UNIQUENESS OF E_∞ STRUCTURES FOR CONNECTIVE COVERS

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Abstract. We refine our earlier work on the existence and uniqueness of E_∞ structures on K-theoretic spectra to show that at each prime p, the connective Adams summand ℓ has a unique structure as a commutative $\mathbb{S}$-algebra. For the p-completion ℓ_p we show that the McClure-Staffeldt model for ℓ_p is equivalent as an E_∞ ring spectrum to the connective cover of the periodic Adams summand L_p. We establish a Bousfield equivalence between the connective cover of the Lubin-Tate spectrum E_n and $BP\langle n\rangle$.

Introduction

The aim of this short note is to establish the uniqueness of E_∞ structures on connective covers of certain periodic commutative $\mathbb{S}$-algebras E, most prominently for the connective p-complete Adams summand. It is clear that the connective cover of an E_∞ ring spectrum inherits a E_∞ structure; there is even a *functorial* way of assigning a connective cover within the category of E_∞ ring spectra [9, VII.3.2]. But it is not obvious in general that this E_∞ multiplication is unique.

Our main concern is with examples in the vicinity of K-theory; we apply our uniqueness theorem to real and complex K-theory and their localizations and completions and to the Adams summand and its completion.

The existence and uniqueness of E_∞ structures on the periodic spectra KU, KO and L was established in [5] by means of the obstruction theory for E_∞ structures developed by Goerss-Hopkins [8] and Robinson [12]. Note however, that obstruction theoretic methods would fail in the connective cases. Let e be a commutative ring spectrum. If e satisfies some Künneth and universal coefficient properties [12, proposition 5.4], then the obstruction groups for E_∞ multiplications consist of André-Quillen cohomology groups in the context of differential graded E_∞ algebras applied to the graded commutative e_*-algebra e_*e. Besides problems with non-projectivity of e_*e over e_*, the algebra structures of ku_*ku, ko_*ko and $\ell_*\ell$ are far from being étale and therefore one would obtain non-trivial obstruction groups. One would then have to identify actual obstruction classes in these obstruction groups in order to establish the uniqueness of the given E_∞ structure – but at the moment, this seems to be an intractable problem. Thus an alternative approach is called for.

In Theorem 1.3 we prove that a unique E_∞ structure on E gives rise to a unique structure on the connective cover if E is obtained from some connective spectrum via a process of Bousfield localization. In particular, we identify the E_∞ structure on the p-completed connective Adams summand ℓ_p provided by McClure and Staffeldt in [10] with the one that arises by taking the unique E_∞ structure on the periodic Adams summand $L = E(1)$ developed in [5] and taking its connective cover.

Our Theorem applies as well to the connective covers of the Lubin-Tate spectra E_n and we prove in section 2 that these spectra are Bousfield equivalent to the truncated Brown-Peterson spectra $BP\langle n\rangle$. Unlike other spectra that are Bousfield equivalent to $BP\langle n\rangle$, such as the connective cover of the completed Johnson-Wilson spectrum, $\widehat{E(n)}$, the connective cover of

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E_n is calculationally convenient. So far, only $BP\langle 1\rangle = \ell$ is known to have an E_∞ structure, and we propose the connective cover of E_n as an E_∞ approximation of $BP\langle n\rangle$.

1. E_∞ structures on connective covers

Let us first make explicit what we mean by uniqueness of E_∞ structures. We admit that this is an *ad hoc* notion, but it suffices for the examples we want to consider.

Definition 1.1. In the following, we will say that an E_∞ structure on some homotopy commutative and associative ring spectrum E is unique if whenever there is a map of ring spectra $\varphi\colon E' \longrightarrow E$ from some other E_∞ ring spectrum E' to E which induces an isomorphism on homotopy groups, then there is a morphism in the homotopy category of E_∞ ring spectra $\varphi'\colon E' \longrightarrow E$ such that $\pi_*(\varphi) = \pi_*(\varphi')$.

