

THE COLLAPSE OF THE PERIODICITY SEQUENCE IN THE STABLE RANGE

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ABSTRACT. The stabilization of Hochschild homology of commutative algebras is Gamma homology. We describe a cyclic variant of Gamma homology and prove that the associated analogue of Connes' periodicity sequence becomes almost trivial, because the cyclic version coincides with the ordinary version from homological degree two on. We offer an alternative explanation for this by proving that the B -operator followed by the stabilization map is trivial from degree one on.

1. INTRODUCTION

Given a commutative algebra A there are several homology theories available that can help to understand A . Hochschild homology and cyclic homology of A are related by Connes' periodicity sequence

$$\dots \rightarrow HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \rightarrow \dots$$

which is a good means for comparing Hochschild homology with its cyclic variant.

Using the commutativity of A we could consider André-Quillen homology as well. Viewing A as an E_∞ -algebra with trivial homotopies for commutativity allows us to consider André-Quillen homology in the category of differential graded E_∞ -algebras as defined by Mike Mandell [M]. This homology theory coincides with Alan Robinson's Gamma homology [Ro, BR] which in turn can be interpreted as stabilization of Hochschild homology of A by [PR, Theorem 1].

This homology theory has the feature that it coincides with André-Quillen homology for \mathbb{Q} -algebras [RoWh, Theorem 6.4].

The Hodge decomposition for Hochschild homology for commutative algebras whose base ring contain the rationals splits André-Quillen homology off as the first summand $HH_*^{(1)} \cong AQ_{*-1}$ in the decomposition (see [GS, L2, NS]). Cyclic homology splits similarly and from degree three on the first summand $HC_*^{(1)}$ of that decomposition is again André-Quillen homology, AQ_{*-1} . It is known that the periodicity sequence passes to a sequence for the decomposition summands [L2, NS],

$$\dots \rightarrow HH_n^{(i)}(A) \xrightarrow{I} HC_n^{(i)}(A) \xrightarrow{S} HC_{n-2}^{(i-1)}(A) \xrightarrow{B} HH_{n-1}^{(i)}(A) \rightarrow \dots$$

Therefore rationally the periodicity sequence collapses in higher degrees for the first decomposition summand, because there the map I becomes an isomorphism. One could guess that this is a defect of working over the rationals, but we will show in the course of this paper that this is not the case.

Robinson and Whitehouse proposed a cyclic variant of Gamma homology of differential graded E_∞ -algebras over a cyclic E_∞ -operad in [RoWh].

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Motivated by the application of Gamma homology to obstruction theories for E_∞ ring structures on ring spectra (see [Ro]) we construct a cyclic variant of Gamma homology in the restricted case of commutative algebras which arises naturally from the interpretation of Gamma homology as stable homotopy of certain Γ -modules.

The aim of this paper is to deduce a periodicity sequence for Gamma homology and its cyclic version. It turns out, however, (*cf.* 4.4) that cyclic Gamma homology coincides with usual Gamma homology from homological degree two on; hence this sequence collapses. We can explicitly describe (see Propositions 5.4 and 5.5) cyclic Gamma homology in small degrees in terms of ordinary cyclic homology and deRham cohomology.

As one explanation for this behavior we show (Theorem 6.6) that the composition of the stabilization map with the B -operator is trivial in degrees bigger than zero. For large enough degrees the stable and unstable periodicity sequence are related as follows:

$$\begin{array}{ccccccccccc}
 \cdots & \longrightarrow & HC_{n+2}(A) & \xrightarrow{S} & HC_n(A) & \xrightarrow{B} & HH_{n+1}(A) & \xrightarrow{I} & HC_{n+1}(A) & \xrightarrow{S} & \cdots \\
 & & & & & \searrow 0 & \downarrow \text{stab} & & & & \\
 \cdots & \longrightarrow & HHC_{n+1}(A) & \longrightarrow & 0 & \longrightarrow & HG_n(A) & \xrightarrow{\cong} & HHC_n(A) & \longrightarrow & \cdots
 \end{array}$$

Our results must disappoint everybody who hoped that a cyclic version of Gamma homology would help to calculate Gamma homology and probably identify obstruction classes in the setting of [Ro]. However, we clarify the rôle Gamma homology plays as a stabilization of Hochschild homology.

The proofs use the extension of the definitions of cyclic, Hochschild and Gamma homology to functor categories. We recall the necessary prerequisites from [L, P, PR].

