The algebraic meaning of genus-zero[1]

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Abstract

The Conway–Norton conjectures unexpectedly related the Monster with certain special modular functions (Hauptmoduls). Their proof by Borcherds *et al* was remarkable for demonstrating the rich mathematics implicit there. Unfortunately Moonshine remained almost as mysterious after the proof as before. In particular, a computer check — as opposed to a general conceptual argument — was used to verify the Monster functions equal the appropriate modular functions. This, the so-called 'conceptual gap', was eventually filled; we review the solution here. We conclude by speculating on the shape of a new proof of the Moonshine conjectures.

1   The conceptual gap

The main Conway–Norton conjecture [5] says:

Theorem 1. *There is an infinite-dimensional graded representation $V = \oplus_{n=-1}^{\infty} V_n$ of the Monster $\mathbb{M}$, such that the McKay–Thompson series*

$$T_g(\tau) := \mathrm{Tr}_V g\, q^{L_0-1} = \sum_{n \geq -1} c_n(g)\, q^n \qquad (1.1)$$

equals the Hauptmodul J_g for some discrete subgroup Γ_g of $SL_2(\mathbb{R})$.

Moreover, each coefficient $c_n(g)$ lies in $\mathbb{Z}$, and Γ_g contains the congruence subgroup $\Gamma_0(N)$ as a normal subgroup, where $N = h\, o(g)$ for some h dividing $\gcd(24, o(g))$ ($o(g)$ is the order of $g \in \mathbb{M}$). In his ICM talk [2], Borcherds outlined the proof of Theorem 1:

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[1] A contribution to the Moonshine Conference at ICMS, Edinburgh, July 2004.

(i) Construction of the Frenkel–Lepowsky–Meurman Moonshine module $V^{\natural}$, which is to equal the space V;

(ii) Derivation of recursions for the McKay–Thompson coefficients $c_n(g)$, such as

$$c_{4n+2}(g) = c_{2k+2}(g^2) + \sum_{j=1}^{k} c_j(g^2)\, c_{2k+1-j}(g^2) \qquad \forall k \geq 1 ; \qquad (1.2)$$

(iii) From these recursions, prove $T_g = J_g$.

The original treatment of step (i) is [11], and is reviewed elsewhere in these proceedings. Borcherds derived (ii) by first constructing a Lie algebra out of $V^{\natural}$, and then computing its twisted denominator identities [1]. It was already known that Hauptmoduls automatically satisfied these recursions, and that any function obeying all those recursions was uniquely determined by its first few coefficients. Thus establishing (iii) merely requires comparing finitely many coefficients of each T_g with J_g — in fact, comparing 5 coefficients for each of the 171 functions suffices [1]. In this way Borcherds accomplished (iii) and with it completed the proof of Theorem 1. The proof successfully established the mathematical richness of the subject, and for his work Borcherds deservedly received a Fields medal (math's highest honour) in 1998.

This quick sketch hides the technical sophistication of the proof of (i) and (ii). More recently, the construction of (i) has been simplified in [22], and the derivations of the recursions have been simplified in [17, 18]. But the biggest weakness of the proof is hidden in the nearly trivial argument of step (iii).

At the risk of sending shivers down Bourbaki's collective spine, the point of mathematics is surely not acquiring proofs (just as the point of theoretical physics is not careful calculations, and that of painting is not the creation of realistic scenes on canvas). The point of mathematics, like that of any intellectual discipline, is to find qualitative truths, to abstract out patterns from the inundation of seemingly disconnected facts. An example is the algebraic notion of group. Another, dear to many of us, is the A-D-E meta-pattern: many different classifications (e.g. finite subgroups of SU_2, subfactors of small index, the simplest conformal field theories) fall unexpectedly into the same pattern. The conceptual explanation for the ubiquity of this meta-pattern — that is, the combinatorial fact underlying its various manifestations — presumably involves the graphs with largest eigenvalue $|\lambda| \leq 2$.

Likewise, the real challenge of Monstrous Moonshine wasn't to prove Theorem 1, but rather to understand what the Monster has to do with modularity and genus-0. The first proof was due to Atkin, Fong and Smith [25], who by studying the first 100 coefficients of the T_g verified (without constructing it)

that there existed a (possibly virtual) representation V of $\mathbb{M}$ obeying Theorem 1. Their proof is forgotten because it didn't explain anything.

