Conformal characters and the modular representation

P. Bantay* **and T. Gannon†**

ABSTRACT: A general procedure is presented to determine, given any suitable representation of the modular group, the characters of all possible Rational Conformal Field Theories whose associated modular representation is the given one. The relevant ideas and methods are illustrated on two non-trivial examples: the Yang-Lee and the Ising models.

KEYWORDS: Rational Conformal Field Theory, modular group, conformal characters.

*Institute for Theoretical Physics, Eötvös University, Budapest
†Department of Mathematical Sciences, University of Alberta, Edmonton

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1. Introduction

Among the many quantities that characterize a Conformal Field Theory, a special role is played by the characters $\chi_\mu(\tau)$ of the primary fields. By determining the degeneracies of the Virasoro generator L_0 in the different sectors, they convey information about how the chiral algebra of the theory is represented on the space of states. As such, they are the basic building blocks of the torus partition function, and from their knowledge one can read off at once the conformal weights h_μ and the central charge c of the theory. Even more, they determine almost uniquely the modular S-matrix of the theory, and thus the fusion rules via Verlinde's formula.

Reversing the logic, one may ask to what extent can the characters be recovered from the knowledge of the conformal weights, central charge and fusion rules. This paper aims to answer that question. An elementary observation is that the fusion rules uniquely determine S, up to at most a permutation of its columns and a multiplication of each column by ± 1 [4]. Together, h_μ, c and S uniquely determine a representation ρ of the modular group $\mathrm{SL}_2(\mathbb{Z})$. The characters χ_μ may be grouped into a *character vector* $\mathbb{X}$ that is holomorphic in the upper half-plane, transforms according to ρ, and - in a suitable sense, to be explained in Section 2 - has only finite order poles at $\tau = i\infty$. Thus the real question is: to what extent does the modular representation ρ determine the characters? We explain that the characters of an RCFT are uniquely determined by its modular representation and the singular terms

$\sum_{s<0} a_s q^s$ of each character, up to perhaps some constant terms. We show how to construct all character vectors compatible with a representation ρ that satisfies some simple conditions holding in any RCFT.

This paper collects our basic results, illustrating them with examples. The follow-up paper [3] will give more details and push our analysis much further. Section 2 introduces the notation and establishes the extent to which the characters are determined by ρ. Sections 3 to 5 describe how to explicitly build the character vectors term-by-term, starting from ρ. We conclude the paper with two concrete examples: the Yang-Lee and Ising models. In the explicit computations, we made heavy use of the Computer Algebra Systems GAP [8] (for the computations of invariants and covariants), PARI/GP [18] (for the computations involving modular forms) and Singular [9] (for the solution of polynomial systems).

2. Admissible representations and the canonical basis

Let's consider a finite dimensional (unitary) representation $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(r, \mathbb{C})$ of the modular group $\mathrm{SL}_2(\mathbb{Z})$. It is known that such a representation is completely characterized by the pair of matrices $T = \rho\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \rho\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, which satisfy the relations $S^4 = 1$ and $STS = T^{-1}ST^{-1}$. As usual, $\Gamma(N)$ will denote the *principal congruence subgroup* of level N, i.e.

$$\Gamma(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid a, d \equiv 1 \pmod{N} \text{ and } b, c \equiv 0 \pmod{N} \right\} . \tag{2.1}$$

We'll denote by W_ρ the subspace of $\mathbb{C}^r$ consisting of those vectors that are invariant under ρ, i.e. $W_\rho = \{v \in \mathbb{C}^r \mid Tv = Sv = v\}$, by $\langle ., . \rangle$ the standard scalar product on $\mathbb{C}^r$, and by $\mathbf{e}_\mu$ the μ-th element of the standard basis of $\mathbb{C}^r$, whose μ-th entry is 1, and all others 0.

We'll call a representation $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(r, \mathbb{C})$ *admissible* if it satisfies the following conditions:

1 $\ker \rho$ is a *congruence subgroup*, i.e. it contains $\Gamma(N)$ for some integer N;

2 T is diagonal and S^2 is a permutation matrix (with respect to the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_r$ of $\mathbb{C}^r$).

Note that the permutation associated to S^2 (*charge conjugation*) is an involution $\mu \mapsto \overline{\mu}$, since $S^4 = 1$. Also, T has finite order dividing N because of $\Gamma(N) < \ker \rho$, and so is of the form $T = \exp(2\pi i \Lambda)$ for some diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix} , \tag{2.2}$$

where $0 \leq \lambda_1, \ldots, \lambda_r < 1$ are rational numbers - the *exponents* of ρ - whose denominator divides the level N. Because S^2 commutes with T, one also has $\lambda_{\overline{\mu}} = \lambda_\mu$. This means, denoting by $\mathcal{O}$ the orbits of charge conjugation, that the exponent is the same for each element of an orbit

$\xi \in \mathcal{O}$: we'll denote by λ_ξ this common value. We'll also use the notation $\mathbf{e}_\xi = \sum_{\mu\in\xi} \mathbf{e}_\mu$ for an orbit $\xi \in \mathcal{O}$.

It is known that the above admissibility conditions are always satisfied by the modular representation associated to a Rational Conformal Field Theory. Indeed, the primary fields of the RCFT provide a distinguished basis in which T is diagonal and S^2 is a permutation matrix, while the first admissibility condition follows from the results of [2]. More generally, an analogue will hold for certain (as yet undetermined) classes of Vertex Operator Algebras and Modular Tensor Categories.