Lars Hesselholt proved an analogous phenomenon in the setting of topological Hochschild homology. Fix an arbitrary prime p . In [H] he showed that the equivalence between stable K -theory and topological Hochschild homology is reflected in an equivalence between the p -completions of the stabilization of topological cyclic homology and p -completed topological Hochschild homology.

2. THE CATEGORY \mathcal{F} AND \mathcal{F} -MODULES

We recall the definition of cyclic homology from [L, §6] (see also [P, §3]). Let \mathcal{F} denote the skeleton of the category of finite unpointed sets and let \underline{n} be the object $\{0, \dots, n\}$ in \mathcal{F} . We call functors from \mathcal{F} to the category of k -modules \mathcal{F} -modules. Here k is an arbitrary commutative ring with unit. For a set S we denote by $k[S]$ the free k -module generated by S .

The projective generators for the category of \mathcal{F} -modules are the functors \mathcal{F}^n given by

$$\mathcal{F}^n(\underline{m}) := k[\mathcal{F}(\underline{n}, \underline{m})];$$

whereas the category of contravariant functors from \mathcal{F} to k -modules has the family \mathcal{F}_n with

$$\mathcal{F}_n(\underline{m}) := k[\mathcal{F}(\underline{m}, \underline{n})]$$

as generators.

For two \mathcal{F} -modules F and F' let $F \otimes F'$ be the pointwise tensor product of F and F' , i.e., $F \otimes F'(\underline{n}) = F(\underline{n}) \otimes F'(\underline{n})$. As a map in \mathcal{F} from the object $\underline{0}$ to an object \underline{m} just picks an arbitrary element, one obtains that $(\mathcal{F}^0)^{\otimes n} \cong \mathcal{F}^{n-1}$. The functor \mathcal{F}^0 is an analog of the functor L from [PR] in the unpointed setting and the tensor powers $(\mathcal{F}^0)^{\otimes n} \cong \mathcal{F}^{n-1}$ for $n > 1$ correspond to $L^{\otimes n}$.

Given a unital commutative k -algebra A , the \mathcal{F} -module which gives rise to cyclic homology of A is the functor $\mathcal{L}(A)$ that sends \underline{n} to $A^{\otimes n+1}$. A map $f: \underline{n} \rightarrow \underline{m}$ induces $f_*: \mathcal{L}(A)(\underline{n}) \rightarrow \mathcal{L}(A)(\underline{m})$ via

$$f_*(a_0 \otimes \dots \otimes a_n) = b_0 \otimes \dots \otimes b_m, \text{ with } b_i = \prod_{j \in f^{-1}(i)} a_j.$$

Here we set $b_i = 1$ if the preimage of i is empty.

For any \mathcal{F} -module F , cyclic homology of F , $HC_*(F)$, can be defined [P, 3.4] as the homology of the total complex associated to the bicomplex

$$\begin{array}{ccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow b & & \downarrow b & & \downarrow b \\ & & F(\underline{2}) & \xleftarrow{B} & F(\underline{1}) & \xleftarrow{B} & F(\underline{0}) \\ & & \downarrow b & & \downarrow b & & \\ & & F(\underline{1}) & \xleftarrow{B} & F(\underline{0}) & & \\ & & \downarrow b & & & & \\ & & F(\underline{0}) & & & & \end{array}$$

In particular, cyclic homology of A , $HC_*(A)$, is the homology of this total complex applied to the functor $\mathcal{L}(A)$. We recall the definition of b in (1) and the one of B in Definition 5.1.

3. THE RELATIONSHIP TO THE CATEGORY Γ

Let Γ be the skeleton of the category of pointed finite sets and let $[n]$ be the object $[n] = \{0, \dots, n\}$ with 0 as basepoint. The projective generators of the category of Γ -modules are the functors Γ^n given by

$$\Gamma^n[m] = k[\Gamma([n], [m])].$$

There is a natural forgetful functor $\mu: \Gamma \rightarrow \mathcal{F}$ and the left adjoint to μ , $\nu: \mathcal{F} \rightarrow \Gamma$, which adds an extra basepoint $\nu(\underline{m}) = [m+1]$. Pulling back with these functors transforms Γ -modules into \mathcal{F} -modules and vice versa:

$$\mu^*: \mathcal{F}\text{-modules} \rightleftarrows \Gamma\text{-modules}: \nu^*$$

