and analogously for B . But $Z(A \# B)$ is given by the permutation $\pi_{A \# B} = \pi_A \circ \pi_B$, which has order 2 only if π_A and π_B commute.

As a consequence, an equivalence relation (such as (16)) between Jandl algebras is not (in general) compatible with $\#$. In particular, $\#$ does not induce a group structure on the set of equivalence classes of Azumaya–Jandl algebras. Thus there appears to be no braided analogue of the *involutive Brauer group* that can be defined for algebras [PS] or for symmetric monoidal categories [VV].

As already pointed out, Morita equivalent symmetric special Frobenius algebras yield, via the TFT construction, the same *oriented* full CFT. Comparing the correlators of the *unoriented* full CFTs that are obtained from different Jandl algebras is a much more difficult task than in the oriented case, and we have not yet fully investigated this issue. But we plan to establish in a subsequent paper [FFRS4] that it is indeed the equivalence relation (16) that takes over the role of Morita equivalence in unoriented CFT. There is, in fact, a special situation in which the comparison of the CFTs obtained from two Jandl algebras (A, σ) and (B, τ) is much simplified, namely when $B = A$. In this case one can use the fact that for any two reversion σ and σ' of A , $\omega := \sigma^{-1} \circ \sigma'$ is an algebra automorphism of A . Conversely, while the composition of a reversion and an algebra automorphism is not, in general, again a reversion, we still have the following

PROPOSITION 15. *Let (A, σ) be a Jandl algebra and ω an algebra automorphism of A .*

(i) *$(A, \sigma \circ \omega)$ is a Jandl algebra iff*

$$(17) \quad \omega \circ \sigma \circ \omega = \sigma.$$

(ii) *If A is simple, $\omega = \varpi_\alpha$ is inner, and $\varpi_\alpha \circ \sigma \circ \varpi_\alpha = \sigma$, then*

$$(18) \quad (A, \sigma) \smile (A, \sigma \circ \varpi_\alpha).$$

PROOF. (i) is shown in proposition 2.3 of [FRS2].

(ii) Consider the morphism

$$g_\alpha := \epsilon_\alpha \Phi \circ \sigma^{-1} \circ m \circ (id_A \otimes \alpha^{-1}) \in \text{Hom}(A, A^\vee),$$

with Φ as in (4) and $\epsilon_\alpha \in \{\pm 1\}$. By using the defining properties of A (being a symmetric special Frobenius algebra) and of ω and σ , it is not difficult to see that g_α intertwines the left A -modules A and A^σ . Similarly, making in addition use of the equality (17) one shows that g_α satisfies (14) (with $\nu = 1$). Finally, regarding $A \cong A^\vee \otimes_A A$ as a retract of $A^\vee \otimes A$ via $e := (id_{A^\vee} \otimes m) \circ (\tilde{b}_A \otimes id_A)$ and $r := (d \otimes id_A) \circ (id_{A^\vee} \otimes \Delta)$, one can show that the morphism in (15) becomes $\sigma_{g_\alpha} = \sigma \circ \varpi_\alpha$, provided that

$$\sigma \circ \alpha = \epsilon_\alpha \alpha.$$

As will be seen in proposition 19 below, this equality is equivalent to $\varpi_\alpha \circ \sigma \circ \varpi_\alpha = \sigma$. Thus indeed $(A, \sigma) \smile (A, \sigma \circ \varpi_\alpha)$, via $M = A$ and $g = g_\alpha$. \square

According to proposition 15, reversion of a symmetric special Frobenius algebra A that endow A with inequivalent Jandl structures are related by *outer* algebra automorphisms of A . Thus the number of inequivalent Jandl structures on A is bounded by the order of the group $\text{Aut}(A)/\text{Inn}(A)$ of outer automorphisms, which in turn by proposition 7 is bounded by the order of $\text{Pic}(\mathcal{C}_{A|A})$. Now for any symmetric special Frobenius algebra in a modular tensor category \mathcal{C} , $\text{Pic}(\mathcal{C}_{A|A})$ is a finite group. When combined with (the proof of) proposition 11 and with the results of section 2.5 of [FRS2], this yields

COROLLARY 16. *The number of equivalence classes, with respect to the relation (16), of simple Jandl algebras in a modular tensor category is finite.*

In accordance with our remarks about the equivalence of unoriented CFTs above, part (ii) of proposition 15 leads to the following counterpart in CFT, which will be demonstrated in [FFRS4]:

