arxiv:0705.0452
Hamburger Beiträge zur Mathematik Nr. 269
ZMP-HH/07-5

Parallel Transport and Functors

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Abstract

Parallel transport of a connection in a smooth fibre bundle yields a functor from the path groupoid of the base manifold into a category that describes the fibres of the bundle. We characterize functors obtained like this by two notions we introduce: local trivializations and smooth descent data. This provides a way to substitute categories of functors for categories of smooth fibre bundles with connection. We indicate that this concept can be generalized to connections in categorified bundles, and how this generalization improves the understanding of higher dimensional parallel transport.

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1 Introduction

Higher dimensional parallel transport generalizes parallel transport along curves to parallel transport along higher dimensional objects, for instance surfaces. One motivation to consider parallel transport along surfaces comes from two-dimensional conformal field theories, where so-called Wess-Zumino terms have been recognized as surface holonomies [Gaw88, CJM02, SSW07].

Several mathematical objects have have been used to define higher dimensional parallel transport, among them classes in Deligne cohomology [Del91], bundle gerbes with connection and curving [Mur96], or 2-bundles with 2-connections [BS04, BS07]. The development of such definitions often occurs in two steps: an appropriate definition of parallel transport along curves, followed by a generalization to higher dimensions. For instance, bundle gerbes with connection can be obtained as a generalization of principal bundles with connection. However, in the case of both bundle gerbes and Deligne classes one encounters the obstruction that the structure group has to be abelian. It is hence desirable to find a reformulation of fibre bundles with connection, that brings along a natural generalization for arbitrary structure group.

A candidate for such a reformulation are holonomy maps [Bar91, CP94]. These are group homomorphisms

$$\mathcal{H} : \pi_1^1(M, *) \longrightarrow G$$

from the group of thin homotopy classes of based loops in a smooth manifold M into a Lie group G. Any principal G-bundle with connection over M defines a group homomorphism $\mathcal{H}$, but the crucial point is to distinguish those from arbitrary ones. By imposing a certain smoothness condition on $\mathcal{H}$, these holonomy maps correspond – for connected manifolds – bijectively to principal G-bundles with connection [Bar91, CP94]. On the other hand, they have a natural generalization from loops to surfaces. However, the obstruction for M being connected becomes even stronger: only if the manifold M is connected and simply-connected, holonomy maps generalized to surfaces capture all aspects of surface holonomy [MP02]. Especially the second obstruction erases one of the most interesting of these aspects, see, for example, [GR02].

In order to obtain a formulation of parallel transport along curves without topological assumptions on the base manifold M, one considers functors

$$F : \mathcal{P}_1(M) \longrightarrow T$$

from the path groupoid $\mathcal{P}_1(M)$ of M into another category T [Mac87, MP02]. The set of objects of the path groupoid $\mathcal{P}_1(M)$ is the manifold M itself, and the set of morphisms between two points x and y is the set of thin homotopy classes of curves starting at x and ending at y. A functor $F : \mathcal{P}_1(M) \longrightarrow T$ is a generalization of a group homomorphism $\mathcal{H} : \pi_1^1(M, *) \longrightarrow G$, but it is not clear how the smoothness condition for holonomy maps has to be generalized to these functors.

Let us first review how a functor $F : \mathcal{P}_1(M) \longrightarrow T$ arises from parallel transport in a, say, principal G-bundle P with connection. In this case, the category T is the category G-Tor of smooth manifolds with smooth, free and transitive G-action from the right, and smooth equivariant maps between those. Now, the connection on P associates to any smooth curve $\gamma : [0, 1] \longrightarrow M$ and any element in the fibre $P_{\gamma(0)}$ over the starting point, a unique horizontal lift $\tilde{\gamma} : [0, 1] \longrightarrow P$. Evaluating this lift at its endpoint defines a smooth map

$$\tau_\gamma : P_{\gamma(0)} \longrightarrow P_{\gamma(1)},$$

the parallel transport in P along the curve γ. It is G-equivariant with respect to the G-action on the fibres of P, and it is invariant under thin homotopies. Moreover, it satisfies

$$\tau_{\mathrm{id}_x} = \mathrm{id}_{P_x} \quad \text{and} \quad \tau_{\gamma' \circ \gamma} = \tau_{\gamma'} \circ \tau_\gamma,$$

where id_x is the constant curve and γ and γ' are smoothly composable curves. These are the axioms of a functor

$$\mathrm{tra}_P : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$$

which sends an object x of $\mathcal{P}_1(M)$ to the object P_x of G-Tor and a morphism γ of $\mathcal{P}_1(M)$ to the morphism τ_γ of G-Tor. Summarizing, every principal G-bundle with connection over M defines a functor tra_P. Now the crucial point is to characterize these functors among all functors from $\mathcal{P}_1(M)$ to G-Tor.

In this article we describe such a characterization. For this purpose, we introduce, for general target categories T, the notion of a transport functor. These are certain functors

$$\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow T,$$

such that the category they form is – in the case of $T = G$-Tor – equivalent to the category of principal G-bundles with connection.

The defining properties of a transport functor capture two important concepts: the existence of local trivializations and the smoothness of associated descent data. Just as for fibre bundles, local trivializations are specified with respect to an open cover of the base manifold M and to a choice of a typical fibre. Here, we represent an open cover by a surjective submersion $\pi : Y \longrightarrow M$, and encode the typical fibre in the notion of a structure groupoid: this is a Lie groupoid Gr together with a functor

$$i : \mathrm{Gr} \longrightarrow T.$$

Now, a π-local i-trivialization of a functor $F : \mathcal{P}_1(M) \longrightarrow T$ is another functor

$$\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$$

together with a natural equivalence

$$t : F \circ \pi_* \longrightarrow i \circ \mathrm{triv},$$

where $\pi_* : \mathcal{P}_1(Y) \longrightarrow \mathcal{P}_1(M)$ is the induced functor between path groupoids. In detail, the natural equivalence t gives for every point $y \in Y$ an isomorphism $F(\pi(y)) \cong i(\mathrm{triv}(y))$ that identifies the "fibre" $F(\pi(y))$ of F over $\pi(y)$ with the image of a "typical fibre" $\mathrm{triv}(y)$ under the functor i. In other words, a functor is π-locally i-trivializable, if its pullback to the cover Y factors through the functor i up to a natural equivalence. Functors with a chosen π-local i-trivialization (triv, t) form a category $\mathrm{Triv}^1_\pi(i)$.

The second concept we introduce is that of smooth descent data. Descent data is specified with respect to a surjective submersion π and a structure groupoid $i : \mathrm{Gr} \longrightarrow T$. While descent data for a fibre bundle with connection is a collection of transition functions and local connection 1-forms, descent data for a functor $F : \mathcal{P}_1(M) \longrightarrow T$ is a pair (triv, g) consisting of a functor $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$ like the one from a local trivializations and of a certain natural equivalence g that compares triv on the two-fold fibre product of Y with itself. Such pairs define a descent category $\mathfrak{Des}^1_\pi(i)$. The first result of this article (Theorem 2.9) is to prove the descent property: extracting descent data and, conversely, reconstructing a functor from descent data, are equivalences of categories

$$\mathrm{Triv}^1_\pi(i) \cong \mathfrak{Des}^1_\pi(i).$$

We introduce descent data because one can precisely decide whether a pair (triv, g) is smooth or not (Definition 3.1). The smoothness conditions we introduce can be expressed in basic terms of smooth maps between smooth manifolds, and arises from the theory of diffeological spaces [Che77]. The concept of smooth descent data is our generalization of the smoothness condition for holonomy maps to functors.

Combining both concepts we have introduced, we call a functor that allows – for some surjective submersion π – a π-local i-trivialization whose corresponding descend data is smooth, a transport functor on M in T with Gr-structure. The category formed by these transport functors is denoted by $\mathrm{Trans}^1_{\mathrm{Gr}}(M,T)$.

Let us return to the particular target category $T = G\text{-Tor}$. As described above, one obtains a functor $\mathrm{tra}_P : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$ from any principal G-bundle P with connection. We consider the Lie groupoid $\mathrm{Gr} = \mathcal{B}G$, which has only one object, and where every group element $g \in G$ is an automorphism of this object. The notation indicates the fact that the geometric realization of the nerve of this category yields the classifying space BG of the group G. The Lie groupoid $\mathcal{B}G$ can be embedded in the category G-Tor via the functor $i_G : \mathcal{B}G \longrightarrow G\text{-Tor}$ which sends the object of $\mathcal{B}G$ to the group G regarded as a G-space, and a morphism $g \in G$ to the equivariant smooth map which multiplies with g from the left.

The descent category $\mathfrak{Des}^1_\pi(i_G)$ for the structure groupoid $\mathcal{B}G$ and some surjective submersion π is closely related to differential geometric objects: we derive a one-to-one correspondence between smooth functors $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathcal{B}G$, which are part of the objects of $\mathfrak{Des}^1_\pi(i_G)$, and 1-forms A on Y with values in the Lie algebra of G (Proposition 4.7). The correspondence can be symbolically expressed as the path-ordered exponential

$$\mathrm{triv}(\gamma) = \mathcal{P}\exp\left(\int_\gamma A\right)$$

for a path γ. Using this relation between smooth functors and differential forms, we show that a functor $\mathrm{tra}_P : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$ obtained from a principal G-bundle with connection, is a transport functor on M in G-Tor with $\mathcal{B}G$-structure. The main result of this article (Theorem 5.4) is that this establishes an equivalence of categories

$$\mathfrak{Bun}^\nabla_G(M) \cong \mathrm{Trans}^1_{\mathcal{B}G}(M, G\text{-Tor})$$

between the category of principal G-bundles with connection over M and the category of transport functors on M in G-Tor with $\mathcal{B}G$-structure. In other words, these transport functors provide a proper reformulation of principal bundles with connection, emphasizing the aspect of parallel transport.

This article is organized as follows. In Section 2 we review the path groupoid of a smooth manifold and describe some properties of functors defined on it. We introduce local trivializations for functors and the descent category $\mathfrak{Des}^1_\pi(i)$. In Section 3 we define the category $\mathrm{Trans}^1_{\mathrm{Gr}}(M,T)$ of transport functors on M in T with Gr-structure and discuss several properties.

In Section 4 we derive the result that relates the descent category $\mathfrak{Des}^1_\pi(i_G)$ for the particular functor $i_G : \mathcal{B}G \longrightarrow G\text{-Tor}$ to differential forms. In Section 5 we provide examples that show that the theory of transport functors applies well to several situations: we prove our main result concerning principal G-bundles with connection, show a similar statement for vector bundles with connection, and also discuss holonomy maps. In Section 6 we discuss principal groupoid bundles and show how transport functors can be used to derive the definition of a connection on such groupoid bundles. Section 7 contains various directions in which the concept of transport functors can be generalized. In particular, we outline a possible generalization of transport functors to transport n-functors

$$\mathrm{tra} : \mathcal{P}_n(M) \longrightarrow T,$$

which provide an implementation for higher dimensional parallel transport. The discussion of the interesting case $n = 2$ is the subject of a separate publication [SW08a].

2 Functors and local Trivializations

We give the definition of the path groupoid of a smooth manifold and describe functors defined on it. We introduce local trivializations and descent data of such functors.

2.1 The Path Groupoid of a smooth Manifold

We start by setting up the basic definitions around the path groupoid of a smooth manifold M. We use the conventions of [CP94, MP02], generalized from loops to paths.

Definition 2.1. *A* <u>*path*</u> $\gamma : x \longrightarrow y$ *between two points* $x, y \in M$ *is a smooth map* $\gamma : [0,1] \longrightarrow M$ *which has a sitting instant: a number* $0 < \epsilon < \frac{1}{2}$ *such that* $\gamma(t) = x$ *for* $0 \leq t < \epsilon$ *and* $\gamma(t) = y$ *for* $1 - \epsilon < t \leq 1$.

Let us denote the set of such paths by PM. For example, for any point $x \in M$ there is the constant path id_x defined by $\mathrm{id}_x(t) := x$. Given a path $\gamma_1 : x \longrightarrow y$ and another path $\gamma_2 : y \longrightarrow z$ we define their composition to be the path $\gamma_2 \circ \gamma_1 : x \longrightarrow z$ defined by

$$(\gamma_2 \circ \gamma_1)(t) := \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

This gives a smooth map since γ_1 and γ_2 are both constant near the gluing point, due to their sitting instants. We also define the inverse $\gamma^{-1} : y \longrightarrow x$ of a path $\gamma : x \longrightarrow y$ by $\gamma^{-1}(t) := \gamma(1-t)$.

Definition 2.2. *Two paths $\gamma_1 : x \longrightarrow y$ and $\gamma_2 : x \longrightarrow y$ are called thin homotopy equivalent, if there exists a smooth map $h : [0,1] \times [0,1] \longrightarrow M$ such that*

1. *there exists a number $0 < \epsilon < \frac{1}{2}$ with*
   (a) *$h(s,t) = x$ for $0 \leq t < \epsilon$ and $h(s,t) = y$ for $1-\epsilon < t \leq 1$.*
   (b) *$h(s,t) = \gamma_1(t)$ for $0 \leq s < \epsilon$ and $h(s,t) = \gamma_2(t)$ for $1-\epsilon < s \leq 1$.*
2. *the differential of h has at most rank 1 everywhere, i.e.*
$$\mathrm{rank}(\mathrm{d}h|_{(s,t)}) \leq 1$$
   for all $(s,t) \in [0,1] \times [0,1]$.

Due to condition (b), thin homotopy defines an equivalence relation on PM. The set of thin homotopy classes of paths is denoted by P^1M, and the projection to classes is denoted by

$$\mathrm{pr} : PM \longrightarrow P^1M.$$

We denote a thin homotopy class of a path $\gamma : x \longrightarrow y$ by $\overline{\gamma} : x \longrightarrow y$. Notice that thin homotopies include the following type of reparameterizations: let $\beta : [0,1] \longrightarrow [0,1]$ be a path $\beta : 0 \longrightarrow 1$, in particular with $\beta(0) = 0$ and $\beta(1) = 1$. Then, for any path $\gamma : x \longrightarrow y$, also $\gamma \circ \beta : x \longrightarrow y$ is a path and

$$h(s,t) := \gamma(t\beta(1-s) + \beta(t)\beta(s))$$

defines a thin homotopy between them.

The composition of paths defined above on PM descends to P^1M due to condition (a), which admits a smooth composition of smooth homotopies. The composition of thin homotopy classes of paths obeys the following rules:

Lemma 2.3. *For any path $\gamma : x \longrightarrow y$,*

a) $\overline{\gamma} \circ \overline{\mathrm{id}_x} = \overline{\gamma} = \overline{\mathrm{id}_y} \circ \overline{\gamma}$,

b) *For further paths $\gamma' : y \longrightarrow z$ and $\gamma'' : z \longrightarrow w$,*
$$(\overline{\gamma''} \circ \overline{\gamma'}) \circ \overline{\gamma} = \overline{\gamma''} \circ (\overline{\gamma'} \circ \overline{\gamma}).$$

c) $\overline{\gamma} \circ \overline{\gamma^{-1}} = \overline{\mathrm{id}_y}$ *and* $\overline{\gamma^{-1}} \circ \overline{\gamma} = \overline{\mathrm{id}_x}$.

These three properties lead us to the following

Definition 2.4. *For a smooth manifold M, we consider the category whose set of objects is M, whose set of morphisms is P^1M, where a class $\overline{\gamma} : x \longrightarrow y$ is a morphism from x to y, and the composition is as described above. Lemma 2.3 a) and b) are the axioms of a category and c) says that every morphism is invertible. Hence we have defined a groupoid, called the* <u>*path groupoid*</u> *of M, and denoted by $\mathcal{P}_1(M)$.*

For a smooth map $f : M \longrightarrow N$, we denote by

$$f_* : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1(N)$$

the functor with $f_*(x) = f(x)$ and $(f_*)(\overline{\gamma}) := \overline{f \circ \gamma}$. The latter is well-defined, since a thin homotopy h between paths γ and γ' induces a thin homotopy $f \circ h$ between $f \circ \gamma$ and $f \circ \gamma'$.

In the following we consider functors

$$F : \mathcal{P}_1(M) \longrightarrow T \tag{2.1}$$

for some arbitrary category T. Such a functor sends each point $p \in M$ to an object $F(p)$ in T, and each thin homotopy class $\overline{\gamma} : x \longrightarrow y$ of paths to a morphism $F(\overline{\gamma}) : F(x) \longrightarrow F(y)$ in T. We use the following notation: we call M the *base space* of the functor F, and the object $F(p)$ the *fibre* of F over p. In the remainder of this section we give examples of natural constructions with functors (2.1).

Additional Structure on T. Any additional structure for the category T can be applied pointwise to functors into T, for instance,

a) if T has direct sums, we can take the direct sum $F_1 \oplus F_2$ of two functors.

b) if T is a monoidal category, we can take tensor products $F_1 \otimes F_2$ of functors.

c) if T is monoidal and has a duality regarded as a functor $d : T \longrightarrow T^{\mathrm{op}}$, we can form the dual $F^* := d \circ F$ of a functor F.

Pullback. If $f : M \longrightarrow N$ is a smooth map and $F : \mathcal{P}_1(N) \longrightarrow T$ is a functor, we define

$$f^*F := F \circ f_* : \mathcal{P}_1(M) \longrightarrow T$$

to be the pullback of F along f.

Flat Functors. Instead of the path groupoid, one can also consider the fundamental groupoid $\Pi_1(M)$ of a smooth manifold M, whose objects are points in M, just like for $\mathcal{P}_1(M)$, but whose morphisms are smooth homotopy classes of paths (whose differential may have arbitrary rank). The projection from thin homotopy classes to smooth homotopy classes provides a functor

$$p : \mathcal{P}_1(M) \longrightarrow \Pi_1(M).$$

We call a functor $F : \mathcal{P}_1(M) \longrightarrow T$ *flat*, if there exists a functor $\tilde{F} : \Pi_1(M) \longrightarrow T$ with $F \cong \tilde{F} \circ p$. This is motivated by parallel transport in principal G-bundles: while it is invariant under *thin* homotopy, it is only homotopy invariant if the bundle is flat, i.e. has vanishing curvature. However, aside from Section 7.2 we will not discuss the flat case any further in this article.

Restriction to Paths between fixed Points. Finally, let us consider the restriction of a functor $F : \mathcal{P}_1(M) \longrightarrow T$ to paths between two fixed points. This yields a map

$$F_{x,y} : \mathrm{Mor}_{\mathcal{P}_1(M)}(x, y) \longrightarrow \mathrm{Mor}_T(F(x), F(y)).$$

Of particular interest is the case $x = y$, in which $\mathrm{Mor}_{\mathcal{P}_1(M)}(x, x)$ forms a group under composition, which is called the thin homotopy group of M at x, and is denoted by $\pi_1^1(M, x)$ [CP94, MP02]. Even more particular, we consider the target category G-Tor: by choosing a diffeomorphism $F(x) \cong G$, we obtain an identification

$$\mathrm{Mor}_{G\text{-Tor}}(F(x), F(x)) = G,$$

and the restriction $F_{x,x}$ of a functor $F : \mathcal{P}_1(M) \longrightarrow G$-Tor to the thin homotopy group of M at x gives a group homomorphism

$$F_{x,x} : \pi_1^1(M, x) \longrightarrow G.$$

This way one obtains the setup of [Bar91, CP94] and [MP02] for the case $G = U(1)$ as a particular case of our setup. A further question is, whether the group homomorphism $F_{x,x}$ is smooth in the sense used in [Bar91, CP94, MP02]. An answer is given in Section 5.2.

2.2 Extracting Descent Data from a Functor

To define local trivializations of a functor $F : \mathcal{P}_1(M) \longrightarrow T$, we fix three attributes:

1. A surjective submersion $\pi : Y \longrightarrow M$. Compared to local trivializations of fibre bundles, the surjective submersion replaces an open cover of the manifold. Indeed, given an open cover $\{U_\alpha\}_{\alpha \in A}$ of M, one obtains a surjective submersion by taking Y to be the disjoint union of the U_α and $\pi : Y \longrightarrow M$ to be the union of the inclusions $U_\alpha \hookrightarrow M$.

2. A Lie groupoid Gr, i.e. a groupoid whose sets of objects and morphisms are smooth manifolds, whose source and target maps

$$s, t : \mathrm{Mor}(\mathrm{Gr}) \longrightarrow \mathrm{Obj}(\mathrm{Gr})$$

are surjective submersions, and whose composition

$$\circ : \mathrm{Mor}(\mathrm{Gr}) \;{}_s\!\times_t\; \mathrm{Mor}(\mathrm{Gr}) \longrightarrow \mathrm{Mor}(\mathrm{Gr})$$

and the identity $\mathrm{id} : \mathrm{Obj}(\mathrm{Gr}) \longrightarrow \mathrm{Mor}(\mathrm{Gr})$ are smooth maps. The Lie groupoid Gr plays the role of the typical fibre of the functor F.

3. A functor $i : \mathrm{Gr} \longrightarrow T$, which relates the typical fibre Gr to the target category T of the functor F. In all of our examples, i will be an equivalence of categories. This is important for some results derived in Section 3.2.

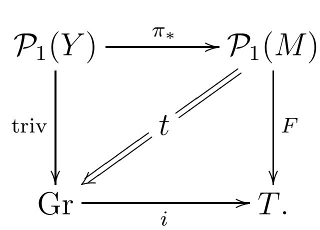
Definition 2.5. *Given a Lie groupoid* Gr*, a functor* $i : \mathrm{Gr} \longrightarrow T$ *and a surjective submersion* $\pi : Y \longrightarrow M$*, a* *$\pi$-local i-trivialization* *of a functor*

$$F : \mathcal{P}_1(M) \longrightarrow T$$

is a pair (triv, t) *of a functor* $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$ *and a natural equivalence*

$$t : \pi^* F \longrightarrow i \circ \mathrm{triv}.$$

The natural equivalence t is also depicted by the diagram



To set up the familiar terminology, we call a functor *locally i-trivializable*, if it admits a π-local i-trivialization for some choice of π. We call a functor

i-trivial, if it admits an id_M-local i-trivialization, i.e. if it is naturally equivalent to the functor $i \circ \mathrm{triv}$. To abbreviate the notation, we will often write triv_i instead of $i \circ \mathrm{triv}$.

