A COUNTEREXAMPLE TO KING'S CONJECTURE

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Abstract. King's conjecture states that on every smooth complete toric variety X there exists a strongly exceptional collection which generates the bounded derived category of X and which consists of line bundles. We give a counterexample to this conjecture. This example is just the Hirzebruch surface $\mathbb{F}_2$ iteratively blown up three times, and we show by explicit computation of cohomology vanishing that there exist no strongly exceptional sequences of length 7.

1. Introduction

It is a widely open question whether on a given smooth algebraic variety X (say, complete and smooth), there exists a *tilting sheaf*. A tilting sheaf is a sheaf $\mathcal{T}$ which generates the bounded derived category $\mathcal{D}^b(X)$ of X and $\mathrm{Ext}^k(\mathcal{T}, \mathcal{T}) = 0$ for all $k > 0$. For such $\mathcal{T}$, the functor

$$\mathbf{R}\mathrm{Hom}(\mathcal{T}, \, . \,) : D^b(X) \longrightarrow D^b(A - \mathrm{mod}),$$

where $A := \mathrm{End}(\mathcal{T})$ is the endomorphism algebra, induces an equivalence of categories (see [Rud90], [Bon90], [Bei78]). The existence of a tilting sheaf implies that the Grothendieck group of X is finitely generated and free, so that in general such sheaves can not exist. However, so far there are a number of positive examples known, including projective spaces, del Pezzos, certain homogeneous spaces, and some higher dimensional Fanos. An obvious testbed for the existence of tilting sheaves are the toric varieties. There is a quite strong conjecture which was first stated by King:

Conjecture [Kin97] **:** Let X be a smooth complete toric variety. Then X has a tilting sheaf which is a direct sum of line bundles.

If a tilting sheaf decomposes into a direct sum of line bundles, its direct summands $\mathcal{T} = \bigoplus_{i=1}^t \mathcal{L}_i$ form a so-called *strongly exceptional sequence*, i.e. $\mathrm{Ext}^k(\mathcal{L}_i, \mathcal{L}_j) = 0$ for all i, j and all $k > 0$, and — after eventually reordering the $\mathcal{L}_i$ — $\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j) = 0$ for $i > j$. Moreover, t is the rank of the Grothendieck group of X.

It would be very nice if there existed easy-computable tilting sheaves on toric varieties, and indeed there are known a lot of positive examples in favor of the conjecture (see [CM04], [Kaw05], [Hil04], [CS05]). Computer experiments also look promising in many directions. However, the conjecture remained somewhat mysterious so far and, as it turns out, it is false in general. It is the purpose of this paper to present a counterexample.

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Our counterexample is the toric surface X as shown in figure 1, which can be obtained by iteratively blowing up the Hirzebruch surface $\mathbb{F}_2$ three times. In coordinates, the primitive vectors of its rays are given by $l_1 = (1, -1)$, $l_2 = (2, -1)$, $l_3 = (3, -1)$, $l_4 = (1, 0)$, $l_5 = (0, 1)$, $l_6 = (-1, 2)$, $l_7 = (0, -1)$. Note that the rank of the Grothendieck group of X is 7.

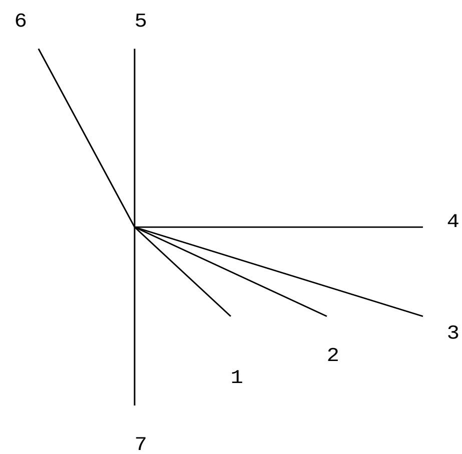

FIGURE 1. The fan

To show that there do not exist any strongly exceptional sequences of length 7 on this surface, we will perform explicit computations in the Picard group to determine cohomology vanishing. More precisely, note that if $\mathcal{L}_1, \ldots, \mathcal{L}_t$ is a strongly exceptional sequence, then also $\mathcal{L}_1 \otimes \mathcal{L}', \ldots, \mathcal{L}_t \otimes \mathcal{L}'$ is strongly exact, where $\mathcal{L}'$ is any line bundle. So one can assume without loss of generality that the sequence contains the structure sheaf. Then a necessary condition for the bundles in the sequence is that all the higher cohomology groups of the bundles and of their dual bundles vanish, i.e. $H^k(X, \mathcal{L}_i) = H^k(X, \mathcal{L}_i^*) = 0$ for all i and all $k > 0$. This is a rather strong condition on the sheaves and our main computation will be to compile a complete list of such bundles for our surface X. After having obtained this classification, we deduce by simple inspection that a strongly exceptional sequence of length 7 and consisting of line bundles does not exist.

Overview: In section 2 we state everything we need to know about cohomology of line bundles on toric surfaces and we describe in more detail our method of computation. In section 3 all bundles are classified which have the property that the higher cohomologies of the bundles themselves and of their dual bundles vanish. In section 4 we present the complete classification obtained in section 3 and we show by inspection that there exist no strongly exceptional sequences of length 7 on X.

2. The setup

In this section we recall basic facts on cohomology of line bundles on a toric surface and we describe our method of computation. For general information about toric varieties we refer to the books [Oda88], [Ful93].