If E and F are spectra whose E_∞ structure was provided by the obstruction theory of Goerss and Hopkins [8], then we can compare our uniqueness notion with theirs. Note that examples of such E_∞ ring spectra include E_n [8, 7.6], KO, KU, L and $\widehat{E(n)}$ [5]. In such cases the Hurewicz map

$$\mathrm{Hom}_{E_\infty}(E, F) \xrightarrow{h} \mathrm{Hom}_{F_*-\mathrm{alg}}(F_*E, F_*) \tag{1.1}$$

is an isomorphism. Assume that we have a mere ring map φ as above between E and F. This gives rise to a map of F_*-algebras from F_*E to F_* by composing $F_*(\varphi)$ with the multiplication μ in F_*F. The left hand side in (1.1) denotes the derived space of E_∞ maps from E to F. In presence of a universal coefficient theorem we have $\mathrm{Hom}_{F_*-\mathrm{hom}}(F_*E, F_*) = [E, F]$, therefore the element $\mu \circ F_*(\varphi)$ gives rise to a homotopy class of maps of ring spectra $\widetilde{\varphi}$ from E to F. We can assume that we have functorial cofibrant replacement $Q(-)$, hence we obtain a ring map $Q(\widetilde{\varphi})$ from $Q(E)$ to $Q(F)$. Via the isomorphism (1.1) this gives a map of E_∞ spectra from $Q(E)$ to $Q(F)$, Φ, therefore we obtain a zigzag

$$\begin{array}{ccc} Q(E) & \overset{\Phi}{\dashrightarrow} & Q(F) \\ \Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} \\ E & \xrightarrow{\varphi} & F \end{array}$$

of weak equivalences of E_∞ spectra from E to F. Thus in such cases our definition agrees with the uniqueness notion that is natural in the Goerss-Hopkins setting.

For the rest of the paper we assume the following.

Assumption 1.2. Let E be a periodic commutative $\mathbb{S}$-algebra with periodicity element $v \in E_*$ of positive degree. We will view E as being obtained from a connective commutative $\mathbb{S}$-algebra e by Bousfield localization at $e[v^{-1}]$ in the category of e-modules. Furthermore we assume that the localization map induces an isomorphism between the homotopy groups of e and the homotopy groups of the connective cover of E.

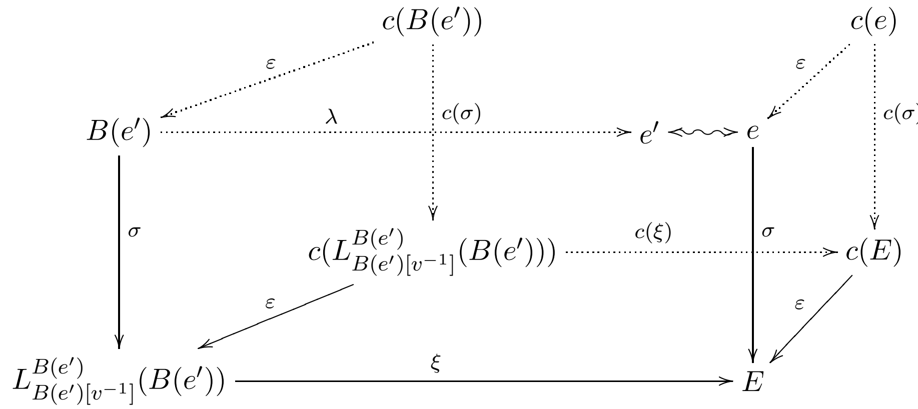
Let us denote the connective cover functor from [9, VII.3.2] by $c(-)$. For any E_∞ ring spectrum A, there is a weakly equivalent commutative $\mathbb{S}$-algebra $B(\mathbb{P}, \mathbb{P}, \mathbb{L})(A)$, with equivalence

$$\lambda\colon B(\mathbb{P}, \mathbb{P}, \mathbb{L})(A) \xrightarrow{\simeq} A,$$

in the E_∞ category [7, XII.1.4]. Here $B(\mathbb{P}, \mathbb{P}, \mathbb{L})$ is a bar construction with respect to the monad associated to the linear isometries operad L and the monad for commutative monoids in the category of $\mathbb{S}$-algebras $\mathbb{P}$. We will denote the composite $B(\mathbb{P}, \mathbb{P}, \mathbb{L}) \circ c$ by $\bar{c}$. For a commutative $\mathbb{S}$-algebra R and an R-module M, let $L^R_M(-)$ denote Bousfield localization at M in the category of R-modules and we denote the localization map by $\sigma\colon E \longrightarrow L^R_M(E)$ for any R-module E.