For an admissible representation $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(r, \mathbb{C})$, we'll denote by $\mathcal{M}(\rho)$ the space of vector-valued complex functions $\mathbb{X} : \mathbf{H} \to \mathbb{C}^r$ which are holomorphic in the upper half-plane $\mathbf{H} = \{\tau \,|\, \mathrm{Im}\,\tau > 0\}$, satisfy the transformation rule

$$\mathbb{X}\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{X}(\tau) \tag{2.3}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and have only finite order poles at the cusps. To explain this last condition, note that Eqs.(2.3) and (2.2) imply that $q^{-\Lambda}\mathbb{X}(\tau)$ is invariant under $\tau \mapsto \tau + 1$, and so may be expanded into a power series in $q = \exp(2\pi i\tau)$:

$$q^{-\Lambda}\mathbb{X} = \sum_{n\in\mathbb{Z}} \mathbb{X}[n]\, q^n \ , \tag{2.4}$$

where $\mathbb{X}[n] \in \mathbb{C}^r$ for all n. We define

$$\mathscr{P}\mathbb{X} = \sum_{n<0} \mathbb{X}[n]\, q^n \tag{2.5}$$

to denote the sum of the negative powers of q in Eq.(2.4), i.e. the singular (or principal) part of $\mathbb{X}$. Then the requirement of having only finite order poles at the cusps means that each component of $\mathscr{P}\mathbb{X}$ is a polynomial in q^{-1} for $\mathbb{X} \in \mathcal{M}(\rho)$, i.e. we get a linear map $\mathscr{P} : \mathcal{M}(\rho) \to V$, where $V = \oplus_{\mu=1}^{r} \mathfrak{m}\mathbf{e}_\mu$, for the set $\mathfrak{m}$ of all polynomials in q^{-1} with vanishing constant term. It follows from the basic principles of Rational Conformal Field Theory, that the characters $\chi_\mu(\tau)$ of the primary fields form a vector that belongs to the space $\mathcal{M}(\rho)$, where ρ is the modular representation associated to the given model (see [21] for a more general statement). [1]

Let's note that Eq.(2.3) implies

$$\mathbb{X}(\tau) = \rho\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbb{X}(\tau) \ ,$$

in other words any $\mathbb{X}(\tau) \in \mathcal{M}(\rho)$ is invariant under charge conjugation S^2. This means that the image of $\mathscr{P}$ lies in the subspace V_+ of V whose elements are left invariant by S^2. Actually,

[1] We note that a closely related theory of vector-valued modular forms had been developed in [11, 12].

it follows from the Riemann-Roch theorem for vector bundles [16] (or the arguments on p.15–16 of [10]), that $\operatorname{im}\mathscr{P} = V_+$, since the (compactified) quotient of the upper half-plane $\mathbf{H}$ by $\mathrm{SL}_2(\mathbb{Z})$ is a sphere (see [3] for details). A basis of V_+ is provided by the vectors $q^{-m}\mathbf{e}_\xi$, for any positive integer m and S^2-orbit $\xi \in \mathcal{O}$.

The kernel of the map $\mathscr{P}$ is also easy to describe. Indeed, suppose that $\mathbb{X} \in \mathcal{M}(\rho)$ has vanishing singular part: then all of its vector components are holomorphic on the upper half-plane, including the cusps (since $\mathrm{SL}_2(\mathbb{Z})$ can map any cusp to $i\infty$). Since the kernel of ρ is a congruence subgroup, the components of $\mathbb{X}$ are holomorphic functions on the compact Riemann surface $\overline{\mathbf{H}/\ker\rho}$ uniformized by $\ker\rho$, hence they are all constant. But this constant vector $\mathbb{X}$ should also satisfy Eq.(2.3), and thus it should belong to W_ρ, the invariant subspace of ρ. This shows that $\ker\mathscr{P} = W_\rho$, and is then finite dimensional (over $\mathbb{C}$). Thus, the induced map $\mathcal{M}(\rho)/W_\rho \to V_+$ is a bijection.

The outcome of the above considerations is that any element $v \in V_+$ determines a unique coset $\mathbb{X} \in \mathcal{M}(\rho)/W_\rho$ such that $\mathscr{P}\mathbb{X} = v$. We'll denote by $\mathbb{X}^{(\xi;m)}$ the coset for which $\mathscr{P}\mathbb{X}^{(\xi;m)} = q^{-m}\mathbf{e}_\xi$. Since the $q^{-m}\mathbf{e}_\xi$ form a basis of V_+, the cosets $\mathbb{X}^{(\xi;m)}$ form a basis of the linear space $\mathcal{M}(\rho)/W_\rho$, which we'll call the *canonical basis*. Our main concern will be to determine explicitly the $\mathbb{X}^{(\xi;m)}$ for a given representation ρ.

Strictly speaking, the canonical basis vectors $\mathbb{X}^{(\xi;m)}$ are not elements of $\mathcal{M}(\rho)$, although they come close to it, since their Laurent expansions are completely determined up to addition of a constant term from W_ρ. To simplify the ensuing presentation, we'll make the assumption that $W_\rho = 0$: this small loss of generality is amply compensated by the gain in clarity and brevity. The general case can be worked out without too much effort. Indeed, $W_\rho = 0$ will generically hold, since $W_\rho \neq 0$ can only occur when some $\lambda_\mu = 0$ in Eq.(2.2). A familiar example when $W_\rho \neq 0$ is provided by the trivial one-dimensional ρ - e.g. the $c = 24$ theories with only 1 primary field have identical character, up to an additive constant (which counts the number of spin-1 fields) running from 0 to 1128 [20].

3. The Hauptmodul and the recursion relations

According to the results of the previous section, the space $\mathcal{M}(\rho)$ is an infinite dimensional linear space over $\mathbb{C}$. In particular, the elements $\mathbb{X}^{(\xi;m)}$ of the canonical basis are linearly independent, and thus one needs seemingly an infinite amount of data to describe the structure of $\mathcal{M}(\rho)$. As it turns out, the situation is much better, and this is related to the existence of the Hauptmodul J (or absolute invariant) for $\mathrm{SL}_2(\mathbb{Z})$.

Indeed, when ρ is the trivial one-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$, the space $\mathcal{M}(\rho)$ is known [1, 13] to be the polynomial algebra $\mathbb{C}[J]$, where J is the Hauptmodul of $\mathrm{SL}_2(\mathbb{Z})$, i.e. the unique holomorphic function on $\mathbf{H}$ that is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, i.e. satisfies the functional equation

$$J\left(\frac{a\tau+b}{c\tau+d}\right) = J(\tau) \tag{3.1}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and whose Laurent series around $q=0$ reads

$$J(\tau) = q^{-1} + \sum_{n=1}^{\infty} c(n)\, q^n \ . \tag{3.2}$$

We note that the coefficients $c(n)$ of the Laurent expansion are all positive integers (since J is the character of the Moonshine module), equal to the dimensions of specific representations of the Monster, the largest sporadic simple group [7].