2.1. **Generalities on toric line bundles.** Let X be a complete smooth toric surface on which the torus T acts. The variety X is described by a fan Δ which is contained in a 2-dimensional vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, where $N \cong \mathbb{Z}^2$. We denote by $\Delta(1)$ the set of rays, that is, of one-dimensional cones of Δ. As X is a complete surface, the fan is completely determined by the rays. We denote the rays by $\rho_1, \ldots, \rho_n$, enumerated in counterclockwise order, and $l_1, \ldots, l_n$ the primitive vectors of the rays. To any ρ_i there is associated a T-invariant divisor D_i, and every divisor D can, up to rational equivalence, written as a sum of these invariant divisors, i.e. $D = \sum_{i=1}^n c_i D_i$. We denote $M \cong \mathbb{Z}^2$ the character group of the torus acting on X and we set $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. The lattice N is in a natural way dual to M, and the primitive vectors l_i are integral linear forms on M (and on $M_{\mathbb{R}}$, respectively). There is a short exact sequence

$$0 \longrightarrow M \stackrel{A}{\longrightarrow} \mathbb{Z}^{\Delta(1)} \longrightarrow \operatorname{Pic}(X) \longrightarrow 0,$$

where the matrix A is composed of the l_i as row vectors. This sequence is split exact. More precisely, if we choose two of the l_i, for instance l_{n-1} and l_n, which form a $\mathbb{Z}$-basis of N, then the divisors $D_1, \ldots, D_{n-2}$ form a $\mathbb{Z}$-basis of $\operatorname{Pic}(X)$. So every divisor D has a unique representation $D = \sum_{i=1}^{n-2} c_i D_i$.

Now let $D = \sum_{i=1}^n c_i D_i$ be any T-invariant divisor. D in a natural way defines an affine hyperplane arrangement $\mathcal{H}_D = \{H_1, \ldots, H_n\}$ in the vector space $M_{\mathbb{R}}$, where

$$H_i = \{m \in M_{\mathbb{R}} \mid l_i(m) = -c_i\}.$$

All information on the cohomology of the line bundle $\mathcal{O}(D)$ is contained in the chamber structure $\mathcal{H}_D$ (or more precisely, in the intersection of this chamber structure with the lattice M). Recall that the T-action induces an eigenspace decomposition on the cohomology groups of $\mathcal{O}(D)$:

$$H^i\big(X, \mathcal{O}(D)\big) = \bigoplus_{m \in M} H^i\big(X, \mathcal{O}(D)\big)_m.$$

The dimension of $H^i\big(X, \mathcal{O}(D)\big)_m$ as a k-vector space is determined by the *signature* of m with respect to the arrangement $\mathcal{H}_D$:

Definition 2.1: Let $D = \sum_{i=1}^n c_i D_i$ be a T-invariant divisor on X. Then for every $i = 1, \ldots, n$ we define a signature

$$\Sigma_i^D : M \longrightarrow \{+, -, 0\},$$

where $\Sigma_i^D(m) = +$ if $l_i(m) > -c_i$, $\Sigma_i^D(m) = -$ if $l_i(m) < -c_i$ and $\Sigma_i^D(m) = 0$ if $l_i(m) = -c_i$. Moreover, we denote

$$\Sigma^D : M \longrightarrow \{+, -, 0\}^n,$$

where $\Sigma^D(m)$ is the tuple $\big(\Sigma_1^D(m), \ldots, \Sigma_n^D(m)\big)$.

Below we will mostly work with only one D at a time, which will be clear from the context. So usually we will omit the reference to D in the notation, i.e. we will mostly write $\Sigma(m)$ instead of $\Sigma^D(m)$.

Given the signature $\Sigma^D(m)$, the computation of $H^i\big(X, \mathcal{O}(D)\big)_m$ is straightforward. For H^2, we have:

$$\dim H^2\big(X, \mathcal{O}(D)\big)_m = \begin{cases} 1 & \text{if } \Sigma^D(m) = \{-\}^n \\ 0 & \text{else.} \end{cases}$$

For H^1, we have to consider the $-$*-intervals*. For a given signature $\Sigma^D(m)$, a $-$-interval is a connected sequence of $-$ with respect to the circular order of the ρ_i. For example, assume that $\Delta(1)$ consists of 7 elements enumerated in circular order. Then the signature $+--++-+$ has two $-$-intervals. Note that due to the *circular* ordering of the rays, the signature $--+++--$ has only one $-$-interval. We have:

$$\dim H^1\big(X, \mathcal{O}(D)\big)_m = \begin{cases} \text{the number of } -\text{-intervals } -1 \text{ if there exists at least one } -\text{-interval,} \\ 0 \text{ else.} \end{cases}$$

Thus $H^1\big(X, \mathcal{O}(D)\big)$ vanishes if and only if the signatures $\Sigma^D(m)$, as m runs through M, have at most one $-$-interval.

2.2. **Method of computation.** Let $\mathcal{L}_1, \ldots, \mathcal{L}_t$ be a strongly exceptional sequence of line bundles, i.e. we have $\operatorname{Ext}^k(\mathcal{L}_i, \mathcal{L}_j) = 0$ for all i, j and all $k > 0$. There is a natural isomorphism $\operatorname{Ext}^k(\mathcal{L}_i, \mathcal{L}_j) \cong H^k(X, \mathcal{L}_i^* \otimes \mathcal{L}_j)$, where $\mathcal{L}_i^* = \operatorname{Hom}(\mathcal{L}_i, \mathcal{O}_X)$ denotes the dual bundle. By this we can assume without loss of generality that one of the $\mathcal{L}_i$ is just the structure sheaf $\mathcal{O}_X$, i.e. $\mathcal{L}_1, \ldots, \mathcal{L}_t$ is a strongly exceptional sequence if and only if $\mathcal{L}_i^* \otimes \mathcal{L}_1, \ldots, \mathcal{L}_i^* \otimes \mathcal{L}_t$ is a strongly exceptional sequence. If $\mathcal{O}_X$ is part of the sequence, this in turn implies a rather strong condition on the cohomologies of the other bundles. Namely, for every $\mathcal{L}_i$ we have:

$$H^k(X, \mathcal{L}_i) = H^k(X, \mathcal{L}_i^*) = 0 \text{ for all } k > 0.$$

Thus, to show that our toric surface does not have a strongly exceptional sequence of length 7, we proceed in 2 steps:

(i) We classify all line bundles where higher cohomologies of the bundle itself as well as of its dual vanish. It turns out that the list of such bundles has a rather short description, although it is not finite.
(ii) After having obtained the list, we show by exclusion that there are no strongly exceptional sequences of length 7.

