

DUALITY AND DEFECTS IN RATIONAL CONFORMAL FIELD THEORY

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Abstract

We study topological defect lines in two-dimensional rational conformal field theory. Continuous variation of the location of such a defect does not change the value of a correlator. Defects separating different phases of local CFTs with the same chiral symmetry are included in our discussion. We show how the resulting one-dimensional phase boundaries can be used to extract symmetries and order-disorder dualities of the CFT.

The case of central charge $c = 4/5$, for which there are two inequivalent world sheet phases corresponding to the tetra-critical Ising model and the critical three-states Potts model, is treated as an illustrative example.

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1 Introduction and summary

Symmetries in general, and symmetry groups in particular, provide significant insight in the structure of physical theories. It is thus of much interest to gain a firm conceptual and computational handle on symmetries of conformal field theories. A recent formulation of rational CFT, the so-called TFT approach (see [0–V] and sections 2.1, 2.2 and 4.1 below) allows one to address these issues from a novel point of view. The TFT approach accounts for chiral symmetries which lead to conserved chiral quantities by working with the representation category of the chiral symmetry algebra; in rational CFT, this is a modular tensor category \mathcal{C} . One can typically construct several full CFTs which have a given chiral algebra as part of their chiral symmetry; the free boson compactified at different radii (squaring to a rational number) and the A-D-E modular invariants of the $\mathfrak{sl}(2)$ WZW model are examples of this. Therefore, beyond \mathcal{C} one also needs an additional datum to characterise a full CFT model. This turns out to be

a (Morita class of) Frobenius algebra(s) A in \mathcal{C} . Accordingly, also other symmetries than the chiral symmetries can be present; e.g. for the critical Ising model the \mathbb{Z}_2 transformation which exchanges spin up and spin down, or the S_3 -symmetry for the critical three-states Potts model. We refer to such additional symmetries as *internal* symmetries of the full CFT.

On the basis of the TFT approach we show that the internal symmetries of a CFT (\mathcal{C}, A) are closely related to a certain class of conformal defect lines that preserve all the chiral symmetries. These defect lines can be used to construct nontrivial mappings of CFT data like boundary conditions and bulk or boundary fields, and to establish invariance of correlation functions under those transformations. The construction uses a generalisation of contour deformation arguments familiar from complex analysis.

In more technical terms, the group of internal symmetries that are described by defects turns out to be the Picard group $\text{Pic}(\mathcal{C}_{A|A})$ of the tensor category of A -bimodules. Naively one might instead have expected the symmetry group to be the group of automorphisms of the algebra A . This would, however, be incompatible with the result that the CFT depends only on the Morita class of A [0], for a proof see section 3.3 of the present paper): Morita equivalent algebras have, in general, non-isomorphic automorphism groups. It turns out (see proposition 7 of [1], which extends a result in [2]) that in fact the group of outer automorphisms of A is a subgroup of $\text{Pic}(\mathcal{C}_{A|A})$.

This is not the end of the story: Besides internal symmetries, it is known that certain CFTs exhibit order-disorder dualities of Kramers–Wannier type. We show that their presence can be deduced with similar arguments from the existence of defect lines of a more general type. A relation between Kramers–Wannier dualities and defects might actually have been suspected, since such dualities relate bulk fields to disorder fields which, after all, are located at the end points of defect lines. However, the defect lines that we show to be relevant for obtaining the dualities are of a different type than the ones created by the dual disorder fields.

Let us give a general description of defects in CFT in somewhat more detail. Consider two CFTs defined on the upper and lower half of the complex plane, respectively, which are joined together along an interface – or defect – on the real line. To characterise the defect, one must consider the behaviour of the holomorphic and antiholomorphic components $T^1(z)$, $\bar{T}^1(z)$ and $T^2(z)$, $\bar{T}^2(z)$ of the stress tensors of the two CFTs. If the stress tensors obey the conservation law

$$T^1(x) - \bar{T}^1(x) = T^2(x) - \bar{T}^2(x) \quad \text{for all } x \in \mathbb{R} \quad (1.1)$$

then the defect is called *conformal*. This condition can be implemented by the so-called ‘folding trick’, in which one identifies the lower and upper complex half planes, thereby giving rise to a tensor product of two CFTs on the upper half plane [3]. The condition (1.1) then states that the resulting boundary condition on the real line is conformal in the sense of [4]. Conformal defects have been investigated with the help of the folding trick in e.g. [5, 6, 7, 8].

There are two obvious solutions to the condition (1.1), namely $T^1(x) = \bar{T}^1(x)$, $T^2(x) = \bar{T}^2(x)$ and $T^1(x) = T^2(x)$, $\bar{T}^1(x) = \bar{T}^2(x)$. The former is the *totally reflective* case, for which the defect can be regarded as describing a conformal boundary for each of the two CFTs individually, so that the two theories are completely decoupled. In the second case the two theories are maximally coupled in the sense that the defect is *totally transmissive* for momentum. If $T^1(x) = T^2(x)$ on the real line, then the operator products $T^1(x)T^1(y)$ and $T^2(x)T^2(y)$ must agree, so that in particular totally transmissive defects can only exist if the two CFTs have equal central charge.

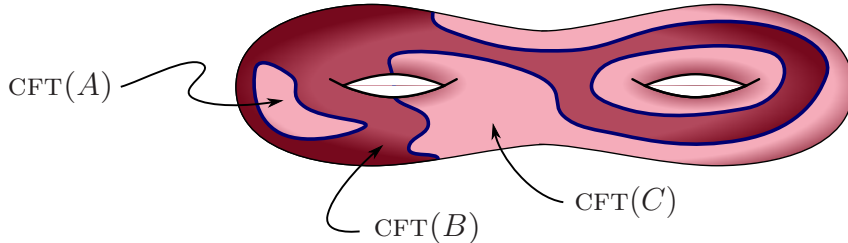


Figure 1: A genus two world sheet with different phases of full CFT. The phases meet along one-dimensional phase boundaries which are topological defects.

In this article we shall investigate the totally transmissive case. Such defects have been studied in the framework of rational CFT in e.g. [9, 10, 11, 12, I, II, 13, III, 14, IV]. Integrable lattice realisations have been found in [15, 16]. Totally transmissive defects commute by definition with the action of the generators L_m, \bar{L}_m of conformal transformations; they can be deformed without affecting the value of correlators, as long as they are not taken across a field insertion. Accordingly we will follow [17] in referring to totally transmissive defects also as *topological defects*.

Topological defects appear in fact very naturally in the context of lattice models when one studies order-disorder dualities [18, 19, 20]. In that context, a correlator of local fields is equated to a correlator of lines of inhomogeneity in the lattice couplings which end at the positions of the field insertions. The path by which these lines connect the various insertion points is a gauge choice and can be varied without changing the value of the correlator. The continuum limit of such lines of inhomogeneity provides one example of topological defects.

Correlators of disorder fields related to the internal \mathbb{Z}_2 symmetry of certain minimal models have been considered in [21, 22], while correlators of disorder fields for a \mathbb{Z}_2 symmetry which in addition obey a level-2 null vector relation have recently been given a statistical interpretation in terms of winding numbers in a loop gas model [23].

Topological defects also appear when compactifying topologically twisted four-dimensional $N=4$ super Yang-Mills theory to two dimensions [24]. They arise as the images of certain topological Wilson line operators in four dimensions.