The importance of the Hauptmodul J for our considerations stems from the fact that for an admissible representation ρ and any $\mathbb{X} \in \mathcal{M}(\rho)$, the product $J\mathbb{X}$ is still an element of $\mathcal{M}(\rho)$. In other words, the linear space $\mathcal{M}(\rho)$ is a module over the polynomial algebra $\mathbb{C}[J]$. The all-important result is that this module is finitely generated by the $\mathbb{X}^{(\xi;1)}$-s. To see this, let's consider the product $J\mathbb{X}^{(\xi;m)}$. Since the map $\mathscr{P}$ is linear, the knowledge of the Laurent expansions of J and $\mathbb{X}^{(\xi;m)}$ determines the singular part of the product, and this is enough to determine $J\mathbb{X}^{(\xi;m)}$ uniquely as a combination of the canonical basis vectors. In particular, from Eq.(3.2) and the Laurent expansion

$$q^{-\Lambda}\mathbb{X}^{(\xi;m)} = q^{-m}\mathbf{e}_{\xi} + \sum_{n=0}^{\infty} \mathbb{X}^{(\xi;m)}[n]\, q^n \tag{3.3}$$

one gets

$$\mathscr{P}\left(J\mathbb{X}^{(\xi;m)}\right) = \left(q^{-(m+1)} + \sum_{n=1}^{m-1} c(n)\, q^{n-m}\right)\mathbf{e}_{\xi} + q^{-1}\mathbb{X}^{(\xi;m)}[0] \ , \tag{3.4}$$

from which one reads off the following recursion relation

$$J\mathbb{X}^{(\xi;m)} = \mathbb{X}^{(\xi;m+1)} + \sum_{n=1}^{m-1} c(n)\, \mathbb{X}^{(\xi;m-n)} + \sum_{\eta\in\mathcal{O}} \frac{1}{|\eta|}\left\langle \mathbb{X}^{(\xi;m)}[0], \mathbf{e}_{\eta}\right\rangle \mathbb{X}^{(\eta;1)} \ , \tag{3.5}$$

where $|\eta|$ denotes the length of the orbit $\eta \in \mathcal{O}$ (which is either 1 or 2, because charge conjugation is an involution). This may be rearranged as

$$\mathbb{X}^{(\xi;m+1)} = J\mathbb{X}^{(\xi;m)} - \sum_{n=1}^{m-1} c(n)\, \mathbb{X}^{(\xi;m-n)} - \sum_{\eta\in\mathcal{O}} \mathcal{X}_{\eta}^{(\xi;m)}\mathbb{X}^{(\eta;1)} \ , \tag{3.6}$$

where we have introduced the notation

$$\mathcal{X}_{\eta}^{(\xi;m)} = \frac{1}{|\eta|}\left\langle \mathbb{X}^{(\xi;m)}[0], \mathbf{e}_{\eta}\right\rangle \ . \tag{3.7}$$

Clearly, Eq.(3.6) and the knowledge of the $\mathbb{X}^{(\eta;1)}$-s for all $\eta \in \mathcal{O}$ allows to compute the canonical basis elements $\mathbb{X}^{(\xi;2)}$, and inductively all $\mathbb{X}^{(\xi;m)}$-s for $m > 1$, as linear combinations of the $\mathbb{X}^{(\eta;1)}$-s, with coefficients which are polynomials in J, proving the claim that $\mathcal{M}(\rho)$ is a finitely generated $\mathbb{C}[J]$-module. Note that the form of the recursion relations Eq.(3.6) does not depend explicitly on the representation ρ, only implicitly, through the values $\mathcal{X}_{\eta}^{(\xi;m)}$ (of course, the latter are determined by ρ).

4. Eisenstein series and the differential relations

It is a well known result of the theory of modular forms that some differential operators map modular forms to modular forms. In the present context, appropriate linear differential operators may be found that map the space $\mathcal{M}(\rho)$ to itself, providing us with differential relations between the different elements of the canonical basis.

To begin with, let's recall the definition of the (normalized) Eisenstein series and the discriminant form [1, 5, 13]. For a positive integer k let $\sigma_k(n)$ denote the sum of the k-th powers of the divisors of the integer n, i.e.

$$\sigma_k(n) = \sum_{d|n} d^k \ . \tag{4.1}$$

The Eisenstein series of weight $2k$ is

$$E_{2k}(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, q^n \ , \tag{4.2}$$

where B_k denotes the k-th Bernoulli number, and $q = \exp(2\pi i \tau)$ as usual. For $1 \leq k \leq 5$ the coefficient $-2k/B_k$ equals $-24, +240, -504, +480, -264$, respectively.

The Eisenstein series are holomorphic in the upper half-plane and at the cusps, and for $k > 1$ they are modular forms of weight $2k$, i.e. they satisfy the transformation rule

$$E_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} E_{2k}(\tau) \tag{4.3}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. There are many relations among the Eisenstein series, e.g.

$$E_8 = E_4^2 \tag{4.4a}$$

$$E_{10} = E_4 E_6 \ . \tag{4.4b}$$

The discriminant form Δ is defined by the infinite product

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \ . \tag{4.5}$$

It is a cusp form of weight 12, i.e. it is holomorphic in the upper half-plane, has a (first order) zero at the cusp $\tau = i\infty$, and satisfies

$$\Delta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12} \Delta(\tau) \tag{4.6}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. It is related to the Eisenstein series E_4 and E_6 by the formula

$$1728\Delta = E_4^3 - E_6^2 \ . \tag{4.7}$$

An important property of Δ is that it doesn't vanish on the upper half-plane $\mathbf{H}$.

Finally, let's note that the Hauptmodul J may be expressed through the above quantities as

$$J = \frac{E_4^3}{\Delta} - 744 \ . \tag{4.8}$$

Let's consider the linear differential operator

$$\nabla = \frac{E_{10}}{2\pi i \Delta} \frac{d}{d\tau} \ . \tag{4.9}$$

This operator maps the space $\mathcal{M}(\rho)$ to itself. Indeed, since the ratio E_{10}/Δ is holomorphic in $\mathbf{H}$, with a first order pole at $\tau = i\infty$, it follows that $\nabla\mathbb{X}$ is also holomorphic in $\mathbf{H}$ and has only finite order poles at $\tau = i\infty$ for $\mathbb{X} \in \mathcal{M}(\rho)$. On the other hand, differentiating both sides of Eq.(2.3), and taking into account the transformation rules Eqs.(4.3) and (4.6), one gets

$$\nabla\mathbb{X}\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} \nabla\mathbb{X}(\tau) \ ,$$

showing that $\nabla\mathbb{X} \in \mathcal{M}(\rho)$, as claimed.

