# Unoriented WZW Models and Holonomy of Bundle Gerbes

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## Abstract

The Wess-Zumino term in two-dimensional conformal field theory is best understood as a surface holonomy of a bundle gerbe. We define additional structure for a bundle gerbe that allows to extend the notion of surface holonomy to unoriented surfaces. This provides a candidate for the Wess-Zumino term for WZW models on unoriented surfaces. Our ansatz reproduces some results known from the algebraic approach to WZW models.

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manche meinen
lechts und rinks
kann man nicht velwechsern
werch ein illtum

Ernst Jandl [Jan95]

# Contents

# 1 Introduction

Wess-Zumino-Witten (WZW) models are one of the most important classes of (two-dimensional) rational conformal field theories. They describe physical systems with (non-abelian) current symmetries, provide gauge sectors in heterotic string compactifications and are the starting point for other constructions of conformal field theories, e.g. the coset construction. Moreover, they have played a crucial role as a bridge between Lie theory and conformal field theory.

It is well-known that for the Lagrangian description of such a model, a Wess-Zumino term is needed to get a conformally invariant theory [Wit84]. Later, the relation of this term to Deligne hypercohomology has been realized [Gaw88] and its nature as a surface holonomy has been identified [Gaw88, Alv85]. More recently, the appropriate differential-geometric object for the holonomy has been identified as a hermitian $U(1)$ bundle gerbe with connection and curving [CJM02].

Already the case of non-simply connected Lie groups with non–cyclic fundamental group, such as $G := Spin(4n)/\mathbb{Z}_2 \times \mathbb{Z}_2$ shows that gerbes and their holonomy are really indispensable, even when one restricts one's attention to oriented surfaces without boundary. The original definition of the Wess-Zumino term as the integral of a three form $H$ over a suitable three-manifold cannot be applied to such groups; moreover, it could not explain the well-established fact that to such a group *two* different rational conformal field theories that differ by "discrete torsion" can be associated.

Bundle gerbes will be central for the problem we address in this paper. A long series of algebraic results indicate that the WZW model can be consistently considered on unorientable surfaces. Early results include a detailed study of the abelian case [BPS92] and of $SU(2)$ [PSS95b, PSS95a]. Sewing constraints for unoriented surfaces have been derived in [FPS94].

Already the abelian case [BPS92] shows that not every rational conformal field theory that is well-defined on oriented surfaces can be considered on unoriented surfaces. A necessary condition is that the bulk partition function is symmetric under exchange of left and right movers. This restricts, for example, the values of the Kalb-Ramond field in toroidal compactifications [BPS92]. Moreover, if the theory can be extended to unoriented surfaces, there can be different extensions that yield inequivalent correlation functions. This has been studied in detail for WZW theories based on $SU(2)$ in [PSS95b, PSS95a]; later on, this has been systematically described with simple current techniques [HS00, HSS99]. Unifying general formulae have been proposed in [FHS$^+$00]; the structure has been studied at the level of NIMreps in [SS03]. Aspects of these results have been proven in [FRS04] combining topological

field theory in three-dimensions with algebra and representation theory in modular tensor categories. As a crucial ingredient, a generalization of the notion of an algebra with involution, i.e. an algebra together with an algebra-isomorphism to the opposed algebra, has been identified in [FRS04]; the isomorphism is not an involution any longer, but squares to the twist on the algebra. An algebra with such an isomorphism has been called Jandl algebra. A similar structure, in a geometric setting, will be the subject of the present article.

The success of the algebraic theory leads, in the Lagrangian description, to the quest for corresponding geometric structures on the target space. From previous work [BCW01, HSS02, Bru02] it is clear that a map $k : M \to M$ on the target space with the additional property that $k^*H = -H$ will be one ingredient. Examples like the Lie group $SO(3)$, for which two different unoriented WZW models with the same map $k$ are known, already show that this structure does not suffice.

We are thus looking for an additional structure on a hermitian bundle gerbe which allows to define a Wess-Zumino term, i.e. which allows to define holonomy for unoriented surfaces. For a general bundle gerbe, such a structure need not exist; if it exists, it will not be unique.

In the present article, we make a proposal for such a structure. It exists whenever there are sufficiently well-behaved stable isomorphisms between the pullback gerbe $k^*\mathcal{G}$ and the dual gerbe $\mathcal{G}^*$. If one thinks about a gerbe as a sheaf of groupoids, the formal similarity to the Jandl structures in [FRS04] becomes apparent, if one realizes that the dual gerbe plays the role of the opposed algebra. For this reason, we term the relevant structure a Jandl structure on the gerbe. We show that the Jandl structures on a gerbe on the target space $M$, if they exist at all, form a torsor over the group of flat equivariant hermitian line bundles on $M$. As explained in section 4.3, this group always contains an element $L_{-1}^k$ of order two. We show that two Jandl structures that are related by the action of $L_{-1}^k$ provide amplitudes that just differ by a sign that depends only on the topology of the worldsheet. Such Jandl structures are considered to be essentially equivalent. We finally show that a Jandl structure allows to extend the definition of the usual gerbe holonomy from oriented surfaces to unoriented surfaces. We derive formulae for these holonomies in local data that generalize the formulae of [GR02, Alv85] for oriented surfaces.

To give a concrete impression of a Jandl structure, we write out the local data of a Jandl structure for a given gerbe $\mathcal{G}$ on the target space $M$. To this end, we first recall the local data of a hermitian bundle gerbe in a good open cover $\{V_i\}_{i \in I}$ of $M$: we have a 2-form $B_i$ for each open set $V_i$, a 1-form $A_{ij}$

on each intersection $V_i \cap V_j$ and a $U(1)$-valued function $g_{ijk}$ on each triple intersection $V_i \cap V_j \cap V_k$. They are required to satisfy the following constraints:

$$
\begin{aligned}
g_{jkl} \cdot g_{ikl}^{-1} \cdot g_{ijl} \cdot g_{ijk}^{-1} &= 1 \\
A_{jk} - A_{ik} + A_{ij} + \mathrm{dlog}\,(g_{ijk}) &= 0 \\
-\mathrm{d}A_{ij} + B_j - B_i &= 0.
\end{aligned}
$$

To write down the local data of a Jandl structure for a given involution $k : M \to M$ in a succinct manner, we make the simplifying assumption that we have a cover $\{V_i\}_{i \in I}$ that is invariant under $k$, $k(V_i) = V_i$, and that is still good enough to provide local data. The local data of a Jandl structure then consist of a $U(1)$-valued function $j_i : V_i \to U(1)$ for each open subset, a $U(1)$-valued function $t_{ij} : V_i \cap V_j \to U(1)$ on two-fold intersections and a 1-form $W_i \in \Omega^1(V_i)$.

They relate the pullbacks of the gerbe data under $k$ to the local data of the dual gerbe as follows:

$$
\begin{aligned}
k^* B_i &= -B_i + \mathrm{d}W_i \\
k^* A_{ij} &= -A_{ij} - \mathrm{dlog}(t_{ij}) + W_j - W_i \\
k^* g_{ijk} &= g_{ijk}^{-1} \cdot t_{jk} \cdot t_{ik}^{-1} \cdot t_{ij}
\end{aligned}
$$

The local data of a Jandl structure are required to be equivariant under $k$ in the sense that

$$
\begin{aligned}
k^* W_i &= W_i - \mathrm{dlog}(j_i) \\
k^* t_{ij} &= t_{ij} \cdot j_j^{-1} \cdot j_i \\
k^* j_i &= j_i^{-1}.
\end{aligned}
$$

It should be appreciated that the functions $t_{ij}$ are *not* transition functions of some line bundle; as we will explain in section 2.4, they are rather the local data describing an isomorphism of line bundles appearing in the Jandl structure.

The notion of a Jandl structure naturally explains algebraic results for specific classes of rational conformal field theories. It is well-known that both the Lie group $SU(2)$ and its quotient $SO(3)$ admit two Jandl structures that are essentially different (i.e. that do not just differ by a sign depending on the topology of the surface). In the case of $SU(2)$, this is explained by the fact that two different involutions are relevant: $g \mapsto g^{-1}$ and $g \mapsto zg^{-1}$, where $z$ is the non-trivial element in the center of $SU(2)$. Indeed, since $SU(2)$ is simply-connected, we have a single flat line bundle and hence for each involution only two Jandl structures which are essentially the same.

5

The two involutions of $SU(2)$ descend to the same involution of the quotient $SO(3)$. The latter manifold, however, has fundamental group $\mathbb{Z}_2$ and thus twice as many equivariant flat line bundles as $SU(2)$. The different Jandl structures of $SO(3)$ are therefore not explained by different involutions on the target space but rather by the fact that one involution admits two essentially different Jandl structures.

Needless to say, there remain many open questions. A discussion of surfaces with boundaries is beyond the scope of this article. The results of [FRS04] suggest, however, that a Jandl structure leads to an involution on gerbe modules. Most importantly, it remains to be shown that, in the Wess-Zumino-Witten path integral for a surface $\Sigma$, the holonomy we introduced yields amplitudes that take their values in the space of conformal blocks associated to the complex double of $\Sigma$, which ensures that the relevant chiral Ward identities are obeyed. To this end, it will be important to have a suitable reformulation of Jandl structures at our disposal. Indeed, the holonomy we propose in this article also arises as the surface holonomy of a 2-vector bundle with a certain 2-group; this issue will be the subject of a separate publication.

# 2 Bundle Gerbes with Jandl Structures

## 2.1 Bundle Gerbes and stable Isomorphisms

In preparation of the following sections, in this section we define an equivalence relation on the set of stable isomorphisms between two fixed bundle gerbes. To this end, we first set up the notation concerning bundle gerbes and stable isomorphisms. We mainly adopt the formalism used by Murray and collaborators, see [CJM02] for example, as well as by Gawędzki and Reis [GR02].

**Definition 1.** *A hermitian $U(1)$ bundle gerbe $\mathcal{G}$ with connection and curving over a smooth manifold $M$ consists of the following data: a surjective submersion $\pi : Y \to M$, a hermitian line bundle $p : L \to Y^{[2]}$ with connection, an associative isomorphism*

$$\mu \quad : \quad \pi_{12}^* L \otimes \pi_{23}^* L \quad \longrightarrow \quad \pi_{13}^* L \tag{1}$$

*of hermitian line bundles with connection over $Y^{[3]}$, and a 2-form $C \in \Omega^2(Y)$ which satisfies*

$$\pi_2^* C - \pi_1^* C = \mathrm{curv}(L). \tag{2}$$

Here $Y^{[p]}$ denotes the $p$-fold fiber product of $\pi : Y \to M$, which is a smooth manifold since $\pi$ is a surjective submersion. For example $\pi_{12} : Y^{[3]} \to Y^{[2]}$ is the projection on the first two factors.

**Remark 1.** *From now we will use the following conventions: the term line bundle refers to a hermitian line bundle with connection, and an isomorphism of line bundles refers to an isomorphism of hermitian line bundles with connection. Accordingly, we refer to Definition 1 by the term gerbe. The 2-form $C$ is called curving, and the isomorphism $\mu$ is called multiplication.*

One can show that there is a unique 3-form $H \in \Omega^3(M)$ with $\pi^* H = \mathrm{d}C$; this 3-form is called the curvature of the gerbe and is denoted by $H = \mathrm{curv}(\mathcal{G})$.

To each gerbe $\mathcal{G}$, we associate the dual gerbe $\mathcal{G}^*$. It has the same surjective submersion $\pi : Y \to M$, but the dual line bundle $L^* \to Y^{[2]}$ with multiplication

$$(\mu^*)^{-1} \quad : \quad \pi_{12}^* L^* \otimes \pi_{23}^* L^* \quad \longrightarrow \quad \pi_{13}^* L^*, \tag{3}$$

and the negative curving $-C$. Accordingly, the curvature of the dual gerbe satisfies

$$\mathrm{curv}(\mathcal{G}^*) = -\mathrm{curv}(\mathcal{G}). \tag{4}$$

Even more, the classes of $\mathcal{G}$ and the one of $\mathcal{G}^*$ in Deligne hypercohomology are inverses.

For a smooth map $f : N \to M$ and a pullback diagram

$$
\begin{array}{ccc}
Y_f & \xrightarrow{\tilde{f}} & Y \\
{\scriptstyle \pi_f}\downarrow & & \downarrow{\scriptstyle \pi} \\
N & \xrightarrow{f} & M
\end{array}
\quad , \tag{5}
$$

$\pi_f : Y_f \to N$ is a surjective submersion, and together with the line bundle $\tilde{f}^* L$ over $Y_f^{[2]}$, the multiplication $\tilde{f}^* \mu$ and the curving $\tilde{f}^* C$, we have defined a gerbe $f^* \mathcal{G}$. If $f : M \to M$ is a diffeomorphism, $Y_f$ is canonically isomorphic to $Y$, such that $\tilde{f} = \mathrm{id}_Y$ and $\pi_f = f^{-1} \circ \pi$. The curvature of the pullback gerbe is

$$\mathrm{curv}(f^* \mathcal{G}) = f^* \mathrm{curv}(\mathcal{G}). \tag{6}$$

**Remark 2.** *As we did in the last paragraph, whenever there is a map $\tilde{f} : Y_f \to Y$, we will use the same letter for the induced map on higher fiber products.*

**Definition 2.** *A trivialization* $\mathcal{T} = (T, \tau)$ *of a gerbe* $\mathcal{G}$ *is a line bundle* $T \to Y$, *together with an isomorphism*

$$\tau \ : \ L \otimes \pi_2^* T \ \longrightarrow \ \pi_1^* T \tag{7}$$

*of line bundles over* $Y^{[2]}$, *which is compatible with the isomorphism* $\mu$ *of the gerbe.*

We call a gerbe $\mathcal{G}$ trivial, if it admits a trivialization. A choice of a trivialization $\mathcal{T}$ gives the 2-form $C - \mathrm{curv}(T) \in \Omega^2(Y)$, which descends to a unique 2-form $\rho \in \Omega^2(M)$ with $\pi^* \rho = C - \mathrm{curv}(T)$. This 2-form satisfies $\mathrm{d}\rho = H$, so the curvature $H$ of a trivial gerbe is an exact form.