We are interested in situations like the one in figure 1, where different parts of a world sheet can be in different ‘phases’ which are joined by one-dimensional phase boundaries. By this we mean the following. The chiral symmetry of a rational CFT can be encoded in a rational vertex algebra \mathcal{V} . As mentioned above, one can fix a full CFT with symmetry \mathcal{V} by giving a symmetric special Frobenius algebra A in $\mathcal{C} = \mathcal{R}ep(\mathcal{V})$. If the field content, OPEs, possible boundary conditions, etc. in some region of the world sheet are that of a full CFT obtained from an algebra A , we say that this part of the world sheet is in *phase* $CFT(A)$. Different regions, with (generically different, but possibly also equal) phases, meet along phase boundaries. These phase boundaries can carry their own fields, called defect fields, which can in particular change the type of phase boundary. Here we will only consider phase boundaries that are transparent to all fields in \mathcal{V} , so that they are described by topological defects. An important aspect of the TFT approach is that CFT quantities have a natural counterpart in the category \mathcal{C} . For example, boundary conditions preserving \mathcal{V} correspond to modules of the algebra A , and topological defects that separate two phases $CFT(A)$ and $CFT(B)$ on a world

sheet are given by A - B -bimodules. If a defect cannot be written as a superposition of other defects, it corresponds to a simple bimodule; accordingly we refer to such a defect also as a ‘simple’ defect.

Consider two simple defects running parallel to each other. In the limit of vanishing distance they fuse to a single defect which is, in general, not a simple defect. This gives rise to a fusion algebra of defects [10, 25, 15, 12]. In terms of the category \mathcal{C} , this corresponds to the tensor product of bimodules [I, 14]. Specifically, if the two defects are labelled by an A - B -bimodule X and a B - C -bimodule Y , then the fused defect is labelled by the A - C -bimodule $X \otimes_B Y$.

While there is a version of the TFT approach also for unoriented world sheets, apart from section 3.4 we will in this paper only consider oriented world sheets. Some further information on defects in the unoriented case can be found in section 3.8 of [II].

The main results of this paper are the following.

- To distinguish two topological defects labelled by non-isomorphic bimodules it suffices to compare their action on bulk fields (as opposed to considering e.g. also the action on disorder fields or on boundary conditions); the action on a bulk field consists in surrounding the field with a little defect loop and contracting that loop to zero size. This applies in particular to defects which separate different phases $\text{CFT}(A)$ and $\text{CFT}(B)$, so that their action turns a bulk field of $\text{CFT}(A)$ into a bulk field of $\text{CFT}(B)$ (which may be zero). This is shown in proposition 2.8.

- While every symmetric special Frobenius algebra A in $\mathcal{C} = \mathcal{R}ep(\mathcal{V})$ gives rise to a full CFT, not all of these full CFTs are distinct. As already announced in [0], Morita equivalent algebras give equivalent full CFTs on oriented world sheets. The proof of this claim, together with an explanation of the relevant concept of equivalence of full CFTs, is provided in section 3.3; it requires appropriate manipulations of phase-changing defects. In section 3.4 analogous considerations are carried out for unoriented world sheets, leading to the notion of ‘Jandl-Morita equivalence’ of algebras.

- In addition to being labelled by a bimodule, defect lines also carry an orientation. A defect X which has the same full CFT on either side is called *group-like* iff, upon fusing two defects which are both labelled by X but which have opposite orientation, one is left only with the invisible defect, i.e. the two defects disappear from the world sheet. Such defects form a group. It turns out that this is a group of internal symmetries for CFT correlators on world sheets of arbitrary genus; this is established in section 3.1 (see also [14]). Owing to the result mentioned in the first point, two different defect-induced symmetries can already be distinguished by their action on bulk fields.

- In theorem 3.9 we extend these results to order-disorder dualities. We establish a simple characterisation of the defects that lead to such dualities: A defect Y is a duality defect iff there is at least one other defect, labelled by Y' , say, such that fusing Y and Y' results in a superposition of only group-like defects. Moreover, in this case one can simply take $Y' = Y$, but with orientation opposite to that of the defect Y . This allows one to read off duality symmetries of the CFT from the fusion algebra of defects. Such dualities can relate correlators of different full CFTs, too. This is shown in the example of the tetracritical Ising model and the three-states Potts model in section 6.4.

- In proposition 3.13 we show that if two phases $\text{CFT}(A)$ and $\text{CFT}(B)$ can be separated by a duality defect Y , then the torus partition function of $\text{CFT}(A)$ can be written in terms of partition

functions with defect lines of $\text{CFT}(B)$ in a manner that resembles the way the partition function of an orbifold is expressed as a sum over twisted partition functions. The relevant group elements are the group-like defects appearing in the fusion of the defect Y with an orientation-reversed copy of itself. In the case $A = B$ this is precisely the ‘auto-orbifold property’ observed in [26].

■ When the full CFT is of simple current type, one can use tools developed in [III] to study topological defects; this is done in section 5. For instance, one can make general statements about the symmetry group, and in particular determine a subgroup that is present generically in all models of a given simple current type, see section 5.3.

■ In proposition 4.7 we establish a result for bimodules of simple symmetric Frobenius algebras in modular tensor categories, namely that every A - A -bimodule is a submodule of the tensor product of two α -induced bimodules with opposite braiding in the two α -inductions. An analogous statement has been proved in [27] in the context of subfactors, and has been conjectured for modular tensor categories in general in [28]. This result has a simple interpretation in terms of the CFT associated to the algebra A : it is equivalent to the statement that every defect line that has the phase $\text{CFT}(A)$ on both sides can end in the bulk at an appropriate disorder field.

2 Algebras and CFT

As mentioned in the introduction, in the TFT approach to RCFT a full CFT is given by two pieces of data: a rational vertex algebra \mathcal{V} , which encodes the chiral symmetries of the CFT, and a symmetric special Frobenius algebra A in the representation category $\mathcal{C} = \mathcal{R}ep(\mathcal{V})$, which determines the full CFT whose chiral symmetries include \mathcal{V} .

In section 2.1 we review some aspects of algebra in braided tensor categories, in section 2.2 the relation of these quantities to CFT is described, and in section 2.3 we describe some identities that are useful in calculations with correlators involving topological defect lines. These rules will be derived later in section 4.1 with the help of the TFT approach. Finally, in section 2.4 it is established that all defects can be distinguished by their action on bulk fields.

2.1 Algebraic preliminaries

In order to investigate the properties of topological defects we will need the notion of algebras, modules and bimodules in tensor categories and, more specifically, in modular tensor categories. We adopt the following definition, which is slightly more restrictive than the original one in [29].

Definition 2.1 :

A tensor category is called *modular* iff

- (i) The tensor unit is simple.¹
- (ii) \mathcal{C} is abelian, \mathbb{C} -linear and semisimple.
- (iii) \mathcal{C} is ribbon: There are families $\{c_{U,V}\}$ of braiding, $\{\theta_U\}$ of twist, and $\{d_U, b_U\}$ of evaluation and coevaluation morphisms satisfying the relevant properties (see e.g. definition 2.1 in [30]).
- (iv) \mathcal{C} is Artinian (or ‘finite’), i.e. the number of isomorphism classes of simple objects is finite.

¹ In a semisimple \mathbb{C} -linear tensor category every object U that is simple (i.e., does not have any proper subobject), is also absolutely simple, i.e. satisfies $\text{End}(U) \cong \mathbb{C} id_U$.

(v) The braiding is maximally non-degenerate: the numerical matrix s with entries $s_{i,j} := (d_{U_j} \otimes \tilde{d}_{U_i}) \circ [id_{U_j^\vee} \otimes (c_{U_i, U_j} \circ c_{U_j, U_i}) \otimes id_{U_i^\vee}] \circ (\tilde{b}_{U_j} \otimes b_{U_i})$ for $i, j \in \mathcal{I}$ is invertible.

Here we denote by $\{U_i \mid i \in \mathcal{I}\}$ a (finite) set of representatives of isomorphism classes of simple objects; we also take $U_0 := \mathbf{1}$ as the representative for the class of the tensor unit. Further, \tilde{d}_U and \tilde{b}_U denote the left duality morphisms constructed from right duality, braiding and twist.

