# Topological and conformal field theory as Frobenius algebras

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ABSTRACT. Two-dimensional conformal field theory (CFT) can be defined through its correlation functions. These must satisfy certain consistency conditions which arise from the cutting of world sheets along circles or intervals. The construction of a (rational) CFT can be divided into two steps, of which one is complex-analytic and one purely algebraic. We realise the algebraic part of the construction with the help of three-dimensional topological field theory and show that any symmetric special Frobenius algebra in the appropriate braided monoidal category gives rise to a solution. A special class of examples is provided by two-dimensional topological field theories, for which the relevant monoidal category is the category of vector spaces.

### 1. Introduction

It has been known for some time [**Di**, **Ab**, **Ko**] that two-dimensional topological field theories are the same as finite-dimensional commutative Frobenius algebras over a field k. This correspondence can be extended to so-called open/closed topological field theories [**Lz**, **Mo**, **Se3**, **Ld**, **LP**]; the Frobenius algebra is then no longer, in general, commutative. It has also been known for some time [**FHK**] that a subclass of such theories, so called lattice topological field theories, can be constructed from a separable non-commutative Frobenius algebra A. In this case, the commutative Frobenius algebra just mentioned is the centre of A, and only the Morita class of A matters. Intriguingly, an analogous relationship holds in the much more involved situation of two-dimensional conformal field theory [**FRS1**]: the relevant structure is now (a Morita class of) a symmetric special Frobenius algebra in a braided monoidal category. It is this latter relation that is further investigated in this article.

The appearance of Frobenius algebras in two-dimensional conformal field theory (in CFT, for short) is based on the remarkable fact that the problem of constructing a CFT can be separated into a "complex-analytic" and an "algebraic" part. Here we are mainly concerned with the algebraic part. We formulate it (in section 3)

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as Problem 1, and show in Theorem 4 that indeed a symmetric special Frobenius algebra provides a solution.

The purpose of section 2 is to motivate Problem 1 and to explain how it relates to other aspects of CFT. Accordingly, sections 2.1-2.3 contain a brief outline of CFT. The relation between the complex-analytic and the algebraic aspects relies on a number of physical ideas which so far have only partially been cast into a precise mathematical language. If, however, one accepts those ideas, then solving Problem 1 amounts to the construction of a CFT with a prescribed rational conformal vertex algebra as its chiral symmetry.

Two-dimensional conformal field theory has its origins in several areas of physics. In all of these it arises as an effective theory, that is, it applies to experiments once a suitable limit is taken, like the limit of large system size or of low energy. For example, the long range behaviour of two-dimensional statistical systems at equilibrium [C2] is described by a (full, local) CFT, and the properties of edge states in quantum Hall systems (see e.g [FPSW]) by chiral conformal field theories.

CFT is also a basic ingredient in string theory, which is a candidate for a fundamental theory of matter and gravity [Po]. However, string theory is not yet sufficiently well understood to unambiguously reproduce or contradict known experimental results. In this sense the only established appearance of CFT in physics to date is via effective theories. Nonetheless the strongest incentive for the mathematical development of CFT came from its applications to string theory.

## 2. Two-dimensional conformal field theory

2.1. What is a two-dimensional CFT? There are several versions of two-dimensional CFT, such as formulations in terms of nets of von Neumann algebras on two-dimensional Minkowski space (see e.g. [Mü1, Re]), or in terms of correlation functions in the euclidean plane (see e.g. [Gb]). The formulation of CFT that will be reviewed below has been developed to fit the needs of string theory [FS, Va]. It has been cast in a more axiomatic language in [Se1, Se2]; some recent treatments are [Gw1, Gw2, HKr1].

In this setting, a CFT is based on two pieces of data: First, a  $\mathbb{Z}_+$ -graded complex vector space H with finite-dimensional homogeneous subspaces  $H^{(n)}$  ("the nth energy eigenspace in H"). H is called the space of states of the CFT. By  $H^\vee$  we denote the graded dual of H. Assuming the existence of a discrete grading is actually a simplification; there are examples of CFTs which lack such a grading, most importantly Liouville theory [Te]. A more appropriate name for the class of theories we study here might thus be compact CFTs. Also, in the light of the open/closed CFT that will be considered in section 2.3 below, what we are discussing is full (as opposed to chiral, see section 2.4 below) closed CFT, and accordingly H is the space of closed CFT states.

To describe the second piece of data we need the notion of a world sheet. A world sheet X is a smooth, compact two-manifold with parametrised and labelled boundary components and a Riemannian metric; in this paper we also require that X is oriented. An example is shown in figure 1. The components of the boundary  $\partial X$  are either 'incoming' or 'outgoing'. We number the components in the two subclasses separately, and denote by |in| and |out| the number of incoming and outgoing components of  $\partial X$ , respectively. Further, every incoming boundary component is parametrised by an embedding  $f: A_{\varepsilon}^+ \to X$  of a small annulus

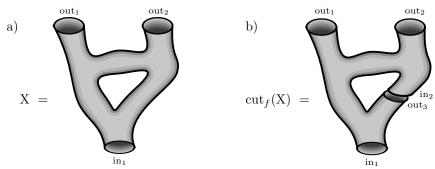


FIGURE 1. a) An example of a world sheet and b) the result of cutting that world sheet along a circle.

 $\mathbf{A}_{\varepsilon}^+ = \{z \in \mathbb{C} \mid 1 \leq |z| < 1 + \epsilon\}$ , which is is required to be conformal and orientation preserving. Similarly, outgoing boundary components are parametrised by embeddings  $f \colon \mathbf{A}_{\varepsilon}^- \to \mathbf{X}$  of a small annulus  $\mathbf{A}_{\varepsilon}^- = \{z \in \mathbb{C} \mid 1 - \varepsilon < |z| \leq 1\}$ . Two world sheets  $\mathbf{X}$  and  $\mathbf{X}'$  are *isomorphic* iff there exists an isometry which is compatible with the boundary labelling and parametrisations.

The second piece of data is a mapping Cor that assigns to an isomorphism class [X] of world sheets an element of the vector space  $(H^{|\text{in}|} \times (H^{\vee})^{|\text{out}|})^*$  of multilinear maps  $H \times \cdots \times H^{\vee} \to \mathbb{C}$ . For instance, the world sheet X in figure 1 a) results in a multilinear map

(2.1) 
$$\operatorname{Cor}(X): H \times H^{\vee} \times H^{\vee} \longrightarrow \mathbb{C}.$$

Cor(X) is called the amplitude, or correlation function, or correlator for X.

The assignment Cor has to fulfil a number of consistency requirements. To formulate them we need the notion of  $cutting\ a\ world\ sheet\ along\ a\ curve$ . More precisely, let X be a world sheet,  $A_{\varepsilon} = \{z \in \mathbb{C} \mid 1-\varepsilon < |z| < 1+\varepsilon\}$  a small annulus, and  $f \colon A_{\varepsilon} \to X$  a conformal orientation preserving embedding. Then the world sheet  $\operatorname{cut}_f(X)$  is defined by cutting along the image of the unit circle, which results in one additional incoming and one additional outgoing boundary component. For example, from the world sheet X in figure 1 a) one can obtain the world sheet  $\operatorname{cut}_f(X)$  shown in figure 1 b). Thus the correlator of  $\operatorname{cut}_f(X)$  is a multilinear map

(2.2) 
$$\operatorname{Cor}(\operatorname{cut}_f(X)): H \times H \times H^{\vee} \times H^{\vee} \times H^{\vee} \longrightarrow \mathbb{C}.$$

We also define a partial evaluation (or 'trace') operation  $\operatorname{tr}_{\operatorname{last}}$  from  $(H^n \times (H^{\vee})^m)^*$  to  $(H^{n-1} \times (H^{\vee})^{m-1})^*$  as evaluation of the last out- on the last in-component. <sup>1</sup> The data H, Cor define a closed CFT iff the following conditions are satisfied:

(C1) Factorisation:

$$Cor(X) = tr_{last} Cor(cut_f(X))$$

for every world sheet X and every embedding f.