PROPOSITION 17. *Let ϖ_α be an inner automorphism of a simple symmetric special Frobenius algebra A in a modular tensor category \mathcal{C} and σ a reversion of A such that $\sigma \circ \varpi_\alpha$ is a reversion, too. Then the correlation functions of the full CFTs based on the Jandl algebras (A, σ) and $(A, \sigma \circ \varpi_\alpha)$ differ at most by a sign:*

$$\text{Cor}_{(A, \sigma)}(X) = (\epsilon_\alpha)^{\text{cr}(X)} \text{Cor}_{(A, \sigma \circ \varpi_\alpha)}(X).$$

$\text{cr}(X)$ is the ‘number of crosscaps’ of the world sheet X , which is defined modulo 2.

The TFT construction of the correlators of unoriented full CFTs involves, as compared to the oriented case, in addition at each vertex of the triangulation of $\mathfrak{I}(X)$ a choice of local orientation. Further, the reversion σ enters the prescription only via those edges of the triangulation for which the chosen orientation at the vertex on one end, when transported along the edge to its other end, is different from the orientation chosen for the vertex on that end. The proof then relies on the fact that the local orientations can be chosen in such a way that there are at most two such particular edges (e.g. none of them if X is orientable). This way the effects of replacing σ by $\varpi_\alpha \circ \sigma$ are confined to neighborhoods of at most two A -ribbons in $\mathfrak{I}(X)$, and can therefore be analyzed relatively easily.

For completing the proof of proposition 15(ii), we must still study the question of when together with σ also $\sigma \circ \varpi_\alpha$ is a reversion.

LEMMA 18. *Given a reversion σ of a symmetric special Frobenius algebra A and an invertible $\alpha \in A_\circ$, the morphism $\sigma \circ \varpi_\alpha$ is again a reversion of A iff $\varpi_{\sigma \circ \alpha} = \varpi_\alpha$.*

PROOF. According to proposition 15(i), $\sigma \circ \varpi_\alpha$ is a reversion iff the equality $\sigma \circ \varpi_\alpha \circ \sigma^{-1} = \varpi_\alpha^{-1} = \varpi_{\alpha^{-1}}$ holds. On the other hand, by using the defining properties (7) of a reversion one shows $\sigma \circ \varpi_\alpha \circ \sigma^{-1} = \varpi_{\sigma \circ \alpha^{-1}}$. Thus $\varpi_{\alpha^{-1}} = \varpi_{\sigma \circ \alpha^{-1}}$, which is equivalent to $\varpi_{\sigma \circ \alpha} = \varpi_\alpha$. \square

Note that, using lemma 5, $\varpi_{\sigma \circ \alpha} = \varpi_\alpha$ iff $\alpha^{-1} * (\sigma \circ \alpha) \in C_\circ^\times$. Also, for any symmetric special Frobenius algebra A one has $\text{Hom}(\mathbf{1}, C_l(A)) \cong \text{Hom}_{A|A}(A, A)$ (proposition 2.36 of [FFRS1]). Now assume that A is simple. Then $\text{Hom}_{A|A}(A, A) \cong \mathbb{k}$, and it follows that $C_\circ^\times = \mathbb{k}^\times \eta$. Hence $\varpi_{\sigma \circ \alpha} = \varpi_\alpha$ means that $\sigma \circ \alpha$ is a nonzero multiple of α . Furthermore, owing to $\sigma \circ \sigma \circ \alpha = \theta_A \circ \alpha = \alpha$, this multiple must be ± 1 . Further, because of $C_\circ^\times = \mathbb{k}^\times \eta$ we also learn that, for simple A , $\varpi_\alpha = \varpi_{\alpha'}$ iff $\alpha = \zeta \alpha'$ for some $\zeta \in \mathbb{k}^\times$. Thus we have proven

PROPOSITION 19. *Let (A, σ) be a simple Jandl algebra and $\alpha \in A_\circ^\times$.*

- (i) *The endomorphism $\sigma \circ \varpi_\alpha$ is a reversion of A iff $\sigma \circ \alpha = \epsilon_\alpha \alpha$ with $\epsilon_\alpha \in \{\pm 1\}$.*
- (ii) *If $\varpi_\alpha = \varpi_{\alpha'}$, then $\epsilon_\alpha = \epsilon_{\alpha'}$.*

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