Remark 2.2:

(i) The relevance of modular tensor categories for the present discussion derives from the fact that for certain conformal vertex algebras \mathcal{V} , the category $\mathcal{R}ep(\mathcal{V})$ of representations is modular. It is this class of vertex algebras to which we refer as being ‘rational’; sufficient conditions on \mathcal{V} for $\mathcal{R}ep(\mathcal{V})$ to be modular have been derived in [31].

However, it is important to stress that for the algebraic computations involving modular categories we do not assume that the category in question can be realised as $\mathcal{R}ep(\mathcal{V})$ for some \mathcal{V} .

(ii) We do not require the modular tensor categories we consider to be unitary (in the sense of [29, Sect. II:5.5]). In particular, we allow for the possibility that some simple objects have negative quantum dimension, and that some non-simple objects have zero quantum dimension.

We will make ample use of the graphical notation of [32] to represent morphisms in braided tensor categories. Our conventions, as well as more references, are given in [I, sect. 2] and [30, sect. 2.1].

Let \mathcal{C} be a tensor category. A *Frobenius algebra* $A = (A, m, \eta, \Delta, \varepsilon)$ in \mathcal{C} is, by definition, an object of \mathcal{C} carrying the structures of a unital associative algebra (A, m, η) and of a counital coassociative coalgebra (A, Δ, ε) in \mathcal{C} , with the algebra and coalgebra structures satisfying the compatibility requirement that the coproduct $\Delta: A \rightarrow A \otimes A$ is a morphism of A -bimodules (or, equivalently, that the product $m: A \otimes A \rightarrow A$ is a morphism of A -bi-comodules). A Frobenius algebra is called *special* iff a scalar multiple of the coproduct is a right-inverse to the product – this means in particular that the algebra is separable – and a multiple of the unit $\eta: \mathbf{1} \rightarrow A$ is a right-inverse to the counit $\varepsilon: A \rightarrow \mathbf{1}$. Suppose now that \mathcal{C} is also sovereign (i.e. it has compatible left/right dualities). There are two isomorphisms $A \rightarrow A^\vee$ that are naturally induced by product, counit and duality; A is called *symmetric* iff these two isomorphisms coincide. For more details and references, as well as graphical representations of the above conditions and morphisms we refer the reader to [I, sect. 3] and [30, sect. 2.3].

Actually a special Frobenius algebra has a one-parameter family of coproducts, obtained by rescaling coproduct and counit by an invertible scalar; out of this family, we always take the element for which

$$m \circ \Delta = id_A \quad \text{and} \quad \varepsilon \circ \eta = \dim(A) id_{\mathbf{1}} \tag{2.1}$$

holds.

Modules and bimodules of an algebra are defined analogously to the case of vector spaces, too. Since bimodules will be used frequently let us state explicitly

Definition 2.3:

Let $A = (A, m_A, \eta_A)$ and $B = (B, m_B, \eta_B)$ be (unital, associative) algebras in a strict² tensor category \mathcal{C} . An *A-B-bimodule* is a triple $X = (\dot{X}, \rho_l, \rho_r)$, where \dot{X} is an object of \mathcal{C} ,

² Recall that by Mac Lane’s coherence theorem, every tensor category is equivalent to a strict one.

$\rho_l \in \text{Hom}(A \otimes \dot{X}, \dot{X})$, and $\rho_r \in \text{Hom}(\dot{X} \otimes B, \dot{X})$ such that the following equalities hold:

- (i) Unit property: $\rho_l \circ (\eta_A \otimes id_{\dot{X}}) = id_{\dot{X}}$ and $\rho_r \circ (id_{\dot{X}} \otimes \eta_B) = id_{\dot{X}}$.
- (ii) Representation property: $\rho_l \circ (m_A \otimes id_{\dot{X}}) = \rho_l \circ (id_A \otimes \rho_l)$ and $\rho_r \circ (id_{\dot{X}} \otimes m_B) = \rho_r \circ (\rho_r \otimes id_B)$.
- (iii) Left and right action commute: $\rho_l \circ (id_A \otimes \rho_r) = \rho_r \circ (\rho_l \otimes id_B)$.

The definition for non-strict tensor categories involves associators and unit constraints in the appropriate places. Note also that an A -left module is the same as an A -1-bimodule.

Given two algebras A, B , and two bimodules X, Y , the space of bimodule intertwiners from X to Y is given by

$$\text{Hom}_{A|B}(X, Y) = \{f \in \text{Hom}(\dot{X}, \dot{Y}) \mid f \circ \rho_l = \rho_l \circ (id_A \otimes f), f \circ \rho_r = \rho_r \circ (f \otimes id_B)\}. \quad (2.2)$$

In this way we obtain the category $\mathcal{C}_{A|B}$ whose objects are A - B -bimodules and whose morphism spaces are given by $\text{Hom}_{A|B}(\cdot, \cdot)$. If the algebras are also special Frobenius, many properties pass from \mathcal{C} to $\mathcal{C}_{A|B}$.

Proposition 2.4:

Let A and B be special Frobenius algebras in a strict tensor category \mathcal{C} .

- (i) If \mathcal{C} is idempotent complete, then so is $\mathcal{C}_{A|B}$.
- (ii) If \mathcal{C} is abelian, then so is $\mathcal{C}_{A|B}$.
- (iii) If \mathcal{C} is semisimple, then so is $\mathcal{C}_{A|B}$.

If in addition, \mathcal{C} only has a finite number of non-isomorphic simple objects, then so has $\mathcal{C}_{A|B}$.

The proof is analogous to the proof of the corresponding statements for \mathcal{C}_A in section 5 of [33].

If the category \mathcal{C} is ribbon,³ then we get a contravariant functor $(\cdot)^\vee: \mathcal{C}_{A|B} \rightarrow \mathcal{C}_{B|A}$. For an object X of $\mathcal{C}_{A|B}$ we set

$$X^\vee := (\dot{X}^\vee, \tilde{\rho}_l, \tilde{\rho}_r) \quad (2.3)$$

where \dot{X}^\vee is the dual in \mathcal{C} and

$$\tilde{\rho}_l := (\tilde{d}_B \otimes id_{\dot{X}^\vee}) \circ (id_B \otimes \rho_r^\vee), \quad \tilde{\rho}_r := (id_{\dot{X}^\vee} \otimes d_A) \circ (\rho_l^\vee \otimes id_A). \quad (2.4)$$

On morphisms, $(\cdot)^\vee$ just takes f to f^\vee , where we understand the morphism spaces of $\mathcal{C}_{A|B}$ and $\mathcal{C}_{B|A}$ as subspaces of morphism spaces in \mathcal{C} and use the action of the duality of \mathcal{C} on morphisms. One verifies that X^\vee is indeed a B - A -bimodule. In graphical notation, the left/right action (2.3) looks as follows (compare equation (2.40) of [II])



³ In fact it is enough to require that \mathcal{C} is spherical. But later on we will only be interested in categories \mathcal{C} that are ribbon, and even modular.