If there are two trivializations $\mathcal{T}_1 = (T_1, \tau_1)$ and $\mathcal{T}_2 = (T_2, \tau_2)$ of the same gerbe $\mathcal{G}$, one obtains an isomorphism

$$\alpha := \tau_1^{-1} \otimes \tau_2^* \ : \ \pi_1^*(T_1 \otimes T_2^*) \ \longrightarrow \ \pi_2^*(T_1 \otimes T_2^*), \tag{8}$$

of line bundles over $Y^{[2]}$. From the compatibility condition between the multiplication $\mu$ and both $\tau_1$ and $\tau_2$ the cocycle condition

$$\pi_{23}^* \alpha \circ \pi_{12}^* \alpha = \pi_{13}^* \alpha \tag{9}$$

follows. Such an isomorphism determines a unique descent line bundle $N \to M$ with connection together with an isomorphism $\nu : \pi^* N \to T_1 \otimes T_2^*$ [Bry93]. The two 2-forms $\rho_1$ and $\rho_2$ coming from the two trivializations are related by

$$\rho_2 = \rho_1 + \mathrm{curv}(N). \tag{10}$$

**Definition 3.** *Let* $\mathcal{G}$ *and* $\mathcal{G}'$ *be two gerbes. A stable isomorphism*

$$\mathcal{A} \ : \ \mathcal{G} \ \longrightarrow \ \mathcal{G}' \tag{11}$$

*consists of a line bundle* $A \to Z$ *over the fiber product* $Z := Y' \times_M Y$ *with curvature*

$$\mathrm{curv}(A) = p'^* C' - p^* C, \tag{12}$$

*and an isomorphism*

$$\alpha \ : \ p^* L \otimes p'^* L'^* \otimes \pi_2^* A \ \longrightarrow \ \pi_1^* A \tag{13}$$

*of line bundles over* $Z^{[2]}$, *which is compatible with the multiplications* $\mu$ *and* $\mu'$ *of both gerbes.*

Here $p$ and $p'$ denote the projections from $Z$ to $Y$ and to $Y'$ respectively. Since the pullbacks of the curvings $C$ and $C'$ to $Z$ differ by a closed 2-form, the curvatures of stably isomorphic gerbes, defined by the differential of $C$, are equal.

8

**Definition 4.** *Let $\mathcal{G}$ and $\mathcal{G}'$ be two gerbes, and $\mathcal{A}_1$ and $\mathcal{A}_2$ two stable iso-morphisms from $\mathcal{G}$ to $\mathcal{G}'$. A morphism*

$$\beta \; : \; \mathcal{A}_1 \; \implies \; \mathcal{A}_2 \tag{14}$$

*is an isomorphism $\beta : A_1 \to A_2$ of line bundles over $Z$, which is compatible with $\alpha_1$ and $\alpha_2$ in the sense that the diagram*

$$
\begin{array}{ccc}
p^*L \otimes p'^*L'^* \otimes \pi_2^*A_1 & \xrightarrow{\quad \alpha_1 \quad} & \pi_1^*A_1 \\
{\scriptstyle 1 \otimes 1 \otimes \pi_2^*\beta} \Big\downarrow & & \Big\downarrow {\scriptstyle \pi_1^*\beta} \\
p^*L \otimes p'^*L'^* \otimes \pi_2^*A_2 & \xrightarrow{\quad \alpha_2 \quad} & \pi_1^*A_2
\end{array}
\tag{15}
$$

*of isomorphisms of line bundles over $Z^{[2]}$ commutes.*

The definition of such a morphism of stable isomorphisms already ap-peared in [Ste00]. We call two stable isomorphisms equivalent, if there is a morphism between them. This defines an equivalence relation on the set of stable isomorphisms between two fixed gerbes $\mathcal{G}$ and $\mathcal{G}'$.

## 2.2   Jandl Structures

Recall that for a group $K$ acting on a manifold $M$ by diffeomorphisms $k : M \to M$, a $K$-equivariant structure on a line bundle $L \to M$ is a family $\left\{\varphi^k\right\}_{k \in K}$ of isomorphisms

$$\varphi^k \; : \; k^*L \; \longrightarrow \; L \tag{16}$$

of line bundles, which respect the group structure of $K$ in the sense that $\varphi^1 : L \to L$ is the identity, and the multiplication law

$$\varphi^{k_1 k_2} = \varphi^{k_2} \circ k_2^* \varphi^{k_1} \tag{17}$$

is satisfied. Remember that according to our convention in Remark 1 all line bundles have connections, and all isomorphisms of line bundles preserve them. In this article, we only consider the group $K = \mathbb{Z}_2$ for the sake of simplicity.

Let $\mathcal{G}$ be a gerbe over $M$ and let $K = \mathbb{Z}_2$ act on $M$. Denote the action of the non-trivial element $k$ by $k : M \to M$. Assume that there is a stable isomorphism $\mathcal{A} = (A, \alpha) : k^*\mathcal{G} \to \mathcal{G}^*$. Recall that in this particular situation, $A$ is a line bundle over the space $Z = Y_k \times_M Y$, where $Y_k := Y$ and $\pi_k := k^{-1} \circ \pi$ as in our discussion of the pullback of $\mathcal{G}$ by a diffeomorphism $k$. We still denote the projections from $Z$ to $Y$ and to $Y_k$ by $p$ and $p'$ respectively.

Define the surjective submersion $\pi_Z := \pi \circ p : Z \to M$. As $k^2 = \mathrm{id}_M$, the permutation map

$$\tilde{k} \;:\; Z \;\longrightarrow\; Z \;:\; (y_k, y) \;\longmapsto\; (y, y_k) \tag{18}$$

gives the following commuting diagram:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \tilde{k}\ } & Z \\
{\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle \pi_Z} \\
M & \xrightarrow[\ k\ ]{} & M
\end{array}
\tag{19}
$$

Furthermore, since also $\tilde{k}^2 = \mathrm{id}_Z$, we even have a lift of the action of $K$ into $Z$.

**Definition 5.** *A Jandl structure on $\mathcal{G}$ is a collection $\mathcal{J} = (k, \mathcal{A}, \varphi)$ consisting of*

- *a smooth action of $K = \mathbb{Z}_2$ on $M$, where we denote the non-trivial element and the diffeomorphism associated to that non-trivial element by $k : M \to M$.*

- *a stable isomorphism of gerbes $\mathcal{A} = (A, \alpha) : k^*\mathcal{G} \to \mathcal{G}^*$.*

- *a $K$-equivariant structure $\varphi := \varphi^k$ on the line bundle $A$, which is compatible with the stable isomorphism $\mathcal{A}$ in the sense that the diagram*

$$
\begin{array}{ccc}
p'^*L \otimes p^*L \otimes \pi_2^*A & \xrightarrow{\quad\quad \alpha \quad\quad} & \pi_1^*A \\
{\scriptstyle 1\otimes 1\otimes \pi_2^*\varphi}\downarrow & & \downarrow{\scriptstyle \pi_1^*\varphi} \\
p'^*L \otimes p^*L \otimes k^*\pi_2^*A & \xrightarrow{\quad k^*\alpha \quad} & k^*\pi_1^*A
\end{array}
\tag{20}
$$

*of isomorphisms of line bundles over $Z^{[2]}$ commutes.*

We can immediately deduce a necessary condition for the existence of a Jandl structure for a given gerbe $\mathcal{G}$, namely the condition, that the gerbes $k^*\mathcal{G}$ and $\mathcal{G}^*$ are stably isomorphic. Since the curvatures of stably isomorphic gerbes are equal, this in turn demands

$$k^*H = -H \tag{21}$$

for the curvature $H = \mathrm{curv}(\mathcal{G})$ of $\mathcal{G}$. In particular, there will be gerbes on manifolds with involution which do not admit a Jandl structure.

10

**Definition 6.** *Two Jandl structures $\mathcal{J}$ and $\mathcal{J}'$ on the same gerbe $\mathcal{G}$ are equivalent, if the following conditions are satisfied:*

- *the actions are the same, i.e. $k$ and $k'$ are the same diffeomorphisms,*

- *there is a morphism $\beta : \mathcal{A} \Rightarrow \mathcal{A}'$ of stable isomorphisms in the sense of Definition 4 such that*

- *$\beta : A \to A'$ is even an isomorphism of $K$-equivariant line bundles on $Z$.*

Next, we show that Jandl structures behave well under the pullback of gerbes along a smooth map $f : N \to M$. Let $\mathcal{J} = (k, \mathcal{A}, \varphi)$ be a Jandl structure on $\mathcal{G}$. Assume, that there is an action of $K = \mathbb{Z}_2$ on $N$ by a diffeomorphism $g$, such that the diagram

$$
\begin{array}{ccc}
N & \xrightarrow{\ f\ } & M \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\
N & \xrightarrow[\ f\ ]{} & M
\end{array}
\tag{22}
$$

commutes. Consider the pullback of $\mathcal{G}$ by $f$ as discussed before, and define

$$
Z_f := (Y_f)_g \times_N Y_f
\tag{23}
$$

and the permutation map $\tilde{g} : Z_f \to Z_f$. Then

$$
\tag{24}
$$

is a cube with commuting faces. It follows that $f^*\mathcal{A} := (\tilde{f}^*A, \tilde{f}^*\alpha)$ is a stable isomorphism from $g^*f^*\mathcal{G}$ to $f^*\mathcal{G}^*$. Furthermore, $\tilde{f}^*\varphi$ is a $K$-equivariant structure on $\tilde{f}^*A$, where $K$ acts by $\tilde{g}$. In summary,

$$
f^*\mathcal{J} := (g, \tilde{f}^*\mathcal{A}, \tilde{f}^*\varphi)
\tag{25}
$$

defines a pullback Jandl structure on $f^*\mathcal{G}$.

## 2.3 Classification of Jandl Structures

If a gerbe $\mathcal{G}$ admits a Jandl structure, it is natural to ask, how many inequivalent choices exist. So we are interested in the set $\mathrm{Jdl}(\mathcal{G}, k)$ of equivalence classes of Jandl structures $\mathcal{J} = (k, -, -)$ with a fixed action of $K = \mathbb{Z}_2$ via $k$. This will be crucial in the discussion of the unoriented WZW model in section 4.

To approach this task, we first investigate the set $\mathrm{Hom}(\mathcal{G}, \mathcal{G}')$ of equivalence classes of stable isomorphisms between $\mathcal{G}$ and $\mathcal{G}'$. We start by recalling the following

**Lemma 1** ([CJM02]).

(i) *If $N \to M$ is a flat line bundle and $\mathcal{A} = (A, \alpha)$ is a stable isomorphism, then $N.\mathcal{A} := (A \otimes \pi_Z^* N, \alpha \otimes 1)$ is also a stable isomorphism.*

(ii) *If $\mathcal{A}_1 = (A_1, \alpha_1)$ and $\mathcal{A}_2 = (A_2, \alpha_2)$ are two stable isomorphisms, then there is a unique flat line bundle $N \to M$ such that $\mathcal{A}_1$ and $N.\mathcal{A}_2$ are equivalent as stable isomorphisms.*

Proof. For the first part we note that because $N$ is flat, $A$ and $A \otimes \pi_Z^* N$ have the same curvature, so that (12) is satisfied. For the second part, we use the isomorphism

$$\alpha_1^{-1} \otimes \alpha_2^* \ : \ \pi_1^*(A_1 \otimes A_2^*) \ \longrightarrow \ \pi_2^*(A_1 \otimes A_2^*) \tag{26}$$

which satisfies the cocycle condition because of the compatibility of $\alpha_1$ and $\alpha_2$ with $\mu$ and $\mu'$. This determines a unique line bundle $N \to M$ with connection together with an isomorphism $\nu : \pi_Z^* N \to A_1 \otimes A_2^*$. Because (12) requires the curvatures of both $A_1$ and $A_2$ to be the same, $N$ is flat. Now $\nu$ determines an isomorphism $A_1 \to A_2 \otimes \pi_Z^* N$, which is a morphism $\mathcal{A}_1 \Rightarrow N.\mathcal{A}_2$. $\square$

We denote the group of isomorphism classes of flat line bundles over $M$ by $\mathrm{Pic}_0(M)$. It is a subgroup of the Picard group $\mathrm{Pic}(M)$ of isomorphism classes of hermitian line bundles with connection over $M$.

**Lemma 2.** *The set $\mathrm{Hom}(\mathcal{G}, \mathcal{G}')$ of equivalence classes of stable isomorphisms is a torsor over the flat Picard group $\mathrm{Pic}_0(M)$.*

Proof. We will (a) define the action and show, that it is (b) transitive and (c) free.

(a) We act $[N].[\mathcal{A}] := [N.\mathcal{A}]$, where the right hand side was defined in Lemma 1 (i). This definition is independent of the choice of representatives $N$ and $\mathcal{A}$: an isomorphism $N \to N'$ gives an isomorphism

$N.A \to N'.A$, which in fact is a morphism of stable isomorphisms $N.\mathcal{A} \Rightarrow N'.\mathcal{A}$. On the other hand, a morphism $\mathcal{A} \Rightarrow \mathcal{A}'$ of stable isomorphisms induces a morphism $N.\mathcal{A} \Rightarrow N.\mathcal{A}'$.

Because $N.\mathcal{A}$ is defined using the group structure on the group of isomorphism classes of line bundles with connection, it respects the group structure on $\mathrm{Pic}_0(M)$, and hence defines an action.

(b) The transitivity follows directly from Lemma 1 (ii).

(c) Let $[\mathcal{A}]$ be an element in $\mathrm{Hom}(\mathcal{G}, \mathcal{G}')$, let $N$ be a flat line bundle and let us assume that $N.\mathcal{A}$ and $\mathcal{A}$ are equivalent, in particular $A \otimes \pi_Z^* N$ is isomorphic to $A$. Since $N$ is unique by Lemma 1 (ii), it is the trivial line bundle. Hence the action is free. $\qquad\square$

This lemma allows us to make use of the flat Picard group $\mathrm{Pic}_0(M)$. Remember that line bundles are, according to our convention in Remark 1, line bundles with connection. It is well understood [Bry93], that the Picard group $\mathrm{Pic}(M)$ of isomorphism classes of line bundles fits into the exact sequence

$$0 \longrightarrow \mathrm{H}^1(M, U(1)) \longrightarrow \mathrm{Pic}(M) \xrightarrow{\mathrm{curv}} \Omega^2(M) \ . \tag{27}$$

In particular this means $\mathrm{Pic}_0(M) \cong \mathrm{H}^1(M, U(1))$. This cohomology group can be computed using the universal coefficient theorem

$$0 \longrightarrow \mathrm{Ext}(\mathrm{H}_0(M), U(1)) \longrightarrow \mathrm{H}^1(M, U(1)) \longrightarrow \mathrm{Hom}(\mathrm{H}_1(M), U(1)) \longrightarrow 0 \tag{28}$$

If $M$ is connected, the Ext-group is trivial and we obtain

$$\mathrm{Pic}_0(M) \cong \mathrm{Hom}(\pi_1(M), U(1)). \tag{29}$$

An equivariant version of Lemma 2 applies to Jandl structures. We denote the group of isomorphism classes of flat $K$-equivariant line bundles by $\mathrm{Pic}_0^K(M)$ and call it the flat $K$-equivariant Picard group. In this equivalence relation isomorphisms are isomorphisms of equivariant line bundles with connection.