Figure 2 shows the arrangement which belongs to the structure sheaf. We see that this arrangement is central and induces a chamber decomposition of the space $M_{\mathbb{R}}$, consisting of unbounded chambers. To every chamber there is associated a signature which we have indicated in the picture. Note that in fact there are some more signatures which are not shown. For instance, the points lying on the line between the chambers with signatures $++++++-$ and $-+++++-$ have signature $0+++++-$. The origin has signature 0000000. Figure 3 shows a deformation of this central arrangement which belongs to the divisor $D = -(4D_1 + 7D_2 + 11D_3 + 4D_4 + 2D_5)$.

As we can see, moving the hyperplanes creates new chambers with new signatures. There are two new unbounded chambers with signatures $----+++$ and $+++++-+$, respectively, which obviously have no influence on the comohology of $\mathcal{O}(D)$. The other chambers are all bounded and thus contain only a finite number of lattice points (i.e. points in M). We have indicated the signatures of some of these points in the picture. As one can check, most of these signatures give not rise to nonvanishing cohomology, the only exception being the point with signature $+++++++$. Recall that we are interested in the classification of line bundles which have no higher cohomology and whose duals have also no higher cohomology. So, if there is an inequality $l_i(m) < -c_i$ (or $l_i(m) > -c_i$, respectively), then we have $l_i(-m) > c_i$ ($l_i(-m) < c_i$,

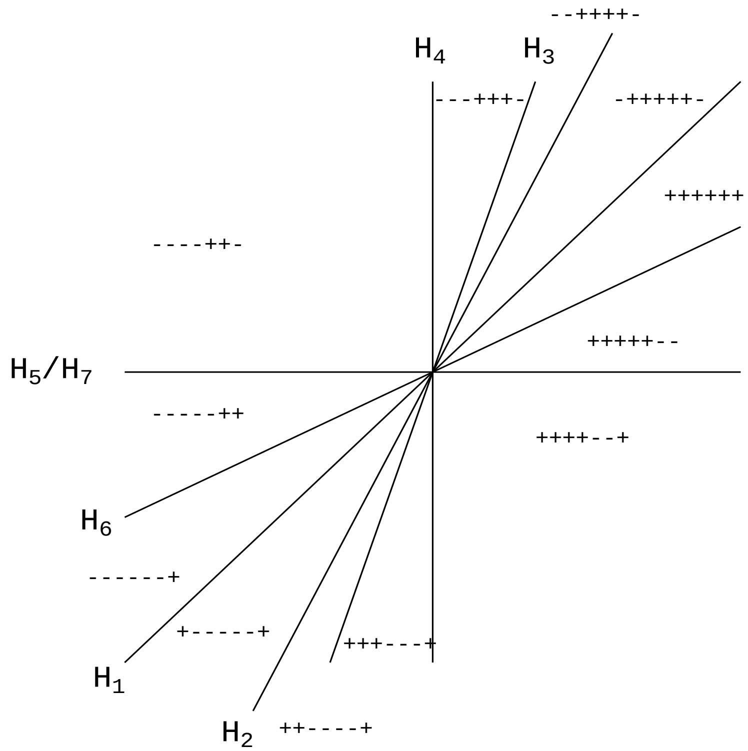

FIGURE 2. The central arrangement

respectively), whereas for $l_i(m) = -c_i$ we have $l_i(-m) = c_i$. In our example the signature of m with $\Sigma^D(m) = +++++++$ becomes $\Sigma^{-D}(-m) = -------$ for the dual bundle, which therefore has nonvanishing H^2.

We give one more example and some more notation. In many situation it will not be necessary to know the complete signature of some point $m \in M$. Therefore we define:

Definition 2.2: A *partial* signature is given by

$$\Sigma^D : M \longrightarrow \{+, -, 0, *\}^n$$

which is a signature for some subset I of $\{1, \ldots, n\}$ such that $\big(\Sigma^D(m)\big)_i = \Sigma_i^D(m)$ for $i \in I$ and $\big(\Sigma^D(m)\big)_i = *$ for $i \notin I$.

For us it is convenient to use the same symbol for signatures and partial signatures. Let us give an explicit example. Assume $D = \sum_{i=1}^5 c_i D_i$ and $c_5 > 0$. Now consider the point m in M which has the coordinates $(1 - 2c_5, -c_5)$ (see figure 4). Its partial signature with respect to the linear forms l_5, l_6, l_7 is $\Sigma(m) = ****0-+$. Our aim is to derive conditions on the values of the c_i. Evidently, any complete signature which is obtained by filling the $*$'s has at least one $-$-interval. Moreover, if any of the $*$'s becomes a $-$, the signature has at least two $-$-intervals, and any corresponding line bundle will have nonvanishing H^1. So, a necessary condition is that