By contrast, the proof of Theorem 1 by Borcherds *et al* is clearly superior: it explicitly constructs $V = V^\natural$, and emphasises the remarkable mathematical richness saturating the problem. On the other hand, it also fails to explain modularity and the Hauptmodul property. The problem is step (iii): precisely at the point where we want to identify the algebraically defined T_g's with the topologically defined J_g's, a conceptually empty computer check of a few hundred coefficients is done. This is called the conceptual gap of Monstrous Moonshine, and it has an analogue in Borcherds' proof of Modular Moonshine [3] and in Höhn's proof of 'generalised Moonshine' for the Baby Monster [15]. Clearly preferable would be to replace the numerical check of [1] with a more general theorem.

Next section we review the standard definition of Hauptmodul. In Section 3 we describe the solution [9] to the conceptual gap: it replaces that topological definition of Hauptmodul with an algebraic one. We conclude the paper with some speculations. Even with the improvements [17, 18, 22] and especially [9] to the original proof of Theorem 1, the resulting argument still does a poor job explaining Monstrous Moonshine. Moonshine remains mysterious to this day. There is a lot left to do — for example establishing Norton's generalised Moonshine [24], or finding the Moonshine manifold [14]. But the greatest task for Moonshiners is to find a second independent proof of Theorem 1. It would (hopefully) clarify some things that the original proof leaves murky. In particular, we still don't know what really is so important about the Monster, that it has such a rich genus-0 moonshine. To what extent does Monstrous Moonshine determine the Monster? We turn to this open problem in Section 4.

2 The topological meaning of the Hauptmodul property

Just as a *periodic* function is a function on a compact *real* curve (i.e. on a circle), a *modular* function is a function on a compact *complex* curve. More precisely, let Σ be a compact surface. We can regard this as a complex curve, and thus put on it a complex analytic structure. Up to (biholomorphic or conformal) equivalence, there is a unique genus 0 surface (which we can take to be the Riemann sphere $\mathbb{C}\cup\{\infty\}$), but there is a continuum (moduli space) of inequivalent complex analytic structures which can be placed on a torus (genus 1), a double-torus, etc. For example, this moduli space for the torus can be naturally identified with $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$, where $\mathbb{H}$ is the upper half-plane $\{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$: any torus is equivalent to one of the form $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$

for some $\tau \in \mathbb{H}$; the tori corresponding to τ and τ' are themselves equivalent iff $\tau' = (a\tau + b)/(c\tau + d)$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

We learn from both geometry and physics that we should study a space through the functions (fields) that live on it, which respect the relevant properties of the space. Therefore we should consider meromorphic functions $f : \Sigma \to \mathbb{C}$ (a meromorphic function is holomorphic everywhere, except for isolated finite poles; we would have preferred f to be holomorphic, but then f would be constant). These f are called *modular functions*, and as they are as important as complex curves, they should be central to mathematics.

We know all about meromorphic functions for $\mathbb{C}$: these include rational functions (i.e. quotients of polynomials), together with transcendental functions such as $\exp(z)$ and $\cos(z)$. But both $\exp(z)$ and $\cos(z)$ have an essential singularity at ∞. In fact, the modular functions for the Riemann sphere are the rational functions in z. By contrast, the modular functions for the other compact surfaces will be rational functions in two generators, where those two generators satisfy a polynomial relation. For example, the modular functions for a torus are generated by the Weierstrass $\mathfrak{p}$-function and its derivative, and $\mathfrak{p}$ and $\mathfrak{p}'$ satisfy the cubic equation defining the torus. In this way, the sphere is distinguished from all other compact surfaces.

There are three possible geometries in two dimensions: Euclidean, spherical and hyperbolic. The most important of these, in any sense, is the hyperbolic one. The upper half-plane is a model for it: its 'lines' consist of vertical half-lines and semi-circles, its infinitesimal metric is $ds = |d\tau|/\mathrm{Im}(z)$, etc. As with Euclidean geometry, 'lines' are the paths of shortest distance. Just as the Euclidean plane $\mathbb{R}^2$ has a circular horizon (one infinite point for every angle θ), so does the hyperbolic plane $\mathbb{H}$, and it can be identified with $\mathbb{R} \cup \{i\infty\}$. The group of isometries (geometry-preserving transformations $\mathbb{H} \to \mathbb{H}$) is $\mathrm{SL}_2(\mathbb{R})$, which acts on $\mathbb{H}$ as fractional linear transformations $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, and sends the horizon to itself.

