WZW orientifolds and finite group cohomology

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ABSTRACT: The simplest orientifolds of the WZW models are obtained by gauging a $\mathbb{Z}_2$ symmetry group generated by a combined involution of the target Lie group G and of the worldsheet. The action of the involution on the target is by a twisted inversion $g \mapsto (\zeta g)^{-1}$, where ζ is an element of the center of G. It reverses the sign of the Kalb-Ramond torsion field H given by a bi-invariant closed 3-form on G. The action on the worldsheet reverses its orientation. An unambiguous definition of Feynman amplitudes of the orientifold theory requires a choice of a gerbe with curvature H on the target group G, together with a so-called Jandl structure introduced in [31]. More generally, one may gauge orientifold symmetry groups $\Gamma = \mathbb{Z}_2 \ltimes Z$ that combine the $\mathbb{Z}_2$-action described above with the target symmetry induced by a subgroup Z of the center of G. To define the orientifold theory in such a situation, one needs a gerbe on G with a Z-equivariant Jandl structure. We reduce the study of the existence of such structures and of their inequivalent choices to a problem in group-Γ cohomology that we solve for all simple simply-connected compact Lie groups G and all orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes Z$.

Contents

1. Introduction

Unoriented string theory, both in the closed and in the open sector, has a long history [32, 25]. From the two-dimensional point of view, it involves conformal field theory defined on unoriented worldsheets. Such a theory may be viewed as an "orientifold" obtained from a conformal field model defined on oriented surfaces by gauging a discrete symmetry containing transformations reversing the worldsheet orientation. If the conformal theory is a sigma model whose target space carries a background Kalb-Ramond 2-form field B, the worldsheet orientation-changing transformations have to be combined with target-space transformations that change the sign of B so that the B-field contribution to the sigma model action functional stays invariant. This leads to subtle issues if the B-field is topologically non-trivial, like the one present in the Wess-Zumino-Witten (WZW) sigma models with Lie group targets [35]. In those models, only the closed torsion 3-form $H = dB$, a right-left invariant 3-form on the group manifold, is globally defined. Orientifolds of the WZW models have been studied intensively within the algebraic approach, following the

pioneering work of the Rome group [26, 27]. The main tool in this approach was the use of sewing and modular duality constraints in order to find consistent expressions for the crosscap states encoding the action of the orientation inversion in the closed string sector. The algebraic approach was further developed in the context of more general orientifolds combining simple-current orbifolds and orientation reversal in [17, 18, 16, 9, 5]. It gave rise to an abstract formulation of the relevant topological structures in the language of tensor categories [29]. The interpretation of the results of the algebraic approach in terms of the target geometry was the subject of papers [2] and [4] that studied orientifolds of the $SU(2)$ and $SO(3)$ WZW theories.

In general, one may expect that the intricacies appearing in the algebraic studies of WZW orientifolds have their source in the classical target geometry, more precisely in non-triviality of the B-field background, similarly to the ones involved in the simple-current orbifolds of the WZW models. In the latter case, it was argued in [11] that the proper treatment of the non-trivial B-field background in the closed string sector may be achieved by employing the third (real) Deligne cohomology. This approach lay behind the classification of the WZW models on non-simply connected simple compact groups obtained in [8]. The third Deligne cohomology classifies geometric structures called bundle gerbes with connections introduced in [23, 24]. The latter are in a similar relation to the closed 3-forms H as line bundles with connection are to their curvature 2-forms F. Consequently, the 3-form H corresponding to a gerbe is called its curvature. The geometric language of bundle gerbes is sometimes more convenient than the cohomological one of Deligne cohomology. For general simple groups, in particular, it appeared to be easier to construct the bundle gerbes with the curvature given by a bi-invariant 3-form H than the corresponding Deligne cohomology classes. Such a construction was accomplished for the simply connected groups in [22] and was generalized to the non-simply connected ones in [14]. An extension of the geometric analysis including open strings and D-branes required studying gerbe-modules carrying Chan-Paton gauge fields twisted by the gerbe [19]. In the algebraic language, the WZW models with non-simply connected target groups are simple-current orbifolds of the models with simply connected targets. The geometric analysis of [13, 12], employing (bundle) gerbes and gerbe modules, permitted a systematic classification of symmetric D-branes in the WZW models and exposed the classical origin of the finite group cohomology that appeared in the algebraic analysis of the simple-current orbifolds. Indeed, the relevant cohomological aspects pass undeformed to the quantum theory that is obtained by geometric quantization of the classical one [12].

The recent paper [31] introduced additional data, called a Jandl structure on a gerbe, that are required to define Feynman amplitudes for closed unoriented worldsheets in the presence of a topologically non-trivial B-field. A Jandl structure may be viewed as a symmetry of the gerbe under a transformation of the underlying space that changes the sign of the curvature 3-form. In this paper, we classify such structures on all gerbes on simple compact groups with the gerbe curvature equal to a bi-invariant torsion 3-form H. More precisely, on the simply connected group targets G, we consider the action of orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes Z$. This action combines the involutive twisted inversion $g \mapsto (\zeta g)^{-1}$, where the twist element ζ belongs to center $Z(G)$ of G, with the multiplication by elements of the "orbifold" subgroup $Z \subset Z(G)$. The action of Z preserves the bi-invariant 3-form H, whereas the action of the twisted inversion changes its sign. We introduce the notion of a Γ-equivariant structure on the gerbe with curvature H on group G. Such a

structure may be regarded as a Z-equivariant Jandl structure on that gerbe. It determines a genuine Jandl structure on the quotient gerbe on the non-simply connected group G/Z and enables to define unambiguously the contribution of the B-field to Feynman amplitudes of unoriented string world histories represented by maps from unoriented closed surfaces to the target G/Γ.

We show that obstructions to existence of Γ-equivariant structures are contained in the cohomology group $H^3(\Gamma, U(1)_\epsilon)$, where the subscript ϵ indicates that $U(1)$ is considered with the action $\lambda \mapsto \lambda^{-1}$ of the elements of Γ that reverse the sign of H. If the obstruction class vanishes, non-equivalent Γ-equivariant structures may be labeled by elements of the cohomology group $H^2(\Gamma, U(1)_\epsilon)$. Each choice gives a different (closed-string) orientifold theory. Let us recall that obstructions to existence of the quotient gerbe on G/Z (and of the Z-orbifold theory) lie in $H^3(Z, U(1))$ and that ambiguities in its construction (the "discrete torsion" of [33]) take values in $H^2(Z, U(1))$, see [12]. The present paper is devoted to the study of obstruction 3-cocycles for all simple simply connected groups G of the Cartan series and all choices of the orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes Z$. We find the conditions under which the obstruction cocycles are coboundaries, i.e. the obstruction cohomology class is trivial. This provides an extension of the work of [14] from the orbifold to the orientifold case. Similarly as in the orbifold case analyzed in [14, 12], the cochains trivializing the obstruction cocycles enter directly the construction of Γ-equivarient structures on the gerbes on groups G and the analysis of the symmetric D-branes in the WZW orientifolds. These topics, involving more geometric considerations as well as a discussion of the relation between our approach and the algebraic one of [5], are postponed to a later publication [15]. In the present paper, we shall avoid geometry by sticking to a local description of gerbes, staying close to the Deligne cohomology approach of [11].

The paper is organized as follows. In Sect. 2, we summarize the description of gerbes by local data and the relation of gerbes on discrete quotients to finite group cohomology. The application to gerbes on simple simply connected compact groups G and their non-simply connected quotients G/Z is recalled from [14]. Finally, we extend the construction to the case of quotients by orientifold groups Γ and describe a 3-cocycle whose cohomology class obstructs existence of Γ-equivariant structures on gerbes on the simply connected groups G for $\Gamma = \mathbb{Z}_2 \ltimes Z$. In Sect. 3, we study the relevant cohomology groups: the one containing the obstruction classes: $H^3(\Gamma, U(1)_\epsilon)$ and the one classifying non-equivalent Γ-equivariant structures: $H^2(\Gamma, U(1)_\epsilon)$. They are more difficult to calculate than the corresponding orbifold cohomologies but information about those groups may be obtained from the Lyndon-Hochschild-Serre spectral sequence that we discuss in some detail. In particular, we are able to calculate the classifying group $H^2(\Gamma, U(1)_\epsilon)$ in all relevant cases. Sect. 4 is the most technical part of the paper. It analyzes the obstruction 3-cocycles for all simple groups G of the Cartan series and all choices of the twisted orientifold group actions and finds cohomologically inequivalent trivializing cochains whenever the obstruction cohomology class is trivial. The results are tabulated in Appendix. In Sect. 5, we collect our conclusions.

2. Bundle gerbes and orientifolds

2.1 Local description of bundle gerbes

(Bundle) **gerbes** (with hermitian structure and unitary connection) are geometric structures that allow to define the contribution of the Kalb-Ramond torsion 3-form H to closed-string Feynman amplitudes. A simple, although not always convenient, way to present a gerbe on a manifold M is via its local data. In this paper, we shall stick to such a local description of bundle gerbes that reduces the geometric structures to the cohomological ones described already in [11]. A discussion, in relation to orientifolds, of the geometric structures underlying the notion of bundle gerbes [23, 24] is postponed to [15].

Gerbe local data subordinate to a good open covering[1] (O_i) of M are a collection (B_i, A_{ij}, g_{ijk}) where B_i are 2-forms on the sets O_i, $A_{ij} = -A_{ji}$ are 1-forms on O_{ij} and $g_{ijk} = g_{jik}^{-1} = g_{jki} = g_{ikj}^{-1}$ are $U(1)$-valued functions on O_{ijk} such that the following descent equations hold:

$$
\begin{aligned}
B_j - B_i &= dA_{ij} & &\text{on} \quad O_{ij}, \\
A_{ij} - A_{ik} + A_{jk} &= i\, g_{ijk}^{-1} dg_{ijk} & &\text{on} \quad O_{ijk}, \\
g_{ijk}\, g_{ijl}^{-1} g_{ikl}\, g_{jkl}^{-1} &= 1 & &\text{on} \quad O_{ijkl}.
\end{aligned}
$$

The global closed 3-form H equal to dB_i on the sets O_i is called the curvature of the gerbe. The necessary and sufficient condition for existence of a gerbe with curvature H (and of the corresponding local data) is that the periods of the 3-form $\frac{1}{2\pi}H$ be integers. The local data $(B_i', A_{ij}', g_{ijk}')$ and (B_i, A_{ij}, g_{ijk}) are considered equivalent if there exist 1-forms Π_i on O_i and $U(1)$-valued functions $\chi_{ij} = \chi_{ji}^{-1}$ on O_{ij} such that

$$
\begin{aligned}
B_i' &= B_i + d\Pi_i, \\
A_{ij}' &= A_{ij} + \Pi_j - \Pi_i - i\, \chi_{ij}^{-1} d\chi_{ij}, \\
g_{ijk}' &= g_{ijk}\, \chi_{ij}^{-1} \chi_{ik}\, \chi_{jk}^{-1}.
\end{aligned}
$$

Equivalent local data correspond to gerbes that are called stably isomorphic [24]. Clearly, such gerbes have the same curvature 3-form H. In general, two gerbes with the same curvature differ by a flat gerbe with vanishing curvature. Up to equivalence, the local data of a flat gerbe are of the form $(0, 0, u_{ijk})$ with $u_{ijk} \in U(1)$ [11]. Their equivalence classes are in a one-to-one correspondence with elements of the cohomology group $H^2(M, U(1))$. In particular, if $H^2(M, U(1))$ is trivial then all gerbes with the same curvature are stably isomorphic. If there is no torsion in $H^3(M, \mathbb{Z})$ then one may also put the flat gerbe local data into an equivalent form $(B, 0, 1)$ where B is a global closed 2-form.

If Σ is an oriented closed connected surface and X maps Σ to M then, pulling back the gerbe by X to Σ, one obtains a flat gerbe on Σ which, up to a stable isomorphism, is characterized by a cohomology class in $H^2(\Sigma, U(1)) = U(1)$. The corresponding number in $U(1)$ is called the holonomy of the gerbe on M along X. If the local data for the pulled-back gerbe are taken in the form $(B, 0, 1)$ then the holonomy along X is given by $\exp[i \int_\Sigma B]$. It enters as a factor in the Feynman amplitude of the closed-string world history X.

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[1] In a good open covering, the sets O_i and all their (non-empty) intersections $O_{i_1 i_2 \dots i_k} = O_{i_1} \cap O_{i_2} \cap \cdots \cap O_{i_k}$ are contractible.

It is convenient to use the cohomological language to describe gerbe local data and their equivalence classes [11]. We shall denote by $\check{C}^p(\mathcal{S})$ the Abelian group of Čech p-cochains with values in an (Abelian) sheaf $\mathcal{S}$. An element $c \in \check{C}^p(\mathcal{S})$ is a collection of sections $c_{i_0 \cdots i_p}$ of $\mathcal{S}$ over the sets $O_{i_0 \cdots i_p}$ that is antisymmetric in the indices $i_0, \ldots, i_p$. The groups $\check{C}^p(\mathcal{S})$ form the Čech complex $\check{C}(S)$,

$$0 \longrightarrow \check{C}^0(\mathcal{S}) \xrightarrow{\check{\delta}} \check{C}^1(\mathcal{S}) \xrightarrow{\check{\delta}} \check{C}^2(\mathcal{S}) \xrightarrow{\check{\delta}} \cdots, \tag{2.1}$$

where the Čech coboundary $\check{\delta}$ is defined by

$$(\check{\delta}c)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j c_{i_0 \cdots i_{j-1} i_{j+1} \cdots i_{p+1}}.$$

The Čech cohomology groups $H^p(M, \mathcal{S})$ are composed of Čech p-cocycles modulo p-coboundaries. In particular, taking the sheaf of locally constant $U(1)$-valued functions, one obtains the cohomology groups $H^p(M, U(1))$. Given a complex $\mathcal{D}$ of sheaves

$$0 \longrightarrow \mathcal{S}^0 \xrightarrow{d_0} \mathcal{S}^1 \xrightarrow{d_1} \mathcal{S}^2 \xrightarrow{d_2} \cdots, \tag{2.2}$$

one may build a double complex

$$
\begin{array}{ccccccccc}
& & \downarrow \check{\delta} & & \downarrow \check{\delta} & & \downarrow \check{\delta} & & \\
0 & \longrightarrow & \check{C}^p(\mathcal{S}^0) & \xrightarrow{d_0} & \check{C}^p(\mathcal{S}^1) & \xrightarrow{d_1} & \check{C}^p(\mathcal{S}^2) & \xrightarrow{d_2} & \cdots \\
& & \downarrow \check{\delta} & & \downarrow \check{\delta} & & \downarrow \check{\delta} & & \\
0 & \longrightarrow & \check{C}^{p+1}(\mathcal{S}^0) & \xrightarrow{d_0} & \check{C}^{p+1}(\mathcal{S}^1) & \xrightarrow{d_1} & \check{C}^{p+1}(\mathcal{S}^2) & \xrightarrow{d_2} & \cdots \\
& & \downarrow \check{\delta} & & \downarrow \check{\delta} & & \downarrow \check{\delta} & &
\end{array}
$$

The hypercohomology groups $\mathbb{H}^s(M, \mathcal{D})$ of the complex $\mathcal{D}$ are defined as the cohomology groups of the diagonal complex $K(\mathcal{D})$

$$0 \longrightarrow A^0 \xrightarrow{D_0} A^1 \xrightarrow{D_1} A^2 \xrightarrow{D_2} A^3 \xrightarrow{D_3} \cdots \tag{2.3}$$

where

$$A^s = \bigoplus_{p+q=s} \check{C}^p(\mathcal{S}^q) \tag{2.4}$$

and $D_s = (-1)^{q+1}\check{\delta} + d_q$ on $\check{C}^p(\mathcal{S}^q)$.

We shall denote by $\mathcal{U}$ the sheaf of local (smooth) $U(1)$-valued functions on M and by Λ^q the sheaves of (smooth) q-forms on M. For the complex $\mathcal{D}(2)$,

$$0 \longrightarrow \mathcal{U} \xrightarrow{\frac{1}{i} d \log} \Lambda^1 \xrightarrow{d} \Lambda^2 \tag{2.5}$$

where d is the exterior derivative, the groups A^s of (2.4) are:

$$A^0 = \check{C}^0(\mathcal{U}) = \{(f_i)\}, \tag{2.6}$$
$$A^1 = \check{C}^0(\Lambda^1) \oplus \check{C}^1(\mathcal{U}) = \{(\Pi_i, \chi_{ij})\}, \tag{2.7}$$
$$A^2 = \check{C}^0(\Lambda^2) \oplus \check{C}^1(\Lambda^1) \oplus \check{C}^2(\mathcal{U}) = \{(B_i, A_{ij}, g_{ijk})\}, \tag{2.8}$$
$$A^3 = \check{C}^1(\Lambda^2) \oplus \check{C}^2(\Lambda^1) \oplus \check{C}^3(\mathcal{U}) = \{(F_{ij}, D_{ijk}, \sigma_{ijkl})\}$$

where f_i, χ_{ij}, g_{ijk}, σ_{ijkl} are $U(1)$-valued functions on O_i, O_{ij}, O_{ijk} and O_{ijkl}, respectively, Π_i, A_{ij} and D_{ijk} are 1-forms on O_i, O_{ij} and O_{ijk} and B_i, F_{ij} are 2-forms on O_i and O_{ij}. The differentials D_i combine the exterior derivative with the Čech coboundary:

$$
\begin{aligned}
D_0(f_i) &= (-\mathsf{i}\, f_i^{-1} df_i,\ f_j^{-1} f_i), \\
D_1(\Pi_i, \chi_{ij}) &= (d\Pi_i,\ -\mathsf{i}\, \chi_{ij}^{-1} d\chi_{ij} + \Pi_j - \Pi_i,\ \chi_{jk}^{-1} \chi_{ik} \chi_{ij}^{-1}) \\
D_2(B_i, A_{ij}, g_{ijk}) &= (dA_{ij} - B_j + B_i,\ -\mathsf{i}\, g_{ijk}^{-1} dg_{ijk} + A_{jk} - A_{ik} + A_{ij}, \\
&\qquad\qquad\qquad\qquad g_{jkl}^{-1} g_{ikl}\, g_{ijl}^{-1} g_{ijk}).
\end{aligned}
$$

The hypercohomology of the complex $\mathcal{D}(2)$ of (2.5), i.e. the cohomology of the complex $K(\mathcal{D}(2))$, see (2.3), is

$$
\mathbb{H}^0(M, \mathcal{D}(2)) = \ker D_0 \cong H^0(M, U(1)),
$$

$$
\mathbb{H}^1(M, \mathcal{D}(2)) = \frac{\ker D_1}{\operatorname{im} D_0} \cong H^1(M, U(1)) \tag{2.9}
$$

and, in the second degree,

$$
\mathbb{H}^2(M, \mathcal{D}(2)) = \frac{\ker D_2}{\operatorname{im} D_1}.
$$

$H^0(M, U(1))$ is the group of constant $U(1)$-valued functions on M, $H^1(M, U(1))$ is the one of the isomorphism classes of flat line bundles on M and $H^2(M, \mathcal{D}(2))$ is the third real Deligne cohomology group [6, 10]. The local data of a gerbe $c = (B_i, A_{ij}, g_{ijk})$ satisfy the cocycle condition[2] $D_2 c = 0$ and equivalent local data differ by a coboundary $D_1 \beta$ with $\beta = (\Pi_i, \chi_{ij})$ so that the elements of the hypercohomology group $\mathbb{H}^2(M, \mathcal{D}(2))$ are in a one-to-one correspondence with stable isomorphism classes of gerbes.

2.2 Gerbes on orbifolds and group cohomology

Suppose now that a discrete group Γ acts on M preserving the closed 3-form H. Let us assume that the open covering (O_i) is such that $\gamma(O_i) = O_{\gamma i}$ for an action $(\gamma, i) \mapsto \gamma i$ of Γ on the index set. We shall call Γ the orbifold group. In a natural way, we may lift its action to the Abelian groups A^n of (2.6)-(2.9) by defining

$$
\gamma f_i = \gamma^{-1*} f_{\gamma^{-1}i}, \qquad \gamma \Pi_i = \gamma^{-1*} \Pi_{\gamma^{-1}i}, \tag{2.10}
$$

etc. This turns the complex $K(\mathcal{D}(2))$ of (2.3) induced from the sheaf complex (2.5) into one of Γ-modules.

Below, we shall employ the language of the group Γ cohomology, see e.g. [3] or Appendix A of [13], defining p-cochains on Γ with values in a Γ-module N as maps from Γ^p to N, and the coboundary operator δ by

$$
\begin{aligned}
(\delta n)_{\gamma, \gamma', \dots, \gamma^{(p)}} &= \gamma n_{\gamma', \dots, \gamma^{(p)}} - n_{\gamma\gamma', \gamma'', \dots, \gamma^{(p)}} + \cdots + (-1)^p n_{\gamma, \dots, \gamma^{(p-1)}\gamma^p} \\
&\qquad\qquad + (-1)^{p+1} n_{\gamma, \gamma', \dots, \gamma^{(p-1)}}.
\end{aligned}
$$

The Abelian groups $C^p(N)$ of p-cochains on Γ form the complex $C(N)$,

$$
0 \longrightarrow C^0(N) \xrightarrow{\delta} C^1(N) \xrightarrow{\delta} C^2(N) \xrightarrow{\delta} \cdots.
$$

---

[2] We use the additive notation for the Abelian groups A^n.

The cohomology groups $H^p(\Gamma, N)$ are composed of p-cocycles on Γ modulo p-coboundaries. Given a complex K of Γ-modules

$$0 \longrightarrow N^0 \xrightarrow{d_0} N^1 \xrightarrow{d_1} N^2 \xrightarrow{d_2} \cdots, \tag{2.11}$$

we may consider again a double complex formed from the groups $C^p(N^q)$ and the induced diagonal complex. The cohomology groups of the latter define the hypercohomology groups $\mathbb{H}^s(\Gamma, K)$.