(C2) Weyl transformations: For any two metrics g and g' on X that are related by  $g'_p = e^{\sigma(p)}g_p$  with some smooth function  $\sigma: X \to \mathbb{R}$ , one has

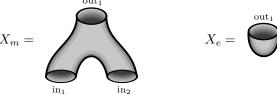
$$Cor(X \text{ with metric } g) = e^{cS[\sigma]}Cor(X \text{ with metric } g'),$$

<sup>&</sup>lt;sup>1</sup> For doing so, one first performs the evaluation up to some fixed grade N and then takes the limit  $N \to \infty$ . Property (C1) implies that for  $Cor(\operatorname{cut}_f(X))$  this limit exists. Since the embedding  $f \colon A_\varepsilon \to X$  does not appear on the left hand side of the equality in (C1), it also follows that the trace is independent of the parametrisation of the embedded annulus.

where  $c \in \mathbb{C}$  and  $S[\sigma]$  is the Liouville action (see [**Gw1**, **Gw2**] for details). The number c appearing here is called the *central charge* of the CFT.

2.2. Special case: Two-dimensional topological field theory. A class of examples for closed CFTs is provided by two-dimensional topological field theories [At]; detailed expositions can be found in [Qu, BK, Ko].

Let B be a finite-dimensional commutative Frobenius algebra over  $\mathbb{C}$ . Then a CFT in the sense described above is obtained upon setting  $H = H^{(0)} = B$  and constructing Cor by cutting a world sheet into elementary building blocks on which Cor is defined in terms of the defining data of B. For example, consider the world sheets

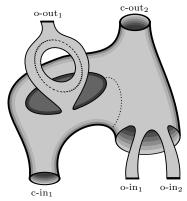


The associated correlators  $Cor(X_m): B \times B \times B^{\vee} \to \mathbb{C}$  and  $Cor(X_e): B^{\vee} \to \mathbb{C}$  are then defined in terms of the multiplication  $m: B \times B \to B$  of B, and by the unit e of B (via the map  $\eta: \lambda \mapsto \lambda e$  from  $\mathbb{C}$  to B), respectively.

The pair B, Cor obeys condition (C2), with c=0, for the trivial reason that Cor does not depend on the metric at all. Independence of the world sheet metric also explains the name topological field theory. That B and Cor obey (C1) follows from, and is indeed equivalent to, the defining properties of the commutative Frobenius algebra B; this result is due to [Di] and [Ab], see also [Vo, Qu, BK, Ko].

2.3. Open/closed CFT. The point of view from which open/closed CFT is described below is again inspired by string theory, namely from models that include both closed strings, which are circles, and open strings, which are intervals [Po, AS, SS]. In the physics literature, the study of open/closed CFT reaches back to [C1, CL, Lw]. The presentation below is in the spirit of the one given in section 2.1; other mathematical approaches along similar lines have recently been developed in [Hu1, HKo1, HKr2].

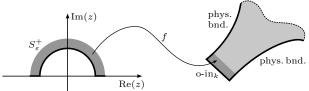
An open/closed CFT is defined similarly to section 2.1, but for a wider class of world sheets. Namely, the manifolds are now allowed to have corners, for example



(2.3)

In particular there are now three kinds of boundaries, to which we will refer as *closed* state boundaries, open state boundaries, and physical (or external) boundaries,

respectively. Closed state boundaries occurred already in closed CFT in section 2.1. They are circles with a parametrised neighbourhood, and they are labelled either as incoming or as outgoing and are numbered. Open state boundaries are intervals that lie between two corners; they, too, are labelled either as incoming or as outgoing and are numbered. In addition they are parametrised by a (conformal, orientation and boundary preserving) embedding of a small half-annulus  $S_{\varepsilon}^+ = A_{\varepsilon}^+ \cap \mathbb{H}$  for incoming boundaries and  $S_{\varepsilon}^- = A_{\varepsilon}^- \cap \mathbb{H}$  for outgoing boundaries, with  $\mathbb{H}$  the upper half plane. For example,



Physical boundaries are not labelled <sup>2</sup> and not parametrised. They are either circles or intervals between two corners. At each corner, a physical and an open state boundary meet. Two open/closed world sheets are isomorphic if they are isometric in a way compatible with labelling and parametrisation.

The basic data for an open/closed CFT are two graded  $\mathbb{Z}_+$ -graded vector spaces  $H_{\rm cl}$  and  $H_{\rm op}$  (with finite-dimensional homogeneous subspaces) and a mapping Cor, called again correlator or amplitude, that assigns to an isomorphism class of open/closed world sheets a multilinear map. For example, to the world sheet (2.3) a multilinear map

(2.4) 
$$Cor(X): H_{cl} \times H_{op} \times H_{op} \times H_{cl}^{\vee} \times H_{op}^{\vee} \to \mathbb{C}$$

is assigned. The conditions for the triple  $H_{\rm cl}$ ,  $H_{\rm op}$ , Cor to define an open/closed CFT are the same as **(C1)** and **(C2)** in section 2.1, except that now there are two distinct ways of cutting a surface. One can either embed a small annulus  $A_{\varepsilon}$ , in which case cutting along the image of the unit circle results in two new closed state boundaries, or one can embed a small half-annulus  $S_{\varepsilon} = A_{\varepsilon} \cap \mathbb{H}$ , so that cutting along the image of the unit half-circle results in two new open state boundaries. The dashed lines in (2.3) show two possible cutting paths.

As there are now two distinct cutting procedures, there are also two partial trace operations  $\operatorname{tr}_{\operatorname{last,cl}}$  and  $\operatorname{tr}_{\operatorname{last,op}}$ ; the former is the evaluation for the last pair of  $H_{\operatorname{cl}}$  and  $H_{\operatorname{cl}}^{\vee}$ , while the latter is the one for the last pair of  $H_{\operatorname{op}}$  and  $H_{\operatorname{op}}^{\vee}$ . Condition (C1) is imposed with respect to each of these two operations.

Again, two-dimensional topological field theory provides examples for such a structure. In this particular class of theories, one considers a finite-dimensional symmetric special Frobenius algebra A over  $\mathbb{C}$ , and obtains an open/closed CFT with  $H_{\text{op}} := A$ ; the results of [FRS1] take a particularly simple form in this case, and one finds  $H_{\text{cl}} := Z(A)$ , the centre of A. Open/closed 2-d TFT has been studied in [Lz, Mo, Se3] and in the detailed work [LP]. As before, the correlators Cor are constructed by decomposing world sheets into simple building blocks; we refrain

 $<sup>^2</sup>$  Here we make again a simplification: a physical boundary must actually be labelled by a so-called boundary condition. To keep the exposition short, for simplicity we think of all physical boundaries as having one and the same boundary condition, and suppress the label for that boundary condition in our notation.

from going into any more detail here. Again, the open/closed CFT obtained in this way does not depend on the world sheet metric.

As we will see in section 3, by considering a symmetric special Frobenius algebra not in the category  $\mathcal{V}ect_{\mathbb{C}}$  of finite-dimensional complex vector spaces, but rather in a more general braided monoidal category, we are able to describe also open/closed CFTs that are not topological.