This result allows to derive (first order) differential relations between the canonical basis vectors $\mathbb{X}^{(\xi;m)}$. To illustrate this point, let's apply ∇ to the Laurent expansion Eq.(3.3) of the canonical basis vector $\mathbb{X}^{(\xi;m)}$. A straightforward computation shows that

$$\mathscr{P}\left(\nabla\mathbb{X}^{(\xi;m)}\right) = (\lambda_\xi - m) \sum_{n=-1}^{m-1} \mathcal{E}_n q^{n-m} \mathbf{e}_\xi + q^{-1} \sum_{\eta\in\mathcal{O}} \lambda_\eta \mathcal{X}_\eta^{(\xi;m)} \mathbf{e}_\eta \ , \tag{4.10}$$

where $\mathcal{X}_\eta^{(\xi;m)}$ is the constant matrix introduced in Eq.(3.7), and the integers $\mathcal{E}_n$ are the Laurent coefficients of E_{10}/Δ:

$$\frac{E_{10}}{\Delta} = \sum_{n=-1}^{\infty} \mathcal{E}_n q^n = q^{-1} - 240 - 141444q - 8529280q^2 + \dots \ . \tag{4.11}$$

Since the singular part determines the whole series, it follows from Eq.(4.10) that

$$\nabla\mathbb{X}^{(\xi;m)} = (\lambda_\xi - m) \sum_{n=-1}^{m-1} \mathcal{E}_n \mathbb{X}^{(\xi;m-n)} + \sum_{\eta\in\mathcal{O}} \lambda_\eta \mathcal{X}_\eta^{(\xi;m)} \mathbb{X}^{(\eta;1)} \ . \tag{4.12}$$

In particular, for $m = 1$ Eq.(4.12) reduces to

$$\nabla\mathbb{X}^{(\xi;1)} = (\lambda_\xi - 1)\left(\mathbb{X}^{(\xi;2)} - 240\mathbb{X}^{(\xi;1)}\right) + \sum_{\eta\in\mathcal{O}} \lambda_\eta \mathcal{X}_\eta^{(\xi;1)} \mathbb{X}^{(\eta;1)} \ . \tag{4.13}$$

But $\mathbb{X}^{(\xi;2)}$ is determined by the recursion relation Eq.(3.6), as a linear combination of the $\mathbb{X}^{(\eta;1)}$-s. Substituting its expression into Eq.(4.13) leads to

$$\nabla\mathbb{X}^{(\xi;1)} = (\lambda_\xi - 1)(J - 240)\mathbb{X}^{(\xi;1)} + \sum_{\eta\in\mathcal{O}} (1 + \lambda_\eta - \lambda_\xi) \mathcal{X}_\eta^{(\xi;1)} \mathbb{X}^{(\eta;1)} \ . \tag{4.14}$$

Thus we get a system of first order linear differential equations satisfied by the canonical basis vectors $\mathbb{X}^{(\xi;1)}$, which may be rewritten as

$$\frac{1}{2\pi i}\frac{d\mathbb{X}^{(\xi;1)}}{d\tau} = \sum_{\eta} \mathcal{D}_{\eta}^{\xi}(q)\, \mathbb{X}^{(\eta;1)} \tag{4.15}$$

upon introducing the square matrix

$$\mathcal{D}_{\eta}^{\xi}(q) = \frac{\Delta}{E_{10}} \left\{ (J-240)(\lambda_{\xi}-1)\,\delta_{\eta}^{\xi} + (1+\lambda_{\eta}-\lambda_{\xi})\,\mathcal{X}_{\eta}^{(\xi;1)} \right\} \ . \tag{4.16}$$

Note that this matrix is meromorphic in the upper half-plane (it has first order poles at $\tau = \exp(2\pi i/3)$ and $\tau = i$), and holomorphic at the cusp $\tau = i\infty$, i.e. it has a Laurent expansion

$$\mathcal{D}_{\eta}^{\xi}(q) = \sum_{n=0}^{\infty} \mathcal{D}_{\eta}^{\xi}[n]\, q^{n} \tag{4.17}$$

without negative powers of q. The first few coefficients of the above expansion read

$$\begin{aligned}
\mathcal{D}_{\eta}^{\xi}[0] &= (\lambda_{\xi}-1)\,\delta_{\eta}^{\xi} \\
\mathcal{D}_{\eta}^{\xi}[1] &= (1+\lambda_{\eta}-\lambda_{\xi})\,\mathcal{X}_{\eta}^{(\xi;1)} \\
\mathcal{D}_{\eta}^{\xi}[2] &= 338328\,(\lambda_{\xi}-1)\,\delta_{\eta}^{\xi} + 240\,(1+\lambda_{\eta}-\lambda_{\xi})\,\mathcal{X}_{\eta}^{(\xi;1)} \ .
\end{aligned} \tag{4.18}$$

Differential equations obeyed by the characters of Rational Conformal Field Theories have been studied elsewhere, where they have been used for e.g. classification purposes and studying modularity (see [15, 21], resp.).

The differential equation Eq.(4.15), supplemented with the boundary conditions $\mathscr{P}\left(\mathbb{X}^{(\xi;1)}\right) = q^{-1}\mathbf{e}_{\xi}$ and $\mathbb{X}^{(\xi;1)}[0] = \sum_{\eta} \mathcal{X}_{\eta}^{(\xi;1)}\mathbf{e}_{\eta}$, allows in principle to determine recursively the coefficients $\mathbb{X}^{(\xi;1)}[n]$ for $n>0$. Indeed, substituting into Eq.(4.15) the expansions Eqs.(4.17) and (3.3), and taking into account that $\frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$, one gets, after comparing powers of q, the relation

$$(n+1-\lambda_{\xi}+\Lambda)\,\mathbb{X}^{(\xi;1)}[n] = \sum_{\eta\in\mathcal{O}} \left(\mathcal{D}_{\eta}^{\xi}[n+1]\,\mathbf{e}_{\eta} + \sum_{m=1}^{n} \mathcal{D}_{\eta}^{\xi}[m]\,\mathbb{X}^{(\eta;1)}[n-m] \right) \ . \tag{4.19}$$

For example, after some rearrangements the above formula gives for $n=1$

$$\mathbb{X}^{(\xi;1)}[1] = \sum_{\eta} \frac{\mathcal{D}_{\eta}^{\xi}[2] + \sum_{\nu}(1+\lambda_{\nu}-\lambda_{\xi})\,\mathcal{X}_{\nu}^{(\xi;1)}\mathcal{X}_{\eta}^{(\nu;1)}}{2+\lambda_{\eta}-\lambda_{\xi}}\mathbf{e}_{\eta} \ ,$$

and similar expressions may be obtained for the higher terms. This means that the knowledge of the exponents and of the matrix $\mathcal{X}_{\eta}^{(\xi;1)}$ of Eq.(3.7) allows the explicit computation of the canonical basis vectors $\mathbb{X}^{(\xi;1)}$, and hence - via the recursion relations Eq.(3.6) - of all the elements of the canonical basis.