Another important concept is the tensor product of two bimodules over the intermediate algebra. We will define it in terms of idempotents. Suppose that \mathcal{C} is an idempotent complete ribbon category³. Let A , B and C be special Frobenius algebras, let X be an A - B -bimodule and let Y be a B - C bimodule. Consider the morphism $P_{X,Y} \in \text{Hom}(X \otimes Y, X \otimes Y)$ given by

$$P_{X,Y} = \begin{array}{c} \begin{array}{c} X \quad Y \\ \downarrow \quad \downarrow \\ \text{---} \text{---} \text{---} \\ \downarrow \quad \downarrow \\ X \quad Y \end{array} \end{array} \quad (2.6)$$

Noting that $X \otimes Y$ is an A - C -bimodule, one checks that in fact $P_{X,Y} \in \text{Hom}_{A|C}(X \otimes Y, X \otimes Y)$. Furthermore, using that B is special Frobenius, one can verify that $P_{X,Y}$ is an idempotent (this is analogous to the calculation in e.g. equation (5.127) of [I]). By proposition 2.4, the category $\mathcal{C}_{A|C}$ is idempotent complete, so there exists an object $\text{Im}(P_{X,Y})$ in $\mathcal{C}_{A|C}$, a monomorphism $e_{X,Y} \in \text{Hom}_{A|C}(\text{Im}(P_{X,Y}), X \otimes Y)$ and an epimorphism $r_{X,Y} \in \text{Hom}_{A|C}(X \otimes Y, \text{Im}(P_{X,Y}))$ such that $P_{X,Y} = e_{X,Y} \circ r_{X,Y}$. We define

$$X \otimes_B Y := \text{Im}(P_{X,Y}). \quad (2.7)$$

In fact we get a bifunctor $\otimes_B : \mathcal{C}_{A|B} \times \mathcal{C}_{B|C} \rightarrow \mathcal{C}_{A|C}$, by also defining the action on morphisms $f \in \text{Hom}_{A|B}(X, X')$ and $g \in \text{Hom}_{B|C}(Y, Y')$ as

$$f \otimes_B g := r_{X',Y'} \circ (f \otimes g) \circ e_{X,Y}. \quad (2.8)$$

When verifying functoriality one needs that $(f \otimes g) \circ P_{X,Y} = P_{X',Y'} \circ (f \otimes g)$, which follows from the fact that f and g commute with the action B . This bifunctor also admits natural associativity constraints and unit constraints, see e.g. [34, 35] for more details.

Given objects U and V of a category and idempotents $p \in \text{End}(U)$ and $q \in \text{End}(V)$, the following subspaces of $\text{Hom}(U, V)$ will be of interest to us:

$$\begin{aligned} \text{Hom}_{(p)}(U, V) &:= \{f \in \text{Hom}(U, V) \mid f \circ p = f\} \quad \text{and} \\ \text{Hom}^{(q)}(U, V) &:= \{f \in \text{Hom}(U, V) \mid q \circ f = f\}. \end{aligned} \quad (2.9)$$

Note that the embedding and restriction morphisms defined before (2.7) give isomorphisms

$$\begin{aligned} \text{Hom}_{A|C}^{(P_{X,Y})}(X \otimes Y, Z) &\xrightarrow{\cong} \text{Hom}_{A|C}(X \otimes_B Y, Z) \quad \text{and} \\ \text{Hom}_{A|C}^{(P_{X,Y})}(Z, X \otimes Y) &\xrightarrow{\cong} \text{Hom}_{A|C}(Z, X \otimes_B Y), \end{aligned} \quad (2.10)$$

for X, Y bimodules as above and Z an A - C -bimodule. The first isomorphism is given by $f \mapsto f \circ e_{X,Y}$, and the second by $g \mapsto r_{X,Y} \circ g$.

Remark 2.5:

(i) We have seen in (2.3) that, given an A - B -bimodule X , the duality of \mathcal{C} can be used to define a dual bimodule X^\vee that is a B - A -bimodule. For these bimodule dualities analogous rules are valid as for ordinary dualities (where the dual object lies in the same category). In particular,

for A and B simple symmetric special Frobenius algebras and X a simple A - B -bimodule, the tensor product $X \otimes_B X^\vee$ contains A with multiplicity one, while $X^\vee \otimes_A X$ contains B with multiplicity one, see lemma 3.3 below.

(ii) Note that in this way $\mathcal{C}_{A|A}$ becomes a tensor category with dualities. If A is simple, the quantum dimension $\dim_A(\cdot)$ in $\mathcal{C}_{A|A}$ is given in terms of the quantum dimension $\dim(\cdot)$ of \mathcal{C} by $\dim_A(X) = \dim(X)/\dim(A)$ (this follows for example from writing out the definition of $\dim_A(\cdot)$ and using lemma 4.1 below).

(iii) For a modular tensor category \mathcal{C} one can consider the 2-category $\mathcal{Frob}_{\mathcal{C}}$ whose objects are symmetric special Frobenius algebras in \mathcal{C} (compare also [36, 37, 35] and section 3 of [38]). The categories $\mathcal{Frob}_{\mathcal{C}}(A, B)$ (whose objects are the 1-morphisms $A \rightarrow B$) are the categories $\mathcal{C}_{A|B}$ of A - B -bimodules and the 2-morphisms are morphisms of bimodules. As we will see in the next section, the objects of $\mathcal{Frob}_{\mathcal{C}}$ can be interpreted as possible phases of a CFT with given chiral symmetry, and the 1-morphisms as the allowed types of topological boundaries between different phases of the theory. In fact, from this point of view it is not surprising that CFT world sheets with topological phase boundaries (but without field insertions) look very similar to 2-categorical string diagrams, see e.g. [39] and section 2.2 of [40].

Given an A - B -bimodule $X = (\dot{X}, \rho_l, \rho_r)$ in a ribbon category \mathcal{C} , and two objects U, V of \mathcal{C} , we can use the braiding to define several bimodules structures on the object $U \otimes \dot{X} \otimes V$ of \mathcal{C} . The one which we will use is

$$U \otimes^+ X \otimes^- V := (U \otimes \dot{X} \otimes V, (id_U \otimes \rho_l \otimes id_V) \circ (c_{U,A}^{-1} \otimes id_{\dot{X}} \otimes id_V), (id_U \otimes \rho_r \otimes id_V) \circ (id_U \otimes id_{\dot{X}} \otimes c_{B,V}^{-1})). \quad (2.11)$$

Pictorially, the left and right actions of A and B , respectively, on $U \otimes^+ X \otimes^- V$ are given by



$$(2.12)$$

2.2 Relation between algebras and CFT quantities

The construction of correlation functions of a full rational CFT on oriented world sheets with (possibly empty) boundary can be described in terms of a modular tensor category \mathcal{C} and a symmetric special Frobenius algebra A in \mathcal{C} . These data can be realised using a rational conformal vertex algebra \mathcal{V} with $\mathcal{C} \simeq \mathcal{Rep}(\mathcal{V})$ and for A an ‘open-string vertex algebra’ in the sense of [41]. We denote this full CFT by $\text{CFT}(A)$. Recall that we denote by $\{U_i \mid i \in \mathcal{I}\}$ a set of representatives for the isomorphism classes of simple objects in \mathcal{C} .

Quantities of interest for the CFT do have natural algebraic counterparts. For example, conformal boundary conditions (which preserve also \mathcal{V}) are labelled by A -left modules,⁴ see [1,

⁴ The use of left modules rather than right modules is dictated by the conventions for the relative orientation of bulk and boundary.

sect. 4.4] for details. By proposition 2.4, the category \mathcal{C}_A of A -left modules (which are the same as A -1-bimodules) is again semisimple and has a finite number of simple objects. We denote by

$$\{M_\mu \mid \mu \in \mathcal{J}_A\} \quad (2.13)$$

a set of representatives of isomorphism classes of simple A -left modules. Simple A -modules correspond to boundary conditions that cannot be written as a direct sum of other boundary conditions.

The multiplicity space of boundary fields and boundary changing fields which join a segment of boundary labelled M to a segment labelled N and that transform in the representation U_k of \mathcal{V} is given by $\text{Hom}_A(M \otimes U_k, N)$, more details can be found in [IV, sect. 3.2]. In fact, the algebra A is a left module over itself and thus a preferred boundary condition. The multiplicity spaces for the boundary fields are then $\text{Hom}_A(A \otimes U_k, A) \cong \text{Hom}(U_k, A)$ and the OPE of boundary fields is just given by the multiplication on A (see [I, sect. 3.2] and [IV, remark 6.2]).

Thus, the pair (\mathcal{V}, A) does in fact correspond to a full CFT defined on oriented surfaces with boundary, *together with a distinguished boundary condition*, namely the one labelled by the A -module A . The algebras of boundary fields on other boundary conditions lead to symmetric special Frobenius algebras, too. These are all Morita equivalent to A (see section 4.1 of [I] and theorem 2.14 in [II]). In section 3.3 we will show, by manipulating topological defects, that, conversely, two Morita equivalent algebras lead to equivalent full CFTs.