**2.4.** Chiral CFT and full CFT. One way to achieve a construction of open/closed CFT is to proceed in two steps. As far as the correlators are concerned, the first step may be stated as

"find all possible functions consistent with a given symmetry," while the second step can be summarised as

"among all functions with the correct symmetry, find the correlators."

This approach has proven particularly powerful in what is known (see below) as rational CFT; it has its roots in [BPZ, MS]. Let us describe the basic ingredients of this approach.

The symmetries of a CFT can be described by a conformal vertex algebra  $\mathcal{V}$ . To a tuple  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  of  $\mathcal{V}$ -modules and a complex curve C with m marked points one can associate a vector space  $\mathcal{B}_C(\lambda_1, \lambda_2, \dots, \lambda_m)$  of conformal blocks, which is a subspace of the space of multilinear maps from  $\lambda_1 \times \lambda_2 \times \dots \times \lambda_m$  to  $\mathbb{C}$ . This subspace can be constructed explicitly as the space of invariants in the algebraic dual of the algebraic tensor product  $\lambda_1 \otimes_{\mathbb{C}} \lambda_2 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} \lambda_m$  with respect to the dual of the action of  $\mathcal{V}$  on this tensor product. For a fixed m-tuple  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  and fixed genus g of G the spaces  $\mathcal{B}_C(\lambda_1, \lambda_2, \dots, \lambda_m)$  of conformal blocks form a vector bundle  $\mathcal{B}(\lambda_1, \lambda_2, \dots, \lambda_m)$  over the moduli space  $\mathcal{M}_{g,m}$  of complex curves of genus g with g marked points. Each such bundle of conformal blocks is equipped with a projectively flat connection.

CFT on complex curves, with the associated system of bundles of conformal blocks, is called *chiral* CFT. Step 1 in the construction of a *full* CFT, i.e. of an open/closed CFT in the sense of section 2.3, consists in determining the corresponding chiral CFT. The chiral CFT does not fix the correlators uniquely, but it does determine, for each world sheet X, a subspace

$$(2.5) V(X) \subset (H_{cl} \times \cdots \times H_{op}^{\vee})^*$$

to which the correlator must belong. This subspace V(X) is given as a space of conformal blocks. The relevant surface is, however, not the world sheet X itself (which e.g. may have a non-empty boundary), but rather its complex double  $\widehat{X}$ , which is a complex curve, having two marked points for each bulk insertion and one marked point for each boundary insertion (see section 3.3 for the topological analogue of the complex double). V(X) is the space of conformal blocks on  $\widehat{X}$ . That bulk insertions result in two marked points while each boundary insertion only leads to one marked point is in accordance with the fact that one finds an action of  $\mathcal{V} \times \mathcal{V}$  on  $H_{cl}$  and an action of  $\mathcal{V}$  on  $H_{op}$ .

<sup>&</sup>lt;sup>3</sup> In the particular case that  $\mathcal{V}$  is the vertex algebra associated with an untwisted affine Lie algebra  $\mathfrak{g}^{(1)}$ , the action of  $\mathcal{V}$  on  $\lambda_1 \otimes_{\mathbb{C}} \lambda_2 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \lambda_m$  is through the Lie algebra  $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{F}$ , with  $\mathfrak{g}$  the finite-dimensional simple Lie algebra underlying  $\mathfrak{g}^{(1)}$  and  $\mathcal{F}$  the associative algebra of meromorphic functions on C that have at most finite order poles at the marked points and are holomorphic everywhere else [TUY]. Details for the general case can be found in [FB].

The symmetries of the CFT, as encoded in the conformal vertex algebra  $\mathcal{V}$ , force the correlator Cor(X) of the full CFT to be an element of the space V(X),

(2.6) 
$$Cor(X) \in V(X)$$
.

In quantum field theoretic terms, the symmetries give rise to partial differential equations, known as chiral Ward identities, that must be obeyed by the correlators; they can be formulated as the condition on the sections of the bundles  $\mathcal{B}(\lambda_1, \lambda_2, ..., \lambda_m)$  to be covariantly constant.

In step 2 of the procedure one constructs a full CFT from the chiral CFT obtained in step 1. The crucial point is now that for achieving this, everything that is needed as input from step 1 is already encoded in the representation category  $\mathcal{R}ep(\mathcal{V})$  of  $\mathcal{V}$ . Under suitable conditions on  $\mathcal{V}$ ,  $\mathcal{R}ep(\mathcal{V})$  is ribbon [HL] and even a modular tensor category [Hu2]. We refer to a chiral CFT for which  $\mathcal{R}ep(\mathcal{V})$  is modular as a rational chiral CFT; it is this class of theories to which the construction in section 3 applies. Carrying out step 2 can then be formulated entirely as an algebraic problem in the modular tensor category  $\mathcal{C}$ , without the need to make any further reference to the complex-analytic considerations involved in step 1.

That the algebraic problem in step 2 does indeed result in an open/closed CFT as defined in section 2.3 relies on several properties of the spaces V(X) under cutting of X and under deformation of the metric (and hence of the complex structure) on X. These are natural from the physical perspective, but a proof that they are indeed fulfilled in any rational CFT is still missing, even though some pertinent issues have recently been clarified [FB, HKo2]. For a more detailed account and references see e.g. section 5 of [FRS4].

In section 3 we formulate the algebraic problem to be solved in the category  $\mathcal{C}$ . We do so without making any further reference to the underlying chiral CFT. Indeed, we do not need to assume that  $\mathcal{C}$  is the category of representations of a vertex algebra.

## 3. The construction of full CFT as an algebraic problem

In this section C is a modular tensor category; its defining properties (which we take to be slightly stronger than in [Tu2]) are e.g. listed in section 2 of [FRS6]. We take the ground ring Hom(1,1) to be an algebraically closed field k.

**3.1. Statement of the problem.** We define a topological world sheet X in the same way as a world sheet in sections 2.1 and 2.3, except that X does not come equipped with a metric. Accordingly, the parametrisations are just required to be orientation preserving and continuous, and an isomorphism  $X \xrightarrow{\cong} Y$  of topological world sheets is a continuous map compatible with orientations and parametrisations. For  $f: A_{\varepsilon} \to X$  or  $f: S_{\varepsilon} \to X$  continuous orientation preserving embeddings, the cut topological world sheet  $\operatorname{cut}_f(X)$  is defined in the same way as in sections 2.1 and 2.3, too. From now on we will only consider topological world sheets; for brevity we will therefore omit the explicit mentioning of the qualification 'topological.'

We denote by  $\overline{\mathcal{C}}$  the dual modular tensor category of  $\mathcal{C}$ , that is,  $\mathcal{C}$  with the braiding replaced by the inverse braiding, see [Mü2, section 7] and [FFRS1, section 6.2]. Further, let  $\mathcal{C} \boxtimes \overline{\mathcal{C}}$  be the product of  $\mathcal{C}$  and  $\overline{\mathcal{C}}$  in the sense of enriched (over  $\mathcal{V}ect_{\mathbb{k}}$ ) category theory, i.e. the modular tensor category obtained by idempotent

completion of the category whose objects are pairs of objects of  $\mathcal{C}$  and  $\overline{\mathcal{C}}$  and whose morphism spaces are tensor products over  $\mathbb{k}$  of the morphism spaces of  $\mathcal{C}$  and  $\overline{\mathcal{C}}$ .

With these concepts, and using mappings  $\varphi^{\sharp}$ ,  $\operatorname{tr}_{\operatorname{last}}$  and vector spaces V(X) to be defined below, the task of constructing a full CFT can be summarised as follows.