In [P, Proposition 3.3] Pirashvili shows that

$$\mathrm{Tor}_*^{\mathcal{F}}(\nu^*F, G) \cong \mathrm{Tor}_*^{\Gamma}(F, \mu^*G).$$

Lemma 3.1. *The functor \mathcal{F}^n pulled back along μ is isomorphic to Γ^{n+1} .*

Proof. We first show that the Γ -module $\mu^*(\mathcal{F}^0)$ is isomorphic to Γ^1 : On every object $[n]$ we obtain that

$$\mu^*(\mathcal{F}^0)[n] = \mathcal{F}^0(\underline{n}) = k[\mathcal{F}(\underline{0}, \underline{n})] \cong k^{n+1}$$

because the value of a function $f \in \mathcal{F}(\underline{0}, \underline{n})$ on 0 can be an arbitrary element $i \in \underline{n}$. The Γ -module Γ^1 has the same value on $[n]$, because a function $g \in \Gamma([1], [n])$ has an arbitrary value on 1 but sends zero to zero. As we just allow pointed maps, the two functors are isomorphic.

The general case easily follows by direct considerations or by using the decompositions of \mathcal{F}^n and Γ^{n+1} as $(n+1)$ -fold tensor products $\mathcal{F}^n \cong (\mathcal{F}^0)^{\otimes n+1}$ and $\Gamma^{n+1} \cong (\Gamma^1)^{\otimes n+1}$. \square

Recall, that Hochschild homology of a Γ -module G , $HH_*(G)$, can be defined as the homology of the complex

$$(1) \quad G[0] \xleftarrow{b} G[1] \xleftarrow{b} \dots$$

where $b = \sum_{i=0}^n G(d_i)$ and d_i is the map of pointed sets that for $i < n$ sends i and $i + 1$ to i and is bijective and order preserving on the other values in $[n]$. The last map, d_n , maps 0 and n to 0 and is the identity for all other elements of $[n]$.

Later, we will need the following auxiliary result.

Lemma 3.2. *Hochschild homology of a Gamma module G is isomorphic to the homology of the normalized complex which consist of $G[n]/D_n$ in chain degree n where $D_n \subset G[n]$ consists of all elements of the form $(s_i)_*F[n-1]$ where s_i is the order preserving injection from $[n-1]$ to $[n]$ which misses i .*

Proof. This result just uses the standard fact that the Hochschild complex is the chain complex associated to a simplicial k -module and the elements in D_n correspond to the degenerate elements; therefore the complex D_* is acyclic. \square

A similar result applies to cyclic homology of \mathcal{F} -modules.

Let $\mathbb{S}^1 = \Delta^1/\partial\Delta^1$ denote the standard model of the simplicial 1-sphere. Recall from [L, P] that Hochschild homology of a commutative unital k -algebra A , $HH_*(A)$, coincides with the homotopy groups of the simplicial k -module $\mu^*\mathcal{L}(A)(\mathbb{S}^1)$. Here, we evaluate $\mu^*\mathcal{L}(A)$ degreewise. More general, Hochschild homology of any Γ -module G coincides with $\pi_*G(\mathbb{S}^1)$.

4. GAMMA HOMOLOGY AND ITS CYCLIC VERSION

Let t be the contravariant functor from Γ to k -modules which is defined as

$$t[n] = \text{Hom}_{\text{Sets}_*}([n], k)$$

where Sets_* denotes the category of pointed sets. Pirashvili and the author proved in [PR] that Gamma homology of any Γ -module G , $H\Gamma_*(G)$, is isomorphic to $\text{Tor}_*^\Gamma(t, G)$. In particular, Gamma homology of the algebra A , $H\Gamma_*(A)$ is isomorphic to $\text{Tor}_*^\Gamma(t, \mu^*\mathcal{L}(A))$.

For a cyclic variant of Gamma homology, we have to transform the functor t into a contravariant \mathcal{F} -module. Choosing ν^*t does this, but it inserts an extra basepoint. Killing the value on an additional point amounts to define the \mathcal{F} -module \bar{t} by the following exact sequence:

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \nu^*(t) \longrightarrow \bar{t} \longrightarrow 0.$$

The transformation from \mathcal{F}_0 to ν^*t is given by sending a scalar multiple λf of a map $f: \underline{n} \rightarrow \underline{0}$ to the function in $\text{Hom}_{\text{Sets}_*}([n+1], k)$ which sends the points $1, \dots, n+1$ to λ .