**Theorem 1.** *The set* $\mathrm{Jdl}(\mathcal{G}, k)$ *of equivalence classes of Jandl structures on $\mathcal{G}$ with involution $k$ is a torsor over the flat $K$-equivariant Picard group* $\mathrm{Pic}_0^K(M)$.

Proof.

(a) We first describe the action of a flat line bundle $N$ over $M$ with equivariant structure $\nu$ on a Jandl structure $\mathcal{J} = (k, \mathcal{A}, \varphi)$. According to diagram (19), $\pi_Z^* \nu : \pi_Z^* N \to \tilde{k}^* \pi_Z^* N$ is a $K$-equivariant structure on $\pi_Z^* N$. Now, by taking the tensor product of $A$ and $\pi_Z^* N$ as $K$-equivariant line bundles, we obtain an equivariant structure $\varphi \otimes \pi_Z^* \nu$ on the line bundle of $N.\mathcal{A}$. So we define

$$N.\mathcal{J} := (k, N.\mathcal{A}, \varphi \otimes \pi_Z^* \nu). \tag{30}$$

Since

$$\begin{array}{ccc} Z^{[2]} & \xrightarrow{\pi_1} & Z \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle \pi_Z} \\ Z & \xrightarrow{\pi_Z} & M \end{array} \tag{31}$$

commutes, we have $\pi_1^* \pi_Z^* \nu = \pi_2^* \pi_Z^* \nu$. This shows that condition (20) for Jandl structures is satisfied for $N.\mathcal{J}$. The arguments in the proof of Lemma 2 (a) apply here too and show that this defines an action on equivalence classes.

(b) Let two equivalence classes of Jandl structures be represented by $\mathcal{J}_1$ and $\mathcal{J}_2$. We already know from Lemma 1 (ii) that there is a flat line bundle $N \to M$ together with an isomorphism $\beta : A_1 \to A_2 \otimes \pi_Z^* N$, which is a morphism of stable isomorphism $\beta : N.\mathcal{A}_1 \Rightarrow \mathcal{A}_2$. We have to show that there is an equivariant structure on $N$ such that $\beta$ is an isomorphism of equivariant line bundles. Remember that we defined $N$ by a descent isomorphism $\alpha_1^{-1} \otimes \alpha_2^*$ in (26). Because the equivariant structures on $A_1$ and $A_2$ are compatible with $\alpha_1$ and $\alpha_2$ respectively due to the property (20) of Jandl structures, the descent isomorphism is an isomorphism of equivariant line bundles. Thus $N$ is an equivariant line bundle, and $\beta$ is an isomorphism of equivariant line bundles.

(c) Let $\mathcal{J} = (k, \mathcal{A}, \varphi)$ represent a Jandl structure on $\mathcal{G}$, and let $N$ be a flat line bundle over $M$ with equivariant structure $\nu$, such that $N.\mathcal{J}$ and $\mathcal{J}$ are equivalent. It follows from Lemma 2 that $N$ is the trivial line bundle. Furthermore, $\pi_Z^* \nu$ is the trivial equivariant structure on $\pi_Z^* N$, so that $\nu$ is the trivial equivariant structure on $N$. $\square$

For an action of a discrete group $K$ on $M$, an equivariant version of the sequence (27) is derived in [Gom03], namely

$$0 \longrightarrow \mathrm{H}_K^1(M, U(1)) \longrightarrow \mathrm{Pic}^K(M) \xrightarrow{\mathrm{curv}} \Omega^2(M)^K. \tag{32}$$

14

Here, $\mathrm{H}_K^1(M, U(1))$ is the equivariant cohomology of $M$, i.e. the cohomology of the associated Borel space. In particular, we get for flat equivariant line bundles

$$\mathrm{Pic}_0^K(M) \cong \mathrm{H}_K^1(M, U(1)). \tag{33}$$

## 2.4 Local Data

Let $\mathcal{G}$ be a gerbe over $M$ and $\mathfrak{V} = \{V_i\}_{i \in I}$ be a good open cover of $M$. Let $M_{\mathfrak{V}}$ be the disjoint union of all the $V_i$'s. The $p$-fold fiber product of $M_{\mathfrak{V}}$ over $M$ is just the disjoint union of all $p$-fold intersections of the $V_i$'s. Recall from [CJM02] how to extract local data from $\mathcal{G}$:

A choice of local sections $s_i : V_i \to Y$ gives a fiber preserving map $s : M_{\mathfrak{V}} \to Y$ by $(x, i) \mapsto s_i(x)$. Pull back the line bundle $L \to Y^{[2]}$ with its connection $\nabla$ along $s$ to a line bundle on the double intersections, and choose local sections $\sigma_{ij} : V_i \cap V_j \to s^*L$. Pull back the isomorphism $\mu$ of the gerbe, too. Then define local data, namely smooth functions $g_{ijk} : V_i \cap V_j \cap V_k \to U(1)$, real-valued 1-forms $A_{ij} \in \Omega^1(V_i \cap V_j)$ and 2-forms $B_i \in \Omega^2(V_i)$ by the following relations

$$s^*\mu \left( \pi_{12}^* \sigma_{ij} \otimes \pi_{23}^* \sigma_{jk} \right) = g_{ijk} \cdot \pi_{13}^* \sigma_{ik} \tag{34}$$

$$s^*\nabla(\sigma_{ij}) = \frac{1}{\mathrm{i}} A_{ij} \otimes \sigma_{ij} \tag{35}$$

$$B_i = s_i^*C. \tag{36}$$

These local data give elements $g$, $A$, $B$ in the Čech-Deligne double complex for the cover $\mathfrak{V}$, and the cochain $(g, A, B)$ satisfies the Deligne cocycle condition

$$\mathrm{D}\,(g, A, B) = (1, 0, 0)\,, \tag{37}$$

or equivalently in components

$$g_{jkl} \cdot g_{ikl}^{-1} \cdot g_{ijl} \cdot g_{ijk}^{-1} = 1 \tag{38}$$

$$A_{jk} - A_{ik} + A_{ij} + \mathrm{dlog}\,(g_{ijk}) = 0 \tag{39}$$

$$-\mathrm{d}A_{ij} + B_j - B_i = 0. \tag{40}$$

Furthermore, it satisfies

$$\mathrm{d}B_i = H|_{V_i}, \tag{41}$$

where the 3-form $H$ is the curvature of the gerbe.

The dual gerbe and the pullback gerbe $f^*\mathcal{G}$ along some map $f : N \to M$ can be conveniently expressed in local data as follows: by choosing the same $s_i$ and the dual sections $\sigma_{ij}^*$, one gets $(g^{-1}, -A, -B) = -(g, A, B)$ as

15

local data of $\mathcal{G}^*$. Furthermore, if we induce a cover $\{f^{-1}V_i\}_{i \in I}$ of $N$, and choose the pullback sections $f^*s_i$ and $\tilde{f}^*\sigma_{ij}$, then we obtain $(f^*g, f^*A, f^*B) = f^*(g, A, B)$ as local data of $f^*\mathcal{G}$.

We next need to derive local data of trivializations and stable isomorphisms. So, let $\mathcal{T} = (T, \tau)$ be a trivialization of $\mathcal{G}$. Since $T$ is a line bundle over $Y$, we can pull it back with $s : M_{\mathfrak{V}} \to Y$ to a line bundle over the open subsets, and choose local sections $\sigma_i : V_i \to s^*T$. We also pull back the isomorphism $\tau$ to an isomorphism

$$s^*\tau \quad : \quad s^*L \otimes \pi_2^* s^*T \quad \longrightarrow \quad \pi_1^* s^*T. \tag{42}$$

Then we obtain smooth functions $h_{ij} : V_i \cap V_j \to U(1)$ by

$$s^*\tau \left( \sigma_{ij} \otimes \pi_2^* \sigma_j \right) = h_{ij} \cdot \pi_1^* \sigma_i. \tag{43}$$

Let $\blacktriangledown$ be the connection of $T$. It defines connection 1-forms $M_i \in \Omega^1(V_i)$ by

$$s^*\blacktriangledown (\sigma_i) = \frac{1}{i} M_i \otimes \sigma_i. \tag{44}$$

The local data $h$ and $M$ are again elements in the Čech-Deligne double complex. Now the compatibility of $\tau$ and $\mu$ in Definition 2 is equivalent to

$$g_{ijk} = h_{ij} \cdot h_{ik}^{-1} \cdot h_{jk}, \tag{45}$$

and the condition, that the isomorphism $\tau$ respect connections, is equivalent to

$$A_{ij} = -\mathrm{dlog}\left(h_{ij}\right) + M_j - M_i. \tag{46}$$

Furthermore, the local 2-form $\rho = B_i + \mathrm{d}M_i$ coincides with the 2-form $\rho$ obtained from Definition 2. The last three properties of $h$ and $M$ are equivalent to the Deligne coboundary equation

$$(g, A, B) = (1, 0, \rho) + \mathrm{D}\left(h, M\right). \tag{47}$$

Now consider a stable isomorphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ of gerbes over $M$. With respect to the good open cover $\{V_i\}_{i \in I}$ we may have chosen local sections $s_i$, $\sigma_{ij}$ and $s_i'$, $\sigma_{ij}'$ to get local data $(g, A, B)$ and $(g', A', B')$ of $\mathcal{G}$ and $\mathcal{G}'$ respectively. We construct a map

$$\tilde{s} \quad : \quad M_{\mathfrak{V}} \quad \longrightarrow \quad Y \times_M Y' \quad : \quad (x, i) \quad \longmapsto \quad (s_i(x), s_i'(x)), \tag{48}$$

and pull the line bundle $A \to Y \times_M Y'$ of the stable isomorphism together with its connection $\blacktriangledown$ back to $M_{\mathfrak{V}}$. We also pull back the isomorphism $\alpha$ and get an isomorphism

$$\tilde{s}^*\alpha \quad : \quad s^*L \otimes s'^*L'^* \otimes \pi_2^* \tilde{s}^*A \quad \longrightarrow \quad \pi_1^* \tilde{s}^*A. \tag{49}$$

16

Then we choose local sections $\sigma_i : V_i \to \tilde{s}^*A$. We obtain local data in form of smooth functions $t_{ij} : V_i \cap V_j \to U(1)$ and connection 1-forms $W_i \in \Omega^1(V_i)$ by the following relations:

$$\tilde{s}^*\alpha \left( \sigma_{ij} \otimes \sigma_{ij}'^* \otimes \pi_2^*\sigma_j^* \right) \;=\; t_{ij} \cdot \pi_1^*\sigma_i \tag{50}$$

$$\tilde{s}^*\blacktriangledown(\sigma_i) \;=\; \frac{1}{i}W_i \otimes \sigma_i. \tag{51}$$

Note that the functions $t_{ij}$ are not transition functions of some bundle but are defined by the isomorphism $\alpha$.

These local data $t$ and $W$ are elements in the Čech-Deligne double complex. The compatibility of $\alpha$ with the isomorphisms $\mu$ and $\mu'$ of both gerbes as isomorphisms of hermitian line bundles with connection according to Definition 3 is equivalent to

$$g_{ijk} \cdot g_{ijk}'^{-1} \;=\; t_{jk} \cdot t_{ik}^{-1} \cdot t_{ij} \tag{52}$$

$$A_{ij} - A_{ij}' \;=\; -\mathrm{dlog}(t_{ij}) + W_j - W_i \tag{53}$$

while the condition (12) on the curvature of $A$ is equivalent to

$$B_i - B_i' = \mathrm{d}W_i. \tag{54}$$

The three last equations are in turn equivalent to the Deligne coboundary equation

$$(g, A, B) - (g', A', B') = \mathrm{D}\left(t, W\right). \tag{55}$$

This formalism of local data reproduces results on bundle gerbes and their stable isomorphisms, for example Lemma 1 (ii). Consider again two gerbes $\mathcal{G}$ and $\mathcal{G}'$, and now two stable isomorphisms $\mathcal{A}_1$ and $\mathcal{A}_2$ both from $\mathcal{G}$ to $\mathcal{G}'$. We may have extracted local data $(t_1, W_1)$ of $\mathcal{A}_1$ and $(t_2, W_2)$ of $\mathcal{A}_2$ such that equation (55) holds for both. It follows

$$\mathrm{D}(t \cdot t'^{-1}, W - W') = (1, 0, 0), \tag{56}$$

which is the Deligne cocycle condition for a flat hermitian line bundle over $M$. This is the bundle $N$ constructed in Lemma 1 (ii).

We are now in a position to derive the local data of a Jandl structure $\mathcal{J} = (k, \mathcal{A}, \varphi)$ on a gerbe $\mathcal{G}$. Recall that $k : M \to M$ is the action of the non-trivial element of $K = \mathbb{Z}_2$ acting on $M$, in particular $k^2 = \mathrm{id}_M$. We simplify the situation by considering an open cover $\mathfrak{V} = \{V_i\}_{i \in I}$ of $M$, which is invariant under $k$, i.e. $k(V_i) = V_i$, and which is still good enough to enable us to extract local data. The generalization other covers is straightforward, but makes the notation somewhat more cumbersome.