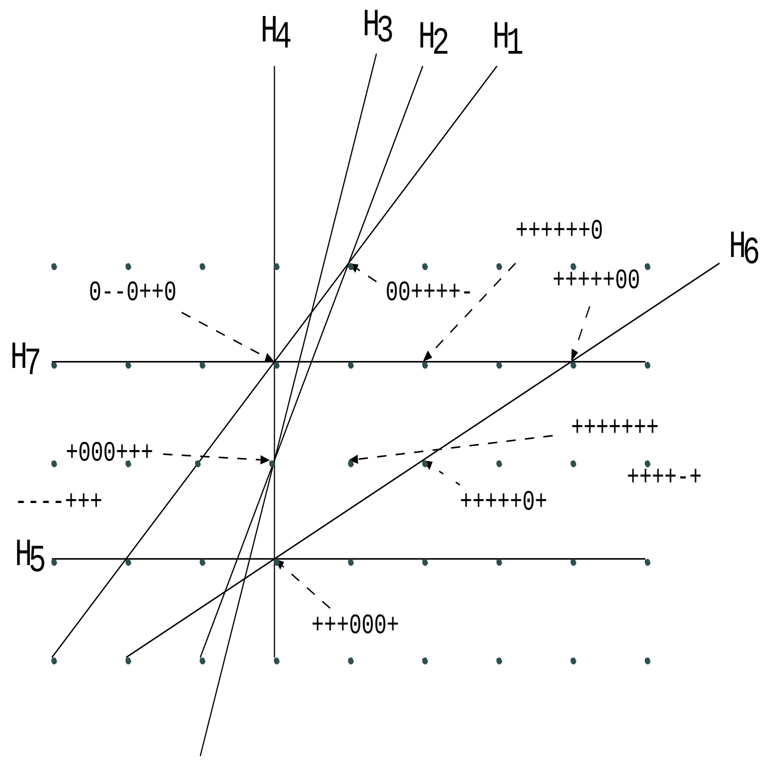

FIGURE 3. A deformation of the central arrangement

$\Sigma_i^D(m) \in \{+, 0\}$ for $i = 1, \ldots, 4$ and any valid divisor D. This in turn implies:

$$\begin{aligned} c_1 &\geq c_5 - 1 \\ c_2 &\geq 3c_5 - 2 \\ c_3 &\geq 5c_5 - 3 \\ c_4 &\geq 2c_5 - 1. \end{aligned} \tag{1}$$

Now the point $(-3, -1)$ has partial signature $\Sigma(-3, -1) = * * * * + + +$, and the above conditions on $c_1, \ldots, c_4$ imply that for $c_5 > 3$ this point always has signature $+ + + + + + +$, and thus we have nonvanishing H^2. Hence, we conclude $c_5 \leq 3$.

3. Classification of line bundles without higher cohomology

In this section we do the complete classification of line bundles for our toric surface which have the property that the higher cohomologies vanish for both, the bundle itself and its dual. As explained in the previous section, we can always assume that a line bundle $\mathcal{L}$ is uniquely represented by an invariant divisor $D = \sum_{i=1}^5 c_i D_i$, and every tuple of numbers $(c_1, \ldots, c_5)$ represents a unique isomorphism class in $\mathrm{Pic}(X)$. As we already have seen, a necessary condition is that $c_5 \leq 3$. Moreover, as it does not matter if we deal with a bundle or its dual, we can assume without loss of generality that $c_5 \geq 0$. So, this leaves us with four possible values for

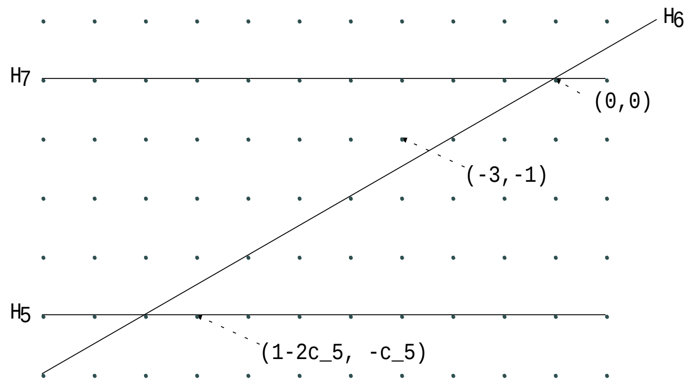

FIGURE 4. A partial arrangement

c_5. Our classification will be done by subsequential case distinctions which on the toplevel are guided by the four possible values of c_5.

Note that in the sequel for a given bundle we will use phrases like "has cohomology" if either the bundle itself or its dual has a nonvanishing higher cohomology group.

3.1. $c_5 = 3$. Recall that the partial signature of the point $(-3,-1)$ is $\Sigma^D(-3,-1) = ****+++$. By the conditions (1), we immediately obtain the partial signature $*++++++$. So, the only way to prevent H^2 to show up in the dual bundle is $\Sigma_1^D(-3,-1) = 0$ (then, for the dual bundle, we have the signature $\Sigma^{-D}(3,1) = 0------$). This in turn means that $c_1 = 2$. But then, we have $\Sigma^D(-4,-2) = 0++++0+$, and thus $\Sigma^{-D}(4,2) = 0----0-$, hence we get H^1. We conclude that there are no divisors with $c_5 = 3$ and vanishing cohomology.

3.2. $c_5 = 2$. Here, conditions (1) read:

$$
\begin{aligned}
c_1 &\geq 1, \\
c_2 &\geq 4, \\
c_3 &\geq 7, \\
c_4 &\geq 3.
\end{aligned}
$$

We first consider $c_4 = 3$. Then $\Sigma(-3,0) = ***0++0$. If one of the $*$'s is replaced by $+$, this implies that the dual signature will have at least two $-$-intervals, independent on the other substitutions. So we obtain:

$$
\begin{aligned}
c_1 &\leq 3, \\
c_2 &\leq 6, \\
c_3 &\leq 9.
\end{aligned}
$$

We treat these 27 possibilities case by case. First, let $c_1 = 1$. Then we have $\Sigma(-2,-1) = 0**++0+$, and so we have H^1, leaving only 18 more cases. For these we write a table:

c_1	c_2	c_3	m	$\Sigma(m)$
2	4	8	$(-3,-2)$	$+0+00-+$
2	6	—	$(-3,-1)$	$0+*0+++$
2	—	9	$(-3,-1)$	$0*+0+++$
3	4	—	$(-2,0)$	$+0*+++0$
3	5	9	$(-3,-1)$	$+0+0+++$
3	6	7	$(-2,1)$	$0+0+++-$

This table contains a list of all values which have cohomology. For given values of c_1, c_2, c_3, the fourth columns contains a lattice point $m \in M$ with bad signature, which is displayed in the fifth column. Sometimes it suffices to display only a partial signature. Then the box for the corresponding c_i contains a dash (—). All tuples which are not displayed in the above table represent cohomology free line bundles, namely:

c_1	c_2	c_3	c_4	c_5
2	4	7	3	2
2	5	7	3	2
2	5	8	3	2
3	5	7	3	2
3	5	8	3	2
3	6	8	3	2
3	6	9	3	2