Proposition 4.1. *On the family of projective generators $(\mathcal{F}^n)_{n \geq 0}$ the torsion groups with respect to \bar{t} are as follows:*

$$\text{Tor}_*^{\mathcal{F}}(\bar{t}, \mathcal{F}^n) \cong \begin{cases} 0 & \text{for } * > 0 \\ k^n & \text{for } * = 0 \end{cases}$$

Proof. It is clear that the torsion groups vanish in positive degrees because the functors \mathcal{F}^n are projective. We have to prove the claim in degree zero, but the tensor products in question are easy to calculate:

$$\bar{t} \otimes_{\mathcal{F}} \mathcal{F}^n \cong \bar{t}(\underline{n}) \cong k^n.$$

\square

Definition 4.2. We call the group $\mathrm{Tor}_n^{\mathcal{F}}(\bar{t}, F)$ the n th cyclic Gamma homology group of the \mathcal{F} -module F and denote it by $HFC_n(F)$.

Remark 4.3. We will see in 5.4 that cyclic Gamma homology of an algebra A in degree zero behaves analogously to usual Gamma homology whose value in homological degree zero gives Hochschild homology of degree one.

As the functor \mathcal{F}_0 is projective, the calculation in proposition 4.1 allows us to draw the following conclusion.

Corollary 4.4. Cyclic Gamma homology of any \mathcal{F} -module F coincides with Gamma homology of the induced Gamma module $\mu^*(F)$ in degrees higher than 1, i.e.,

$$HFC_*(F) = \mathrm{Tor}_*^{\mathcal{F}}(\bar{t}, F) \cong \mathrm{Tor}_*^{\Gamma}(t, \mu^*F) \cong H\Gamma_*(\mu^*(F)) \quad \forall * > 1.$$

In low degrees the difference between cyclic and ordinary Gamma homology is measured by the following exact sequence:

$$0 \rightarrow \mathrm{Tor}_1^{\mathcal{F}}(\nu^*t, F) \rightarrow \mathrm{Tor}_1^{\mathcal{F}}(\bar{t}, F) \xrightarrow{\delta} F(\underline{0}) \rightarrow \nu^*t \otimes_{\mathcal{F}} F \rightarrow \bar{t} \otimes_{\mathcal{F}} F \rightarrow 0$$

which is nothing but

$$0 \rightarrow H\Gamma_1(\mu^*F) \rightarrow HFC_1(F) \rightarrow \mu^*F(\underline{0}) \xrightarrow{\delta} H\Gamma_0(\mu^*F) \rightarrow HFC_0(F) \rightarrow 0.$$

We will obtain more explicit descriptions in the algebraic case in the next section.

5. THE B OPERATOR

In the unstable situation there is a map B which connects cyclic homology and Hochschild homology and which gives rise to Connes' important periodicity sequence

$$\cdots \longrightarrow HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \longrightarrow \cdots$$

In low degrees the map B sends the zeroth cyclic homology of a k -algebra A which is nothing but A again to the first Hochschild homology group of A which consists of the module of Kähler differentials $\Omega_{A|k}^1$ and the map is given by $B(a) = da$. If we consider the first nontrivial parts in the long exact sequence of Tor-groups as above, arising from the short exact sequence $0 \rightarrow \mathcal{F}_0 \rightarrow \nu^*t \rightarrow \bar{t} \rightarrow 0$ then, for the functor $\mathcal{L}(A)$, we obtain

$$\cdots \rightarrow A \rightarrow \nu^*t \otimes_{\mathcal{F}} \mathcal{L}(A) \rightarrow \bar{t} \otimes_{\mathcal{F}} \mathcal{L}(A) \rightarrow 0$$

and $\nu^*t \otimes_{\mathcal{F}} \mathcal{L}(A)$ is isomorphic to the zeroth Gamma homology group of A which is the module of Kähler differentials. The map is induced by the natural transformation from \mathcal{F}_0 to ν^*t . The aim of this section is to prove that this map is given by the B -map.

Let us recall the general definition of the B -map for cyclic and Hochschild homology of functors. The B -map from cyclic homology to Hochschild homology can be viewed as a map from the n th generator \mathcal{F}_n to the $(n+1)$ st in the following manner:

Definition 5.1. Let τ be the generator of the cyclic group on $n+1$ (resp $n+2$) elements and let s be the map of finite sets which sends i to $i+1$. Then the B -map is defined as a map $B: \mathcal{F}_n \rightarrow \mathcal{F}_{n+1}$. On a generator $f: \underline{m} \rightarrow \underline{n}$ it is $B(f) := (-1)^n(1-\tau) \circ s \circ N \circ f$ where N is the norm map $N = \sum_{i=1}^{n+1} (-1)^i \tau^i$.