Problem 1.

- (i) Select objects  $H_{\rm cl} \in \mathcal{O}bj(\mathcal{C} \boxtimes \overline{\mathcal{C}})$  and  $H_{\rm op} \in \mathcal{O}bj(\mathcal{C})$ .
- (ii) Find an assignment  $X \mapsto Cor(X) \in V(X)$  such that
  - (A1) For any isomorphism  $\varphi \colon X \xrightarrow{\cong} Y$  one has  $Cor(Y) = \varphi^{\sharp} Cor(X)$ .
  - (A2) For any embedding f of  $A_{\varepsilon}$  or  $S_{\varepsilon}$  into X,  $Cor(X) = tr_{last}(Cor(cut_f(X)))$ .

Once we have chosen  $H_{\rm cl}$ , we further select objects  $B_l$  and  $B_r$  of  $\mathcal C$  in such a way that  $H_{\rm cl}$  is a subobject of  $B_l \times \overline{B_r}$ . With the help of these objects we can give a prescription, to be detailed in sections 3.3 and 3.4 below, how to obtain from  $\mathcal C, H_{\rm cl}$  and  $H_{\rm op}$  the quantities  $\varphi^{\sharp}$ , tr<sub>last</sub> and V(X) entering in the formulation of part (ii) of the problem, i.e. how to obtain an assignment of

- a  $\mathbb{k}$ -vector space V(X) to every world sheet X;
- a vector space isomorphism  $\varphi^{\sharp}: V(X) \xrightarrow{\cong} V(Y)$  to every isomorphism  $\varphi: X \xrightarrow{\cong} Y$ ;
- a map  $\operatorname{tr}_{\operatorname{last}}: V(\operatorname{cut}_f(X)) \to V(X)$  to every embedding f of  $A_{\varepsilon}$  or  $S_{\varepsilon}$  into X.

REMARK 2. Let us make a few comments on how Problem 1 relates to the discussion in section 2. As the notation suggests,  $H_{cl}$  and  $H_{op}$  correspond the spaces of states of an open/closed CFT. But the space V(X) in Problem 1 does not correspond to the space of conformal blocks on the double of X that appeared in the discussion in section 2.4 (this would require a complex structure), but rather to the space of flat sections in the relevant vector bundle of conformal blocks <sup>4</sup>. The assignment Cor in Problem 1 is the analogue of the correlator on the CFT side; as in equation (2.6) it is an element in the vector space V(X).

Note that in Problem 1 we have been careful to *not* regard Cor(X) as a morphism between tensor products of state spaces, as one might be tempted to do for having an analogue of (2.4). (In such an approach the morphisms could only depend on the topology of X, and hence one could only describe two-dimensional topological field theories.) Instead, Cor(X) is an element of V(X) and thus, in conjunction with step 1, is a section of a bundle over  $\mathcal{M}_{g,m}$ . When endowing the topological world sheet X with a complex structure one selects a point in  $\mathcal{M}_{g,m}$  and thereby a specific conformal block. In this way the dependence of the correlators on the complex structure on the world sheet is recovered.

**3.2.** Three-dimensional topological field theory. To obtain the prescription announced after Problem 1, and to arrive (in section 3.6) at the solution to the Problem, we use as a crucial tool three-dimensional topological field theory (3-dTFT). It appeared originally in the guise of Chern-Simons field theory [Sc, W, FK], and was mathematically developed in [RT1, RT2, Tu1, Tu2]; for reviews see [BK, KRT] or section 2 of [FRS1].

Let us give a brief outline following [Tu2, BK] to set the notation. A 3-d TFT furnishes a monoidal functor  $tft_{\mathcal{C}}$  from a cobordism category  $3cob_{\mathcal{C}}$  to the category

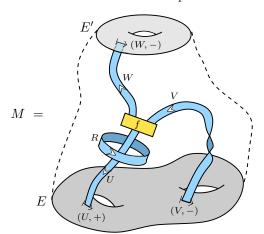
<sup>&</sup>lt;sup>4</sup> More precisely, as the bundle  $\mathcal{B}(\lambda_1, \lambda_2, \dots, \lambda_m)$  of conformal blocks over  $\mathcal{M}_{g,m}$  is equipped only with a projectively flat connection, one must consider flat sections in the projective bundle  $\mathbb{P}\mathcal{B}(\lambda_1, \lambda_2, \dots, \lambda_m)$ .

 $Vect_{\mathbb{k}}$  of finite-dimensional  $\mathbb{k}$ -vector spaces. The objects of  $3cob_{\mathcal{C}}$  are extended surfaces E, that is, oriented, compact, closed two-manifolds with a finite number of disjoint marked arcs and a distinguished Lagrangian subspace of  $H_1(E,\mathbb{R})$ . The arcs are marked by pairs  $(U,\pm)$  with U an object of  $\mathcal{C}$ . The following is an example of an extended surface:

$$E = (U, +) (V, -)$$

There is a tensor product on  $3cob_{\mathcal{C}}$ , which on objects is given by disjoint union; the tensor unit is the empty set.

The morphisms of  $3cob_{\mathcal{C}}$  are homotopy classes of extended cobordisms  $M: E \to E'$ . An extended cobordism is a compact, oriented three-manifolds with boundary given by the disjoint union of E and E', with an embedded oriented ribbon graph, and with a weight  $m \in \mathbb{Z}$ . The ribbon graph consists of ribbons labelled by objects of  $\mathcal{C}$  and of coupons labelled by morphisms of  $\mathcal{C}$ . The ribbons end either on coupons or on the marked arcs of E or E'. Here is an example:

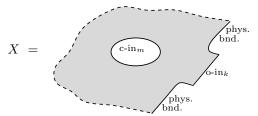


Here R, U, V, W are objects of  $\mathcal{C}$  and  $f \in \text{Hom}(U^{\vee}, W \otimes V^{\vee})$ . The identity morphism  $id_E$  is given by  $E \times [0,1]$ , the composition of morphisms is just the gluing of the three-manifolds, together with a rule to compute the new integer weight (see [**Tu2**] for details). The tensor product of morphisms is again given by disjoint union.

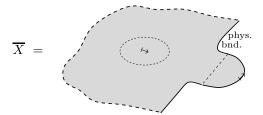
**3.3.** The assignment  $X \mapsto V(X)$ . After these preliminaries on 3-d TFT we can present the first part of the prescription announced after Problem 1, namely the assignment  $X \mapsto V(X)$ . We first need to select  $B_l$ ,  $B_r \in \mathcal{O}bj(\mathcal{C})$  such that  $H_{cl}$  is a subobject of  $B_l \times \overline{B_r} \in \mathcal{O}bj(\mathcal{C} \boxtimes \overline{\mathcal{C}})$ . This is a technical point: it is indeed not necessary to introduce these objects, but, as it turns out, passing to a larger object in  $\mathcal{C} \boxtimes \overline{\mathcal{C}}$  that has a factorised form simplifies the presentation. <sup>5</sup> Anticipating the result of Theorem 4 below, let us also abbreviate  $A \equiv H_{op} \in \mathcal{O}bj(\mathcal{C})$ .

<sup>&</sup>lt;sup>5</sup> In [**FjFRS**] a different approach has been followed, in which the object  $B_l \times \overline{B_r}$  does not appear directly. To recover that approach from the present one, one has to decompose  $B_l \times \overline{B_r}$  into a direct sum of simple subobjects and retain only those which also appear in  $H_{\rm cl}$ .