We shall be interested in gerbes on M with Γ-**equivariant structures** (Γ-**gerbes** for short) that permit to define the contribution of the torsion field H to Feynman amplitudes of closed strings moving in the orbifold M/Γ. Γ-gerbes may be presented by their local data $(c, b_\gamma, a_{\gamma,\gamma'})$, where $c = (B_i, A_{ij}, g_{ijk}) \in A^2$, $b_\gamma = (\Pi_i^\gamma, \chi_{ij}^\gamma) \in A^1$ and $a_{\gamma,\gamma'} = (f_i^{\gamma,\gamma'}) \in A^0$ satisfy the relations

$$D_2 c = 0, \tag{2.12}$$

$$(\delta c)_\gamma \equiv \gamma c - c = D_1 b_\gamma, \tag{2.13}$$

$$(\delta b)_{\gamma,\gamma'} \equiv \gamma b_{\gamma'} - b_{\gamma\gamma'} + b_\gamma = -D_0 a_{\gamma,\gamma'}, \tag{2.14}$$

$$(\delta a)_{\gamma,\gamma',\gamma''} \equiv \gamma a_{\gamma',\gamma''} - a_{\gamma\gamma',\gamma''} + a_{\gamma,\gamma'\gamma''} - a_{\gamma,\gamma'} = 0. \tag{2.15}$$

The Γ-gerbe local data $(c', b_\gamma', a_{\gamma,\gamma'}')$ and $(c, b_\gamma, a_{\gamma,\gamma'})$ will be considered equivalent if there exist $\beta \in A^1$ and $\phi_\gamma \in A^0$ such that

$$c' = c + D_1 \beta, \tag{2.16}$$

$$b_\gamma' = b_\gamma + \gamma\beta - \beta + D_0\phi_\gamma \equiv b_\gamma + (\delta\beta)_\gamma + D_0\phi_\gamma, \tag{2.17}$$

$$a_{\gamma,\gamma'}' = a_{\gamma,\gamma'} - \gamma\phi_{\gamma'} + \phi_{\gamma\gamma'} - \phi_\gamma \equiv a_{\gamma,\gamma'} - (\delta\phi)_{\gamma,\gamma'}. \tag{2.18}$$

In particular, c' and c are equivalent local data for gerbes on M. Γ-gerbes with equivalent local data will be called stably isomorphic. Equivalence classes of local data $(c, b_\gamma, a_{\gamma,\gamma'})$ form the hypercohomology group $\mathbb{H}^2(\Gamma, K(\mathcal{D}(2)))$.

It is easy to see that, up to equivalence, the local data $(c, b_\gamma, a_{\gamma,\gamma'})$ of a flat Γ-gerbe are of the form

$$c = (0, 0, u_{ijk}), \qquad b_\gamma = (0, v_{\gamma;ij}^{-1}), \qquad a_{\gamma,\gamma'} = (w_{\gamma,\gamma';i}), \tag{2.19}$$

where $u_{ijk}, v_{\gamma;ij}, w_{\gamma,\gamma';i} \in U(1)$. This form is preserved by the transformations (2.16)-(2.18) with

$$\beta = (0, v_{ij}^{-1}), \qquad \phi_\gamma = (w_{\gamma;i}) \tag{2.20}$$

with $v_{ij}, w_{\gamma;i} \in U(1)$. The equivalence classes of local data for a flat Γ-gerbe form the hypercohomology group $\mathbb{H}^2(\Gamma, \check{C}(U(1)))$ where $\check{C}(U(1))$ is the Čech complex (2.1) for the sheaf of locally constant $U(1)$-valued functions (viewed as a complex of Γ-modules).

In general, there are obstructions to existence of a Γ-equivariant structure on a gerbe with local data c. First, existence of $b_\gamma \in \mathcal{A}^1$ such that (2.13) holds requires that the equivalence class of the flat-gerbe local data

$$[\gamma c - c] \in H^2(M, U(1))$$

be trivial (or, geometrically, that the pullback of the gerbe by γ stays in the same stable isomorphism class). This is automatically assured if $H^2(M, U(1)) = 0$. Suppose then that $\gamma c - c = D_1 b_\gamma$ for $b_\gamma \in A^1$. It follows that $D_1(\delta b)_{\gamma,\gamma'} = 0$ so that $(\delta b)_{\gamma,\gamma'}$ defines a 2-cocycle $r_{\gamma,\gamma'}$ on Γ with values in $H^1(M, U(1)) \equiv H^1$, see (2.9). Its cohomology class

$$[r_{\gamma,\gamma'}] \in H^2(\Gamma, H^1)$$

is the next obstruction to existence of a Γ-equivariant structure. If it is trivial, which holds automatically if $H^1 = 0$, then there exist $e_\gamma \in A^1$ with $D_1 e_\gamma = 0$ and $a_{\gamma,\gamma'} \in A^0$ such that

$$(\delta b)_{\gamma,\gamma'} = (\delta e)_{\gamma,\gamma'} - D_0 a_{\gamma,\gamma'}.$$

Note that $D_0(\delta a)_{\gamma,\gamma',\gamma''} = 0$. Hence $u_{\gamma,\gamma',\gamma''} = (\delta a)_{\gamma,\gamma',\gamma''}$ is a 3-cocycle on Γ with values in $\ker D_0 = H^0(M, U(1)) \equiv H^0$. Its cohomology class

$$[u_{\gamma,\gamma',\gamma''}] \in H^3(\Gamma, H^0) \tag{2.21}$$

is the last obstruction to existence of a Γ-equivariant structure. If it is trivial, i.e. if $u_{\gamma,\gamma',\gamma''} = (\delta v)_{\gamma,\gamma',\gamma''}$ for some $v_{\gamma,\gamma'} \in H^0$, then taking $b_\gamma - e_\gamma$ as a new b_γ and $a_{\gamma,\gamma'} - v_{\gamma,\gamma'}$ as a new $a_{\gamma,\gamma'}$, we obtain the relations (2.13)-(2.15). Note that in (2.21), the group H^0 of locally constant $U(1)$-valued functions f should be viewed as a Γ-module with $\gamma f = \gamma^{-1*} f$. If M is connected then $H^0 = U(1)$ with the trivial action of Γ.

An important question arises as to how many inequivalent Γ-equivariant structures exist on a gerbe on M if all obstructions vanish. Two sets of local data for a Γ-gerbe with the same underlying gerbe local data c differ by $(b_\gamma, a_{\gamma,\gamma'})$ such that

$$D_1 b_\gamma = 0, \qquad (\delta b)_{\gamma,\gamma'} = -D_0 a_{\gamma,\gamma'}, \qquad (\delta a)_{\gamma,\gamma',\gamma''} = 0. \tag{2.22}$$

The equivalence classes of $(b_\gamma, a_{\gamma,\gamma'})$ satisfying (2.22) modulo

$$((\delta\beta)_\gamma + D_0\phi_\gamma, \ -(\delta\phi)_{\gamma,\gamma'})$$

with $D_1\beta = 0$ label then inequivalent Γ-equivariant structures on the gerbe with local data c. Note that b_γ and $a_{\gamma,\gamma'}$ above may be taken in the form (2.19) and β and ϕ_γ in the form (2.20). The set of equivalence classes $[b_\gamma, a_{\gamma,\gamma'}]$ forms an Abelian group that we shall denote H_Γ. It may be interpreted as the hypercohomology group $\mathbb{H}^2(\Gamma, K(U(1)))$ where $K(U(1))$ is the complex

$$0 \longrightarrow \check{C}^0(U(1)) \xrightarrow{\check{\delta}} \check{Z}^1(U(1))$$

of Γ-modules with $\check{Z}^1(U(1)) = \ker \check{\delta}|_{\check{C}^1(U(1))}$. There is a natural map from H_Γ to $H^1(\Gamma, H^1)$ that assigns to $[b_\gamma, a_{\gamma,\gamma'}]$ the cohomology class $[b_\gamma]$ of the image of b_γ in H^1.

If $H^1(\Gamma, H^1) = 0$, e.g. if $H^1 = 0$, then $[b_\gamma] = 0$ and there exist (β, ϕ_γ) such that $D_1\beta = 0$ and $b_\gamma = (\delta\beta)_\gamma + D_0\phi_\gamma$. For $\alpha_{\gamma,\gamma'} = a_{\gamma,\gamma'} + (\delta\phi)_{\gamma,\gamma'}$, one has the relation $D_0\alpha_{\gamma,\gamma'} = 0$. It follows that ϕ_γ may be modified so that $a_{\gamma,\gamma'} = -(\delta\phi)_\gamma$ if and only if the cohomology class $[\alpha_{\gamma,\gamma'}] \in H^2(\Gamma, H^0)$ is trivial. This results in the isomorphism

$$H_\Gamma \ni [b_\gamma, a_{\gamma,\gamma'}] \longmapsto [\alpha_{\gamma,\gamma'}] \in H^2(\Gamma, H^0)$$

of Abelian groups. We infer this way that if $(c, b_\gamma, a_{\gamma,\gamma'})$ are local data for a Γ-gerbe then, for 2-cocycles $v_{\gamma,\gamma'}$ on Γ with values in H^0,

$$(c, b_\gamma, a_{\gamma,\gamma'} + v_{\gamma,\gamma'})$$

are also local data for a Γ-gerbe and, up to equivalence, all Γ-gerbe local data with the same gerbe local data c are obtained in such a way. The local data $(c, b_\gamma, a_{\gamma,\gamma'})$ and $(c, b_\gamma, a_{\gamma,\gamma'} + v_{\gamma,\gamma'})$ are equivalent if and only if $v_{\gamma,\gamma'} = (\delta w)_{\gamma,\gamma'}$ for $w_\gamma \in H^0$. Hence elements of $H^2(\Gamma, H^0)$ label inequivalent Γ-structures on a gerbe on M provided that $H^1(\Gamma, H^1) = 0$.

Suppose now that Γ acts on M without fixed points and that $M/\Gamma \equiv M'$ is a manifold. Under the assumption that the open covering (O_i) of M is such that $O_{i(\gamma i)} \neq \emptyset$ only if $\gamma = 1$, the sets $O_i' = \pi(O_i)$, where $\pi : M \to M'$ is the canonical projection, form a good covering of M' and

$$O_{ij'}' \equiv O_i' \cap O_{j'}' = \bigsqcup_{j=\gamma j'} \pi(O_{ij}), \qquad O_{ij'k'}' = \bigsqcup_{\substack{j=\gamma j' \\ k=\gamma\gamma' k'}} \pi(O_{ijk})$$

etc. In that situation, a Γ-gerbe with local data $(c, b_\gamma, a_{\gamma,\gamma'})$, where $c = (B_i, A_{ij}, g_{ijk})$, $b_\gamma = (\Pi_i^\gamma, \chi_{ij}^\gamma)$ and $a_{\gamma,\gamma'} = (f_i^{\gamma,\gamma'})$, induces in a canonical way a gerbe on M' with local data $(B_i', A_{ij'}', g_{ij'k'}')$ given by the relations [28]:

$$
\begin{aligned}
\pi^* B_i' &= B_i & &\text{on } O_i, \\
\pi^* A_{ij'}' &= A_{ij} + \Pi_j^\gamma & &\text{on } O_{ij} \text{ for } j = \gamma j', \\
\pi^* g_{ij'k'}' &= g_{ijk}(\chi_{jk}^\gamma f_k^{\gamma,\gamma'})^{-1} & &\text{on } O_{ijk} \text{ for } j = \gamma j', \ k = \gamma\gamma' k'.
\end{aligned}
$$

Equivalent Γ-gerbe local data on M are associated with equivalent gerbe local data on M'. Note that the latter correspond to the curvature 3-form H' such that $\pi^* H' = H$. In the more general context where Γ acts on M with fixed points, we shall sometimes talk, by an abuse of language, of Γ-gerbes on M as gerbes on the orbifold M/Γ. A more sophisticated approach to gerbes on orbifolds may be found in [20].

Let $\Sigma = \tilde{\Sigma}/\pi_1$ be an oriented closed connected surface with π_1 its fundamental group and $\tilde{\Sigma}$ its universal covering space. The maps $X : \tilde{\Sigma} \to M$ such that there exists a homomorphism $x : \pi_1 \to \Gamma$ for which

$$X(a\tilde{\sigma}) = x(a)X(\tilde{\sigma})$$

if $a \in \pi_1$ and $\tilde{\sigma} \in \tilde{\Sigma}$ describe world histories of the closed string moving in the orbifold M/Γ. The pullback by X of the local data for a Γ-gerbe on M defines local data for a flat π_1-gerbe on $\tilde{\Sigma}$. Those, in turn, determine local data for a flat gerbe on Σ by the construction described above and an element in $H^2(\Sigma, U(1)) = U(1)$ called the holonomy along X that represents the contribution of the Kalb-Ramond field to the Feynman amplitude of X.

2.3 Gerbes on simple compact Lie groups

Gerbes on Lie groups have been studied in the context of the Wess-Zumino-Witten (WZW) models [35] of conformal field theory describing the motion of strings in group manifolds.

Let G be a connected and simply connected compact simple Lie group and let H_{k} be the bi-invariant closed 3-form on G,

$$H_{\mathsf{k}} = \frac{\mathsf{k}}{12\pi}\, \mathrm{tr}\,(g^{-1}dg)^3.$$

Here, tr denotes the *ad*-invariant positive bilinear symmetric (Killing) form on the Lie algebra $\mathbf{g}$, normalized so that the 3-form $\frac{1}{2\pi}H_{\mathsf{k}}$ has integer periods if and only if k (called the level) is an integer. For such k, there exists a gerbe on G with curvature H_{k} and it is unique up to stable isomorphisms since $H^2(G, U(1)) = 0$. We shall call it the level k gerbe on G. An explicit construction of such gerbes was given in [11] for $G = SU(2)$, in [7] for $G = SU(N)$ and in [22] for all simple simply-connected compact Lie groups. In the last two cases, the construction used a more geometric description of gerbes along the lines of [23, 24] rather than the one employing local data.

Let $Z(G)$ be the center of the simply connected group G and let $\Gamma = Z \subset Z(G)$ be its subgroup. The case of non-simply connected quotients $G/Z \equiv G'$ was studied in [13] for $G = SU(N)$ and in [14] for other groups G. In those references, gerbes on groups G' with curvature H'_{k} were explicitly constructed whenever possible. Equivalently, the construction provides Z-equivariant structures on the level k gerbe on G. Since the groups $H^2(G, U(1))$ and $H^1(G, U(1))$ are trivial and $H^0(G, U(1)) = U(1)$ with the trivial action of Z, the only obstruction to existence of such Z-equivariant structures is the cohomology class $[u_{z,z',z''}] \in H^3(Z, U(1))$, see (2.21). The main part of the construction of [13, 14] consisted in analyzing the cohomological equation

$$u_{z,z',z''} = (\delta v)_{z,z',z''} \tag{2.23}$$

and finding its solutions for all levels k for which the obstruction cohomology class (2.21) is trivial. In agreement with the analysis of the last subsection, solutions $v_{z,z'}$ differing by non-cohomologous 2-cocycles gave rise to inequivalent Z-equivariant structures and hence to stably non-isomorphic gerbes on G' with curvature H'_{k}. The levels k for which the obstruction class is trivial are the ones for which the 3-form $\frac{1}{2\pi}H'_{\mathsf{k}}$ on G' has integer periods. They were identified for the first time in [8].

Let us recall here the form of the obstruction 3-cocycle $u_{z,z',z''}$ obtained in [14]. The cocycle was related to the action of the center $Z(G)$ on the set of conjugacy classes of G. Each conjugacy class has a single representative of the form $\mathrm{e}^{2\pi i \tau}$ where τ belongs to the positive Weyl alcove $\mathcal{A}$, a simplex in the Cartan algebra $\mathbf{t} \subset \mathbf{g}$ with the vertices

$$\tau_0 = 0, \qquad \tau_i = \frac{1}{k_i}\lambda_i^\vee \quad \text{for} \quad i = 1, \ldots, r$$

where $r = \dim \mathbf{t}$ is the rank of G, $\lambda_i^\vee$ are the simple coweights in $\mathbf{t}$ and k_i are the corresponding Coxeter labels. The latter are defined by the relations

$$\mathrm{tr}\,\lambda_i^\vee \alpha_j = \delta_{ij}, \qquad \phi = \sum_{i=1}^r k_i \alpha_i$$

where α_i, $i = 1, \ldots, r$, are the simple roots and ϕ is the highest root of the Lie algebra $\mathbf{g}$. Multiplication by an element $z \in Z(G)$ sends conjugacy classes into conjugacy classes and induces an affine map $\tau \mapsto \gamma\tau$ of the positive Weyl alcove. More exactly,

$$z\, \mathrm{e}^{2\pi i \tau} = w_z^{-1}\, \mathrm{e}^{2\pi i (z\tau)}\, w_z \tag{2.24}$$

for some w_z belonging to the normalizer $N(T)$ of the Cartan subgroup $T \subset G$. For the vertices of $\mathcal{A}$, we have

$$z\tau_i = \tau_{zi} \qquad \text{for} \quad i = 0, \ldots, r.$$

Upon identification of the set of indices $i = 0, 1, \ldots, r$ with the set of nodes of the extended Dynkin diagram of $\mathbf{g}$, the action $i \mapsto zi$ induces a symmetry of the diagram. The group elements w_z are determined up to left multiplication by elements of T and, in general, cannot be chosen to depend multiplicatively on z but $w_z w_{z'} w_{zz'}^{-1} \in T$. Let $b_{z,z'} \in \mathbf{t}$ be such that

$$w_z w_{z'} w_{zz'}^{-1} = \mathrm{e}^{2\pi \mathrm{i}\, b_{z,z'}}.$$

For $z \in Z(G)$, the vertex $\tau_{z^{-1}0}$ of $\mathcal{A}$ is a simple coweight such that $z = \mathrm{e}^{-2\pi \mathrm{i}\, \tau_{z^{-1}0}}$. The formula

$$u_{z,z',z''} = \mathrm{e}^{-2\pi \mathrm{i} \mathsf{k}\, \mathrm{tr}\, \tau_{z^{-1}0}\, b_{z',z''}} \tag{2.25}$$

defines a 3-cocycle on $Z(G)$ whose cohomology class does not depend on the choices made in the definition[3]. The restriction of $u_{z,z',z''}$ to $z, z', z'' \in Z \subset Z(G)$ gives the 3-cocycle whose cohomology class in $H^3(Z, U(1))$ is the obstruction (2.21) to existence of a Z-equivariant structure on the level k gerbe on G. The cohomological equation (2.23) was discussed case by case in [14].

2.4 Gerbes on orientifolds

A simple generalization of the notion of a Γ-gerbe developed in Sect. 2.2 is to admit a more general action of the discrete group Γ on M such that $\gamma^* H = \epsilon(\gamma) H$ for a homomorphism $\epsilon : \Gamma \to \{\pm 1\}$. The only modification required is in the definition (2.10) of the action of Γ on the groups A^n of (2.6)-(2.9) that should read:

$$\gamma f_i = (\gamma^{-1*} f_{\gamma^{-1}i})^{\epsilon(\gamma)}, \qquad \gamma \Pi_i = \epsilon(\gamma)\, \gamma^{-1*} \Pi_{\gamma^{-1}i},$$

etc. The change assures, for example, that if $c \in \mathcal{A}^2$ with $D_2 c = 0$ gives local data for a gerbe with curvature H then so does γc. Let $\Gamma_0 \subset \Gamma$ denote the kernel of ϵ so that one has the exact sequence of groups:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \xrightarrow{\epsilon} \mathbb{Z}_2 \longrightarrow 1.$$

We shall call Γ an orientifold group if $\Gamma_0 \neq \Gamma$. The whole discussion of Sect. 2.2 except for the two end paragraphs about gerbes on non-singular quotients and about the holonomy extends to the case of orientifold group actions generalizing the notions of Γ-equivariant structures and of Γ-gerbes to that case. We shall loosely talk of Γ-gerbes for Γ an orientifold group as gerbes on the orientifold M/Γ. As before, if $H^1(M, U(1)) = 0 = H^2(M, U(1))$ then the only obstruction to existence of a Γ-equivariant structure on the gerbe with local data c is the class (2.21), where the group $H^0 \equiv H^0(M, U(1))$ of locally constant $U(1)$-valued functions is viewed now as a Γ-module with $\gamma f = (\gamma^{-1*} f)^{\epsilon(\gamma)}$. For M connected, $H^0 = U(1)$ with the action $\gamma\lambda = \lambda^{\epsilon(\gamma)}$. If the obstruction (2.21) is trivial

---

[3] The 3-cocycle analyzed in [14] differed by a coboundary from the one of (2.25).

then inequivalent Γ-equivariant structures are labeled by elements of $H^2(\Gamma, H^0_\epsilon)$, where the subscript ϵ indicates that H^0 is taken with the Γ-module structure just described.

The simplest example is that of the **inversion group** $\Gamma = \{\pm 1\} \equiv \mathbb{Z}_2$ with $\epsilon(\pm 1) = \pm 1$. Γ-equivariant structures on a gerbe on M for such Γ were introduced (in an equivalent formulation) in [31] under the name of **Jandl structures**. A particular case is when M is the orientation double $\hat{\Sigma}$ of an unoriented closed connected surface $\Sigma = \hat{\Sigma}/\mathbb{Z}_2$. $\hat{\Sigma}$ is an oriented closed surface (connected if Σ is non-orientable and with two components otherwise). The group of stable isomorphism classes of $\mathbb{Z}_2$-gerbes on $\hat{\Sigma}$ is $\mathbb{H}^2(\mathbb{Z}_2, \check{C}(U(1)))$, as described in Sect. 2.2. In [31], a natural group homomorphism ι was constructed that renders the diagram

$$\begin{array}{ccc} \mathbb{H}^2(\mathbb{Z}_2, C(U(1))) & \xrightarrow{\iota} & U(1) \\ & \searrow & \nearrow \text{ sq} \\ & H^2(\hat{\Sigma}, U(1)) = U(1) & \end{array} \qquad (2.26)$$

commutative. In the diagram, the south-east arrow is induced by forgetting the $\mathbb{Z}_2$-equivariant structure and the south-west one by $\mathrm{sq}(\lambda) = \lambda^2$. Let $X : \hat{\Sigma} \to M$ be such that

$$X(-1 \cdot \hat{\sigma}) = -1 \cdot X(\hat{\sigma}).$$

for $\hat{\sigma} \in \hat{\Sigma}$. Such maps X, invariant under the combined worldsheet orientation reversal and a target map that changes the sign of the torsion field, describe world histories of the closed unoriented string moving in the orientifold $M/\mathbb{Z}_2$. The pullback by X of a $\mathbb{Z}_2$-gerbe on M to $\hat{\Sigma}$ defines a $\mathbb{Z}_2$-gerbe on $\hat{\Sigma}$. The number in $U(1)$ associated to the stable isomorphism class of the latter by the homomorphism ι is called the holonomy of the $\mathbb{Z}_2$-gerbe on M along X. It represents the contribution of the Kalb-Ramond field to the Feynman amplitude of the world history X.