On the part $F(\underline{n}) \cong \mathcal{F}_n \otimes_{\mathcal{F}} F$ of the complex for cyclic homology of F this induces the usual B -map known from the algebraic case $F = \mathcal{L}(A)$, for a commutative algebra A . By the very definition of the map it is clear that it is well-defined on the tensor product.

In our situation we apply the B -map to the first column of the double complex for cyclic homology of F

$$\begin{array}{ccccc}
 \vdots & & \vdots & & \vdots \\
 \downarrow b & & \downarrow b & & \downarrow b \\
 F(\underline{2}) & \xleftarrow{B} & F(\underline{1}) & \xleftarrow{B} & F(\underline{0}) \\
 \downarrow b & & \downarrow b & & \\
 F(\underline{1}) & \xleftarrow{B} & F(\underline{0}) & & \\
 \downarrow b & & & & \\
 F(\underline{0}) & & & &
 \end{array}$$

and send all other columns to zero. In [P, 3.2] it is shown that $\nu^*\Gamma_n \cong \mathcal{F}_n$. Using this we obtain an isomorphism $F(\underline{n}) \cong \mathcal{F}_n \otimes_{\mathcal{F}} F \cong \nu^*\Gamma_n \otimes_{\mathcal{F}} F \cong \Gamma_n \otimes_{\Gamma} \mu^*F$ and see that B gives rise to a map from the total complex for cyclic homology of F to the complex for Hochschild homology of $\mu^*(F)$.

A verbatim translation of the proof for ([L, 2.5.10, 2.1]) in the case of a cyclic module to our setting gives the following result:

Lemma 5.2. *The map B is a map of chain complexes and therefore induces a map from $HC_*(F)$ to $HH_{*+1}(\mu^*F)$.*

Remark 5.3. In degree zero, the B -map from \mathcal{F}_0 to \mathcal{F}_1 applied to an $f \in \mathcal{F}_0(\underline{n})$ reduces to $(1 - \tau) \circ s \circ f$

We should first make sure that cyclic Gamma homology has the right value in homological dimension zero.

Proposition 5.4. *Cyclic Gamma homology in degree zero is isomorphic to cyclic homology in degree one. In particular, $HIC_0(A) \cong HC_1(A)$.*

Proof. The cokernel of the map $F(\underline{0}) \rightarrow \nu^*(t) \otimes_{\mathcal{F}} F$ can be determined by a map from $F(\underline{0})$ to $F(\underline{1})$: similar to the beginning of the resolution $\dots \rightarrow \Gamma_2 \rightarrow \Gamma_1$ of t , the exactness of ν^* turns this into a resolution $\dots \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0$ of ν^*t . The projectivity of \mathcal{F}_0 therefore gives us a lift $F(\underline{0}) \rightarrow F(\underline{1})$. In this lift only one summand of the B -map arises: instead of the sum $(1 - \tau) \circ s \circ f$ a generator $f \in \mathcal{F}_0(\underline{n})$ is sent to $s \circ f$. But Hochschild and cyclic homology coincide with their normalized version (see Lemma 3.2) and the second summand $\tau \circ s \circ f$ has an image in the degenerate part.

In the algebraic case, this lift induces a map from A to $A \otimes A$ which sends a to $1 \otimes a$. The projection to the Kähler differentials is then just the map $a \mapsto d(a)$ which is the same as the B -map in this dimension. \square

Cyclic Gamma homology in dimension one can be explicitly described as well. In small degrees our Tor-exact sequence looks as follows:

$$0 \rightarrow H\Gamma_1(\mu^*F) \rightarrow HIC_1(F) \xrightarrow{\delta} F(\underline{0}) \xrightarrow{B} HH_1(F).$$

Therefore we obtain the following.

Proposition 5.5. *The difference between cyclic Gamma homology and ordinary Gamma homology in degree one is measured by the kernel of the B -map.*

In the case of the functor $\mathcal{L}(A)$ the exact sequence is

$$0 \rightarrow H\Gamma_1(A) \rightarrow HIC_1(A) \xrightarrow{\delta} A \xrightarrow{d} \Omega_{A|k}^1.$$

Thus in degree one the difference between Gamma homology and its cyclic version is measured by the zeroth deRham cohomology of A . For instance, if A is étale, then $H\Gamma_1(A) = 0 = \Omega_{A|k}^1$ and therefore $HFC_1(A) \cong A$.