Recall further that $\mathcal{A}$ is a stable isomorphism from $k^*\mathcal{G} \to \mathcal{G}^*$. Let $(t, W)$ be local data of $\mathcal{A}$, obtained by pulling back the line bundle $A \to Z$ by $\tilde{s} : M_{\mathfrak{Y}} \to Z$ from equation (48) and choosing local sections $\sigma_i : V_i \to \tilde{s}^*A$. As we derived for the local data of the dual gerbe and the pullback gerbe, equation (55) here appears as

$$k^*(g, A, B) = -(g, A, B) + \mathrm{D}(t, W), \tag{57}$$

or equivalently:

$$\begin{align}
k^*B_i &= -B_i + \mathrm{d}W_i \tag{58} \\
k^*A_{ij} &= -A_{ij} - \mathrm{dlog}(t_{ij}) + W_j - W_i \tag{59} \\
k^*g_{ijk} &= g_{ijk}^{-1} \cdot t_{jk} \cdot t_{ik}^{-1} \cdot t_{ij} \tag{60}
\end{align}$$

Now recall that a part of a Jandl structure is a $K$-equivariant structure $\varphi : k^*A \to A$ on $A$. By pullback with $\tilde{s}$, we obtain

$$\tilde{s}^*\varphi \quad : \quad k^*\tilde{s}^*A \quad \longrightarrow \quad \tilde{s}^*A. \tag{61}$$

Now, because $\sigma_i$ is a section of $\tilde{s}^*A$, $k^*\sigma_i = \sigma_i \circ k$ is a section of $k^*\tilde{s}^*A$ on the same patch $V_i$, since the latter is invariant under $k$. This allows us to extract a local $U(1)$-valued functions $j_i : V_i \to U(1)$, defined by

$$\tilde{s}^*\varphi(\sigma_i) = j_i \cdot \sigma_i \circ k. \tag{62}$$

The compatibility of $\varphi$ with $\alpha$ in the sense of diagram (20) is equivalent to

$$k^*(t, W) = (t, W) - \mathrm{D}(j), \tag{63}$$

or in turn equivalently

$$\begin{align}
k^*W_i &= W_i - \mathrm{dlog}(j_i) \tag{64} \\
k^*t_{ij} &= t_{ij} \cdot j_j^{-1} \cdot j_i. \tag{65}
\end{align}$$

By definition of an equivariant structure, the $K = \mathbb{Z}_2$ group law (17) is satisfied. In terms of local data, this is equivalent to

$$k^*j_i = j_i^{-1}. \tag{66}$$

In summary, the Jandl structure $\mathcal{J} = (k, \mathcal{A}, \varphi)$ gives rise to local data $(t, W)$ and $j$ which satisfy the following three conditions:

$$\begin{align}
k^*(g, A, B) &= -(g, A, B) + \mathrm{D}(t, W) \tag{67} \\
k^*(t, W) &= (t, W) - \mathrm{D}(j) \tag{68} \\
k^*j_i &= j_i^{-1} \tag{69}
\end{align}$$

18

Again, using local data, we can reproduce results on Jandl structures like Theorem 1. In detail, let $\mathcal{J}$ be a Jandl structure on $\mathcal{G}$ with local data $(t, W)$ and $j$. Let $N$ be a flat $K$-equivariant hermitian line bundle over $M$ with transition functions $n_{ij} : V_i \cap V_j \to U(1)$ and local connection 1-forms $N_i \in \Omega^1(V_i)$ with

$$\mathrm{D}(n, N) = (1, 0, 0). \tag{70}$$

The equivariant structure on $N$ determines smooth functions $\nu_i : V_i \to U(1)$ with

$$k^*(n, N) = (n, N) - \mathrm{D}(\nu) \tag{71}$$

and $k^*\nu = \nu^{-1}$. Then,

$$
\begin{aligned}
(t', W') &:= (t, W) + (n, N) \tag{72} \\
j' &:= j \cdot \nu \tag{73}
\end{aligned}
$$

are local data of the Jandl structure $N.\mathcal{J}$. Indeed, equation (67) is satisfied because of the Deligne cocycle condition (70). Compute

$$
\begin{aligned}
k^*(t', W') &= k^*(t, W) + k^*(n, N) \\
&= (t, W) - \mathrm{D}(j) + (n, N) - \mathrm{D}(\nu) \\
&= (t', W') - \mathrm{D}(j'), \tag{74}
\end{aligned}
$$

this is equation (68), and the last equation (69) for $j'$ is just a consequence from the conditions on $j$ and $\nu$.

Let now $\mathcal{J}$ and $\mathcal{J}'$ be two Jandl structures on $\mathcal{G}$ with local data $(t, W), j$ and $(t', W'), j'$ respectively.

$$(n, N) := (t, W) - (t', W') \tag{75}$$

are the local data of the flat descent line bundle $N$, and using equation (67), we get its cocycle condition

$$\mathrm{D}(n, N) = (1, 0, 0). \tag{76}$$

Now compute

$$
\begin{aligned}
k^*(n, N) &= k^*(t, W) - k^*(t', W') \\
&= (t, W) - \mathrm{D}(j) - (t', W') + \mathrm{D}(j') \\
&= (n, N) - \mathrm{D}(\nu), \tag{77}
\end{aligned}
$$

where we defined $\nu := j \cdot j'^{-1}$. Hence, $N$ and $k^*N$ are isomorphic as hermitian line bundles with connection via an isomorphism represented by $\nu$. By definition, we have $k^*\nu = \nu^{-1}$, this means, that $\nu$ is a $K$-equivariant structure.

# 3 Holonomy of Gerbes with Jandl Structure

## 3.1 Double Coverings, Fundamental Domains and Orientations

Let us first recall the setup that allows to define holonomy around closed oriented surfaces. This is a gerbe $\mathcal{G}$ over $M$ and a closed oriented surface $\Sigma$ together with a smooth map $\phi : \Sigma \to M$. Following [CJM02], we pull back $\mathcal{G}$ along $\phi$ to a gerbe over $\Sigma$. For dimensional reasons, $\phi^*\mathcal{G}$ is trivial. As explained in section 2.1, a trivialization $\mathcal{T}$ determines a 2-form $\rho \in \Omega^2(\Sigma)$, while another trivialization $\mathcal{T}'$ determines a 2-form $\rho' = \rho + \mathrm{curv}(N)$. Since $\mathrm{curv}(N)$ defines an integral class in cohomology, we have

$$\int_\Sigma \rho' = \int_\Sigma \rho \mod 2\pi\mathbb{Z}. \tag{78}$$

So the integral is independent of the choice of a trivialization up to $2\pi\mathbb{Z}$, and admits therefore the following

**Definition 7.** *The holonomy of $\mathcal{G}$ around the closed oriented surface $\phi : \Sigma \to M$ is defined as*

$$\mathrm{hol}_\mathcal{G}(\phi, \Sigma) := \exp\left(\mathrm{i}\int_\Sigma \rho\right) \in U(1). \tag{79}$$

We state three important properties of this definition:

- The dual gerbe has inverse holonomy,

$$\mathrm{hol}_\mathcal{G}(\phi, \Sigma) = \mathrm{hol}_{\mathcal{G}^*}(\phi, \Sigma)^{-1}. \tag{80}$$

- If $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ is a stable isomorphism, we have

$$\mathrm{hol}_\mathcal{G}(\phi, \Sigma) = \mathrm{hol}_{\mathcal{G}'}(\phi, \Sigma). \tag{81}$$

- By $\bar{\Sigma}$ we denote the same manifold $\Sigma$ with the opposite orientation; then we obtain

$$\mathrm{hol}_\mathcal{G}(\phi, \Sigma) = \mathrm{hol}_\mathcal{G}(\phi, \bar{\Sigma})^{-1}. \tag{82}$$

Obviously, the orientation on $\Sigma$ is essential for this definition. In this section we will define the holonomy around unoriented or even unorientable surfaces. The most important property of this definition will be, that it reduces to Definition 7 if $\Sigma$ is orientable and an orientation is chosen. One of the main tools will be an orientation covering.

Let $\Sigma$ be a smooth manifold (without orientation).

**Definition 8.** *An orientation covering of $\Sigma$ is a double covering* $\mathrm{pr} : \hat{\Sigma} \to \Sigma$ *with an oriented manifold $\hat{\Sigma}$, such that the canonical involution $\sigma : \hat{\Sigma} \to \hat{\Sigma}$ is orientation-reversing.*

Recall three basic properties of orientation coverings (some of them can be found for example in [BG88]):

- it is unique up to orientation-preserving diffeomorphisms of covering spaces.

- the canonical involution $\sigma : \hat{\Sigma} \to \hat{\Sigma}$ preserves fibers and permutes the the sheets.

- under the assumption that $\Sigma$ is connected, $\hat{\Sigma}$ is connected if and only if $\Sigma$ is not orientable.

Due to the first point, by $\hat{\Sigma}$ we will from now refer to this unique orientation cover. Let $k : M \to M$ be an involution on $M$. By $C^\infty(\hat{\Sigma}, M)^{\sigma,k}$ we denote the space of smooth maps $\hat{\phi} : \hat{\Sigma} \to M$ for which the diagram

$$
\begin{array}{ccc}
\hat{\Sigma} & \overset{\hat{\phi}}{\longrightarrow} & M \\
{\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle k} \\
\hat{\Sigma} & \underset{\hat{\phi}}{\longrightarrow} & M
\end{array}
\tag{83}
$$

commutes in the category of smooth manifolds (neglecting orientations).

Let $\Sigma$ be orientable.

**Lemma 3.** *An orientation on $\Sigma$ defines a bijection*

$$
C^\infty(\hat{\Sigma}, M)^{\sigma,k} \quad \longrightarrow \quad C^\infty(\Sigma, M).
\tag{84}
$$

Proof. Since $\Sigma$ is orientable, $\hat{\Sigma}$ consists of two disjoint copies of $\Sigma$ with opposite orientations. An orientation on $\Sigma$ is a global section $\mathrm{or} : \Sigma \to \hat{\Sigma}$ in the covering $\mathrm{pr} : \hat{\Sigma} \to \Sigma$. Now let $\hat{\phi} : \hat{\Sigma} \to M$ be a map. Define its image as $\phi := \hat{\phi} \circ \mathrm{or}$. On the other hand, given a map $\phi : \Sigma \to M$, we define the preimage $\hat{\phi}$ on the two sheets of $\hat{\Sigma}$ separately as $\hat{\phi}|_{\mathrm{or}(\Sigma)} := \phi$ and $\hat{\phi}|_{\sigma \mathrm{or}(\Sigma)} := k \circ \phi$ respectively. $\qquad\square$

If $\Sigma$ is not orientable or no orientation of $\Sigma$ is chosen, we will make use of the following generalization of an orientation.

**Definition 9.** *A fundamental domain for $\Sigma$ in $\hat{\Sigma}$ is a submanifold $F \subset \hat{\Sigma}$ possibly with (piecewise smooth) boundary, satisfying the following two conditions as sets:*

*(i)* $F \cap \sigma(F) = \partial F$

*(ii)* $F \cup \sigma(F) = \hat{\Sigma}$

This is a generalization of an orientation on $\Sigma$ in the sense, that any orientation on $\Sigma$ gives a global section or $: \Sigma \to \hat{\Sigma}$ which in turn defines a fundamental domain, namely $F := \text{or}(\Sigma)$, one of the two copies of $\Sigma$ in $\hat{\Sigma}$.

We show the existence of such a fundamental domain for an arbitrary closed surface $\Sigma$ by an explicit construction, which we will also use in section 3.3. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of $\Sigma$, which admits local sections $\text{or}_i : U_i \to \hat{\Sigma}$. One can think of such sections as local orientations. Choose a dual triangulation $T$ of $\Sigma$, subordinate to the cover $\mathfrak{U}$, together with a subordinating map $i : T \to I$. So, for each face $f \in T$ there is an index $i(f)$ with $f \subset U_{i(f)}$, as well as for each edge $e \in T$ and for each vertex $v \in T$. Because we have a dual triangulation, each vertex is trivalent.

Consider a common edge $e = f_1 \cap f_2$ of two faces $f_1$ and $f_2$. We call the edge $e$ orientation-preserving, if

$$\text{or}_{i(f_1)}(e) = \text{or}_{i(f_2)}(e), \tag{85}$$

otherwise we call it orientation-reversing. So the set of edges splits in a set $E$ of orientation-preserving, and a set $\bar{E}$ of orientation-reversing edges. If $v$ is a vertex, the number of orientation-reversing edges ending in $v$ must be even, and since we started with a dual triangulation, it is either zero or two. Hence, the edges in $\bar{E}$ form non-intersecting closed lines in $\Sigma$.
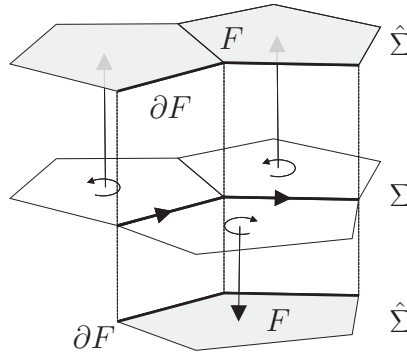


**Figure 1**: The construction of a fundamental domain by local orientations for a dual triangulation.

Define the subset

$$F := \bigcup_{f \in T} \text{or}_{i(f)}(f). \tag{86}$$

22

of $\hat{\Sigma}$ and endow it with the subspace topology. The boundary of $F$ is exactly the union of the preimages of orientation-reversing edges under the covering map,

$$\partial F = \bigcup_{e \in \bar{E}} \mathrm{pr}^{-1}(e), \tag{87}$$

and hence a disjoint union of piecewise smooth circles. This shows that $F$ is a submanifold of $\hat{\Sigma}$ with piecewise smooth boundary. It satisfies the two properties of a fundamental domain, and hence shows the existence of such a fundamental domain.

Let now $F$ be any fundamental domain for $\Sigma$ in $\hat{\Sigma}$. The following observation will be essential.

**Lemma 4.** *The quotient $\overline{\partial F} := \partial F / \sigma$ is a 1-dimensional oriented closed submanifold of $\Sigma$.*

Proof. We act with $\sigma$ on property (i) of the fundamental domain $F$:

$$\sigma(\partial F) = \sigma(F \cap \sigma(F)) = F \cap \sigma(F) = \partial F \tag{88}$$

This shows that $\sigma$ restricts to an involution on $\partial F$. Since $\sigma$ acts on $\hat{\Sigma}$ without fixed points, the quotient $\partial F / \sigma$ is a submanifold of $\Sigma$, and as $\partial F$ is closed, so is the quotient. The orientation of $\hat{\Sigma}$ induces an orientation on $F$. Because $\sigma$ is orientation-reversing, the orientation of $\sigma(F)$ is opposite to the one induced on $\sigma(F)$ as a submanifold of $\hat{\Sigma}$. Hence, $\partial F$ and $\partial(\sigma(F))$ are equal as sets as well as as oriented submanifolds. Thus $\sigma$ preserves the orientation on $\partial F$. $\qquad\square$



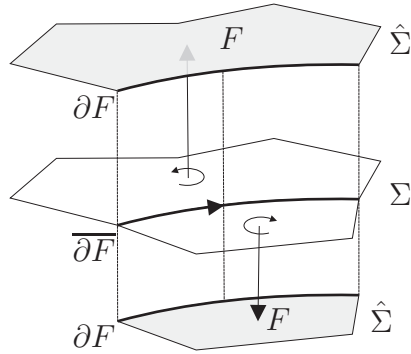**Figure 2**: The orientation on $\overline{\partial F}$.