5. Invariants and covariants

The notion of invariants and covariants (aka. equivariant polynomial maps) will play an important role in what follows, so let's sketch their definition. Consider the polynomial algebra in r variables $R = \mathbb{C}[x_1, \ldots, x_r]$: to each matrix $A \in \mathrm{GL}(r, \mathbb{C})$ is associated the algebra map

$$\begin{aligned} \hat{A} : R &\to R \\ x_i &\mapsto \sum_j A_{ij} x_j \ . \end{aligned} \tag{5.1}$$

If G is a subgroup of $\mathrm{GL}(r, \mathbb{C})$, an invariant of G is a polynomial $\mathcal{I} \in R$ left fixed by $\hat{A}$ for all $A \in G$. The invariants of G form a subring

$$Inv\,(G) = \left\{ \mathcal{I} \in R \,|\, \hat{A}\,(\mathcal{I}) = \mathcal{I} \text{ for all } A \in G \right\} \tag{5.2}$$

of the algebra R, which inherits the natural grading from R. It is known that under mild conditions (e.g. if G is linearly reductive) the algebra $Inv\,(G)$ is finitely generated [16].

A covariant of the subgroup $G < \mathrm{GL}(r, \mathbb{C})$ is an algebra map $\phi : R \to R$ such that $\hat{A} \circ \phi = \phi \circ \hat{A}$ for all $A \in G$. The set of covariants of G is graded by degree (as algebra maps) and, besides forming a linear space over $\mathbb{C}$, is a (graded) $Inv\,(G)$ module, since for a covariant ϕ and an invariant $\mathcal{I}$ the map $\mathcal{I}\phi : x_i \mapsto \mathcal{I}\phi\,(x_i)$ is again a covariant of G.

Covariants of degree 0 are related to the commutant of G: indeed, if ϕ is a covariant of degree 0, then there exists an r-by-r matrix M such that $\phi\,(x_i) = \sum_j M_{ij} x_j$ (because ϕ has degree 0), and $AM = MA$ for all $A \in G$ (because ϕ is a covariant). Conversely, for a matrix M in the commutant of G, the map $\phi : x_i \mapsto \sum_j M_{ij} x_j$ is a covariant of degree 0.

By an invariant (covariant) of a finite dimensional matrix representation ρ of $\mathrm{SL}_2(\mathbb{Z})$, we'll mean an invariant (resp. covariant) of the image of ρ. The importance of covariants stems from the fact that for $\mathbb{X} \in \mathcal{M}(\rho)$ and ϕ a covariant of ρ, one has $\phi\,(\mathbb{X}) \in \mathcal{M}(\rho)$, since $\phi\,(\mathbb{X})$ is holomorphic - being a polynomial expression in holomorphic functions -, has only finite order poles at the cusps, and transforms according to the representation ρ. Similarly, if $\mathcal{I}$ is an invariant and $\mathbb{X} \in \mathcal{M}(\rho)$, then $\mathcal{I}\,(\mathbb{X}) \in \mathbb{C}\,[J]$, because it is holomorphic, has only finite order poles at the cusps, and is invariant under all modular transformations.

6. A worked-out example: the Yang-Lee model

In the present section we'll illustrate, on the example of the Yang-Lee model, how the results of the previous sections may be used to determine explicitly the elements $\mathbb{X}^{(\xi;m)}$ of the canonical basis of $\mathcal{M}(\rho)\,/W_\rho$, and hence all solutions to Eq.(2.3) holomorphic in $\mathbf{H}$ and meromorphic at the cusps.

The Yang-Lee model is the Virasoro minimal model $M\,(5, 2)$ of central charge $c = -\frac{22}{5}$. Its exponents are

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{60} \begin{pmatrix} 11 \\ 59 \end{pmatrix} \ ,$$

while its S matrix reads

$$S = \sqrt{\frac{2}{5+\sqrt{5}}}\begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & -1 \end{pmatrix} .$$

The corresponding representation ρ is admissible (since it comes from an RCFT), has trivial charge conjugation, and trivial invariant subspace $W_\rho = 0$.

The map

$$\begin{aligned} \phi : \mathbb{C}\left[x_1, x_2\right] &\to \mathbb{C}\left[x_1, x_2\right] \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\mapsto \begin{pmatrix} p\left(x_1, x_2\right) \\ p\left(x_2, -x_1\right) \end{pmatrix} \end{aligned} \tag{6.1}$$

is a covariant of degree 48, where

$$\begin{aligned} p\left(x_1, x_2\right) = x_2^{49} &- 114464 x_1^{10} x_2^{39} - 1586424 x_1^{15} x_2^{34} - 4273878 x_1^{20} x_2^{29} \\ &+ 3491397 x_1^{25} x_2^{24} - 559580 x^{30} x_2^{19} + 952812 x_1^{35} x_2^{14} - 14063 x_1^{40} x_2^{9} + 294 x_1^{45} x_2^{4}, \end{aligned} \tag{6.2}$$

while

$$\mathcal{I} = x_1 x_2 \left(x_1^{10} + 11 x_1^5 x_2^5 - x_2^{10}\right) \tag{6.3}$$

is an invariant of degree 12.