The above calculations in small dimensions suggest that one should view the sequence of Tor-groups coming from the sequence $0 \rightarrow \mathcal{F}_0 \rightarrow \nu^*t \rightarrow \bar{t} \rightarrow 0$ as the stable version of the periodicity sequence. In the algebraic case the two sequences are nicely related in the following way.

$$\begin{array}{ccccccccccccccc}
 HC_1(A) & \xrightarrow{B} & HH_2(A) & \xrightarrow{I} & HC_2(A) & \xrightarrow{S} & HC_0(A) & \xrightarrow{B} & HH_1(A) & \longrightarrow & HC_1(A) & \longrightarrow & 0 \\
 & & \downarrow \text{stab} & & & & \parallel & & \cong \downarrow \text{stab} & & \cong \downarrow & & \\
 0 & \longrightarrow & H\Gamma_1(A) & \longrightarrow & HFC_1(A) & \xrightarrow{\delta} & A & \xrightarrow{B} & H\Gamma_0(A) & \longrightarrow & HFC_0(A) & \longrightarrow & 0
 \end{array}$$

But in higher dimensions the transformation I from the periodicity sequence becomes an isomorphism. The term $\mathcal{F}_0 \otimes_{\mathcal{F}} F \cong F(\underline{0})$ plays the role of cyclic Gamma homology in dimension -1 .

6. TRIVIALITY OF THE B -MAP AFTER STABILIZATION

We will prove a result which we like to think of as an explanation of the collapsing of the periodicity sequence in the stable world: of course one could say that the isomorphism of cyclic and ordinary Gamma homology in dimensions different from zero and one explains this phenomenon, but we would like to relate the unstable periodicity sequence to the stable one by an explicit stabilization process.

Similar to the B -map, we define a stabilization map $\text{stab}: HH_{n+1}(G) \rightarrow H\Gamma_n(G)$ for a Γ -module G on the corresponding generator Γ_{n+1} . A Gamma module G is called *reduced* if $G[0] = 0$. As we have a unique pair of maps $[0] \rightarrow [n] \rightarrow [0]$ for every $[n]$ we can split any Gamma module G as $G \cong G[0] \oplus G'$ such that G' is reduced.

Gamma homology of a reduced functor G' has a description as the homology of the cubical construction $Q_*(G')$ of the functor G (see [Ri, Theorem 4.5] and [PR, Theorem 1]) and $Q_*(G')$ is a tensor product $SQ_* \otimes_{\Gamma} G'$ where SQ_* is an analog of the cubical construction of Eilenberg and MacLane on pointed sets. Gamma homology of an unreduced functor G , e.g., $\mathcal{L}(A)$, is then just given as

$$H\Gamma_*(G) = \begin{cases} G[0] & \text{if } * = 0 \\ H_*(Q_*(G')) & \text{if } * > 0. \end{cases}$$

This particular cubical model for a chain complex for Gamma homology will give us an explicit way of describing the stabilization map.

For each finite pointed set X_+ the chain-complex $SQ'_*(X_+)$ in degree n is the free k -module generated by all families $\chi(\varepsilon_1, \dots, \varepsilon_n)$ of pairwise disjoint subsets of X indexed by n -tuples of elements $\varepsilon_i \in \{0, 1\}$. We divide out all elements that map a face or a diagonal of the cube to the empty set. The result of this normalization process is $SQ_*(X_+)$. The boundary is $\delta := \sum_{i=1}^n (-1)^i (P_i - R_i - S_i)$. Here

$$\begin{aligned}
 R_i(\chi)(\varepsilon_1, \dots, \varepsilon_{n-1}) &= \chi(\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \varepsilon_i, \dots, \varepsilon_{n-1}), \\
 S_i(\chi)(\varepsilon_1, \dots, \varepsilon_{n-1}) &= \chi(\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_i, \dots, \varepsilon_{n-1})
 \end{aligned}$$

and $P_i(\chi)$ is given by the pointwise union of $R_i(\chi)$ and $S_i(\chi)$.