Note that local trivializations can be pulled back: if $\zeta : Z \longrightarrow Y$ and $\pi : Y \longrightarrow M$ are surjective submersions, and (triv, t) is a π-local i-trivialization of a functor F, we obtain a $(\pi \circ \zeta)$-local i-trivialization $(\zeta^*\mathrm{triv}, \zeta^*t)$ of F. In terms of open covers, this corresponds to a refinement of the cover.

Definition 2.6. *Let* Gr *be a Lie groupoid and let* $i : \mathrm{Gr} \longrightarrow T$ *be a functor. The category* $\mathrm{Triv}^1_\pi(i)$ *of functors with π-local i-trivialization is defined as follows:*

1. *its objects are triples* (F, triv, t) *consisting of a functor* $F : \mathcal{P}_1(M) \longrightarrow T$ *and a π-local i-trivialization* (triv, t) *of* F*.*

2. *a morphism*
$$(F, \mathrm{triv}, t) \xrightarrow{\alpha} (F', \mathrm{triv}', t')$$
is a natural transformation $\alpha : F \longrightarrow F'$*. Composition of morphisms is simply composition of these natural transformations.*

Motivated by transition functions of fibre bundles, we extract a similar datum from a functor F with π-local i-trivialization (triv, t); this datum is a natural equivalence
$$g : \pi_1^*\mathrm{triv}_i \longrightarrow \pi_2^*\mathrm{triv}_i$$
between the two functors $\pi_1^*\mathrm{triv}_i$ and $\pi_2^*\mathrm{triv}_i$ from $\mathcal{P}_1(Y^{[2]})$ to T, where π_1 and π_2 are the projections from the two-fold fibre product $Y^{[2]} := Y \times_M Y$ of Y to the components. In the case that the surjective submersion comes from an open cover of M, $Y^{[2]}$ is the disjoint union of all two-fold intersections of open subsets. The natural equivalence g is defined by
$$g := \pi_2^*t \circ \pi_1^*t^{-1};$$
its component at a point $\alpha \in Y^{[2]}$ is the morphism $t(\pi_2(\alpha)) \circ t(\pi_1(\alpha))^{-1}$ in T. The composition is well-defined because $\pi \circ \pi_1 = \pi \circ \pi_2$.

Transition functions of fibre bundles satisfy a cocycle condition over three-fold intersections. The natural equivalence g has a similar property when pulled back to the three-fold fibre product $Y^{[3]} := Y \times_M Y \times_M Y$.

Proposition 2.7. *The diagram*

$$\begin{array}{ccccc} & & \pi_2^*\mathrm{triv}_i & & \\ & {\scriptstyle \pi_{12}^*g}\nearrow & & \searrow{\scriptstyle \pi_{23}^*g} & \\ \pi_1^*\mathrm{triv}_i & & \xrightarrow[\pi_{13}^*g]{} & & \pi_3^*\mathrm{triv}_i \end{array}$$

of natural equivalences between functors from $\mathcal{P}_1(Y^{[3]})$ *to* T *is commutative.*

Now that we have defined the data (triv, g) associated to an object (F, triv, t) in $\mathrm{Triv}^1_\pi(i)$, we consider a morphism

$$\alpha : (F, \mathrm{triv}, t) \longrightarrow (F', \mathrm{triv}', t')$$

between two functors with π-local i-trivializations, i.e. a natural transformation $\alpha : F \longrightarrow F'$. We define a natural transformation

$$h : \mathrm{triv}_i \longrightarrow \mathrm{triv}'_i$$

by $h := t' \circ \pi^*\alpha \circ t^{-1}$, whose component at $x \in Y$ is the morphism $t'(x) \circ \alpha(\pi(x)) \circ t(x)^{-1}$ in T. From the definitions of g, g' and h one obtains the commutative diagram

$$\begin{array}{ccc} \pi_1^*\mathrm{triv}_i & \xrightarrow{g} & \pi_2^*\mathrm{triv}_i \\ {\scriptstyle \pi_1^*h}\downarrow & & \downarrow{\scriptstyle \pi_2^*h} \\ \pi_1^*\mathrm{triv}'_i & \xrightarrow[g']{} & \pi_2^*\mathrm{triv}'_i . \end{array} \tag{2.2}$$

The behaviour of the natural equivalences data g and h leads to the following definition of a category $\mathfrak{Des}^1_\pi(i)$ of descent data. This terminology will be explained in the next section.

Definition 2.8. *The category* $\mathfrak{Des}^1_\pi(i)$ *of* *descent data* *of* π*-locally* i*-trivialized functors is defined as follows:*

1. *its objects are pairs* (triv, g) *of a functor* $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$ *and a natural equivalence*

$$g : \pi_1^*\mathrm{triv}_i \longrightarrow \pi_2^*\mathrm{triv}_i,$$

such that the diagram

$$\begin{array}{ccccc} & & \pi_2^*\mathrm{triv}_i & & \\ & \overset{\pi_{12}^*g}{\nearrow} & & \overset{\pi_{23}^*g}{\searrow} & \\ \pi_1^*\mathrm{triv}_i & & \xrightarrow[\pi_{13}^*g]{} & & \pi_3^*\mathrm{triv}_i \end{array} \tag{2.3}$$

is commutative.

2. *a morphism* $(\mathrm{triv}, g) \longrightarrow (\mathrm{triv}', g')$ *is a natural transformation*

$$h : \mathrm{triv}_i \longrightarrow \mathrm{triv}'_i$$

such that the diagram

$$\begin{array}{ccc} \pi_1^*\mathrm{triv}_i & \xrightarrow{g} & \pi_2^*\mathrm{triv}_i \\ {\scriptstyle \pi_1^*h}\downarrow & & \downarrow{\scriptstyle \pi_2^*h} \\ \pi_1^*\mathrm{triv}'_i & \xrightarrow[g']{} & \pi_2^*\mathrm{triv}'_i. \end{array} \tag{2.4}$$

is commutative. The composition is the composition of these natural transformations.

Summarizing, we have defined a functor

$$\mathrm{Ex}_\pi : \mathrm{Triv}^1_\pi(i) \longrightarrow \mathfrak{Des}^1_\pi(i), \tag{2.5}$$

that extracts descent data from functors with local trivialization and of morphisms of those in the way described above.

2.3 Reconstructing a Functor from Descent Data

In this section we show that extracting descent data from a functor F preserves all information about F. We also justify the terminology *descent data*, see Remark 2.10 below.

Theorem 2.9. *The functor*

$$\mathrm{Ex}_\pi : \mathrm{Triv}^1_\pi(i) \longrightarrow \mathfrak{Des}^1_\pi(i)$$

is an equivalence of categories.

For the proof we define a weak inverse functor

$$\mathrm{Rec}_\pi : \mathfrak{Des}^1_\pi(i) \longrightarrow \mathrm{Triv}^1_\pi(i) \tag{2.6}$$

that reconstructs a functor (and a π-local i-trivialization) from given descent data. The definition of Rec_π is given in three steps:

1. We construct a groupoid $\mathcal{P}_1^\pi(M)$ covering the path groupoid $\mathcal{P}_1(M)$ by means of a surjective functor $p^\pi : \mathcal{P}_1^\pi(M) \longrightarrow \mathcal{P}_1(M)$, and show that any object (triv, g) in $\mathfrak{Des}^1_\pi(i)$ gives rise to a functor

   $$R_{(\mathrm{triv},g)} : \mathcal{P}_1^\pi(M) \longrightarrow T.$$

   We enhance this to a functor

   $$R : \mathfrak{Des}^1_\pi(i) \longrightarrow \mathrm{Funct}(\mathcal{P}_1^\pi(M), T), \tag{2.7}$$

   where $\mathrm{Funct}(\mathcal{P}_1^\pi(M), T)$ is the category of functors from $\mathcal{P}_1^\pi(M)$ to T and natural transformations between those.

2. We show that the functor $p^\pi : \mathcal{P}_1^\pi(M) \longrightarrow \mathcal{P}_1(M)$ is an equivalence of categories and construct a weak inverse

   $$s : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M).$$

   The pullback along s is the functor

   $$s^* : \mathrm{Funct}(\mathcal{P}_1^\pi(M), T) \longrightarrow \mathrm{Funct}(\mathcal{P}_1(M), T) \tag{2.8}$$

   obtained by pre-composition with s.

3. By constructing canonical π-local i-trivializations of functors in the image of the composition $s^* \circ R$ of the functors (2.7) and (2.8), we extend this composition to a functor

   $$\mathrm{Rec}_\pi := s^* \circ R : \mathfrak{Des}^1_\pi(i) \longrightarrow \mathrm{Triv}^1_\pi(i).$$

   Finally, we give in Appendix B.1 the proof that Rec_π is a weak inverse of the functor Ex_π and thus show that Ex_π is an equivalence of categories.

Before we perform the steps 1 to 3, let us make the following remark about the nature of the category $\mathfrak{Des}^1_\pi(i)$ and the functor Rec_π.

Remark 2.10. We consider the case $i := \mathrm{id}_{\mathrm{Gr}}$. Now, the forgetful functor $v : \mathrm{Triv}^1_\pi(i) \longrightarrow \mathrm{Funct}(\mathcal{P}_1(M), \mathrm{Gr})$ has a canonical weak inverse, which associates to a functor $F : \mathcal{P}_1(M) \longrightarrow \mathrm{Gr}$ the π-local i-trivialization $(\pi^*F, \mathrm{id}_{\pi^*F})$. Under this identification, $\mathfrak{Des}^1_\pi(i)$ is the descent category of the functor category $\mathrm{Funct}(M, \mathrm{Gr})$ with respect to π in the sense of a stack [Moe02, Str04]. The functor

$$\mathrm{Rec}_\pi : \mathfrak{Des}^1_\pi(i) \longrightarrow \mathrm{Funct}(\mathcal{P}_1(M), \mathrm{Gr})$$

realizes the descent.

Step 1: The Groupoid $\mathcal{P}_1^\pi(M)$. The groupoid $\mathcal{P}_1^\pi(M)$ we introduce is the *universal path pushout* associated to the surjective submersion $\pi : Y \longrightarrow M$. Heuristically, $\mathcal{P}_1^\pi(M)$ is the path groupoid of the covering Y combined with "jumps" in the fibres of π. We explain its universality in Appendix A.1 for completeness and introduce here a concrete realization (see Lemma A.4).

Definition 2.11. *The groupoid $\mathcal{P}_1^\pi(M)$ is defined as follows. Its objects are points $x \in Y$ and its morphisms are formal (finite) compositions of two types of basic morphisms: thin homotopy classes $\overline{\gamma} : x \longrightarrow y$ of paths in Y, and points $\alpha \in Y^{[2]}$ regarded as morphisms $\alpha : \pi_1(\alpha) \longrightarrow \pi_2(\alpha)$. Among the morphisms, we impose three relations:*

(1) for any thin homotopy class $\overline{\Theta} : \alpha \longrightarrow \beta$ of paths in $Y^{[2]}$, we demand that the diagram

$$\begin{array}{ccc} \pi_1(\alpha) & \xrightarrow{\alpha} & \pi_2(\alpha) \\ {\scriptstyle (\pi_1)_*(\overline{\Theta})}\Big\downarrow & & \Big\downarrow{\scriptstyle (\pi_2)_*(\overline{\Theta})} \\ \pi_1(\beta) & \xrightarrow[\beta]{} & \pi_2(\beta). \end{array}$$

of morphisms in $\mathcal{P}_1^\pi(M)$ is commutative.

(2) for any point $\Xi \in Y^{[3]}$, we demand that the diagram

$$\begin{array}{ccccc} & & \pi_2(\Xi) & & \\ & {\scriptstyle \pi_{12}(\Xi)}\nearrow & & \searrow{\scriptstyle \pi_{23}(\Xi)} & \\ \pi_1(\Xi) & & \xrightarrow[\pi_{13}(\Xi)]{} & & \pi_3(\Xi) \end{array}$$

of morphisms in $\mathcal{P}_1^\pi(M)$ is commutative.

(3) we impose the equation $\overline{\mathrm{id}_x} = (x, x) \in Y^{[2]}$ for any $x \in Y$.

It is clear that this definition indeed gives a groupoid. It is important for us because it provides the two following natural definitions.

Definition 2.12. *For an object (triv, g) in $\mathfrak{Des}_\pi^1(i)$, we have a functor*

$$R_{(\mathrm{triv},g)} : \mathcal{P}_1^\pi(M) \longrightarrow T$$

that sends an object $x \in Y$ to $\mathrm{triv}_i(x)$, a basic morphism $\overline{\gamma} : x \longrightarrow y$ to $\mathrm{triv}_i(\overline{\gamma})$ and a basic morphism α to $g(\alpha)$.

The definition is well-defined since it respects the relations among the morphisms: (1) is respected due to the commutative diagram for the natural transformation g, (2) is the cocycle condition (2.3) for g and (3) follows from the latter since g is invertible.

Definition 2.13. *For a morphism $h : (\mathrm{triv}, g) \longrightarrow (\mathrm{triv}', g')$ in $\mathfrak{Des}^1_\pi(i)$ we have a natural transformation*

$$R_h : R_{(\mathrm{triv},g)} \longrightarrow R_{(\mathrm{triv}',g')}$$

that sends an object $x \in Y$ to the morphism $h(x)$ in T.

The commutative diagram for the natural transformation R_h for a basic morphism $\overline{\gamma} : x \longrightarrow y$ follows from the one of h, and for a basic morphism $\alpha \in Y^{[2]}$ from the condition (2.4) on the morphisms of $\mathfrak{Des}^1_\pi(i)$.

We explain in Appendix A.1 that Definitions (2.12) and (2.13) are consequences of the universal property of the groupoid $\mathcal{P}^\pi_1(M)$, as specified in Definition A.1 and calculated in Lemma A.4. Here we summarize the definitions above in the following way:

Lemma 2.14. *Definitions (2.12) and (2.13) yield a functor*

$$R : \mathfrak{Des}^1_\pi(i) \longrightarrow \mathrm{Funct}(\mathcal{P}^\pi_1(M), T). \tag{2.9}$$

Step 2: Pullback to M. To continue the reconstruction of a functor from given descent data let us introduce the projection functor

$$p^\pi : \mathcal{P}^\pi_1(M) \longrightarrow \mathcal{P}_1(M) \tag{2.10}$$

sending an object $x \in Y$ to $\pi(x)$, a basic morphism $\overline{\gamma} : x \longrightarrow y$ to $\pi_*(\overline{\gamma})$ and a basic morphism $\alpha \in Y^{[2]}$ to $\mathrm{id}_{\pi(\pi_1(\alpha))}$ $(= \mathrm{id}_{\pi(\pi_2(\alpha))})$. In other words, it is just the functor π_* and forgets the jumps in the fibres of π. More precisely,

$$p^\pi \circ \iota = \pi_*,$$

where $\iota : \mathcal{P}_1(Y) \longrightarrow \mathcal{P}^\pi_1(M)$ is the obvious inclusion functor.

Lemma 2.15. *The projection functor $p^\pi : \mathcal{P}^\pi_1(M) \longrightarrow \mathcal{P}_1(M)$ is a surjective equivalence of categories.*

Proof. Since $\pi : Y \longrightarrow M$ is surjective, it is clear that p^π is surjective on objects. It remains to show that the map

$$(p^\pi)_1 : \mathrm{Mor}_{\mathcal{P}_1^\pi(M)}(x,y) \longrightarrow \mathrm{Mor}_{\mathcal{P}_1(M)}(\pi(x),\pi(y)) \tag{2.11}$$

is bijective for all $x, y \in Y$. Let $\gamma : \pi(x) \longrightarrow \pi(y)$ be any path in M. Let $\{U_i\}_{i\in I}$ an open cover of M with sections $s_i : U_i \longrightarrow Y$. Since the image of $\gamma : [0,1] \longrightarrow M$ is compact, there exists a *finite* subset $J \subset I$ such that $\{U_i\}_{i\in J}$ covers the image. Let $\gamma = \gamma_n \circ ... \circ \gamma_1$ be a decomposition of γ such that $\gamma_i \in PU_{j(i)}$ for some assignment $j : \{1, ..., n\} \longrightarrow J$. Let $\tilde{\gamma}_i := (s_{j(i)})_*\gamma_i \in PY$ be lifts of the pieces, $\tilde{\gamma}_i : a_i \longrightarrow b_i$ with $a_i, b_i \in Y$. Now we consider the path

$$\tilde{\gamma} := (b_n, y) \circ \tilde{\gamma}_n \circ (b_{n-1}, a_n) \circ ... \circ \tilde{\gamma}_2 \circ (b_1, a_2) \circ \tilde{\gamma}_1 \circ (x, a_1),$$

whose thin homotopy class is evidently a preimage of the thin homotopy class of γ under $(p^\pi)_1$. The injectivity of (2.11) follows from the identifications (1), (2) and (3) of morphisms in the groupoid $\mathcal{P}_1^\pi(M)$. ☐

Since p^π is an equivalence of categories, there exists a (up to natural isomorphism) unique weak inverse functor $s : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M)$ together with natural equivalences $\lambda : s \circ p^\pi \longrightarrow \mathrm{id}_{\mathcal{P}_1^\pi(M)}$ and $\rho : p^\pi \circ s \longrightarrow \mathrm{id}_{\mathcal{P}_1(M)}$. The inverse functor s can be constructed explicitly: for a fixed choice of lifts $s(x) \in Y$ for every point $x \in M$, and a fixed choice of an open cover, each path can be lifted as described in the proof of Lemma 2.15. In this case we have $\rho = \mathrm{id}$, and the component of λ at $x \in Y$ is the morphism $(s(\pi(x)), x)$ in $\mathcal{P}_1^\pi(M)$. Now we have a canonical functor

$$s^* \circ R : \mathfrak{Des}_\pi^1(i) \longrightarrow \mathrm{Funct}(\mathcal{P}_1(M), T).$$

It reconstructs a functor $s^*R_{(\mathrm{triv},g)}$ from a given object (triv, g) in $\mathfrak{Des}_\pi^1(i)$ and a natural transformation s^*R_h from a given morphism h in $\mathfrak{Des}_\pi^1(i)$.

Step 3: Local Trivialization. What remains to enhance the functor $s^* \circ R$ to a functor

$$\mathrm{Rec}_\pi : \mathfrak{Des}_\pi^1(i) \longrightarrow \mathrm{Triv}_\pi^1(i)$$

is finding a π-local i-trivialization (triv, t) of each reconstructed functor $s^*R_{(\mathrm{triv},g)}$. Of course the given functor $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$ serves as the first component of the trivialization, and it remains to define the natural equivalence

$$t : \pi^* s^* R_{(\mathrm{triv},g)} \longrightarrow \mathrm{triv}_i. \tag{2.12}$$

We use the natural equivalence $\lambda : s \circ p^\pi \longrightarrow \mathrm{id}_{\mathcal{P}_1^\pi(M)}$ associated to the functor s and obtain a natural equivalence

$$\iota^*\lambda : s \circ \pi_* \longrightarrow \iota$$

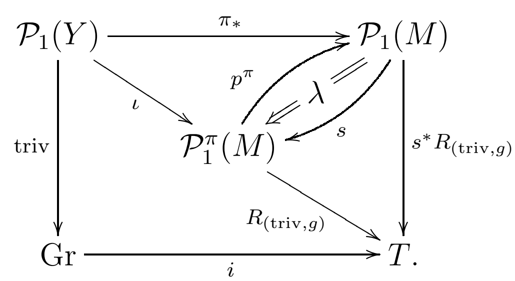
between functors from $\mathcal{P}_1(Y)$ to $\mathcal{P}_1^\pi(M)$. Its component at $x \in Y$ is the morphism $(s(\pi(x)), x)$ going from $s(\pi(x))$ to x. Using

$$\pi^* s^* R_{(\mathrm{triv},g)} = (s \circ \pi_*)^* R_{(\mathrm{triv},g)} \quad \text{and} \quad \mathrm{triv}_i = \iota^* R_{(\mathrm{triv},g)},$$

we define by

$$t := g \circ \iota^*\lambda$$

the natural equivalence (2.12). Indeed, its component at $x \in Y$ is the morphism $g((s(\pi(x)), x)) : \mathrm{triv}_i(s(\pi(x))) \longrightarrow \mathrm{triv}_i(x)$, these are natural in x and isomorphisms because g is one. Diagrammatically, it is



This shows

Lemma 2.16. *The pair* (triv, t) *is a* π*-local* i*-trivialization of the functor* $s^*R_{(\mathrm{triv},g)}$.

This finishes the definition of the reconstruction functor Rec_π. The remaining proof that Rec_π is a weak inverse of Ex_π is postponed to Appendix B.1.

3 Transport Functors

Transport functors are locally trivializable functors whose descent data is smooth. Wilson lines are restrictions of a functor to paths between two fixed points. We deduce a characterization of transport functors by the smoothness of their Wilson lines.

3.1 Smooth Descent Data

In this section we specify a subcategory $\mathfrak{Des}^1_\pi(i)^\infty$ of the category $\mathfrak{Des}^1_\pi(i)$ of descent data we have defined in the previous section. This subcategory is supposed to contain *smooth* descent data. The main issue is to decide, when a functor $F : \mathcal{P}_1(X) \longrightarrow \mathrm{Gr}$ is smooth: in contrast to the objects and the morphisms of the Lie groupoid Gr, the set P^1X of morphisms of $\mathcal{P}_1(X)$ is not a smooth manifold.

Definition 3.1. *Let* Gr *be a Lie groupoid and let X be a smooth manifold. A functor $F : \mathcal{P}_1(X) \longrightarrow \mathrm{Gr}$ is called* <u>smooth</u>*, if the following two conditions are satisfied:*

1. *On objects, $F : X \longrightarrow \mathrm{Obj(Gr)}$ is a smooth map.*
2. *For every $k \in \mathbb{N}_0$, every open subset $U \subset \mathbb{R}^k$ and every map $c : U \longrightarrow PX$ such that the composite*

$$U \times [0,1] \xrightarrow{c\times\mathrm{id}} PX \times [0,1] \xrightarrow{\mathrm{ev}} X \tag{3.1}$$

is smooth, also

$$U \xrightarrow{c} PX \xrightarrow{\mathrm{pr}} P^1X \xrightarrow{F} \mathrm{Mor(Gr)}$$

is smooth.

In (3.1), ev is the evaluation map $\mathrm{ev}(\gamma, t) := \gamma(t)$. Similar definitions of smooth maps defined on thin homotopy classes of paths have also been used in [Bar91, CP94, MP02]. We explain in Appendix A.2 how Definition 3.1 is motivated and how it arises from the general concept of diffeological spaces [Che77], a generalization of the concept of a smooth manifold, cf. Proposition A.7 i).