Theorem 1.3. *Assume that we know that the E_∞ structure on E is unique. Then the E_∞ structure on $c(E)$ is unique.*

Proof. Each commutative $\mathbb{S}$-algebra can be viewed as an E_∞ ring spectrum. Let e' be a model for the connective cover $c(E)$, *i.e.*, e' is an E_∞ ring spectrum with a map of ring spectra φ to $c(E)$, such that $\pi_*(\varphi)$ is an isomorphism. Write $v \in e'_*$ for the isomorphic image of v under the inverse of $\pi_*(\varphi)$. As φ is a ring map it will induce a ring map on the corresponding Bousfield localizations. But as the E_∞ structure on E is unique by assumption, this map can be replaced by an equivalent equivalence, ξ, of E_∞ ring spectra. We abbreviate $B(\mathbb{P}, \mathbb{P}, \mathbb{L})(e')$ to $B(e')$. We consider the following diagram whose dotted lines provide a zigzag of E_∞ equivalences and hence a map in the homotopy category of E_∞ ring spectra.



□

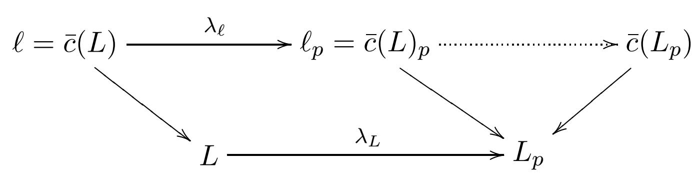
Real and complex K-theory, ko and ku, have E_∞ structures obtained using algebraic K-theory models [9, VIII, §2]. The connective Adams summand ℓ has an E_∞ structure because it is the connective cover of $E(1)$. In the following we will refer to these models as the standard ones. The E_∞ structures on KO, KU and $E(1)$ are unique by [5, theorems 7.2, 6.2]. In all of these cases, the periodic versions are obtained by Bousfield localization [7, VIII.4.3].

Corollary 1.4. *The E_∞ structures on ko, ku and ℓ are unique.*

In [10], McClure and Staffeldt construct a model for the p-completed connective Adams summand using algebraic K-theory of fields. Let $\widetilde{\ell} = K(\mathbf{k}')$, the algebraic K-theory spectrum of $\mathbf{k}' = \bigcup_i \mathbb{F}_{q^{p^i}}$, where q is a prime which generates the p-adic units $\mathbb{Z}_p^\times$. Then the p-completion of $\widetilde{\ell}$ is additively equivalent to the p-completed connective Adams summand ℓ_p [10, proposition 9.2]. For further details see also [2, §1]. Note that the p-completion ℓ_p inherits an E_∞ structure from ℓ because p-completion is Bousfield localization with respect to $H\mathbb{F}_p$ and therefore preserves commutative $\mathbb{S}$-algebras [7, VIII.2.2]. An *a priori* different model for the p-completion of the connective Adams summand can be obtained by taking the connective cover of the p-complete periodic version $L = E(1)$. This is consistent with the statement of Corollary 1.4 because p-completion and Bousfield localization of ℓ in the category of ℓ-modules with respect to L are compatible in the following sense. Consider $\ell = \bar{c}(L)$ and its p-completion

$$\lambda_\ell \colon \ell \longrightarrow \ell_p = (\bar{c}(L))_p.$$

The p-completion map λ is functorial in the spectrum, therefore the following diagram of solid arrows commutes.



The universal property of the connective cover functor ensures that there is a map in the homotopy category of commutative $\mathbb{S}$-algebras from ℓ_p to $\bar{c}(L_p)$ which is a weak equivalence. In the following we will not distinguish ℓ_p from $\bar{c}(L_p)$ anymore and denote this model simply by ℓ_p.