As mentioned in the introduction, we are interested in situations in which different parts of a world sheet can be in different *phases*, i.e. different full CFTs are assigned to different parts of a world sheet. We refer to the lines along which such phases meet as phase boundaries or defect lines. We demand all these full CFTs to have the same underlying rational vertex algebra \mathcal{V} , and we demand that correlators are continuous when a bulk field for which both the left moving and right moving degrees of freedom transform in the defining representation \mathcal{V} of the vertex algebra is taken across a phase boundary. In particular, the phase boundaries we consider are transparent to the holomorphic and anti-holomorphic components of the stress tensor and are therefore realised by topological defect lines. We can then label the various parts of a world sheet by symmetric special Frobenius algebras A_1, A_2, A_3, \dots in $\mathcal{R}ep(\mathcal{V})$. A phase boundary between phases $\text{CFT}(A)$ and $\text{CFT}(B)$ is labelled by an A - B -bimodule X . We call such a phase boundary an *A - B -defect line of type X* , or an *A - B -defect X* , for short. The *invisible defect* in phase $\text{CFT}(A)$ is labelled by the algebra A itself. The category $\mathcal{C}_{A|B}$ of A - B -bimodules is again semisimple with a finite number of simple objects [33]. We denote by

$$\{X_\mu \mid \mu \in \mathcal{K}_{AB}\} \quad (2.14)$$

a set of representatives of isomorphism classes of simple A - B -bimodules.

The multiplicity space of *bulk fields* that transform in the representation $U_i \times U_j$ of the left- and right-moving copies of \mathcal{V} (or in short, in the left/right representation (i, j)) is given by $\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$. Analogously, the space of *defect fields*, which join a defect labelled by an A - B -bimodule X to a defect labelled by another A - B -bimodule Y is given by $\text{Hom}_{A|B}(U_i \otimes^+ X \otimes^- U_j, Y)$, see [IV, sect. 3.3 & 3.4] for details in the case $A = B$. By the term *disorder field* we refer to a defect field that changes a defect labelled by an A - A -bimodule X to the invisible defect, labelled by A . In other words, a defect line starts (or ends) at a disorder field.

Topological defects can be deformed continuously on the world sheet with the field insertion points removed without changing the value of a correlator. For brevity, we will refer in the sequel to the situation when two defect lines can be taken close to each other as dealing with two *parallel* defect lines. The dimensions of the multiplicity spaces are encoded in the partition function of a torus on which two parallel defect lines of opposite orientation, labelled X and Y , are inserted along a non-contractible cycle. We will only need the case $A = B$, in which case the coefficients of the expansion of the partition function in terms of simple characters are given by

$$Z(A)_{ij}^{X|Y} = \dim_{\mathbb{C}} \text{Hom}_{A|A}(U_i \otimes^+ X \otimes^- U_j, Y). \quad (2.15)$$

Such partition functions were first investigated in [10, 11]. Further details can be found in section 5.10 of [I] and section 2.2 of [IV]. Note also that the matrix $Z(A) \equiv Z(A)^{A|A}$, which gives the multiplicities of bulk fields, describes the modular invariant torus partition of the CFT in phase $\text{CFT}(A)$.

For some of the calculations later on we need to choose bases in various morphism spaces. First, given an A - B -bimodule X , we choose a basis for the morphisms which describe its decomposition into a finite direct sum of simple bimodules. Let

$$e_{X,\mu}^\alpha \in \text{Hom}_{A|B}(X_\mu, X) \quad \text{and} \quad r_{X,\mu}^\alpha \in \text{Hom}_{A|B}(X, X_\mu), \quad (2.16)$$

with $\mu \in \mathcal{K}_{AB}$ and $\alpha = 1, \dots, \dim_{\mathbb{C}} \text{Hom}_{A|B}(X_\mu, X)$, be elements of dual bases in the sense that $r_{X,\mu}^\alpha \circ e_{X,\mu}^\beta = \delta_{\alpha,\beta} \text{id}_{X_\mu}$. In particular, $(X_\mu, e_{X,\mu}^\alpha, r_{X,\mu}^\alpha)$ is a retract of X . Similarly we choose

$$\Lambda_{(\rho\sigma)\mu}^\alpha \in \text{Hom}_{A|C}(X_\rho \otimes_B X_\sigma, X_\mu) \quad \text{and} \quad \bar{\Lambda}_{(\rho\sigma)\mu}^\alpha \in \text{Hom}_{A|C}(X_\mu, X_\rho \otimes_B X_\sigma), \quad (2.17)$$

where $\mu \in \mathcal{K}_{AC}$, $\rho \in \mathcal{K}_{AB}$, $\sigma \in \mathcal{K}_{BC}$ and $\alpha = 1, \dots, \dim_{\mathbb{C}} \text{Hom}_{A|C}(X_\mu, X_\rho \otimes_B X_\sigma)$. Again we demand that $\Lambda_{(\rho\sigma)\mu}^\alpha \circ \bar{\Lambda}_{(\rho\sigma)\mu}^\beta = \delta_{\alpha,\beta} \text{id}_{X_\mu}$. Via the isomorphisms (2.10), the bases (2.17) also yield bases

$$\Lambda_{(\rho\sigma)\mu}^\alpha \in \text{Hom}_{A|C}(P_{X_\rho, X_\sigma})(X_\rho \otimes X_\sigma, X_\mu) \quad \text{and} \quad \bar{\Lambda}_{(\rho\sigma)\mu}^\alpha \in \text{Hom}_{A|C}(P_{X_\rho, X_\sigma})(X_\mu, X_\rho \otimes X_\sigma). \quad (2.18)$$

We will use the same symbols as in (2.17) for these bases; it will be clear from the context to which morphism space we refer. Finally we need a basis for the multiplicity space of defect fields; we denote them by

$$\xi_{(i\mu j)\nu}^\alpha \in \text{Hom}_{A|B}(U_i \otimes^+ X_\mu \otimes^- U_j, X_\nu) \quad \text{and} \quad \bar{\xi}_{(i\mu j)\nu}^\alpha \in \text{Hom}_{A|B}(X_\nu, U_i \otimes^+ X_\mu \otimes^- U_j), \quad (2.19)$$

respectively, where $i, j \in \mathcal{I}$, $\mu, \nu \in \mathcal{K}_{AB}$ and α runs from 1 to the dimension of the space. And again we require the two sets of vectors to be dual to each other, $\xi_{(i\mu j)\nu}^\alpha \circ \bar{\xi}_{(i\mu j)\nu}^\beta = \delta_{\alpha,\beta} \text{id}_{X_\nu}$.

Since we allow for categories \mathcal{C} in which non-simple objects can have zero quantum dimension, the following result is useful.

Lemma 2.6:

Let A and B be simple symmetric special Frobenius algebras in \mathcal{C} , and let X be a simple A - B -bimodule. Then $\dim(X) \neq 0$ (as an object in \mathcal{C}).