Definition 3.2. *A natural transformation $\eta : F \longrightarrow G$ between smooth functors $F, G : \mathcal{P}_1(X) \longrightarrow \mathrm{Gr}$ is called* <u>smooth</u>*, if its components form a smooth map $X \longrightarrow \mathrm{Mor(Gr)} : X \longmapsto \eta(X)$.*

Because the composition in the Lie groupoid Gr is smooth, compositions of smooth natural transformations are again smooth. Hence, smooth functors and smooth natural transformations form a category $\mathrm{Funct}^\infty(\mathcal{P}_1(X), \mathrm{Gr})$. Notice that if $f : M \longrightarrow X$ is a smooth map, and $F : \mathcal{P}_1(X) \longrightarrow \mathrm{Gr}$ is a smooth functor, the pullback f^*F is also smooth. Similarly, pullbacks of smooth natural transformations are smooth.

Definition 3.3. *Let* Gr *be a Lie groupoid and let* $i : \mathrm{Gr} \longrightarrow T$ *be a functor. An object* (triv, g) *in* $\mathfrak{Des}^1_\pi(i)$ *is called* smooth*, if the following two conditions are satisfied:*

1. *The functor*
$$\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$$
is smooth in the sense of Definition 3.1.

2. *The natural equivalence*
$$g : \pi_1^*\mathrm{triv}_i \longrightarrow \pi_2^*\mathrm{triv}_i$$
factors through i *by a natural equivalence* $\tilde{g} : \pi_1^*\mathrm{triv} \longrightarrow \pi_2^*\mathrm{triv}$ *which is smooth in the sense of Definition 3.2. For the components at a point* $\alpha \in Y^{[2]}$*, the factorization means* $g(\alpha) = i(\tilde{g}(\alpha))$*.*

In the same sense, a morphism
$$h : (\mathrm{triv}, g) \longrightarrow (\mathrm{triv}', g')$$
between smooth objects is called smooth, if it factors through i *by a smooth natural equivalence* $\tilde{h} : \mathrm{triv} \longrightarrow \mathrm{triv}'$*.*

Remark 3.4. If i is faithful, the natural equivalences $\tilde{g}$ and $\tilde{h}$ in Definition 3.3 are uniquely determined, provided that they exist. If i is additionally full, also the existence of g and h is guaranteed.

Smooth objects and morphisms in $\mathfrak{Des}^1_\pi(i)$ form the subcategory $\mathfrak{Des}^1_\pi(i)^\infty$. Using the equivalence Ex_π defined in Section 2.2, we obtain a subcategory $\mathrm{Triv}^1_\pi(i)^\infty$ of $\mathrm{Triv}^1_\pi(i)$ consisting of those objects (F, triv, t) for which $\mathrm{Ex}_\pi(F, \mathrm{triv}, t)$ is smooth and of those morphisms h for which $\mathrm{Ex}_\pi(h)$ is smooth.

Proposition 3.5. *The functor* $\mathrm{Rec}_\pi : \mathfrak{Des}^1_\pi(i) \longrightarrow \mathrm{Triv}^1_\pi(i)$ *restricts to an equivalence of categories*
$$\mathrm{Rec}_\pi : \mathfrak{Des}^1_\pi(i)^\infty \longrightarrow \mathrm{Triv}^1_\pi(i)^\infty.$$

Proof. This follows from the fact that $\mathrm{Ex}_\pi \circ \mathrm{Rec}_\pi = \mathrm{id}_{\mathfrak{Des}^1_\pi(i)}$, see the proof of Theorem 2.9 in Appendix B.1. □

Now we are ready to define transport functors.

Definition 3.6. *Let M be a smooth manifold, T a category,* Gr *a Lie groupoid and $i : \mathrm{Gr} \longrightarrow T$ a functor.*

1. *A transport functor on M in T with* Gr*-structure is a functor*

$$\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow T$$

such that there exists a surjective submersion $\pi : Y \longrightarrow M$ and a π-local i-trivialization (triv, t), such that $\mathrm{Ex}_\pi(\mathrm{tra}, \mathrm{triv}, t)$ is smooth.

2. *A morphism between transport functors on M in T with* Gr*-structure is a natural equivalence $\eta : \mathrm{tra} \longrightarrow \mathrm{tra}'$ such that there exists a surjective submersion $\pi : Y \longrightarrow M$ together with π-local i-trivializations of* tra *and* tra'*, such that $\mathrm{Ex}_\pi(\eta)$ is smooth.*

It is clear that the identity natural transformation of a transport functor tra is a morphism in the above sense. To show that the composition of morphisms between transport functors is possible, note that if $\pi : Y \longrightarrow M$ is a surjective submersion for which $\mathrm{Ex}_\pi(\eta)$ is smooth, and $\zeta : Z \longrightarrow Y$ is another surjective submersion, then also $\mathrm{Ex}_{\pi\circ\zeta}(\eta)$ is smooth. If now $\eta : \mathrm{tra} \longrightarrow \mathrm{tra}'$ and $\eta' : \mathrm{tra}' \longrightarrow \mathrm{tra}''$ are morphisms of transport functors, and $\pi : Y \longrightarrow M$ and $\pi' : Y' \longrightarrow M$ are surjective submersions for which $\mathrm{Ex}_\pi(\eta)$ and $\mathrm{Ex}_{\pi'}(\eta')$ are smooth, the fibre product $\tilde{\pi} : Y \times_M Y' \longrightarrow M$ is a surjective submersion and factors through π and π' by surjective submersions. Hence, $\mathrm{Ex}_{\tilde{\pi}}(\eta' \circ \eta)$ is smooth.

Definition 3.7. *The category of all transport functors on M in T with* Gr*-structure and all morphisms between those is denoted by* $\mathrm{Trans}^1_{\mathrm{Gr}}(M, T)$*.*

From the definition of a transport functor with Gr-structure it is not clear that, for a fixed surjective submersion $\pi : Y \longrightarrow M$, all choices of π-local i-trivializations (triv, t) with smooth descent data give rise to isomorphic objects in $\mathfrak{Des}^1_\pi(i)^\infty$. This is at least true for full functors $i : \mathrm{Gr} \longrightarrow T$ and contractible surjective submersions: a surjective submersion $\pi : Y \longrightarrow M$ is called *contractible*, if there exists a smooth map $c : Y \times [0,1] \longrightarrow Y$ such that $c(y, 0) = y$ for all $y \in Y$ and $c(y, 1) = y_k$ for some fixed choice of $y_k \in Y_k$ for each connected component Y_k of Y. We may assume without loss of generality, that c has a sitting instant with respect to the second parameter, so that we can regard c also as a map $c : Y \longrightarrow PY$. For example, if Y is the disjoint union of the open sets of a good open cover of M, $\pi : Y \longrightarrow M$ is contractible.

Lemma 3.8. *Let $i : \mathrm{Gr} \longrightarrow T$ be a full functor, let $\pi : Y \longrightarrow M$ be a contractible surjective submersion and let (triv, t) and (triv', t') be two π-local i-trivializations of a transport functor $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow T$ with Gr-structure. Then, the identity natural transformation $\mathrm{id}_{\mathrm{tra}} : \mathrm{tra} \longrightarrow \mathrm{tra}$ defines a morphism*

$$\mathrm{id}_{\mathrm{tra}} : (\mathrm{tra}, \mathrm{triv}, t) \longrightarrow (\mathrm{tra}, \mathrm{triv}', t')$$

in $\mathrm{Triv}^1_\pi(i)^\infty$, in particular, $\mathrm{Ex}_\pi(\mathrm{tra}, \mathrm{triv}, t)$ and $\mathrm{Ex}_\pi(\mathrm{tra}, \mathrm{triv}', t')$ are isomorphic objects in $\mathfrak{Des}^1_\pi(i)^\infty$.

Proof. Let $c : Y \times [0,1] \longrightarrow Y$ be a smooth contraction, regarded as a map $c : Y \longrightarrow PY$. For each $y \in Y_k$ we have a path $c(y) : y \longrightarrow y_k$, and the commutative diagram for the natural transformation t gives

$$t(y) = \mathrm{triv}_i(\overline{c(y)})^{-1} \circ t(y_k) \circ \mathrm{tra}(\pi_*(\overline{c(y)})),$$

and analogously for t'. The descent datum of the natural equivalence $\mathrm{id}_{\mathrm{tra}}$ is the natural equivalence

$$h := \mathrm{Ex}_\pi(\mathrm{id}) = t' \circ t^{-1} : \mathrm{triv}_i \longrightarrow \mathrm{triv}'_i.$$

Its component at $y \in Y_k$ is the morphism

$$h(y) = \mathrm{triv}'_i(\overline{c(y)})^{-1} \circ t'(y_k) \circ t(y_k)^{-1} \circ \mathrm{triv}_i(\overline{c(y)}) : \mathrm{triv}_i(y) \longrightarrow \mathrm{triv}'_i(y) \quad (3.2)$$

in T. Since i is full, $t'(y_k) \circ t(y_k)^{-1} = i(\kappa_k)$ for some morphism $\kappa_k : \mathrm{triv}(y_k) \longrightarrow \mathrm{triv}'(y_k)$, so that h factors through i by

$$\tilde{h}(y) := \mathrm{triv}'(\overline{c(y)})^{-1} \circ \kappa_k \circ \mathrm{triv}(\overline{c(y)}) \in \mathrm{Mor}(\mathrm{Gr}).$$

Since triv and triv' are smooth functors, $\mathrm{triv} \circ \mathrm{pr} \circ c$ and $\mathrm{triv}' \circ \mathrm{pr} \circ c$ are smooth maps, so that the components of $\tilde{h}$ form a smooth map $Y \longrightarrow \mathrm{Mor}(\mathrm{Gr})$. Hence, h is a morphism in $\mathfrak{Des}^1_\pi(i)^\infty$. ☐

To keep track of all the categories we have defined, consider the following diagram of functors which is strictly commutative:

$$\begin{array}{ccccc} \mathfrak{Des}^1_\pi(i)^\infty & \xrightarrow{\mathrm{Rec}_\pi} & \mathrm{Triv}^1_\pi(i)^\infty & \xrightarrow{v^\infty} & \mathrm{Trans}^1_{\mathrm{Gr}}(M,T) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{Des}^1_\pi(i) & \xrightarrow[\mathrm{Rec}_\pi]{} & \mathrm{Triv}^1_\pi(i) & \xrightarrow[v]{} & \mathrm{Funct}(M,T) \end{array} \quad (3.3)$$

The vertical arrows are the inclusion functors, and v^∞ and v are forgetful functors. In the next subsection we show that the functor v^∞ is an equivalence of categories.

3.2 Wilson Lines of Transport Functors

We restrict functors to paths between two fixed points and study the smoothness of these restrictions. For this purpose we assume that the functor $i : \mathrm{Gr} \longrightarrow T$ is an equivalence of categories; this is the case in all examples of transport functors we give in Section 5.

Definition 3.9. *Let $F : \mathcal{P}_1(M) \longrightarrow T$ be a functor, let* Gr *be a Lie groupoid and let $i : \mathrm{Gr} \longrightarrow T$ be an equivalence of categories. Consider two points $x_1, x_2 \in M$ together with a choice of objects G_k in* Gr *and isomorphisms $t_k : F(x_k) \longrightarrow i(G_k)$ in T for $k = 1, 2$. Then, the map*

$$\mathcal{W}^{F,i}_{x_1,x_2} : \mathrm{Mor}_{\mathcal{P}_1(M)}(x, y) \longrightarrow \mathrm{Mor}_{\mathrm{Gr}}(G_1, G_2) : \overline{\gamma} \longmapsto i^{-1}(t_2 \circ F(\overline{\gamma}) \circ t_1^{-1})$$

is called the Wilson line of F from x_1 to x_2.

Note that because i is essentially surjective, the choices of objects G_k and morphisms $t_k : F(x_k) \longrightarrow G_k$ exist for all points $x_k \in M$. Because i is full and faithful, the morphism $t_2 \circ F(\overline{\gamma}) \circ t_1^{-1} : i(G_1) \longrightarrow i(G_2)$ has a unique preimage under i, which is the Wilson line. For a different choice $t'_k : F(x_k) \longrightarrow i(G'_k)$ of objects in Gr and isomorphisms in T the Wilson line changes like

$$\mathcal{W}^{F,i}_{x_1,x_2} \longmapsto \tau_2^{-1} \circ \mathcal{W}^{F,i}_{x_1,x_2} \circ \tau_1$$

for $\tau_k : G'_k \longrightarrow G_k$ defined by $i(\tau_k) = t_k \circ t'^{-1}_k$.

Definition 3.10. *A Wilson line $\mathcal{W}^{F,i}_{x_1,x_2}$ is called smooth, if for every $k \in \mathbb{N}_0$, every open subset $U \subset \mathbb{R}^k$ and every map $c : U \longrightarrow PM$ such that $c(u)(t) \in M$ is smooth on $U \times [0, 1]$, $c(u, 0) = x_1$ and $c(u, 1) = x_2$ for all $u \in U$, also the map*

$$\mathcal{W}^{F,i}_{x_1,x_2} \circ \mathrm{pr} \circ c : U \longrightarrow \mathrm{Mor}_{\mathrm{Gr}}(G_1, G_2)$$

is smooth.

This definition of smoothness arises again from the context of diffeological spaces, see Proposition A.6 i) in Appendix A.2. Notice that if a Wilson line is smooth for some choice of objects G_k and isomorphisms t_k, it is smooth for any other choice. For this reason we have not labelled Wilson lines with additional indices G_1, G_2, t_1, t_2.

Lemma 3.11. *Let $i : \mathrm{Gr} \longrightarrow T$ be an equivalence of categories, let*

$$F : \mathcal{P}_1(M) \longrightarrow T$$

be a functor whose Wilson lines $\mathcal{W}^{F,i}_{x_1,x_2}$ are smooth for all points $x_1, x_2 \in M$, and let $\pi : Y \longrightarrow M$ be a contractible surjective submersion. Then, F admits a π-local i-trivialization (triv, t) *whose descent data* $\mathrm{Ex}_\pi(\mathrm{triv}, t)$ *is smooth.*

Proof. We choose a smooth contraction $r : Y \longrightarrow PY$ and make, for every connected component Y_k of Y, a choice of objects G_k in Gr and isomorphisms $t_k : F(\pi(y_k)) \longrightarrow i(G_k)$. First we set $\mathrm{triv}(y) := G_k$ for all $y \in Y_k$, and define morphisms

$$t(y) := t_k \circ F(\pi_*(\overline{r(y)})) : F(\pi(y)) \longrightarrow i(G_k)$$

in T. For a path $\gamma : y \longrightarrow y'$, we define the morphism

$$\mathrm{triv}(\overline{\gamma}) := i^{-1}(t(y') \circ F(\pi_*(\overline{\gamma})) \circ t(y)^{-1}) : G_k \longrightarrow G_k$$

in Gr. By construction, the morphisms $t(y)$ are the components of a natural equivalence $t : \pi^*F \longrightarrow \mathrm{triv}_i$, so that we have defined a π-local i-trivialization (triv, t) of F. Since triv is locally constant on objects, it satisfies condition 1 of Definition 3.1. To check condition 2, notice that, for any path $\gamma : y \longrightarrow y'$,

$$\mathrm{triv}(\overline{\gamma}) = \mathcal{W}^{F,i}_{y_k,y_k}(\pi_*(\overline{r(y')} \circ \overline{\gamma} \circ \overline{r(y)}^{-1})). \tag{3.4}$$

More generally, if $c : U \longrightarrow PY$ is a map, we have, for every $u \in U$, a path

$$\tilde{c}(u) := \pi_*(r(c(u)(1)) \circ c(u) \circ r(c(u)(0))^{-1})$$

in M. Then, equation (3.4) becomes

$$\mathrm{triv} \circ \mathrm{pr} \circ c = \mathcal{W}^{F,i}_{y_k,y_k} \circ \mathrm{pr} \circ \tilde{c}.$$

Since the right hand side is by assumption a smooth function; triv is a smooth functor. The component of the natural equivalence $g := \pi_2^*t \circ \pi_1^*t^{-1}$ at a point $\alpha = (y, y') \in Y^{[2]}$ with $y \in Y_k$ and $y' \in Y_l$ is the morphism

$$g(\alpha) = t_l \circ F(\pi(c(y'))) \circ F(\pi(c(y)))^{-1} \circ t_k^{-1} : i(G_k) \longrightarrow i(G_l),$$

and hence of the form $g(\alpha) = i(\tilde{g}(\alpha))$. Now consider a chart $\varphi : V \longrightarrow Y^{[2]}$ with an open subset $V \in \mathbb{R}^n$, and the path $c(u) := r(\pi_2(\varphi(u))) \circ r(\pi_1(\varphi(u)))^{-1}$ in Y. We find

$$\tilde{g} \circ \varphi = \mathcal{W}^{F,i}_{y_k,y_l} \circ \mathrm{pr} \circ c$$

as functions from U to $\mathrm{Mor}(G_k, G_l)$. Because the right hand side is by assumption a smooth function, $\tilde{g}$ is smooth on every chart, and hence also a smooth function. □

Theorem 3.12. *Let $i : \mathrm{Gr} \longrightarrow T$ be an equivalence of categories. A functor*

$$F : \mathcal{P}_1(M) \longrightarrow T$$

is a transport functor with Gr*-structure if and only if for every pair (x_1, x_2) of points in M the Wilson line $\mathcal{W}^{F,i}_{x_1,x_2}$ is smooth.*

Proof. One implication is shown by Lemma 3.11, using the fact that contractible surjective submersions always exist. To prove the other implication we express the Wilson line of the transport functor locally in terms of the functor $R_{(\mathrm{triv},g)} : \mathcal{P}_1^\pi(M) \longrightarrow T$ from Section 2.3. We postpone this construction to Appendix B.2. $\square$

Theorem 3.12 makes it possible to check explicitly, whether a given functor F is a transport functor or not. Furthermore, because every transport functor has smooth Wilson lines, we can apply Lemma 3.11 and have

Corollary 3.13. *Every transport functor* $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow T$ *with* Gr-*structure (with* $i : \mathrm{Gr} \longrightarrow T$ *an equivalence of categories) admits a* π*-local* i*-trivialization with smooth descent data for any contractible surjective submersion* π*.*

This corollary can be understood analogously to the fact, that every fibre bundle over M is trivializable over every good open cover of M.

Proposition 3.14. *For an equivalence of categories* $i : \mathrm{Gr} \longrightarrow T$ *and a contractible surjective submersion* $\pi : Y \longrightarrow M$*, the forgetful functor*

$$v^\infty : \mathrm{Triv}^1_\pi(i)^\infty \longrightarrow \mathrm{Trans}^1_{\mathrm{Gr}}(M,T)$$

is a surjective equivalence of categories.

Proof. By Corollary 3.13 v^∞ is surjective. Since it is certainly faithful, it remains to prove that it is full. Let η be a morphism of transport functors with π-local i-trivialization, i.e. there exists a surjective submersion $\pi' : Y' \longrightarrow M$ such that $\mathrm{Ex}_{\pi'}(\eta)$ is smooth. Going to a contractible surjective submersion $Z \longrightarrow Y \times_M Y'$ shows that also $\mathrm{Ex}_\pi(\eta)$ is smooth. $\square$

Summarizing, we have for i an equivalence of categories and π a contractible surjective submersion, the following equivalences of categories:

$$\mathfrak{Des}^1_\pi(i)^\infty \underset{\mathrm{Ex}_\pi}{\overset{\mathrm{Rec}_\pi}{\rightleftarrows}} \mathrm{Triv}^1_\pi(i)^\infty \xrightarrow{v^\infty} \mathrm{Trans}^1_{\mathrm{Gr}}(M,T).$$

4 Differential Forms and smooth Functors

We establish a relation between smooth descent data we have defined in the previous section and more familiar geometric objects like differential forms, motivated by [BS04] and [Bae07]. The relation we find can be expressed as a path ordered exponential, understood as the solution of an initial value problem.

Lemma 4.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. There is a canonical bijection between the set $\Omega^1(\mathbb{R},\mathfrak{g})$ of $\mathfrak{g}$-valued 1-forms on $\mathbb{R}$ and the set of smooth maps*

$$f : \mathbb{R} \times \mathbb{R} \longrightarrow G$$

satisfying the cocycle condition

$$f(y,z) \cdot f(x,y) = f(x,z). \tag{4.1}$$

Proof. The idea behind this bijection is that f is the path-ordered exponential of a 1-form A,

$$f(x,y) = \mathcal{P}\exp\left(\int_x^y A\right).$$

Let us explain in detail what that means. Given the 1-form A, we pose the initial value problem

$$\frac{\partial}{\partial t}u(t) = -\mathrm{d}r_{u(t)}|_1(A_t\left(\frac{\partial}{\partial t}\right)) \quad \text{and} \quad u(t_0) = 1 \tag{4.2}$$

for a smooth function $u : \mathbb{R} \longrightarrow G$ and a number $t_0 \in \mathbb{R}$. Here, $r_{u(t)}$ is the right multiplication in G and $\mathrm{d}r_{u(t)}|_1 : \mathfrak{g} \longrightarrow T_{u(t)}G$ is its differential evaluated at $1 \in G$. The sign in (4.2) is a convention well-adapted to the examples in Section 5. Differential equations of this type have a unique solution $u(t)$ defined on all of $\mathbb{R}$, such that $f(t_0,t) := u(t)$ depends smoothly on both parameters. To see that f satisfies the cocycle condition (4.1), define for fixed $x, y \in \mathbb{R}$ the function $\Psi(t) := f(y,t) \cdot f(x,y)$. Its derivative is

$$\begin{aligned} \frac{\partial}{\partial t}\Psi(t) &= \mathrm{d}r_{f(x,y)}|_1\left(\frac{\partial}{\partial t}f(y,t)\right) \\ &= -\mathrm{d}r_{f(x,y)}|_1(\mathrm{d}r_{f(y,t)}|_1(A_t\left(\frac{\partial}{\partial t}\right))) \\ &= -\mathrm{d}r_{\Psi(t)}(A_t\left(\frac{\partial}{\partial t}\right)) \end{aligned}$$

and furthermore $\Psi(y) = f(x, y)$. So, by uniqueness

$$f(y, t) \cdot f(x, y) = \Psi(t) = f(x, t).$$

Conversely, for a smooth function $f : \mathbb{R} \times \mathbb{R} \longrightarrow G$, let $u(t) := f(t_0, t)$ for some $t_0 \in \mathbb{R}$, and define

$$A_t\left(\frac{\partial}{\partial t}\right) := -\mathrm{d}r_{u(t)}|_1^{-1}\frac{\partial}{\partial t}u(t), \tag{4.3}$$

which yields a 1-form on $\mathbb{R}$. If f satisfies the cocycle condition, this 1-form is independent of the choice of t_0. The definition of the 1-form A is obviously inverse to (4.2) and thus establishes the claimed bijection. □

We also need a relation between the functions f_A and $f_{A'}$ corresponding to 1-forms A and A', when A and A' are related by a gauge transformation. In the following we denote the left and right invariant Maurer-Cartan forms on G forms by θ and $\bar{\theta}$ respectively.