Proposition 1.5. *The McClure-Staffeldt model $\widetilde{\ell}_p$ of the p-complete connective Adams summand is equivalent as an E_∞ ring spectrum to ℓ_p.*

Remark 1.6. If E is a commutative $\mathbb{S}$-algebra with naive G-action for some group G, then neither the connective cover functor $\bar{c}(-)$ nor Bousfield localization of E has to commute with taking homotopy fixed points. As an example, consider connective complex K-theory ku with the conjugation action by C_2. The homotopy fixed points ku^{hC_2} are not equivalent to ko, but on the periodic versions we obtain $KU^{hC_2} \simeq KO$.

of Proposition 1.5. Consider the algebraic K-theory model for connective complex K-theory, $ku = K(\mathbf{k})$, with $\mathbf{k} = \bigcup_i \mathbb{F}_{q^{p^i(p-1)}}$. The canonical inclusions $\mathbb{F}_{q^{p^i}} \hookrightarrow \mathbb{F}_{q^{p^i(p-1)}}$ assemble into a map $j\colon \mathbf{k}' \longrightarrow \mathbf{k}$. The Galois group C_{p-1} of $\mathbf{k}$ over $\mathbf{k}'$ acts on $\mathbf{k}$ and induces an action on algebraic K-theory. As $\mathbf{k}'$ is fixed under the action of C_{p-1} there is a factorization of $K(j)_p$ as

$$\begin{array}{ccc} K(\mathbf{k}')_p & \xrightarrow{\quad K(j)_p \quad} & K(\mathbf{k})_p \\ & {}_{i}\searrow \qquad \nearrow & \\ & K(\mathbf{k})_p^{hC_{p-1}} & \end{array}$$

and i yields a weak equivalence of commutative $\mathbb{S}$-algebras, where $K(\mathbf{k})_p^{hC_{p-1}}$ is a model for the connective p-complete Adams summand which is weakly equivalent to $\widetilde{\ell}_p$ (see [2, §1]).

Consider the composition of the following chain of maps between commutative $\mathbb{S}$-algebras:

$$K(\mathbf{k}')_p \xrightarrow{i} (K(\mathbf{k})_p)^{hC_{p-1}} \longrightarrow K(\mathbf{k})_p \longrightarrow KU_p.$$

The target KU_p is as well the target of the map $\bar{c}(KU_p) \longrightarrow KU_p$. Note that the universal property of $\bar{c}(-)$ yields a zigzag $\varsigma\colon K(\mathbf{k})_p \leftrightsquigarrow \bar{c}(KU_p)$ of equivalences between $K(\mathbf{k})_p$ and $\bar{c}(KU_p)$ in the category of commutative $\mathbb{S}$-algebras.

As KU_p is the Bousfield localization of $K(\mathbf{k})_p$ in the category of $K(\mathbf{k})_p$-modules with respect to the Bott element,

$$KU_p = L^{K(\mathbf{k})_p}_{K(\mathbf{k})_p[\beta^{-1}]} K(\mathbf{k})_p,$$

it inherits the C_{p-1}-action on $K(\mathbf{k})_p$. The functoriality of the connective cover lifts this action to an action on $\bar{c}(KU_p)$.

The connective cover functor is in fact a functor in the category of commutative $\mathbb{S}$-algebras with multiplicative naive G-action for any group G. To see this we have to show that the map $\bar{c}(A) \longrightarrow A$ is G-equivariant if A is a commutative $\mathbb{S}$-algebra with an underlying naive G-spectrum. The functor $B(\mathbb{P}, \mathbb{P}, \mathbb{L})$ does not cause any problems. Proving the claim for the functor c involves chasing the definition given in [9, VII, §3].