Now for $c_4 = 4$. We have $\Sigma(-3,-2) = ***+0-+$ and thus we get the bounds

$$c_1 \geq 2,$$
$$c_2 \geq 5,$$
$$c_3 \geq 8.$$

Moreover, we have $\Sigma(-4,0) = ***0++0$ and thus

$$c_1 \leq 4,$$
$$c_2 \leq 8,$$
$$c_3 \leq 12.$$

Further, $\Sigma(-3,-1) = ***++++$ and hence

$$c_1 \leq 2 \text{ or}$$
$$c_2 \leq 5 \text{ or}$$
$$c_3 \leq 8.$$

We have $\Sigma(-4,-3) = ***0--+$ which implies

$$c_3 \geq 9$$

and so the case $c_3 \leq 8$ cannot occur. Also, the conditions imply that either $c_1 = 2$ or $c_2 = 5$, thus leaving 24 possibilities.

We first consider the case $c_1 = 2$. Then we have $\Sigma(-3,-2) = 0**000+$, which implies $c_2 \leq 6$ and $c_3 \leq 10$. Now we take $c_2 = 5$. Then $\Sigma(-4,-3) = +0*0--+$, so that we must have $c_3 = 9$. For $c_2 = 6$, we have $\Sigma(-4,-2) = 00*000+$, which implies $c_3 = 10$. Indeed, we have found:

c_1	c_2	c_3	c_4	c_5
2	5	9	4	2
2	6	10	4	2

Now we consider $c_2 = 5$. We can assume that $c_1 \geq 3$. Assume that $c_3 \geq 10$. Then $\Sigma(-4,-3) = +0+0--+$, so we have cohomology, hence $c_3 = 9$. For $c_1 = 4$, we have $\Sigma(-4,-3) = +0+0--+$ and thus cohomology, hence $c_1 = 3$, and indeed we have found:

c_1	c_2	c_3	c_4	c_5
3	5	9	4	2

Now we go on with $c_4 \geq 5$. Then we have $\Sigma(-4,-2) = ***+00+$, which yields the conditions

$$c_1 \geq 4,$$
$$c_2 \geq 7,$$
$$c_3 \geq 11.$$

The signature $\Sigma(-3,-1)$ as before implies

$$c_1 \leq 2 \text{ or}$$
$$c_2 \leq 5 \text{ or}$$
$$c_3 \leq 8.$$

both conditions cannot be fulfilled simultaneously, and hence, for $c_4 \geq 5$ there are no cohomologyfree bundles.

3.3. $c_5 = 1$. Again, we start with the conditions (1), which read

$$c_1 \geq 0,$$
$$c_2 \geq 1,$$
$$c_3 \geq 2,$$
$$c_4 \geq 1.$$

Now we go for the different cases for c_1.

$c_1 = 0$. Then we have $\Sigma(-1,-1) = 0***0-+$ so that all of the $*$'s can only be substituted by 0's, and thus $c_2 = 1$, $c_3 = 2$, $c_4 = 1$ and indeed we have found:

c_1	c_2	c_3	c_4	c_5
0	1	2	1	1

with no other possibilities left.

$c_1 = 1$. We have $\Sigma(-2,-1) = 0***00+$ which implies

$$c_2 \leq 3,$$
$$c_3 \leq 5,$$
$$c_4 \leq 2.$$

Let $c_4 = 1$, then $\Sigma(-1,0) = 0**0++0$, which implies

$$c_2 \leq 2,$$
$$c_3 \leq 3.$$

From these four cases, only $c_2 = 1, c_3 = 2$ has cohomology, as in this case $\Sigma(-1,-1) = +0+00-+$. We have found:

c_1	c_2	c_3	c_4	c_5
1	1	2	1	1
1	2	2	1	1
1	2	3	1	1

Now let $c_4 = 2$. Then $\Sigma(-2,-1) = + * * + 0 - 0$, so that

$$c_2 \geq 2,$$
$$c_3 \geq 3,$$

leaving six cases. We write a table as before:

c_2	c_3	m	$\Sigma(m)$
2	3	$(-3,-2)$	$+0-0--+$
2	5	$(-3,-2)$	$+0+0--+$
3	3	$(-1,0)$	$0+0+++0$

and thus we have found:

c_1	c_2	c_3	c_4	c_5
1	2	4	2	1
1	3	4	2	1
1	3	5	2	1

$c_1 \geq 2$. Now for any $c_1 \geq 2$, the point $(1-c_1, 0)$ has signature $\Sigma(1-c_1,0) = + * * * + + 0$. So, we obtain general conditions

$$c_2 \geq 2c_1 - 1,$$
$$c_3 \geq 3c_1 - 2,$$
$$c_4 \geq c_1.$$

We obtain another general condition as follows. Consider the signature $\Sigma(-c_1-1,0) = - * * * + + 0$. Assume that $c_4 \geq c_1 + 2$. Then $\Sigma(-c_1,0) = - + * * + + 0$ and so the $*$'s can only be replaced by $+$'s, hence $c_2 \geq 2c_1 + 3$. The signature $\Sigma(-c_2-2,-1)$ then becomes either $-0*-0+0$ or $-+*-0+0$ which both are bad. Thus c_4 must be strictly smaller than c_1+2, and we have:

$$c_4 \in \{c_1, c_1+1\} \text{ for any value of } c_1 \geq 2.$$

Now consider the signature $\Sigma(-c_1-1,-1) = 0***0++$, which yields the following restrictions:

$$c_2 \leq 2c_1 + 1,$$
$$c_3 \leq 3c_1 + 2.$$

Now assume that $c_4 = c_1$. From the signature $\Sigma(-c_1,0) = 0**0++0$ we get immediately the conditions