Lemma 4.2. *Let $A \in \Omega^1(\mathbb{R}, \mathfrak{g})$ be a $\mathfrak{g}$-valued 1-form on $\mathbb{R}$, let $g : \mathbb{R} \longrightarrow G$ be a smooth function and let $A' := \mathrm{Ad}_g(A) - g^*\bar{\theta}$. If f_A and $f_{A'}$ are the smooth functions corresponding to A and A' by Lemma 4.1, we have*

$$g(y) \cdot f_A(x, y) = f_{A'}(x, y) \cdot g(x).$$

Proof. By direct verification, the function $g(y) \cdot f_A(x, y) \cdot g(x)^{-1}$ solves the initial value problem (4.2) for the 1-form A'. Uniqueness gives the claimed equality. □

In the following we use the two lemmata above for 1-forms on $\mathbb{R}$ to obtain a similar correspondence between 1-forms on an arbitrary smooth manifold X and certain smooth functors defined on the path groupoid $\mathcal{P}_1(X)$. For a given 1-form $A \in \Omega^1(X, \mathfrak{g})$, we first define a map

$$k_A : PX \longrightarrow G$$

in the following way: a path $\gamma : x \longrightarrow y$ in X can be continued to a smooth function $\gamma : \mathbb{R} \longrightarrow X$ with $\gamma(t) = x$ for $t < 0$ and $\gamma(t) = y$ for $t > 1$, due to its sitting instants. Then, the pullback $\gamma^*A \in \Omega^1(\mathbb{R}, \mathfrak{g})$ corresponds by Lemma 4.1 to a smooth function $f_{\gamma^*A} : \mathbb{R} \times \mathbb{R} \longrightarrow G$. Now we define

$$k_A(\gamma) := f_{\gamma^*A}(0, 1).$$

The map k_A defined like this comes with the following properties:

a) For the constant path id_x we obtain the constant function $f_{\mathrm{id}_x^* A}(x,y) = 1$ and thus

$$k_A(\mathrm{id}_x) = 1. \tag{4.4}$$

b) For two paths $\gamma_1 : x \longrightarrow y$ and $\gamma_2 : y \longrightarrow z$, we have

$$f_{(\gamma_2\circ\gamma_1)^*A}(0,1) = f_{(\gamma_2\circ\gamma_1)^*A}(\frac{1}{2},1)\cdot f_{(\gamma_2\circ\gamma_1)^*A}(0,\frac{1}{2}) = f_{\gamma_1^*A}(0,1)\cdot f_{\gamma_2^*A}(0,1)$$

and thus

$$k_A(\gamma_2\circ\gamma_1) = k_A(\gamma_2)\cdot k_A(\gamma_1). \tag{4.5}$$

c) If $g : X \longrightarrow G$ is a smooth function and $A' := \mathrm{Ad}_g(A) - g^*\bar{\theta}$,

$$g(y)\cdot k_A(\gamma) = k_{A'}(\gamma)\cdot g(x) \tag{4.6}$$

for any path $\gamma : x \longrightarrow y$.

The next proposition shows that the definition of $k_A(\gamma)$ depends only on the thin homotopy class of γ.

Proposition 4.3. *The map $k_A : PX \longrightarrow G$ factors in a unique way through the set P^1X of thin homotopy classes of paths, i.e. there is a unique map*

$$F_A : P^1X \longrightarrow G$$

such that $k_A = F_A \circ \mathrm{pr}$ with $\mathrm{pr} : PX \longrightarrow P^1X$ the projection.

Proof. If k_A factors through the surjective map $\mathrm{pr} : PX \longrightarrow P^1X$, the map F_A is determined uniquely. So we only have to show that two thin homotopy equivalent paths $\gamma_0 : x \longrightarrow y$ and $\gamma_1 : x \longrightarrow y$ are mapped to the same group element, $k_A(\gamma_0) = k_A(\gamma_1)$. We have moved this issue to Appendix B.3. □

In fact, the map $F_A : P^1X \longrightarrow G$ is not just a map. To understand it correctly, we need the following category:

Definition 4.4. *Let G be a Lie group. We denote by $\mathcal{B}G$ the following Lie groupoid: it has only one object, and G is its set of morphisms. The unit element $1 \in G$ is the identity morphism, and group multiplication is the composition, i.e. $g_2 \circ g_1 := g_2 \cdot g_1$.*

To understand the notation, notice that the geometric realization of the nerve of $\mathcal{B}G$ yields the classifying space of the group G, i.e. $|N(\mathcal{B}G)| = BG$. We claim that the map F_A defined by Proposition 4.3 defines a functor

$$F_A : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G.$$

Indeed, since $\mathcal{B}G$ has only one object one only has to check that F_A respects the composition (which is shown by (4.4)) and the identity morphisms (shown in (4.5)).

Lemma 4.5. *The functor F_A is smooth in the sense of Definition 3.1.*

Proof. Let $U \subset \mathbb{R}^k$ be an open subset of some $\mathbb{R}^k$ and let $c : U \longrightarrow PX$ be a map such that $c(u)(t)$ is smooth on $U \times [0,1]$. We denote the path associated to a point $x \in U$ and extended smoothly to $\mathbb{R}$ by $\gamma_x := c(x) : \mathbb{R} \longrightarrow X$. This means that $U \longrightarrow \Omega^1(\mathbb{R}, \mathfrak{g}) : x \longmapsto \gamma_x^* A$ is a smooth family of $\mathfrak{g}$-valued 1-forms on $\mathbb{R}$. We recall that

$$(k_A \circ c)(x) = k_A(\gamma_x) = f_{\gamma_x^* A}(0,1)$$

is defined to be the solution of a differential equation, which now depends smoothly on x. Hence, $k_A \circ c = F_A \circ \mathrm{pr} \circ c : U \longrightarrow G$ is a smooth function. $\square$

Let us summarize the correspondence between 1-forms on X and smooth functors developed in the Lemmata above in terms of an equivalence between categories. One category is a category $\mathrm{Funct}^\infty(\mathcal{P}_1(X), \mathcal{B}G)$ of smooth functors and smooth natural transformations. The second category is the category of differential G-cocycles on X:

Definition 4.6. *Let X be a smooth manifold and G be a Lie group with Lie algebra $\mathfrak{g}$. We consider the following category $Z_X^1(G)^\infty$: objects are all $\mathfrak{g}$-valued 1-forms A on X, and a morphism $A \longrightarrow A'$ is a smooth function $g : X \longrightarrow G$ such that*

$$A' = \mathrm{Ad}_g(A) - g^*\bar{\theta}.$$

The composition is the multiplication of functions, $g_2 \circ g_1 = g_2 g_1$.

We claim that the Lemmata above provide the structure of a functor

$$\mathcal{P} : Z_X^1(G)^\infty \longrightarrow \mathrm{Funct}^\infty(\mathcal{P}_1(X), \mathcal{B}G).$$

It sends a $\mathfrak{g}$-valued 1-form A on X to the functor F_A defined uniquely in Proposition 4.3 and which is shown by Lemma 4.5. It sends a function $g : X \longrightarrow G$ regarded as a morphism $A \longrightarrow A'$ to the smooth natural transformation $F_A \longrightarrow F_{A'}$ whose component at a point x is $g(x)$. This is natural in x due to (4.6).

Proposition 4.7. *The functor*

$$\mathcal{P} : Z^1_X(G)^\infty \longrightarrow \mathrm{Funct}^\infty(X, \mathcal{B}G).$$

is an isomorphism of categories, which reduces on the level of objects to a bijection

$$\Omega^1(X, \mathfrak{g}) \cong \{\textit{Smooth functors } F : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G\}.$$

Proof. If A and A' are two $\mathfrak{g}$-valued 1-forms on X, the set of morphisms between them is the set of smooth functions $g : X \longrightarrow G$ satisfying the condition $A' = \mathrm{Ad}_g(A) - g^*\bar{\theta}$. The set of morphisms between the functors F_A and $F_{A'}$ are smooth natural transformations, i.e. smooth maps $g : X \longrightarrow G$, whose naturality square is equivalent to the same condition. So, the functor $\mathcal{P}$ is manifestly full and faithful. It remains to show that it is a bijection on the level of objects. This is done in Appendix B.4 by an explicit construction of a 1-form A to a given smooth functor F. □

One can also enhance the category $Z^1_X(G)^\infty$ in such a way that it becomes the familiar category of local data of principal G-bundles with connection.

Definition 4.8. *The category $Z^1_\pi(G)^\infty$ of differential G-cocycles of the surjective submersion π is the category whose objects are pairs (g, A) consisting of a 1-form $A \in \Omega^1(Y, \mathfrak{g})$ and a smooth function $g : Y^{[2]} \longrightarrow G$ such that*

$$\pi_{13}^* g = \pi_{23}^* g \cdot \pi_{12}^* g \quad \textit{and} \quad \pi_2^* A = \mathrm{Ad}_g(\pi_1^* A) - g^*\bar{\theta}.$$

A morphism

$$h : (g, A) \longrightarrow (g', A')$$

is a smooth function $h : Y \longrightarrow G$ such that

$$A' = \mathrm{Ad}_h(A) - h^*\bar{\theta} \quad \textit{and} \quad \pi_2^* h \cdot g = g' \cdot \pi_1^* h.$$

Composition of morphisms is given by the product of these functions, $h_2 \circ h_1 = h_2 h_1$.

To explain the notation, notice that for $\pi = \mathrm{id}_X$ we obtain $Z^1_X(G)^\infty = Z^1_\pi(G)^\infty$. As an example, we consider the group $G = U(1)$ and a surjective submersion $\pi : Y \longrightarrow M$ coming from a good open cover $\mathfrak{U}$ of M. Then, the group of isomorphism classes of $Z^1_\pi(U(1))^\infty$ is the Deligne hypercohomology group $H^1(\mathfrak{U}, \mathcal{D}(1))$, where $\mathcal{D}(1)$ is the Deligne sheaf complex $0 \longrightarrow \underline{U(1)} \longrightarrow \underline{\Omega}^1$.

Corollary 4.9. *The functor $\mathcal{P}$ extends to an equivalence of categories*

$$Z^1_\pi(G)^\infty \cong \mathfrak{Des}^1_\pi(i_G)^\infty,$$

where $i_G : \mathcal{B}G \longrightarrow G\text{-Tor}$ sends the object of $\mathcal{B}G$ to the group G regarded as a G-space, and a morphism $g \in G$ to the equivariant smooth map which multiplies with g from the left.

This corollary is an important step towards our main theorem, to which we come in the next section.

5 Examples

Various structures in the theory of bundles with connection are special cases of transport functors with Gr-structure for particular choices of the structure groupoid Gr. In this section we spell out some prominent examples.

5.1 Principal Bundles with Connection

In this section, we fix a Lie group G. Associated to this Lie group, we have the Lie groupoid $\mathcal{B}G$ from Definition 4.4, the category G-Tor of smooth manifolds with right G-action and G-equivariant smooth maps between those, and the functor $i_G : \mathcal{B}G \longrightarrow G\text{-Tor}$ that sends the object of $\mathcal{B}G$ to the G-space G and a morphism $g \in G$ to the G-equivariant diffeomorphism that multiplies from the left by g. The functor i_G is an equivalence of categories.

As we have outlined in the introduction, a principal G-bundle P with connection over M defines a functor

$$\mathrm{tra}_P : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}.$$

Before we show that tra_P is a transport functor with $\mathcal{B}G$-structure, let us recall its definition in detail. To an object $x \in M$ it assigns the fibre P_x of the bundle P over the point x. To a path $\gamma : x \longrightarrow y$, it assigns the parallel transport map $\tau_\gamma : P_x \longrightarrow P_y$.

For preparation, we recall the basic definitions concerning local trivializations of principal bundles with connections. In the spirit of this article, we use surjective submersions instead of coverings by open sets. In this language, a local trivialization of the principal bundle P is a surjective submersion $\pi : Y \longrightarrow M$ together with a G-equivariant diffeomorphism

$$\phi : \pi^*P \longrightarrow Y \times G$$

that covers the identity on Y. Here, the fibre product $\pi^*P = Y \times_M P$ comes with the projection $p : \pi^*P \longrightarrow P$ on the second factor. It induces a section

$$s : Y \longrightarrow P : y \longmapsto p(\phi^{-1}(y, 1)).$$

The transition function $\tilde{g}_\phi : Y^{[2]} \longrightarrow G$ associated to the local trivialization ϕ is defined by

$$s(\pi_1(\alpha)) = s(\pi_2(\alpha)) \cdot \tilde{g}_\phi(\alpha) \tag{5.1}$$

for every point $\alpha \in Y^{[2]}$. A connection on P is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ that obeys

$$\omega_{\rho g}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\rho g)\right) = \mathrm{Ad}_g^{-1}\left(\omega_\rho\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)\right) + \theta_g\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right) \tag{5.2}$$

for smooth maps $\rho : [0, 1] \longrightarrow P$ and $g : [0, 1] \longrightarrow G$. In this setup, a tangent vector $v \in T_pP$ is called horizontal, if it is in the kernel of ω.

Notice that all our conventions are chosen such that the transition function $\tilde{g}_\phi : Y^{[2]} \longrightarrow G$ and the local connection 1-form $\tilde{A}_\phi := s^*\omega \in \Omega^1(Y, \mathfrak{g})$ define an object in the category $Z^1_\pi(G)^\infty$ from Definition 4.8.

To define the parallel transport map τ_γ associated to a path $\gamma : x \longrightarrow y$ in M, we assume first that γ has a lift $\tilde{\gamma} : \tilde{x} \longrightarrow \tilde{y}$ in Y, that is, $\pi_*\tilde{\gamma} = \gamma$. Consider then the path $s_*\tilde{\gamma}$ in P, which can be modified by the pointwise action of a path g in G from the right, $(s_*\tilde{\gamma})g$. This modification has now to be chosen such that every tangent vector to $(s_*\tilde{\gamma})g$ is horizontal, i.e.

$$0 = \omega_{(s_*\tilde{\gamma})g}\left(\frac{\mathrm{d}}{\mathrm{d}t}((s_*\tilde{\gamma})g)\right) \overset{(5.2)}{=} \mathrm{Ad}_g^{-1}\left(\omega_{s_*\tilde{\gamma}}\left(\frac{\mathrm{d}(s_*\tilde{\gamma})}{\mathrm{d}t}\right)\right) + \theta_g\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)$$

This is a linear differential equation for g, which has together with the initial condition $g(0) = 1$ a unique solution $g = g(\tilde{\gamma})$. Then, for any $p \in P_x$,

$$\tau_\gamma(p) := s(y)(g(1) \cdot h), \tag{5.3}$$

where h is the unique group element with $s(x)h = p$. It is evidently smooth in p and G-equivariant. Paths γ in M which do not have a lift to Y have to be split up in pieces which admit lifts; τ_γ is then the composition of the parallel transport maps of those.

Lemma 5.1. *Let P be a principal G-bundle over M with connection $\omega \in \Omega^1(P, \mathfrak{g})$. For a surjective submersion $\pi : Y \longrightarrow M$ and a trivialization ϕ with associated section $s : Y \longrightarrow P$, we consider the smooth functor*

$$F_\omega := \mathcal{P}(s^*\omega) : \mathcal{P}_1(Y) \longrightarrow \mathcal{B}G$$

associated to the 1-form $s^*\omega \in \Omega^1(Y, \mathfrak{g})$ by Proposition 4.7. Then,

$$i_G(F_\omega(\overline{\gamma})) = \phi_y \circ \tau_{\pi_*\gamma} \circ \phi_x^{-1} \tag{5.4}$$

for any path $\gamma : x \longrightarrow y$ in PY.

Proof. Recall the definition of the functor F_ω: for a path $\gamma : x \longrightarrow y$, we have to consider the 1-form $\gamma^* s^*\omega \in \Omega^1(\mathbb{R}, \mathfrak{g})$, which defines a smooth function $f_\omega : \mathbb{R} \times \mathbb{R} \longrightarrow G$. Then, $F_\omega(\overline{\gamma}) := f_\omega(0, 1)$. We claim the equation

$$f_\omega(0, t) = g(t). \tag{5.5}$$

This comes from the fact that both functions are solutions of the same differential equation, with the same initial value for $t = 0$. Using (5.5),

$$i_G(F_\omega(\overline{\gamma}))(h) = F_\omega(\overline{\gamma}) \cdot h = g(1) \cdot h$$

for some $h \in G$. On the other hand,

$$\phi_y(\tau_{\pi_*\gamma}(\phi_x^{-1}(h))) = \phi_y(\tau_{\pi_*\gamma}(s(x)h)) \overset{(5.3)}{=} \phi_y(s(y)(g(1) \cdot h)) = g(1) \cdot h.$$

This proves equation (5.4). □

Now we are ready to formulate the basic relation between principal G-bundles with connection and transport functors with $\mathcal{B}G$-structure.

Proposition 5.2. *The functor*

$$\mathrm{tra}_P : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$$

obtained from parallel transport in a principal G-bundle P, is a transport functor with $\mathcal{B}G$-structure in the sense of Definition 3.6.

Proof. The essential ingredient is, that P is locally trivializable: we choose a surjective submersion $\pi : Y \longrightarrow M$ and a trivialization ϕ. The construction of a functor $\mathrm{triv}_\phi : \mathcal{P}_1(Y) \longrightarrow \mathcal{B}G$ and a natural equivalence

$$\begin{array}{ccc} \mathcal{P}_1(Y) & \xrightarrow{\pi_*} & \mathcal{P}_1(M) \\ {\scriptstyle \mathrm{triv}_\phi}\downarrow & \overset{t_\phi}{\swarrow} & \downarrow{\scriptstyle \mathrm{tra}_P} \\ \mathcal{B}G & \xrightarrow[i_G]{} & G\text{-Tor} \end{array}$$

is as follows. We let $\mathrm{triv}_\phi := \mathcal{P}(s^*\omega)$ be the smooth functor associated to the 1-form $s^*\omega$ by Proposition 4.7. To define the natural equivalence t_ϕ, consider a point $x \in Y$. We find $\pi^*\mathrm{tra}_P(x) = P_{\pi(x)}$ and $(i_G \circ \mathrm{triv}_\phi)(x) = G$. So we define the component of t_ϕ at x by

$$t_\phi(x) := \phi_x : P_{\pi(x)} \longrightarrow G.$$

This is natural in x since the diagram

$$\begin{array}{ccc} P_{\pi(x)} & \xrightarrow{\phi_x} & G \\ {\scriptstyle \tau_{\pi_*\overline{\gamma}}}\downarrow & & \downarrow{\scriptstyle i_G(\mathrm{triv}_\phi(\overline{\gamma}))} \\ P_{\pi(y)} & \xrightarrow[\phi_y]{} & G \end{array}$$

is commutative by Lemma 5.1. Notice that the natural equivalence

$$g_\phi := \pi_2^* t_\phi \circ \pi_1^* t_\phi \tag{5.6}$$

factors through the smooth transition function $\tilde{g}_\phi$ from (5.1), i.e. $g_\phi = i_G(\tilde{g}_\phi)$. Hence, the pair $(\mathrm{triv}_\phi, g_\phi)$ is a smooth object in $\mathfrak{Des}^1_\pi(i)^\infty$. $\square$

Now we consider the morphisms. Let $\varphi : P \longrightarrow P'$ be a morphism of principal G-bundles over M (covering the identity on M) which respects the connections, i.e. $\omega = \varphi^*\omega'$. For any point $p \in M$, its restriction $\varphi_x : P_x \longrightarrow P'_x$ is a smooth G-equivariant map. For any path $\gamma : x \longrightarrow y$, the parallel transport map satisfies

$$\varphi_y \circ \tau_\gamma = \tau'_\gamma \circ \varphi_x.$$

This is nothing but the commutative diagram for the components $\eta_\varphi(x) := \varphi_x$ natural transformation $\eta_\varphi : \mathrm{tra}_P \longrightarrow \mathrm{tra}_{P'}$.

Proposition 5.3. *The natural transformation*

$$\eta_\varphi : \mathrm{tra}_P \longrightarrow \mathrm{tra}_{P'}$$

obtained from a morphism $\varphi : P \longrightarrow P'$ of principal G-bundles, is a morphism of transport functors in the sense of Definition 3.6.

Proof. Consider a surjective submersion $\pi : Y \longrightarrow M$ such that π^*P and π^*P' are trivializable, and choose trivializations ϕ and ϕ'. The descent datum of η_φ is the natural equivalence $h := t'_\phi \circ \pi^*\eta_\varphi \circ t_\phi^{-1}$. Now define the map

$$\tilde{h} : Y \longrightarrow G : x \longmapsto p_G(\phi'(x, \varphi(s(x))))$$

where p_G is the projection to G. This map is smooth and satisfies $h = i_G(\tilde{h})$. Thus, η_φ is a morphism of transport functors. $\square$

Taking the Propositions 5.2 and 5.3 together, we have defined a functor

$$\mathfrak{Bun}_G^\nabla(M) \longrightarrow \mathrm{Trans}^1_{\mathrm{Gr}}(M, G\text{-Tor}) \tag{5.7}$$

from the category of principal G-bundles over M with connection to the category of transport functors on M in G-Tor with $\mathcal{B}G$-structure. In particular, this functor provides us with lots of examples of transport functors.

Theorem 5.4. *The functor*

$$\mathfrak{Bun}_G^\nabla(M) \longrightarrow \mathrm{Trans}^1_{\mathcal{B}G}(M, G\text{-Tor}) \tag{5.8}$$

is an equivalence of categories.

We give two proofs of this Theorem: the first is short and the second is explicit.