Proof:

The argument is analogous to the one that shows that the quantum dimension $\dim(U)$ of a

simple object in a semisimple tensor category with dualities cannot be zero, see e.g. section 2 of [42]. Consider the two morphisms

$$\beta_X := \text{[diagram]} \quad \text{and} \quad \tilde{\delta}_X := \text{[diagram]} \quad (2.20)$$

The morphism in the dashed boxes is the idempotent P_{X, X^\vee} introduced in (2.6) whose image is the tensor product over B . Note that the morphisms

$$b_X = r_{X, X^\vee} \circ \beta_X \in \text{Hom}_{A|A}(A, X \otimes_B X^\vee) \quad \text{and} \quad \tilde{d}_X = \tilde{\delta}_X \circ e_{X, X^\vee} \in \text{Hom}_{A|A}(X \otimes_B X^\vee, A) \quad (2.21)$$

are the duality morphisms in the bimodule categories, cf. remark 2.5. Since X is simple, it follows from lemma 3.3 below that both morphism spaces in (2.21) are one-dimensional. Thus they are spanned by b_X and by \tilde{d}_X , respectively, provided that these morphisms are nonzero. Writing $P_{X, X^\vee} = e_{X, X^\vee} \circ r_{X, X^\vee}$ we see that β_X and $\tilde{\delta}_X$ can be written as $\beta_X = e_{X, X^\vee} \circ b_X$ and $\tilde{\delta}_X = \tilde{d}_X \circ r_{X, X^\vee}$. To show that e.g. b_X is nonzero, it is therefore enough to show that β_X is nonzero. This in turn follows from

$$\text{[diagram]} = \text{[diagram]} \quad (2.22)$$

where the morphism in the dashed box is β_X and we used properties of A and B as well as of the duality in \mathcal{C} . It follows that $\tilde{d}_X \circ b_X = \lambda \text{id}_A$ for some $\lambda \in \mathbb{C}^\times$. By taking the trace one determines $\lambda = \dim(X)/\dim(A)$, so that in particular $\dim(X) \neq 0$. (The dimension of a special algebra A is required to be nonzero, see definition 3.4 (i) of [I].) \checkmark

2.3 Calculating with defects

In this section we present a number of rules for how to compute with topological defects. Later on, these rules will be deduced with the help of the TFT construction, see section 4 below.

First of all, two defects can fuse to produce another defect, and a defect can act on a boundary, thereby changing the boundary condition. These notions were first introduced (for A - A -defects, in our terminology) from the CFT point of view in [10, 25, 13] and from a lattice perspective in [15, 43]⁵.

Consider an A - B -defect labelled X and a B - C -defect labelled Y running parallel to each other. We can move these defects close to each other and then replace them by a single new defect of type A - C . This defect is then labelled by the A - C -bimodule $X \otimes_B Y$. Similarly, if an A - B -defect X is running close to a B -boundary M , it can be replaced by an A -boundary labelled by the A -module $X \otimes_B M$. Graphically,

(2.23)

The algebra A is a bimodule over itself, and it obeys $A \otimes_A X \cong X$ for any A - B -bimodule X . In fact, A labels the *invisible defect* in $\text{CFT}(A)$; defect lines labelled by A can be omitted without changing the value of a correlator.

Defects can also be joined. The junction is labelled by an element of the relevant morphism space of bimodules. For example, when joining two A - B -defects X and X' , or an A - B -defect X and a B - C -defect Y to an A - C -defect Z , according to

(2.24)

the junctions get labelled by morphisms $\alpha \in \text{Hom}_{A|B}(X', X)$ and $\beta \in \text{Hom}_{A|C}(Z, X \otimes_B Y)$, respectively.⁶ Note also that a junction linking an A - A -defect X to the invisible defect A is labelled by an element α of $\text{Hom}_{A|A}(A, X)$. In particular,

(2.25)

As a consequence, a nontrivial simple defect X cannot just end in the interior of a world sheet.

An arbitrary defect can be decomposed into a sum of simple defects by using the direct sum decomposition of the corresponding bimodules (recall from section 2.2 that $\mathcal{C}_{A|A}$ is semisimple).

⁵ Topological defects joining different phases can also be realised in integrable lattice models [44].

⁶ Note that e.g. in the first picture in (2.24), the arrows on the defect lines point from X to X' , while the bimodule morphism labelling the point where they join goes from X' to X . This is just a choice of convention entering the TFT construction of an RCFT correlator (see sections 3.1 and 3.4 of [IV] as well as section 4.1 below for more details), and does not have any deeper significance.

Via the fusion procedure, this applies likewise to the situation that defects are running parallel to one another. In particular, we have

$$\begin{array}{c} \boxed{A} \quad \boxed{B} \\ \downarrow \quad \downarrow \\ X \end{array} = \sum_{\mu, \alpha} \begin{array}{c} \boxed{A} \quad \boxed{B} \\ \downarrow \quad \downarrow \\ X \\ \uparrow \quad \uparrow \\ \alpha \\ \uparrow \quad \uparrow \\ X_\mu \quad X \\ \uparrow \quad \uparrow \\ X \end{array} \quad \text{and} \quad \begin{array}{c} \boxed{A} \quad \boxed{B} \quad \boxed{C} \\ \downarrow \quad \downarrow \quad \downarrow \\ X_\rho \quad X_\sigma \end{array} = \sum_{\nu, \beta} \begin{array}{c} \boxed{B} \\ \downarrow \quad \downarrow \\ X_\rho \quad X_\sigma \\ \uparrow \quad \uparrow \\ \beta \\ \uparrow \quad \uparrow \\ X_\nu \quad X_\sigma \\ \uparrow \quad \uparrow \\ \bar{\beta} \\ \uparrow \quad \uparrow \\ X_\rho \quad X_\sigma \\ \uparrow \quad \uparrow \\ \boxed{B} \end{array} \quad (2.26)$$

where the μ -summation is over the label set \mathcal{K}_{AB} of (isomorphism classes of) simple A - B -bimodules, α runs over the basis (2.16) of $\text{Hom}_{A|B}(X_\mu, X)$, while ν runs over \mathcal{K}_{AC} , and β over the basis (2.17) of $\text{Hom}_{A|B}(X_\rho \otimes_B X_\sigma, X_\nu)$. Another useful identity is

$$\begin{array}{c} \boxed{A} \quad \boxed{B} \quad \boxed{C} \\ \downarrow \quad \downarrow \quad \downarrow \\ X_\rho \quad X_\sigma \end{array} = \sum_{\mu, \gamma} \frac{\dim X_\mu}{\dim X_\rho} \begin{array}{c} X_\rho \quad X_\sigma \\ \uparrow \quad \uparrow \\ \bar{\gamma} \\ \uparrow \quad \uparrow \\ X_\mu \quad X_\sigma \\ \uparrow \quad \uparrow \\ \gamma \\ \uparrow \quad \uparrow \\ X_\rho \quad X_\sigma \end{array} \quad (2.27)$$

An A - B -defect X_ν can also wrap around a B -bulk field, changing it into a disorder field of $\text{CFT}(A)$ by shrinking the defect loop:

$$\begin{array}{c} \boxed{B} \\ \downarrow \\ X_\nu \\ \uparrow \\ \alpha \\ \uparrow \\ \phi \\ \downarrow \\ \boxed{A} \end{array} \quad X_\mu \quad = \quad \begin{array}{c} X_\mu \\ \downarrow \\ \boxed{A} \\ \uparrow \\ D_{\mu\nu\alpha}(\phi) \end{array} \quad (2.28)$$

Here α is an element of $\text{Hom}_{A|B}(X_\mu \otimes_A X_\nu, X_\nu)$, and the bulk field is labelled by the morphism $\phi \in \text{Hom}_{B|B}(U \otimes^+ B \otimes^- V, B)$. The resulting disorder field starts the A - A -defect X_μ and is labelled by $D_{\mu\nu\alpha}(\phi)$, where $D_{\mu\nu\alpha}$ is a linear map

$$D_{\mu\nu\alpha} : \quad \text{Hom}_{B|B}(U \otimes^+ B \otimes^- V, B) \longrightarrow \text{Hom}_{A|A}(U \otimes^+ X_\mu \otimes^- V, A). \quad (2.29)$$

This map can be obtained explicitly in the TFT construction, see equation (4.14) below. In the special case that $X_\mu = A$ and that $\alpha = \rho_{X_\nu}$ is given by the representation morphism, we abbreviate $D_\nu \equiv D_{\mu\nu\alpha}$, i.e. write

$$\begin{array}{c} \boxed{B} \\ \downarrow \\ X_\nu \\ \uparrow \\ \alpha \\ \uparrow \\ \phi \\ \downarrow \\ \boxed{A} \end{array} = \begin{array}{c} \boxed{A} \\ \downarrow \\ \bullet \\ \uparrow \\ D_\nu(\phi) \end{array} \quad (2.30)$$

If, still for $X_\mu = A$ and α the representation morphism, the defect line that wraps around the bulk field is labelled by an arbitrary (not necessarily simple) A - B -bimodule X instead of X_ν , we write analogously D_X for the resulting linear map. These maps obey, for Y a B - C -defect and ϕ a bulk field of $\text{CFT}(C)$,

$$D_X \circ D_Y(\phi) = D_{X \otimes_B Y}(\phi). \quad (2.31)$$

When $A = B = C$, this gives a representation of the fusion algebra of A - A -defects on each of the spaces $\text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A)$.