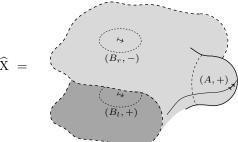
Consider a world sheet X part of which looks as follows:



The aim is to construct from X an extended surface  $\widehat{X}$ , the *double* of X. This will be done in two steps. First, using the parametrisation of the state boundaries, we glue a disc with an embedded arc to each closed state boundary, and a semi-disc with an embedded arc on the boundary to each open state boundary. This results in a surface  $\overline{X}$  with only physical boundaries. In the example above,



Second, the double  $\widehat{X}$  is defined to be the orientation bundle  $Or(\overline{X})$  over  $\overline{X}$ , divided by an equivalence relation that identifies the two points of fibres over the boundary of  $\overline{X}$ ,



An arc embedded in the interior of  $\overline{X}$  has two preimages in  $\widehat{X}$ ; we label them by  $(B_l, +)$  and by  $(B_r, -)$ , respectively. An arc on the boundary  $\partial \overline{X}$  has one preimage in  $\widehat{X}$ , which we label by (A, +). If a state boundary we start from is outgoing rather than incoming, then the corresponding plus sign in this prescription is replaced by a minus sign, and vice versa. There is also a natural choice for the Lagrangian subspace, see appendix A.1 of [**FjFRS**]. In this way,  $\widehat{X}$  becomes an extended surface in  $3cob_{\mathcal{C}}$ . We define the vector space V(X) as the image of  $\widehat{X}$  under the functor  $tft_{\mathcal{C}}$ ,

$$(3.1) V(X) := tft_{\mathcal{C}}(\widehat{X}) \in \mathcal{O}bj(\mathcal{V}ect_{\mathbb{k}}).$$

<sup>&</sup>lt;sup>6</sup> Both the world sheet X (and with it  $\overline{X}$ ) and the double  $\widehat{X}$  are oriented. The labelling is such that the projection from  $\widehat{X}$  to  $\overline{X}$  is orientation preserving in a neighbourhood of the arc labelled  $(B_l, +)$  and orientation reversing in a neighbourhood of the other.

From an isomorphism  $\varphi \colon X \xrightarrow{\cong} Y$  we can construct a cobordism  $M_{\varphi} \colon X \to Y$  by taking the two cylinders  $X \times [-1, 0]$  and  $Y \times [0, 1]$  and identifying  $X \times \{0\}$  with  $Y \times \{0\}$  using  $\varphi$ . Applying the 3-d TFT to this cobordism  $M_{\varphi}$  results in the isomorphism

(3.2) 
$$\varphi^{\sharp} := tft_{\mathcal{C}}(\mathcal{M}_{\varphi}) : V(\mathcal{X}) \xrightarrow{\cong} V(\mathcal{Y}).$$

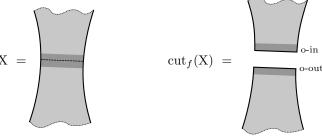
More details can again be found in [Tu2].

**3.4.** Cobordism for  $\operatorname{tr_{last}}$ . The final ingredient in the statement of Problem 1 is the map  $\operatorname{tr_{last}}: V(\operatorname{cut}_f(X)) \to V(X)$ . A precise notation would be  $\operatorname{tr_{last}}(X,f)$  so as to keep track of the world sheet and the parametrised cut locus, but we will use the short hand  $\operatorname{tr_{last}}$ . It is the analogue of both  $\operatorname{tr_{last,cl}}$  and  $\operatorname{tr_{last,op}}$  from section 2.3, depending on whether  $f \colon A_{\varepsilon} \to X$  or  $f \colon S_{\varepsilon} \to X$ .

By definition,  $V(X) = tft_{\mathcal{C}}(\widehat{X})$ . The map  $tr_{last}$  will be expressed as

$$(3.3) \hspace{1cm} \mathrm{tr}_{\mathrm{last}} = \mathit{tft}_{\mathcal{C}}(\mathrm{M}_f) \,, \hspace{5mm} \mathrm{where} \hspace{5mm} \mathrm{M}_f : \hspace{1mm} \widehat{\mathrm{cut}_f(\mathrm{X})} \,{\to}\, \widehat{\mathrm{X}}$$

is a suitable cobordism. Let us start with the case that f is an orientation preserving embedding  $S_{\varepsilon} \to X$ . Then locally the world sheet and the cut world sheet look as follows:



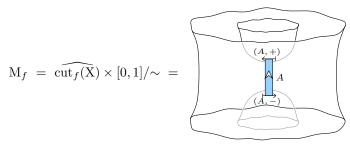
(3.4)

(3.5)

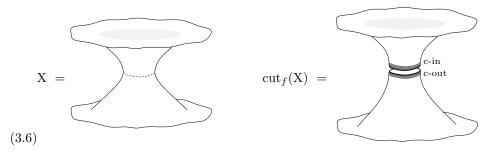
Their doubles are constructed as described in section 3.3, leading to

$$\widehat{\operatorname{Cut}_f(X)} = \bigcap_{A, -1} \widehat{\operatorname{Cut}_f(X)} = \bigcap_{A, -1} \widehat{\operatorname{Cut}_f(X)} = \bigcap_{A, -1} \widehat{\operatorname{Cut}_f(X)} = \bigcap_{A, -1} \widehat{\operatorname{$$

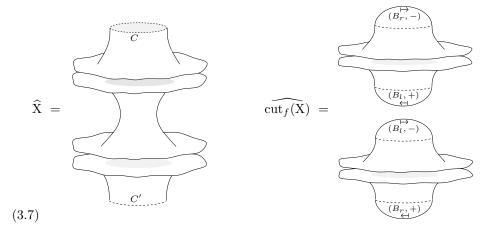
Finally, the cobordism  $M_f$  is obtained by taking the cylinder over  $\widetilde{\operatorname{cut}_f(X)}$  and identifying a disc (in fact, the disc resulting from taking the double of the half-discs glued to the open state boundaries) around the marked arcs at one of its ends:



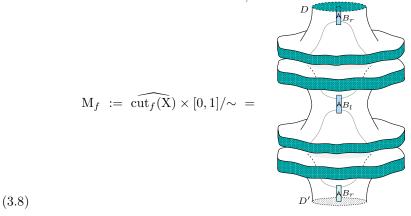
When cutting along a circle, we essentially have to duplicate the above construction. Let  $f \colon A_{\varepsilon} \to X$  be an orientation preserving embedding. Locally, the world sheet and the cut world sheet now look as



while their doubles are



Here for  $\widehat{X}$  the circles marked C and C' are to be identified. As before, the cobordism  $M_f$  is obtained by taking the cylinder over  $\widehat{\operatorname{cut}_f(X)}$  and identifying the discs around the marked arcs at one of its ends,

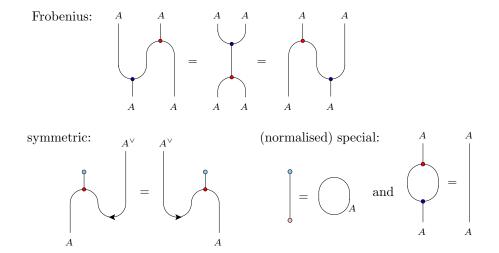


In this picture, the discs marked D and D' are to be identified. Note that  $M_f$  is indeed a cobordism from  $\widehat{\operatorname{cut}_f(X)}$  to  $\widehat{X}$ .