23

## 3.2 Unoriented Surface Holonomy

The setup for the definition of holonomy around closed unoriented surfaces is

- a gerbe $\mathcal{G}$ over a smooth manifold $M$ with Jandl structure $\mathcal{J} = (k, \mathcal{A}, \varphi)$

- a closed surface $\Sigma$

- a map $\hat{\phi} \in C^\infty(\hat{\Sigma}, M)^{\sigma,k}$

The idea of the definition is the following: Pull back the gerbe $\mathcal{G}$ to $\hat{\Sigma}$ along $\hat{\phi}$, choose a trivialization and determine the 2-form $\hat{\rho} \in \Omega^2(\hat{\Sigma})$ as in Definition 7. Choose a fundamental domain $F$ for $\Sigma$ in $\hat{\Sigma}$. The integral

$$\exp i \int_F \hat{\rho} \tag{89}$$

is independent neither of the choice of the trivialization – which enters in $\hat{\rho}$ – nor of the choice of the fundamental domain $F$. The Jandl structure, however, allows to correct (89) by a boundary term in such a way that the holonomy becomes well-defined.

We will now give a detailed definition of this boundary term, and then show that it gives rise to a well-defined holonomy.

Recall that a gerbe $\mathcal{G}$ consist of the following data: a surjective submersion $\pi : Y \to M$, a line bundle $L \to Y^{[2]}$, an isomorphism $\mu$, and a 2-form $C \in \Omega^2(Y)$. Recall that the pullback gerbe $\hat{\phi}^*\mathcal{G}$ consists of a pullback

$$
\begin{array}{ccc}
Y_\phi & \xrightarrow{\tilde{\phi}} & Y \\
{\scriptstyle \pi_\phi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\hat{\Sigma} & \xrightarrow{\hat{\phi}} & M
\end{array}
\quad , \tag{90}
$$

the pullback line bundle $\tilde{\phi}^*L$, isomorphism $\tilde{\phi}^*\mu$ and 2-form $\tilde{\phi}^*C$. Accordingly, a trivialization $\mathcal{T}$ of $\hat{\phi}^*\mathcal{G}$ is a line bundle $T \to Y_\phi$ together with an isomorphism

$$\tau \; : \; \tilde{\phi}^*L \otimes \pi_{\phi_2}^*T \; \longrightarrow \; \pi_{\phi_1}^*T \tag{91}$$

of line bundles over $Y_\phi^{[2]}$. It determines a 2-form $\hat{\rho} \in \Omega^2(\hat{\Sigma})$ with

$$\pi_\phi^*\hat{\rho} = \tilde{\phi}^*C - \mathrm{curv}(T). \tag{92}$$

24

Due to the commutativity of diagram (83), $\hat{\phi}^*\mathcal{J} = (\sigma, \tilde{\phi}^*\mathcal{A}, \tilde{\phi}^*\varphi)$ is a Jandl structure on $\hat{\phi}^*\mathcal{G}$. Recall that part of the data are a line bundle $\tilde{\phi}^*A \to Z_\phi$ over the space $Z_\phi := (Y_\phi)_\sigma \times_{\hat{\Sigma}} Y_\phi$, and an isomorphism

$$\tilde{\phi}^*\alpha \; : \; p'^*\tilde{\phi}^*L \otimes p^*\tilde{\phi}^*L^* \otimes \pi_{\phi 2}^*\tilde{\phi}^*A \; \longrightarrow \; \pi_{\phi 1}^*\tilde{\phi}^*A \tag{93}$$

of line bundles over $Z_\phi^{[2]}$, where $p$ and $p'$ are the projections in

$$\begin{array}{ccc} Z_\phi & \xrightarrow{\;p\;} & Y_\phi \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle \pi_\phi} \\ Y_{\phi_\sigma} & \xrightarrow[\sigma \circ \pi_\phi]{} & \hat{\Sigma} \end{array} \quad . \tag{94}$$

Further, the action of $K$ by $\sigma$ lifts to $Z_\phi$ via the permutation map $\tilde{\sigma}$, and $\hat{\phi}^*\mathcal{J}$ contains an $K$-equivariant structure $\tilde{\phi}^*\varphi$ on $\tilde{\phi}^*A$.

Combining the trivialization with the Jandl structure, we define a line bundle

$$R := \tilde{\phi}^*A \otimes p'^*T^* \otimes p^*T^* \tag{95}$$

over $Z_\phi$. In addition, we define an isomorphism

$$r := \tilde{\phi}^*\alpha^{-1} \otimes p'^*\tau^* \otimes p^*\tau^* \; : \; \pi_{\phi 1}^*R \; \longrightarrow \; \pi_{\phi 2}^*R \tag{96}$$

of line bundles over $Z_\phi^{[2]}$. The compatibility of $\tau$ and $\alpha$ with the isomorphism $\mu$ of $\mathcal{G}$ guarantees the cocycle condition

$$\pi_{\phi 23}^*r \circ \pi_{\phi 12}^*r = \pi_{\phi 13}^*r \tag{97}$$

over $Z_\phi^{[3]}$, hence $R$ determines a unique descent line bundle $\hat{R} \to \hat{\Sigma}$, together with an isomorphism $\pi_{Z_\phi}^*\hat{R} \to R$. We shall compute the curvature of these bundles, namely

$$\begin{aligned} \mathrm{curv}\,(R) \; &\overset{(95)}{=} \; \tilde{\phi}^*\mathrm{curv}\,(A) - p'^*\mathrm{curv}\,(T) - p^*\mathrm{curv}\,(T) && (98) \\ &\overset{(12)}{=} \; p'^*(\tilde{\phi}^*C - \mathrm{curv}(T)) + p^*(\tilde{\phi}^*C - \mathrm{curv}(T)) && (99) \\ &\overset{(92)}{=} \; p'^*\pi_\phi^*\hat{\rho} + p^*\pi_\phi^*\hat{\rho} && (100) \\ &\overset{(94)}{=} \; \pi_{Z_\phi}^*(\sigma^*\hat{\rho} + \hat{\rho}). && (101) \end{aligned}$$

Hence the curvature of $\hat{R}$ is

$$\mathrm{curv}(\hat{R}) = \sigma^*\hat{\rho} + \hat{\rho}. \tag{102}$$

25

The next step is to define $\sigma$-equivariant structure on $\hat{R}$. Note that the canonical permutation of tensor products is an equivariant structure on $p'^*T^* \otimes p^*T^*$, since the permutation map $\tilde{\sigma}$ exchanges $p$ and $p'$. Together with the equivariant structure $\tilde{\phi}^*\varphi$ on $\tilde{\phi}^*A$, the tensor product (95) is the tensor product of two equivariant line bundles. By definition of a Jandl structure $\varphi$ is compatible with $\alpha$, which means that the descent isomorphism $r$ is an isomorphism of equivariant line bundles. Hence, also the descent bundle $\hat{R}$ over $\hat{\Sigma}$ is endowed with an equivariant structure.

It is a standard fact [Gom03, Bry00], that if $K$ is discrete and acts freely, a $K$-equivariant line bundle $\hat{R} \to \hat{\Sigma}$ defines a unique line bundle $Q$ on the quotient $\hat{\Sigma}/K = \Sigma$.

Now choose a fundamental domain $F$ of $\Sigma$ in $\hat{\Sigma}$.

**Definition 10.** *The holonomy of the gerbe $\mathcal{G}$ with Jandl structure $\mathcal{J}$ around the unoriented closed surface $\Sigma$ is defined as*

$$\mathrm{hol}_{\mathcal{G},\mathcal{J}}(\hat{\phi}, \Sigma) := \exp\left(\mathrm{i}\int_F \hat{\rho}\right) \cdot \mathrm{hol}_Q(\overline{\partial F})^{-1}. \tag{103}$$

In this definition, the compensating term $\mathrm{hol}_Q(\overline{\partial F})$ is the holonomy of the line bundle $Q$ around the one-dimensional closed oriented submanifold $\overline{\partial F}$.

**Theorem 2.** *The holonomy defined in Definition 10 depends neither on the choice of the fundamental domain $F$ nor on the choice of the trivialization $\mathcal{T}$.*

Proof. Let $F'$ be another fundamental domain. We define the set

$$B := \mathrm{Int}(F) \cap \sigma(\mathrm{Int}(F')), \tag{104}$$

where Int denotes the interior. As the intersection of two open sets, $B$ is open and hence a submanifold of $\hat{\Sigma}$. It contains those parts of $F$, which are not contained in $F'$ (cf. Figure 3). Because we excluded the boundaries of $F$ and $F'$, we have

$$B \cap \sigma(B) = \emptyset, \tag{105}$$

such that there is a unique section $\mathrm{or}_B : \mathrm{pr}(B) \to \hat{\Sigma}$ with image $B$.

From Figure 3, we have

$$\int_{F'} \hat{\rho} = \int_F \hat{\rho} - \int_B \hat{\rho} + \int_{\sigma(B)} \hat{\rho} = \int_F \hat{\rho} - \int_B \mathrm{curv}(\hat{R}), \tag{106}$$
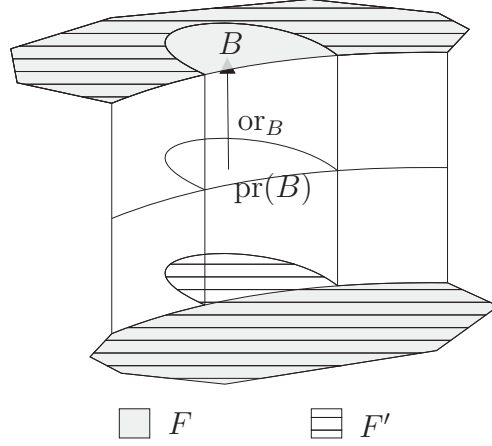
26

**Figure 3**: The difference between two fundamental domains.

since $\sigma$ is orientation-reversing. By Stoke's theorem, the exponential of the integral of the curvature of $\hat{R}$ over $B$ is nothing but the holonomy of that line bundle around $\partial B$. Thus,

$$\exp\left(-\mathrm{i}\int_B \mathrm{curv}(\hat{R})\right) = \mathrm{hol}_{\hat{R}}(\partial B)^{-1} = \mathrm{hol}_Q(\mathrm{pr}(\partial B))^{-1}.$$

This is the term which is compensated by the boundary term, which is

$$\mathrm{hol}_Q(\overline{\partial F'})^{-1} = \mathrm{hol}_Q(\overline{\partial F})^{-1} \cdot \mathrm{hol}_Q(\mathrm{pr}(\partial B)). \tag{107}$$

In summary

$$\exp\left(\mathrm{i}\int_{F'} \hat{\rho}\right) \cdot \mathrm{hol}_Q(\overline{\partial F'})^{-1} = \exp\left(\mathrm{i}\int_F \hat{\rho}\right) \cdot \mathrm{hol}_Q(\overline{\partial F})^{-1}, \tag{108}$$

i.e. the holonomy is independent of the choice of the fundamental domain.

Now let $\mathcal{T}' = (\tau', T')$ be another trivialization of $\hat{\phi}^*\mathcal{G}$. As discussed in section 2.1, there is a line bundle $N \to \hat{\Sigma}$ together with an isomorphism $\nu : \pi_\phi^* N \otimes T' \to T$, such that the 2-forms $\hat{\rho}$ and $\hat{\rho}'$ are related by

$$\hat{\rho}' = \hat{\rho} + \mathrm{curv}(N). \tag{109}$$

For the line bundle $\hat{R}$ defined in (95) this means

$$R' = R \otimes \pi_Z^* \sigma^* N \otimes \pi_Z^* N, \tag{110}$$

27

and its descent line bundle $\hat{R}'$ is

$$\hat{R}' = \hat{R} \otimes \sigma^* N \otimes N. \tag{111}$$

This is an equation of $\sigma$-equivariant line bundles, where $\hat{R}$ and $\hat{R}'$ obtain equivariant structures from the Jandl structure as described before, and $K := \sigma^* N \otimes N$ carries the canonical $\sigma$-equivariant structure by permuting the order in the tensor product. Hence, equation (111) pushes into the quotient, namely

$$Q' = Q \otimes \bar{K}. \tag{112}$$

The holonomy of the descent bundle $\bar{K}$ satisfies

$$\mathrm{hol}_{\bar{K}}(\overline{\partial F}) = \mathrm{hol}_N(\partial F) = \mathrm{hol}_{\sigma^* N}(\partial F). \tag{113}$$

This finally means

$$\exp\left(\mathrm{i}\int_F \hat{\rho}'\right) \cdot \mathrm{hol}_{Q'}(\overline{\partial F})^{-1}$$

$$\stackrel{(109)}{=} \exp\left(\mathrm{i}\int_F \hat{\rho} + \mathrm{curv}(N)\right) \cdot \mathrm{hol}_{Q \otimes \bar{K}}(\overline{\partial F})^{-1} \tag{114}$$

$$\stackrel{(113)}{=} \exp\left(\mathrm{i}\int_F \hat{\rho}\right) \cdot \mathrm{hol}_N(\partial F) \cdot \mathrm{hol}_N(\partial F)^{-1} \cdot \mathrm{hol}_Q(\overline{\partial F})^{-1} \tag{115}$$

$$= \exp\left(\mathrm{i}\int_F \hat{\rho}\right) \cdot \mathrm{hol}_Q(\overline{\partial F})^{-1}, \tag{116}$$

thus the holonomy is independent of the choice of the trivialization. $\qquad\square$

The following Lemma asserts that the definition of holonomy is compatible with the definition of equivalence of Jandl structures.