An A - B -defect X can also start and end on the boundary of the world sheet. The corresponding junctions are again labelled by appropriate morphisms,

(2.32)

respectively, where M is an A -module, i.e. a boundary condition for the phase $\text{CFT}(A)$, N a B -module, α a morphism in $\text{Hom}_A(M, X \otimes_B N)$, and $\beta \in \text{Hom}_A(X \otimes_B N, M)$. In this way one also obtains an action of defects on boundary fields, in the same spirit as in (2.28):

(2.33)

We will not use this transformation explicitly in the present paper, though, and accordingly we do not introduce a separate notation for the resulting boundary field (labelled $\tilde{\psi}$), unlike what we did in (2.29) for the bulk case. The fusion of defect lines to boundaries, and in particular their action on boundary fields, has also been studied in [13], where topological defects were used to deduce relations between boundary renormalisation group flows.

Finally, in a region of the world sheet in the phase $\text{CFT}(A)$ we can insert a little bubble of phase $\text{CFT}(B)$, via a small circular B - A -defect Y . If the algebra A is simple, then this merely changes the correlator by a factor $\dim(A)/\dim(Y)$. Hence in this case we obtain the identity

$$\left[\begin{array}{c} \boxed{A} \\ \blacksquare \end{array} \right] = \frac{\dim(A)}{\dim(Y)} \cdot \left[\begin{array}{c} \boxed{A} \\ \blacksquare \end{array} \right] \circlearrowleft \begin{array}{c} \boxed{B} \\ \blacksquare \end{array} \begin{array}{c} Y \end{array} \quad (2.34)$$

This leads to the notion of “*inflating a B - A -defect in a world sheet in phase $\text{CFT}(A)$ ”*, by which we refer to the following procedure. Let now both algebras A and B be simple, let Y be a B - A -bimodule, and let X be a connected world sheet in phase $\text{CFT}(A)$. We start by inserting a little circular defect labelled Y as in (2.34). The Y -loop separates X in regions ‘ A ’ and ‘ B ’. Now deform the loop until the ‘ A ’ area has shrunk to zero (this is only possible if X is connected). For example, on a genus one world sheet with connected boundary and one bulk insertion, we

have

$$(2.35)$$

In more detail, here we used the deformations

$$(2.36)$$

for the handle. The concept of inflating a topological defect in a world sheet will play a central role in the discussions below.

Remark 2.7:

Topological defects also play an important role in the recently established connection between the geometric Langlands program and dimensionally reduced topologically twisted $N = 4$ four-dimensional super Yang–Mills theory [24]. These Yang–Mills theories contain topological Wilson loop operators or, at different points in moduli space, topological ’t Hooft operators, both supported on oriented lines embedded in a four-manifold. One can consider such a theory on a product $\Sigma \times C$ of two two-manifolds, where one thinks of C as being ‘small’, so that at low energies one deals with an effective two-dimensional sigma-model on Σ . The image of the Wilson and ’t Hooft operators under the projection $\Sigma \times C \rightarrow \Sigma$ gives rise to (possibly point-like) topological defect lines on Σ .

Other phenomena discussed above, most prominently the action of defects on boundary conditions and boundary fields, play a crucial role in the identification of Hecke eigensheaves in the setting of [24], see section 6 there. In our context, a more general notion of ‘eigenbrane’ seems to be natural: an A -module M – corresponding to a boundary condition of the conformal field theory – is called an eigenbrane for the A - A -bimodule X – describing a defect – iff there exists an object U of \mathcal{C} such that

$$X \otimes_A M \cong M \otimes U. \tag{2.37}$$

Notice that here the role of the eigenvalues in [24] is taken over by objects of \mathcal{C} ; the case of Chan–Paton multiplicities considered in [24] amounts to requiring U to be of the form $\mathbf{1}^{\oplus n}$. While the generalised eigenvalue equation above can be formulated for all A - A -bimodules X , at the present stage it is not clear to us whether for a general CFT there is a distinguished subset of bimodules that play the role of Hecke operators.

2.4 Non-isomorphic bimodules label distinct defects

In this section we establish the following result. Let Y and Y' be B - A -bimodules. If labelling a given defect line by Y or by Y' gives the same result for all correlators, then already $Y \cong Y'$ as bimodules. This is a consequence of the next proposition, which makes the stronger statement that non-isomorphic B - A -defects differ in their action on at least one bulk field of $\text{CFT}(A)$, where the action is given by the maps D_X and D_Y defined below (2.30).

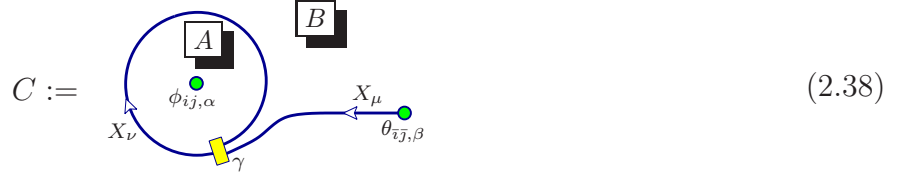
Proposition 2.8:

Let X and Y be B - A -bimodules. If for all $i, j \in \mathcal{I}$ and all $\phi \in \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$ we have $D_X(\phi) = D_Y(\phi)$, then already $X \cong Y$ as bimodules.

Remark 2.9:

Topological defects labelled by non-isomorphic bimodules can also be distinguished by their action on the collection of all boundary conditions and boundary fields. To see this one uses the fact that a B - A -bimodule gives rise to a module functor from \mathcal{C}_A to \mathcal{C}_B . Note that a module functor also acts on morphisms, which corresponds to the action of the B - A -defect on boundary fields. The important point in proposition 2.8 is that knowing the action on boundary fields is not required, but rather it suffices to consider the action on bulk fields.

As a preparation for the proof of proposition 2.8, we need to consider a certain two-point correlator on the Riemann sphere (which we identify with $\mathbb{C} \cup \{\infty\}$). The correlator we are interested in is



$$C := \text{Diagram} \quad (2.38)$$

where the bulk field in $\text{CFT}(A)$, inserted at some point $z \in \mathbb{C}$, is labelled by a morphism $\phi_{ij, \alpha} \in \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$, and the disorder field in $\text{CFT}(B)$, inserted at $w \in \mathbb{C}$, is labelled by $\theta_{\bar{i}, \bar{j}, \beta} \in \text{Hom}_{B|B}(U_{\bar{i}} \otimes^+ X_\mu \otimes^- U_{\bar{j}}, B)$. The label γ stands for a basis element of the morphism space $\text{Hom}_{B|A}(X_\nu, X_\mu \otimes_B X_\nu)$, and we take $\phi_{ij, \alpha}$ and $\theta_{\bar{i}, \bar{j}, \beta}$ to be given by the basis elements (2.19), i.e. $\phi_{ij, \alpha} = \xi_{(i0j)0}^\alpha$ and $\theta_{\bar{i}, \bar{j}, \beta} = \xi_{(\bar{i}\bar{\mu}\bar{j})0}^\beta$. As a two-point correlator on the Riemann sphere, C can be written as a product of conformal two-point blocks

$$C = \Gamma(\mu)_{ij\alpha\beta}^{\nu\gamma} \beta[\bar{i}, \bar{j}](z, w) \beta[j, \bar{j}](z^*, w^*), \quad (2.39)$$

where $\Gamma(\mu)_{ij\alpha\beta}^{\nu\gamma} \in \mathbb{C}$ is a constant and $\beta[k, \bar{k}](\zeta_1, \zeta_2)$ denotes a conformal two-point block with insertions at ζ_1, ζ_2 , i.e. a bilinear map from the two representation spaces $S_k \times S_{\bar{k}}$ of the chiral algebra to the complex numbers; see sections 5 and 6 of [IV] for more details. Here we are only interested in the prefactor $\Gamma(\mu)_{ij\alpha\beta}^{\nu\gamma}$.