**3.5. Frobenius algebras in**  $\mathcal{C}$ **.** Essential for the solution to Problem 1 that we will present below is the concept of a symmetric special Frobenius algebra. We briefly review its definition; for more details and references consult sections 1.2 and 2.3 of [**FFRS1**]. A symmetric special Frobenius algebra in  $\mathcal{C}$  is a quintuple  $A = (A, m, \eta, \Delta, \varepsilon)$ , where  $A \in \mathcal{O}bj(\mathcal{C})$  and  $m, \eta, \Delta, \varepsilon$  are the morphisms of multiplication, unit, co-multiplication and co-unit, respectively. Since  $\mathcal{C}$  is strict monoidal, these can be visualised in the form of Joyal-Street type diagrams [**JS**] (to be read from bottom to top) as follows:

$$m = \bigwedge_{A = A}^{A}$$
 ,  $\eta = \bigwedge_{A}^{A}$  ,  $\Delta = \bigwedge_{A}^{A}$  ,  $\varepsilon = \bigwedge_{A}^{A}$ 

In terms of this diagrammatic notation, the properties that  $m, \eta, \Delta, \varepsilon$  must possess – apart from (co)associativity and the (co)unit property – in order that A is a symmetric special Frobenius algebra look as



A Frobenius algebra is special iff  $\varepsilon \circ \eta = \gamma id_1$  and  $m \circ \Delta = \gamma' id_A$  for  $\gamma, \gamma' \neq 0$ . By rescaling  $\Delta$  and  $\varepsilon$  we can always achieve  $\gamma = \dim(A)$  and  $\gamma' = 1$ . To emphasise that this is the choice made above we called the last property 'normalised special'. Below, when we say 'special', we shall always mean 'normalised special'.

Given an algebra A we can consider bimodules over A, and intertwiners between them. For A-bimodules B, B', we denote by  $\operatorname{Hom}_{A|A}(B,B')$  the space of bimodule intertwiners from B to B'. Of particular interest will be the bimodules  $U \otimes^+ A \otimes^- V$  for  $U, V \in \mathcal{O}bj(\mathcal{C})$ , which are defined as follows. The underlying object is  $U \otimes A \otimes V$ , while the left and right A-actions are given by  $(id_U \otimes m \otimes id_V) \circ (c_{U,A}^{-1} \otimes id_A \otimes id_V)$  and  $(id_U \otimes m \otimes id_V) \circ (id_U \otimes id_A \otimes c_{A,V}^{-1})$ , respectively (with  $c_{X,Y}$  the braiding of  $\mathcal{C}$ ).

Denote by  $\mathcal{I}$  the label set for isomorphism classes of simple objects of  $\mathcal{C}$  and by  $U_i$  a representative of the class with label  $i \in \mathcal{I}$ . We define the object

(3.9) 
$$Z(A) := \bigoplus_{i,j \in \mathcal{I}} \left( U_i \times \overline{U_j} \right)^{\oplus \tilde{Z}(A)_{ij}}$$

of  $C \boxtimes \overline{C}$ , with  $\widetilde{Z}(A)_{ij} := \dim_{\mathbb{K}} \operatorname{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j^{\vee}, A)$ . The object Z(A) will play the same role as the centre of the  $\mathbb{C}$ -algebra did in the open/closed two-dimensional topological field theory mentioned at the end of section 2.3.

The integers  $\tilde{\mathbf{Z}}(A)_{ij}$  and the object Z(A) have a number of interesting properties. For instance, according to Theorem 5.1 of [FRS1] the matrix  $\tilde{\mathbf{Z}}(A)$  commutes with the the modular group representation that can be constructed [Tu2] from the structural morphisms of a modular tensor category (this property of the matrix  $\tilde{\mathbf{Z}}(A)$  was in fact first established in the subfactor context [BEK]), while as shown in Proposition 5.3 of [FRS1], the matrix  $\tilde{\mathbf{Z}}$  for the product A # A' of two symmetric special Frobenius algebras A and A' is the matrix product of those of A and A',  $\tilde{\mathbf{Z}}(A\# A') = \tilde{\mathbf{Z}}(A) \tilde{\mathbf{Z}}(A')$  (our convention for the product # of algebras is stated in Proposition 3.22 of [FRS1]). One of the properties of the object Z(A) is given in

PROPOSITION 3. Let A be a symmetric special Frobenius algebra in C. Then the object Z(A) of  $C \boxtimes \overline{C}$  inherits from A the structure of a <u>commutative</u> symmetric Frobenius algebra.

The proof of this assertion requires several results from  $[\mathbf{FFRS1}]$ ; we postpone it to appendix A.

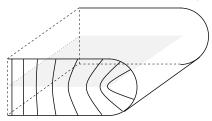
**3.6.** Construction of the correlator  $Cor_A(X) \in V(X)$ . Fix a symmetric special Frobenius algebra A in C. Given a world sheet X, we will construct an element  $Cor_A(X) \in V(X)$  using again the 3-d TFT associated to C. As a first step we choose a directed dual triangulation T of  $\overline{X}$  (recall from section 3.3 that  $\overline{X}$  is obtained by gluing discs and semi-discs to the closed and open state boundaries of X, respectively). We demand that the arcs embedded in  $\overline{X}$  are covered by edges of T.

Next we introduce the *connecting manifold*  $M_A(X,T)$ . It is a cobordism from the empty set to  $\widehat{X}$ , defined as follows.

■ As a manifold,  $M_A(X,T)$  is given by <sup>8</sup>

(3.10) 
$$M_A(X,T) = \overline{X} \times [-1,1]/\sim \text{ where } (x,t) \sim (x,-t) \text{ for } x \in \partial \overline{X}.$$

For example, close to a stretch of physical boundary  $M_A(X,T)$  looks as follows:



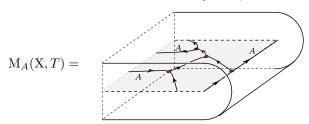
It is easy to see that indeed  $\partial M_A(X,T) = \widehat{X}$ , as is required if  $M_A(X,T)$  is to yield a cobordism  $\emptyset \to \widehat{X}$ . There is also a natural embedding  $\overline{X} \hookrightarrow M_A(X,T)$  sending  $x \in \overline{X}$  to  $(x,0) \in M_A(X,T)$ .

■ The ribbon graph embedded in  $M_A(X,T)$  depends on the triangulation T. Thinking of  $\overline{X}$  as embedded in  $M_A(X,T)$ , ribbons labelled by A are placed on the edges

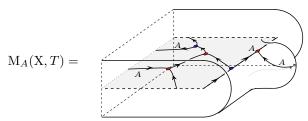
<sup>&</sup>lt;sup>7</sup> That is, at every vertex precisely three edges meet, while faces are allowed to have an arbitrary number of edges. Also, for each edge a direction must be chosen, in such a way that of the three edges meeting at any vertex, at least one is incoming and at least one outgoing.

<sup>&</sup>lt;sup>8</sup> The formula presented here applies to oriented world sheets only. For the unoriented case consult [FjFRS, appendix A.1].

of the triangulation, and the vertices are built from the multiplication and comultiplication morphism of A. Some examples should suffice to illustrate how the ribbon graph is constructed. Close to the boundary of  $\overline{X}$ , we have



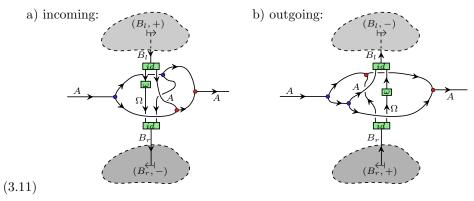
In this picture and in the following ones we simplify the graphical presentation by drawing ribbons as lines. Close to an arc on the boundary (which results in this example from an incoming open state boundary of X, see section 3.3) of  $\overline{X}$  we have



For an arc in the bulk, the construction is more complicated. Consider the object

$$\Omega := \bigoplus_{i \in \mathcal{I}} U_i.$$

Then close to an incoming (respectively outgoing) closed state boundary of X the ribbon graph in  $\mathcal{M}_A(\mathcal{X},T)$  looks as follows:



That is, we choose  $B_l = B_r = A \otimes \Omega$  (and hence it makes sense to label the coupons joining  $B_{l/r}$  and  $A \otimes \Omega$  by the identity morphism). The morphism  $\omega \in \operatorname{End}(\Omega)$  appearing in (3.11) is defined as

(3.12) 
$$\omega := \sum_{i \in \mathcal{I}} (S_{0,0} \dim(U_i))^{1/2} P_i$$
 with  $S_{0,0} = (\sum_{i \in \mathcal{I}} \dim(U_i)^2)^{-1/2}$ .