Let's turn to the computation of the canonical basis. By definition, $\mathbb{X}^{(2;1)}$ has a Laurent expansion of the form

$$\mathbb{X}^{(2;1)}(\tau) = q^{\Lambda}\begin{pmatrix} a_0 + a_1 q + \dots \\ q^{-1} + b_0 + b_1 q + \dots \end{pmatrix} , \tag{6.4}$$

where the coefficients $a_0, b_0, \dots$ have to be determined. Inserting this expression into Eq.(6.3), we have

$$\mathcal{I}\left(\mathbb{X}^{(2;1)}\right) = -a_0 + \left(11 a_0^6 - 11 b_0 a_0 - a_1\right) q + \dots \tag{6.5}$$

This means that $\mathcal{I}\left(\mathbb{X}^{(2;1)}\right)$ is holomorphic at $q = 0$, hence it should be a constant, i.e. the coefficients of the positive powers of q in the expansion Eq.(6.5) should vanish[2]. In particular, one has

$$a_1 = 11 a_0 \left(a_0^5 - b_0\right) , \tag{6.6}$$

$$a_2 = 727 a_0^{11} - 781 b_0 a_0^6 + (66 b_0^2 - 11 b_1) a_0 . \tag{6.7}$$

On the other hand, because ϕ is a covariant, we know that $\phi\left(\mathbb{X}^{(2;1)}\right)$ belongs to $\mathcal{M}(\rho)$. Inserting Eq.(6.4) into ϕ, one gets

$$\phi\left(\mathbb{X}^{(2;1)}\right) = q^{\Lambda}\begin{pmatrix} q^{-1} + 49 b_0 + (1176 b_0^2 + 49 b_1 - 114464 a_0^{10}) q + \dots \\ 294 a_0^4 q^{-1} + 26999 a_0^9 + 294 b_0 a_0^4 + \dots \end{pmatrix} , \tag{6.8}$$

[2]Note that $a_0 = 0$ is not possible, since Eq.(6.3) would then imply that the first component of $\mathbb{X}^{(2;1)}$ vanishes identically, which is incompatible with Eq.(2.3).

from which one reads off

$$\phi\left(\mathbb{X}^{(2;1)}\right) = \mathbb{X}^{(1;1)} + 294a_0^4\mathbb{X}^{(2;1)} \tag{6.9}$$

by comparing the singular parts. This leads at once to the following q-expansion, taking into account Eq.(6.6)

$$\mathbb{X}^{(1;1)}(\tau) = q^{\Lambda}\begin{pmatrix} q^{-1} + 49b_0 - 294a_0^5 + \left(1176b_0^2 + 49b_1 - 117698a_0^{10} + 3234b_0a_0^5\right)q + \dots \\ 26999a_0^9 + \left(3413445a_0^{14} - 1592941b_0a_0^9\right)q + \dots \end{pmatrix} . \tag{6.10}$$

Note that from the above we can read off the entries of the matrix of Eq.(3.7):

$$\mathcal{X}_\eta^{(\xi;1)} = \begin{pmatrix} 49(b_0 - 6a_0^5) & 26999a_0^9 \\ a_0 & b_0 \end{pmatrix} . \tag{6.11}$$

This shows that, should we know the values of a_0 and b_0, we could compute the q-expansion of the $\mathbb{X}^{(\xi;1)}$-s recursively via Eq.(4.19). To determine these parameters, let's plug the expression Eq.(6.4) into Eq.(4.14), and equate the coefficients of the constant terms on each side (the coefficients of the negative q powers are equal by construction), which gives

$$72a_0^6 - (24b_0 + 48)a_0 = 0 , \tag{6.12a}$$

$$-\frac{26999}{5}a_0^{10} - b_0^2 - 240b_0 + 2b_1 + \frac{28194}{5} = 0 , \tag{6.12b}$$

which can be solved to give (recall that $a_0 \neq 0$)

$$b_0 = 3a_0^5 - 2 , \tag{6.13a}$$

$$b_1 = \frac{1}{5}\left(13522a_0^{10} + 3540a_0^5 - 15287\right) . \tag{6.13b}$$

Equating the coefficients of the terms linear in q, and taking into account Eqs.(6.13a) and (6.13b), one gets

$$-\frac{373248}{5}a_0^{11} + \frac{373248}{5}a_0 = 0 , \tag{6.14}$$

$$\frac{1324812}{5}a_0^{15} - \frac{9654708}{5}a_0^{10} - \frac{2837277}{5}a_0^5 + 3b_2 + \frac{11167158}{5} = 0 , \tag{6.15}$$

from which follows that $a_0^{10} = 1$, i.e. a_0 is a tenth root of unity. The other expansion coefficients may be expressed in terms of a_0:

$$\begin{aligned} b_0 &= 3a_0^5 - 2 \\ a_1 &= 22a_0(1 - a_0^5) \\ b_1 &= 354a_0^5 - 353 \\ a_2 &= a_0(3125 - 3124a_0^5) \\ b_2 &= 100831a_0^5 - 100830 \end{aligned} \tag{6.16}$$

and so on.

All in all, we got 10 different possibilities for $\mathbb{X}^{(2;1)}$, according to the precise value of a_0. Only one of these does solve our original problem, i.e. only one of them transforms according to the representation ρ: it can be selected by e.g. determining the corresponding solution of the differential equation Eq.(4.15), and checking its transformation law under $\tau \mapsto -\frac{1}{\tau}$. But in our case there is a shortcut: the character vector of the Yang-Lee model has a first order pole in its second component, i.e. it equals $\mathbb{X}^{(2;1)}$, and being a character vector, its q-expansion coefficients are all non-negative integers (being eigenvalue multiplicities). In particular, a_0 should be a non-negative integer: the only 10-th root of unity that satisfies this is $a_0 = 1$. Indeed, with this value of a_0 we get the following q-expansion

$$\mathbb{X}^{(2;1)}(\tau) = q^{\Lambda} \begin{pmatrix} 1 + q^2 + \dots \\ q^{-1} + 1 + q + q^2 + \dots \end{pmatrix} , \tag{6.17}$$

recovering the well-known result for the character vector of the Yang-Lee model (see e.g. [19]). From this and Eq.(6.10) one gets

$$\mathbb{X}^{(1;1)}(\tau) = q^{\Lambda} \begin{pmatrix} q^{-1} - 245 - 113239q - 6029989q^2 + \dots \\ 26999 + 1820504q + \dots \end{pmatrix} . \tag{6.18}$$

Note that this cannot be the character vector of a RCFT, since some of its expansion coefficients are negative (though still integers). This shows that the modular representation in an RCFT does constrain the singular part $\mathscr{P}\mathbb{X}$ of character vectors (e.g. the conformal weights h_μ) in a nontrivial way. These constraints go far beyond the inequality $\sum_\mu (h_\mu - c/24) \leq r(r-1)/12$ of [15], which is satisfied by Eq.(6.18) and indeed by any $\mathbb{X} \in \mathcal{M}(\rho)$.