The prespectrum underlying $c(A)$ applied to an inner product space V is defined as $T(A_0)(V)$, where A_0 is the zeroth space of the spectrum A and T is a certain bar construction involving suspensions and a monad consisting of the product of a fixed E_∞ operad with the partial operad of little convex bodies $\mathcal{K}$. For a fixed V the suspension Σ^V and the operadic term $\mathcal{K}_V$ are used. As the G-action is compatible with the E_∞ and the additive structure of A, the evaluation map $T(A_0)(V) \longrightarrow A(V)$ is G-equivariant. For varying V, these maps constitute a map of prespectra and its adjoint on the level of spectra is $c(A) \longrightarrow A$. As the spectrification functor preserves G-equivariance, the claim follows. Therefore the resulting zigzag $\varsigma\colon K(\mathbf{k})_p \leftrightsquigarrow \bar{c}(KU_p)$ is C_{p-1}-equivariant and we obtain an induced zigzag on homotopy fixed points,

$$\varsigma^{hC_{p-1}}\colon (K(\mathbf{k})_p)^{hC_{p-1}} \leftrightsquigarrow (\bar{c}(KU_p))^{hC_{p-1}}.$$

As ς is an isomorphism in the homotopy category and is C_{p-1}-equivariant, $\varsigma^{hC_{p-1}}$ yields an isomorphism as well. $\square$

Goerss and Hopkins proved in [8] that the Lubin-Tate spectra E_n with

$$(E_n)_* = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^{\pm 1}] \quad \text{with } |u_i| = 0 \text{ and } |u| = -2$$

possess unique E_∞ structures for all primes p and all $n \geqslant 1$. The connective cover $c(E_n)$ has coefficients

$$(c(E_n))_* = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^{-1}] \quad \text{with } |u_i| = 0 \text{ and } |u| = -2.$$

Of course $\bar{c}(E_n)[(u^{-1})^{-1}] \sim E_n$.

The spectra $BP\langle n\rangle$ can be built from the Brown-Peterson spectrum BP by killing all generators of the form v_m with $m > n$ in $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$. Using for instance Angeltveit's result [1, theorem 4.2] one can prove that the $BP\langle n\rangle$ are A_∞ spectra and from [4] it is known that this $\mathbb{S}$-algebra structure can be improved to an MU-algebra structure. On the other hand, Strickland showed in [13] that $BP\langle n\rangle$ with $n \geqslant 2$ is not a homotopy commutative MU-ring spectrum for $p = 2$. We offer $c(E_n)$ as a replacement for the p-completion $BP\langle n\rangle_p$ of $BP\langle n\rangle$.

We also need to recall that in the category of MU-modules, $E(n)$ is the Bousfield localization of $BP\langle n\rangle$ with respect to $BP\langle n\rangle[v_n^{-1}]$, hence by [7] it inherits the structure of an MU-algebra and the natural map $BP\langle n\rangle \longrightarrow E(n)$ is a morphism of MU-algebras. Furthermore, the Bousfield localization of $E(n)$ with respect to the MU-algebra $K(n)$ is the I_n-adic completion $\widehat{E(n)}$, which was shown to be a commutative $\mathbb{S}$-algebra in [5], and the natural map $\widehat{E(n)} \longrightarrow E_n$ is a morphism of commutative $\mathbb{S}$-algebras, see for example [6, example 2.2.6]. Thus there is a morphism of ring spectra $BP\langle n\rangle \longrightarrow E_n$ which lifts to a map $BP\langle n\rangle \longrightarrow c(E_n)$.

Proposition 2.1. *The spectra $BP\langle n\rangle$ and $BP\langle n\rangle_p$ are Bousfield equivalent to $c(E_n)$.*

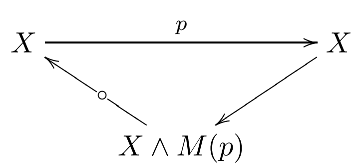
Proof. On coefficients, we obtain a ring homomorphism $(BP\langle n\rangle_p)_* \longrightarrow (c(E_n))_*$ which on homotopy is given by

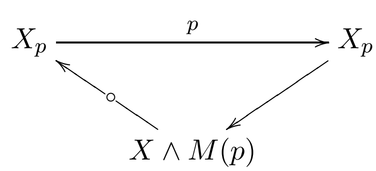
$$v_k \longmapsto \begin{cases} u^{1-p^k} u_k & \text{for } 1 \leqslant k \leqslant n-1, \\ u^{1-p^n} & \text{for } k = n. \end{cases}$$

extending the natural inclusion of the p-adic integers $\mathbb{Z}_p = W(\mathbb{F}_p)$ into $W(\mathbb{F}_{p^n})$. This homomorphism is induced by a map of ring spectra.