For more general orientifold groups Γ, the restriction of the Γ-equivariant structure to the Γ_0-equivariant one may be used to define a quotient gerbe on $M' = M/\Gamma_0$ if Γ_0 acts without fixed points. The full Γ-equivariant structure induces then a Jandl structure on the quotient gerbe, see [15]. We could, more correctly, call a Γ-equivariant structure a Γ_0-**equivariant Jandl structure**, but we shall stick in what follows to the former name. The construction of holonomy of gerbes with Jandl structures described above may be extended to the equivariant case. Let Σ be an unoriented closed connected surface, $\hat{\Sigma}$ its orientation double, $\hat{\pi}_1$ the fundamental group of $\hat{\Sigma}$, and $\tilde{\Sigma}$ the universal covering of $\hat{\Sigma}$. There is a natural group $\tilde{\pi}$ entering the exact sequence of groups

$$1 \longrightarrow \hat{\pi}_1 \longrightarrow \tilde{\pi} \xrightarrow{\tilde{\epsilon}} \mathbb{Z}_2 \longrightarrow 1$$

and acting on $\tilde{\Sigma}$ without fixed points in a way that extends the action of $\hat{\pi}_1$, projects to the natural action of $\mathbb{Z}_2$ on $\hat{\Sigma}$ and to the identity on Σ. Suppose that $X : \tilde{\Sigma} \to M$ is such that, for a homomorphism $x : \tilde{\pi} \to \Gamma$ with $\epsilon(x(\tilde{a})) = \tilde{\epsilon}(\tilde{a})$, one has:

$$X(\hat{a}\tilde{\sigma}) = x(\tilde{a})X(\tilde{\sigma})$$

for $\tilde{a} \in \tilde{\pi}$ and $\tilde{\sigma} \in \tilde{\Sigma}$. Such maps X, covariant with respect to the action of the orientifold groups, describe world histories of the closed unoriented string moving in the orientifold

M/Γ. The pullback of a Γ-gerbe on M by X defines a flat $\tilde{\pi}$-gerbe on $\tilde{\Sigma}$. Using the restriction of the $\tilde{\pi}$-equivariant structure to $\hat{\pi}_1 \subset \tilde{\pi}$, one may then obtain a flat gerbe on $\hat{\Sigma}$. The $\tilde{\pi}$-equivariant structure descends to a Jandl structure on it. Applying the homomorphism ι of (2.26) to the $\mathbb{Z}_2$-gerbe on $\hat{\Sigma}$ obtained this way, one finds then the holonomy of the Γ-gerbe along X that contributes to the Feynman amplitude of the closed unoriented string moving in the orientifold M/Γ.

2.5 Orientifolds of simple compact Lie groups

We may consider the inversion group $\Gamma = \{\pm 1\} \cong \mathbb{Z}_2$ with $\epsilon(\pm 1) = \pm 1$ and $-1 \equiv z_0$ acting on a connected simply connected simple compact Lie group G by

$$G \ni g \to z_0 g = (\zeta g)^{-1} \in G \tag{2.27}$$

for $\zeta \in Z(G)$ that we shall call the **twist element**. The action of z_0 changes, indeed, the sign of H_k so that the relation $\gamma^* H_k = \epsilon(\gamma) H_k$ holds. More generally, we shall consider orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes Z$ for $Z \subset Z(G)$ with the multiplication table

$$(1, z) \cdot (1, z') = (1, zz'), \qquad\qquad (-1, z) \cdot (1, z') = (-1, zz'),$$
$$(1, z) \cdot (-1, z') = (-1, z^{-1} z'), \qquad\qquad (-1, z) \cdot (-1, z') = (1, z^{-1} z')$$

and $\epsilon(\pm 1, z) = \pm 1$ so that $\Gamma_0 = Z$. Note that Γ is a non-Abelian group if Z is non-trivial and different from $\mathbb{Z}_2$ or from a direct product of $\mathbb{Z}_2$ factors. To simplify the notation, we shall write $(1, z) \equiv z$ and $(-1, z) \equiv z_0 z$. For the action of Γ on G we shall take the one that combines (2.27) with the action of Z by multiplication so that $z_0 z g = (\zeta z g)^{-1}$. Note that if $h_{\zeta'}$ for $\zeta' \in Z$ denotes the automorphism of $\Gamma = \mathbb{Z}_2 \ltimes Z$ defined by the relations

$$h_{\zeta'}(z) = z, \qquad h_{\zeta'}(z_0 z) = z_0 \zeta' z \tag{2.28}$$

then the composition of the action of Γ on G with $h_{\zeta'}$ induces the change $\zeta \mapsto \zeta\zeta'$ of the twist element. Hence twist elements in the same coset of $Z(G)/Z$ give rise to equivalent orientifold group actions. This is in agreement with the observation [15] that the restriction of a Γ-equivariant structure on a gerbe on G to the Z-equivariant structure induces a gerbe on the non-simply connected group $G' = G/Z$ and the full Γ-equivariant structure gives rise to a Jandl structure on that gerbe. Indeed, actions of Γ on G corresponding to twist elements in the same coset of $Z(G)/Z$ induce the same action of $\mathbb{Z}_2$ on G'.

As discussed in the previous subsection, the sole obstruction to existence of a Γ-equivariant structure on the level k gerbe on G is given by the cohomology class $[u_{\gamma, \gamma', \gamma''}] \in H^3(\Gamma, U(1)_\epsilon)$, where the subscript ϵ indicates that $U(1)$ is taken with the action $\gamma\lambda = \lambda^{\epsilon(\gamma)}$ of Γ. The 3-cocycle $u_{\gamma, \gamma', \gamma''}$ may be found by a straightforward generalization of the work done in [14] where the case of orbifold groups Z was treated, see Sect. 2.3 above. We shall only describe the result here, postponing a more detailed exposition to [15].

First, let us observe that the inversion map $g \overset{\kappa}{\mapsto} g^{-1}$ sends conjugacy classes to conjugacy classes. Upon identification of the set of conjugacy classes in G with the positive Weyl alcove $\mathcal{A} \subset \mathbf{t}$, see Sect. 2.3, it induces an affine map $\tau \mapsto \kappa\tau$ on $\mathcal{A}$ such that

$$\kappa\tau_i = \tau_{\kappa i}$$

for $\kappa 0 = 0$ and $i \mapsto \kappa i$ for $i = 1, \ldots, r$, giving rise to a symmetry of the (unextended) Dynkin diagram of $\mathbf{g}$. More precisely,

$$e^{-2\pi i \tau} = w_\kappa^{-1} e^{2\pi i (\kappa\tau)} w_\kappa \tag{2.29}$$

for some w_κ belonging to the normalizer $N(T)$ of the Cartan subgroup $T \subset G$. The element w_κ is determined up to left multiplication by elements of T. Combining the relations (2.29) and (2.24), we infer that for any $\gamma \in \Gamma = \mathbb{Z}_2 \ltimes Z$ there exist an affine map $\tau \mapsto \gamma\tau$ of the Weyl alcove $\mathcal{A}$ and $w_\gamma \in N(T)$ such that

$$\gamma \, e^{2\pi i \tau} \;=\; w_\gamma^{-1} \, e^{2\pi i(\gamma\tau)} \, w_\gamma \, . \tag{2.30}$$

One has $\gamma\tau = z\tau$ for $\gamma = z$ and $\gamma\tau = \kappa\zeta z\tau$ for $\gamma = z_0 z$ and one may take

$$w_\gamma \;=\; w_z \quad \text{for} \quad \gamma = w \, , \qquad w_\gamma \;=\; w_\kappa w_\zeta w_z \quad \text{for} \quad \gamma = z_0 z \, . \tag{2.31}$$

The action of Γ on the vertices of $\mathcal{A}$,

$$\gamma\tau_i \;=\; \tau_{\gamma i} \qquad \text{for} \quad i = 0, \dots, r \tag{2.32}$$

induces a symmetry $i \mapsto \gamma i$ of the extended Dynkin diagram of $\mathbf{g}$. This symmetry preserves the Coxeter labels: $k_{\gamma i} = k_i$ if one sets $k_0 = 1$. From the relations (2.30) and (2.32), one obtains the formula:

$$\gamma\tau \;=\; \epsilon(\gamma) \, w_\gamma \tau w_\gamma^{-1} \;+\; \tau_{\gamma 0}$$

for the action of γ on $\mathcal{A}$. As before, it is easy to see that $w_\gamma w_{\gamma'} w_{\gamma\gamma'}^{-1} \in T$ so that one may choose $b_{\gamma,\gamma'} \in \mathbf{t}$ such that

$$w_\gamma w_{\gamma'} w_{\gamma\gamma'}^{-1} \;=\; e^{2\pi i \, b_{\gamma,\gamma'}} \, . \tag{2.33}$$

The 3-cocycle on Γ, whose cohomology class defines the obstruction to existence of a Γ-equivariant structure on the level k gerbe on G, takes the form:

$$u_{\gamma,\gamma',\gamma''} \;=\; e^{-2\pi i \mathsf{k} \, \epsilon(\gamma) \, \mathrm{tr} \, \tau_{\gamma^{-1}0} \, b_{\gamma',\gamma''}} \, . \tag{2.34}$$

The cocycle condition means that

$$(\delta u)_{\gamma,\gamma',\gamma'',\gamma'''} \;\equiv\; u^{\epsilon(\gamma)}_{\gamma',\gamma'',\gamma'''} \, u^{-1}_{\gamma\gamma',\gamma'',\gamma'''} u_{\gamma,\gamma'\gamma'',\gamma'''} \, u^{-1}_{\gamma,\gamma',\gamma''\gamma'''} \, u_{\gamma,\gamma',\gamma''} \;=\; 1 \, .$$

It may be verified by a direct calculation. The cohomology class of $u_{\gamma,\gamma',\gamma''}$ is independent of the choices made in its definition. Note that the 3-cocycle (2.34) on Γ restricts to the 3-cocycle (2.25) on $Z = \Gamma_0$.

Let us finally remark that, since the orientifold action (2.27) of $\Gamma = \mathbb{Z}_2 \ltimes Z$ with the twist element $\zeta\zeta'$ for $\zeta' \in Z$ may be obtained from that with the twist element ζ by composing with the automorphism (2.28) of Γ, the cocycle $u_{\gamma,\gamma',\gamma''}$ for the new action defines the same cohomology class in $H^3(\Gamma, U(1)_\epsilon)$ as the one for the original action composed with the automorphism $h_{\zeta'}$. The composition with an automorphism of Γ that leaves the homomorphism ϵ invariant commutes with the coboundary δ and induces an automorphism of the cohomology groups $H^n(\Gamma, U(1)_\epsilon)$.

3. Lyndon-Hochschild-Serre spectral sequence

Recall that the cohomology class

$$[u_{\gamma,\gamma',\gamma''}] \;\in\; H^3(\Gamma, U(1)_\epsilon)$$

is the obstruction to existence of a Γ-equivariant structure on the level k gerbe on the simply connected group G. The purpose of the present paper is to discuss in detail the cohomological equation

$$u_{\gamma,\gamma',\gamma''} \;=\; v_{\gamma',\gamma''}^{\epsilon(\gamma)}\, v_{\gamma\gamma',\gamma''}^{-1}\, v_{\gamma,\gamma'\gamma''}\, v_{\gamma,\gamma'}^{-1} \;\equiv\; (\delta v)_{\gamma,\gamma',\gamma''}. \tag{3.1}$$

which is solvable if and only if the cohomology class $[u_{\gamma,\gamma',\gamma''}]$ is trivial. Knowledge of the general structure of the cohomology group $H^3(\Gamma,U(1)_\epsilon)$ will be useful in checking the latter condition. In what follows, we shall call $u_{\gamma,\gamma',\gamma''}$ the **obstruction cocycle** and $v_{\gamma,\gamma'}$ a **trivializing cochain**. As will be shown in [15], trivializing cochains enter directly the construction of a Γ-equivariant structure on the level k gerbe on G, similarly as in the case of orbifold groups that was discussed in [14]. The classification of inequivalent Γ-gerbes may, likewise, be formulated in the cohomological language, with inequivalent Γ-gerbes corresponding to trivializing cochains differing by non-cohomologous 2-cocycles $v_{\gamma,\gamma'}$,

$$[v_{\gamma,\gamma'}] \;\in\; H^2(\Gamma,U(1)_\epsilon). \tag{3.2}$$

This way $H^2(\Gamma,U(1)_\epsilon)$ plays the role of the classifying group for inequivalent Γ-gerbes on G. Its structure will provide valuable insights into certain algebraic properties and the origin of trivializing cochains prior to entering the straightforward yet tedious computations of Sect.4. It should be stressed at this point that while obstructions to existence of orientifold gerbes do not, in general, exhaust the obstruction cohomology group $H^3(\Gamma,U(1)_\epsilon)$, it is the entire classifying group $H^2(\Gamma,U(1)_\epsilon)$ that captures inequivalent orientifold gerbe structures.

In consequence of the semi-direct product nature of the orientifold group $\Gamma = \mathbb{Z}_2 \ltimes Z$, the main tool which will be used in exploring the $U(1)_\epsilon$-valued cohomology of Γ is the Lyndon-Hochschild-Serre (LHS) spectral sequence [34]

$$E_r^{p,q} \Longrightarrow H^{p+q}(\Gamma,U(1)_\epsilon) \tag{3.3}$$

associated with the short exact sequence of groups:

$$1 \longrightarrow Z \longrightarrow \Gamma \longrightarrow \mathbb{Z}_2 \longrightarrow 1. \tag{3.4}$$

Recall [21] that the rth page of a spectral sequence with $r \geq 0$ is a collection of Abelian groups $E_r^{p,q}$ vanishing for negative p or q, together with the coboundary homomorphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} d_r^{p,q} = 0$. The groups of the next page are defined by setting $E_{r+1}^{p,q} = \ker d_r^{p,q}/\operatorname{im} d^{p-r,q+r-1}$. The second page of the LHS spectral sequence is composed of the groups

$$E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(Z,U(1))_\epsilon), \tag{3.5}$$

with the action of $\mathbb{Z}_2$ on $H^q(Z,U(1))$ induced by the one on the q-cochains on Z:

$$(-1 \cdot c)_{z,z',\dots,z^{(q)}} = c_{z^{-1},z'^{-1},\dots,z^{(q)-1}}^{-1}. \tag{3.6}$$

The relation (3.3) of the LHS sequence to the cohomology groups $H^n(\Gamma,U(1)_\epsilon)$ is established with the help of a filtration

$$0 = H_{n+1}^n \subset H_n^n \subset \cdots \subset H_1^n \subset H_0^n = H^n(\Gamma,U(1)_\epsilon)$$

such that

$$H_p^n / H_{p+1}^n \cong E_\infty^{p,n-p}$$

where $E_\infty^{p,q}$ denotes the group at which $E_r^{p,q}$ stabilize for (p,q) fixed and r sufficiently large.

Already the second page of the LHS spectral sequence provides a great deal of information on the possible structure of the cohomology groups $H^n(\Gamma, U(1)_\epsilon)$, at least for Z cyclic to which case we shall specialize first, taking $Z = \mathbb{Z}_m$ with $m > 0$. The cyclic group cohomology is well known, see [3]:

$$H^q(\mathbb{Z}_m, U(1)) = \begin{cases} U(1) & \text{if} \quad q = 0, \\ 0 & \text{if} \quad q > 0 \text{ is even,} \\ \mathbb{Z}_m & \text{if} \quad q \text{ is odd} \end{cases}$$

for the trivial action of the orbifold group $\mathbb{Z}_m$ on $U(1)$. The action of the generator -1 of the orientifold group $\mathbb{Z}_2$ on $H^q(\mathbb{Z}_m, U(1))$ induced by (3.6) reduces to the inversion for q even and to the trivial action for q odd. One further has:

$$H^p(\mathbb{Z}_2, U(1)_\epsilon) = \begin{cases} \mathbb{Z}_2 & \text{if} \quad p \text{ is even,} \\ 0 & \text{if} \quad p \text{ is odd} \end{cases} \tag{3.7}$$

for the action of -1 on $U(1)$ by inversion, and
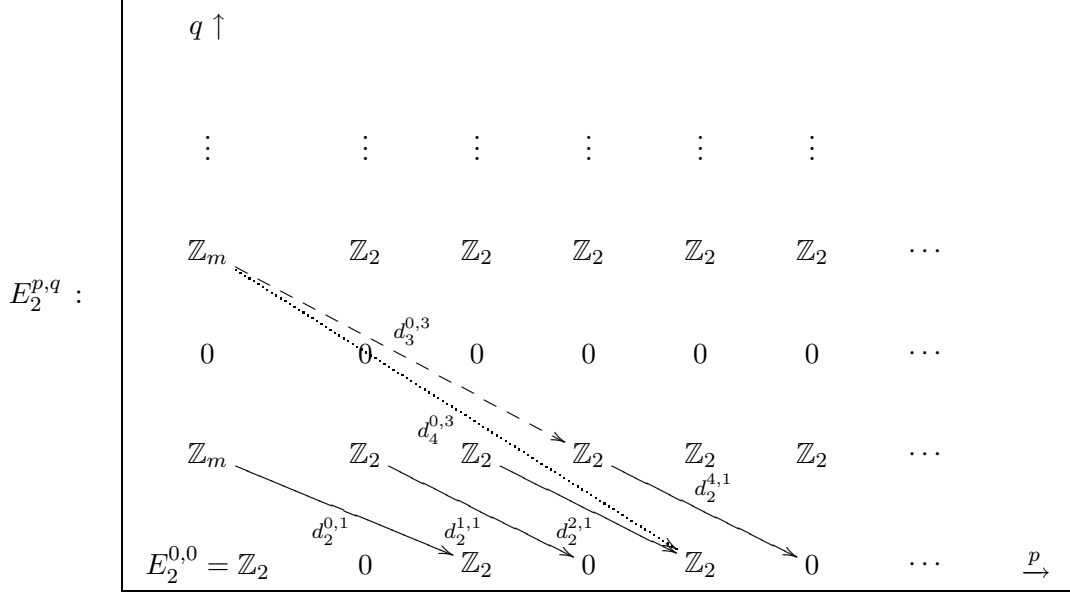
$$H^p(\mathbb{Z}_2, \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & \text{if} \quad p = 0, \\ \mathbb{Z}_2 & \text{if} \quad p > 0 \text{ and } m \text{ is even,} \\ 0 & \text{if} \quad p > 0 \text{ and } m \text{ is odd} \end{cases} \tag{3.8}$$

for the trivial action of $\mathbb{Z}_2$ on $\mathbb{Z}_m$. This gives for the second page of the spectral sequence:

for m odd, and

$E_2^{p,q}$:

$q \uparrow$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbb{Z}_m$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
0	0	0	0	0	0	$\cdots$
$\mathbb{Z}_m$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
$E_2^{0,0} = \mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\cdots$ $\xrightarrow{p}$

with differentials $d_3^{0,3}$, $d_4^{0,3}$, $d_2^{4,1}$, $d_2^{0,1}$, $d_2^{1,1}$, $d_2^{2,1}$.

for m even. The images of the coboundary homomorphisms $d_r^{p,q}$ for the second page (the continuous lines) and of the higher ones (the dotted and dashed lines), together with the definition of the groups entering next pages, lead us to the conclusion that the LHS spectral sequence stabilizes quickly for the cohomology groups of interest: the classifying group $H^2(\Gamma, U(1)_\epsilon)$, and the obstruction group $H^3(\Gamma, U(1)_\epsilon)$.