This arbitrariness up to a 10-th root of unity is not surprising in hindsight, and certainly does not contradict our earlier claim that $\ker \mathscr{P} = W_\rho$ (which vanishes here). The only ingredients which went into Eqs.(6.16) and the constraint $a_0^{10} = 1$ were the exponents λ_μ, the covariant ϕ in Eq.(6.1), and invariant $\mathcal{I}$ in Eq.(6.3). Of course these are all determined by the modular representation ρ, but it is easy to find other modular representations having the same $\lambda_\mu, \phi, \mathcal{I}$ which correspond to the 9 other values of a_0.

7. A second example: the Ising model

The Ising model is the Virasoro minimal model $M(4,3)$ of central charge $c = \frac{1}{2}$. It has 3 primary fields, and the modular representation ρ associated to it is characterized by

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

and the exponents

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \frac{1}{48} \begin{pmatrix} 47 \\ 23 \\ 2 \end{pmatrix} .$$

Note that ρ - besides being admissible - is irreducible, has trivial charge conjugation, and $W_\rho = 0$.

The map

$$\begin{aligned} \phi_n : \mathbb{C}\left[x_1, x_2, x_3\right] &\to \mathbb{C}\left[x_1, x_2, x_3\right] \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\mapsto \frac{1}{2}\begin{pmatrix} \left(x_1+x_2\right)^{24n+1} + (-1)^n\left(x_1-x_2\right)^{24n+1} \\ \left(x_1+x_2\right)^{24n+1} - (-1)^n\left(x_1-x_2\right)^{24n+1} \\ (-1)^n\, 2^{12n+1} x_3^{24n+1} \end{pmatrix} \end{aligned} \tag{7.1}$$

is a covariant (of degree $24n$) for any non-negative integer n.

The above information is already enough to determine the canonical basis of $\mathcal{M}(\rho)$ along the lines presented in the previous section. Instead of going through this lengthy calculation, which doesn't present any difficulties, we'll exploit the fact that the character vector of the Ising model is known (see e.g. [19]):

$$\mathbb{X}_{\text{Ising}} = \frac{1}{2}\begin{pmatrix} \mathfrak{f} + \mathfrak{f}_1 \\ \mathfrak{f} - \mathfrak{f}_1 \\ \sqrt{2}\mathfrak{f}_2 \end{pmatrix} \ , \tag{7.2}$$

where

$$\begin{aligned} \mathfrak{f}(\tau) &= q^{-1/48}\prod_{n=0}^{\infty}\left(1+q^{n+\frac{1}{2}}\right) \ , \\ \mathfrak{f}_1(\tau) &= q^{-1/48}\prod_{n=0}^{\infty}\left(1-q^{n+\frac{1}{2}}\right) \ , \\ \mathfrak{f}_2(\tau) &= \sqrt{2}q^{1/24}\prod_{n=1}^{\infty}\left(1+q^{n}\right) \end{aligned} \tag{7.3}$$

are the Weber functions. Note that, while linearly independent, the Weber functions are not algebraically independent, for they satisfy the identities

$$\mathfrak{f}_1^8 + \mathfrak{f}_2^8 = \mathfrak{f}^8 \ , \tag{7.4a}$$

$$\mathfrak{f}\mathfrak{f}_1\mathfrak{f}_2 = \sqrt{2} \ . \tag{7.4b}$$

Moreover, they are related to the Hauptmodul through

$$J + 744 = \frac{\left(\mathfrak{f}^{24}-16\right)^3}{\mathfrak{f}^{24}} = \frac{\left(\mathfrak{f}_1^{24}+16\right)^3}{\mathfrak{f}_1^{24}} = \frac{\left(\mathfrak{f}_2^{24}+16\right)^3}{\mathfrak{f}_2^{24}} \ . \tag{7.5}$$

$\mathbb{X}_{\text{Ising}}$ has a pole of order 1 in its first component, which means that $\mathbb{X}_{\text{Ising}} = \mathbb{X}^{(1;1)}$, i.e.

$$\mathbb{X}^{(1;1)} = \frac{1}{2}\begin{pmatrix} \mathfrak{f} + \mathfrak{f}_1 \\ \mathfrak{f} - \mathfrak{f}_1 \\ \sqrt{2}\mathfrak{f}_2 \end{pmatrix} \ . \tag{7.6}$$

Applying ϕ_1 and ϕ_2 to $\mathbb{X}_{\text{Ising}}$, and comparing the singular parts, one gets the relations

$$\phi_1(\mathbb{X}_{\text{Ising}}) = 25\mathbb{X}^{(1;1)} + \mathbb{X}^{(2;1)} , \tag{7.7}$$

$$\phi_2(\mathbb{X}_{\text{Ising}}) = \mathbb{X}^{(1;2)} + 1176\mathbb{X}^{(1;1)} + 49\mathbb{X}^{(2;1)} , \tag{7.8}$$

from which one deduces

$$\mathbb{X}^{(2;1)} = \frac{1}{2}\begin{pmatrix} \mathfrak{f}^{25} - \mathfrak{f}_1^{25} - 25\mathfrak{f} - 25\mathfrak{f}_1 \\ \mathfrak{f}^{25} + \mathfrak{f}_1^{25} - 25\mathfrak{f} + 25\mathfrak{f}_1 \\ -\sqrt{2}\mathfrak{f}_2\left(25 + \mathfrak{f}_2^{24}\right) \end{pmatrix} \tag{7.9}$$

and

$$\mathbb{X}^{(1;2)} = \frac{1}{2}\begin{pmatrix} \mathfrak{f}^{49} - 49\mathfrak{f}^{25} + 49\mathfrak{f} + \mathfrak{f}_1^{49} + 49\mathfrak{f}_1^{25} + 49\mathfrak{f}_1 \\ \mathfrak{f}^{49} - 49\mathfrak{f}^{25} + 49\mathfrak{f} - \mathfrak{f}_1^{49} - 49\mathfrak{f}_1^{25} - 49\mathfrak{f}_1 \\ \sqrt{2}\mathfrak{f}_2\left(49 + 49\mathfrak{f}_2^{24} + \mathfrak{f}_2^{48}\right) \end{pmatrix} . \tag{7.10}$$