Here  $P_i \in \text{End}(\Omega)$  is the idempotent whose image is the subobject  $U_i$  of  $\Omega$ , and it is understood that we make once and for all a choice for all square roots involved.

■ By construction,  $tft_{\mathcal{C}}(M_A(X,T))$  is a linear map from k to V(X). Let T' be another directed dual triangulation of  $\overline{X}$ . One can prove [FjFRS, Proposition 3.1] that

$$(3.13) tft_{\mathcal{C}}(M_A(X,T)) = tft_{\mathcal{C}}(M_A(X,T')).$$

The basic idea of the proof is that the properties of A, as depicted in section 3.5 are similar to the two-dimensional Matveev moves by which one can relate any two dual triangulations of a given surface. Because of (3.13) it makes sense to set

$$(3.14) Cor_A(X) := tft_{\mathcal{C}}(M_A(X,T))1 \in V(X),$$

as the right hand side independent of the choice of T.

**3.7. Main theorem.** We have now gathered all ingredients needed to state our main result.

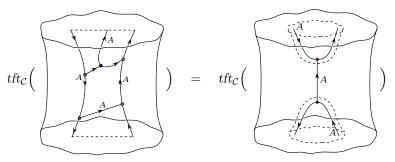
Theorem 4. Let A be a symmetric special Frobenius algebra in C. Then a solution to Problem 1 is obtained by setting  $H_{\rm op} = A$ ,  $H_{\rm cl} = Z(A)$  and taking the assignment  $X \mapsto Cor_A(X)$  as in (3.14).

By formulating the theorem in terms of  $Cor_A$  it is understood that in the prescription below Problem 1 one chooses  $B_l = B_r = A \otimes \Omega$ .

Theorem 4 has been proven, in a formulation more oriented towards using simple objects instead of the direct sum  $\Omega$ , in Theorems 2.1, 2.6 and 2.9 of [**FjFRS**]. Let us give a sketch of the proof. First, the requirement (**A1**) in Problem 1 follows form the triangulation independence of  $Cor_A(X)$ . Given an isomorphism  $\varphi \colon X \to Y$ , we can transport a triangulation T from  $\overline{X}$  to a triangulation T' of  $\overline{Y}$ . It is then not difficult to see that  $M_{\varphi} \circ M_A(X,T)$  and  $M_A(Y,T')$  are in the same homotopy class of cobordisms. It follows that

(3.15) 
$$\varphi^{\sharp} Cor(\mathbf{X}) = tft_{\mathcal{C}}(\mathbf{M}_{\varphi}) \circ tft_{\mathcal{C}}(\mathbf{M}_{A}(\mathbf{X}, T)) = tft_{\mathcal{C}}(\mathbf{M}_{\varphi} \circ \mathbf{M}_{A}(\mathbf{X}, T)) \\ = tft_{\mathcal{C}}(\mathbf{M}_{A}(\mathbf{Y}, T')) = Cor(\mathbf{Y}).$$

For an embedding  $f \colon S_{\varepsilon} \to X$  also property (A2) is easy to check. Just recall that cutting a world sheet along an interval looks locally as in (3.4) and that the cobordism for  $\operatorname{tr}_{\operatorname{last}}$  takes the form (3.5). Combining with the prescription to construct the connecting manifold, the desired equality  $\operatorname{Cor}_A(X) = \operatorname{tr}_{\operatorname{last}}(\operatorname{cut}_f(X))$  amounts to verifying that



That this equality indeed holds can be shown by making use of the properties of A given in section 3.5 [FjFRS].

For an embedding  $f: A_{\varepsilon} \to X$  the geometry is more involved. The proof of property (A2) for this case is sketched in appendix B.

#### 4. Comments

We have argued in section 2 that solving Problem 1 is equivalent to constructing an open/closed CFT whose underlying chiral symmetry is a given rational vertex algebra. Section 3 was devoted to the precise formulation of Problem 1 and to proving that solutions can be obtained from symmetric special Frobenius algebras in the relevant modular tensor category  $\mathcal{C}$ . To set these considerations into perspective, let us mention a number of issues that we did not address in the discussions above.

- From a quantum field theoretic point of view it is natural to allow different physical boundary segments to carry different boundary conditions. In the algebraic setting, boundary conditions are left modules of the algebra A, see [FRS1].
- We have *not* claimed that *every* solution to Problem 1 gives rise to a symmetric special Frobenius algebra in  $\mathcal{C}$ . To address this point is an obvious next step in our future work.
- Another question is whether different symmetric special Frobenius algebras in  $\mathcal{C}$  yield different CFTs. One finds that an oriented open/closed CFT together with all its symmetry preserving boundary conditions is encoded not by a single such algebra, but by a Morita class of symmetric special Frobenius algebras, or, equivalently, by an appropriate module category over  $\mathcal{C}$ , see section 4.1 of [FRS1] for details and references. On the other hand, an oriented open/closed CFT with one preferred boundary condition the situation discussed in the present article corresponds to an isomorphism class of symmetric (normalised) special Frobenius algebras.
- We have only treated open/closed CFTs on oriented world sheets. Results similar to those discussed here can also be obtained for *unoriented* world sheets, in which case suitable equivalence classes of so-called Jandl algebras are the relevant algebraic structure, see [FRS2] and section 11 of [FRS6].
- In another approach [BPPZ, PZ] to the algebraic part of the construction of a CFT it is argued that the structure of a weak Hopf algebra can be extracted from a rational open/closed CFT. The connection to the present approach is via module categories, see section 4 of [FFRS2] for more details.

As a closing remark, let us emphasise the following point. We have seen that algebras in braided monoidal categories appear naturally in the study of CFTs. Various physical quantities on the CFT side correspond to a standard construction on the algebraic side. Boundary conditions versus modules over an algebra provide just one example, but there are many more; see e.g. the dictionary in section 7 of [FRS6]. One can thus try and use CFT as a guiding principle in generalising other aspects of algebra from vector spaces to a truly braided setting.

The centre of an algebra A provides a nice illustration of this idea. In the braided setting one must consider two centres, the left centre  $C_l$  and the right centre  $C_r$  [VZ, Os, FFRS1]. Both are subobjects of A, and they coincide if the braiding is symmetric. Comparing the example at the end of section 2.3 and the statement of Theorem 4, another natural object to consider is Z(A), which unlike A and  $C_{l/r}$  is not an object of C, but of  $C \boxtimes \overline{C}$ . Actually this subsumes the notion of left and right centre in the sense that both  $C_l \times \mathbf{1}$  and  $\mathbf{1} \times C_r$  are subobjects of Z(A). Further, there is a notion of a centre Z of a monoidal category, which for a modular tensor category C is given by  $Z(C) \cong C \boxtimes \overline{C}$  [Mü2]. Thus for a symmetric

special Frobenius algebra A in a modular tensor category C, the object Z(A) is a commutative symmetric Frobenius algebra in the centre  $\mathcal{Z}(C)$ .