First Proof. Let $\pi : Y \longrightarrow M$ be a contractible surjective submersion, over which every principal G-bundle is trivializable. Extracting a connection 1-form $\tilde{A}_\phi \in \Omega^1(Y, \mathfrak{g})$ and the transition function (5.1) yields a functor

$$\mathfrak{Bun}_G^\nabla(M) \longrightarrow Z^1_\pi(G)^\infty$$

to the category of differential G-cocycles for π, which is in fact an equivalence of categories. We claim that the composition of this equivalence with the sequence

$$Z^1_\pi(G)^\infty \xrightarrow{\mathcal{P}} \mathfrak{Des}^1_\pi(i)^\infty \xrightarrow{\mathrm{Rec}_\pi} \mathrm{Triv}^1_\pi(i)^\infty \xrightarrow{v^\infty} \mathrm{Trans}^1_{\mathrm{Gr}}(M, G\text{-Tor}) \tag{5.9}$$

of functors is naturally equivalent to the functor (5.8). By Corollary 4.9, Theorem 2.9 and Proposition 3.14 all functors in (5.9) are equivalences of categories, and so is (5.8). To show the claim recall that in the proof of Proposition 5.2 we have defined a local trivialization of tra_P, whose descent data $(\mathrm{triv}_\phi, g_\phi)$ is the image of the local data $(\tilde{A}_\phi, \tilde{g}_\phi)$ of the principal G-bundle under the functor $\mathcal{P}$. This reproduces exactly the steps in the sequence (5.9). $\square$

Second proof. We show that the functor (5.8) is faithful, full and essentially surjective. In fact, this proof shows that it is even surjective. So let P and P' two principal G-bundles with connection over M, and let tra_P and $\mathrm{tra}_{P'}$ be the associated transport functors.

Faithfulness follows directly from the definition, so assume now that $\eta : \mathrm{tra}_P \longrightarrow \mathrm{tra}_{P'}$ is a morphism of transport functors. We define a morphism $\varphi : P \longrightarrow P$ pointwise as $\varphi(x) := \eta(p(x))(x)$ for any $x \in P$, where $p : P \longrightarrow M$ is the projection of the bundle P. This is clearly a preimage of η under the functor (5.8), so that we only have to show that φ is a smooth map. We choose a surjective submersion such that P and P' are trivializable and such that $h := \mathrm{Ex}_\pi(\eta) = t_{\phi'} \circ \pi^*\eta \circ t_\phi^{-1}$ is a smooth morphism in $\mathfrak{Des}^1_\pi(i)^\infty$. Hence it factors through a smooth map $\tilde{h} : Y \longrightarrow G$, and from the definitions of t_ϕ and $t_{\phi'}$ it follows that $\pi^*\varphi$ is the function

$$\pi^*\varphi : \pi^*P \longrightarrow \pi^*P' : (y,p) \longmapsto \phi'^{-1}(\phi(y,p)\tilde{h}(y)),$$

and thus smooth. Finally, since π is a surjective submersion, φ is smooth.

It remains to prove that the functor (5.8) is essentially surjective. First we construct, for a given transport functor $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$ a principal G-bundle P with connection over M, performing exactly the inverse steps of (5.9). We choose a surjective submersion $\pi : Y \longrightarrow M$ and a π-local i-trivialization (triv, t) of the transport functor tra. By construction, its descent data $(\mathrm{triv}, g) := \mathrm{Ex}_\pi(\mathrm{triv}, g)$ is an object in $\mathfrak{Des}^1_\pi(i)^\infty$. By Corollary 4.9, there exists a 1-form $A \in \Omega^1(Y, \mathfrak{g})$, and a smooth function $\tilde{g} : Y^{[2]} \longrightarrow G$, forming an object $(A, \tilde{g})$ in the category $Z^1_\pi(G)^\infty$ of differential cocycles such that

$$\mathcal{P}(A, \tilde{g}) = (\mathrm{triv}, g) \tag{5.10}$$

in $\mathfrak{Des}^1_\pi(i)^\infty$. In particular $g = i_G(\tilde{g})$. The pair $(A, \tilde{g})$ is local data for a principal G-bundle P with connection ω. The reconstructed bundle comes with a canonical trivialization $\phi : \pi^*P \longrightarrow Y \times G$, for which the associated section $s : Y \longrightarrow P$ is such that $A = s^*\omega$, and whose transition function is $\tilde{g}_\phi = \tilde{g}$.

Let us extract descent data of the transport functor tra_P of P: as described in the proof of Proposition 5.4, the trivialization ϕ of the bundle P gives rise to a π-local i_G-trivialization $(\mathrm{triv}_\phi, t_\phi)$ of the transport functor tra_P, namely

$$\mathrm{triv}_\phi := F_\omega := \mathcal{P}(s^*\omega) = \mathcal{P}(A) \tag{5.11}$$

and $t_\phi(x) := \phi_x$. Its natural equivalence g_ϕ from (5.6) is just $g_\phi = i_G(\tilde{g}_\phi)$.

Finally we construct an isomorphism $\eta : \mathrm{tra}_P \longrightarrow \mathrm{tra}$ of transport functors. Consider the natural equivalence

$$\zeta := t^{-1} \circ t_\phi : \pi^*\mathrm{tra}_P \longrightarrow \pi^*\mathrm{tra}.$$

From condition (2.4) it follows that $\zeta(\pi_1(\alpha)) = \zeta(\pi_2(\alpha))$ for every point $\alpha \in Y^{[2]}$. So ζ descends to a natural equivalence

$$\eta(x) := \zeta(\tilde{x})$$

for $x \in M$ and any $\tilde{x} \in Y$ with $\pi(\tilde{x}) = x$. An easy computation shows that $\mathrm{Ex}_\pi(\eta) = t \circ \zeta \circ t_\phi^{-1} = \mathrm{id}$, which is in particular smooth and thus proves that η is an isomorphism in $\mathfrak{Des}_\pi^1(i)^\infty$. $\square$

5.2 Holonomy Maps

In this section, we show that important results of [Bar91, CP94] on holonomy maps of principal G-bundles with connection can be reproduced as particular cases.

Definition 5.5 ([CP94])**.** *A holonomy map on a smooth manifold M at a point $x \in M$ is a group homomorphism*

$$\mathcal{H}_x : \pi_1^1(M,x) \longrightarrow G,$$

which is smooth in the following sense: for every open subset $U \subset \mathbb{R}^k$ and every map $c : U \longrightarrow L_xM$ such that $\Gamma(u,t) := c(u)(t)$ is smooth on $U \times [0,1]$, also

$$U \xrightarrow{c} L_xM \xrightarrow{\mathrm{pr}} \pi_1^1(M,x) \xrightarrow{\mathcal{H}} G$$

is smooth.

Here, $L_xM \subset PM$ is the set of paths $\gamma : x \longrightarrow x$, whose image under the projection $\mathrm{pr} : PM \longrightarrow P^1M$ is, by definition, the thin homotopy group $\pi_1^1(M,x)$ of M at x. Also notice, that

- in the context of diffeological spaces reviewed in Appendix A.2, the definition of smoothness given here just means that $\mathcal{H}$ is a morphism between diffeological spaces, cf. Proposition A.6 ii).
- the notion of *intimate paths* from [CP94] and the notion of thin homotopy from [MP02] coincides with our notion of thin homotopy, while the notion of thin homotopy used in [Bar91] is different from ours.

In [CP94] it has been shown that parallel transport in a principal G-bundle over M around based loops defines a holonomy map. For connected manifolds M it was also shown how to reconstruct a principal G-bundle with connection from a given holonomy map $\mathcal{H}$ at x, such that the holonomy of this bundle around loops based at x equals $\mathcal{H}$. This establishes a bijection between holonomy maps and principal G-bundles with connection over connected manifolds. The same result has been proven (with the before mentioned different notion of thin homotopy) in [Bar91].

To relate these results to Theorem 5.4, we consider again transport functors $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$ with $\mathcal{B}G$-structure. Recall from Section 2.1 that for any point $x \in M$ and any identification $F(x) \cong G$ the functor tra produces a group homomorphism $F_{x,x} : \pi_1^1(M, x) \longrightarrow G$.

Proposition 5.6. *Let* $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$ *be a transport functor on* M *with* $\mathcal{B}G$*-structure. Then, for any point* $x \in M$ *and any identification* $F(x) \cong G$*, the group homomorphism*

$$\mathrm{tra}_{x,x} : \pi_1^1(M, x) \longrightarrow G$$

is a holonomy map.

Proof. The group homomorphism $\mathrm{tra}_{x,x}$ is a Wilson line of the transport functor tra, and hence smooth by Theorem 3.12. □

For illustration, let us combine Theorem 5.4 and Proposition 5.6 to the following diagram, which is evidently commutative:

$$\begin{array}{ccc} \mathfrak{Bun}_G^\nabla(M) & \xrightarrow{\text{[CP94]}} & \left\{\begin{array}{c}\text{Holonomy maps}\\ \text{on } M \text{ at } x\end{array}\right\} \\ {\scriptstyle \text{Theorem 5.4}} \searrow & & \nearrow {\scriptstyle \text{Proposition 5.6}} \\ & \mathrm{Trans}^1_{\mathrm{Gr}}(M, G\text{-Tor}). & \end{array}$$

5.3 Associated Bundles and Vector Bundles with Connection

Recall that a principal G-bundle P together with a faithful representation $\rho : G \longrightarrow \mathrm{Gl}(V)$ of the Lie group G on a vector space V defines a vector bundle $P \times_\rho V$ with structure group G, called the vector bundle associated to P by the representation ρ. One can regard a (say, complex) representation of a group G conveniently as a functor $\rho : \mathcal{B}G \longrightarrow \mathrm{Vect}(\mathbb{C})$ from the one-point-category $\mathcal{B}G$ into the category of complex vector spaces: the object of $\mathcal{B}G$ is sent to the vector space V of the representation, and a group element $g \in G$ is sent to an isomorphism $g : V \longrightarrow V$ of this vector space. The axioms of a functor are precisely the axioms one demands for a representation. Furthermore, the representation is faithful, if and only if the functor is faithful.

Definition 5.7. *Let*

$$\rho : \mathcal{B}G \longrightarrow \mathrm{Vect}(\mathbb{C})$$

be any representation of the Lie group G. A transport functor

$$\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow \mathrm{Vect}(\mathbb{C})$$

with $\mathcal{B}G$-structure is called associated transport functor.

As an example, we consider the defining representation of the Lie group $U(n)$ on the vector space $\mathbb{C}^n$, considered as a functor

$$\rho_n : \mathcal{B}U(n) \longrightarrow \mathrm{Vect}(\mathbb{C}^n_h) \tag{5.12}$$

to the category of n-dimensional hermitian vector spaces and isometries between those. Because we only include isometries in $\mathrm{Vect}(\mathbb{C}^n_h)$, the functor ρ_n is an equivalence of categories.

Similarly to Theorem 5.4, we find a geometric interpretation for associated transport functors on M with $\mathcal{B}U(n)$-structure, namely hermitian vector bundles of rank n with (unitary) connection over M. We denote the category of those vector bundles by $\mathrm{VB}(\mathbb{C}^n_h)^{\nabla}_M$. Let us just outline the very basics: given such a vector bundle E, we associate a functor

$$\mathrm{tra}_E : \mathcal{P}_1(M) \longrightarrow \mathrm{Vect}(\mathbb{C}^n_h),$$

which sends a point $x \in M$ to the vector space E_x, the fibre of E over x, and a path $\gamma : x \longrightarrow y$ to the parallel transport map $\tau : E_x \longrightarrow E_y$, which is linear and an isometry.

Theorem 5.8. *The functor tra_E obtained from a hermitian vector bundle E with connection over M is a transport functor on M with $\mathcal{B}U(n)$-structure; furthermore, the assignment $E \longmapsto \mathrm{tra}_E$ yields a functor*

$$\mathrm{VB}(\mathbb{C}^n_h)^{\nabla}_M \longrightarrow \mathrm{Trans}^1_{\mathcal{B}U(n)}(M, \mathrm{Vect}(\mathbb{C}^n_h)), \tag{5.13}$$

which is an equivalence of categories.

Proof. We proceed like in the first proof of Theorem 5.4. Here we use the correspondence between hermitian vector bundles with connection and their local data in $Z^1_\pi(U(n))^\infty$, for contractible surjective submersions π. Under this correspondence the functor (5.13) becomes naturally equivalent to the composite

$$Z^1_\pi(U(n))^\infty \xrightarrow{\Xi} \mathfrak{Des}^1_\pi(i)^\infty \xrightarrow{\mathrm{Rec}_\pi} \mathrm{Triv}^1_\pi(i)^\infty \xrightarrow{v^\infty} \mathrm{Trans}^1_{\mathcal{B}U(n)}(M, \mathrm{Vect}(\mathbb{C}^n_h))$$

which is, by Corollary 4.9, Theorem 2.9 and Proposition 3.14, an equivalence of categories. $\square$

Let us also consider the Lie groupoid $\mathrm{Gr}_U := \bigsqcup_{n\in\mathbb{N}} \mathcal{B}U(n)$, whose set of objects is $\mathbb{N}$ (with the discrete smooth structure) and whose morphisms are

$$\mathrm{Mor}_{\mathrm{Gr}_U}(n,m) = \begin{cases} U(n) & \text{if } n = m \\ \emptyset & \text{if } n \neq m \end{cases}$$

so that $\mathrm{Mor}(\mathrm{Gr}_U)$ is a disjoint union of Lie groups. The functors ρ_n from (5.12) induce a functor

$$\rho_U : \mathrm{Gr}_U \longrightarrow \mathrm{Vect}(\mathbb{C}_h)$$

to the category of hermitian vector spaces (without a fixed dimension) and isometries between those.

The category $\mathrm{Vect}(\mathbb{C}_h)$ in fact a monoidal category, and its monoidal structure induces monoidal structures on the category $\mathrm{VB}(\mathbb{C}_h)_M^\nabla$ of hermitian vector bundles with connection over M as well as on the category of transport functors $\mathrm{Trans}^1_{\mathrm{Gr}_U}(M, \mathrm{Vect}(\mathbb{C}_h))$, as outlined in Section 2.1. Since parallel transport in vector bundles is compatible with tensor products, we have

Corollary 5.9. *The functor*

$$\mathrm{VB}(\mathbb{C}_h)_M^\nabla \longrightarrow \mathrm{Trans}^1_{\mathrm{Gr}_U}(M, \mathrm{Vect}(\mathbb{C}_h))$$

is a monoidal equivalence of monoidal categories.

In particular, we have the unit transport functor $\mathbf{I}_\mathbb{C}$ which sends every point to the complex numbers $\mathbb{C}$, and every path to the identity $\mathrm{id}_\mathbb{C}$. The following fact is easy to verify:

Lemma 5.10. *Let* $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow \mathrm{Vect}(\mathbb{C}_h)$ *be a transport functor with* Gr_U*-structure, corresponding to a hermitian vector bundle* E *with connection over* M*. Then, there is a canonical bijection between morphisms*

$$\eta : \mathbf{I}_\mathbb{C} \longrightarrow \mathrm{tra}$$

of transport functors with Gr_U*-structure and smooth flat section of* E*.*

5.4 Generalized Connections

In this section we consider functors

$$F : \mathcal{P}_1(M) \longrightarrow \mathcal{B}G.$$

By now, we can arrange such functors in three types:

1. We demand nothing of F: such functors are addressed as *generalized connections* [AI92].

2. We demand that F is a transport functor with $\mathcal{B}G$-structure: it corresponds to an ordinary principal G-bundle with connection.

3. We demand that F is smooth in the sense of Definition 3.1: by Proposition 4.7, one can replace such functors by 1-forms $A \in \Omega^1(M, \mathfrak{g})$, so that we can speak of a trivial G-bundle.

Note that for a functor $F : \mathcal{P}_1(M) \longrightarrow \mathcal{B}G$ and the identity functor $\mathrm{id}_{\mathcal{B}G}$ on $\mathcal{B}G$ the Wilson line

$$\mathcal{W}_{x_1,x_2}^{F,\mathrm{id}_{\mathcal{B}G}} : \mathrm{Mor}_{\mathcal{P}_1(M)}(x_1, x_2) \longrightarrow G$$

does not depend on choices of objects G_1, G_2 and morphisms t_k : $i(G_k) \longrightarrow F(x_k)$ as in the general setup described in Section 3.2, since $\mathcal{B}G$ has only one object and one can canonically choose $t_k = \mathrm{id}$. So, generalized connections have a particularly good Wilson lines. Theorem 3.12 provides a precise criterion to decide when a generalized connection is regular: if and only if all its Wilson lines are smooth.

6 Groupoid Bundles with Connection

In all examples we have discussed so far the Lie groupoid Gr is of the form $\mathcal{B}G$, or a union of those. In this section we discuss transport functors with Gr-structure for a general Lie groupoid Gr. We start with the local aspects of such transport functors, and then discuss two examples of target categories. Our main example is related to the notion of principal groupoid bundles [MM03]. In contrast to the examples in Section 5, transport functors with Gr-structure do not only reproduce the existing definition of a principal groupoid bundle, but also reveal precisely what a connection on such a bundle must be.

We start with the local aspects of transport functors with Gr-structure by considering smooth functors

$$F : \mathcal{P}_1(X) \longrightarrow \mathrm{Gr}. \tag{6.1}$$

Our aim is to obtain a correspondence between such functors and certain 1-forms, generalizing the one derived in Section 4. If we denote the objects of Gr by Gr_0 and the morphisms by Gr_1, F defines in the first place a smooth map $f : X \longrightarrow \mathrm{Gr}_0$. Using the technique introduced in Section 4, we obtain further a 1-form A on X with values in the vector bundle $f^*\mathrm{id}^*T\mathrm{Gr}_1$ over X. Only the fact that F respects targets and sources imposes two new conditions:

$$f^*\mathrm{d}s \circ A = 0 \quad \text{ and } \quad f^*\mathrm{d}t \circ A + \mathrm{d}f = 0.$$

Here we regard $\mathrm{d}f$ as a 1-form on X with values in $f^*T\mathrm{Gr}_0$, and $\mathrm{d}s$ and $\mathrm{d}t$ are the differentials of the source and target maps.

Now we recall that the *Lie algebroid* E of Gr is the vector bundle

$$E := \mathrm{id}^*\mathrm{ker}(\mathrm{d}s)$$

over Gr_0 where $\mathrm{id} : \mathrm{Gr}_0 \longrightarrow \mathrm{Gr}_1$ is the identity embedding. The *anchor* is the morphism $a := \mathrm{d}t : E \longrightarrow T\mathrm{Gr}_0$ of vector bundles over Gr_0. Using this terminology, we see that the smooth functor (6.1) defines a smooth map $f : X \longrightarrow \mathrm{Gr}_0$ plus a 1-form $A \in \Omega^1(X, f^*E)$ such that $f^*a \circ A + \mathrm{d}f = 0$.

In order to deal with smooth natural transformations, we introduce the following notation. We denote by

$$c : \mathrm{Gr}_1 \;{}_s\!\times_t \mathrm{Gr}_1 \longrightarrow \mathrm{Gr}_1 : (h, g) \longmapsto h \circ g$$

the composition in the Lie groupoid Gr, and for $g : x \longrightarrow y$ a morphism by

$$r_g : s^{-1}(y) \longrightarrow s^{-1}(x) : h \longmapsto h \circ g$$

the composition by g from the right. Notice that c and r_g are smooth maps. It is straightforward to check that one has a well-defined map

$$\mathrm{AD}_g : T_g\Gamma_1 \;{}_{\mathrm{d}s}\!\times_a E_{s(g)} \longrightarrow E_{t(g)}$$

which is defined by

$$\mathrm{AD}_g(X, Y) := \mathrm{d}r_{g^{-1}}|_g(\mathrm{d}c|_{g,\mathrm{id}_{s(g)}}(X, Y)). \tag{6.2}$$

For example, if $\mathrm{Gr} = \mathcal{B}G$ for a Lie group G, the Lie algebroid is the trivial bundle $E = \Gamma_0 \times \mathfrak{g}$, the composition c is the multiplication of G, and (6.2) reduces to

$$\mathrm{AD}_g(X, Y) = \bar{\theta}_g(X) + \mathrm{Ad}_g(Y) \in \mathfrak{g}.$$

Suppose now that

$$\eta : F \Longrightarrow F'$$

is a smooth natural transformation between smooth functors F and F' which correspond to pairs (f, A) and (f', A'), respectively. It defines a smooth map $g : X \longrightarrow \mathrm{Gr}_1$ such that

$$s \circ g = f \quad \text{and} \quad t \circ g = f'. \tag{6.3}$$

Generalizing Lemma 4.2, the naturality of η implies additionally

$$A' + \mathrm{AD}_g(\mathrm{d}g, -A) = 0. \tag{6.4}$$

The structure obtained like this forms a category $Z^1_X(\mathrm{Gr})$ of Gr-*connections*: its objects are pairs (f, A) of smooth functions $f : X \longrightarrow \mathrm{Gr}_0$ and 1-forms $A \in \Omega^1(X, f^*E)$ satisfying $f^*\mathrm{d}t \circ A + \mathrm{d}f = 0$, and its morphisms are smooth maps $g : X \longrightarrow \mathrm{Gr}_1$ satisfying (6.3) and (6.4). The category $Z^1_X(\mathrm{Gr})$ generalizes the category of G-connections from Definition 4.6 in the sense that $Z^1_X(\mathcal{B}G) = Z^1_X(G)$ for G a Lie group. We obtain the following generalization of Proposition 4.7.

Proposition 6.1. *There is a canonical isomorphism of categories*

$$\mathrm{Funct}^\infty(X, \mathrm{Gr}) \cong Z^1_X(\mathrm{Gr}).$$

We remark that examples of smooth of functors with values in a Lie groupoid naturally appear in the discussion of transgression to loop spaces, see Section 4 of the forthcoming paper [SW08b].

Now we come to the global aspects of transport functors with Gr-structure. We introduce the category of Gr-torsors as an interesting target category of such transport functors. A *smooth* Gr-*manifold* [MM03] is a triple (P, λ, ρ) consisting a smooth manifold P, a surjective submersion $\lambda : P \longrightarrow \mathrm{Gr}_0$ and a smooth map

$$\rho : P \;{}_\lambda\!\times_t \mathrm{Gr}_1 \longrightarrow P$$

such that

1. ρ respects λ in the sense that $\lambda(\rho(p, \varphi)) = s(\varphi)$ for all $p \in P$ and $\varphi \in \mathrm{Gr}_1$ with $\lambda(p) = t(\varphi)$,

2. ρ respects the composition $\circ$ of morphisms of Gr.