It turns out that any compact surface Σ can be realised (in infinitely many different ways) as the compactification of the space $\mathbb{H}/\Gamma$ of orbits, for some discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$. The compactification amounts to including finitely many Γ-orbits of horizon points. The most important example is $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$, which can be identified with the sphere with one puncture, the puncture corresponding to the single compactification orbit $\mathbb{Q} \cup \{i\infty\}$. Those points $\mathbb{Q} \cup \{i\infty\}$ are called cusps. The groups Γ of greatest interest in number theory, and to us, are those commensurable to $\mathrm{SL}_2(\mathbb{Z})$: i.e. $\Gamma \cap \mathrm{SL}_2(\mathbb{Z})$ is also an infinite discrete group with finite index in both Γ and $\mathrm{SL}_2(\mathbb{Z})$. Their compactification points will again be the cusps $\mathbb{Q} \cup \{i\infty\}$. Examples of such groups are $\mathrm{SL}_2(\mathbb{Z})$ itself, as well as its subgroups

$$\Gamma(N) = \{A \in \mathrm{SL}_2(\mathbb{Z}) \,|\, A \equiv \pm I \pmod{N}\}, \tag{2.1}$$

$$\Gamma_0(N) \;=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid N \text{ divides } c \right\}, \qquad (2.2)$$

$$\Gamma_1(N) \;=\; \left\langle \Gamma(N), \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle. \qquad (2.3)$$

For example, $\mathbb{H}/\Gamma_0(2)$ and $\mathbb{H}/\Gamma(2)$ are spheres with 2 and 3 punctures, respectively, while e.g. $\mathbb{H}/\Gamma_0(24)$ is a torus with 7 punctures.

The modular functions for the compact surface $\Sigma = \overline{\mathbb{H}/\Gamma}$ are easy to describe: they are the meromorphic functions f on $\mathbb{H}$, which are also meromorphic at the cusps $\mathbb{Q} \cup \{i\infty\}$, and which obey the symmetry $f(\frac{a\tau+b}{c\tau+d}) = f(\tau)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. The precise definition of 'meromorphic at the cusps' isn't important here. For example, for any group Γ obeying

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \Gamma \text{ iff } t \in \mathbb{Z}, \qquad (2.4)$$

a meromorphic function $f(\tau)$ with symmetry Γ will have a Fourier expansion $\sum_{n=-\infty}^{\infty} a_n q^n$ for $q = e^{2\pi i \tau}$ (q is a local coordinate for $\tau = i\infty$); then we say f is meromorphic at the cusp $i\infty$ iff all but finitely many a_n, for $n < 0$, are nonzero.

We say a group Γ is genus-0 when $\Sigma = \overline{\mathbb{H}/\Gamma}$ is a sphere. For these Γ, there will be a uniformising function $f_\Gamma(\tau)$ identifying Σ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. That is, f_Γ will be the mother-of-all modular functions; i.e., it is a modular function for Γ, and any other modular function $f(\tau)$ for Γ can be written uniquely as a rational function $poly(f_\Gamma(\tau))/poly(f_\Gamma(\tau))$. This function f_Γ is not quite unique ($\mathrm{SL}_2(\mathbb{R})$ permutes these generating functions). By contrast, in genus > 0 two (non-canonical) generating functions will be needed.

The groups Γ we are interested in are genus-0, contain some $\Gamma_0(N)$, and obey (2.4). We call such Γ *genus-0 groups of moonshine-type*. Cummins [8] has classified all of these — there are precisely 6486 of them. For these groups (and more generally a group containing a $\Gamma_1(N)$ rather than a $\Gamma_0(N)$) there is a canonical choice of generator f_Γ: we can always choose it uniquely so that it has a q-expansion of the form $q^{-1} + \sum_{n=1}^{\infty} a_n q^n$. This choice of generator is called the *Hauptmodul*, and will be denoted $J_\Gamma(\tau)$. Some examples are

$$\begin{aligned}
J_{\Gamma(1)}(\tau) &= q^{-1} + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + \cdots & (2.5) \\
J_{\Gamma_0(2)}(\tau) &= q^{-1} + 276q - 2048q^2 + 11202q^3 - 49152q^4 + 184024q^5 + \cdots & (2.6) \\
J_{\Gamma_0(13)}(\tau) &= q^{-1} - q + 2\,q^2 + q^3 + 2\,q^4 - 2\,q^5 - 2\,q^7 - 2\,q^8 + q^9 + \cdots & (2.7) \\
J_{\Gamma_0(25)}(\tau) &= q^{-1} - q + q^4 + q^6 - q^{11} - q^{14} + q^{21} + q^{24} - q^{26} + \cdots. & (2.8)
\end{aligned}$$

Of course $J_{\Gamma(1)}$ is the famous J-function. Exactly 616 of these Hauptmoduls

have integer coefficients (171 of which are the McKay–Thompson series); the remainder have cyclotomic integer coefficients.