**Lemma 5.** *The holonomy of a gerbe $\mathcal{G}$ with Jandl structure $\mathcal{J}$ only depends on the equivalence class of $\mathcal{J}$.*

Proof. Let $\mathcal{J} = (k, \mathcal{A}, \varphi)$ and $\mathcal{J}' = (k, \mathcal{A}', \varphi')$ be two equivalent Jandl structures on $\mathcal{G}$. It is shown in Theorem 1 that there is a unique flat equivariant line bundle $N$ on $M$, such that $N.\mathcal{A} \cong \mathcal{A}'$ as equivariant line bundles. Because the action of $\mathrm{Pic}_0^K(M)$ is free, and $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic, $N$ is the trivial equivariant line bundle. Remember the definition of the bundle $R \to Z$ in equation (95). For the two Jandl structures we get $R' = R \otimes \pi_Z^* N$, and hence the descent bundles $\hat{R}' = \hat{R} \otimes N$ over $\hat{\Sigma}$. Since $N$ is the trivial equivariant line bundle, $\hat{R}'$ and $\hat{R}$ are isomorphic as equivariant line bundles, and thus define isomorphic line bundles $Q'$ and $Q$ over $\Sigma$. Isomorphic line

28

bundles have the same holonomies, so Definition 10 is independent of the equivalence class of $\mathcal{J}$. $\qquad\square$

An important condition for any notion of unoriented surface holonomy is its compatibility with ordinary surface holonomy for oriented surfaces:

**Theorem 3.** *If $\Sigma$ is orientable, for any choice of an orientation, the holonomy defined in Definition 10 reduces to the ordinary holonomy defined in Definition 7,*

$$\mathrm{hol}_{\mathcal{G},\mathcal{J}}(\hat{\phi},\Sigma) = \mathrm{hol}_{\mathcal{G}}(\phi,\Sigma), \qquad (117)$$

*where $\phi$ and $\hat{\phi}$ are related by the bijection of Lemma 3. In particular, if $\mathcal{G}$ admits a Jandl structure, the holonomy of $\mathcal{G}$ does not depend on the orientation.*

Proof. Let or $: \Sigma \to \hat{\Sigma}$ be a choice of an orientation on $\Sigma$. Then $F := \mathrm{or}(\Sigma)$ is a fundamental domain with empty boundary $\partial F = \emptyset$. Choose a trivialization $\mathcal{T}$ of $\hat{\phi}^*\mathcal{G}$ to obtain the 2-form $\hat{\rho} \in \Omega^2(\hat{\Sigma})$. Then the left hand side is equal to $\exp \mathrm{i} \int_{\mathrm{or}(\Sigma)} \hat{\rho}$, because of Theorem 2. Because $\hat{\phi}$ and $\phi$ correspond to each other, $\mathrm{or}^*\hat{\phi}^*\mathcal{G}$ is the same gerbe as $\phi^*\mathcal{G}$, and $\mathrm{or}^*\mathcal{T}$ is a trivialization with 2-form $\rho = \mathrm{or}^*\hat{\rho}$. Thus, the right hand side is equal to $\exp \mathrm{i} \int_{\Sigma} \rho$ and therefore equals the ordinary holonomy. $\square$

## 3.3 Holonomy in Local Data

Let $\{V_i\}_{i \in I}$ be an open cover of $M$. To avoid notation, we assume that it is invariant under $k$ and still good enough to admit all the local sections necessary to extract local data $(g, A, B)$ of the gerbe $\mathcal{G}$ and $(t, W, j)$ of the Jandl structure $\mathcal{J}$, as we explained in section 2.4. We pull back the cover $\{V_i\}_{i \in I}$ along $\hat{\phi} : \hat{\Sigma} \to M$ and obtain a cover $\{\hat{U}_i\}_{i \in I}$ with $\hat{U}_i := \hat{\phi}^{-1}(V_i)$, together with pullback local data. Next, choose local data $(h, M)$ of the trivialization $\mathcal{T}$ of the pullback gerbe and a 2-form $\hat{\rho} \in \Omega^2(\hat{\Sigma})$, so that

$$\left(\hat{\phi}^*g, \hat{\phi}^*A, \hat{\phi}^*B\right) = (1, 0, \hat{\rho}) + \mathrm{D}\,(h, M) \qquad (118)$$

holds. Following the definition of the bundle $R \to Z$ in equation (95), the bundle $\hat{R} \to \hat{\Sigma}$ has local data

$$(r, R) := \hat{\phi}^*(t, W) - \sigma^*(h, M) - (h, M); \qquad (119)$$

the condition that $\hat{R}$ descends is equivalent to the Deligne cocycle condition

$$\mathrm{D}(r, R) = (1, 0), \qquad (120)$$

29

which follows from equations (118) and (67).

Because $\hat{\phi}$ is an element of $C^\infty(\hat{\Sigma}, M)^{\sigma,k}$, the pullback cover is invariant under $\sigma$. Hence it projects to a cover of $\Sigma$ with open sets $U_i := \mathrm{pr}(\hat{U}_i)$. Choose local sections $\mathrm{or}_i : U_i \to \hat{\Sigma}$ and a dual triangulation $T$ of $\Sigma$, subordinate to the cover $\{U_i\}_{i \in I}$, together with a subordinating map $i : T \to I$. As we did in section 3.1 we choose the fundamental domain

$$F := \bigcup_{f \in T} \mathrm{or}_{i(f)}(f), \tag{121}$$

where the $f$'s are the faces of the triangulation.

We now introduce three abbreviations. Let $\omega_i^2 \in \Omega^2(\hat{U}_i)$, $\omega_{ij}^1 \in \Omega^1(\hat{U}_i \cap \hat{U}_j)$ and $\omega_{ijk} : \hat{U}_i \cap \hat{U}_j \cap \hat{U}_k \to U(1)$ be some local data. First we denote the integral over a face $f$ by

$$\begin{aligned} I_f(\omega, \omega^1, \omega^2) &:= \exp\left(\mathrm{i} \int_{\mathrm{or}_{i(f)}(f)} \omega_{i(f)}^2 + \mathrm{i} \sum_{e \in \partial f} \int_{\mathrm{or}_{i(f)}(e)} \omega_{i(f)i(e)}^1\right) \\ &\quad \cdot \prod_{v \in \partial e} \omega_{i(f)i(e)i(v)}^{\varepsilon(f,e,v)}(\mathrm{or}_{i(f)}(v)), \end{aligned} \tag{122}$$

where $\varepsilon(f,e,v) \in \{1, -1\}$ indicates, whether $v$ is the end or the starting point of the edge $e$ with respect to the orientation $\mathrm{or}_{i(f)}$.

Second, we denote the integral of some local data $\omega_i^1 \in \Omega^1(\hat{U}_i)$ and $\omega_{ij} : \hat{U}_i \cap \hat{U}_j \to U(1)$ along an edge $e$ of a face $f$ by

$$I_{e,f}(\omega, \omega^1) := \exp\left(\mathrm{i} \int_{\mathrm{or}_{i(f)}(e)} \omega_{i(e)}^1\right) \cdot \prod_{v \in \partial e} \omega_{i(e)i(v)}^{\varepsilon(f,e,v)}(\mathrm{or}_{i(f)}(v)). \tag{123}$$

Recall that the set of edges in $T$ splits into the set $E$ of orientation-preserving edges and the set $\bar{E}$ of orientation-reversing edges. For an orientation-preserving edge $e \in f_1 \cap f_2$ we have

$$I_{e,f_1}(\omega, \omega^1) = I_{e,f_2}(\omega, \omega^1)^{-1}, \tag{124}$$

while for an orientation-reversing edge

$$I_{e,f_1}(\omega, \omega^1) = I_{e,f_2}(\sigma^*\omega, \sigma^*\omega^1) \tag{125}$$

holds. In the latter case, since $e$ is orientation-reversing, we have either $\mathrm{or}_{i(e)}(e) = \mathrm{or}_{i(f_1)}(e)$ or $\mathrm{or}_{i(e)}(e) = \mathrm{or}_{i(f_2)}(e)$, so that we can write just $I_e(\omega, \omega^1)$, where the for $f$ the choice of the face with the coinciding orientation is understood.

Third, if $v$ is a vertex of an edge $e$, we define for some smooth function $\omega_i : \hat{U}_i \to U(1)$

$$I_{v,e,f}(\omega) := \omega_{i(v)}^{\varepsilon(f,e,v)}(\mathrm{or}_{i(f)}(v)). \qquad (126)$$

Now if $v$ is the common vertex of two orientation-reversing edges $e_1, e_2 \in \bar{E}$, we call $v$ orientation-preserving, if $\mathrm{or}_{i(e_1)}(v) = \mathrm{or}_{i(e_2)}(v)$ and orientation-reversing otherwise. Let us denote the set of orientation-reversing vertices by $\bar{V}$. If $v$ is such a vertex, we just write $I_v(\omega)$ instead of $I_{v,e,f}(\omega)$, where for $e$ the choice of the edge as well as for $f$ the face with the coinciding orientation is understood.

Now the first factor in the holonomy formula (103) is

$$\exp\left(\mathrm{i} \int_F \hat{\rho}\right) = \exp\left(\mathrm{i} \sum_{f \in T} \int_{\mathrm{or}_{i(f)}(f)} \hat{\phi}^* B_{i(f)} + \mathrm{d}M_{i(f)}\right). \qquad (127)$$

Following [CJM02], by using Stoke's theorem, equation (118) and our abbreviations, we end up with

$$\exp\left(\mathrm{i} \int_F \hat{\rho}\right) = \prod_{f \in T} I_f(\hat{\phi}^* g, \hat{\phi}^* A, \hat{\phi}^* B) \cdot \prod_{f \in T} \prod_{e \in \partial f} I_{e,f}(h, M)^{-1}. \qquad (128)$$

Here the second factor collects the boundary contributions that appear in the application of Stoke's theorem.

Let us assume for the moment that $\Sigma$ the oriented, and all sections $\mathrm{or}_i$ coincide with the global orientation restricted to $U_i$. In this situation, we have only orientation preserving edges, and each of them appears twice in the second factor. Since the contributions are inverse by (124), the second factor vanishes. We obtain the local holonomy formula expressed only by the local data of the gerbe, as it appeared originally in [Alv85].

If $\Sigma$ is not oriented, the second factor still consists of two contributions for each orientation-reversing edge $e \in \bar{E}$, which are

$$I_{e,f_1}(h, M) \cdot I_{e,f_2}(h, M) = I_e(h \cdot \sigma^* h, M + \sigma^* M). \qquad (129)$$

Hence, in the general case, the second factor of (128) is

$$\prod_{f \in T} \prod_{e \in \partial f} I_{e,f}(h, M)^{-1} = \prod_{e \in \bar{E}} I_e(h \cdot \sigma^* h, M + \sigma^* M)^{-1}. \qquad (130)$$

For the second factor of the holonomy formula (103) we have to compute the holonomy of the descent line bundle $Q$ around $\overline{\partial F}$. Note that

$$\hat{\bar{E}} := \bigcup_{e \in \bar{E}} \mathrm{or}_{i(e)}(e) \qquad (131)$$

is a fundamental domain of $\overline{\partial F}$ in $\partial F$ with boundary consisting of the preimages of the orientation-reversing vertices $v \in \bar{V}$. Now the holonomy of $Q$ around $\overline{\partial F}$ is equal to the the holonomy of $\hat{R}$ around $\hat{E}$, where at the boundary points the equivariant structure of $\hat{R}$ is used, this is

$$\mathrm{hol}_Q(\overline{\partial F}) = \prod_{e \in \bar{E}} I_e(r, R) \cdot \prod_{v \in \bar{V}} I_v(\hat{\phi}^* j). \tag{132}$$

Since $e$ is orientation-reversing,

$$\begin{aligned} I_e(r, R) &= I_e(\hat{\phi}^* t \cdot \sigma^* h^{-1} \cdot h^{-1}, \hat{\phi}^* W - \sigma^* M - M) \tag{133}\\ &= I_e(\hat{\phi}^* t, \hat{\phi}^* W) \cdot I_e(h \cdot \sigma^* h, M + \sigma^* M)^{-1}. \tag{134} \end{aligned}$$

The second factor of (134) cancels (130) so that all the local data coming from the trivialization drops out. It remains

$$\mathrm{hol}_{\mathcal{G},\mathcal{J}}(\Sigma, \hat{\phi}) = \prod_{f \in T} I_f(\hat{\phi}^* g, \hat{\phi}^* A, \hat{\phi}^* B) \cdot \prod_{e \in \bar{E}} I_e(\hat{\phi}^* t, \hat{\phi}^* W)^{-1} \cdot \prod_{v \in \bar{V}} I_v(\hat{\phi}^* j), \tag{135}$$

depending only on the local data of the gerbe and of the Jandl structure. We visualize this formula in Figure 4.

## 3.4   Examples

In the next two subsections we will apply the general formula (135) to some examples of surfaces $\Sigma$, and we will simplify the situation considerably by starting with the pullback gerbe $\hat{\phi}^* \mathcal{G}$ which allows us to choose a triangulation adapted to $\Sigma$.

### 3.4.1   Klein Bottle

Think of the Klein bottle as a rectangle with the identifications of the boundary indicated by arrows as in Figure 5. The identification by the vertical arrows is orientation-preserving, while the one by the horizontal arrows is orientation-reversing. A dual triangulation is shown in Figure 6. Note that this is a triangulation with only one face. We choose a local section from that face into the double cover, and define the fundamental domain $F$ as its image, as indicated in Figure 7. Here we dropped the arrows, but the identifications are still to be understood, so that both points labelled by $v$ are identified. This means, that we can choose the local orientations of the edges such that the orientation-reversing edges form a closed line, as indicated by the thick line. So there is no orientation-reversing vertex, and the local datum $j$ of the Jandl structure is not relevant for the holonomy around the Klein bottle.
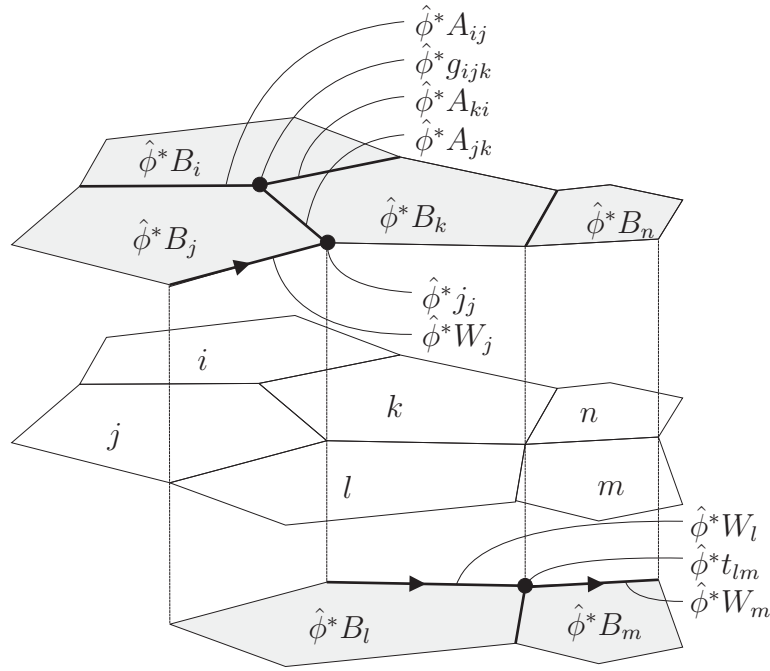
**Figure 4**: Assignment of local data. The middle layer shows
$\Sigma$ and the subordinated indices; the top and lower layer show
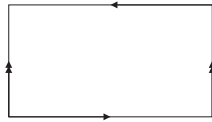parts of the two sheets of $\hat{\Sigma}$.
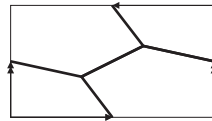


**Figure 5**: Klein Bottle.



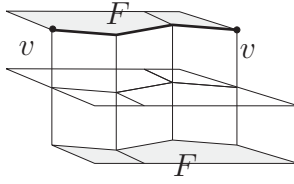**Figure 6**: Klein Bottle with a dual triangulation.