Recall from [3] that $E(n)$ and $\widehat{E(n)}$ are Bousfield equivalent as $\mathbb{S}$-modules, and it follows that E_n is Bousfield equivalent to these since it is a finite wedge of suspensions of $\widehat{E(n)}$.

If X is a p-local spectrum with torsion free homotopy groups then its p-completion X_p is Bousfield equivalent to X, *i.e.*, $\langle X_p\rangle = \langle X\rangle$. This follows using the cofibre triangles (in which $M(p)$ is the mod p Moore spectrum and the circled arrow indicates a map of degree one)



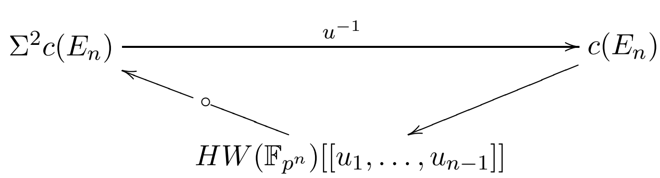


together with the fact that the rationalization $p^{-1}X$ is a retract of $p^{-1}(X_p)$. In particular, we have $\langle BP\langle n\rangle_p\rangle = \langle BP\langle n\rangle\rangle$ and $\langle E(n)_p\rangle = \langle E(n)\rangle$.

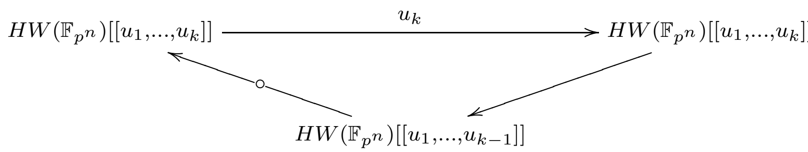
From [11, theorem 2.1], the Bousfield class of $BP\langle n\rangle$ is

$$\langle BP\langle n\rangle\rangle = \langle E(n)\rangle \vee \langle H\mathbb{F}_p\rangle.$$

There is a cofibre triangle



in which $HW(\mathbb{F}_{p^n})[[u_1,\dots,u_{n-1}]]$ is the Eilenberg-MacLane spectrum. More generally we can construct a family of Eilenberg-MacLane spectra with $W(\mathbb{F}_{p^n})[[u_1,\dots,u_k]]$ as coefficients for $k=0,\dots,n-1$ which are related by cofibre triangles



such that for $k=0$ we obtain $HW(\mathbb{F}_{p^n})$. With the help of these cofibre sequences we can identify $\langle HW(\mathbb{F}_{p^n})[[u_1,\dots,u_k]]\rangle$ with $\langle HW(\mathbb{F}_{p^n})[[u_1,\dots,u_{k-1}]]\rangle \vee \langle HW(\mathbb{F}_{p^n})[[u_1,\dots,u_k]][u_k^{-1}]\rangle$.

In general, if R is a commutative ring, then the ring of finite tailed Laurent series $R((x))$ is faithfully flat over R and therefore we have

$$\langle HR((x))\rangle = \langle HR\rangle.$$

Using this auxiliary fact we inductively get that

$$\langle HW(\mathbb{F}_{p^n})[[u_1,\dots,u_k]]\rangle = \langle HW(\mathbb{F}_{p^n})[[u_1,\dots,u_{k-1}]]\rangle.$$

This reduces the Bousfield class of $c(E_n)$ to $\langle E_n\rangle \vee \langle HW(\mathbb{F}_{p^n})\rangle$. As $W(\mathbb{F}_{p^n})$ is a finitely generated free $\mathbb{Z}_p$-module and as $\langle H\mathbb{Z}_p\rangle = \langle H\mathbb{Q}\rangle \vee \langle H\mathbb{F}_p\rangle$ this leads to

$$\begin{aligned}\langle c(E_n)\rangle &= \langle E(n)\vee H\mathbb{Q}\vee H\mathbb{F}_p\rangle\\ &= \langle E(n)\vee H\mathbb{F}_p\rangle = \langle BP\langle n\rangle\rangle.\end{aligned}$$ □

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