Other examples of structures and results arising when generalising algebra to the braided setting, and which become tautological when the braiding is symmetric, can be found in [FFRS1]. In fact, the issues investigated in [FFRS1] were motivated from problems in CFT, too.

## Appendix A. Proof of Proposition 3

In the following we freely use notation from [FFRS1]. We will show that, as objects in  $C \boxtimes \overline{C}$ ,

$$(A.1) Z(A) \cong C_l((A \times 1) \otimes T_C),$$

where  $T_{\mathcal{C}} \cong \bigoplus_{i} U_{i} \times \overline{U_{i}}$  as an object of  $\mathcal{C} \boxtimes \overline{\mathcal{C}}$  (see sections 2.4 and 6.3 of [FFRS1]) and  $C_{l}(\cdot)$  denotes the left centre. Since both A and  $T_{\mathcal{C}}$  are symmetric special Frobenius algebras, so is  $(A \times \mathbf{1}) \otimes T_{\mathcal{C}}$ . Then, by Proposition 2.37(i) of [FFRS1], the left centre  $C_{l}((A \times \mathbf{1}) \otimes T_{\mathcal{C}})$  is a commutative symmetric Frobenius algebra. Together with the isomorphism (A.1) this proves Proposition 3.

To establish the isomorphism (A.1) we combine Propositions 3.14(i) and 3.6 of [FFRS1] so as to find

$$(A.2) \qquad \begin{array}{l} \operatorname{Hom}(C_{l}((A \times \mathbf{1}) \otimes T_{\mathcal{C}}), U_{p} \times \overline{U_{q}}) \\ \cong \operatorname{Hom}(E_{A \times \mathbf{1}}^{l}(T_{\mathcal{C}}), U_{p} \times \overline{U_{q}}) \\ \cong \operatorname{Hom}_{A|A}((A \times \mathbf{1}) \otimes^{-} T_{\mathcal{C}}, (A \times \mathbf{1}) \otimes^{+} (U_{p} \times \overline{U_{q}})) \\ \cong \bigoplus_{k \in \mathcal{I}} \operatorname{Hom}_{A|A}(A \otimes^{-} U_{k}, A \otimes^{+} U_{p}) \otimes_{\mathbb{k}} \operatorname{Hom}(\overline{U_{k}}, \overline{U_{q}}) \end{array}$$

Together with [FRS4, Lemma 2.2] and [FRS1, Theorem 5.23(iii)] this results in

$$\dim \operatorname{Hom}(C_{l}((A \times \mathbf{1}) \otimes T_{\mathcal{C}}), U_{p} \times \overline{U_{q}})$$

$$= \dim \operatorname{Hom}_{A|A}(A \otimes^{-} U_{q}, A \otimes^{+} U_{p})$$

$$= \dim \operatorname{Hom}_{A|A}(U_{p}^{\vee} \otimes^{+} A \otimes^{-} U_{q}, A)$$

$$= \tilde{\operatorname{Z}}(A)_{\overline{p}, \overline{q}} = \tilde{\operatorname{Z}}(A)_{p, q}$$

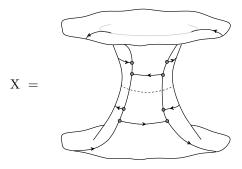
$$= \dim \operatorname{Hom}(Z(A), U_{p} \times \overline{U_{q}}).$$

Since  $U_p \times \overline{U_q}$  are representatives for the isomorphism classes of simple objects of the semisimple category  $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ , this shows that Z(A) and  $C_l((A \times \mathbf{1}) \otimes T_{\mathcal{C}})$  are indeed isomorphic as objects in  $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ , and thus completes the proof.

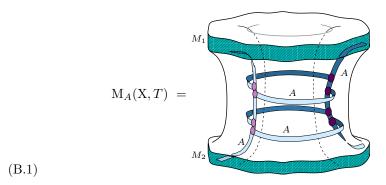
## Appendix B. Bulk factorisation

Here we present some details of the proof that the construction presented in section 3.6 solves the requirement (A2) also when cutting along a circle, i.e. for an embedding  $f \colon A_{\varepsilon} \to X$ . To this end we start from the situation depicted in (3.6).

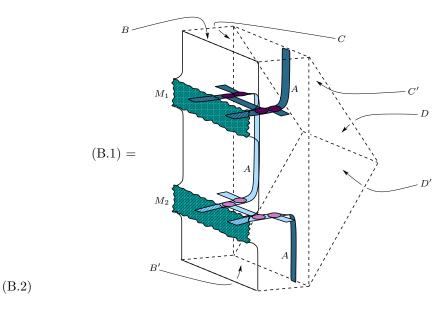
Close to the image of the embedding  $f : A_{\varepsilon} \to X$  the world sheet X looks as follows.



Here we have also indicated the dual triangulation T we will use. The corresponding fragment of the connecting manifold takes the form

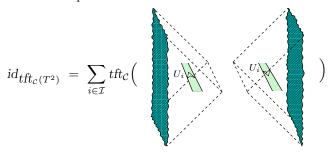


The two cylinders marked  $M_1$  and  $M_2$  indicate where the part of  $M_A(X, T)$  that is drawn connects to the rest of  $M_A(X, T)$ . It is convenient to draw this fragment of cobordism in 'wedge representation' (see [FjFRS, section 5.1] for details),

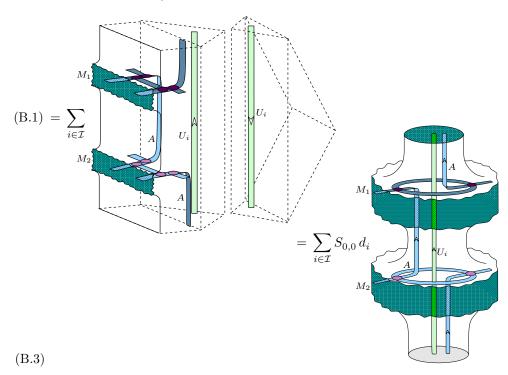


In this drawing, the surface labelled B is to be identified with B' and similarly C with C' and D with D' It is then not too difficult to see that (B.2) indeed describes the same piece of cobordism as (B.1).

Next we note that the identity map on the vector space  $tft_{\mathcal{C}}(T^2)$  assigned to a two-torus can be decomposed as



Applying this identity to (B.2) gives (here we suppress the  $tft_{\mathcal{C}}(\cdots)$  that is to be taken for each cobordism)



with  $d_i := \dim(U_i)$ . For the second step, note that the connected component that contains only a  $U_i$ -ribbon is in fact an  $S^3$ , and its invariant is  $S_{0,0} \dim(U_i)$ .

The picture (B.1) shows a fragment of the cobordism for Cor(X) close to the circle along which X is cut. The right hand side of (B.3) is nothing but the relevant fragment of the cobordism obtained from applying  $tr_{last}$  (as given in (3.3) with  $M_f$  as in (3.8)) to  $Cor(cut_f(X))$ . The relevant fragment of the cobordism for  $Cor(cut_f(X))$ , in turn, is obtained by considering the second picture in (3.7) and inserting the ribbon graphs (3.11) arising from an incoming and outgoing closed

state boundary. The prefactor on the right hand side of (B.3) is produced by evaluating the two morphisms  $\omega$  in (3.11). This establishes property (A2), i.e. that  $Cor(X) = \operatorname{tr}_{last}(Cor(\operatorname{cut}_f(X)))$ , also for cutting along a circle.

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