**Figure 7**: A fundamental domain for the Klein Bottle in its double covering.

### 3.4.2 The real projective Plane

We proceed in the same way as for the Klein bottle, so think of the real projective plane $\mathbb{R}P^2$ as a two-gon with the identification on the boundary
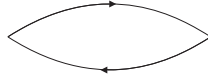


**Figure 8**: The real projective plane.

indicated by arrows in Figure 8. The identification is orientation-reversing. An example of a dual triangulation is for example shown in Figure 9. Now we
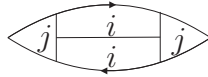


**Figure 9**: A dual triangulation of the real projective plane with two faces.

choose local sections from these two faces into the double cover, for example as shown in Figure 10. Note that here the thick line is not a closed line in $\hat{\Sigma}$, and $v$ is an orientation-reversing vertex. According to the local holonomy formula (135) here the local datum $j$ of the Jandl structure enters in the holonomy.
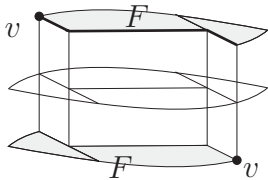
34

**Figure 10**: A fundamental domain of the real projective plane in its double covering.

# 4  Gerbes and Jandl Structures in WZW Models

## 4.1  Oriented and orientable WZW Models

In the following we are concerned with Lie groups $M$, and we will use the following notation. The left multiplication with a group element $h$ is denoted by $l_h : M \to M$, and the map which assigns to $h$ the inverse group element $h^{-1}$ is denoted by Inv $: M \to M$. The left invariant Maurer-Cartan form is denoted by $\theta$, and the right invariant form by $\bar{\theta}$. We call a gerbe $\mathcal{G}$ over $M$ left invariant, if it is stably isomorphic to the gerbe $l_h^* \mathcal{G}$ for each $h \in M$, and similar for right and bi-invariance.

A WZW model is a theory of maps $\phi : \Sigma \to M$ from a worldsheet $\Sigma$ into a target space $M$, which is a Lie group together with additional structure, called the background fields. It assigns to each map $\phi$ an amplitude, i.e. a number in $U(1)$, as the weight of this map in a path integral. To be more precise:

**Definition 11.** *An oriented WZW model consists of a compact connected Lie group $M$, which is equipped with an* Ad-*invariant metric $g = \langle -, - \rangle$ on its Lie algebra and a bi-invariant gerbe $\mathcal{G}$. It assigns an amplitude*

$$A_{g,\mathcal{G}}^{ortd}(\phi, \Sigma) := \exp\left(\mathrm{i}S_{kin}(\phi)\right) \cdot \mathrm{hol}_{\mathcal{G}}(\Sigma, \phi) \tag{136}$$

*to a map $\phi : \Sigma \to M$ from a closed oriented conformal worldsheet $\Sigma$ to $M$, where the kinetic term is*

$$S_{kin}(\phi) := \frac{1}{2} \int_{\Sigma} \langle \phi^* \theta \wedge \star \phi^* \theta \rangle . \tag{137}$$

Note that the conformal structure and the orientation on $\Sigma$ determine the Hodge star.

In [Wit84] Witten discussed this theory for $M = SU(2)$, which is an example for a compact, simple, connected and simply-connected Lie group. In this particular situation, the holonomy can be written as the exponential of the Wess-Zumino term,

$$\mathrm{hol}_{\mathcal{G}}(\Sigma, \phi) = \exp\left(\mathrm{i}\int_B \tilde{\phi}^* H\right), \tag{138}$$

so that we can express the amplitudes as

$$A_{g,\mathcal{G}}^{\mathrm{ortd}}(\phi, \Sigma) = \exp(\mathrm{i}S_{\mathrm{WZW}}(\phi)) \tag{139}$$

with the action functional

$$S_{\mathrm{WZW}}(\phi) := S_{\mathrm{kin}}(\phi) + \int_B \tilde{\phi}^* H. \tag{140}$$

Here $B$ is a 3-dimensional manifold with boundary $\Sigma$, $\tilde{\phi}$ is an extension of $\phi$ on $B$, and $H$ is the curvature of the gerbe $\mathcal{G}$.

Witten observed two symmetries of the WZW model on the type of Lie groups he considered. The first is translation symmetry: the action functional $S_{\mathrm{WZW}}(\phi)$ is invariant under the translation $\phi \mapsto l_h \circ \phi$. The associated conserved Noether current is given by

$$J(\phi) := -(1 + \star)\phi^*\theta, \tag{141}$$

which is a 1-form on $\Sigma$ with values in the Lie algebra of $M$. To obtain this conserved, non-abelian current, Witten derived a specific relative normalization of the kinetic and the Wess-Zumino term, which was also adapted here.

The second symmetry Witten observed is the invariance of the action functional $S_{\mathrm{WZW}}(\phi)$ under what he called parity transformation: reverse the orientation on $\Sigma$ and replace $\phi$ by $\bar{\phi} := \mathrm{Inv} \circ \phi$. Accordingly, the conserved current $J(\phi)$ for $\Sigma$ and the one for $\bar{\Sigma}$, the manifold $\Sigma$ with reversed orientation, namely

$$\bar{J}(\bar{\phi}) = (1 - \star)\phi^*\bar{\theta}, \tag{142}$$

are often called equivalent. Note that here the right invariant Maurer-Cartan form appears. In that sense, the parity transformation exchanges left and right movers.

We now want to generalize this equivalence to any compact connected Lie group $M$. It is a simple consequence of the properties of the holonomy of $\mathcal{G}$, that the parity symmetry

$$A_{g,\mathcal{G}}^{\mathrm{ortd}}(\phi, \Sigma) = A_{g,\mathcal{G}}^{\mathrm{ortd}}(\mathrm{Inv} \circ \phi, \bar{\Sigma}) \tag{143}$$

holds, if the gerbes $\mathrm{Inv}^*\mathcal{G}$ and $\mathcal{G}^*$ are stably isomorphic. Note that this is a condition on the gerbe $\mathcal{G}$. It should not come as a surprise that in Witten's discussion there is no such condition:

**Lemma 6.** *If $\mathcal{G}$ is a bi-invariant gerbe over a compact, simple, connected and simply connected Lie group, then $\mathrm{Inv}^*\mathcal{G}$ and $\mathcal{G}^*$ are stably isomorphic.*

Proof. Because stably isomorphic gerbes have the same curvatures, the curvature $H$ of the bi-invariant gerbe $\mathcal{G}$ is a bi-invariant 3-form. It is a theorem by Cartan, that on compact, simple, connected, simply connected Lie groups $M$ the space of bi-invariant 3-forms is the span of the canonical 3-form $\nu$, which satisfies $\mathrm{Inv}^*\nu = -\nu$. Hence $\mathrm{Inv}^*\mathcal{G}$ and $\mathcal{G}^*$ have the same curvature. Because the set of stable isomorphism classes of gerbes of same curvature form a torsor over $\mathrm{H}^2(M, U(1))$ [GR02], which here is the trivial group, the gerbes $\mathrm{Inv}^*\mathcal{G}$ and $\mathcal{G}^*$ are stably isomorphic. □

We now give an even more general definition of parity transformations of a target space $M$ with metric $g$ and gerbe $\mathcal{G}$.

**Definition 12.** *A parity transformation map is an isometry $k : M \to M$ of the metric $g$ of order two, such that $k^*\mathcal{G}$ and $\mathcal{G}^*$ are stably isomorphic. We denote the set of parity transformation maps by $P(M, g, \mathcal{G})$.*

Consider an oriented WZW model with target space $M$, Ad-invariant metric $g$ and bi-invariant gerbe $\mathcal{G}$. If $k \in P(M, g, \mathcal{G})$ is a parity transformation map, we obtain the parity symmetry

$$A_{g,\mathcal{G}}^{\mathrm{ortd}}(\phi, \Sigma) = A_{g,\mathcal{G}}^{\mathrm{ortd}}(k \circ \phi, \bar{\Sigma}). \tag{144}$$

We already discussed that $k = \mathrm{Inv}$ is a parity transformation map in the sense of Definition 12, if the gerbes $\mathrm{Inv}^*\mathcal{G}$ and $\mathcal{G}^*$ are stably isomorphic. However, for oriented WZW models on compact connected Lie groups there are more such parity transformation maps. Because the gerbe $\mathcal{G}$ is supposed to be bi-invariant, we try an ansatz $k := l_h \circ \mathrm{Inv}$ for some group element $h \in M$. The condition $k^2 = \mathrm{id}_M$ restricts $h$ to be an element of the center $Z(M)$. So, the set $P(M, g, \mathcal{G})$ of parity transformation maps for a compact connected Lie group $M$ and a bi-invariant gerbe $\mathcal{G}$, such that $\mathcal{G}^*$ is stably isomorphic to $\mathrm{Inv}^*\mathcal{G}$, contains at least

$$\{l_z \circ \mathrm{Inv} \mid z \in Z(M)\} \subset P(M, g, \mathcal{G}). \tag{145}$$

In particular, $P(M, g, \mathcal{G})$ is not empty in the situation we are interested in.

As a preparation for the unoriented case, we now relate parity symmetry to the orientation cover $\hat{\Sigma}$: Start with an oriented WZW model on $\Sigma$ together with a parity transformation map $k$. Let $\phi : \Sigma \to M$ be a map. By Lemma 3, there is a unique map $\hat{\phi} \in C^\infty(\hat{\Sigma}, M)^{k,\sigma}$. Once we have the orientation cover $\hat{\Sigma}$ and the map $\hat{\phi}$, we may forget their origin, in particular the orientation on $\Sigma$. Then we may give the following

**Definition 13.** *An orientable WZW model consists of a compact connected Lie group $M$, which is equipped with an* Ad-*invariant metric $g$ on its Lie algebra, a bi-invariant gerbe $\mathcal{G}$ and a parity transformation map $k \in P(M, g, \mathcal{G})$. To a closed orientable conformal surface $\Sigma$ and a map $\hat{\phi} \in C^\infty(\hat{\Sigma}, M)^{k,\sigma}$, the following amplitude $A_{g,\mathcal{G}}^{orble}(\hat{\phi}, \Sigma)$ is assigned. Choose any orientation on $\Sigma$, and obtain a map $\phi : \Sigma \to M$ by Lemma 3. Define*

$$A_{g,\mathcal{G}}^{orble}(\hat{\phi}, \Sigma) := A_{g,\mathcal{G}}^{ortd}(\phi, \Sigma). \tag{146}$$

The amplitude is well-defined: if we had chosen the other orientation, we would get the same amplitudes, due to the fact that $k$ is a parity transformation map and satisfies equation (144).

## 4.2 Unoriented WZW Models

In the last section we gave the definition of an orientable WZW model. The derivation of the amplitude of a map $\hat{\phi} \in C^\infty(\hat{\Sigma}, M)^{k,\sigma}$ makes use of the existence of an orientation on $\Sigma$ both in the kinetic term and in the holonomy term. In this section, we want to overcome this obstruction.

Let us first discuss the kinetic term. We want to define the kinetic term $S_{\text{kin}}(\hat{\phi})$ for a map $\hat{\phi} \in C^\infty(\hat{\Sigma}, M)^{k,\sigma}$ in such a way that if $\Sigma$ is orientable, it reduces to the kinetic term $S_{\text{kin}}(\phi)$ of the corresponding map $\phi$. Note that

$$\mathcal{L}(\hat{\phi}) := \frac{1}{2} \left\langle \hat{\phi}^*\theta \wedge \star\hat{\phi}^*\theta \right\rangle \tag{147}$$

is a 2-form on $\hat{\Sigma}$, which satisfies

$$\sigma^*\mathcal{L}(\hat{\phi}) = -\mathcal{L}(\hat{\phi}). \tag{148}$$

This property tells us that $\mathcal{L}(\hat{\phi})$ defines a 2-density $\mathcal{L}_{\text{den}}(\hat{\phi})$ [BT82, BG88] on $\Sigma$. The integral of a 2-density over a surface is defined without respect to the orientability of this surface, so we define

$$S_{\text{kin}}(\hat{\phi}) := \int_\Sigma \mathcal{L}_{\text{den}}(\hat{\phi}). \tag{149}$$

38

To make the integral (149) more explicit, choose a triangulation $T$ of $\Sigma$, and for each face $f \in T$ a local section $\mathrm{or}_f : U_f \to \hat{\Sigma}$, where $U_f$ is some open neighborhood of $f$ in $\Sigma$. By definition of the integral of a density,

$$S_{\mathrm{kin}}(\hat{\phi}) = \sum_{f \in T} \int_{\mathrm{or}_f(f)} \mathcal{L}(\hat{\phi}). \tag{150}$$

One immediately checks that this definition is independent of the choice of the local sections: if one chooses for one face $f$ the other orientation, namely $\sigma(\mathrm{or}_f)$, the corresponding term in the sum (150),

$$\int_{\sigma(\mathrm{or}_f(f))} \mathcal{L}(\hat{\phi}) = - \int_{\mathrm{or}_f(f)} \sigma^* \mathcal{L}(\hat{\phi}) = \int_{\mathrm{or}_f(f)} \mathcal{L}(\hat{\phi}), \tag{151}$$

gives the same contribution. It is also independent of the choice of the triangulation. Furthermore, if $\Sigma$ is orientable, we can choose a triangulation with a single face $f = \Sigma$ and get $S_{\mathrm{kin}}(\hat{\phi}) = S_{\mathrm{kin}}(\phi)$, which was precisely our requirement on $S_{\mathrm{kin}}(\hat{\phi})$.