$$c_2 \leq 2c_1,$$
$$c_3 \leq 3c_1.$$

If $c_2 = 2c_1 - 1$, we have the signature $\Sigma(-c_1, -1) = +0*000+$, respectively $\Sigma(-2,-1) = +0*00++$, for the case $c_1 = 2$. In either case, we get:

$$c_3 \leq 3c_1 - 1.$$

For $c_2 = 2c_1$, we have the signature $\Sigma(1-c_1, 1) = 0+*+++-$, hence the $*$ cannot be replaced by $-$ or 0, thus we get $c_3 \geq 2c_1 - 1$. We cannot find any more restrictions and in fact we have found inifinite series of cohomology-free line bundles:

c_1	c_2	c_3	c_4	c_5
$k \geq 2$	$2k-1$	$3k-2$	k	1
$k \geq 2$	$2k-1$	$3k-1$	k	1
$k \geq 2$	$2k$	$3k-1$	k	1
$k \geq 2$	$2k$	$3k$	k	1

Now let $c_4 = c_1 + 1$. The signature $\Sigma(-c_4, -1) = +**0--+$ yields

$$c_2 \geq 2c_1,$$
$$c_3 \geq 3c_1 + 1.$$

This leaves four possibilities of which we can only exclude the case $c_2 = 2c_1$, $c_2 = 3c_1 + 2$. Here we distinguish cases $c_1 = 2, 3, \geq 4$. For $c_1 = 2$, we have $\Sigma(-3,-2) = +0+0--+$, for $c_1 = 3$ we have $\Sigma(-4,-2) = +0+00-+$ and for $c_1 \geq 4$ we have $\Sigma(-c_1-1,-2) = +0+0+-+$, all of which are bad signatures. So, we have extracted three more series:

c_1	c_2	c_3	c_4	c_5
$k \geq 2$	$2k$	$3k+1$	$k+1$	1
$k \geq 2$	$2k+1$	$3k+1$	$k+1$	1
$k \geq 2$	$2k+1$	$3k+2$	$k+1$	1

3.4. $c_5 = 0$. . We have the signatures $\Sigma(-1,0) = ****0+0$ and $\Sigma(1,0) = ****0-0$ which imply:

$$-1 \leq c_1 \leq 1,$$
$$-2 \leq c_2 \leq 2,$$
$$-3 \leq c_3 \leq 3,$$
$$-1 \leq c_4 \leq 1.$$

As $c_5 = 0$, we can assume without loss of generality $c_1 \geq 0$. We refine by case distinction by the values of c_1.

$c_1 = 0$. Here we can assume without loss of generality that $c_4 \geq 0$. Let $c_4 = 0$ and thus without loss of generality $c_2 \geq 0$. We have the following table

c_2	c_3	m	$\Sigma(m)$
0	2	$(-1,-1)$	$0-0---+$
0	3	$(-1,-1)$	$0-+---+$
1	≤ 0	$(1,0)$	$-0-0++-$
1	≥ 2	$(0,1)$	$-0+0++-$

Thus we have found:

c_1	c_2	c_3	c_4	c_5
0	0	0	0	0
0	0	1	0	0
0	1	1	0	0

Now let $c_4 = 1$. Then $\Sigma(-1,-1) = 0{*}{*}0--+$ and hence $c_2 = 1$ and $c_3 = 2$. We have found:

c_1	c_2	c_3	c_4	c_5
0	1	2	1	0

$c_1 = 1$. Assume first that $c_4 = -1$. Then $\Sigma(0,1) = 0{*}{*}-++-$ which makes $c_4 = -1$ impossible.

Now let $c_4 = 0$. We have $\Sigma(0,1) = 0{*}{*}0++-$ which implies that $c_2 = 1$ and $c_3 = 1$. We have found

c_1	c_2	c_3	c_4	c_5
1	1	1	0	0

Finally, let $c_4 = 1$. Then $\Sigma(0,0) = +{*}{*}+000$, so

$$c_2 \geq 1,$$
$$c_3 \geq 1.$$

So we have reduced to six possibilities. Consider the table

c_2	c_3	m	$\Sigma(m)$
1	1	$(-1,-1)$	$+0-0--+$
1	3	$(-1,-1)$	$+0+0--+$
2	1	$(0,2)$	$-0-+++-$

The remaining cases are:

c_1	c_2	c_3	c_4	c_5
1	1	2	1	0
1	2	2	1	0
1	2	3	1	0

which finishes the classification.

4. Table of cohomology-free line bundles and theorem

We represent the classification obtained in the previous section in a table at the end of this section. We distinguish three types of line bundles, named by the letters A to C, where the B-type bundles form infinite series. For a given cohomology-free bundle $\mathcal{L}$ the table shows the tuple $(c_1, c_2, c_3, c_4, c_5)$ and a list all cohomology-free bundles $\mathcal{L}'$ which have the property that $H^k(X, \mathcal{L}^* \otimes \mathcal{L}) = H^k(X, \mathcal{L} \otimes (\mathcal{L}')^*) = 0$ for all $k > 0$, which is a necessary condition for $\mathcal{L}$ and $\mathcal{L}'$ for being part of the same strongly exceptional sequence. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *compatible*. For notation, $-A_4$ for instance means the line bundle $(-1,-1,-1,0,0)$. Now we state and proof our main result. Let X be the toric surface as given in the introduction.

Theorem 4.1: *On X there are no strongly exceptional sequences of length 7 which consist of line bundles.*

Proof. The proof is done by inspection of the table and exclusion principle. For example, assume that we have a strongly exceptional sequence of length 7 which contains C_{10}. Then the rest of the sequence can at most be selected from A_1, A_2, A_4, C_3, C_7, C_9, $B_{4,1}$, $B_{4,2}$. We see from the corresponding rows that at most one of the A_i and at most one of the C_i can be selected simultaneously. Hence we can chose at most four elements from the list to complete the sequence. We conclude that a strongly exceptional sequence of length 7 which contains C_{10} cannot exist. Thus we can eliminate C_{10} from the table.