For m odd, taking into account that there are no non-trivial homomorphisms from $\mathbb{Z}_m$ to $\mathbb{Z}_2$, we conclude that the sequence collapses to the second page giving

$$H^n(\mathbb{Z}_2 \ltimes \mathbb{Z}_m, U(1)_\epsilon) = \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even,} \\ \mathbb{Z}_m & \text{if } n \text{ is odd.} \end{cases}$$

The case of m even is somewhat more complicated. We shall argue that $d_2^{0,1} = 0$ also in this case. It is shown in [30] that for $\Gamma = \mathbb{Z}_2 \ltimes Z$ there exists a 7-term exact sequence :

$$0 \longrightarrow H^1(\mathbb{Z}_2, H^0(Z, U(1))_\epsilon) \longrightarrow H^1(\Gamma, U(1)_\epsilon) \xrightarrow{\rho} H^0(\mathbb{Z}_2, H^1(Z, U(1))_\epsilon)$$

$$\xrightarrow{d_2^{0,1}} H^2(\mathbb{Z}_2, H^0(Z, U(1))_\epsilon) \longrightarrow H^2(\Gamma, U(1)_\epsilon)_1 \longrightarrow H^1(\mathbb{Z}_2, H^1(Z, U(1))_\epsilon)$$

$$\xrightarrow{d_2^{1,1}} H^3(\mathbb{Z}_2, H^0(Z, U(1))_\epsilon)$$

with ρ denoting the restriction map and $H^2(\Gamma, U(1)_\epsilon)_1$ entering the exact sequence

$$0 \longrightarrow H^2(\Gamma, U(1)_\epsilon)_1 \longrightarrow H^2(\Gamma, U(1)_\epsilon) \longrightarrow H^2(Z, U(1))^{\mathbb{Z}_2}$$

where the last group is the subgroup of $\mathbb{Z}_2$-invariant elements of $H^2(Z, U(1))$. Since every 1-cocycle w_z on Z with values in $U(1)$ (i.e. a character of Z) extends to a 1-cocycle on Γ with values in $U(1)_\epsilon$ upon setting $w_{z_0 z} = w_z^{-1}$, the restriction map is surjective. Besides, $H^2(Z, U(1)) = 0$ for Z cyclic so that $H^2(\Gamma, U(1)_\epsilon)_1 \cong H^2(\Gamma, U(1)_\epsilon)$ in this case. If $Z = \mathbb{Z}_m$ with m even then

$$H^1(\mathbb{Z}_2, H^0(Z, U(1))_\epsilon) = H^1(\mathbb{Z}_2, U(1)_\epsilon) = 0,$$

$$H^0(\mathbb{Z}_2, H^1(Z, U(1))_\epsilon) = H^0(\mathbb{Z}_2, Z) = Z,$$
$$H^2(\mathbb{Z}_2, H^0(Z, U(1))_\epsilon) = H^2(\mathbb{Z}_2, U(1)_\epsilon) = \mathbb{Z}_2,$$
$$H^1(\mathbb{Z}_2, H^1(Z, U(1))_\epsilon) = H^1(\mathbb{Z}_2, Z) = \mathbb{Z}_2,$$
$$H^3(\mathbb{Z}_2, H^0(Z, U(1))_\epsilon) = H^3(\mathbb{Z}_2, U(1)_\epsilon) = 0$$

so that the 7-term exact sequence reduces to

$$0 \longrightarrow H^1(\Gamma, U(1)_\epsilon) \xrightarrow{\rho} Z \xrightarrow{d_2^{0,1}} \mathbb{Z}_2 \longrightarrow H^2(\Gamma, U(1)_\epsilon)) \longrightarrow \mathbb{Z}_2 \xrightarrow{d_2^{1,1}} 0.$$

It follows, in particular, that ρ is an isomorphism and $d_2^{0,1} = 0$. Finally, using this information in the LHS spectral sequence, we infer that for m even,

$$H^n(\mathbb{Z}_2 \ltimes \mathbb{Z}_m, U(1)_\epsilon) = \begin{cases} \mathbb{Z}_2 & \text{if} \quad n = 0, \\ \mathbb{Z}_m & \text{if} \quad n = 1, \\ \mathbb{Z}_4 \quad \text{or} \quad \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if} \quad n = 2, \\ \mathbb{Z}_{2k} \quad \text{or} \quad \mathbb{Z}_2 \times \mathbb{Z}_k, \quad \frac{m}{k} = 1, 2, 4, & \text{if} \quad n = 3. \end{cases} \quad (3.9)$$

The ambiguity in (3.9) can actually be resolved for the group $H^2(\mathbb{Z}_2 \ltimes \mathbb{Z}_m, U(1)_\epsilon)$. Indeed, consider its element defined by the cocycle

$$v^{(1)}_{z_0^n z, z_0^{n'} z'} = (-1)^{nn'} \quad (3.10)$$

for $z, z' \in \mathbb{Z}_m$. Suppose that $v^{(1)}$ is a coboundary, $v^{(1)}_\gamma = (\delta w)_\gamma$, from which it would follow, in particular, that

$$w_{z_0 z'} (w_{z_0 z^{-1} z'})^{-1} w_z = 1 \quad \text{and} \quad (w_{z_0 z'})^{-1} (w_{z^{-1} z'})^{-1} w_{z_0 z} = -1. \quad (3.11)$$

The two conditions are, however, contradictory as can be verified by replacing z by $z^{-1} z'$ in the second one. Hence the class $[v^{(1)}_{\gamma,\gamma'}]$ generates a $\mathbb{Z}_2$ subgroup of $H^2(\mathbb{Z}_2 \ltimes \mathbb{Z}_m, U(1)_\epsilon)$. For m odd, this is the whole group but for m even we may repeat the same reasoning with respect to the 2-cocycle

$$v^{(2)}_{z_0^n z, z_0^{n'} z'} = \begin{cases} 1 & \text{if} \quad (z')^{m/2} = 1, \\ (-1)^n & \text{if} \quad (z')^{m/2} \neq 1 \end{cases} \quad (3.12)$$

and $v^{(2)}_{\gamma,\gamma'} (v^{(1)}_{\gamma,\gamma'})^{-1}$, establishing that the class $[v^{(2)}_{\gamma,\gamma'}]$ in $H^2(\mathbb{Z}_2 \ltimes \mathbb{Z}_m, U(1)_\epsilon)$ is non-trivial and different from $[v^{(1)}_{\gamma,\gamma'}]$. This immediately implies that

$$H^2(\mathbb{Z}_2 \ltimes \mathbb{Z}_m, U(1)_\epsilon) = \mathbb{Z}_2 \times \mathbb{Z}_2 \quad (3.13)$$

for m even and it is generated by the cohomology classes $[v^{(1)}_{\gamma,\gamma'}]$ and $[v^{(2)}_{\gamma,\gamma'}]$.

Finally, we give, for the sake of completeness, the classifying cohomology group for the case of the non-cyclic orbifold subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ that will be encountered in the study of the Cartan series D_{2s} of simple groups. Since

$$H^q(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \begin{cases} U(1) & \text{if} \quad q = 0, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if} \quad q = 1, \\ \mathbb{Z}_2 & \text{if} \quad q = 2, \end{cases}$$

see [14], in the LHS sequence

$$E^{0,2} = \mathbb{Z}_2, \qquad E^{1,1} = \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad E^{2,0} = \mathbb{Z}_2.$$

It follows that $H^2(\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2), U(1)_\epsilon)$ has rank smaller or equal to 16. In Sect. 4.4.2, we shall exhibit 16 cohomologically inequivalent 2-cocycles on $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ taking values ± 1. This will establish the equality

$$H^2(\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2), U(1)_\epsilon) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{3.14}$$

Prior to refining the tools of analysis of the obstruction cocycles, let us make one general comment. In the cyclic case $Z = \mathbb{Z}_m$, the (possibly non-factorizing) component of the obstruction group $H^3(\Gamma, U(1)_\epsilon)$ coming from $H^0(\mathbb{Z}_2, H^3(\mathbb{Z}_m, U(1))_\epsilon)$ is determined completely by the orbifold subgroup. After imposing conditions that allow a trivialization of the obstruction cocycle $u_{z,z',z''}$ restricted to Z, we are left with at most a sign obstruction. We shall encounter such sign obstructions for m even in the cases considered below. On the other hand, for m odd, the restriction to Z is clearly the sole source of obstruction. In particular, there are no obstructions to the trivializability of $u_{\gamma,\gamma,\gamma''}$ for trivial Z and, consequently, no obstruction to existence of Jandl structures on the gerbes on simply connected groups G.

4. Case-by-case analysis

Trivializability of the obstruction 3-cocycle $u_{\gamma,\gamma',\gamma''}$ given by (2.34) constrains the admissible values of the level k in terms of the other elements such as the structure of the group G, the choice of the orbifold subgroup $Z \subset Z(G)$, and that of the twist element $\zeta \in Z$ entering the action (2.27) on G of the $\mathbb{Z}_2$ component of the orientifold group $\Gamma = \mathbb{Z}_2 \ltimes Z$. Below, we shall calculate the cocycles $u_{\gamma,\gamma',\gamma''}$ on Γ and classify the cases when they may be trivialized, giving also an explicit form of trivializing cochains. The latter provide the main input in the explicit construction of Γ-equivariant structures on the level k gerbe on G that will be described in [15]. Cohomologically inequivalent trivializing cochains give rise to inequivalent Γ-equivariant structures. The construction of [15] is a direct generalization of the one of gerbes on non-simply connected groups G/Z described in [14].

Below, we shall denote by z a fixed generator of $Z(G)$ for the groups G with cyclic center $Z(G)$. The essential input in the calculations of the obstruction cocycle $u_{\gamma,\gamma',\gamma''}$ is the identification of the elements w_z and w_κ in the normalizer $N(T)$ of the Cartan subgroup of $T \subset G$ and of the maps $\tau \mapsto z\tau$ and $\tau \mapsto \kappa z$ of the positive Weyl alcove that satisfy (2.24) and (2.29). To simplify the notation, we shall abbreviate $z^n \equiv n$ and $z_0 z^n \equiv \underline{n}$, where $n = 0, \ldots, |Z(G)| - 1$ for the elements of the maximal orientifold group $\Gamma = \mathbb{Z}_2 \ltimes Z(G)$. For any integer n, we shall denote by $[n]$ the number equal to n modulo $|Z(G)|$ and such that $0 \le [n] < |Z(G)|$. In accordance with (2.31), for the general elements $\gamma = z^n, z_0 z^n$ of Γ with $n = 0, \ldots, |Z(G)| - 1$, one may set:

$$w_n \equiv w_{z^n} = w_z^n, \qquad w_{\underline{n}} \equiv w_{z_0 z^n} = w_\kappa w_z^{n_0} w_z^n \tag{4.1}$$

if the twist element ζ entering the action (2.27) of z_0 on G is equal to $z^{n_0} \equiv n_0$. With these choices of w_γ, the calculation of the obstruction cocycle $u_{\gamma,\gamma',\gamma''}$ will follow (2.33)

and (2.34). For smaller orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes Z \subset \mathbb{Z}_2 \ltimes Z(G)$ with $Z \subset Z(G)$, the obstruction cocycle will be obtained by restriction of the one for the maximal Γ.

Obstructions to existence of a trivializing cochain coming from the orbifold subgroup $Z \subset \Gamma$ were analyzed in [14]. To look for a further obstruction associated with the subgroup $\mathbb{Z}_2 \ltimes \mathbb{Z}_2 \subset \Gamma$ for $Z(G)$ cyclic with $|Z(G)|$ even, we shall consider, for $n = \frac{1}{2}|Z(G)|$, the combination

$$X \;=\; u_{\underline{n},n,\underline{0}}^{-2}\, u_{0,\underline{n},\underline{n}}^{2}\, u_{\underline{0},\underline{0},\underline{0}}^{2}\, u_{0,\underline{0},\underline{0}}^{-2}\, u_{\underline{n},\underline{n},\underline{n}}^{-1}\, u_{\underline{n},0,\underline{n}}^{-1}\, u_{n,\underline{0},n}\, u_{\underline{0},n,n}\, u_{n,\underline{n},\underline{0}}^{-1}\, u_{\underline{0},\underline{0},\underline{0}} \tag{4.2}$$

By direct substitution, one may check that $X = 1$ if $u_{\gamma,\gamma',\gamma''}$ satisfies (2.23) (or its restriction to $\mathbb{Z}_2 \ltimes \mathbb{Z}_2$). In a few cases, this equality will impose further non-trivial conditions on existence of trivializing cochains.

Recall from Sect. 3 that, for Z cyclic, the cohomology groups $H^2(\mathbb{Z}_2 \ltimes Z) = \mathbb{Z}_2$ if the rank $|Z|$ of Z is odd, and $H^2(\mathbb{Z}_2 \ltimes Z) = \mathbb{Z}_2 \times \mathbb{Z}_2$ if $|Z|$ is even. If a 2-cochain $v_{\gamma,\gamma'}$ solves the cohomological equation (2.23) then

$$v_{\gamma,\gamma'} \quad \text{and} \quad v_{\gamma,\gamma'}\, v_{\gamma,\gamma'}^{(1)}$$

give two cohomologically inequivalent solutions if the rank of $|Z|$ is odd and

$$v_{\gamma,\gamma'}, \quad v_{\gamma,\gamma'}\, v_{\gamma,\gamma'}^{(1)}, \quad v_{\gamma,\gamma}\, v_{\gamma,\gamma'}^{(2)}, \quad v_{\gamma,\gamma'}\, v_{\gamma,\gamma'}^{(1)}\, v_{\gamma,\gamma'}^{(2)}$$

give four cohomologically inequivalent solutions if $|Z|$ is even. All other solutions of (2.23) differ from those by 2-coboundaries. In the notation introduced above, the 2-cocycles $v_{\gamma,\gamma'}^{(1)}$ and $v_{\gamma,\gamma'}^{(2)}$ of (3.10) and (3.12) read

$$v_{n,n'}^{(1)} \;=\; v_{n,\underline{n}'}^{(1)} \;=\; v_{\underline{n},n'}^{(1)} \;=\; 1, \quad v_{\underline{n},\underline{n}'}^{(1)} \;=\; -1, \tag{4.3}$$

$$v_{n,n'}^{(2)} \;=\; v_{n,\underline{n}'}^{(2)} \;=\; 1, \quad v_{\underline{n},n'}^{(2)} \;=\; v_{\underline{n},\underline{n}'}^{(2)} \;=\; (-1)^{n'|Z|/|Z(G)|}. \tag{4.4}$$

4.1 The case of $G = A_r = SU(r+1)$

The Lie algebra $\mathbf{g} = \mathfrak{su}(r+1)$ is composed of the hermitian traceless $(r+1) \times (r+1)$-matrices. The Cartan algebra $\mathbf{t} \subset \mathbf{g}$ is chosen in the standard way as composed from the diagonal matrices in $\mathbf{g}$. We shall denote by e_i, $i = 1, \ldots, r+1$, the diagonal matrices with the matrix elements $(e_i)_{j,j'} = \delta_{i,j}\delta_{i,j'}$. The scalar product $\operatorname{tr} e_i e_{i'} = \delta_{i,i'}$ defines the Killing form on $\mathbf{t}$ with the required normalization. The center $Z(G) \cong \mathbb{Z}_{r+1}$ is generated by the element $z = \mathrm{e}^{-2\pi \mathrm{i}\, \lambda_r^\vee} = \mathrm{e}^{-\frac{2\pi \mathrm{i}}{r+1}}$, with the simple (co)weights

$$\lambda_i^\vee \;=\; \sum_{j=1}^{i} e_j - \frac{i}{r+1} \sum_{j=1}^{r+1} e_j$$

corresponding to the standard choice of the simple (co)roots[4] $\alpha_i = e_i - e_{i+1}$. The positive Weyl alcove $\mathcal{A}$ is the simplex in $\mathbf{t}$ with the vertices $\tau_0 = 0$ and $\tau_i = \lambda_i^\vee$. For $\tau \in \mathcal{A}$, the relations (2.24) and (2.29) hold for

$$(w_z)_{j,j'} \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}\, r}{r+1}}\, \delta_{j-1,[j']}\,, \qquad (w_\kappa)_{j,j'} \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}\, r}{2}}\, \delta_{j,r+2-j'}$$

---

[4]We shall always identify the Cartan algebra $\mathbf{t}$ with its dual using the Killing form tr.

and for the transformations of the positive Weyl alcove acting on the vertices of $\mathcal{A}$ by

$$z\tau_i = \tau_{[i+1]}, \qquad \kappa\tau_i = \tau_{[r-i+1]}.$$

The corresponding index transformations induce, respectively, the symmetry of the extended Dynkin diagram and the symmetry of the unextended one that are depicted in Fig.1 and Fig.2.
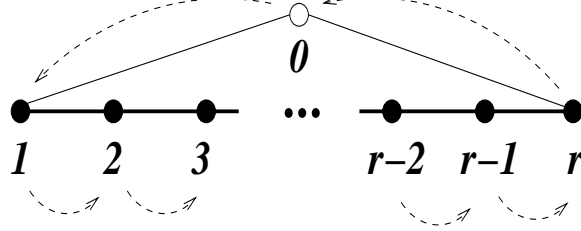

Figure 1: The rotation of the extended Dynkin diagram of A_r under z.
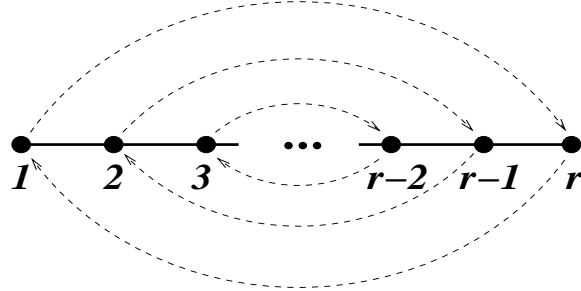
Figure 2: The reflection of the Dynkin diagram of A_r under κ.

For the maximal orientifold group $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_{r+1}$ with the action of the generator z_0 of $\mathbb{Z}_2$ given by (2.27), we shall define w_n and $w_{\underline{n}}$ according to (4.1). To satisfy the relation (2.33) for $\gamma, \gamma' = n, \underline{n}$, one may take

$$b_{n,n'} = b_{\underline{n},n'} = \begin{cases} 0 & \text{if } n+n' < r+1, \\ \frac{r(r+1)}{2}\lambda_r^\vee & \text{if } n+n' \geq r+1, \end{cases} \tag{4.5}$$

$$b_{n,\underline{n}'} = \begin{cases} rn\lambda_r^\vee & \text{if } n' \geq n, \\ r\left(n+\frac{r+1}{2}\right)\lambda_r^\vee & \text{if } n' < n, \end{cases} \tag{4.6}$$

$$b_{\underline{n},\underline{n}'} = \begin{cases} r\left(n+n_0+\frac{r+1}{2}\right)\lambda_r^\vee & \text{if } n' \geq n, \\ r\left(n+n_0\right)\lambda_r^\vee & \text{if } n' < n. \end{cases}$$

Using the identity

$$\mathrm{tr}\left(\lambda_i^\vee \lambda_r^\vee\right) = \frac{i}{r+1}, \tag{4.7}$$

one obtains from (2.34) the explicit form of the obstruction cocycle on the group $\Gamma =$

$\mathbb{Z}_2 \ltimes \mathbb{Z}_{r+1}$:

$$u_{n,n',n''} = \Phi^{n\frac{n'+n''-[n'+n'']}{r+1}} = u_{n,\underline{n}',n''}, \tag{4.8}$$

$$u_{\underline{n},n',n''} = \Phi^{(n_0+n)\frac{n'+n''-[n'+n'']}{r+1}} = u_{\underline{n},\underline{n}',n''}, \tag{4.9}$$

$$u_{n,n',\underline{n}''} = \Phi^{n\frac{n''-n'-[n''-n']}{r+1}}\Psi^{-nn'}, \tag{4.10}$$

$$u_{\underline{n},n',\underline{n}''} = \Phi^{(n_0+n)\frac{n''-n'-[n''-n']}{r+1}}\Psi^{(n_0+n)n'}, \tag{4.11}$$

$$u_{n,\underline{n}',\underline{n}''} = \Phi^{n\left(1+\frac{n''-n'-[n''-n']}{r+1}\right)}\Psi^{-n(n_0+n')}, \tag{4.12}$$

$$u_{\underline{n},\underline{n}',\underline{n}''} = \Phi^{(n_0+n)\left(1+\frac{n''-n'-[n''-n']}{r+1}\right)}\Psi^{(n_0+n)(n_0+n')}, \tag{4.13}$$

where $\Phi \equiv (-1)^{kr}$ and $\Psi \equiv e^{\frac{2\pi i k}{r+1}}$. For the case when $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_m$ for a proper subgroup $\mathbb{Z}_m \subset \mathbb{Z}_{r+1}$ composed of elements n that are multiples of $\frac{r+1}{m}$, the obstruction cocycle is obtained by restriction of n, n' and n'' to such values.

A necessary condition for the solvability of the cohomological equation (3.1) is the triviality of the cohomology class $[u_{n,n',n''}] \in H^3(Z, U(1))$. This condition was analyzed in [13, 14] where it was shown that it implies that

$$\text{k is even if } m \text{ is even and } \frac{r+1}{m} \text{ is odd.} \tag{4.14}$$

The latter restriction means that $kr\frac{r+1}{m}$ is even so that the factors Φ^n in the expression for $u_{\gamma,\gamma',\gamma''}$ may be replaced by 1 and that $u_{n,n',n''} \equiv 1$, in particular.

Another necessary condition for the solvability of (3.1) is the trivializability of the restriction of $u_{\gamma,\gamma',\gamma''}$ to $\mathbb{Z}_2 \subset \Gamma$, i.e. to $\gamma, \gamma', \gamma'' = 0, \underline{0}$. This, however, always holds because of the triviality of the cohomology group $H^3(\mathbb{Z}_2, U(1)_\epsilon)$, see (3.7). The 2-cochain on $\mathbb{Z}_2$ which trivializes the restricted 3-cocycle is

$$\tilde{v}_{0,0} = \tilde{v}_{\underline{0},0} = \tilde{v}_{0,\underline{0}} = 1, \qquad \tilde{v}_{\underline{0},\underline{0}} = \pm\Psi^{-\frac{1}{2}n_0\left(n_0+\frac{r(r+1)}{2}\right)},$$

with the two signs giving cohomologically inequivalent 2-cochains. All other trivializing 2-cochains differ from them by 2-coboundaries (recall that $H^2(\mathbb{Z}_2, U(1)_\epsilon) = \mathbb{Z}_2$). Note again that the triviality of $H^3(\mathbb{Z}_2, U(1)_\epsilon)$ implies that if the orbifold subgroup Z is trivial then the cohomological equation (3.1) is always solvable.

Returning to the case of non-trivial $Z \cong \mathbb{Z}_m$, further simplification of the 3-cocycle (4.8)-(4.13) may be achieved by extracting from it the coboundary $(\delta v')_{\gamma,\gamma',\gamma''}$ for

$$v'_{n,n'} = \Psi^{nn'}, \qquad v'_{\underline{n},n'} = \Psi^{-nn'}, \tag{4.15}$$

$$v'_{n,\underline{n}'} = \Psi^{nn'}c_n, \qquad v'_{\underline{n},\underline{n}'} = \pm\Psi^{-\frac{1}{2}n_0\left(n_0+\frac{r(r+1)}{2}\right)}\Psi^{-n(n_0+n')}c_n^{-1}, \tag{4.16}$$

where $c_n = \Psi^{-\frac{1}{2}\left(n^2+(r+1)n\right)}$ satisfies

$$c_n\, c_{[n+n']}^{-1}\, c_{n'} = \Psi^{nn'}, \qquad c_{[-n]} = c_n. \tag{4.17}$$

Note that the lift of the 2-cochain $\tilde{v}$ to Γ appears as an explicit factor in v'. Writing

$$u_{\gamma,\gamma',\gamma''} = u'_{\gamma,\gamma',\gamma''} \, (\delta v')_{\gamma,\gamma',\gamma''} \,,$$

we obtain the following formulae by a straightforward calculation using (4.15)-(4.17):

$$u'_{n,n',n''} = u'_{n,\underline{n}',n''} = u'_{n,n',\underline{n}''} = u'_{n,\underline{n}',\underline{n}''} = 1 \,,$$

$$u'_{\underline{n},n',n''} = u'_{\underline{n},\underline{n}',n''} = \Phi^{n_0 \frac{n'+n''-[n'+n'']}{r+1}} \,,$$

$$u'_{\underline{n},n',\underline{n}''} = u'_{\underline{n},\underline{n}',\underline{n}''} = \Phi^{n_0 \frac{n''-n'-[n''-n']}{r+1}} \,.$$

If m is odd then the cocycle $u'_{\gamma,\gamma',\gamma''}$ may be trivialized by setting

$$v''_{n,n'} = v''_{n,\underline{n}'} = 1 \,, \qquad v''_{\underline{n},n'} = v''_{\underline{n},\underline{n}'} = \Phi^{n_0 \frac{mn'}{r+1}} \,. \tag{4.18}$$

Indeed, using the fact that $\Phi = \pm 1$ and that

$$\Phi^{n_0 \frac{mn}{r+1}} \, \Phi^{-n_0 \frac{m[n+n']}{r+1}} \, \Phi^{n_0 \frac{mn'}{r+1}} = \Phi^{\frac{n+n'-[n+n']}{r+1}} \,,$$

for m odd, one easily verifies that

$$u'_{\gamma,\gamma',\gamma''} = (\delta v'')_{\gamma,\gamma',\gamma''} \,.$$

If m is even and $\frac{r+1}{m}$ is odd then the condition (4.14) implies that $\Phi = 1$ so that the cocycle $u'_{\gamma,\gamma',\gamma''}$ is trivial. For m and $\frac{r+1}{m}$ even, however, there exists a further obstruction to the trivializability of $u'_{\gamma,\gamma',\gamma''}$ that is related to the choice of the twist element $\zeta = n_0$ in the action (2.27). The analysis of the cohomology group $H^3(\Gamma, U(1)_\epsilon)$ done in Sect. 3 showed that such an obstruction has to lie in $\mathbb{Z}_2$ since the part of the obstruction related to the orbifold group has already been removed by the condition (4.14). To identify it, we note that the combination X of (4.2) calculated for the cocycle $u'_{\gamma,\gamma',\gamma''}$ and $n = \frac{r+1}{2}$ is equal to Φ^{n_0} since $u'_{\underline{0},n,n}$ contributes the only non-trivial factor to it. One obtains this way the equality $\Phi^{n_0} = 1$ showing that if $u'_{\gamma,\gamma',\gamma''}$ is a coboundary then

$$\mathsf{k}\, n_0 \quad \text{is even} \quad \text{if} \quad m \quad \text{is even} \quad \text{and} \quad \frac{r+1}{m} \quad \text{is even.} \tag{4.19}$$

In that case, $v''_{\gamma,\gamma'}$ may be taken trivial or, which amounts to the same, given by (4.18). Note that $\mathsf{k}\frac{r+1}{m}$ is even for m even due to the restriction (4.14) so that the condition (4.19) holds or fails simultaneously for all n_0 in the same congruence class modulo $\frac{r+1}{m}$, in agreement with the equivalence of the Γ actions for the twist elements in the same Z-coset that we discussed in Sect. 2.5.