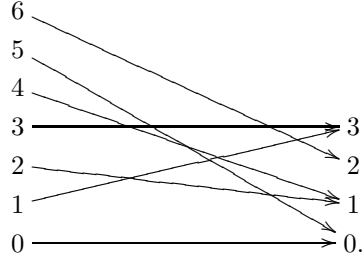
Definition 6.1. *On a generator $f: [m] \rightarrow [n+1]$ with $n \geq 1$ the stabilization map $\text{stab}: \Gamma_{n+1} \rightarrow SQ_n$ is defined as $\text{stab}(f) := \chi(f)$ where $\chi(f)(\varepsilon_1, \dots, \varepsilon_n)$ is the empty set for all n -tuples which are not of the form $(0, \dots, 0, \underbrace{1, \dots, 1}_i)$ for $0 \leq i \leq n$.*

On these tuples the value of $\chi(f)$ is the preimage of $i + 1$ under the map f :

$$\chi(f)(0, \dots, 0, \underbrace{1, \dots, 1}_i) := f^{-1}(i + 1).$$

For $n = 0$ we use the convention that $\chi(f)(0) = f^{-1}(1)$.

Example 6.2. Let f be the following map of pointed sets



Then $\chi(f) \in SQ_2$ is the cube

$$\begin{pmatrix} \{2, 4\} & \{6\} \\ \emptyset & \{1, 3\} \end{pmatrix}.$$

We have to justify that the map **stab** deserves the name ‘stabilization’. First we will show that its image is a subcomplex in the cubical construction.

Lemma 6.3. *The stabilization induces a map of chain complexes from $G(\mathbb{S}^1)_*$ to $Q_{*-1}(G)$.*

Proof. The boundary of **stab**(f) with $f: [m] \rightarrow [n + 1]$ is given as $\delta(\chi(f)) = \sum_{i=1}^n (-1)^i (P_i - R_i - S_i)(\chi(f))$. As $\chi(f)$ gives the empty set on every n -tuple $(\varepsilon_1, \dots, \varepsilon_n)$ which is not of the form $(0, \dots, 0, 1, \dots, 1)$, the summands $R_i(\chi(f))$ and $S_i(\chi(f))$ are degenerate except for $R_0(\chi(f))$ which equals $\chi(d_{n+1}(f))$ and $S_n(\chi(f))$ which corresponds to $\chi(d_0(f))$. The summands $P_i(\chi(f))$ evaluated on an n -tuple $(0, \dots, 0, \underbrace{1, \dots, 1}_j)$ give $\chi(f)(0, \dots, 0, \underbrace{1, \dots, 1}_j)$ for $i \leq n - j$, $\chi(f)(0, \dots, 0, \underbrace{1, \dots, 1}_{j+1})$ for $i > n - j + 1$ and the union of $\chi(f)(0, \dots, 0, \underbrace{1, \dots, 1}_j)$ and $\chi(f)(0, \dots, 0, \underbrace{1, \dots, 1}_{j+1})$ for $i = n - j + 1$. and these are exactly the values of the face maps d_{n-i+1} for $i = 1, \dots, n$. \square

Thus the stabilization induces a well-defined map **stab**: $HH_{n+1}(G) \rightarrow HF_n(G)$ for all n greater or equal to zero and for all Γ -modules G . The construction of **stab** is analogous to the one in [EM], where Eilenberg and MacLane considered the stabilization map from the homology of Eilenberg-MacLane spaces to the homology of the corresponding spectrum. Their result gives one concrete example for the connection of Hochschild homology and Gamma homology via the stabilization map.

Example 6.4. Let C be an abelian group. The $(n + 1)$ st Hochschild homology of the group algebra $k[C]$ with coefficients in the ground ring k is nothing but the group homology of C with coefficients in k , i.e., the k -homology of the Eilenberg-MacLane space $K(C, 1)$. The stabilization map has Gamma homology of $k[C]$ as its target and this is the k -homology of the Eilenberg-MacLane spectrum HC (see [RiRo]). In this case the stabilization map **stab**: $H_{n+1}(K(C, 1); k) \rightarrow Hk_n HC$ coincides with the one from [EM, p.547].

As in [EM, p.546] we denote the subcomplex generated by the image of the stabilization map by $SQ_*^{(0)} \otimes_{\Gamma} G = Q_*^{(0)}(G)$. In the following, we will assume that G is reduced.

Lemma 6.5. *Assume that G is reduced. The homology of the subcomplex $Q_*^{(0)}(G)$ is precisely Hochschild homology of G shifted by one.*

Proof. For reduced functors G , we have that $G[n] \cong \Gamma_n \otimes_{\Gamma} G$ coincides with $G[n]/G[0]$ which is the cokernel of

$$\Gamma_0 \otimes_{\Gamma} G \longrightarrow \Gamma_n \otimes_{\Gamma} G.$$

On the surjective generators of $\Gamma_n[m]/\Gamma_0[m]$ the stabilization map is injective and the surjective maps in $\Gamma_n[m]$ correspond to the normalized chains for Hochschild homology.