As general rules we read off that at most two of the A_i can be part of a strongly exceptional sequence, i.e. we have either $\pm A_i$, $i = 1, \ldots, 7$ alone or A_i, $i = 1, \ldots, 7$, and A_7 (respectively $-A_i$ and $-A_7$), together, or A_i and $-A_{7-i}$, $i = 1, \ldots, 6$ (respectively $-A_i$ and A_{7-i}), together.

Assume that a strongly exceptional sequence contains three bundles of type $B_{r,k}$, $B_{s,l}$, $B_{t,m}$. We read immediately off from the table that this is not possible if r, s, t are pairwise distinct, hence at least two of the r, s, t coincide. We also see that always $B_{r,k+1} - B_{r,k} = A_7$ for all r and $B_{r,k+n} - B_{r,k} = n \cdot A_7$, so if two bundles of the same B-type are contained in a strongly exceptional sequence, these must be of the form $B_{r,k}$, $B_{r,k+1}$. Now given such a pair and assume that there exists one more $B_{s,l}$ together with this pair in a strongly exceptional sequence. Then $B_{r,k+1} - B_{s,l} = A_i$ for some $1 \leq i \leq 6$ and $B_{r,k} - B_{s,l} = -A_{7-i}$. If there exists another $B_{t,m}$ in this sequence, we have $B_{r,k+1} - B_{t,m} = A_j$ for $1 \leq i \leq 6$ and $B_{t,m} - B_{s,l} = A_i - A_j$, which is not possible. So we conclude that a strongly exceptional sequence can contain at most three of the B's. This in turn, together with the above condition on the A's, implies that a strongly exceptional sequence must contain at least one of the C's.

We proceed now with C_9. We can only choose at most three of the compatible B's and at most one of the A's. So we have to choose at least one out of C_2 and C_6. These two are mutually exclusive, so we can choose only one of them. Both choices restrict the choice of the A's to $-A_1$. $-A_1$ in turn is not compatible with $B_{4,k}$, so that we can choose at most two of the B's, which is not enough, hence we can forget about C_9.

For C_8, we can choose at mosts three of the B's and thus to obtain a strongly exceptional sequence, we have to choose both, A_5 and C_1. But A_5 is not compatible with B_2, so we can not complete to a full sequence. Hence we eliminate C_8.

C_7. The bundles C_1 and C_6 are mutually exclusive, so in order to obtain an exceptional sequence of length seven, we have to choose one out of the A's and three out of the B's. The C's leave only one choice for the A's, namely $-A_2$, which in turn is not compatible with $B_{4,k}$, hence we can discard C_7.

C_6. Here we have only the choice of at most one of the A's and of at most three of the B's left, which is not enough. So C_6 goes away.

C_5. Both pairs C_3, C_4 and $B_{1,2}$, $B_{7,2}$ are mutually exclusive, leaving not enough choices to complete the sequences. Bye bye, C_5.

C_4. The sequence must contain C_1 and $-A_2$, where the latter is not compatible with the B_7's, so no C_4.

C_3. We can choose at most one A and at most one C. The C's are not compatible with A_1 and A_2, and B_4 and B_7 are not simultaneously compatible with one of $-A_3$ and $-A_4$, which does not leave enough choices. to choose also $-A_4$, which is not compatible with $B_{4,k}$. So we can also exclude C_3.

In the remaining cases, for C_1 and C_2, we do not have any other C's at our disposal. Therefore we can not complete to a sequence and so we can eliminate C_1 and C_2.

Altogether, we have removed now all C's, and as we have seen above, it is not possible to complete to a strongly exceptional sequence of length 7. □