To summarize, for the orientifold group $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_m$, the triviality of the cohomology class $[u_{\gamma,\gamma',\gamma''}] \in H^3(\Gamma, U(1)_\epsilon)$ imposes the conditions (4.14) and (4.19). If they are satisfied then the 2-cochain trivializing $u_{\gamma,\gamma',\gamma''}$ may be taken in the form

$$v_{\gamma,\gamma'} = v'_{\gamma,\gamma'} \, v''_{\gamma,\gamma'} \,,$$

where v' and v'' are given by (4.15)-(4.16) and (4.18), respectively. For m odd, the two choices of the sign in (4.16) give two cohomologically inequivalent trivializing cochains

from which other trivializing cochains differ by 2-coboundaries. Indeed, the sign change is induced by multiplication of $v_{\gamma,\gamma'}$ by the 2-cocycle $v^{(1)}_{\gamma,\gamma'}$ given by (4.3). For m even, further two cohomologically inequivalent solutions are obtained by additionally multiplying $v_{\gamma,\gamma'}$ by the 2-cocycle $v^{(2)}_{\gamma,\gamma'}$ given by (4.4).

Let us illustrate how the above analysis provides concrete information about the numbers of inequivalent orientifold gerbes on a few examples of the A_r groups of low ranks.

For $G = SU(2)$, if $\Gamma = \mathbb{Z}_2$ with its generator acting by (2.27) then there are two inequivalent Γ-equivariant (or Jandl) structures on the gerbe of level k on $SU(2)$ for each k and each of the two choices of the twist element ζ. For $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2$ with the second factor being the center of $SU(2)$, the condition (4.14) imposes that the level k be even. For each of the two choices of the twist element ζ, there are then 4 inequivalent Γ-equivariant structures. The different choices of the twist element lead to equivalent actions of Γ on $SU(2)$ and there are altogether 4 inequivalent Jandl structures on the induced gerbe on $SO(3)$, see the discussion in Sect. 2.5. These results are in agreement with the analysis of refs. [31] and [26, 27].

For $G = SU(3)$, there are no obstructions. There are two inequivalent Γ-structures on the gerbe on G for $\Gamma = \mathbb{Z}_2$ or $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_3$ for each level k and each of the three choices of the twist element. For $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_3$, different choices of the twist element lead to equivalent Γ-actions. Consequently, there are, altogether, two inequivalent Jandl structures on the induced gerbe on the quotient group $SU(3)/\mathbb{Z}_3$ for each k.

For $G = SU(4)$, there are two inequivalent Jandl structures on the gerbe on G for each level k and each of the four choices of the twist element. For $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2$, there are four inequivalent Γ-equivariant structures for each k even and each choice of the twist element in $\mathbb{Z}_4$, and for each k odd and each twist element in $\mathbb{Z}_2 \subset \mathbb{Z}_4$. There are no Γ-equivariant structures for k odd and twist elements in $\mathbb{Z}_4 \setminus \mathbb{Z}_2$. We get this way eight inequivalent Jandl structures on the induced gerbe on the quotient group $SU(4)/\mathbb{Z}_2$ if k is even and four if k is odd. Finally, if $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_4$ there are four inequivalent Γ-equivariant structures for k even and each of the four choices of the twist element. Overall, they give rise to four inequivalent Jandl structures on the induced gerbe on $SU(4)/\mathbb{Z}_4$. There are no Γ-equivariant structures for k odd.

4.2 The case of $G = B_r = Spin(2r + 1)$

The Lie algebra $\mathbf{g} = \mathfrak{spin}(2r + 1)$ is composed of the imaginary antisymmetric $(2r + 1) \times (2r + 1)$-matrices. We shall denote by e_i, $i = 1, \ldots, r$, the matrices with the matrix elements $(e_i)_{j,j'} = \mathsf{i}(\delta_{j,2i}\delta_{2i-1,j'} - \delta_{j,2i-1}\delta_{2i,j'})$ that span the Cartan algebra $\mathbf{t} \subset \mathbf{g}$. The Killing form is normalized so that $\operatorname{tr} e_i e_{i'} = \delta_{i,i'}$. The center is $Z(G) \cong \mathbb{Z}_2$ with the generator $z = \mathrm{e}^{-2\pi \mathsf{i}\, \lambda_1^\vee}$, where

$$\lambda_i^\vee = \sum_{j=1}^{i} e_i$$

are the simple coweights corresponding to the simple roots $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, r-1$ and $\alpha_r = e_r$. We have $Spin(2r + 1)/\mathbb{Z}_2 = SO(2r + 1)$. The vertices of the positive Weyl alcove are $\tau_0 = 0$, $\tau_1 = \lambda_1^\vee$ and $\tau_i = \frac{1}{2}\lambda_i^\vee$ for $i = 2, \ldots, r$. For $\tau \in \mathcal{A}$, the relations (2.24)

and (2.29) hold for w_z and w_κ that project to the $SO(2r+1)$ matrices

$$(w_z)_{j,j'} = -(-1)^{\delta_{1j}}\delta_{j,j'}, \qquad (4.20)$$

$$(w_\kappa)_{j,j'} = \sum_{i=1}^{r}(\delta_{j,2i}\delta_{2i-1,j'} + \delta_{j,2i-1}\delta_{2i,j'}) + (-1)^r\delta_{j,2r+1}\delta_{2r+1,j'},$$

and for the transformations of the positive Weyl alcove acting on the vertices of $\mathcal{A}$ by

$$z\tau_0 = \tau_1, \quad z\tau_1 = 0, \quad z\tau_i = \tau_i \quad \text{for} \quad i = 2,\ldots,r, \qquad \kappa\tau_i = \tau_i \quad \text{for} \quad i = 0,\ldots,r.$$

The symmetry of the extended Dynkin diagram corresponding to the index transformations under z is represented in Fig.3. The index transformation under κ induces a trivial symmetry of the Dynkin diagram. It is easy to see, by calculating first the eigenvalues of
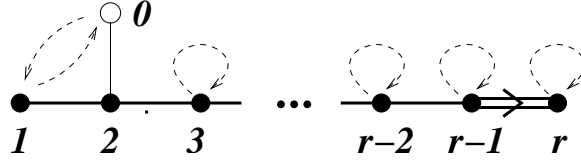

Figure 3: The transformation of the extended Dynkin diagram of B_r under z.

the projections of w_z and w_κ to $SO(2r+1)$, that

$$w_z = z^{n_z} O_z \, \mathrm{e}^{\pi i \lambda_r^\vee} O_z^{-1}, \qquad w_\kappa = z^{n_\kappa} O_\kappa \, \mathrm{e}^{\pi i \lambda_{r'}^\vee} O_\kappa^{-1},$$

where n_z, $n_\kappa = 0$ or 1, O_z, $O_\kappa \in Spin(2r+1)$ and $r' = \frac{r}{2}$ for even r and $r' = \frac{r+1}{2}$ for odd r. The coroot lattice of B_r is spanned by the simple coroots $\alpha_i^\vee = e_i - e_{i+1}$ for $i = 1,\ldots,r-1$, and $\alpha_r^\vee = 2e_r$. By checking that the coweights $\lambda_r^\vee$ and $\lambda_{r'}^\vee$ belong to the coroot lattice if and only if, respectively, r and r' are even, one infers from the above relations that

$$w_z^2 = z^r, \qquad w_\kappa^2 = z^{r'}.$$

As far as $(w_\kappa w_z)^2$ is concerned, we note that it projects to the same matrix in $SO(2r+1)$ as $\mathrm{e}^{\pi i \lambda_1^\vee}$ so that

$$(w_\kappa w_z)^2 = \mathrm{e}^{\pm \pi i \lambda_1^\vee}$$

for some choice of the sign.

For the maximal orientifold group $\mathbb{Z}_2 \ltimes \mathbb{Z}_2$, we define w_n and $w_{\underline{n}}$ for $n = 0, 1$ according to (4.1). One can satisfy the relation (2.33) by taking

$$b_{n,n} = b_{\underline{n},n'} = m_{n,n'}\lambda_1^\vee,$$
$$b_{n,\underline{n}'} = (\tfrac{1}{2}\delta_{n,1} + m_{n,\underline{n}'})\lambda_1^\vee,$$
$$b_{\underline{n},\underline{n}'} = (\tfrac{1}{2}\delta_{[n_0+n],1} + m_{\underline{n},\underline{n}'})\lambda_1^\vee$$

where $m_{n,n'}, m_{n,\underline{n}'}, m_{\underline{n},\underline{n}'}$ are integers. Since

$$\tau_{z^{-n}0} = \delta_{[n],1}\lambda_1^\vee, \qquad \tau_{(z_0 z^n)^{-1}0} = \delta_{[n_0+n],1}\lambda_1^\vee,$$

and $\operatorname{tr}(\lambda_1^\vee)^2 = 1$, one readily sees that the contribution of the integer multiplicities of $\lambda_1^\vee$ to $b_{\gamma,\gamma'}$ drops out from the expression (2.34) for the obstruction cocycle which, accordingly, takes the following form:

$$u_{n,n',n''} = u_{\underline{n},n',n''} = u_{n,\underline{n}',n''} = u_{\underline{n},\underline{n}',n''} = 1, \tag{4.21}$$

$$u_{n,n',\underline{n}''} = (-1)^{k\,nn'}, \qquad u_{\underline{n},n',\underline{n}''} = (-1)^{k\,(n_0+n)n'}, \tag{4.22}$$

$$u_{n,\underline{n}',\underline{n}''} = (-1)^{k\,n(n_0+n')}, \qquad u_{\underline{n},\underline{n}',\underline{n}''} = (-1)^{k\,(n_0+n)(n_0+n')}. \tag{4.23}$$

The cohomological equation (2.23) can be always solved. Two cohomologically inequivalent solutions are obtained by taking

$$v_{n,n'} = v_{\underline{n},n'} = (-1)^{k\,nn'}, \qquad v_{n,\underline{n}'} = (-1)^{k\,nn'}\,\mathrm{e}^{-\frac{3\pi i}{2}k\,n},$$

$$v_{\underline{n},\underline{n}'} = \pm(-1)^{k\,n(n_0+n')}\,\mathrm{e}^{\frac{\pi i}{2}k\,(n_0+3n)}.$$

Two further cohomologically inequivalent solutions for the maximal orientifold group $\mathbb{Z}_2 \ltimes Z(G)$ are obtained by multiplying $v_{\gamma,\gamma'}$ by the 2-cocycle $v_{\gamma,\gamma'}^{(2)}$ given by (4.4).

In summary, there are no obstructions to the trivialization of the 3-cocycle (4.21)-(4.23) on $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2$. For each k and each choice of the twist element $\zeta \in \mathbb{Z}_2$, there are four cohomologically inequivalent trivializing cochains that give rise to inequivalent Γ-equivariant structures on the level k gerbe on $Spin(2r+1)$. The latter induce altogether four inequivalent Jandl structures on the level k gerbe on $SO(2r+1)$. Restriction to the inversion group $\Gamma = \mathbb{Z}_2$ reduces the number of inequivalent trivializing cochains to two for each k and each ζ. Altogether, they induce four inequivalent Jandl structures on the level k gerbe on $Spin(2r+1)$.

4.3 The case of $G = C_r = Sp(2r)$

The group $Sp(2r)$ is composed of the unitary $(2r) \times (2r)$-matrices such that $U^T \Omega U = \Omega$ for

$$(\Omega)_{j,j'} = \sum_{i=1}^r (\delta_{j,2i-1}\delta_{2i,j'} - \delta_{j,2i}\delta_{2i-1,j'}).$$

The Lie algebra $\mathfrak{sp}(2r)$ of $Sp(2r)$ is composed of the hermitian matrices X such that ΩX is symmetric. The Cartan subalgebra $\mathfrak{t} \subset \mathfrak{sp}(2r)$ is spanned by matrices e_i, $i = 1, \ldots, r$, with $(e_i)_{j,j'} = \mathrm{i}(\delta_{j,2i}\delta_{2i-1,j'} - \delta_{j,2i-1}\delta_{2i,j'})$ and the Killing form is normalized so that $\operatorname{tr} e_i e_{i'} = 2\delta_{ij}$. The center $Z(G) \cong \mathbb{Z}_2$ with the generator $z = \mathrm{e}^{-2\pi i\,\lambda_r^\vee} = -1$, where

$$\lambda_i^\vee = \sum_{j=1}^i e_j \quad \text{for} \quad i = 1, \ldots, r-1, \qquad \lambda_r^\vee = \frac{1}{2}\sum_{j=1}^r e_j$$

are the coweights corresponding to the simple roots $\alpha_i = \frac{1}{2}(e_i - e_{i+1})$ for $i = 1, \ldots, r-1$ and $\alpha_r = e_r$. The vertices of the positive Weyl alcove $\mathcal{A}$ are $\tau_0 = 0$, $\tau_i = \frac{1}{2}\lambda_i^\vee$ for $i = 1, \ldots, r-1$ and $\tau_r = \lambda_r^\vee$. To satisfy the relations (2.24) and (2.29) for $\tau \in \mathcal{A}$, we may take for w_z and w_κ the matrices with the elements

$$(w_z)_{j,j'} = \mathrm{i}\,\delta_{j,2r+1-j'}, \qquad (w_\kappa)_{j,j'} = \mathrm{i}\sum_{i=1}^r (\delta_{j,2i-1}\delta_{2i,j'} + \delta_{j,2i}\delta_{2i-1,j'})$$

and the actions of z and κ on the positive Weyl alcove reducing to

$$z\tau_i = \tau_{r-i}, \qquad \kappa\tau_i = \tau_i$$

on the vertices. The symmetry of the extended Dynkin diagram corresponding to the action of z is depicted in Fig.4. Note that $w_z^2 = w_\kappa^2 = -1 = z$ and $(w_\kappa w_z)^2 = 1$. Defining w_n
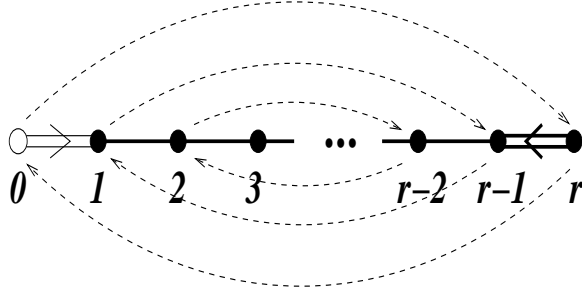

Figure 4: The transformation of the extended Dynkin diagram of C_r under z.

and $w_{\underline{n}}$ for the orientifold group $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2$ by (4.1), we may satisfy (2.33) by taking

$$b_{n,n'} = b_{n,\underline{n}'} = b_{\underline{n},n'} = nn'\lambda_r^\vee, \qquad b_{\underline{n},\underline{n}'} = (1 + n_0 + nn')\lambda_r^\vee.$$

Since

$$\tau_{z^{-n}0} = \delta_{[n],1}\lambda_r^\vee, \qquad \tau_{(z_0 z^n)^{-1}0} = \delta_{[n_0+n],1}\lambda_r^\vee,$$

and $\operatorname{tr}(\lambda_r^\vee)^2 = \frac{r}{2}$ one infers from (2.34) that

$$u_{n,n',n''} = u_{n,\underline{n}',n''} = u_{n,n',\underline{n}''} = (-1)^{\mathsf{k} r\, nn'n''}, \tag{4.24}$$

$$u_{\underline{n},n',n''} = u_{\underline{n},\underline{n}',n''} = u_{\underline{n},n',\underline{n}''} = (-1)^{\mathsf{k} r\, (n_0+n)n'n''}, \tag{4.25}$$

$$u_{n,\underline{n}',\underline{n}''} = (-1)^{\mathsf{k} r\, n(1+n_0+n'n'')}, \qquad u_{\underline{n},\underline{n}',\underline{n}''} = (-1)^{\mathsf{k} r\, (n_0+n)(1+n_0+n'n'')}. \tag{4.26}$$

The restriction $u_{n,n',n''}$ of the cocycle $u_{\gamma,\gamma',\gamma''}$ to the orbifold subgroup $\mathbb{Z}_2$ is trivializable if and only if

$$\mathsf{k} \quad \text{is even} \quad \text{if} \quad r \quad \text{is odd,}$$

see [14]. Under this condition, $u_{\gamma,\gamma',\gamma''} \equiv 1$ and four cohomologically inequivalent solutions of (2.23) may be given by the formulae:

$$v_{n,n'} = v_{n,\underline{n}'} = 1, \qquad v_{\underline{n},n'} = \sigma^{n'}, \qquad v_{\underline{n},\underline{n}'} = \underline{\sigma}\,\sigma^{n'}$$

with $\sigma, \underline{\sigma} = \pm 1$. They lead to four inequivalent $\mathbb{Z}_2 \ltimes \mathbb{Z}_2$-equivariant structures on the level k gerbe on $Sp(2r)$ for each choice of the twist element $\zeta = \mathbb{Z}_2$ and, altogether, to four inequivalent Jandl structures on the quotient gerbe on $Sp(2r)/\mathbb{Z}_2$.

The restriction of the 3-cocycle (4.24)-(4.26) to the inversion group $\Gamma = \mathbb{Z}_2$ is trivial for any level k and any choice of the twist element $\zeta \in \mathbb{Z}_2$. For such a restriction, the two cohomologically inequivalent solutions of (2.23) are given by

$$v_{0,0} = v_{0,\underline{0}} = v_{\underline{0},0} = 1, \qquad v_{\underline{0},\underline{0}} = \pm 1. \tag{4.27}$$

Altogether, they lead to four inequivalent Jandl structures on the level k gerbe on $Sp(2r)$.

4.4 The case of $G = D_r = Spin(2r)$

The Lie algebra $\mathbf{g} = \mathfrak{spin}(2r)$ is composed of the imaginary antisymmetric $(2r) \times (2r)$-matrices, with the Cartan subalgebra $\mathbf{t} \subset \mathbf{g}$ spanned by the matrices e_i, $i = 1, \ldots, r$, with the matrix elements $(e_i)_{j,j'} = \mathrm{i}(\delta_{j,2i}\delta_{2i-1,j'} - \delta_{j,2i-1}\delta_{2i,j'})$ and $\mathrm{tr}\, e_i e_{i'} = \delta_{i,i'}$. The vertices of the positive Weyl alcove are

$$\tau_0 = 0, \quad \tau_1 = \lambda_1^\vee, \quad \tau_i = \tfrac{1}{2}\lambda_i^\vee \quad \text{for} \quad i = 2, \ldots, r-2,$$

$$\tau_{r-1} = \lambda_{r-1}^\vee, \quad \tau_r = \lambda_r^\vee,$$

where

$$\lambda_i^\vee = \sum_{j=1}^{i} e_i \quad \text{for} \quad i = 1, \ldots, r-2,$$

$$\lambda_{r-1}^\vee = \tfrac{1}{2}\sum_{j=1}^{r-1} e_j - \tfrac{1}{2}e_r, \quad \lambda_r^\vee = \tfrac{1}{2}\sum_{j=1}^{r} e_j$$

are the simple coweights corresponding to the simple roots $\alpha_i = e_i - e_{i+1}$, $i = 1, \ldots, r-1$, $\alpha_r = e_{r-1} + e_r$ that coincide with the simple coroots. The subsequent discussion depends on the parity of r and hence will be split into two parts.