Assume that the image of an element $y \in G[n+1]$ in $Q_n^{(0)}(G)$ is a boundary, $\text{stab}(y) = \delta(\rho)$ for one ρ in $Q_{n+1}^{(0)}(G)$. The proof of Lemma 6.3 shows that the terms in the boundary of ρ are in one-to-one correspondence with boundary terms in the preimage of the stabilization map. Therefore there is an element $y' \in G[n+2]$ with $\text{stab}(y') = \rho$ and with boundary exactly y . \square

With the help of the explicit shape of the stabilization map we can now indicate one reason why the periodicity sequence collapses after stabilization. Note that $\text{stab}: HH_1 \rightarrow HH_0$ is an isomorphism.

Theorem 6.6. *For every \mathcal{F} -module F the composition of the B -map with the stabilization map is trivial for all $n \geq 1$*

$$\begin{array}{ccc} HC_n(F) & \xrightarrow{B} & HH_{n+1}(\mu^*F) \\ & \searrow 0 & \downarrow \text{stab} \\ & & HH_n(\mu^*F) \end{array}$$

Proof. In order to prove the claim we will actually show more. We claim that every element $x \in \mathcal{F}_n \otimes_{\mathcal{F}} F$ is sent to a linear combination of degenerate elements in $SQ_n \otimes_{\Gamma} \mu^*F$ for $n \geq 1$. Without loss of generality we may assume that x is a generator, i.e., $x = f \otimes y$ with $f \in \mathcal{F}_n(\underline{m})$. By definition

$$\text{stab} \circ B(x) = \text{stab}((1 - \tau) \circ s \circ N)(f \otimes y).$$

The terms $\text{stab} \circ s \circ \tau^i(f) \otimes y$ are degenerate, because the composition $\text{stab} \circ s \circ \tau^i(f)(\varepsilon_1, \dots, \varepsilon_{n-1}, 0)$ is empty for all $(\varepsilon_1, \dots, \varepsilon_{n-1}) \neq (0, \dots, 0)$ by definition. As precomposition with ν causes a shift by one, the map s causes a trivial preimage of 0 and therefore we obtain \emptyset as a value on the n -tuple $(0, \dots, 0)$ as well.

The other terms $\tau \circ s \circ \tau^i(f)$ are degenerate because they give the empty set on n -tuples $(\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_{n-1} \neq \varepsilon_n$: if ε_{n-1} is 1 and $\varepsilon_n = 0$ then this element gives the empty set by definition of the stabilization map. In the other case, the map $\tau \circ s$ has an empty preimage of 1 and thus ν^* causes \emptyset as a value on $(0, \dots, 0, 1)$. In particular we obtain a degenerate element in the case $n = 1$. \square

Remark 6.7. As the cubical complex SQ_* is a resolution of the functor t and as we know that ν^* is exact, we obtain a resolution $\nu^*SQ_* \rightarrow \nu^*t$. As we have an isomorphism between $\text{Tor}_*^{\mathcal{F}}(\nu^*t, F)$ and cyclic Gamma homology from degrees bigger than one, it suffices to construct a stabilization map from $HC_n(F)$ to $HH_{n-1}(\mu^*F)$ for n bigger than 2. As the stabilization map vanishes on the image of B and as the first

column of the bicomplex for cyclic homology gives rise to $HH_n(\mu^*F)/B(F(\underline{n-1}))$ we define the cyclic stabilization map, stab_C , as

$$\text{stab}_C: HC_n(F) \longrightarrow HIC_{n-1}(F), \quad \text{stab}_C = \text{stab} \circ \pi, \quad \text{for } n > 2$$

where π is the projection from $HC_n(F)$ to $HH_n(\mu^*F)/B(F(\underline{n-1}))$.

Note, that this does *not* give rise to a well-defined map from $HC_2(F)$ to $HIC_1(F)$ that is compatible with both periodicity sequences: we would have to compose $\text{stab} \circ \pi$ with the map from $HI_1(\mu^*F)$ to $HIC_1(F)$, but then δ composed with that map has to be trivial, but the S -map from $HC_2(F)$ to $HC_0(F)$ is non-trivial in general.

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