3   The algebraic meaning of the Hauptmodul property

The conceptual gap will be bridged only when we can directly relate the definition of a Hauptmodul (which is inherently topological), with the recursions, like (1.2), coming from the twisted denominator identities.

The easiest way to produce functions invariant with respect to some symmetry, is to average over the group. For example, given any function $f(x)$, the average $f(x) + f(-x)$ is invariant under $x \leftrightarrow -x$. When the group is infinite, a little more subtlety is required but the same idea can work.

For example, take $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and let p be any prime. Then

$$\Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \{ A \in M_{2 \times 2}(\mathbb{Z}) \,|\, \det(A) = p \}$$

$$= \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cup \bigcup_{k=0}^{p-1} \Gamma \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} . \tag{3.1}$$

This means that, for any modular function $f(\tau)$ of $\mathrm{SL}_2(\mathbb{Z})$, $f(p\tau)$ will no longer be $\mathrm{SL}_2(\mathbb{Z})$-invariant, but

$$s_f^{(p)}(\tau) := f(p\tau) + \sum_{k=0}^{p-1} f\left(\frac{\tau + k}{p} \right) \tag{3.2}$$

is. Considering now f to be the Hauptmodul J, we thus obtain that $s_J^{(p)}(\tau) = P(J(\tau))/Q(J(\tau))$, for polynomials P, Q. By considering poles and the surjectivity of J, we see that Q must be constant, and hence that $s_J^{(p)}$ must be a polynomial in J. The same will hold for any $s_{J^\ell}^{(p)}$.

This implies that there is a monic polynomial $F_p(x, y)$ of degree $p + 1$ in x, y, such that

$$F_p(J(\tau), J(p\tau)) = F_p\left(J(\tau), J\left(\frac{\tau + k}{p} \right) \right) = 0 \tag{3.3}$$

for all $k = 0, \ldots, p - 1$, or equivalently

$$F_p(J(\tau), Y) = (J(p\tau) - Y) \prod_{k=0}^{p-1} \left(J\left(\frac{\tau + k}{p} \right) - Y \right) . \tag{3.4}$$

For example,

$$F_2(x,y) \;=\; (x^2 - y)(y^2 - x) - 393768\,(x^2 + y^2) - 42987520\,xy$$
$$-40491318744\,(x + y) + 12098170833256\;. \qquad (3.5)$$

There is nothing terribly special about p being prime; for a composite number m, the sum in e.g. (3.2) becomes a sum over $\begin{pmatrix} m/d & k \\ 0 & d \end{pmatrix}$, for all divisors d of m and all $0 \le k < d$. Write $\mathcal{A}_m$ for the set of all these pairs (d,k). Note that its cardinality $\|\mathcal{A}_m\|$ is $\psi(m) = m\prod_{p|m}(1 + 1/p)$.

Definition 1. *Let* $h(\tau) = q^{-1} + \sum_{n=1}^{\infty} a_n q^n$. *We say that* $h(\tau)$ *satisfies a modular equation of order* $m > 1$, *if there is a monic polynomial* $F_m(x,y) \in \mathbb{C}[x,y]$ *such that* F_m *is of degree* $\psi(m)$ *in both* x *and* y, *and*

$$F_m(h(\tau), Y) = \prod_{(d,k)\in\mathcal{A}_m} \left(h\left(\frac{m\tau}{d^2} + \frac{k}{d} \right) - Y \right)\;. \qquad (3.6)$$

In the following, it is unnecessary to assume that the series h converges; it is enough to require that (3.6) holds formally at the level of q-series. An easy consequence of this definition is that $F_m(x,y) = F_m(y,x)$.

We've learnt above that the Hauptmodul J satisfies a modular equation of all orders $m > 1$. In fact similar reasoning verifies, more generally, that:

Proposition 1. (a) [10] *If* $J_\Gamma(\tau)$ *is the Hauptmodul of some genus-0 group* Γ *of moonshine-type, with rational coefficients, then* J_Γ *satisfies a modular equation for all* m *coprime to* N.

(b) *Likewise, any McKay–Thompson series* $T_g(\tau)$ *satisfies a modular equation for any* m *coprime to the order of the element* $g \in \mathbb{M}$.

Recall there are 616 such J_Γ, and 171 such T_g. The N in part (a) is the level of any congruence group $\Gamma_0(N)$ contained in Γ. Part (b) involves showing that the recursions such as (1.2) imply the modular equation property q-coefficient-wise.