A *morphism between* Gr-*manifolds* is a smooth map $f : P \longrightarrow P'$ which respects λ, λ' and ρ, ρ'. A Gr-*torsor* is a Gr-manifold for which ρ acts in a free and transitive way. Gr-torsors form a category denoted Gr-Tor. For a fixed object $X \in \mathrm{Obj}(\mathrm{Gr})$, $P_X := t^{-1}(X)$ is a Gr-torsor with $\lambda = s$ and $\rho = \circ$. Furthermore, a morphism $\varphi : X \longrightarrow Y$ in Gr defines a morphism $P_X \longrightarrow P_Y$ of Gr-torsors. Together, this defines a functor

$$i_{\mathrm{Gr}} : \mathrm{Gr} \longrightarrow \text{Gr-Tor}. \tag{6.5}$$

The functor (6.5) allows us to study transport functors

$$\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow \text{Gr-Tor}$$

with Gr-structure. By a straightforward adaption of the Second Proof of Theorem 5.4 one can construct the total space P of a fibre bundle over M from the transition function $\tilde{g} : Y^{[2]} \longrightarrow \mathrm{Gr}_1$ of tra, in such a way that P is fibrewise a Gr-torsor. More precisely, we reproduce the following definition.

Definition 6.2 ([MM03])**.** *A* <u>*principal* Gr-*bundle*</u> *over M is a* Gr-*manifold* (P, λ, ρ) *together with a smooth map $p : P \longrightarrow M$ which is preserved by the action, such that there exists a surjective submersion $\pi : Y \longrightarrow M$ with a smooth map $f : Y \longrightarrow \mathrm{Gr}_1$ and a morphism*

$$\phi : P \times_M Y \longrightarrow Y\,{}_f\!\times_t \mathrm{Gr}_1$$

of Gr-*manifolds that preserves the projections to Y.*

Here we have used surjective submersions instead of open covers, like we already did for principal bundles (see Section 5.1). Principal Gr-bundles over M form a category denoted Gr-$\mathfrak{Bun}^{\nabla}(M)$, whose morphisms are morphisms of Gr-manifolds that preserve the projections to M.

The descent data of the transport functor tra not only consists of the transition function $\tilde{g}$ but also of a smooth functor $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$. Now, Proposition 6.1 *predicts* the notion of a connection 1-form on a principal Gr-bundle:

Definition 6.3. *Let* Gr *be a Lie groupoid and E be its Lie algebroid. A* <u>*connection on a principal* Gr-*bundle*</u> *P is a 1-form $\omega \in \Omega^1(P, \lambda^*E)$ such that*

$$\lambda^*\mathrm{d}t \circ \omega + \mathrm{d}\lambda = 0 \quad \text{and} \quad p_1^*\omega + \mathrm{AD}_g(\mathrm{d}g, -\rho^*\omega) = 0,$$

where $\lambda : P \longrightarrow \mathrm{Gr}_0$ and $\rho : P\,{}_\lambda\!\times_t \mathrm{Gr}_1 \longrightarrow P$ are the structure of the Gr-*manifold P, p_1 and g are the projections to P and* Gr_1*, respectively.*

By construction, we have

Theorem 6.4. *There is a canonical equivalence of categories*

$$\text{Gr-}\mathfrak{Bun}^{\nabla}(M) \cong \mathrm{Trans}_{\mathrm{Gr}}(M, \text{Gr-Tor}).$$

Indeed, choosing a local trivialization (Y, f, ϕ) of a principal Gr-bundle P, one obtains a section $s : Y \longrightarrow P : y \longmapsto p(\phi^{-1}(y, \mathrm{id}_{f(y)}))$. This section satisfies $\lambda \circ s = f$, so that the pullback of a connection 1-form $\omega \in \Omega^1(P, \lambda^* E)$ along s is a 1-form $A := s^*\omega \in \Omega^1(Y, f^*E)$. The first condition in Definition 6.3 implies that (A, f) is an object in $Z^1_Y(\mathrm{Gr})$, and thus by Proposition 6.1 a smooth functor $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$. The second condition implies that the transition function $\tilde{g} : Y^{[2]} \longrightarrow \mathrm{Gr}$ defined by $s(\pi_1(\alpha)) = \rho(s(\pi_2(\alpha)), \tilde{g}(\alpha))$ is a morphism in $Z^1_{Y^{[2]}}(\mathrm{Gr})$ from $\pi_1^* F$ to $\pi_2^* F$. All together, this is descent data for a transport functor on M with Gr-structure.

We remark that this automatically induces a notion of parallel transport for a connection A on a principal Gr-bundle P: let $\mathrm{tra}_{P,A} : \mathcal{P}_1(M) \longrightarrow \text{Gr-Tor}$ be the transport corresponding to (P, A) under the equivalence of Theorem 6.4. Then, the parallel transport of A along a path $\gamma : x \longrightarrow y$ is the Gr-torsor morphism

$$\mathrm{tra}_{P,A}(\gamma) : P_x \longrightarrow P_y.$$

In the remainder of this section we discuss a class of groupoid bundles with connection related to action groupoids. We recall that for V a complex vector space with an action of a Lie group G, the action groupoid $V/\!\!/G$ has V as its objects and $G \times V$ as its morphisms. The source map is the projection to V, and the target map is the action $\rho : G \times V \longrightarrow V$. Every action groupoid $V/\!\!/G$ comes with a canonical functor

$$i_{V/\!\!/G} : V/\!\!/G \longrightarrow \mathrm{Vect}_*(\mathbb{C})$$

to the category of pointed complex vector spaces, which sends an object $v \in V$ to the pointed vector space (V, v) and a morphism (v, g) to the linear map $\rho(g, -)$, which respects the base points.

Proposition 6.5. *A transport functor* $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow \mathrm{Vect}_*(\mathbb{C})$ *with* $V/\!\!/G$*-structure is a complex vector bundle over* M *with structure group* G *and a smooth flat section.*

Proof. We consider the strictly commutative diagram

$$\begin{array}{ccc} V/\!\!/G & \xrightarrow{\;i_{V/\!\!/G}\;} & \mathrm{Vect}_*(\mathbb{C}) \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{B}G & \xrightarrow[\;\rho\;]{} & \mathrm{Vect}(\mathbb{C}) \end{array}$$

of functors, in which f is the functor that forgets the base point, pr is the functor which sends a morphism (g, v) in $V/\!\!/G$ to g, and ρ is the given representation. The diagram shows that the composition

$$f \circ \mathrm{tra} : \mathcal{P}_1(M) \longrightarrow \mathrm{Vect}(\mathbb{C})$$

is a transport functor with $\mathcal{B}G$-structure, and hence the claimed vector bundle E by (a slight generalization of) Theorem 5.8. Remembering the forgotten base point defines a natural transformation

$$\eta : \mathbf{I}_{\mathbb{C}} \longrightarrow f \circ \mathrm{tra}.$$

If we regard the identity transport functor $\mathbf{I}_{\mathbb{C}}$ as a transport functor with $\mathcal{B}G$-structure, the natural transformation η becomes a morphism of transport functors with $\mathcal{B}G$-structure, and thus defines by Lemma 5.10 a smooth flat section in E. □

7 Generalizations and further Topics

The concept of transport functors has generalizations in many aspects, some of which we want to outline in this section.

7.1 Transport n-Functors

The motivation to write this article was to find a formulation of parallel transport along curves, which can be generalized to higher dimensional parallel transport. Transport functors have a natural generalization to transport n-functors. In particular the case $n = 2$ promises relations between transport 2-functors and gerbes with connective structure [BS07], similar to the relation between transport 1-functors and bundles with connections presented in Section 5. We address these issues in a further publication [SW08a].

Let us briefly describe the generalization of the concept of transport functors to transport n-functors. The first generalization is that of the path groupoid $\mathcal{P}_1(M)$ to a path n-groupoid $\mathcal{P}_n(M)$. Here, n-groupoid means that every k-morphism is an equivalence, i.e. invertible up to $(k+1)$-isomorphisms. The set of objects is again the manifold M, the k-morphisms are smooth maps $[0, 1]^k \longrightarrow M$ with sitting instants on each boundary of the k-cube, and the top-level morphisms $k = n$ are additionally taken up to thin homotopy in the appropriate sense.

We then consider n-functors

$$F : \mathcal{P}_n(M) \longrightarrow T \tag{7.1}$$

from the path n-groupoid $\mathcal{P}_n(M)$ to some target n-category T. Local trivializations of such n-functors are considered with respect to an n-functor $i : \mathrm{Gr} \longrightarrow T$, where Gr is a Lie n-groupoid, and to a surjective submersions $\pi : Y \longrightarrow M$. A π-local i-trivialization then consists of an n-functor $\mathrm{triv} : \mathcal{P}_n(Y) \longrightarrow \mathrm{Gr}$ and an equivalence

$$\begin{array}{ccc} \mathcal{P}_n(Y) & \xrightarrow{\pi_*} & \mathcal{P}_n(M) \\ {\scriptstyle \mathrm{triv}}\downarrow & \swarrow_{t} & \downarrow{\scriptstyle F} \\ \mathrm{Gr} & \xrightarrow[i]{} & T \end{array} \tag{7.2}$$

of n-functors. Local trivializations lead to an n-category $\mathfrak{Des}^n_\pi(i)$ of descent data, which are descent n-categories in the sense of [Str04], similar to Remark 2.10 for $n = 1$.

The category $\mathfrak{Des}^n_\pi(i)$ has a natural notion of smooth objects and smooth k-morphisms. Then, n-functors (7.1) which allow local trivializations with smooth descent data will be called transport n-functors, and form an n-category $\mathrm{Trans}^n_{\mathrm{Gr}}(M, T)$.

In the case $n = 1$, the procedure described above reproduces the framework of transport functors described in this article. The case $n = 2$ will be considered in detail in two forthcoming papers. First we settle the local aspects: we derive a correspondence between smooth 2-functors and differential 2-forms (Theorem 2.20 in [SW08b]). Then we continue with the global aspects in [SW08a].

As a further example, we now describe the case $n = 0$. Note that a 0-category is a set, a Lie 0-groupoid is a smooth manifold, and a 0-functor is a map. To start with, we have the set $\mathcal{P}_0(M) = M$, a set T, a smooth manifold G and an injective map $i : G \longrightarrow T$. Now we consider maps $F : M \longrightarrow T$. Following the general concept, such a map is π-locally i-trivializable, if there exists a map $\mathrm{triv} : Y \longrightarrow G$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & M \\ {\scriptstyle \mathrm{triv}}\downarrow & & \downarrow{\scriptstyle F} \\ G & \xrightarrow[i]{} & T \end{array}$$

is commutative. Maps F together with π-local i-trivializations form the set $\mathrm{Triv}^0_\pi(i)$. The set $\mathfrak{Des}^0_\pi(i)$ of descent data is just the set of maps $\mathrm{triv} : Y \longrightarrow G$ satisfying the equation

$$\pi_1^*\mathrm{triv}_i = \pi_2^*\mathrm{triv}_i, \tag{7.3}$$

where we have used the notation $\pi_k^*\mathrm{triv}_i = i \circ \mathrm{triv} \circ \pi_k$ from Section 2. It is easy to see that every π-local i-trivialization triv of a map F satisfies this condition. This defines the map

$$\mathrm{Ex}_\pi : \mathrm{Triv}^0_\pi(i) \longrightarrow \mathfrak{Des}^0_\pi(i).$$

Similar to Theorem 2.9 in the case $n = 1$, this is indeed a bijection: every function $\mathrm{triv} : Y \longrightarrow G$ satisfying (7.3) with i injective factors through π. Now it is easy to say when an element in $\mathfrak{Des}^0_\pi(i)$ is called smooth: if and only if the map $\mathrm{triv} : Y \longrightarrow G$ is smooth. Such maps form the set $\mathfrak{Des}^0_\pi(i)^\infty$, which in turn defines the set $\mathrm{Trans}^0_G(M,T)$ of transport 0-functors with G-structure. Due to (7.3), there is a canonical bijection $\mathfrak{Des}^0_\pi(i)^\infty \cong C^\infty(M,G)$. So, we have

$$\mathrm{Trans}^0_G(M,T) \cong C^\infty(M,G),$$

in other words: transport 0-functors on M with G-structure are smooth functions from M to G.

Let us revisit Definition 3.3 of the category $\mathfrak{Des}^1_\pi(i)^\infty$ of smooth descent data, which now can equivalently be reformulated as follows:

> *Let* Gr *be a Lie groupoid and let* $i : \mathrm{Gr} \longrightarrow T$ *be a functor. An object* (triv, g) *in* $\mathfrak{Des}^1_\pi(i)$ *is called* smooth, *if the functor* $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$ *is smooth the sense of Definition 3.1, and if the natural equivalence* $g : Y^{[2]} \longrightarrow \mathrm{Mor}(T)$ *is a transport* 0*-functor with* $\mathrm{Mor}(T)$*-structure. A morphism*
>
> $$h : (\mathrm{triv}, g) \longrightarrow (\mathrm{triv}', g')$$
>
> *between smooth objects is called* smooth, *if* $h : Y \longrightarrow \mathrm{Mor}(T)$ *is a transport 0-functor with* $\mathrm{Mor}(T)$*-structure.*

This gives an outlook how the definition of the n-category $\mathfrak{Des}^n_\pi(i)^\infty$ of smooth descent data will be for higher n: it will recursively use transport $(n-1)$-functors.

7.2 Curvature of Transport Functors

When we describe parallel transport in terms of functors, it is a natural question how related notions like curvature can be seen in this formulation. Interestingly, it turns out that the curvature of a transport functor is a transport 2-functor. More generally, the curvature of a transport n-functor is a transport $(n+1)$-functor. This becomes evident with a view to Section 4, where we have related smooth functors and differential 1-forms. In a similar way, 2-functors can be related to 2-forms. A comprehensive discussion of the curvature of transport functors is therefore beyond the scope of this article, and has to be postponed until after the discussion of transport 2-functors [SW08a].

We shall briefly indicate the basic ideas. We recall from Section 2.1 when a functor $F : \mathcal{P}_1(M) \longrightarrow T$ is flat: if it factors through the fundamental groupoid $\Pi_1(M)$, whose morphisms are smooth homotopy classes of paths in M. In general, one can associate to a transport functor tra a 2-functor

$$\mathrm{curv}(\mathrm{tra}) : \mathcal{P}_2(M) \longrightarrow \mathrm{Grpd}$$

into the 2-category of groupoids. This 2-functor is particularly trivial if tra is flat. Furthermore, the 2-functor curv(tra) is itself flat in the sense that it factors through the fundamental 2-groupoid of M: this is nothing but the Bianchi identity.

For smooth functors $F : \mathcal{P}_1(M) \longrightarrow \mathcal{B}G$, which corresponding by Proposition 4.7 to 1-forms $A \in \Omega^1(M, \mathfrak{g})$, it turns out that the 2-functor $\mathrm{curv}(F)$ corresponds to a 2-form $K \in \Omega^2(M, \mathfrak{g})$ which is related to A by the usual equality $K = \mathrm{d}A + A \wedge A$.

7.3 Alternatives to smooth Functors

The definition of transport functors concentrates on the smooth aspects of parallel transport. As we have outlined in Appendix A.2, our definition of smooth descent data $\mathfrak{Des}^1_\pi(i)^\infty$ can be regarded as the internalization of functors and natural transformations in the category D^∞ of diffeological spaces and diffeological maps.

Simply by choosing another ambient category C, we obtain possibly weaker notions of parallel transport. Of particular interest is the situation where the ambient category is the category Top of topological spaces and continuous maps. Indicated by results of [Sta74], one would expect that reconstruction theorems as discussed in Section 2.3 should also exist for Top, and also for transport n-functors for $n > 1$. Besides, parallel transport along topological paths of bounded variation can be defined, and is of interest for its own right, see, for example, [Bau05].

7.4 Anafunctors

The notion of smoothly locally trivializable functors is closely related to the concept of anafunctors. Following [Mak96], an anafunctor $F : A \longrightarrow B$ between categories A and B is a category $|F|$ together with a functor $\tilde{F} : |F| \longrightarrow B$ and a surjective equivalence $p : |F| \longrightarrow A$, denoted as a diagram

$$\begin{array}{ccc} |F| & \xrightarrow{\tilde{F}} & B \\ {\scriptstyle p}\downarrow & & \\ A & & \end{array} \tag{7.4}$$

called a span. It has been shown in [Bar04] how to formulate the concept of an anafunctor internally to any category C.

Note that an anafunctor in C gives rise to an ordinary functor $A \longrightarrow B$ in C, if the epimorphism p has a section. In the category of sets, $C = \mathfrak{Set}$, every epimorphism has a section, if one assumes the axiom of choice (this is what we do). The original motivation for introducing anafunctors was, however, to deal with situations where one does not assume the axiom of choice [Mak96]. In the category $C = C^\infty$ of smooth manifolds, surjective submersions are particular epimorphisms, as they arise for example as projections of smooth fibre bundles. Since not every bundle has a global smooth section, an anafunctor in C^∞ does not produce a functor. The same applies to the category $C = D^\infty$ of diffeological spaces described in Appendix A.2.

Let us indicate how anafunctors arise from smoothly locally trivialized functors. Let $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow T$ be a transport functor with Gr-structure. We choose a π-local i-trivialization (triv, t), whose descent data (triv, g) is smooth. Consider the functor

$$R_{(\mathrm{triv},g)} : \mathcal{P}_1^\pi(M) \longrightarrow T$$

that we have defined in Section 2.3 from this descent data. By Definition 3.3 of smooth descent data, the functor $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$ is smooth and the natural equivalence g factors through a smooth natural equivalence $\tilde{g} : Y \longrightarrow \mathrm{Mor}(\mathrm{Gr})$. So, the functor $R_{(\mathrm{triv},g)}$ factors through Gr,

$$R_{(\mathrm{triv},g)} = i \circ A$$

for a functor $A : \mathcal{P}_1^\pi(M) \longrightarrow \mathrm{Gr}$. In fact, the category $\mathcal{P}_1^\pi(M)$ can be considered as a category internal to D^∞, so that the functor A is internal to D^∞ as described in Appendix A.2, Proposition A.7 ii). Hence the reconstructed functor yields a span

$$\begin{array}{ccc} \mathcal{P}_1^\pi(M) & \xrightarrow{A} & \mathrm{Gr} \\ {\scriptstyle p^\pi}\downarrow & & \\ \mathcal{P}_1(M), & & \end{array}$$

internal to D^∞, i.e. an anafunctor $\mathcal{P}_1(M) \longrightarrow \mathrm{Gr}$. Because the epimorphism p^π is not invertible in D^∞, we do not get an ordinary functor $\mathcal{P}_1(M) \longrightarrow \mathrm{Gr}$ internal to D^∞: the weak inverse functor $s : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M)$ we have constructed in Section 2.3 is not internal to D^∞.

Acknowledgements We thank Bruce Bartlett, Uwe Semmelmann, Jim Stasheff and Danny Stevenson for helpful correspondences and Christoph Schweigert for helpful discussions. U.S. thanks John Baez for many valuable suggestions and discussions. We acknowledge support from the Sonderforschungsbereich "Particles, Strings and the Early Universe - the Structure of Matter and Space-Time".

A More Background

A.1 The universal Path Pushout

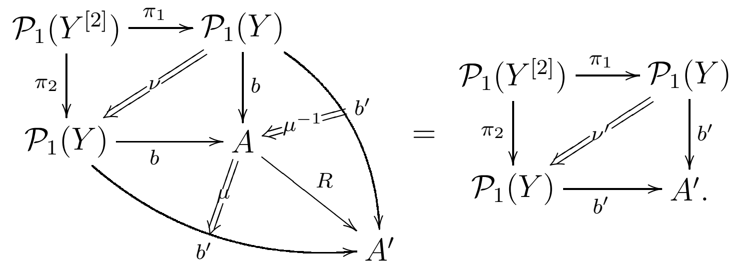
Here we motivate Definition 2.11 of the groupoid $\mathcal{P}_1^\pi(M)$. Let $\pi : Y \longrightarrow M$ be a surjective submersion. A *path pushout of* π is a triple (A, b, ν) consisting of a groupoid A, a functor $b : \mathcal{P}_1(Y) \longrightarrow A$ and a natural equivalence $\nu : \pi_1^*b \longrightarrow \pi_2^*b$ with

$$\pi_{13}^*\nu = \pi_{23}^*\nu \circ \pi_{12}^*\nu.$$

A morphism

$$(R, \mu) : (A, b, \nu) \longrightarrow (A', b', \nu')$$

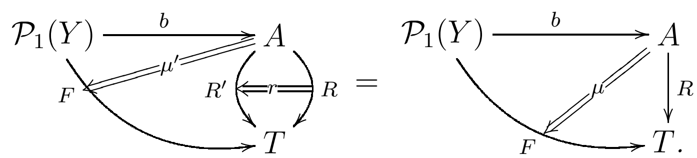
between path pushouts is a functor $R : A \longrightarrow A'$ and a natural equivalence $\mu : R \circ b \longrightarrow b'$ such that



(A.1)

Among all path pushouts of π we distinguish some having a universal property.

Definition A.1. *A path pushout* (A, b, ν) *is* <u>*universal*</u>*, if, given any other path pushout* (T, F, g)*, there exists a morphism* $(R, \mu) : (A, b, \nu) \longrightarrow (T, F, g)$ *such that, given any other such morphism* (R', μ')*, there is a unique natural equivalence* $r : R \longrightarrow R'$ *with*

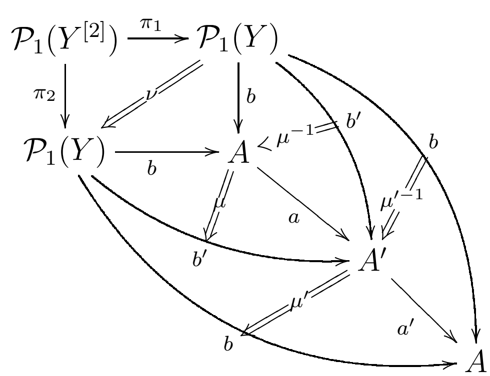


(A.2)

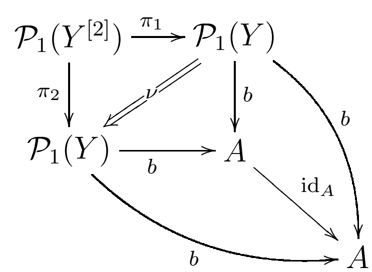
Now we show how two path pushouts having both the universal property, are related.