4.4.1 The subcase of r **odd**

For $r = 2s+1$, the center $Z(G) \cong \mathbb{Z}_4$ is generated by $z = \mathrm{e}^{-2\pi \mathrm{i}\,\lambda_r^\vee}$, with $Spin(2r)/\{1, z^2\} = SO(2r)$. For $\tau \in \mathcal{A}$, the relations (2.24) and (2.29) are satisfied if we take for w_z and w_κ the elements of $Spin(2r)$ that project to matrices in $SO(2r)$ with the elements

$$(w_z)_{j,j'} = (-1)^{\delta_{j,2r}}\delta_{j,2r+1-j'}, \tag{4.28}$$

$$(w_\kappa)_{j,j'} = \sum_{i=1}^{r-1}(\delta_{j,2i-1}\delta_{2i,j'} + \delta_{j,2i}\delta_{2i-1,j'}) + \delta_{j,2r-1}\delta_{2r-1,j'} + \delta_{j,2r}\delta_{2r,j'},$$

with the actions of z and κ on the positive Weyl alcove reducing to

$$z\tau_0 = \tau_{r-1}, \quad z\tau_1 = \tau_r, \quad z\tau_i = \tau_{r-i} \quad \text{for} \quad i = 2, \ldots, r,$$

$$\kappa\tau_i = \tau_i \quad \text{for} \quad i = 0, \ldots, r-2, \quad \kappa\tau_{r-1} = \tau_r, \quad \kappa\tau_r = \tau_{r-1}.$$

on the vertices. The corresponding symmetries of the Dynkin diagrams are depicted in Fig.5 and Fig.6. Note the adjoint action

$$w_z\, e_i\, w_z^{-1} = -(-1)^{\delta_{i,1}}\, e_{r+1-i}. \tag{4.29}$$

It is easy to see, comparing first the eigenvalues of the projections of both sides to $SO(2r)$, that

$$w_z = z^{2n_z}\, O_z\, \mathrm{e}^{2\pi \mathrm{i}\,\tau_z}\, O_z^{-1} \qquad \text{for} \quad \tau_z = \tfrac{1}{2}\sum_{i=1}^{s} e_i + \tfrac{1}{4}e_{s+1},$$

$$w_\kappa = z^{2n_\kappa}\, O_\kappa\, \mathrm{e}^{2\pi \mathrm{i}\,\tau_\kappa}\, O_z^{-1} \qquad \text{for} \quad \tau_\kappa = \tfrac{1}{2}\sum_{i=1}^{s} e_i,$$

$$w_\kappa w_z = z^{2n_{\kappa z}}\, O_{\kappa z}\, \mathrm{e}^{2\pi \mathrm{i}\,\tau_{\kappa z}}\, O_{\kappa z}^{-1} \qquad \text{for} \quad \tau_{\kappa z} = \tfrac{1}{2}\sum_{i=1}^{s-1} e_i + \tfrac{3}{8}e_s + \tfrac{1}{8}e_{s+1}$$
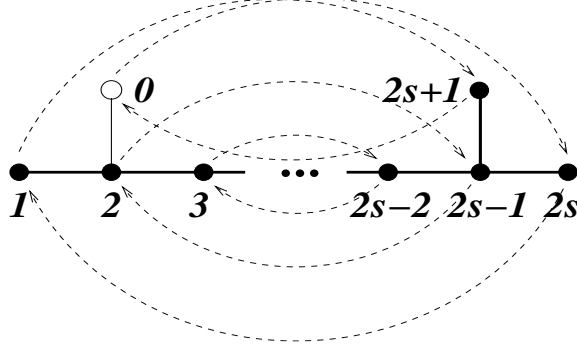
Figure 5: The transformation of the extended Dynkin diagram of D_{2s+1} under z.
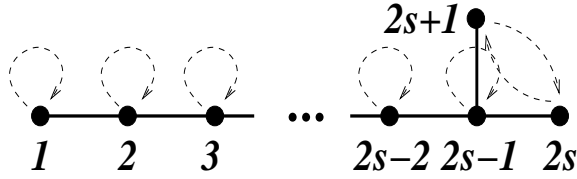

Figure 6: The inversion map κ flips the last nodes of the Dynkin diagram of D_{2s+1}.

for some integers $n_z, n_\kappa, n_{\kappa z}$ and $O_z, O_\kappa, O_{\kappa z} \in Spin(2r)$. The Cartan algebra elements τ_z, τ_κ and $\tau_{\kappa z}$ belong to the positive Weyl alcove $\mathcal{A}$. It is easy to see that $4\tau_z$ belongs also to the coweight lattice but not to the coroot lattice. On the other hand, $2\tau_\kappa = \lambda_s^\vee$ belongs to the coroot lattice if and only if s is even. Since w_z^4 and w_κ^2 project to the identity matrix in $SO(2r)$, it follows that

$$ w_z^4 = z^2, \qquad w_\kappa^2 = z^{2s}. \tag{4.30} $$

We also have

$$ O_{\kappa z}^{-1} (w_\kappa w_z)^2 O_{\kappa z} = e^{2\pi i (2\tau_{\kappa z})} = \begin{cases} e^{2\pi i (e_1 + \frac{3}{4} e_s + \frac{1}{4} e_{s+1})} & \text{for} \quad s \quad \text{even,} \\ e^{2\pi i (\frac{3}{4} e_s + \frac{1}{4} e_{s+1})} & \text{for} \quad s \quad \text{odd} \end{cases} $$

$$ = z^{2(s+1)} e^{\frac{\pi i}{2}(3e_s + e_{s+1})} = z^{2(s+1)} O e^{\frac{\pi i}{2}(3e_1 + e_2)} O^{-1} $$

for $O \in Spin(2r)$ that is straightforward to construct. By the relation (2.24),

$$ z^2 e^{\frac{\pi i}{2}(3e_1 + e_2)} = z^2 e^{2\pi i (\frac{1}{2}\tau_1 + \frac{1}{2}\tau_2)} = w_z^{-2} e^{2\pi i (\frac{1}{2}\tau_0 + \frac{1}{2}\tau_2)} w_z^2 $$

$$ = w_z^{-2} e^{\frac{\pi i}{2}(e_1 + e_2)} w_z^2 = O' e^{\frac{\pi i}{2}(e_1 + e_r)} O'^{-1}. $$

We infer that $(w_\kappa w_z)^2$ is in the same conjugacy class as $z^{2s} e^{\frac{\pi i}{2}(e_1 + e_r)}$ and that the latter is different from the conjugacy class of $z^{2(s+1)} e^{\frac{\pi i}{2}(e_1 + e_r)}$. On the other hand, it is easy to check that $(w_\kappa w_z)^2$ projects to the same matrix in $SO(2r)$ as $e^{\frac{\pi i}{2}(e_1 + e_r)}$. It follows that

$$ (w_\kappa w_z)^2 = z^{2s} e^{\frac{\pi i}{2}(e_1 + e_r)} $$

which, together with the second equality in (4.30), implies that

$$ w_\kappa w_z w_\kappa^{-1} = e^{\frac{\pi i}{2}(e_1 + e_r)} w_z^{-1}. $$

Using (4.29), we obtain the relations:

$$w_\kappa w_z^n w_\kappa^{-1} w_z^n = e^{2\pi i \Delta_n^+}, \qquad w_z^n w_\kappa w_z^n w_\kappa^{-1} = e^{2\pi i \Delta_n^-},$$

where

$$\Delta_n^\pm = \begin{cases} 0 & \text{for} \quad n = 0, \\ \pm\frac{1}{4}(e_1 \pm e_r) & \text{for} \quad n = 1, \\ \pm\frac{1}{2}e_1 & \text{for} \quad n = 2, \\ \pm\frac{1}{4}(e_1 \mp e_r) & \text{for} \quad n = 3. \end{cases}$$

Together with (4.30), they are all what is needed to find $b_{\gamma,\gamma'}$ for γ, γ' in the maximal orientifold group $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_4$. We may set

$$b_{n,n'} = b_{\underline{n},n'} = \frac{n+n'-[n+n']}{4} e_1, \tag{4.31}$$

$$b_{n,\underline{n}'} = \frac{n'-n-[n'-n]}{4} e_1 + \Delta_n^-, \tag{4.32}$$

$$b_{\underline{n},\underline{n}'} = \left(\frac{n'-n-[n'-n]}{4} + s\right) e_1 + \Delta_{[n_0+n]}^+$$

for $n, n' = 0, 1, 2, 3$. We also have

$$\tau_{z^{-1}0} = \lambda_r^\vee, \qquad \tau_{z^{-2}0} = \lambda_1^\vee, \qquad \tau_{z^{-3}0} = \lambda_{r-1}^\vee.$$

Rather than displaying the corresponding obstruction 3-cocycle (2.34) in full, we shall focus on its specific components.

First, for the inversion group $\Gamma = \mathbb{Z}_2$, the only entry of the 3-cocycle different from 1 is

$$u_{\underline{0},\underline{0},\underline{0}} = e^{\frac{\pi i}{2} k r (n_0 + 2\delta_{n_0,3})}.$$

The trivializing cochain may be given by the formulae:

$$v_{0,0} = v_{\underline{0},0} = v_{0,\underline{0}} = 1, \qquad v_{\underline{0},\underline{0}} = \pm e^{-\frac{\pi i}{4} k r (n_0 + 2\delta_{n_0,3})},$$

with the two signs corresponding to cohomologically inequivalent solutions. Next, we pass to the case of orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_m$ with $m = 2, 4$. The restriction of the obstruction 3-cocycle to the orbifold group $\mathbb{Z}_4$ is

$$u_{n,n',n''} = (-1)^{kn\frac{n'+n''-[n'+n'']}{4}}.$$

It is not trivializable if k is odd, see [14]. On the other hand, its further restriction to $\mathbb{Z}_2$ is trivial for all k. In order to proceed further, we note that the scalar product $\operatorname{tr}\tau_{z^{-n}0}e_1$ takes values in integers if n is even and in half-integers if n is odd. It follows that, for even k, only the terms $\Delta^\pm$ in $b_{\gamma,\gamma'}$ contribute to $u_{\gamma,\gamma',\gamma''}$ if $m = 4$. This is still the case if $m = 2$. Indeed, if the twist element $\zeta \in \mathbb{Z}_2 \subset \mathbb{Z}_4$ then $\operatorname{tr}\tau_{z^{-n}0}e_1$ and $\operatorname{tr}\tau_{(z_0 z^n)^{-1}}e_1$ take integral values because $n = 0, 2$. Conversely, if $\zeta \in \mathbb{Z}_4 \setminus \mathbb{Z}_2$, then a straightforward check shows that the combination X of (4.2) is equal to $(-1)^k$ for $n = 2$, thereby contradicting the trivializability of the obstruction cocycle for odd k. Summarizing, we obtain the condition

$$k \quad \text{is even if} \quad m = 4 \quad \text{or} \quad m = 2 \quad \text{and} \quad n_0 \quad \text{is odd}$$

under which only the terms $\Delta^\pm$ in $b_{\gamma,\gamma'}$ contribute to the obstruction cocycle $u_{\gamma,\gamma',\gamma''}$. With this observation in mind, we obtain, for $m = 4$ or for $m = 2$ and n_0 odd, i.e. in both cases in which k has to be even, the following expressions for the obstruction cocycle:

$$u_{n,n',n''} = u_{\underline{n},n',n''} = u_{n,\underline{n}',n''} = u_{\underline{n},\underline{n}',n''} = 1, \tag{4.33}$$

$$u_{n,n',\underline{n}''} = (-1)^{\frac{1}{2}\mathsf{k}(1-\delta_{n,0})(1-\delta_{n',0})(1-\delta_{n,n'})}, \tag{4.34}$$

$$u_{\underline{n},n',\underline{n}''} = (-1)^{\frac{1}{2}\mathsf{k}(1-\delta_{[n_0+n],0})(1-\delta_{n',0})(1-\delta_{[n_0+n],n'})}, \tag{4.35}$$

$$u_{n,\underline{n}',\underline{n}''} = (-1)^{\frac{1}{2}\mathsf{k}(1-\delta_{n,0})(1-\delta_{[n_0+n'],0})(1-\delta_{[n_0+n+n'],0})}, \tag{4.36}$$

$$u_{\underline{n},\underline{n}',\underline{n}''} = (-1)^{\frac{1}{2}\mathsf{k}(1-\delta_{[n_0+n],0})(1-\delta_{[n_0+n'],0})(1-\delta_{[2n_0+n+n'],0})}.$$

Similarly, for $m = 2$ and n_0 even, when k can be any integer,

$$u_{n,n',n''} = u_{\underline{n},n',n''} = u_{n,\underline{n}',n''} = u_{\underline{n},\underline{n}',n''} = 1, \tag{4.37}$$

$$u_{n,n',\underline{n}''} = (-1)^{\mathsf{k}\,\delta_{n,2}\,\delta_{n',2}}, \qquad u_{\underline{n},n',\underline{n}''} = (-1)^{\mathsf{k}\,\delta_{[n_0+n],2}\,\delta_{n',2}}, \tag{4.38}$$

$$u_{n,\underline{n}',\underline{n}''} = (-1)^{\mathsf{k}\,\delta_{n,2}\,\delta_{[n_0+n'],2}}, \qquad u_{\underline{n},\underline{n}',\underline{n}''} = (-1)^{\mathsf{k}\,\delta_{[n_0+n],2}\,\delta_{[n_0+n'],2}}.$$

In all these cases, there exists a trivializing cochain. It may be taken in the form:

$$v_{n,n'} = 1, \qquad v_{\underline{n},n'} = \sigma^{\frac{1}{4}mn'}, \qquad v_{n,\underline{n}'} = e^{\frac{\pi i}{4}\mathsf{k}(n+2\delta_{n,3})}, \tag{4.39}$$

$$v_{\underline{n},\underline{n}'} = \underline{\sigma}\,\sigma^{\frac{1}{4}mn'}\,e^{-\frac{\pi i}{4}\mathsf{k}([n_0+n]+2\delta_{[n_0+n],3})}$$

with different signs $\sigma, \underline{\sigma} = \pm 1$ giving four cohomologically inequivalent solutions.

In summary, for each k and each choice of the twist element $\zeta \in \mathbb{Z}_4$, there are two inequivalent Jandl structures on the level k gerbe on $Spin(2r)$ with r odd. For each k even and each choice of the twist element $\zeta \in \mathbb{Z}_4$ and for each k odd and $\zeta \in \mathbb{Z}_2 \subset \mathbb{Z}_4$, there are four inequivalent $\mathbb{Z}_2 \ltimes \mathbb{Z}_2$-equivariant structures on the level k gerbe on $Spin(2r)$, giving rise, altogether, to eight inequivalent Jandl structures on the induced gerbe on $SO(2r)$ when k is even and to four ones when k is odd. Finally, for each k even and each choice of $\zeta \in \mathbb{Z}_4$, there are four inequivalent $\mathbb{Z}_2 \ltimes \mathbb{Z}_4$-equivariant structures on the level k gerbe on $Spin(2r)$, giving rise, altogether, to four inequivalent Jandl structures on the induced gerbe on $Spin(2r)/\mathbb{Z}_4$. Note that the count is similar to that for the group $SU(4)$.

4.4.2 The subcase of r even

For $r = 2s$, the center $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by $z_1 = e^{-2\pi i \lambda_r^\vee}$ and $z_2 = e^{-2\pi i \lambda_1^\vee}$, with $Spin(2r)/\{1, z_2\} = SO(2r)$. For $\tau \in \mathcal{A}$ and $z = z_1, z_2$, the relations (2.24) are satisfied if we take for w_{z_1} and w_{z_2} the elements of $Spin(2r)$ that project to $SO(2r)$ matrices with the elements[5]

$$(w_{z_1})_{j,j'} = \begin{cases} -\delta_{j,2r+1-j'} & \text{for} \quad j = 1, \ldots, r, \\ \delta_{j,2r+1-j'} & \text{for} \quad j = r+1, \ldots, 2r, \end{cases} \tag{4.40}$$

$$(w_{z_2})_{j,j'} = \delta_{j,1}\delta_{2,j'} + \delta_{j,2}\delta_{1,j'} + \delta_{j,2r-1}\delta_{j',2r} + \delta_{j,2r}\delta_{2r-1,j'} + \sum_{i=3}^{2r-2} \delta_{j,i}\delta_{i,j'},$$

---

[5]For later convenience, we make a different choice from that in [14].

with the actions of z_1 and z_2 on the positive Weyl alcove reducing to

$$z_1 \tau_i = \tau_{r-i}, \qquad z_2 \tau_0 = \tau_1, \quad z_2 \tau_1 = \tau_0,$$
$$z_2 \tau_i = \tau_i \quad \text{for} \quad i = 2, \ldots, r-2, \qquad z_2 \tau_{r-1} = \tau_r, \quad z_2 \tau_r = \tau_{r-1}$$

on the vertices. The corresponding symmetries of the extended Dynkin diagram are depicted in Fig.7 and Fig.8. The adjoint action of w_{z_1} and w_{z_2} on the Cartan algebra is
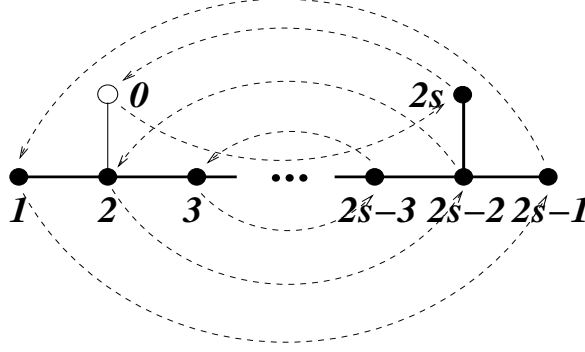
Figure 7: The transformation of the extended Dynkin diagram of D_{2s} under z_1.
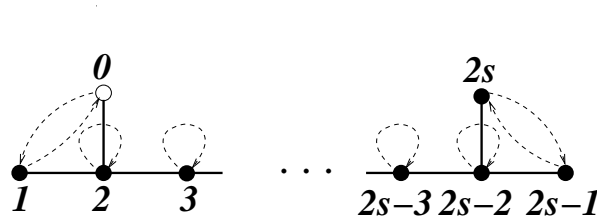
Figure 8: The transformation of the extended Dynkin diagram of D_{2s} under z_2.

given by the equations:

$$w_{z_1} e_i w_{z_1}^{-1} = -e_{r-i+1}, \qquad w_{z_2} e_i w_{z_2}^{-1} = (-1)^{\delta_{i,1} + \delta_{i,r}} e_i.$$

The relations (2.29) are, in turn, satisfied for $\tau \in \mathcal{A}$ if we take for w_κ the element of $Spin(2r)$ that projects to an $SO(2r)$ matrix with the elements

$$(w_\kappa)_{j,j'} = \sum_{i=1}^{r} (\delta_{j,2i-1}\delta_{2i,j'} + \delta_{j,2i}\delta_{2i-1,j'}),$$

with the trivial action of κ on the positive Weyl alcove. We have the relations:

$$w_{z_1} = z_2^{n_{z_1}} O_{z_1} e^{2\pi i \tau_{z_1}} O_{z_1}^{-1} \qquad \text{for} \qquad \tau_{z_1} = -\tfrac{1}{2}\lambda_r^\vee, \tag{4.41}$$

$$w_{z_2} = z_2^{n_{z_2}} O_{z_2} e^{2\pi i \tau_{z_2}} O_{z_2}^{-1} \qquad \text{for} \qquad \tau_{z_2} = -\tfrac{1}{2}\lambda_1^\vee, \tag{4.42}$$

$$w_{z_1} w_{z_2} = z_2^{n_{z_1 z_2}} O_{z_1 z_2} e^{2\pi i \tau_{z_1 z_2}} O_{z_1 z_2}^{-1} \qquad \text{for} \qquad \tau_{z_1 z_2} = \tfrac{1}{2}(\lambda_1^\vee - \lambda_r^\vee), \tag{4.43}$$

$$w_\kappa = z_2^{n_\kappa} O_\kappa e^{2\pi i \tau_\kappa} O_\kappa^{-1} \qquad \text{for} \qquad \tau_\kappa = \tfrac{1}{2}\lambda_s^\vee, \tag{4.44}$$

$$w_\kappa w_{z_1} = z_2^{n_{\kappa z_1}} O_{\kappa z_1} e^{2\pi i \tau_{\kappa z_1}} O_{\kappa z_2}^{-1} \qquad \text{for} \qquad \tau_{\kappa z_1} = \tfrac{1}{2}(\lambda_s^\vee - \lambda_r^\vee), \tag{4.45}$$

$$w_\kappa w_{z_2} = z_2^{n_{\kappa z_2}} O_{\kappa z_2} e^{2\pi i \tau_{\kappa z_2}} O_{\kappa z_2}^{-1} \qquad \text{for} \qquad \tau_{\kappa z_2} = -\tfrac{1}{2}(\lambda_1^\vee - \lambda_s^\vee)$$

from which it follows that

$$w_{z_1}^2 = z_1, \qquad w_{z_2}^2 = z_2, \qquad (w_{z_1}w_{z_2})^2 = z_1 z_2, \qquad w_\kappa^2 = z_2^s, \tag{4.46}$$

$$(w_\kappa w_{z_1})^2 = z_1 z_2^s, \qquad (w_\kappa w_{z_2})^2 = z_2^{s+1}, \qquad (w_\kappa w_{z_1} w_{z_2})^2 = z_1 z_2^{s+1}. \tag{4.47}$$

The last equality is a consequence of the previous ones since

$$(w_\kappa w_{z_1} w_{z_2})^2 (w_\kappa w_{z_2})^{-2} = w_\kappa w_{z_1} w_{z_2} w_\kappa w_{z_1} w_\kappa^{-1} w_{z_2}^{-1} w_\kappa^{-1}$$
$$= z_2^s \, w_\kappa w_{z_1} w_{z_2} (w_\kappa w_{z_1})^2 w_{z_1}^{-1} w_{z_2}^{-1} w_\kappa^{-1} = z_2 \, w_\kappa (w_{z_1} w_{z_2})^2 w_\kappa^{-1} = z_1 \,.$$

Note that the relations (4.46) and (4.47) imply that w_{z_1}, w_{z_2} and w_κ all commute. This will lead to simple expressions for the obstruction cocycle.