Note that the Hauptmodul property of J_Γ plays a crucial role in the proof that they satisfy modular equations. Could the converse of the Proposition hold?

Unfortunately, that is too naive. In particular, $h(\tau) = q^{-1}$ also satisfies a modular equation for any $m > 1$: take $F_m(x,y) = (x^m - y)(y^m - x)$. Using Tchebychev polynomials, it is easy to show that $h(\tau) = q^{-1} + q$ (which is essentially cosine) likewise satisfies a modular equation for any m.

However, Kozlov, in a thesis directed by Meurman, proved the following remarkable fact:

Theorem 2. [19] *If* $h(\tau) = q^{-1} + \sum_{n=1}^{\infty} a_n q^n$ *satisfies a modular equation for all* $m > 1$, *then either* $h = J$, $h(\tau) = q^{-1}$, *or* $h(\tau) = q^{-1} \pm q$.

His proof breaks down when we no longer have all those modular equations, but it gives us confidence to hope that modular equations could provide an algebraic interpretation of what it means to be a Hauptmodul. Indeed that is the case!

Theorem 3. [9] *Suppose a formal series $h(\tau) = q^{-1} + \sum_{n=1}^{\infty} a_n q^n$ satisfies a modular equation for all $m \equiv 1$ (mod K). Then $h(\tau)$ is holomorphic throughout $\mathbb{H}$. Write*

$$\Gamma_h := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \mid h\left(\frac{a\tau + b}{c\tau + d}\right) = h(\tau) \, \forall \tau \in \mathbb{H} \right\} . \qquad (3.7)$$

(a) *If $\Gamma_h \neq \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$, then h is a Hauptmodul for Γ_h, and Γ_h obeys (2.4) and contains $\Gamma_0(N)$ for some $N|K^{\infty}$.*

(b) *If $\Gamma_h = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$, and the coefficients a_n of h are algebraic integers, then $h(z) = q^{-1} + \xi q$ where $\xi = 0$ or $\xi^{\gcd(K,24)} = 1$.*

By '$N|K^{\infty}$' we mean any prime dividing N also divides K. Of course part (a) implies that Γ_h is genus-0 and of moonshine-type. When $h = T_g$, $K = o(g)$ works (see Prop.1(b)), and all coefficients are integers, and so Theorem 3 establishes the Hauptmodul property and fills the conceptual gap. The proof of Theorem 3 is difficult: if $h(\tau_1) = h(\tau_2)$, then it is fairly easy to prove that locally there is an invertible holomorphic map α sending an open disc about τ_1 onto one about τ_2; the hard part of the proof is to show that α extends to a globally invertible map $\mathbb{H} \to \mathbb{H}$ (and hence lies in Γ_h).

The converse of Theorem 3 is also true:

Proposition 2. [9] *If $h(\tau) = q^{-1} + \sum_{n=1}^{\infty} a_n q^n$ is a Hauptmodul for a group Γ_h of moonshine-type, and the coefficients a_n all lie in the cyclotomic field $\mathbb{Q}[\xi_N]$, then there exists a generalised modular equation for any order m coprime to N. Moreover, the field generated over $\mathbb{Q}$ by all coefficients a_n will be a Galois extension of $\mathbb{Q}$, with Galois group of exponent 2.*

The exponent 2 condition means that that field is generated over $\mathbb{Q}$ by a number of square-roots of rationals. We write ξ_N for the root of unity $\exp[2\pi i/N]$. The condition that all a_n lie in the cyclotomic field should be automatically satisfied. By a 'generalised modular equation' of order $m > 1$, we mean that there is a polynomial $F_m(x, y) \in \mathbb{Q}[\xi_N][x, y]$ such that F_m is monic of degree $\psi(m)$ in both x and y, and

$$F_m((\sigma_m.h)(\tau), Y) = \prod_{(d,k) \in \mathcal{A}_m} \left(h\left(\frac{m\tau}{d^2} + \frac{k}{d}\right) - Y \right) . \qquad (3.8)$$

We also have the symmetry condition $F_m(x, y) = (\sigma_m.F_m)(y, x)$. Here, $\sigma_m \in$ $\mathrm{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q}) \cong \mathbb{Z}_N^*$ is the Galois automorphism sending ξ_N to ξ_N^m; it acts on h and F_m coefficient-wise. The beautiful relation of modular functions to cyclotomic fields and their Galois groups is classical and is reviewed in e.g. Chapter 6 of [20].