Lemma A.2. *Given two universal path pushouts* (A, b, ν) *and* (A', b', ν') *of the same surjective submersion* $\pi : Y \longrightarrow M$*, there is an equivalence of categories* $a : A \longrightarrow A'$*.*

Proof. We use the universal properties of both triples applied to each other. We obtain two choices of morphisms (R, μ) and $(\tilde{R}, \tilde{\mu})$, namely



and



The unique natural transformation we get from the universal property is here $r : a' \circ a \longrightarrow \mathrm{id}_A$. Doing the same thing in the other order, we obtain a unique natural transformation $r' : a \circ a' \longrightarrow \mathrm{id}_{A'}$. Hence $a : A \longrightarrow A'$ is an equivalence of categories. $\square$

We also need

Lemma A.3. *Let (A, b, ν) be a universal path pushout of π, let (T, F, g) and (T, F', g') two other path pushouts and let $h : F \longrightarrow F'$ be a natural transformation with $\pi_2^* h \circ g = \pi_1^* h \circ g'$. For any choice of morphisms*

$$(R, \mu) : (A, b, \nu) \longrightarrow (T, F, g) \quad \text{and} \quad (R', \mu') : (A, b, \nu) \longrightarrow (T, F', g')$$

there is a unique natural transformation $r : R \longrightarrow R'$ with $\mu \bullet (\mathrm{id}_b \circ r) = \mu'$.

Proof. Note that the natural equivalence h defines a morphism

$$(\mathrm{id}_T, h) : (T, F, g) \longrightarrow (T, F', g')$$

of path pushouts. The composition $(\mathrm{id}_T, h) \circ (R, \mu)$ gives a morphism

$$(R, h \circ \mu) : (A, b, \nu) \longrightarrow (T', F', g').$$

Since (R', μ') is universal, we obtain a unique natural transformation $r : R \longrightarrow R'$. $\square$

Now consider the groupoid $\mathcal{P}_1^\pi(M)$ from Definition 2.11, together with the inclusion functor $\iota : \mathcal{P}_1(Y) \longrightarrow \mathcal{P}_1^\pi(M)$ and the identity $\mathrm{id}_{Y^{[2]}} : \pi_1^* \iota \longrightarrow \pi_2^* \iota$ whose component at a point $\alpha \in Y^{[2]}$ is the morphism α in $\mathcal{P}_1^\pi(M)$. Its commutative diagram follows from relations (1) and (2), depending on the type of morphism you apply it to. Its cocycle condition follows from (3). So, the triple $(\mathcal{P}_1^\pi(M), \iota, \mathrm{id}_{Y^{[2]}})$ is a path pushout.

Lemma A.4. *The triple $(\mathcal{P}_1^\pi(M), \iota, \mathrm{id}_{Y^{[2]}})$ is universal.*

Proof. Let (T, F, g) any path pushout. We construct the morphism

$$(R, \mu) : (\mathcal{P}_1^\pi(M), \iota, \mathrm{id}_{Y^{[2]}}) \longrightarrow (T, F, g)$$

as follows. The functor

$$R : \mathcal{P}_1^\pi(M) \longrightarrow T$$

sends an object $x \in Y$ to $F(x)$, a morphism $\gamma : x \longrightarrow y$ to $F(\gamma)$ and a morphism α to $g(\alpha)$. This definition is well-defined under the relations among the morphisms: (1) is the commutative diagram for the natural transformation g, (2) is the cocycle condition for g and (3) follows from the latter since g is invertible. The natural equivalence $\mu : R \circ \iota \longrightarrow F$ is the identity. By definition equation (A.1) is satisfied, so that (R, μ) is a morphism of path pushouts. Now we assume that there is another morphism (R', μ'). The component of the natural equivalence $r : R \longrightarrow R'$ at a point $x \in Y$ is $\mu'^{-1}(x)$, its naturality with respect to a morphisms $\gamma : x \longrightarrow y$ is then just the one of μ', and with respect to morphisms $\alpha \in Y^{[2]}$ comes from condition (A.1) on morphisms of path pushouts. It also satisfies the equality (A.2). Since this equation already determines r, it is unique. $\square$

Notice that the construction of the functor R reproduces Definition 2.12. Let us finally apply Lemma A.3 to the universal path pushout $(\mathcal{P}_1^{\pi}(M), \iota, \mathrm{id}_{Y^{[2]}})$. Given the two functors $F, F' : \mathcal{P}_1(Y) \longrightarrow T$, the natural transformation $h : F \longrightarrow F'$, and the universal morphisms (R, μ) and (R', μ') as constructed in the proof of Lemma A.4, the natural transformation $r : R \longrightarrow R'$ has the component $h(x)$ at x. This reproduces Definition 2.13.

A.2 Diffeological Spaces and smooth Functors

This section puts Definition 3.1 of a smooth functor into the wider perspective of functors internal to some category C, here the category D^{∞} of diffeological spaces [Che77, Sou81]. Diffeological spaces generalize the concept of a smooth manifold. While the set $C^{\infty}(X, Y)$ of smooth maps between smooth manifolds X and Y does not form, in general, a smooth manifold itself, the set $D^{\infty}(X, Y)$ of diffeological maps between diffeological spaces is again a diffeological space in a canonical way. In other words, the category D^{∞} of diffeological spaces is closed.

Definition A.5. *A diffeological space is a set X together with a collection of plots: maps*

$$c : U \longrightarrow X,$$

each of them defined on an open subset $U \subset \mathbb{R}^k$ for some $k \in \mathbb{N}_0$, such that

a) for any plot $c : U \longrightarrow X$ and any smooth function $f : V \longrightarrow U$ also

$$c \circ f : V \longrightarrow X$$

is a plot.

b) every constant map $c : U \longrightarrow X$ is a plot.

c) if $f : U \longrightarrow X$ is a map defined on $U \subset \mathbb{R}^k$ and $\{U_i\}_{i\in I}$ is an open cover of U for which all restrictions $f|_{U_i}$ are plots of X, then also f is a plot.

A diffeological map between diffeological spaces X and Y is a map $f : X \longrightarrow Y$ such that for every plot $c : U \longrightarrow X$ of X the map $f \circ c : U \longrightarrow Y$ is a plot of Y. The set of all diffeological maps is denoted by $D^\infty(X, Y)$.

In fact Chen originally used convex subsets $U \subset \mathbb{R}^k$ instead of open ones, but this will not be of any importance for this article. For a comparison of various concepts see [KM97]. The following examples of diffeological spaces are important for us.

(1) First of all, every smooth manifold is a diffeological space, the plots being all smooth maps defined on all open subset of all $\mathbb{R}^n$. A map between two manifolds is smooth if and only if it is diffeological.

(2) For diffeological spaces X and Y the space $D^\infty(X, Y)$ of all diffeological maps from X to Y is a diffeological space in the following way: a map

$$c : U \longrightarrow D^\infty(X, Y)$$

is a plot if and only if for any plot $c' : V \longrightarrow X$ of X the composite

$$U \times V \xrightarrow{c\times c'} D^\infty(X, Y) \times X \xrightarrow{\text{ev}} Y$$

is a plot of Y. Here, ev denotes the evaluation map $\text{ev}(f, x) := f(x)$.

(3) Every subset Y of a diffeological space X is a diffeological space: its plots are those plots of X whose image is contained in Y.

(4) For a diffeological space X, a set Y and a map $p : X \longrightarrow Y$, the set Y is also a diffeological space: a map $c : U \longrightarrow Y$ is a plot if and only if there exists a cover of U by open sets U_α together with plots $c_\alpha : U_\alpha \longrightarrow X$ of X such that $c|_{U_\alpha} = p \circ c_\alpha$.

(5) Combining (1) and (2) we obtain the following important example: for smooth manifolds X and Y the space $C^\infty(X, Y)$ of smooth maps from X to Y is a diffeological space in the following way: a map

$$c : U \longrightarrow C^\infty(X, Y)$$

is a plot if and only if

$$U \times X \xrightarrow{c\times \text{id}_X} C^\infty(X, Y) \times X \xrightarrow{\text{ev}} Y$$

is a smooth map. This applies for instance to the free loop space $LM = C^\infty(S^1, M)$.

(6) Combining (3) and (5), the based loop space L_xM and the path space PM of a smooth manifold are diffeological spaces.

(7) Combining (4) and (6) applied to the projection $\mathrm{pr} : PM \longrightarrow P^1M$ to thin homotopy classes of paths, P^1M is a diffeological space. In the same way, the thin homotopy group $\pi_1^1(M,x)$ is a diffeological space.

From Example (7) we see that diffeological spaces arise naturally in the setup of transport functors introduced in this article.

Proposition A.6. *During this article, we encountered two examples of diffeological maps:*

i) A Wilson line

$$\mathcal{W}^{F,i}_{x_1,x_2} : \mathrm{Mor}_{\mathcal{P}_1(M)}(x_1,x_2) \longrightarrow \mathrm{Mor}_{\mathrm{Gr}}(G_1,G_2)$$

is smooth in the sense of Definition 3.10 if and only if it is diffeological.

ii) A group homomorphism $\mathcal{H} : \pi_1^1(M,x) \longrightarrow G$ is a holonomy map in the sense of Definition 5.5, if and only if it is diffeological.

Diffeological spaces and diffeological maps form a category D^∞ in which we can internalize categories and functors. Examples of such categories are:

- the path groupoid $\mathcal{P}_1(M)$: its set of objects is the smooth manifold M, which is by example (1) a diffeological space. Its set of morphisms P^1X is a diffeological space by example (7).
- the universal path pushout $\mathcal{P}_1^\pi(M)$ of a surjective submersion $\pi : Y \longrightarrow M$: its set of objects is the smooth manifold Y, and hence a diffeological space. A map

  $$\phi : U \longrightarrow \mathrm{Mor}(\mathcal{P}_1^\pi(M))$$

  is a plot if and only if there is a collection of plots $f_i : U \longrightarrow P^1Y$ and a collection of smooth maps $g_i : U \longrightarrow Y^{[2]}$ such that

  $$g_N(x) \circ f_N(x) \cdots g_2(x) \circ f_2(x) \circ g_1(x) \circ f_1(x) = \phi(x).$$

We also have examples of functors internal to D^∞:

Proposition A.7. *During this article, we encountered two examples of functors internal to D^∞:*

i) A functor $F : \mathcal{P}_1(M) \longrightarrow \mathrm{Gr}$ is internal to D^∞ if and only if it is smooth in the sense of Definition 3.1.

ii) For a smooth object (triv, g) in $\mathfrak{Des}^1_\pi(i)$, the functor $R_{(\mathrm{triv},g)}$ factors smoothly through $i : \mathrm{Gr} \longrightarrow T$, i.e. there is a functor $A : \mathcal{P}_1^\pi(M) \longrightarrow \mathrm{Gr}$ internal to D^∞ such that $i \circ A = R_{(\mathrm{triv},g)}$.

B Postponed Proofs

B.1 Proof of Theorem 2.9

Here we prove that the functor

$$\mathrm{Ex}_\pi : \mathrm{Triv}^1_\pi(i) \longrightarrow \mathfrak{Des}^1_\pi(i)$$

is an equivalence of categories. In Section 2.3 we have defined a reconstruction functor Rec_π going in the opposite direction. Now we show that Rec_π is a weak inverse of Ex_π and thus prove that both are equivalences of categories. For this purpose, we show (a) the equation $\mathrm{Ex}_\pi \circ \mathrm{Rec}_\pi = \mathrm{id}_{\mathfrak{Des}^1_\pi(i)}$ and (b) that there exists a natural equivalence

$$\zeta : \mathrm{id}_{\mathrm{Triv}^1_\pi(i)} \longrightarrow \mathrm{Rec}_\pi \circ \mathrm{Ex}_\pi.$$

To see (a), let (triv, g) be an object in $\mathfrak{Des}^1_\pi(i)$, and let $\mathrm{Rec}_\pi(\mathrm{triv}, g) = s^*R_{(\mathrm{triv},g)}$ be the reconstructed functor, coming with the π-local i-trivialization (triv, t) with $t := g \circ \iota^*\lambda$. Extracting descent data as described in Section 2.2, we find

$$(\pi_2^*t \circ \pi_1^*t^{-1})(\alpha) = g((\pi_2^*\lambda \circ \pi_1^*\lambda^{-1})(\alpha)) = g(\alpha)$$

so that $\mathrm{Ex}_\pi(\mathrm{Rec}_\pi(\mathrm{triv}, g)) = (\mathrm{triv}, g)$. Similar, if $h : (\mathrm{triv}, g) \longrightarrow (\mathrm{triv}', g')$ is a morphism in $\mathfrak{Des}^1_\pi(i)$, the reconstructed natural equivalence is $\mathrm{Rec}_\pi(h) := s^*R_h$. Extracting descent data, we obtain for the component at a point $x \in Y$

$$\begin{aligned}(t' \circ \pi^*s^*R_h \circ t^{-1})(x) &= g'(\lambda(x)) \circ R_h(s(\pi(x))) \circ g^{-1}(\lambda(x)) \\ &= g'(x, s(\pi(x))) \circ h(s(\pi(x))) \circ g^{-1}(x, s(\pi(x))) \\ &= h(x)\end{aligned}$$

where we have used Definition 2.13 and the commutativity of diagram (2.2). This shows that $\mathrm{Ex}_\pi(\mathrm{Rec}_\pi(h)) = h$.

To see (b), let $F : \mathcal{P}_1(M) \longrightarrow T$ be a functor with π-local i-trivialization (triv, t). Let us first describe the functor

$$\mathrm{Rec}_\pi(\mathrm{Ex}_\pi(F)) : \mathcal{P}_1(M) \longrightarrow T.$$

We extract descent data (triv, g) in $\mathfrak{Des}^1_\pi(i)$ as described in Section 2.2 by setting

$$g := \pi_2^* t \circ \pi_1^* t^{-1}. \tag{B.1}$$

Then, we have

$$\mathrm{Rec}_\pi(\mathrm{Ex}_\pi(F))(x) = \mathrm{triv}_i(s(x))$$

for any point $x \in M$. A morphism $\overline{\gamma} : x \longrightarrow y$ is mapped by the functor s to some finite composition

$$s(\bar{\gamma}) = \alpha_n \circ \gamma_n \circ \alpha_{n-1} \circ \ldots \circ \gamma_2 \circ \alpha_1 \circ \gamma_1 \circ \alpha_0$$

of basic morphisms $\gamma_i : x_i \longrightarrow y_i$ and $\alpha_i \in Y^{[2]}$, so that we have

$$\mathrm{Rec}_\pi(\mathrm{Ex}_\pi(F))(\overline{\gamma}) = g(\alpha_n) \circ \mathrm{triv}_i(\overline{\gamma_n}) \circ g(\alpha_{n-1}) \circ \ldots \circ \mathrm{triv}_i(\overline{\gamma_1}) \circ g(\alpha_0). \tag{B.2}$$

Now we are ready define the component of the natural equivalence ζ at a functor F. This component is a morphism in $\mathrm{Triv}^1_\pi(i)$ and thus itself a natural equivalence

$$\zeta(F) : F \longrightarrow \mathrm{Rec}_\pi(\mathrm{Ex}_\pi(F)).$$

We define the component of $\zeta(F)$ at a point $x \in M$ by

$$\zeta(F)(x) := t(s(x)) : F(x) \longrightarrow \mathrm{triv}_i(s(x)). \tag{B.3}$$

Now we check that this is natural in x: let $\overline{\gamma} : x \longrightarrow y$ be a morphism like the above one. The diagram whose commutativity we have to show splits along the decomposition (B.2) into diagrams of two types:

$$\begin{array}{ccc} F(\pi(x_i)) & \xrightarrow{t(x_i)} & \mathrm{triv}_i(x_i) \\ {\scriptstyle \pi_*\gamma_i}\downarrow & & \downarrow{\scriptstyle \mathrm{triv}_i(\gamma_i)} \\ F(\pi(y_i)) & \xrightarrow[t(y_i)]{} & \mathrm{triv}_i(y_i) \end{array} \quad \text{and} \quad \begin{array}{ccc} F(\pi(\pi_1(\alpha))) & \xrightarrow{t(\pi_1(\alpha))} & \mathrm{triv}_i(\pi_1(\alpha)) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle g(\gamma_i)} \\ F(\pi(\pi_2(\alpha))) & \xrightarrow[t(\pi_2(\alpha))]{} & \mathrm{triv}_i(\pi_2(\alpha)). \end{array}$$

Both diagrams are indeed commutative, the one on the left because t is natural in $y \in Y$ and the one on the right because of (B.1).

It remains to show that ζ is natural in F, i.e. we have to prove the commutativity of the naturality diagram

$$\begin{array}{ccc} F & \xrightarrow{\zeta(F)} & \mathrm{Rec}_\pi(\mathrm{Ex}_\pi(F)) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\mathrm{Rec}_\pi(\mathrm{Ex}_\pi(\alpha))} \\ F' & \xrightarrow[\zeta(F')]{} & \mathrm{Rec}_\pi(\mathrm{Ex}_\pi(F')) \end{array} \tag{B.4}$$

for any natural transformation $\alpha : F \longrightarrow F'$. Recall that $\mathrm{Ex}_\pi(\alpha)$ is the natural transformation

$$h := t \circ \pi^*\alpha \circ t^{-1} : \mathrm{triv}_i \longrightarrow \mathrm{triv}'_i$$

and that $\mathrm{Rec}_\pi(\mathrm{Ex}_\pi(\alpha))$ is the natural transformation whose component at a point $x \in M$ is the morphism

$$h(s(x)) : \mathrm{triv}_i(s(x)) \longrightarrow \mathrm{triv}'_i(s(x))$$

in T. Then, with definition (B.3), the commutativity of the naturality square (B.4) becomes obvious.

B.2 Proof of Theorem 3.12

We show that a Wilson line $\mathcal{W}^{\mathrm{tra},i}_{x_1,x_2}$ of a transport functor tra with Gr-structure is smooth. Let $c : U \longrightarrow PM$ be a map such that $\Gamma(u,t) := c(u)(t)$ is smooth, let $\pi : Y \longrightarrow M$ be a surjective submersion, and let (triv, t) be a π-local i-trivialization of the transport functor tra, for which $\mathrm{Ex}_\pi(\mathrm{triv}, t)$ is smooth. Consider the pullback diagram

$$\begin{array}{ccc} \Gamma^{-1}Y & \xrightarrow{a} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle\pi} \\ U \times [0,1] & \xrightarrow[\Gamma]{} & M \end{array}$$

with the surjective submersion $p : \Gamma^{-1}Y \longrightarrow U \times [0,1]$. We have to show that

$$\mathcal{W}^{\mathrm{tra},i}_{x_1,x_2} \circ \mathrm{pr} \circ c : U \longrightarrow G \tag{B.5}$$

is a smooth map. This can be checked locally in a neighbourhood of a point $u \in U$. Let $t_j \in I$ for $j = 0, ..., n$ be numbers with $t_{j-1} < t_j$ for $j = 1, ..., n$, and V_j open neighbourhoods of u chosen small enough to admit smooth local sections

$$s_j : V_j \times [t_{j-1}, t_j] \longrightarrow \Gamma^{-1}Y.$$

Then, we restrict all these sections to the intersection V of all the V_j. Let $\beta_j : t_{j-1} \longrightarrow t_j$ be paths through I defining smooth maps

$$\tilde{\Gamma}_j : V \times I \longrightarrow Y : (v, t) \longmapsto a(s_j(V, \beta_j(t))), \tag{B.6}$$

which can be considered as maps $\tilde{c}_j : V \longrightarrow PY$. Additionally, we define the smooth maps

$$\tilde{\alpha}_j : V \longrightarrow Y^{[2]} : v \longmapsto (\tilde{\Gamma}_{j-1}(v, 1), \tilde{\Gamma}_j(v, 0)).$$

Note that for any $v \in V$, both $\mathrm{pr}(\tilde{c}_j(v))$ and $\tilde{\alpha}_j(v)$ are morphisms in the universal path pushout $\mathcal{P}_1^\pi(M)$, namely

$$\mathrm{pr}(\tilde{c}_j(v)) : \tilde{\Gamma}_j(v, 0) \longrightarrow \tilde{\Gamma}_j(v, 1) \quad \text{and} \quad \tilde{\alpha}_j(v) : \tilde{\Gamma}_{j-1}(v, 1) \longrightarrow \tilde{\Gamma}_j(v, 0).$$

Taking their composition, we obtain a map

$$\phi : V \longrightarrow \mathrm{Mor}(\mathcal{P}_1^\pi(M)) : v \longrightarrow \tilde{c}_n(v) \circ \tilde{\alpha}_j(v) \circ ... \circ \tilde{\alpha}_1(v) \circ \tilde{c}_0(v).$$

Now we claim two assertions for the composite

$$i^{-1} \circ (R_{(\mathrm{triv}, g)})_1 \circ \phi : V \longrightarrow \mathrm{Mor}(\mathrm{Gr}) \tag{B.7}$$

of ϕ with the functor $R_{(\mathrm{triv},g)} : \mathcal{P}_1^\pi(M) \longrightarrow \mathrm{Mor}(T)$ we have defined in Section 2.3: first, it is smooth, and second, it coincides with the restriction of $\mathcal{W}_{x_1,x_2}^{\mathrm{tra},i} \circ \mathrm{pr} \circ c$ to V, both assertions together prove the smoothness of (B.5). To show the first assertion, note that (B.7) is the following assignment:

$$v \longmapsto \mathrm{triv}(\tilde{c}_j(v)) \cdot \tilde{g}(\tilde{\alpha}_j(v)) \cdot ... \cdot \tilde{g}(\tilde{\alpha}_1(v)) \circ \mathrm{triv}(\tilde{c}_0(v)). \tag{B.8}$$

By definition, the descent data (triv, g) is smooth. Because triv is a smooth functor, and the maps $\tilde{c}_j$ satisfy the relevant condition (B.6), every factor $\mathrm{triv} \circ \tilde{c}_j : V \longrightarrow G$ is smooth. Furthermore, the maps $\tilde{g} : Y^{[2]} \longrightarrow G$ are smooth, so that also the remaining factors are smooth in v. To show the second assertion, consider a point $v \in V$. If we choose in the definition of the Wilson line $\mathcal{W}_{x_1,x_2}^{\mathrm{tra},i}$ the objects $G_k := \mathrm{triv}(\tilde{x}_k)$ and the isomorphisms $t_k := t(\tilde{x}_k)$ for some lifts $\pi(\tilde{x}_k) = x_k$, where t is the trivialization of tra from the beginning of this section, we find

$$(\mathcal{W}_{x_1,x_2}^{\mathrm{tra},i} \circ \mathrm{pr} \circ c)(v) = \mathrm{tra}(\overline{c(v)}).$$

The right hand side coincides with the right hand side of (B.8).