Similarly as in the case of groups with cyclic centers, we shall use the abbreviated notation:

$$z_1^{n_1} z_2^{n_2} \equiv n_1 n_2, \qquad z_0 z_1^{n_1} z_2^{n_2} \equiv \underline{n_1 n_2}$$

for the elements of the orientifold group $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$, setting

$$w_{n_1 n_2} = w_{z_1}^{n_1} w_{z_2}^{n_2}, \qquad w_{\underline{n_1 n_2}} = w_\kappa w_{z_1}^{n_{01}} w_{z_2}^{n_{02}} w_{z_1}^{n_1} w_{z_2}^{n_2}$$

if $n_1, n_2, n_{01}, n_{02} = 0, 1$ and if the twist element $\zeta = z_1^{n_{01}} z_2^{n_{02}} \equiv n_{01} n_{02}$. It is easy to show with the help of (4.46) and (4.47) that the Cartan algebra elements $b_{\gamma, \gamma'}$ may be taken in the form:

$$b_{n_1 n_2, n_1' n_2'} = b_{\underline{n_1 n_2}, n_1' n_2'} = b_{n_1 n_2, \underline{n_1' n_2'}} = n_1 n_1' \, \lambda_r^\vee + n_2 n_2' \, \lambda_1^\vee,$$
$$b_{\underline{n_1 n_2}, \underline{n_1' n_2'}} = (n_{01} + n_1 n_1') \, \lambda_r^\vee + (s + n_{02} + n_2 n_2') \, \lambda_1^\vee.$$

Employing the relations

$$\tau_{(z_1^{n_1} z_2^{n_2})^{-1} 0} = (1 - n_1) n_2 \, \lambda_1^\vee + n_1 n_2 \, \lambda_{r-1}^\vee + n_1 (1 - n_2) \, \lambda_r^\vee, \tag{4.48}$$

together with

$$\mathrm{tr}\,(\lambda_1^\vee)^2 = 1, \qquad \mathrm{tr}\,\lambda_1^\vee \lambda_{r-1}^\vee = \mathrm{tr}\,\lambda_1^\vee \lambda_r^\vee = \tfrac{1}{2},$$
$$\mathrm{tr}\,\lambda_{r-1}^\vee \lambda_r^\vee = \tfrac{s-1}{2}, \qquad \mathrm{tr}\,(\lambda_r^\vee)^2 = \tfrac{s}{2},$$

we obtain from the definition (2.34) the explicit expressions for the obstruction cocycle

$$u_{n_1 n_2, n_1' n_2', n_1'' n_2''} = u_{\underline{n_1 n_2}, n_1' n_2', n_1'' n_2''} = u_{n_1 n_2, n_1' n_2', \underline{n_1'' n_2''}}$$
$$= (-1)^{\mathsf{k}\left(s\, n_1 n_1' n_1'' + n_1 n_2' n_2'' + n_2 n_1' n_1''\right)}, \tag{4.49}$$

$$u_{n_1 n_2, \underline{n_1' n_2'}, n_1'' n_2''} = (-1)^{\mathsf{k}\left(s\, n_1 (1 + n_{01} + n_1' n_1'') + n_1 (n_{02} + n_2' n_2'') + n_2 (n_{01} + n_1' n_1'')\right)}, \tag{4.50}$$

$$u_{\underline{n_1 n_2}, n_1' n_2', n_1'' n_2''} = u_{n_1 n_2, \underline{n_1' n_2'}, n_1'' n_2''} = u_{n_1 n_2, n_1' n_2', \underline{n_1'' n_2''}}$$
$$= (-1)^{\mathsf{k}\left(s(n_{01} + n_1) n_1' n_1'' + (n_{01} + n_1) n_2' n_2'' + (n_{02} + n_2) n_1' n_1''\right)}, \tag{4.51}$$

$$u_{\underline{n_1 n_2}, \underline{n_1' n_2'}, \underline{n_1'' n_2''}}$$
$$= (-1)^{\mathsf{k}\left(s(n_{01} + n_1)(1 + n_{01} + n_1' n_1'') + (n_{01} + n_1)(n_{02} + n_2' n_2'') + (n_{02} + n_2)(n_{01} + n_1' n_1'')\right)} \tag{4.52}$$

that can easily be analyzed.

First, we note that the restriction of $u_{\gamma,\gamma',\gamma''}$ to the inversion group $\Gamma = \mathbb{Z}_2$ is trivial, with the formulae

$$v_{00,00} = v_{\underline{00},00} = v_{00,\underline{00}} = 1, \qquad v_{\underline{00},\underline{00}} = \pm 1$$

providing two cohomologically inequivalent trivializing cochains.

As the next case, let us consider the restriction of the obstruction cocycle to the orientifold subgroup $\Gamma = \mathbb{Z}_2 \ltimes Z$ with $Z = \{1, z_2\}$. Since the combination X of (4.2) with $n = 01$ is easily calculated to be equal to $(-1)^{k\,n_{01}}$, we infer that the obstruction cocycle restricted to Γ may be trivialized only if

$$\mathsf{k} \quad \text{is even if} \quad n_{01} = 1.$$

Under this condition, the restricted cocycle becomes trivial for all choices of the twist element.

Passing to the the orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes Z$ with $Z = \{1, z_1\}$, $Z = \{1, z_1 z_2\}$ or $Z = \mathbb{Z}_2 \times \mathbb{Z}_2$, we recall from [14] that, in all these three cases, the restriction of the obstruction cocycle to the orbifold group Z may be trivialized only under the condition that

$$\mathsf{k} \quad \text{is even} \quad \text{if} \quad s = \frac{r}{2} \quad \text{is odd.} \tag{4.53}$$

If k is even the whole obstruction cocycle becomes trivial. Suppose then that k is odd but s is even so that the terms multiplied by s may be dropped in the explicit expression for the cocycle. The combinations X of (4.2) with $n = 10$ and $n = 11$ are now easily calculated to take the values $(-1)^{k\,n_{02}}$ and $(-1)^{k(n_{01}+n_{02})}$, respectively. For $Z = \{1, z_1\}$, we then obtain the condition

$$\mathsf{k} \quad \text{is even} \quad \text{if} \quad n_{02} = 1 \tag{4.54}$$

and, for $Z = \{1, z_1 z_2\}$, the condition

$$\mathsf{k} \quad \text{is even} \quad \text{if} \quad n_{01} + n_{02} \quad \text{is odd.}$$

The obstruction cocycle restricted to $\Gamma = \mathbb{Z}_2 \ltimes Z$ with $Z = \{1, z_1\}$ or $Z = \{1, z_1 z_2\}$ becomes trivial under the conditions (4.53) and (4.54) or (4.53) and (4.54), respectively.

Finally, for the maximal orientifold group, the conditions (4.53), (4.54) and (4.53) must hold simultaneously, implying that if the twist element $\zeta \neq 1$ then the obstruction cocycle can be trivialized only if k is even. On the other hand, the trivializability of $u_{\gamma,\gamma',\gamma''}$ cannot depend on the choice of the twist element in this case so that for $\Gamma = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ the cohomological equation (2.23) has a solution only if

$$\mathsf{k} \quad \text{is even} \tag{4.55}$$

whatever the choice of the twist element. Indeed, if $\zeta = 1$ then the combination X of (4.2) with $n = 10$ takes the value $(-1)^{\mathsf{k}}$ for the cocycle $u'_{\gamma,\gamma',\gamma''}$ obtained by composing $u_{\gamma,\gamma',\gamma''}$ with the automorphism h_{z_1} of Γ, see (2.28). Since $u'_{\gamma,\gamma',\gamma''}$ is trivializable if and only if $u_{\gamma,\gamma',\gamma''}$ is, the condition (4.55) for the trivial twist element follows.

For all orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes Z$ with a non-trivial orbifold subgroup Z, the obstruction cocycle $u_{\gamma,\gamma,\gamma''}$ of (4.49)-(4.52) is then trivial whenever it may be trivialized. Sixteen cohomologically inequivalent trivializing 2-cocycles $v_{\gamma,\gamma'}$ on $\Gamma = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ are given by the formulae

$$
\begin{aligned}
v_{n_1 n_2, n_1' n_2'} &= v_{n_1 n_2, \overline{n_1' n_2'}} = \sigma^{n_2 n_1'}, \\
v_{\underline{n_1 n_2}, n_1' n_2'} &= \sigma^{n_2 n_1'} \sigma_1^{n_1'} \sigma_2^{n_2'}, \\
v_{\underline{n_1 n_2}, \overline{n_1' n_2'}} &= \underline{\sigma}\, \sigma^{n_2 n_1'} \sigma_1^{n_1'} \sigma_2^{n_2'}
\end{aligned}
$$

with $\sigma, \sigma_1, \sigma_2, \underline{\sigma} = \pm 1$. In particular, the choice of σ distinguishes two inequivalent restrictions of the 2-cocycle $v_{\gamma,\gamma'}$ to the orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$ that give rise to two inequivalent gerbes on $Spin(2r)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, see [14]. For $\Gamma = \mathbb{Z}_2 \ltimes Z$ with $Z \cong \mathbb{Z}_2$, four inequivalent cohomologically non-trivial 2-cocycles are obtained from the above expressions (with, say, $\sigma = 1$) by restriction.

Let us summarize the results for the $Spin(2r)$ group with even r. First, for each k and each of the four choices of the twist element, there are two inequivalent Jandl structures on the level k gerbe on $Spin(2r)$. Next, for each k even and each choice of the twist element, there are four inequivalent Γ-equivariant structures on the level k gerbe on $Spin(2r)$ for $\Gamma = \mathbb{Z}_2 \ltimes Z$ with $Z \cong \mathbb{Z}_2$. They give rise to eight inequivalent Jandl structures on the induced gerbe on $Spin(2r)/Z$. For such orientifold groups and k odd, there exist four inequivalent Γ-equivariant structures only if the twist belongs to Z and, for $Z = \{1, z_1\}$ or $Z = \{1, z_1 z_2\}$, if, additionally, $s = \frac{r}{2}$ is even. For fixed Z, we thus obtain four inequivalent Jandl structures on the induced gerbe on $Spin(2r)/Z$. Finally, for the maximal orientifold group $\Gamma = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and each k even, there exist sixteen inequivalent Γ-equivariant structures on the level k gerbe on $Spin(2r)$ for each choice of the twist element. They give rise to, altogether, eight inequivalent Jandl structures on each of the two inequivalent gerbes induced on $Spin(2r)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

4.5 The case of $G = E_6$

As in Sect. 4.7 of [14], we identify the Cartan algebra $\mathbf{t}$ of E_6 with the subspace of $\mathbb{R}^7$ with the first six coordinates summing to zero. The Killing form is inherited from the scalar product in $\mathbb{R}^7$. The vertices of the positive Weyl alcove $\mathcal{A}$ are

$$
\tau_0 = 0, \quad \tau_1 = \lambda_1^\vee, \quad \tau_2 = \tfrac{1}{2}\lambda_2^\vee, \quad \tau_3 = \tfrac{1}{3}\lambda_3^\vee, \quad \tau_4 = \tfrac{1}{2}\lambda_4^\vee, \quad \tau_5 = \lambda_5^\vee, \quad \tau_6 = \tfrac{1}{2}\lambda_6^\vee
$$

for the simple coweights $\lambda_i^\vee$ corresponding to the simple roots

$$
\begin{aligned}
\alpha_i &= e_i - e_{i+1} \quad \text{for} \quad i = 1, \dots, 5, \\
\alpha_6 &= \tfrac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6) + \tfrac{1}{\sqrt{2}} e_7,
\end{aligned}
$$

where e_i are the vectors of the canonical basis of $\mathbb{R}^7$. The positive roots have the form $e_i - e_j$ for $1 \le i < j \le 6$, $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6) + \frac{1}{\sqrt{2}} e_7$ with three signs $+$ and three signs $-$, and $\phi = \sqrt{2}\, e_7$ (the highest root). The center $Z(E_6) \cong \mathbb{Z}_3$ is generated by $z = \mathrm{e}^{-2\pi i\, \lambda_5^\vee}$. We shall construct the elements w_z and w_κ entering the relations (2.24) and (2.29) in terms of group elements $w_\alpha = \mathrm{e}^{\frac{\pi i}{2}(e_\alpha + e_{-\alpha})}$ that implement the Weyl reflections r_α in roots α, acting on the Cartan algebra by

$$
\tau \longrightarrow w_\alpha \tau w_\alpha^{-1} = \tau - \alpha^\vee \operatorname{tr} \tau\alpha \equiv r_\alpha(\tau),
$$

where $e_{\pm\alpha}$ and $\alpha^\vee$ stand for the step generators and the coroot associated to α, respectively. One has

$$w_\alpha^2 = e^{\pi i \alpha^\vee}.$$

Besides, since $[e_\alpha, e_\beta]$ does not vanish only if $\alpha + \beta$ is a root, w_α and w_β commute if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. The relation (2.29) is satisfied for $\tau \in \mathcal{A}$ if we take

$$w_\kappa = w_{\alpha_3}\, w_{\alpha_2+\alpha_3+\alpha_4}\, w_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}\, w_\phi\,, \tag{4.56}$$

with the action of κ on $\mathcal{A}$ reducing to

$$\kappa\tau_0 = \tau_0, \qquad \kappa\tau_i = \tau_{6-i} \quad \text{for} \quad i = 1,\ldots,5, \qquad \kappa\tau_6 = \tau_6$$

on the vertices and thereby giving rise to the symmetry of the Dynkin diagram represented in Fig.9. It is easy to check that all the factors on the right hand side of (4.56) commute
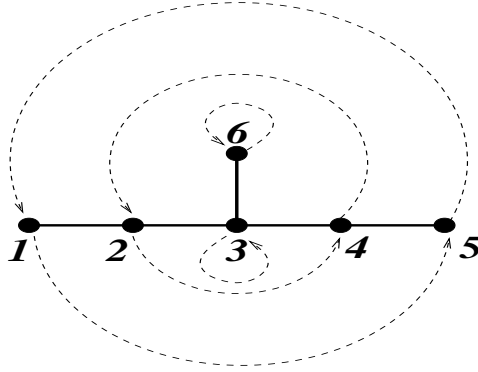
Figure 9: The Weyl reflection of the Dynkin diagram of E_6 under κ.

so that

$$w_\kappa^2 = w_{\alpha_3}^2\, w_{\alpha_2+\alpha_3+\alpha_4}^2\, w_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}^2\, w_\phi^2$$

$$= e^{\pi i(\alpha_3^\vee + \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee + \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee + \alpha_5^\vee + \phi^\vee)} = 1\,.$$

As observed in [14], there is another set of simple roots

$$\begin{aligned}
\beta_1 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\,, & \beta_2 &= \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\,, \\
\beta_3 &= -\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5 - \alpha_6\,, \\
\beta_4 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6\,, & \beta_5 &= \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\,, \\
\beta_6 &= \alpha_3
\end{aligned}$$

such that (2.24) may be satisfied for $\tau \in \mathcal{A}$ if the adjoint action of w_z on the Cartan algebra is given by the product of the Weyl reflections

$$w_z\, \tau\, w_z^{-1} = r_{\beta_1} r_{\beta_4} r_{\beta_5} r_{\beta_2}(\tau)\,,$$

with the action of z on $\mathcal{A}$ reducing to

$$z\tau_0 = \tau_1, \quad z\tau_1 = \tau_5, \quad z\tau_2 = \tau_4,$$
$$z\tau_3 = \tau_3, \quad z\tau_4 = \tau_6, \quad z\tau_5 = \tau_0, \quad z\tau_6 = \tau_2$$
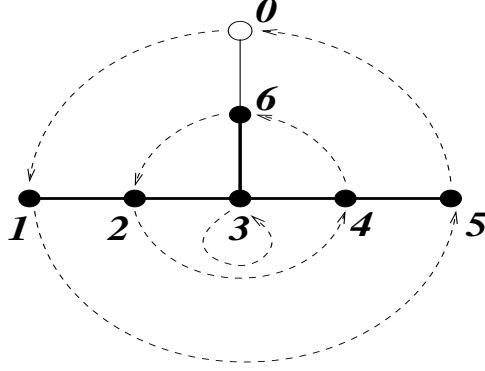
Figure 10: The transformation of the extended Dynkin diagram of E_6 under z.

on the vertices and thus giving rise to the symmetry of the extended Dynkin diagram depicted in Fig.10. Note that $w_\kappa \beta_1^\vee w_\kappa^{-1} = -\beta_5^\vee$ and $w_\kappa \beta_2^\vee w_\kappa^{-1} = -\beta_4^\vee$. It follows that

$$w_\kappa \, e_{\pm\beta_1} \, w_\kappa^{-1} \;=\; \mu_1^{\mp 1} e_{\mp\beta_5}, \qquad w_\kappa \, e_{\pm\beta_2} \, w_\kappa^{-1} \;=\; \mu_2^{\mp 1} e_{\mp\beta_4}$$

for some μ_1 and μ_2 of absolute value 1. Hence

$$w_\kappa \, w_{\beta_1} \, w_\kappa^{-1} \;=\; e^{\frac{\pi i}{2}(\mu_1 e_{\beta_5} + \bar\mu_1 e_{-\beta_5})}, \qquad w_\kappa \, w_{\beta_2} \, w_\kappa^{-1} \;=\; e^{\frac{\pi i}{2}(\mu_2 e_{\beta_4} + \bar\mu_2 e_{-\beta_4})}.$$

Since conjugation with $e^{\frac{\pi i}{2}(\mu e_\alpha + \bar\mu e_{-\alpha})}$ induces the Weyl reflection r_α on the Cartan algebra for all μ with $|\mu| = 1$, we may set

$$w_z \;=\; w_{\beta_1} w_\kappa w_{\beta_2} w_\kappa^{-1} w_\kappa w_{\beta_1} w_\kappa^{-1} w_{\beta_2} \,.$$

The elements $e_{\pm\beta_i}$ with $i = 1, \ldots, 5$ generate an $\mathfrak{su}(6)$ subalgebra of the Lie algebra of E_6. The coroots $\beta_i^\vee$ may be taken as its simple coroots and $e_{\pm\beta_i}$ as its step generators. Clearly, w_z belongs to the $SU(6)$ subgroup of E_6 corresponding to this subalgebra and, with the standard identification of the simple roots and the step generators of $\mathfrak{su}(6)$ in terms of matrices,

$$w_z \;=\; \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu_1\mu_2 \\ 0 & 0 & 0 & i\bar\mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\bar\mu_1 & 0 \end{pmatrix} \;\in\; SU(6) \subset E_6 \,.$$

The relation

$$w_z^3 \;=\; 1$$

follows by raising the above matrix to the third power. Let us further note that, since $[e_{\beta_1}, e_{\pm\beta_4}] = 0$ and $[e_{\beta_2}, e_{\pm\beta_5}] = 0$, we have the commutation relations:

$$w_{\beta_1} w_\kappa w_{\beta_2} w_\kappa^{-1} \;=\; w_\kappa w_{\beta_2} w_\kappa^{-1} w_{\beta_1}, \qquad w_{\beta_2} w_\kappa w_{\beta_1} w_\kappa^{-1} \;=\; w_\kappa w_{\beta_1} w_\kappa^{-1} w_{\beta_2}.$$

Using these identities and the equality $w_\kappa = w_\kappa^{-1}$, we infer that

$$(w_\kappa w_z)^2 \;=\; w_\kappa w_{\beta_1} w_\kappa^{-1} w_{\beta_2} w_{\beta_1}^2 w_\kappa w_{\beta_2}^2 w_\kappa^{-1} w_\kappa w_{\beta_1} w_\kappa^{-1} w_{\beta_2}$$

$$= w_\kappa w_{\beta_1} w_\kappa^{-1} w_{\beta_2} e^{\pi i(\beta_1^\vee + \beta_4^\vee)} w_{\beta_2} w_\kappa w_{\beta_1} w_\kappa^{-1} \,.$$

Next, the relations $[\beta_1^\vee + \beta_4^\vee, e_{\pm\beta_2}] = \mp e_{\pm\beta_2}$ imply that

$$e^{\pi i(\beta_1^\vee + \beta_4^\vee)} w_{\beta_2} e^{-\pi i(\beta_1^\vee + \beta_4^\vee)} = w_{\beta_2}^{-1} \,.$$

Similarly,

$$e^{\pi i(\beta_1^\vee + \beta_4^\vee)} w_k w_{\beta_1} w_k^{-1} e^{-\pi i(\beta_1^\vee + \beta_4^\vee)} = w_k w_{\beta_1}^{-1} w_k^{-1}$$

so that we obtain the identities:

$$\begin{aligned}
(w_k w_z)^2 &= e^{\pi i(\beta_1^\vee + \beta_4^\vee)} = e^{\pi i(\alpha_4^\vee + \alpha_6^\vee)}, \\
(w_z w_k)^2 &= w_k (w_k w_z)^2 w_k^{-1} = e^{\pi i(\alpha_2^\vee + \alpha_6^\vee)} \,.
\end{aligned}$$

It follows easily that we may choose:

$$b_{n,n'} = b_{\underline{n},n'} = 0,$$

$$b_{n,\underline{n}'} = \begin{cases} 0 & \text{for} \quad n = 0, \\ \frac{1}{2}(\alpha_2^\vee + \alpha_6^\vee) & \text{for} \quad n = 1, \\ \frac{1}{2}(\alpha_4^\vee + \alpha_6^\vee) & \text{for} \quad n = 2, \end{cases}$$

$$b_{\underline{n},\underline{n}'} = \begin{cases} 0 & \text{for} \quad [n_0 + n] = 0, \\ \frac{1}{2}(\alpha_4^\vee + \alpha_6^\vee) & \text{for} \quad [n_0 + n] = 1, \\ \frac{1}{2}(\alpha_2^\vee + \alpha_6^\vee) & \text{for} \quad [n_0 + n] = 2. \end{cases}$$

Since $\tau_{z^{-1}0} = \lambda_5^\vee$ and $\tau_{z^{-2}0} = \lambda_1^\vee$, it follows from the definition (2.34) that the obstruction cocycle $u_{\gamma,\gamma',\gamma''}$ is trivial on both orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_3$ and $\Gamma = \mathbb{Z}_2$ so that two cohomologically inequivalent cocycles may be taken in the form

$$v_{n,n'} = v_{\underline{n},n'} = v_{n,\underline{n}'} = 1, \qquad v_{\underline{n},\underline{n}'} = \pm 1.$$

In short, for each orientifold group, each k and each of the three choices of the twist element, there are two inequivalent Γ-equivariant structures on the level k gerbe on E_6. They give rise to six inequivalent Jandl structures on that gerbe and to two inequivalent Jandl structures on the induced gerbe on $E_6/\mathbb{Z}_3$.