Proposition 2 explains why Proposition 1 predicts more modular equations than Theorem 3 assumes: if all coefficients a_n in Theorem 3 are rational, then indeed we'd get that h would satisfy an ordinary modular equation for all m coprime to n.

The lesson of Moonshine is that we probably shouldn't completely ignore the exceptional functions in Theorem 3(b). It is tempting to call those 25 functions *modular fictions* (following John McKay). So a question could be:

Question 1. *What is the question in e.g. vertex algebras, for which the modular fictions are the answer?*

Theorem 3 requires many more modular equations than is probably necessary. In particular, the computer experiments in [4] show that if h has integer coefficients and satisfies modular equations of order 2 and 3, then f is either a Hauptmodul, or a modular fiction. Cummins has made the following conjecture:

Conjecture 1. [7] *Let p_1, p_2 be distinct primes, and $a_i \in \mathbb{C}$. Suppose $h(\tau) = q^{-1} + \sum_{n=1}^{\infty} a_n q^n$ satisfies modular equations of order p_1, p_2. Then*

(a) If $\Gamma_h \neq \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$, then h is a Hauptmodul for Γ_h, and Γ_h obeys (2.4) and contains $\Gamma_1(N)$ for some N coprime to p_1, p_2.

(b) If $\Gamma_h = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$, then $h(z) = q^{-1} + \xi q$ where $\xi = 0$ or $\xi^{\gcd(p_1-1, p_2-1)} = 1$.

We are far from proving this. However, if h obeys a modular equation of order m for all m with the property that all prime divisors p of m obey $p \equiv 1$ (mod K) for some fixed K, then h is either a Hauptmodul for Γ_h containing some $\Gamma_1(N)$, or h is 'trivial' (see [7] for details and a proof). The converse again is known to be true (again provided the a_i are cyclotomic).

It would be interesting to apply similar arguments to fill the related conceptual gaps of Modular Moonshine [3] and Baby Moonshine [15]. Modular equations have many uses in number theory, besides these in Moonshine — see e.g. [6] for important applications to class field theory. Modular equations are also closely related to the notion of replicable functions (see e.g. [23]).

4   The meaning of moonshine

As mentioned earlier, the greatest open challenge for Monstrous Moonshine is to find a second independent proof. In this section we briefly sketch some thoughts on what this proof may involve; see [13] for details.

A powerful guide to Monstrous Moonshine has been rational conformal field theory (RCFT). Modularity arises in RCFT through the conjunction of two standard pictures:

(1) *canonical quantisation* presents us with a state space V, carrying a representation of the symmetries of the theory, a Hamiltonian operator H, etc. In RCFT, the quantum amplitudes involve graded traces such as $\mathrm{Tr}_V q^H$, defining the coefficients of our q-expansions.

(2) The *Feynman picture* interprets the amplitudes using path integrals. In RCFT this permits us to interpret these graded traces as functions (sections) over moduli spaces, and hence they carry actions by the relevant mapping class groups such as $\mathrm{SL}_2(\mathbb{Z})$. This gives us modularity.

In Monstrous Moonshine, canonical quantisation is successfully abstracted into the language of vertex operator algebras (VOAs). The present proof of the Conway–Norton conjectures however ignores the Feynman side, and with it the lesson from RCFT that modularity is ultimately topological. Perhaps this is where to search for a second more conceptual proof. After all, the proof of the modularity of VOA characters [26] — perhaps the deepest result concerning VOAs — follows exactly this RCFT intuition. Let's briefly revisit the RCFT treatment of characters.

In an RCFT, with 'chiral algebra' (i.e. VOA) V, the character of a 'sector' (i.e. V-module) M is essentially the amplitude associated to a torus with one field (state) inserted — we call it a 'one-point function' on the torus. Fix a torus $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, a local parameter $z \in \mathbb{C}$ at the marked point (which we can take to be 0), and the state v which we're inserting at 0 (v can belong to any V-module, but for now we'll take $v \in V$). The local parameter z is needed for sewing surfaces together at the marked points (a fundamental process in RCFT). We get a moduli space $\widehat{\mathcal{M}}_{1,1}$ of 'extended once-marked tori', i.e. tori with a choice of local parameter z at 0. In this language the conformal symmetry of the RCFT becomes actions of the Virasoro algebra; the actions of this infinite-dimensional Lie algebra on the extended moduli spaces $\widehat{\mathcal{M}}_{g,n}$ are responsible for much of the special mathematical features of RCFT.