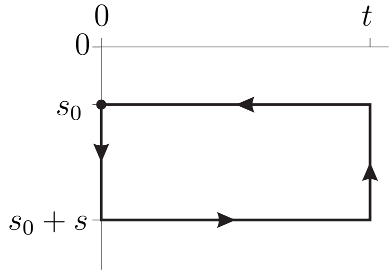


Figure 1: The path $\tau_{s_0}(s,t)$.

B.3 Proof of Proposition 4.3

We are going to prove that the map $k_A : PX \longrightarrow G$, defined by

$$k_A(\gamma) := f_{\gamma^*A}(0,1)$$

for a path $\gamma : [0,1] \longrightarrow X$ depends only on the thin homotopy class of γ. Due to the multiplicative property (4.5) of k_A, it is enough to show $k_A(\gamma_0^{-1} \circ \gamma_1) = 1$ for every thin homotopy equivalent paths γ_0 and γ_1. For this purpose we derive a relation to the pullback of the curvature $K := \mathrm{d}A + [A \wedge A]$ of the 1-form A along a homotopy between γ_0 and γ_1. If this homotopy is thin, the pullback vanishes.

Let us fix the following notation. $Q := [0,1] \times [0,1]$ is the unit square, $\gamma_{(a,b,c,d)} : (a,b) \longrightarrow (c,d)$ is the straight path in Q, and

$$\tau_{s_0} : Q \longrightarrow PQ$$

assigns for fixed $s_0 \in [0,1]$ to a point $(s,t) \in Q$ the closed path

$$\tau_{s_0}(s,t) := \gamma_{(s_0,t,s_0,0)} \circ \gamma_{(s_0+s,t,s_0,t)} \circ \gamma_{(s_0+s,0,s_0+s,t)} \circ \gamma_{(s_0,0,s_0+s,0)},$$

which goes counter-clockwise around the rectangle spanned by the points $(s_0, 0)$ and $(s_0 + s, t)$, see Figure 1. Now consider two paths $\gamma_0, \gamma_1 : x \longrightarrow y$ in X. Without loss of generality we can assume that the paths $\gamma_{(a,b,c,d)}$ used above have sitting instants, such that τ_{s_0} is smooth and

$$\gamma_0(\gamma_{(0,1,0,0)}(t)) = \gamma_0^{-1}(t) \quad \text{and} \quad \gamma_1(\gamma_{(0,1,1,1)}(t)) = \gamma_1(t). \tag{B.9}$$

Lemma B.1. *Let $h : Q \longrightarrow X$ be a smooth homotopy between the paths $\gamma_0, \gamma_1 : x \longrightarrow y$ with $h(0,t) = \gamma_0(t)$ and $h(1,t) = \gamma_1(t)$. Then, the map*

$$u_{A,s_0} := k_A \circ h_* \circ \tau_{s_0} : Q \longrightarrow G$$

is smooth and has the following properties *(a)* $u_{A,0}(1,1) = k_A(\gamma_0^{-1} \circ \gamma_1)$

(b) $u_{A,s_0}(s,1) = u_{A,s_0}(s',1) \cdot u_{A,s_0+s'}(s-s',1)$

(c) with $\gamma_{s,t}$ *the path defined by* $\gamma_{s,t}(\tau) := h(s,\tau t)$ *and* $K := \mathrm{d}A + [A \wedge A]$ *the curvature of* A *we have:*

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t} u_{A,s_0}\bigg|_{(0,t)} = -\mathrm{Ad}^{-1}_{k_A(\gamma_{s_0,t})} (h^*K)_{(s_0,t)}\left(\frac{\partial}{\partial s},\frac{\partial}{\partial t}\right) \tag{B.10}$$

Proof. Since h is constant for $t = 0$ and $t = 1$, (a) follows from (B.9). The multiplicative property (4.5) of k_A implies (b). To prove (c), we define a further path $\gamma_{s_0,s,t}(\tau) := h(s_0 + s\tau, t)$ and write

$$u_{A,s_0}(s,t) = f_{\gamma^*_{s_0,t}A}(0,1)^{-1} \cdot f_{\gamma^*_{s_0,s,t}A}(0,1)^{-1} \cdot f_{\gamma^*_{s_0+s,t}A}(0,1) \tag{B.11}$$

where $f_\varphi : \mathbb{R}\times\mathbb{R} \longrightarrow G$ are the smooth functions that correspond to the a 1-form $\varphi \in \Omega^1(\mathbb{R},\mathfrak{g})$ by Lemma 4.1 as the solution of initial value problems. By a uniqueness argument one can show that $f_{\gamma^*_{s,t}A}(0,1) = f_{\gamma^*_{s,1}A}(0,t)$. Then, we calculate with (B.11) and, for simplicity, in a faithful matrix representation of G,

$$\begin{aligned}\frac{\partial}{\partial t}u_{A,s_0}(s,t) = f^{-1}_{\gamma^*_{s_0,t}A}(0,1)\cdot\Bigg(&(h^*A)_{(s_0,t)}\left(\frac{\partial}{\partial t}\right)\cdot f_{\gamma^*_{s_0,s,t}A}(0,1)^{-1}\\ +\frac{\partial}{\partial t}f_{\gamma^*_{s_0,s,t}A}(0,1)^{-1} - &f_{\gamma^*_{s_0,s,t}A}(0,1)^{-1}\cdot(h^*A)_{(s_0+s,t)}\left(\frac{\partial}{\partial t}\right)\Bigg)\cdot f_{\gamma^*_{s_0+s,t}A}(0,1).\end{aligned}$$

To take the derivatives along s, we use $f_{\gamma^*_{s_0,s,t}A}(0,1) = f_{\gamma_{s_0,1,t}}(0,s)$ and $f_{\gamma_{s_0,0,t}}(0,1) = 1$, both together show

$$\frac{\partial}{\partial s}\bigg|_0 f_{\gamma^*_{s_0,s,t}A}(0,1)^{-1} = (h^*A)_{(s_0,t)}\left(\frac{\partial}{\partial s}\right).$$

Finally,

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial}{\partial t}u_{A,s_0}\bigg|_{s=0} = \;& f^{-1}_{\gamma^*_{s_0,t}A}(0,1)\cdot\Bigg(\Bigg((h^*A)_{(s_0,t)}\left(\frac{\partial}{\partial t}\right)\cdot(h^*A)_{(s_0,t)}\left(\frac{\partial}{\partial s}\right)\\ &+\frac{\partial}{\partial t}(h^*A)_{(s_0,t)}\left(\frac{\partial}{\partial s}\right) - (h^*A)_{(s_0,t)}\left(\frac{\partial}{\partial s}\right)\cdot(h^*A)_{(s_0,t)}\left(\frac{\partial}{\partial t}\right)\\ &-\frac{\partial}{\partial s}\bigg|_0 (h^*A)_{(s_0+s,t)}\left(\frac{\partial}{\partial t}\right)\Bigg)\Bigg)\cdot f_{\gamma^*_{s_0,t}A}(0,1).\end{aligned}$$

This yields the claimed equality. $\square$

Notice that if h is a thin homotopy, $h^*K = 0$, so that the right hand side in (c) vanishes. Then we calculate at $(0,1)$

$$\frac{\partial}{\partial s} u_{A,s_0}\bigg|_{(0,1)} = \int_0^1 \frac{\partial}{\partial s}\frac{\partial}{\partial t} u_{A,s_0}\bigg|_{(0,t)} \mathrm{d}t = 0.$$

Using (b) we obtain the same result for all points $(s_0, 1)$,

$$\frac{\partial}{\partial s} u_{A,0}\bigg|_{(s_0,1)} = u_{A,0}(s_0,1)\cdot \frac{\partial}{\partial s} u_{A,s_0}\bigg|_{(0,1)} = u_{A,0}(s_0,1)\cdot 0 = 0.$$

This means that the function $u_{A,0}(s,1)$ is constant and thus determined by its value at $s = 0$, namely

$$1 = u_{A,0}(0,1) = u_{A,0}(1,1) \overset{\text{(a)}}{=} k_A(\gamma_1^{-1}\circ\gamma_0) = k_A(\gamma_1)^{-1}\cdot k_A(\gamma_0).$$

This finishes the proof.

B.4 Proof of Proposition 4.7

We have to show that the functor $\mathcal{P} : Z^1_X(G)^\infty \longrightarrow \mathrm{Funct}^\infty(\mathcal{P}_1(X), \mathcal{B}G)$ is bijective on objects. For this purpose, we define an inverse map $\mathcal{D}$ that assigns to any smooth functor $F : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G$ a $\mathfrak{g}$-valued 1-form $\mathcal{D}(F)$, such that $\mathcal{P}(\mathcal{D}(F)) = F$ and such that $\mathcal{D}(\mathcal{P}(A)) = A$ for any 1-form $A \in \Omega^1(X, \mathfrak{g})$.

Let $F : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G$ be a smooth functor. We define the 1-form $A := \mathcal{D}(F)$ at a point $p \in X$ and for a tangent vector $v \in T_pX$ in the following way. Let $\gamma : \mathbb{R} \longrightarrow X$ be a smooth curve such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. We consider the map

$$f_\gamma := F_1 \circ (\gamma_*)_1 : \mathbb{R}\times\mathbb{R} \longrightarrow G. \tag{B.12}$$

The evaluation $\mathrm{ev}\circ((\gamma_*)_1 \times \mathrm{id}) : U \times [0,1] \longrightarrow X$ with $U = \mathbb{R}\times\mathbb{R}$ is a smooth map because γ is smooth. Hence, by Definition 3.1, f_γ is smooth. The properties of the functor F further imply the cocycle condition

$$f_\gamma(y,z)\cdot f_\gamma(x,y) = f_\gamma(x,z). \tag{B.13}$$

By Lemma 4.1, the smooth map $f_\gamma : \mathbb{R}\times\mathbb{R} \longrightarrow G$ corresponds to a $\mathfrak{g}$-valued 1-form A_γ on $\mathbb{R}$. We define

$$\alpha_{F,\gamma}(p,v) := A_\gamma|_0\left(\frac{\partial}{\partial t}\right) \in \mathfrak{g}. \tag{B.14}$$

With a view to the definition (4.3) of A_γ, this is

$$\alpha_{F,\gamma}(p,v) = -\left.\frac{\mathrm{d}}{\mathrm{d}t} f_\gamma(0,t)\right|_{t=0}. \tag{B.15}$$

Lemma B.2. *$\alpha_{F,\gamma}(p,v)$ is independent of the choice of the smooth curve γ.*

Proof. Let γ_0 and γ_1 be two choices, both with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v$, for $i = 0,1$. Let $h : [0,1] \times [0,1] \longrightarrow X$ be a smooth homotopy between γ_0 and γ_1, and $\epsilon > 0$ chosen so small that h restricted to $[0,1] \times (0,\epsilon]$ is injective onto its image U. Such an homotopy can always be constructed in a chart of a neighbourhood of x. We construct a map

$$p : [0,1] \times [0,1] \longrightarrow BX$$

by $p(i,\sigma,\tau)(s,t) := h(i,\sigma s,\tau t)$, choose an inverse $\bar h : U \longrightarrow [0,1] \times [0,1]$ of h and obtain a map $q := p \circ \bar h : U \longrightarrow PX$ which satisfies by construction $p = q \circ h$ on $[0,1] \times (0,\epsilon]$. In fact, the domain of q can be enlarged to $U_0 := U \cup \{x\}$ by $q(x) := \mathrm{id}_x$, so that $p = q \circ h$ on all of $[0,1] \times [0,\epsilon]$. The purpose of these constructions is that the map

$$F \circ p : [0,1] \times [0,\epsilon] \longrightarrow G$$

on the one hand satisfies $(F \circ p)(i,t) = f_{\gamma_i}(0,t)$ for $i = 0,1$, and on the other hand factors through two smooth maps $(F \circ q)$ and h, so that we can apply the chain rule:

$$\begin{aligned} -\alpha_{F,\gamma_i}(p,v) = \left.\frac{\mathrm{d}}{\mathrm{d}t} f_{\gamma_i}\right|_{t=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}t}(F \circ p)\right|_{(i,0)} \\ &= \mathrm{d}(F \circ q)|_{h(i,0)} \left(\left.\frac{\partial h}{\partial t}\right|_{(i,0)} \right) = \mathrm{d}(F \circ q)|_x(v) \end{aligned}$$

The right hand side is, in particular, independent of i. ☐

According to the result of Lemma B.2, we drop the index γ, and remain with a map $\alpha_F : TX \longrightarrow \mathfrak{g}$ defined canonically by the functor F. We show next that α_F is linear in v. For a multiple sv of v we can choose the curve γ_s with $\gamma_s(t) := \gamma(st)$. It is easy to see that then $f_{\gamma_s}(x,y) = f_\gamma(sx,sy)$. Again by the chain rule

$$\alpha(p,sv) = -\frac{\mathrm{d}}{\mathrm{d}t} f_{\gamma_s}(0,t)|_{t=0} = -\frac{\mathrm{d}}{\mathrm{d}t} f_\gamma(0,st)|_{t=0} = s\alpha_F(p,v).$$

In the same way one can show that $\alpha(p,v+w) = \alpha(p,v) + \alpha(p,w)$.

Lemma B.3. *The pointwise linear map $\alpha_F : TX \longrightarrow \mathfrak{g}$ is smooth, and thus defines a 1-form $A \in \Omega^1(X, \mathfrak{g})$ by $A|_p(v) := \alpha_F(p, v)$.*

Proof. If X is n-dimensional and $\phi : U \longrightarrow X$ is a coordinate chart with an open subset $U \subset \mathbb{R}^n$, the standard chart for the tangent bundle TX is

$$\phi_{TX} : U \times \mathbb{R}^n \longrightarrow TX : (u, v) \longmapsto \mathrm{d}\phi|_u(v).$$

We prove the smoothness of α_F in the chart ϕ_{TX}, i.e. we show that

$$A \circ \phi_{TX} : U \times \mathbb{R}^n \longrightarrow \mathfrak{g}$$

is smooth. For this purpose, we define the map

$$c : U \times \mathbb{R}^n \times \mathbb{R} \longrightarrow PX : (u, v, \tau) \longmapsto (t \longmapsto \phi(u + \beta(t\tau)v))$$

where β is some smoothing function, i.e. an orientation-preserving diffeomorphism of $[0, 1]$ with sitting instants. Now, $\mathrm{ev} \circ (c \times \mathrm{id})$ is evidently smooth in all parameters, and since F is a smooth functor,

$$f_c := F_1 \circ \mathrm{pr} \circ c : U \times \mathbb{R}^n \times \mathbb{R} \longrightarrow G$$

is a smooth function. Note that $\gamma_{u,v}(t) := c(u, v, t)(1)$ defines a smooth curve in X with the properties

$$\gamma_{u,v}(0) = \phi(u) \quad \text{and} \quad \dot{\gamma}_{u,v} = \mathrm{d}\phi|_u(v), \tag{B.16}$$

and which is in turn related to c by

$$(\gamma_{u,v})_*(0, t) = c(u, v, t). \tag{B.17}$$

Using the path $\gamma_{u,v}$ in the definition of the 1-form A, we find

$$\begin{aligned} (A \circ \phi_{TX})(u, v) &= \alpha_F(\phi(u), \mathrm{d}\phi|_u(v)) \\ &\overset{(B.16)}{=} -\frac{\mathrm{d}}{\mathrm{d}t}(F_1 \circ (\gamma_{u,v})_*)(0, t)\Big|_{t=0} \\ &\overset{(B.17)}{=} -\frac{\mathrm{d}}{\mathrm{d}t} f_c(u, v, t)\Big|_{t=0}. \end{aligned}$$

The last expression is, in particular, smooth in u and v. □

Summarizing, we started with a given smooth functor $F : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G$ and have derived a 1-form $\mathcal{D}(F) := A \in \Omega^1(X, \mathfrak{g})$. Next we show that this 1-form is the preimage of F under the functor

$$\mathcal{P} : Z^1_X(G)^\infty \longrightarrow \mathrm{Funct}^\infty(X, \mathcal{B}G)$$

from Proposition 4.7, i.e. we show

$$\mathcal{P}(A)(\overline{\gamma}) = F(\overline{\gamma})$$

for any path $\gamma \in PX$. We recall that the functor $\mathcal{P}(A)$ was defined by $\mathcal{P}(A)(\overline{\gamma}) := f_{\gamma^*A}(0,1)$, where $f_{\gamma^*A} : \mathbb{R} \times \mathbb{R} \longrightarrow G$ solves the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{\gamma^*A}|_{(0,t)} = -\mathrm{d}r_{f_{\gamma^*A}(0,t)}|_1 \left((\gamma^*A)_t \left(\frac{\partial}{\partial t} \right) \right) \tag{B.18}$$

with the initial value $f_{\gamma^*A}(0,0) = 1$. Now we use the construction of the 1-form A from the given functor F. For the smooth function $f_\gamma : \mathbb{R} \times \mathbb{R} \longrightarrow G$ from (B.12) we obtain using $\gamma_t(\tau) := \gamma(t+\tau)$ with $p := \gamma_t(0)$ and $v := \dot{\gamma}_t(0)$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} f_\gamma(t, t+\tau)|_{\tau=0} &= \frac{\mathrm{d}}{\mathrm{d}\tau} f_{\gamma_t}(0,\tau)|_{\tau=0} \\ &\overset{(B.15)}{=} -\alpha_{F,\gamma_t}(p,v) = -A_p(v) = -(\gamma^*A)_t \left(\frac{\partial}{\partial t} \right). \end{aligned} \tag{B.19}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f_\gamma(0,t) \overset{(B.13)}{=} \frac{\mathrm{d}}{\mathrm{d}\tau} f_\gamma(t,\tau)|_{\tau=t} \cdot f_\gamma(0,t) \overset{(B.19)}{=} -\mathrm{d}r_{f_\gamma(0,t)}|_1 \left((\gamma^*A)_t \left(\frac{\partial}{\partial t} \right) \right).$$

Hence, f_γ solves the initial value problem (B.18). By uniqueness, $f_{\gamma^*A} = f_\gamma$ and finally

$$F(\overline{\gamma}) = f_\gamma(0,1) = f_{\gamma^*A}(0,1) = \mathcal{P}(A)(\overline{\gamma}).$$

It remains to show that, conversely, for a given 1-form $A \in \Omega^1(X, \mathfrak{g})$,

$$\mathcal{D}(\mathcal{P}(A)) = A.$$

We test the 1-form $\mathcal{D}(\mathcal{P}(A))$ at a point $x \in X$ and a tangent vector $v \in T_xX$. Let $\Gamma : \mathbb{R} \longrightarrow X$ be a curve in X with $x = \Gamma(0)$ and $v = \dot{\Gamma}(0)$. If we further denote $\gamma_\tau := \Gamma_*(0,\tau)$ we have

$$-\mathcal{D}(\mathcal{P}(A))|_x(v) \overset{(B.15)}{=} \left. \frac{\partial f_{\gamma_\tau}}{\partial \tau} \right|_{(0,0)} \overset{(B.12)}{=} \left. \frac{\partial}{\partial \tau} \right|_0 \mathcal{P}(A)(\gamma_\tau) = \left. \frac{\partial}{\partial \tau} \right|_0 f_{\gamma_\tau^*A}(0,1)$$

Here, $f_{\gamma_\tau^*A}$ is the unique solution of the initial value problem (B.18) for the given 1-form A and the curve γ_τ. With a uniqueness argument $f_{A,\gamma_\tau}(t_0,t) = f_{A,\gamma_1}(\tau t_0, \tau t)$. Its derivative is

$$\left. \frac{\partial}{\partial \tau} f_{\gamma_\tau^*A}(0,t) \right|_{\tau=0,t=1} = \left. \frac{\partial}{\partial t} f_{\gamma_1^*A}(0,t) \right|_{t=0} = -A_p(v),$$

this yields $\mathcal{D}(\mathcal{P}(A)) = A$.

Table of Notations

$\mathcal{P}_1^\pi(M)$	the universal path pushout of a surjective submersion $\pi : Y \longrightarrow M$.	Page 13
Ex_π	the functor $\mathrm{Ex}_\pi : \mathrm{Triv}_\pi^1(i) \longrightarrow \mathfrak{Des}_\pi^1(i)$ which extracts descent data.	Page 13
f_*	the functor $f_* : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1(N)$ of path groupoids induced by a smooth map $f : M \longrightarrow N$.	Page 8
$\mathrm{Funct}(S,T)$	the category of functors between categories $S$and T.	Page 14
$\mathrm{Funct}^\infty(\mathcal{P}_1(X), \mathrm{Gr})$	the category of smooth functors from $\mathcal{P}_1(X)$ to a Lie groupoid Gr.	Page 29
G-Tor	the category of smooth principal G-spaces and smooth equivariant maps.	Page 2
Gr	a Lie groupoid	Page 26
i	a functor $i : \mathrm{Gr} \longrightarrow T$, which relates the typical fibre Gr of a functor to the target category T.	Page 26
C^∞	the category of smooth manifolds and smooth maps between those.	Page 54
$\pi_1^1(M, x)$	the thin homotopy group of the smooth manifold M at the point $x \in M$.	Page 8
p^π	the projection functor $p^\pi : \mathcal{P}_1^\pi(M) \longrightarrow \mathcal{P}_1(M)$.	Page 16
PM	the set of paths in M	Page 6
P^1M	the set of thin homotopy classes of paths in M	Page 7
$\mathcal{P}_1(M)$	the path groupoid of the smooth manifold M.	Page 8
$\Pi_1(M)$	the fundamental groupoid of a smooth manifold M	Page 9
Rec_π	the functor $\mathrm{Rec}_\pi : \mathfrak{Des}_\pi^1(i) \longrightarrow \mathrm{Triv}_\pi^1(i)$ which reconstructs a functor from descent data.	Page 14
s	the section functor $s : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M)$ associated to a surjective submersion $\pi : Y \longrightarrow M$.	Page 13

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