4.6 The case of $G = E_7$

As in Sect. 4.8 of [14], we identify the Cartan algebra of E_7 with the subspace of the vectors in $\mathbb{R}^8$ whose coordinates sum to zero, with the Killing form inherited from the scalar product in $\mathbb{R}^8$. The vertices of the positive Weyl alcove $\mathcal{A}$ are

$$\begin{aligned}
\tau_0 &= 0, \quad \tau_1 = \lambda_1^\vee, \quad \tau_2 = \tfrac{1}{2}\lambda_2^\vee, \quad \tau_3 = \tfrac{1}{3}\lambda_3^\vee, \\
\tau_4 &= \tfrac{1}{4}\lambda_4^\vee, \quad \tau_5 = \tfrac{1}{3}\lambda_5^\vee, \quad \tau_6 = \tfrac{1}{2}\lambda_6^\vee, \quad \tau_7 = \tfrac{1}{2}\lambda_7^\vee
\end{aligned}$$

for the simple coweights $\lambda_i^\vee$ corresponding to the simple roots

$$\begin{aligned}
\alpha_i &= e_i - e_{i+1} \quad \text{for} \quad i = 1, \ldots, 6, \\
\alpha_7 &= \tfrac{1}{2}(-e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 + e_8),
\end{aligned}$$

where e_i are the vectors of the canonical basis of $\mathbb{R}^8$. Roots have the form $e_i - e_j$ for $i \neq j$ and $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8)$ with four signs $+$ and four signs $-$. The highest root is $\phi = -e_7 + e_8$. The center $Z(E_7) \cong \mathbb{Z}_2$ is generated by $z = e^{-2\pi i \lambda_1^\vee}$, with $\lambda_1^\vee = \frac{1}{4}(3e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + 3e_8)$. The relation (2.29) may be satisfied for $\tau \in \mathcal{A}$ if

$$w_\kappa = w_{\alpha_1}\, w_{\alpha_3}\, w_{\alpha_5}\, w_{\alpha_7}\, w_{\alpha_3+2\alpha_4+\alpha_5+\alpha_7}\, w_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_7}\, w_\phi\,, \tag{4.57}$$

with the trivial action of κ on $\mathcal{A}$. All the factors on the right hand side of (4.57) commute so that

$$w_\kappa^2 = e^{\pi i(\alpha_1^\vee + \alpha_3^\vee + \alpha_7^\vee)} = z\,.$$

The roots

$$
\begin{aligned}
\beta_1 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7,\\
\beta_2 &= -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7),\\
\beta_3 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,\\
\beta_4 &= -(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7),\\
\beta_5 &= \alpha_4,\\
\beta_6 &= \alpha_7,\\
\beta_7 &= \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7,
\end{aligned}
$$

form another system of simple roots such that (2.24) may be satisfied for $\tau \in \mathcal{A}$ if the adjoint action of w_z on the Cartan algebra is given by the product of the Weyl reflections

$$w_z\, \tau\, w_z^{-1} = r_{\beta_1} r_{\beta_3} r_{\beta_7}(\tau)\,,$$

with the action of z on $\mathcal{A}$ reducing to

$$z\tau_0 = \tau_1, \quad z\tau_1 = \tau_0, \quad z\tau_i = \tau_{8-i} \quad \text{for} \quad i = 2,\dots,6, \quad z\tau_7 = \tau_7$$

on the vertices, as illustrated in Fig.11. Since $w_\kappa \beta_i^\vee w_\kappa^{-1} = -\beta_i^\vee$, we must have
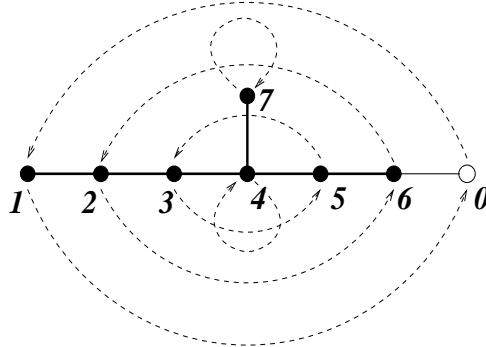

Figure 11: The transformation of the extended Dynkin diagram of E_7 under z.

$$w_\kappa\, e_{\pm\beta_i}\, w_\kappa^{-1} = \mu_i^{\mp 2} e_{\mp\beta_i}$$

for some μ_i with $|\mu_i| = 1$. Let

$$\tilde{w}_{\beta_i} = e^{\frac{\pi i}{2}(\mu_i e_{\beta_i} + \bar\mu_i e_{-\beta_i})}\,.$$

Conjugation with $\tilde{w}_{\beta_i}$ still induces the Weyl reflections r_{β_i} on the Cartan algebra and, similarly as for w_{β_i}, $\tilde{w}_{\beta_i}^2 = e^{\pi i \beta_i^\vee}$. We may then take

$$w_z = \tilde{w}_{\beta_1} \tilde{w}_{\beta_3} \tilde{w}_{\beta_7}.$$

Since $\tilde{w}_{\beta_1}$, $\tilde{w}_{\beta_3}$ and $\tilde{w}_{\beta_7}$ commute, the relation

$$w_z^2 = e^{\pi i (\beta_1^\vee + \beta_3^\vee + \beta_7^\vee)} = e^{\pi i (\alpha_1^\vee + \alpha_3^\vee + \alpha_7^\vee)} = z$$

holds. By construction, $w_\kappa \tilde{w}_{\beta_i} w_\kappa^{-1} = \tilde{w}_{\beta_i}$. Hence

$$(w_\kappa w_z)^2 = w_\kappa w_z w_\kappa^{-1} w_z^{-1} = 1.$$

It follows that we may set:

$$b_{n,n'} = b_{\underline{n},n'} = b_{n,\underline{n}'} = nn'\lambda_1^\vee, \qquad b_{\underline{n},\underline{n}'} = (1 + n_0 + nn')\lambda_1^\vee$$

which, with the help of the relations $\tau_{z^{-n_0}} = \delta_{[n],1}\lambda_1^\vee$ and $\mathrm{tr}(\lambda_1^\vee)^2 = \frac{3}{2}$, gives rise to the obstruction cocycle (2.34) on $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2$ of the form:

$$\begin{aligned}
u_{n,n',n''} &= u_{n,\underline{n}',n''} = u_{n,n',\underline{n}''} = (-1)^{k\,nn'n''}, \\
u_{n,\underline{n}',\underline{n}''} &= (-1)^{k\,n(1+n_0+n'n'')}, \\
u_{\underline{n},n',n''} &= u_{\underline{n},\underline{n}',n''} = u_{\underline{n},n',\underline{n}''} = (-1)^{k\,(n_0+n)n'n''}, \\
u_{\underline{n},\underline{n}',\underline{n}''} &= (-1)^{k\,(n+n_0)(1+n_0+n'n'')}.
\end{aligned}$$

The restriction of this cocycle to the orbifold subgroup $Z(E_7)$ may be trivialized if and only if

$$k \quad \text{is even,}$$

in which case the whole cocycle becomes trivial. As four cohomologically inequivalent trivializing cochains we may take the cocycles

$$v_{n,n'} = 1, \qquad v_{n,\underline{n}'} = 1, \qquad v_{\underline{n},n'} = \sigma^{n'}, \qquad v_{\underline{n},\underline{n}'} = \underline{\sigma}\,\sigma^{n'} \tag{4.58}$$

for $\sigma, \underline{\sigma} = \pm 1$. On the other hand, the restriction of the obstruction cocycle to the inversion group $\mathbb{Z}_2$ is trivial for all k. Two cohomologically nonequivalent trivializing cocycles may be obtained by restriction of (4.58) to $n = n' = 0$.

To summarize, for each k and each of the two choices of the twist element, there are two inequivalent Jandl structures on the level k gerbe on E_7. For k even and each choice of the twist element, there are four inequivalent $\mathbb{Z}_2 \ltimes \mathbb{Z}_2$-equivariant structures on the level k gerbe on E_7, giving rise to, altogether, four Jandl structures on the induced gerbe on $E_7/\mathbb{Z}_2$. There are no $\mathbb{Z}_2 \ltimes \mathbb{Z}_2$-equivariant structures for k odd.

4.7 The cases of $G = E_8$, F_4, G_2

These are the simple groups with a trivial center and no non-trivial Dynkin diagram symmetries. The only possible orientifold group is the inversion group $\Gamma = \mathbb{Z}_2$ and whatever the values of $b_{\gamma,\gamma'}$ the obstruction 3-cocycle (2.34) is trivial since $\tau_{\gamma^{-1}0} = \tau_0 = 0$ for all $\gamma \in \Gamma$. Two cohomologically inequivalent trivializing cochains are given by the 2-cocycles of (4.27). They give rise to two inequivalent Jandl structures on the level k gerbe on G for each k.

5. Conclusions

We have studied orientifolds of the WZW theories with simple compact simply connected groups G as targets. For orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes Z$, where the generator of $\mathbb{Z}_2$ acts by a twisted inversion $g \mapsto (\zeta g)^{-1}$ on G and Z is a subgroup of the center of G, we have classified all inequivalent Γ-equivariant structures on the level k gerbes on groups G. Such structures are required to unambiguously define Feynman amplitudes of classical fields of the orientifold theory. For Z of even order, there may be obstructions to existence of the orientifold theory with a given twist ζ even if the Z-orbifold theory exists. The classification of the Γ-equivariant structures on the level k gerbe on G descends to the classification of the Jandl structures [31] on the induced gerbe on the quotient group G/Z. There exists an even number, at least two, of such induced Jandl structures, giving rise to different orientifold extensions of the Z-orbifold theory, i.e. to different unoriented closed string theories with the G/Z target space. Our results also show that, in all cases except for $G = Spin(8n)$ and $Z = \mathbb{Z}_2 \times \mathbb{Z}_2$, the only obstructions to existence of a Γ-equivariant structure with the trivial twist element $\zeta = 1$ are the ones that obstruct existence of a Z-equivariant structure. In the exceptional case, Z-equivariant structures exist (two inequivalent ones [14]) for all integer levels k, whereas Γ-equivariant ones with the trivial twist element exist only for k even. In [8], an additional condition was imposed on the Z-orbifold theory, see (2.15) therein, that is equivalent to existence of a Γ-equivariant structure with the trivial twist element. This condition, that was unjustly related to unitarity of the Z-orbifold theory, eliminated odd levels k for the $SO(8n)/\mathbb{Z}_2$ WZW theory (in fact, the unitarity holds also for odd k theories; what fails is the left-right symmetry of the toroidal partition functions).

As we shall discuss in [15], our results, based on a systematic geometric approach to the classical orientifold theory, are in agreement with the ones obtained in [5] by studying the sewing and modular invariance constraints for the crosscap states in the simple-current orbifolds of the WZW theory.

6. Appendix

Here is a short list of results, with the signs $\sigma = \pm 1$, $\sigma_1 = \pm 1$, $\sigma_2 \pm 1$ and $\underline{\sigma} = \pm 1$ describing different choices of trivializing cochains.

===

Group $\mathbf{A_r}$

center $\mathbb{Z}_{r+1}$

twist element $n_0 = 0, 1, \ldots, r$

- -

orientifold group $\mathbb{Z}_2 \ltimes \mathbb{Z}_m$, m odd

level $\mathsf{k} \in \mathbb{Z}$

trivializing cochain for $n, n' = 0, \frac{r+1}{m}, \ldots, \frac{r+1}{m}(m-1)$

$$v_{n,n'} = e^{\frac{2\pi i \mathsf{k}}{r+1} nn'}, \qquad\qquad v_{\underline{n},n'} = (-1)^{\mathsf{k} r n_0 \frac{mn'}{r+1}} e^{-\frac{2\pi i \mathsf{k}}{r+1} nn'},$$

$$v_{n,\underline{n}'} = e^{\frac{\pi i \mathsf{k}}{r+1}(2nn' - n^2 - (r+1)n)}, \qquad v_{\underline{n},\underline{n}'} = \underline{\sigma}\, e^{\frac{\pi i \mathsf{k}}{r+1}\left(-n_0(n_0 + \frac{r(r+1)}{2}) - 2n(n_0 + n') + n^2 + (r+1)n\right)}$$

- -

orientifold group $\mathbb{Z}_2 \ltimes \mathbb{Z}_m$, m even

level $\mathsf{k} \in \mathbb{Z}$ if $\frac{r+1}{m}$ and n_0 are even, $\mathsf{k} \in 2\mathbb{Z}$ otherwise

trivializing cochain for $n, n' = 0, \frac{r+1}{m}, \ldots, \frac{r+1}{m}(m-1)$

$$v_{n,n'} = e^{\frac{2\pi i \mathsf{k}}{r+1} nn'}, \qquad\qquad v_{\underline{n},n'} = \sigma^{\frac{mn'}{r+1}} (-1)^{\mathsf{k} r n_0 \frac{mn'}{r+1}} e^{-\frac{2\pi i \mathsf{k}}{r+1} nn'},$$

$$v_{n,\underline{n}'} = e^{\frac{\pi i \mathsf{k}}{r+1}(2nn' - n^2 - (r+1)n)}, \qquad v_{\underline{n},\underline{n}'} = \underline{\sigma}\, \sigma^{\frac{mn'}{r+1}} e^{\frac{\pi i \mathsf{k}}{r+1}\left(-n_0(n_0 + \frac{r(r+1)}{2}) - 2n(n_0 + n') + n^2 + (r+1)n\right)}$$

===

Group $\mathbf{B_r}$

center $\mathbb{Z}_2$

twist element $n_0 = 0, 1$

- -

orientifold group $\mathbb{Z}_2$

level $\mathsf{k} \in \mathbb{Z}$

trivializing cochain

$$v_{0,0} = v_{\underline{0},0} = v_{0,\underline{0}} = 1, \qquad\qquad v_{\underline{0},\underline{0}} = \underline{\sigma}\, e^{\frac{\pi i \mathsf{k}}{2} n_0}$$

- -

orientifold group $\mathbb{Z}_2 \ltimes \mathbb{Z}_2$

level $\mathsf{k} \in \mathbb{Z}$

trivializing cochain for $n, n' = 0, 1$

$$v_{n,n'} = (-1)^{k\,nn'}, \qquad\qquad v_{\underline{n},n'} = \sigma^{n'}\,(-1)^{k\,nn'},$$
$$v_{n,\underline{n'}} = (-1)^{k\,nn'}\,\mathrm{e}^{\frac{-3\pi i k}{2}n}, \qquad v_{\underline{n},\underline{n'}} = \underline{\sigma}\,\sigma^{n'}\,(-1)^{k\,n(n_0+n')}\,\mathrm{e}^{\frac{\pi i k}{2}(n_0+3n)}$$

==

Group $\mathbf{C_r}$

center $\mathbb{Z}_2$

twist element $n_0 = 0, 1$

- -

orientifold group $\mathbb{Z}_2$

level $k \in \mathbb{Z}$

trivializing cochain

$$v_{0,0} = v_{\underline{0},0} = v_{0,\underline{0}} = 1, \qquad\qquad v_{\underline{0},\underline{0}} = \underline{\sigma}$$

- -

orientifold group $\mathbb{Z}_2 \ltimes \mathbb{Z}_2$

level $k \in \mathbb{Z}$ if r is even, $k \in 2\mathbb{Z}$ otherwise

trivializing cochain for $n, n' = 0, 1$

$$v_{n,n'} = 1, \qquad\qquad v_{\underline{n},n'} = \sigma^{n'},$$
$$v_{n,\underline{n'}} = 1, \qquad\qquad v_{\underline{n},\underline{n'}} = \underline{\sigma}\,\sigma^{n'}$$

==

Group $\mathbf{D_r}$ for r odd

center $\mathbb{Z}_4$

twist element $n_0 = 0, 1, 2, 3$

- -

orientifold group $\mathbb{Z}_2$

level $k \in \mathbb{Z}$

trivializing cochain

$$v_{0,0} = v_{\underline{0},0} = v_{0,\underline{0}} = 1, \qquad\qquad v_{\underline{0},\underline{0}} = \underline{\sigma}\,\mathrm{e}^{-\frac{\pi i}{4}kr(n_0+2\delta_{n_0,3})}$$

- -

orientifold group $\mathbb{Z}_2 \ltimes \mathbb{Z}_2$

level $k \in \mathbb{Z}$ if n_0 is even, $k \in 2\mathbb{Z}$ otherwise

trivializing cochain for $n, n' = 0, 2$

$$v_{n,n'} = 1, \qquad\qquad v_{\underline{n},n'} = \sigma^{\frac{1}{2}n'},$$
$$v_{n,\underline{n'}} = \mathrm{e}^{\frac{\pi i k}{4}n}, \qquad\qquad v_{\underline{n},\underline{n'}} = \underline{\sigma}\,\sigma^{\frac{1}{2}n'}\,\mathrm{e}^{-\frac{\pi i k}{4}([n_0+n]+2\delta_{[n_0+n],3})}$$

- -

orientifold group	$\mathbb{Z}_2 \ltimes \mathbb{Z}_4$
level	$\mathsf{k} \in 2\mathbb{Z}$
trivializing cochain for	$n, n' = 0, 1, 2, 3$

$$v_{n,n'} = 1, \qquad\qquad v_{\underline{n},n'} = \sigma^{n'},$$
$$v_{n,\underline{n}'} = e^{\frac{\pi i \mathsf{k}}{4}(n + 2\delta_{n,3})}, \qquad v_{\underline{n},\underline{n}'} = \underline{\sigma}\,\sigma^{n'}\,e^{-\frac{\pi i \mathsf{k}}{4}([n_0+n] + 2\delta_{[n_0+n],3})}$$

===

Group $\mathbf{D_r}$ for r even

| center | $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{n_1 n_2 \,|\, n_1, n_2 = 0, 1\}$ |
| twist element | $n_{01} n_{02} = 00, 10, 01, 11$ |

– –

orientifold group	$\mathbb{Z}_2$
level	$\mathsf{k} \in \mathbb{Z}$
trivializing cochain	

$$v_{00,00} = v_{\underline{00},00} = v_{00,\underline{00}} = 1, \qquad v_{\underline{00},\underline{00}} = \underline{\sigma}$$

– –

orientifold group	$\mathbb{Z}_2 \ltimes \{n0 \,	\, n = 0, 1\}$
level	$\mathsf{k} \in \mathbb{Z}$ if $\frac{r}{2}$ is even and $n_{02} = 0$, $\mathsf{k} \in 2\mathbb{Z}$ otherwise
trivializing cochain for	$n, n' = 0, 1$

$$v_{n0,n'0} = 1, \qquad\qquad v_{\underline{n0},n'0} = \sigma^{n'},$$
$$v_{n0,\underline{n'0}} = 1, \qquad\qquad v_{\underline{n0},\underline{n0'}} = \underline{\sigma}\,\sigma^{n'}$$

– –

orientifold group	$\mathbb{Z}_2 \ltimes \{0n \,	\, n = 0, 1\}$
level	$\mathsf{k} \in \mathbb{Z}$ if $n_{01} = 0$, $\mathsf{k} \in 2\mathbb{Z}$ otherwise
trivializing cochain for	$n, n' = 0, 1$

$$v_{0n,0n'} = 1, \qquad\qquad v_{\underline{0n},0n'} = \sigma^{n'},$$
$$v_{0n,\underline{0n'}} = 1, \qquad\qquad v_{\underline{0n},\underline{0n'}} = \underline{\sigma}\,\sigma^{n'}$$

– –

orientifold group	$\mathbb{Z}_2 \ltimes \{nn \,	\, n = 0, 1\}$
level	$\mathsf{k} \in \mathbb{Z}$ if $\frac{r}{2}$ and $n_{01} + n_{02}$ are even, $\mathsf{k} \in 2\mathbb{Z}$ otherwise
trivializing cochain for	$n, n' = 0, 1$

$$v_{nn,n'n'} = 1, \qquad\qquad v_{\underline{nn},n'n'} = \sigma^{n'},$$
$$v_{nn,\underline{n'n'}} = 1, \qquad\qquad v_{\underline{nn},\underline{n'n'}} = \underline{\sigma}\,\sigma^{n'}$$

----

orientifold group	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$
level	$k \in 2\mathbb{Z}$
trivializing cochain for	$n_1, n_2, n'_1, n'_2 = 0, 1$

$$v_{n_1 n_2, n'_1 n'_2} = \sigma^{n_2 n'_1}, \qquad\qquad v_{\underline{n_1 n_2}, n'_1 n'_2} = \sigma^{n_2 n'_1} \, \sigma_1^{n'_1} \, \sigma_2^{n'_2},$$

$$v_{n_1 n_2, \underline{n'_1 n'_2}} = \sigma^{n_2 n'_1}, \qquad\qquad v_{\underline{n_1 n_2}, \underline{n'_1 n'_2}} = \underline{\sigma} \, \sigma^{n_2 n'_1} \, \sigma_1^{n'_1} \, \sigma_2^{n'_2}$$

===

Group $\mathbf{E_6}$

center	$\mathbb{Z}_3$
twist element	$n_0 = 0, 1, 2$

----

orientifold group	$\mathbb{Z}_2$
level	$k \in \mathbb{Z}$
trivializing cochain	

$$v_{0,0} = v_{\underline{0},0} = v_{0,\underline{0}} = 1, \qquad v_{\underline{0},\underline{0}} = \underline{\sigma}$$

----

orientifold group	$\mathbb{Z}_2 \ltimes \mathbb{Z}_3$
level	$k \in \mathbb{Z}$
trivializing cochain for	$n, n' = 0, 1, 2$

$$v_{n,n'} = v_{\underline{n},n'} = v_{n,\underline{n'}} = 1, \qquad v_{\underline{n},\underline{n'}} = \underline{\sigma}$$

===

Group $\mathbf{E_7}$

center	$\mathbb{Z}_2$
twist element	$n_0 = 0, 1$

----

orientifold group	$\mathbb{Z}_2$
level	$k \in \mathbb{Z}$
trivializing cochain	

$$v_{0,0} = v_{\underline{0},0} = v_{0,\underline{0}} = 1, \qquad v_{\underline{0},\underline{0}} = \underline{\sigma}$$

----

orientifold group	$\mathbb{Z}_2 \ltimes \mathbb{Z}_2$
level	$k \in 2\mathbb{Z}$

trivializing cochain for $\quad n, n' = 0, 1, 2$

$$v_{n,n'} = 1, \qquad\qquad v_{\underline{n},n'} = \sigma^{n'},$$
$$v_{n,\underline{n}'} = 1, \qquad\qquad v_{\underline{n},\underline{n}'} = \underline{\sigma}\,\sigma^{n'}$$

==

Group $\qquad$ **E_8**

center $\qquad\qquad \mathbb{Z}_1$

twist element $\qquad n_0 = 0$

$-\ -$

orientifold group $\qquad \mathbb{Z}_2$

level $\qquad\qquad \mathsf{k} \in \mathbb{Z}$

trivializing cochain

$$v_{0,0} = v_{\underline{0},0} = v_{0,\underline{0}} = 1, \qquad v_{\underline{0},\underline{0}} = \underline{\sigma}$$

==

Group $\qquad$ **F_4**

center $\qquad\qquad \mathbb{Z}_1$

twist element $\qquad n_0 = 0$

$-\ -$

orientifold group $\qquad \mathbb{Z}_2$

level $\qquad\qquad \mathsf{k} \in \mathbb{Z}$

trivializing cochain

$$v_{0,0} = v_{\underline{0},0} = v_{0,\underline{0}} = 1, \qquad v_{\underline{0},\underline{0}} = \underline{\sigma}$$

==

Group $\qquad$ **G_2**

center $\qquad\qquad \mathbb{Z}_1$

twist element $\qquad n_0 = 0$

$-\ -$

orientifold group $\qquad \mathbb{Z}_2$

level $\qquad\qquad \mathsf{k} \in \mathbb{Z}$

trivializing cochain

$$v_{0,0} = v_{\underline{0},0} = v_{0,\underline{0}} = 1, \qquad v_{\underline{0},\underline{0}} = \underline{\sigma}$$

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