Name	$(c_1, c_2, c_3, c_4, c_5)$	Compatible with
A_1	$(0,0,1,0,0)$	$-A_6$, A_7, $-C_2$, C_3, $-C_6$, C_7, $-C_9$, C_{10} $-B_{1,k}$, $B_{2,k}$, $-B_{3,k}$, $B_{4,k}$, $-B_{6,k}$, $B_{7,k}$
A_2	$(0,1,1,0,0)$	$-A_5$, A_7, $-C_1$, C_3, $-C_4$, C_5, $-C_6$, $-C_7$, C_9, C_{10} $-B_{1,k}$, $B_{3,k}$, $-B_{2,k}$, $B_{4,k}$, $-B_{5,k}$, $B_{7,k}$
A_3	$(0,1,2,1,0)$	$-A_4$, A_7, $-C_1$, $-C_3$, C_4, C_5 $-B_{2,k}$, $B_{5,k}$, $-B_{3,k}$, $B_{6,k}$, $-B_{4,k}$, $B_{7,k}$ if $k \geq 1$
A_4	$(1,1,1,0,0)$	$-A_3$, A_7, $-C_1$, $-C_2$, $-C_3$, C_7, C_9, C_{10}, $B_{2,k}$ if $k \geq 2$, $B_{3,k}$ if $k \geq 2$, $B_{4,k}$, $-B_{5,k}$, $-B_{6,k}$, $B_{4,k}$, $-B_{7,k}$
A_5	$(1,1,2,1,0)$	$-A_2$, A_7, $-C_1$, C_8, $B_{1,k}$, $B_{2,k}$ if $k \geq 2$, $-B_{3,k}$, $-B_{4,k}$, $B_{5,k}$, $-B_{6,k}$
A_6	$(1,2,2,1,0)$	$-A_1$, A_7, $B_{1,k}$, $-B_{2,k}$, $B_{3,k}$ if $k \geq 2$, $-B_{4,k}$, $B_{6,k}$, $-B_{7,k}$
A_7	$(1,2,3,1,0)$	A_1, A_2, A_3, A_4, A_5, A_6, $B_{1,k}$ for $k \geq 3$, $B_{7,1}$, $B_{i,k}$ for $i = 2, \ldots, 7$ and $k \geq 2$, $-B_{i,k}$ for $i = 1, \ldots, 7$.
$B_{1,k}$, $k \geq 2$	$(k, 2k-1, 3k-2, k, 1)$	$-A_1$, $-A_2$, A_5, A_6, A_7 if $k \geq 3$, $-A_7$, $B_{1,k-1}$ if $k \geq 2$, $B_{1,k+1}$, $B_{2,k-1}$, $B_{2,k}$, $B_{3,k-1}$, $B_{3,k}$
$B_{2,k}$, $k \geq 1$	$(k, 2k-1, 3k-1, k, 1)$	A_1, $-A_2$, $-A_3$, A_4 if $k \geq 2$, $-A_6$, A_7 if $k \geq 2$, $-A_7$, $B_{1,k}$ if $k \geq 2$, $B_{1,k+1}$, $B_{2,k-1}$ if $k \geq 2$, $B_{2,k+1}$, $B_{4,k-1}$ if $k \geq 2$, $B_{4,k}$, $B_{5,k-1}$ if $k \geq 2$, $B_{5,k}$
$B_{3,k}$, $k \geq 1$	$(k, 2k, 3k-1, k, 1)$	$-A_1$, A_2, A_4 if $k \geq 2$, $-A_5$, A_6 if $k \geq 2$, A_7 if $k \geq 2$, $-A_7$, $B_{1,k}$ if $k \geq 2$, $B_{1,k+1}$, $B_{3,k-1}$ if $k \geq 2$, $B_{3,k+1}$ $B_{4,k-1}$ if $k \geq 2$, $B_{4,k}$, $B_{6,k-1}$ if $k \geq 2$, $B_{6,k}$
$B_{4,k}$, $k \geq 1$	$(k, 2k, 3k, k, 1)$	A_1, A_2, $-A_3$, A_4 if $k \geq 2$, $-A_5$, $-A_6$, A_7 if $k \geq 2$, $-A_7$, $B_{2,k}$, $B_{2,k+1}$, $B_{3,k}$, $B_{3,k+1}$, $B_{4,k-1}$ if $k \geq 2$, $B_{4,k+1}$, $B_{7,k-1}$ if $k \geq 2$, $B_{7,k}$
$B_{5,k}$, $k \geq 1$	$(k, 2k, 3k+1, k+1, 1)$	$-A_2$, $-A_4$, A_5 if $k \geq 2$, A_7 if $k \geq 2$, $-A_7$, $B_{2,k}$, $B_{2,k+1}$, $B_{5,k-1}$ if $k \geq 2$, $B_{5,k+1}$, $B_{7,k-1}$ if $k \geq 2$, $B_{7,k}$
$B_{6,k}$, $k \geq 1$	$(k, 2k+1, 3k+1, k+1, 1)$	$-A_1$, A_3, $-A_4$, $-A_5$, A_6 if $k \geq 2$, A_7 if $k \geq 2$, $-A_7$, $B_{3,k}$, $B_{3,k+1}$, $B_{6,k-1}$ if $k \geq 2$, $B_{6,k+1}$, $B_{7,k-1}$ if $k \geq 2$, $B_{7,k}$
$B_{7,k}$, $k \geq 0$	$(k, 2k+1, 3k+2, k+1, 1)$	A_1 if $k \geq 1$, A_2 if $k \geq 1$, A_3 if $k \geq 1$, $-A_4$, $-A_5$, $-A_6$, A_7 if $k \geq 1$, $-A_7$, $B_{4,k}$, $B_{4,k+1}$, $B_{5,k}$, $B_{5,k+1}$, $B_{6,k}$, $B_{6,k+1}$, $B_{7,k-1}$ if $k \geq 2$, $B_{7,k+1}$

Name	$(c_1, c_2, c_3, c_4, c_5)$	Compatible with
C_1	$(2, 4, 7, 3, 2)$	$-A_2$, $-A_3$, $-A_4$, $-A_5$, C_3, C_4, C_7, C_8, $B_{2,1}$, $B_{2,2}$, $B_{4,1}$, $B_{5,1}$, $B_{7,0}$, $B_{7,1}$
C_2	$(2, 5, 7, 3, 2)$	$-A_1$, $-A_4$, C_3, C_9, $B_{3,1}$, $B_{3,2}$, $B_{4,1}$, $B_{6,1}$, $B_{7,0}$, $B_{7,1}$
C_3	$(2, 5, 8, 3, 2)$	A_1, A_2, $-A_3$, $-A_4$, C_1, C_2, C_5, C_{10}, $B_{4,1}$, $B_{4,2}$, $B_{7,0}$, $B_{7,1}$
C_4	$(2, 5, 9, 4, 2)$	$-A_2$, A_3, C_1, C_5, $B_{5,1}$, $B_{5,2}$, $B_{7,0}$, $B_{7,1}$
C_5	$(2, 6, 10, 4, 2)$	A_2, A_3, C_3, C_4, $B_{7,0}$, $B_{7,2}$
C_6	$(3, 5, 7, 3, 2)$	$-A_1$, $-A_2$, C_7, C_9, $B_{1,2}$, $B_{2,1}$, $B_{2,2}$, $B_{3,1}$, $B_{3,2}$, $B_{4,1}$
C_7	$(3, 5, 8, 3, 2)$	A_1, $-A_2$, A_4, C_1, C_6, C_{10} $B_{2,1}$, $B_{2,2}$, $B_{4,1}$, $B_{4,2}$
C_8	$(3, 5, 9, 4, 2)$	A_5, C_1, $B_{2,1}$, $B_{2,2}$, $B_{5,1}$, $B_{5,2}$
C_9	$(3, 6, 8, 3, 2)$	$-A_1$, A_2, A_4, C_2, C_6, C_{10} $B_{3,1}$, $B_{3,2}$, $B_{4,1}$, $B_{4,2}$
C_{10}	$(3, 6, 9, 3, 2)$	A_1, A_2, A_4, C_3, C_7, C_9 $B_{4,1}$, $B_{4,2}$

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