For convenience assume that $Hv = kv$ (this eigenvalue $k \in \mathbb{Q}$ is called the 'conformal weight' of v). The character is given by

$$\chi_M(\tau, v, z) := \mathrm{Tr}_M Y(v, e^{2\pi \mathrm{i} z}) \, q^{H-c/24} = e^{-2k\pi \mathrm{i} z} \, \mathrm{Tr}_M o(v) \, q^{H-c/24} \,, \qquad (4.1)$$

where c is the 'central charge' ($c = 24$ in Monstrous Moonshine), and $o(v)$ is an endomorphism commuting with H (also called L_0). What naturally acts on these χ_M is the mapping class group $\widehat{\Gamma}_{1,1}$ of $\widehat{\mathcal{M}}_{1,1}$.

This extended moduli space $\widehat{\mathcal{M}}_{1,1}$ is much larger than the usual moduli space $\mathcal{M}_{1,1} = \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ of a torus with one marked point, and the mapping class group $\widehat{\Gamma}_{1,1}$ is larger than the familiar mapping class group $\Gamma_{1,1} = \mathrm{SL}_2(\mathbb{Z})$. In fact, $\widehat{\Gamma}_{1,1}$ can be naturally identified with the braid group

$$\mathcal{B}_3 = \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle \,, \tag{4.2}$$

and acts on the characters by

$$\sigma_1.\chi_M(\tau, v, z) = e^{-2\pi ik/12}\,\chi_M(\tau + 1, v, z) \,, \tag{4.3}$$

$$\sigma_2.\chi_M(\tau, v, z) = e^{-2\pi ik/12}\,\chi_M\left(\frac{\tau}{1-\tau}, \frac{v}{(1-\tau)^k}, z\right) \,. \tag{4.4}$$

Thus in RCFT it is really $\mathcal{B}_3$ and not $\mathrm{SL}_2(\mathbb{Z})$ which acts on the characters. This is usually ignored because we specialise χ_M, and more fundamentally because typically we consider only insertions $v \in V$, and what results is a true action of the modular group $\mathrm{SL}_2(\mathbb{Z})$. But taking v from other V-modules is equally fundamental in the theory, and for those insertions we only get a projective action of $\mathrm{SL}_2(\mathbb{Z})$ (though again a true action of $\mathcal{B}_3$).

This is just a hint of a much more elementary phenomenon. Recall that a modular form f for $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, which is also holomorphic at the cusps, and which obeys

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \mu\begin{pmatrix} a & b \\ c & d \end{pmatrix}(c\tau + d)^k f(\tau) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,, \tag{4.5}$$

for some $k \in \mathbb{Q}$ (called the *weight*) and some function μ (called the *multiplier*) with modulus $|\mu| = 1$. For example, the Eisenstein series

$$E_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2}{}' (m\tau + n)^{-k} \tag{4.6}$$

for even $k > 2$ is a modular form of weight k with trivial multiplier μ, but the Dedekind eta

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n) \tag{4.7}$$

is a modular form of weight $k = 1/2$ with a nontrivial multiplier, given by

$$\mu\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \exp\left(\pi i\left(\frac{a+d}{12c} - \frac{1}{2} - \sum_{i=1}^{c-1}\frac{i}{c}\left(\frac{di}{c} - \left\lfloor\frac{di}{c}\right\rfloor - \frac{1}{2}\right)\right)\right) \tag{4.8}$$

when $c > 0$.

$\mathbb{H}$ can be regarded as a homogeneous space $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. Nowadays we are taught to lift a modular form f from $\mathbb{H}$ to $\mathrm{SL}_2(\mathbb{R})$:

$$\phi_f \begin{pmatrix} a & b \\ c & d \end{pmatrix} := f\left(\frac{a\mathrm{i} + b}{c\mathrm{i} + d}\right) (c\mathrm{i} + d)^{-k} \mu \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* . \tag{4.9}$$

We've sacrificed the implicit $\mathrm{SO}_2(\mathbb{R})$-invariance and explicit Γ-covariance of f, for explicit $\mathrm{SO}_2(\mathbb{R})$-covariance and explicit Γ-invariance of ϕ_f. This is significant, because compact Lie groups like the circle $\mathrm{SO}_2(\mathbb{R})$ are much easier to handle than infinite discrete groups like $\mathrm{SL}_2(\mathbb{Z})$. The result is a much more conceptual and powerful picture.