Finally, from the recursion relation

$$J\mathbb{X}^{(1;1)} = \mathbb{X}^{(1;2)} + \mathbb{X}^{(2;1)} + \mathbb{X}^{(3;1)} , \tag{7.11}$$

taking into account the relations Eqs.(7.4a), (7.4b) and (7.5), one computes

$$\mathbb{X}^{(3;1)} = \begin{pmatrix} 8\mathfrak{f}^{17}\mathfrak{f}_1^{8} - 8\mathfrak{f}^{24}\mathfrak{f}_1 - 128\mathfrak{f} + \frac{\mathfrak{f}_2^7}{\sqrt{2}}\left(\mathfrak{f}^{39} - \mathfrak{f}_1^{39} - 16\mathfrak{f}^{15} - 32\mathfrak{f}_1^{15}\right) \\ 8\mathfrak{f}^{17}\mathfrak{f}_1^{8} + 8\mathfrak{f}^{24}\mathfrak{f}_1 - 128\mathfrak{f} - \frac{\mathfrak{f}_2^7}{\sqrt{2}}\left(\mathfrak{f}^{39} + \mathfrak{f}_1^{39} - 16\mathfrak{f}^{15} + 32\mathfrak{f}_1^{15}\right) \\ \mathfrak{f}^{15}\mathfrak{f}_1^{7}\left(\mathfrak{f}^{24} - 16\right) - 8\sqrt{2}\mathfrak{f}_2\mathfrak{f}^{24} \end{pmatrix} . \tag{7.12}$$

Thus, we have been able to determine explicitly the elements $\mathbb{X}^{(\xi;1)}$ of the canonical basis. Note that from these one may derive explicit expressions for the $\mathbb{X}^{(\xi;m)}$ with $m > 1$ by using the recursion relation Eq.(3.6).

Finally, from the above explicit expressions one gets the q-expansions

$$\mathbb{X}^{(1;1)}(\tau) = q^{\Lambda}\begin{pmatrix} q^{-1} + q + q^2 + 2q^3 + \ldots \\ 1 + q + q^2 + q^3 + \ldots \\ 1 + q + q^2 + 2q^3 + \ldots \end{pmatrix} , \tag{7.13}$$

$$\mathbb{X}^{(2;1)}(\tau) = q^{\Lambda}\begin{pmatrix} 2325 + 60630q + 811950q^2 + 7502125q^3 + \ldots \\ q^{-1} + 275 + 13250q + 235500q^2 + 2558550q^3 + \ldots \\ -25 - 4121q - 102425q^2 - 1331250q^3 + \ldots \end{pmatrix} , \tag{7.14}$$

and

$$\mathbb{X}^{(3;1)}(\tau) = q^{\Lambda} \begin{pmatrix} 94208 + 9515008q + 356765696q^2 + 7853461504q^3 + \dots \\ -4096 - 1130496q - 63401984q^2 - 1763102720q^3 + \dots \\ q^{-1} - 23 + 253q - 1794q^2 + 9384q^3 + \dots \end{pmatrix} . \tag{7.15}$$

Of these, only $\mathbb{X}^{(1;1)}$ may be the character vector of a RCFT, since it is the only one whose q-expansion coefficients are all non-negative integers, illustrating again that the singular parts of character vectors are heavily constrained.

8. Further questions and developments

This paper explains to what extent the $\mathrm{SL}_2(\mathbb{Z})$ representation ρ determines the vector-valued complex function $\mathbb{X}$, and how in practise to construct it. This study suggests a number of additional questions.

We illustrate with examples how to construct the canonical basis vectors $\mathbb{X}^{(\eta;1)}$ coefficient by coefficient, and in principle the differential equation Eq.(4.15) tells us the full series. But is it possible to express these $\mathbb{X}^{(\eta;1)}$ using known transcendental functions, much as we did with the Ising model (and more generally has been done with all the minimal and Wess-Zumino-Witten models)? A reason to think we can is that each component of $\mathbb{X}^{(\eta;1)}$ will be a modular function for some $\Gamma(N)$, and all of these can be expressed in terms of the J function and N^2 Fricke functions $f_{r,s}$ (see e.g. [14]). An alternate approach was followed in [6], who in five specific Conformal Field Theoretic models explained how to write the characters using theta functions, by relating ρ to Weil representations; their method should be quite general.

We are most interested in the modular representation ρ and character vector $\mathbb{X}$ coming from Rational Conformal Field Theory. In this case the components $\chi_\mu(\tau)$ of $\mathbb{X}$ have a q-expansion whose coefficients are non-negative integers. This is quite special; what are the properties of ρ which makes this possible? Given such a ρ, which vectors in V_+ will be the principal parts of such non-negative integer $\mathbb{X}$? Integrality is easy to understand, using Galois methods. In particular, for any admissible ρ, some positive integer multiple of each canonical basis vector $\mathbb{X}^{(\eta;m)}$ (translating by a vector in W_ρ if necessary) will have integer q-expansions, provided all entries of the matrices $\rho(A)$, for all $A \in \mathrm{SL}_2(\mathbb{Z})$, lie in the cyclotomic field $\mathbb{Q}[e^{2\pi i/N}]$, and in addition the ℓ-th Galois automorphism, for all ℓ coprime to N, applied entry by entry to the matrix S, equals $\rho\begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix} S$ (see [3] for more details and the proof). These conditions are automatically satisfied in any Rational Conformal Field Theory [2]. Positivity of those q-expansions seems more difficult to understand, although it is easy to verify that, unless S has a strictly positive eigenvector with eigenvalue 1, no $\mathbb{X} \in \mathcal{M}(\rho)$ can have a non-negative q-expansion. As the examples in sections 6 and 7 illustrate, non-negativity is subtle and would be very interesting to understand.

Curiously, in all examples we've seen, the q-expansions of the canonical basis vectors $\mathbb{X}^{(\eta;1)}$ have been either completely non-negative, completely nonpositive, or alternating in sign (apart from the q^{-1} and constant terms, in some cases). Is this a general phenomenon?

The analysis of the covariants and invariants for holomorphic orbifolds, or equivalently the modular representations coming from quantum-doubles of finite groups, is straightforward, and thus they supply a large family of examples which can be worked out quite explicitly [3].

Some modular representations ρ (involving certain powers of 2) are more exceptional than others [17]. Are these in any way special from the point of view considered here? Can those exceptional modular representations be realized in a Rational Conformal Field Theory?

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