We have already discussed in section 3 how to define surface holonomies for an arbitrary closed surface $\Sigma$ with a map $\hat{\phi} \in C^\infty(\hat{\Sigma}, M)^{k,\sigma}$: we have to choose a Jandl structure $\mathcal{J}$ on $\mathcal{G}$. Then $\mathrm{hol}_{\mathcal{G},\mathcal{J}}(\hat{\phi}, \Sigma)$ is defined in Definition 10 in such a way that if $\Sigma$ is orientable, it coincides by Theorem 3 with $\mathrm{hol}_{\mathcal{G}}(\phi, \Sigma)$. Remember that a necessary condition on the existence of a Jandl structure $\mathcal{J} = (k, -, -)$ was that the gerbes $k^*\mathcal{G}$ and $\mathcal{G}^*$ are stably isomorphic. We already have encountered this condition for the orientable WZW model, so that it does not come as an additional restriction. This leads us to the following

**Definition 14.** *An unoriented WZW model consists of a compact connected Lie group $M$, which is equipped with an $\mathrm{Ad}$-invariant metric $g$ on its Lie-algebra and a bi-invariant gerbe $\mathcal{G}$ with Jandl structure $\mathcal{J}$, whose action of $\mathbb{Z}_2$ on $M$ is a parity transformation map $k \in P(M, g, \mathcal{G})$. To a closed conformal surface $\Sigma$ and a map $\hat{\phi} \in C^\infty(\hat{\Sigma}, M)^{k,\sigma}$ the amplitude*

$$A^{unor}_{g,\mathcal{G},\mathcal{J}}(\hat{\phi}, \Sigma) := \exp\left(\mathrm{i} S_{kin}(\hat{\phi})\right) \cdot \mathrm{hol}_{\mathcal{G},\mathcal{J}}(\hat{\phi}, \Sigma). \tag{152}$$

*is assigned.*

According to the definition of both factors, if $\Sigma$ is orientable, we have

$$A^{\mathrm{unor}}_{g,\mathcal{G},\mathcal{J}}(\hat{\phi}, \Sigma) = A^{\mathrm{orble}}_{g,\mathcal{G}}(\hat{\phi}, \Sigma). \tag{153}$$

If $\Sigma$ is even oriented, by equation (146) we have

$$A^{\mathrm{unor}}_{g,\mathcal{G},\mathcal{J}}(\hat{\phi}, \Sigma) = A^{\mathrm{ort.d}}_{g,\mathcal{G}}(\phi, \Sigma). \tag{154}$$

## 4.3 Crosscaps and the trivial line bundle

In the following two sections we use the classification of Jandl structures to classify unoriented WZW models with a fixed gerbe $\mathcal{G}$ and a fixed parity transformation map $k \in P(M, g, \mathcal{G})$. By Theorem 1, the set of equivalence classes of Jandl structures of $\mathcal{G}$ with the action of $K = \mathbb{Z}_2$ on $M$ defined by $k$ is a torsor over the flat $K$-equivariant Picard group $\mathrm{Pic}_0^K(M)$. In this section we discuss a special element of this group.

On any manifold, there is the trivial line bundle $L_1 := M \times \mathbb{C}$ with the trivial hermitian metric and the trivial connection, which is flat. It represents the unit element of the flat Picard group $\mathrm{Pic}_0(M)$.

Recall the following facts concerning equivariant line bundles [Gom03]. There are two obstructions for a given line bundle to admit equivariant structures: the first depends on the bundle and the group action, namely that

$$k^* L \otimes L^* \cong L_1, \tag{155}$$

which is still to be understood as an equation of hermitian line bundles with connection. The second obstruction is a class in the group cohomology group $\mathrm{H}^2_{\mathrm{Grp}}(K, U(1))$. Now, if both obstructions are absent, the possible equivariant structures are parameterized by the group cohomology group $\mathrm{H}^1_{\mathrm{Grp}}(K, U(1))$ which is just the group of one-dimensional characters of $K$. In our case $K = \mathbb{Z}_2$ we have

$$\mathrm{H}^1_{\mathrm{Grp}}(K, U(1)) \;=\; \mathbb{Z}_2 \tag{156}$$

$$\mathrm{H}^2_{\mathrm{Grp}}(K, U(1)) \;=\; 0 \tag{157}$$

so that the second obstruction vanishes, and every line bundle $L$, which satisfies the remaining obstruction (155) admits exactly two $K$-equivariant structures.

In particular $L_1$ itself satisfies (155). We exhibit its two equivariant structures explicitly. Remember from section 2.2, that we have to choose an isomorphism

$$\varphi : k^* L_1 \to L_1 \tag{158}$$

of line bundles, such that $\varphi \circ k^* \varphi = \mathrm{id}_{L_1}$. So the both choices are either $\varphi_1 = \mathrm{id}_{M \times \mathbb{C}}$ or $\varphi_{-1} : (x, z) \mapsto (x, -z)$. We denote $L_1$ together with the equivariant structure $\varphi_1$ by $L_1^K$. It represents the unit element of $\mathrm{Pic}_0^K(M)$. We denote $L_1$ together with the equivariant structure $\varphi_{-1}$ by $L_{-1}^K$. Note that $L_{-1}^K \otimes L_{-1}^K = L_1^K$ as equivariant line bundles. Hence it represents a non-trivial element of order two in $\mathrm{Pic}_0^K(M)$.

The whole construction is completely independent of $M$, so $\mathrm{Pic}_0^K(M)$ always contains at least these two elements. As a consequence, if a gerbe $\mathcal{G}$

admits a Jandl structure $\mathcal{J}$, then $L_{-1}^K.\mathcal{J}$ is another, inequivalent Jandl structure on $\mathcal{G}$. We will now investigate the difference between the corresponding unoriented WZW models.

We work with local data, so let $\{V_i\}_{i\in I}$ be a good open cover of $M$. Choose all the sections that have been introduced in section 2.4, and extract local data $(t, W)$, $j$ of the Jandl structure $\mathcal{J}$. We also explained how to extract a local datum $\nu_i : V_i \to U(1)$ from an equivariant structure on a line bundle over $M$. The local datum of $L_1^K$ is the constant global function $\nu_1 = 1$, and the local datum of $L_{-1}^K$ is the constant global function $\nu_1 = -1$.

According to the definition of the action of $\mathrm{Pic}_0^K(M)$ on $\mathrm{Jdl}(\mathcal{G}, k)$, the local data of $L_{-1}^K.\mathcal{J}$ are $(t, W)$ and $-j$. Now observe the occurrences of the local datum $j$ in the local holonomy formula (135): it appears for each orientation-reversing vertex $v \in \bar{V}$. Following our example in section 3.4.2, this happens in the presence of a crosscap. We conclude that the amplitudes of both unoriented WZW models with Jandl structures $\mathcal{J}$ and $L_{-1}^K.\mathcal{J}$ differ by a sign for each crosscap in $\Sigma$.

## 4.4 Examples of target spaces

We would like to discuss three examples of target spaces, namely the Lie groups $SU(2)$, $SO(3)$, where the Ad-invariant metric on their Lie algebras is given by their Killing forms, and the two-dimensional torus $T^2 = S^1 \times S^1$ with the euclidean scalar product. The gerbes are supposed to be bi-invariant.

### 4.4.1 The Lie group $SU(2)$

Following our general discussion, the actions of $\mathbb{Z}_2$ on $SU(2)$ we have to consider are given by $k : g \mapsto g^{-1}$ and $k : g \mapsto -g^{-1}$, where $-1 \in Z(SU(2))$ is the non-trivial element in the center. The same maps were considered in [HSS02, Bru02, BCW01].

Fix a bi-invariant gerbe $\mathcal{G}$ over $SU(2)$. Up to stable isomorphism, this is $\mathcal{G} = \mathcal{G}_0^{\otimes n}$, where $\mathcal{G}_0$ is the basic gerbe over $SU(2)$ [Mei02]. By Lemma 6, both $k$'s are parity transformation maps.

The set $\mathrm{Jdl}(\mathcal{G}, k)$ is a torsor over $\mathrm{Pic}_0^K(SU(2))$ by Theorem 1. In order to compute the group of equivariant flat line bundles, we first observe

$$\mathrm{Pic}_0(M) = \mathrm{Hom}(\pi_1(M), U(1)) = 0, \tag{159}$$

since $SU(2)$ is simply connected. So up to isomorphism there is only one flat line bundle, the trivial one. Hence there are exactly two inequivalent Jandl structures for each map $k$ and each bi-invariant gerbe $\mathcal{G}$; this is in agreement with the results of [PSS95a, PSS95b]

### 4.4.2 The Lie group $SO(3)$

The center of $SO(3)$ is trivial, so that we have only one action to consider, namely by $k : g \mapsto g^{-1}$. Let $\mathcal{G}$ be a bi-invariant gerbe over $SO(3)$, such that $k^*\mathcal{G}$ and $\mathcal{G}^*$ are stably isomorphic. Such gerbes for example are constructed up to stable isomorphism in [GR03]. We have to investigate the group $\mathrm{Pic}_0^K(SO(3))$ of flat equivariant line bundles. Again we first consider the group $\mathrm{Pic}_0(SO(3))$ of flat line bundles and classify equivariant structures on them.

By $\pi_1(SO(3)) = \mathbb{Z}_2$ we have

$$\mathrm{Hom}(\pi_1(SO(3)), U(1)) = \mathrm{Hom}(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2, \tag{160}$$

so there are - up to isomorphism - two flat line bundles. We will give them explicitly: As $SO(3)$ is the quotient of $SU(2)$ by $q : g \mapsto -g$, the two flat line bundles over $SO(3)$ correspond to the two equivariant flat line bundles over $SU(2)$, namely $L_1^K$ and $L_{-1}^K$ .

Clearly, $L_1^K$ descends to the trivial flat line bundle $\tilde{L}_1 \to SO(3)$, which admits equivariant structures, more precisely, according to the discussion in section 4.3, there are two of them. $L_{-1}^K$ descends to a non-trivial flat line bundle $\tilde{L}_{-1} \to SO(3)$, and we have to ask whether it admits equivariant structures, which is equivalent to the condition, that

$$\mathrm{d}\tilde{L}_{-1} := k^*\tilde{L}_{-1} \otimes \tilde{L}_{-1}^* \cong \tilde{L}_1. \tag{161}$$

Now $\mathrm{d}\tilde{L}_{-1}$ is a flat line bundle, and hence either isomorphic to $\tilde{L}_{-1}$ or to $\tilde{L}_1$. Because $\mathrm{Pic}_0(SO(3))$ is a group of order two, we have $\tilde{L}_{-1} \otimes \tilde{L}_{-1} = \tilde{L}_1$. The assumption $\mathrm{d}\tilde{L}_{-1} \cong \tilde{L}_{-1}$ would therefore mean $k^*\tilde{L}_{-1} \cong \tilde{L}_1$ which is a contradiction since $\tilde{L}_1$ is the trivial bundle and $k^*\tilde{L}_{-1}$ is not. Hence (161) is true, and $\tilde{L}_{-1}^*$ admits two equivariant structures.

All together, there are four equivariant flat line bundles over $SO(3)$ and hence four Jandl structures on $\mathcal{G}$; again, this is in agreement with [PSS95a, PSS95b].

### 4.4.3 The two-dimensional Torus $T^2$

For dimensional reasons, all gerbes over $T^2$ are trivial and have curvature $H = 0$. This allows us to discuss an example with a parity transformation map $k$, which is not of the form $k = l_z \circ \mathrm{Inv}$ but simply the identity map $k = \mathrm{id}$. This allows us to make contact with [BPS92].

Now let $\mathcal{G}$ be a bi-invariant gerbe over $T^2$. The set $\mathrm{Jdl}(\mathcal{G}, \mathrm{id})$ is a torsor over $\mathrm{Pic}_0^K(T^2)$ by Theorem 1, which is isomorphic to $\mathrm{H}_K^1(T^2, U(1))$

by equation (33). The Borel space associated to the trivial $K$-action is $T_K^2 = E\mathbb{Z}_2 \times T^2$. With $E\mathbb{Z}_2 = \mathbb{R}P^\infty$ we have

$$
\begin{aligned}
\mathrm{H}_K^1(T^2, U(1)) &= H^1(T_K^2, U(1)) \\
&= \mathrm{H}^1(\mathbb{R}P^\infty, U(1)) \oplus \mathrm{H}^1(T^2, U(1)) \\
&= \mathbb{Z}_2 \oplus U(1) \oplus U(1) \\
&= \mathbb{Z}_2 \oplus T^2.
\end{aligned}
$$

We now assume that the gerbe $\mathcal{G}$ admits a Jandl structure $\mathcal{J} = (\mathrm{id}, \mathcal{A}, \varphi)$. In particular, $\mathcal{A} = (A, \alpha)$ is a stable isomorphism from $\mathcal{G}$ to $\mathcal{G}^*$. Recall that a gerbe $\mathcal{G}$ consist of the following data: a surjective submersion $\pi : Y \to M$, a line bundle $L \to Y^{[2]}$, an isomorphism $\mu$, and a 2-form $C \in \Omega^2(Y)$. Recall further that here $A$ is a line bundle over $Z = Y^{[2]}$, and both projections $p$ and $p'$ from $Z$ to $Y$ coincide with $\pi_2, \pi_1 : Y^{[2]} \to Y$.

The condition on the curvature of $A$ in Definition 3 now reads

$$
\mathrm{curv}(A) = \pi_1^* C + \pi_2^* C. \tag{162}
$$

Furthermore, since for all gerbes the curving $C$ satisfies $-\pi_2^* C + \pi_1^* C = \mathrm{curv}(L)$, we have

$$
2\pi_2^* C = \mathrm{curv}(A) - \mathrm{curv}(L), \tag{163}
$$

which is an equation of 2-forms on $Y^{[2]}$. On the right hand side we have a closed 2-form which defines an integral class in cohomology. Since $\pi_2$ is a surjective submersion, also $2C$ defines a class in $\mathrm{H}^2(Y, \mathbb{Z})$.

Because the gerbe $\mathcal{G}$ is trivial, we can choose a trivialization $\mathcal{T}$ and obtain the 2-form $B \in \Omega^2(M)$ as in Definition 7, which satisfies $\pi^* B = C + \mathrm{curv}(T)$ and $\mathrm{d}B = H = 0$. Usually one chooses $\mathcal{T}$ such that $B$ is constant, then it is nothing but the Kalb-Ramond "B-Field". Because $\pi$ is also a surjective submersion it follows that $2B$ defines a class in $\mathrm{H}^2(M, \mathbb{Z})$. Thus we have derived the quantization condition that the $B$-Field has half integer valued periods. This condition was originally found in [BPS92] by an analysis of the bulk spectrum of right and left movers.

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