Thus a modular form should be regarded as a function on the orbit space $X := \Gamma \backslash \mathrm{SL}_2(\mathbb{R})$. Remarkably, this 3-space X can be naturally identified with the complement of the trefoil! We are thus led to ask:

Question 2. *Do modular forms for* $\mathrm{SL}_2(\mathbb{Z})$ *see the trefoil?*

An easy calculation shows that the fundamental group $\pi_1(X)$ is in fact the braid group $\mathcal{B}_3$! It is a central extension of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$. In particular, the quotient of $\mathcal{B}_3$ by its centre $\langle(\sigma_1\sigma_2\sigma_1)^2\rangle$ is $\mathrm{PSL}_2(\mathbb{Z})$; the isomorphism $\mathcal{B}_3/\langle(\sigma_1\sigma_2\sigma_1)^4\rangle \cong \mathrm{SL}_2(\mathbb{Z})$ is defined by the (reduced and specialised) Burau representation

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{4.10}$$

Through this map, which is implicit in (4.3) and (4.4), $\mathcal{B}_3$ acts on modular forms, and the multiplier μ can be lifted to $\mathcal{B}_3$. For example, the multiplier of the Dedekind eta becomes

$$\mu(\beta) = \xi_{24}^{\deg \beta} \qquad \forall \beta \in \mathcal{B}_3 , \tag{4.11}$$

where 'deg β' denotes the crossing number or degree of a braid. This is vastly simpler than (4.8)!

In hindsight it isn't so surprising that the multiplier is simpler as a function of braids than of 2×2 matrices. The multiplier μ will be a true representation of $\mathrm{SL}_2(\mathbb{Z})$ iff the weight k is integral; otherwise it is only a projective representation. And the standard way to handle projective representations is to centrally extend. Of course number theorists know this, but have preferred using the minimal necessary extension; as half-integer weights are the most common, they typically only look at a $\mathbb{Z}_2$-extension of $\mathrm{SL}_2(\mathbb{Z})$ called the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$. But unlike $\mathcal{B}_3$, $\mathrm{Mp}_2(\mathbb{Z})$ isn't much different from the modular group and the multipliers don't simplify much when lifted to $\mathrm{Mp}_2(\mathbb{Z})$. At least in the context of modular forms, *the braid group can*

be regarded as the universal central extension of the modular group, and the universal symmetry of its modular forms.

Topologically, $\mathrm{SL}_2(\mathbb{R})$ is the interior of the solid torus, so its universal covering group $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ will be the interior of the solid helix, and a central extension by $\pi_1 \cong \mathbb{Z}$ of $\mathrm{SL}_2(\mathbb{R})$. $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ can be realised [21] as the set of all pairs $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, n \right)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $n \equiv 0, 1, 2, 3 \pmod{4}$ depending on whether $c = 0$ and $a > 0$, $c < 0$, $c = 0$ and $a < 0$, or $c > 0$, respectively. The group operation is $(A, m)(B, n) = (AB, m + n + \tau)$, where $\tau \in \{0, \pm 1\}$ is called the Maslov index. Just as $\mathrm{SL}_2(\mathbb{Z})$ is the set of all integral points in $\mathrm{SL}_2(\mathbb{R})$, the braid group $\mathcal{B}_3$ is the set of all integral points in $\widetilde{\mathrm{SL}_2}(\mathbb{R})$.

Incidentally, similar comments apply when $\mathrm{SL}_2(\mathbb{Z})$ is replaced with other discrete groups — e.g. for $\Gamma(2)$ the relevant central extension is the pure braid group $\mathcal{P}_3$. It would be interesting to topologically identify the central extension for all the genus-0 groups Γ_g of Monstrous Moonshine.

So far we have only addressed the issue of modularity. A more subtle question in Moonshine is the relation of the Monster to the genus-0 property. Our best attempt at answering this is that the Monster is probably the largest exceptional 6-transposition group [24]. This relates to Norton's generalised Moonshine through the notion of quilts (see e.g. [16]). The relation of '6' to genus-0 is that $\mathbb{H}/\Gamma(n)$ is genus-0 iff $n < 6$, while $\mathbb{H}/\Gamma(6)$ is 'barely' genus 1. The notion of quilts, and indeed the notion of generalised Moonshine and orbifolds in RCFT, is related to braids through the right action of $\mathcal{B}_3$ on any $G \times G$ (for any group G) given by

$$(g, h).\sigma_1 = (g, gh) , \qquad (g, h).\sigma_2 = (gh^{-1}, h) . \tag{4.12}$$

Limited space has forced us to be very sketchy here. For more on all these topics, see [13] (which you are urged to purchase). We suggest that the braid group $\mathcal{B}_3$ and related central extensions may play a central role in a new, more conceptual proof of the Monstrous Moonshine conjectures.

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