Let R_1 be the set of all tuples $(ij\alpha\beta)$ for which α and β have a non-empty range, i.e. for which $Z(A)_{ij} > 0$ and $Z(B)_{\bar{i}\bar{j}}^{X_\mu|B} > 0$ (these numbers were defined in (2.15) above). Further, let R_2 be the set of all tuples $(\nu\gamma)$ for which the index γ has a nonzero range, i.e. for which $\dim \text{Hom}_{A|A}(X_\nu, X_\mu \otimes_B X_\nu) > 0$.

Lemma 2.10:

For each choice of $\mu \in \mathcal{K}_{BB}$, either $R_1 = \emptyset = R_2$, or else the $|R_1| \times |R_2|$ -matrix $\Gamma(\mu)$ with entries $\Gamma(\mu)_{ij\alpha\beta}^{\nu\gamma}$ is nondegenerate. In particular, we have $|R_1| = |R_2|$.

This assertion will be proved in section 4.2 with the help of the TFT approach. It is actually a corollary to the more general theorem 4.2.

Proof of proposition 2.8:

We need the special case $\mu = 0$, for which $R_2 = \mathcal{K}_{BA}$. Lemma 2.10 then implies that $|R_2| = |R_1|$, and that the $|\mathcal{K}_{BA}| \times |\mathcal{K}_{BA}|$ -matrix $\Gamma(0)_{ij\alpha\beta}^\nu$ is non-degenerate (the index γ in (2.39) can only take a single value as $\dim_{\mathbb{C}} \text{Hom}_{BA}(B \otimes_B X_\nu, X_\nu) = 1$, and has been omitted). Suppose now that

$$X \cong \bigoplus_{\mu \in \mathcal{K}_{BA}} X_\mu^{\oplus n(X)_\mu} \quad \text{and} \quad Y \cong \bigoplus_{\mu \in \mathcal{K}_{BA}} X_\mu^{\oplus n(Y)_\mu} \quad (2.40)$$

as B - A -bimodules for some $n(X)_\mu, n(Y)_\mu \in \mathbb{Z}_{\geq 0}$. We abbreviate $\phi_{ij,\alpha} \equiv \xi_{(i0j)0}^\alpha$ and $\theta_{\bar{i}\bar{j},\beta} \equiv \xi_{(\bar{i}\mu\bar{j})0}^\beta$. If $D_X(\phi_{ij,\alpha}) = D_Y(\phi_{ij,\alpha})$, then also $\sum_{\mu \in \mathcal{K}_{BA}} n(X)_\mu D_\mu(\phi_{ij,\alpha}) = \sum_{\mu \in \mathcal{K}_{BA}} n(Y)_\mu D_\mu(\phi_{ij,\alpha})$, i.e.

$$\sum_{\mu \in \mathcal{K}_{BA}} (n(X)_\mu - n(Y)_\mu) D_\mu(\phi_{ij,\alpha}) = 0. \quad (2.41)$$

Inserting this identity in the two-point correlator (2.38) (with $\mu = 0$) leads to

$$\sum_{\nu \in \mathcal{K}_{BA}} (n(X)_\nu - n(Y)_\nu) \Gamma(0)_{ij\alpha\beta}^\nu = 0. \quad (2.42)$$

Since the matrix $\Gamma(0)_{ij\alpha\beta}^\nu$ is non-degenerate, this can only hold if $n(X)_\nu = n(Y)_\nu$ for all $\nu \in \mathcal{K}_{BA}$, i.e. if $X \cong Y$ as bimodules. ✓

3 Group-like and duality generating defects

In this section we investigate two interesting subclasses of topological defects, the *group-like defects* and the *duality defects*. The former can be seen to give symmetries of CFT correlators. The latter, which include the former as a special case, result in order-disorder duality relations between correlators of a single phase $\text{CFT}(A)$, or of two different phases $\text{CFT}(A)$ and $\text{CFT}(B)$.

Throughout this section we assume that the symmetric special Frobenius algebras under consideration are in addition simple. That an algebra A is simple means, by definition, that it is a simple A - A -bimodule; in this case we reserve for it the label $0 \in \mathcal{K}_{AA}$, i.e. write $X_0 = A$. In CFT terms, simplicity of A means that the coefficient $Z_{0,0}$ of the torus partition function is equal to 1, i.e. that there is a unique bulk vacuum. In other words, non-simple algebras correspond to a superposition of several CFTs rather than a single CFT (see definition 2.26 and remark 2.28(i) of [30], as well as section 3.2 of [I] for details).

3.1 Defects generating symmetries

Let us start by defining the notion of a group-like defect. In the remainder of this section we then study some of their properties.

Definition 3.1 :

- (i) An A - A -bimodule X is called *group-like* iff $X \otimes_A X^\vee \cong A$ as bimodules.
- (ii) A topological defect is called group-like iff it is labelled by a group-like A - A -bimodule.

In other words, group-like bimodules are the invertible objects of $\mathcal{C}_{A|A}$, see e.g. definition 2.1 in [III]. In the fusion of two group-like A - A -defects labelled by X which have opposite orientation, only the invisible defect A appears:

$$(3.1)$$

Note that we do not include the possibility of phase-changing defects in the definition of group-like defects. As will be explained in section 3.3 below, for an A - B -defect the corresponding property implies that the algebras A and B are Morita equivalent. We will show, also in section 3.3, that $\text{CFT}(A)$ and $\text{CFT}(B)$ are then equivalent, too.

The following three lemmas will be useful in the discussion of group-like defects and of Morita equivalences. The proofs of lemma 3.2 and lemma 3.4 use the semisimplicity of the relevant categories.

Lemma 3.2 :

Let U and V be two objects in a semisimple \mathbb{C} -linear tensor category such that $U \otimes V$ is simple. Then also U and V are simple.

Proof:

Since $U \otimes V$ is simple, we have $\text{End}(U \otimes V) \cong \mathbb{C}$. Let $U \cong \bigoplus m_i U_i$ and $V \cong \bigoplus n_j U_j$ be decompositions of U and V into simple objects U_i . Then (canonically)

$$\begin{aligned} \dim_{\mathbb{C}} \text{End}(U \otimes V) &= \sum_{i,j,k,l} m_i m_k n_j n_l \dim_{\mathbb{C}} \text{Hom}(U_i \otimes U_j, U_k \otimes U_l) \\ &\geq \sum_{i,j} m_i^2 n_j^2 \dim_{\mathbb{C}} \text{Hom}(U_i \otimes U_j, U_i \otimes U_j) \geq \sum_{i,j} m_i^2 n_j^2. \end{aligned} \quad (3.2)$$

Thus in order that $\dim_{\mathbb{C}} \text{End}(U \otimes V)$ is equal to 1, precisely one of the coefficients m_i and one of the n_j must be equal to 1, while all other coefficients must be zero. This means that U and V are simple. \checkmark

Lemma 3.3 :

Let A , B and C be (not necessarily simple) symmetric special Frobenius algebras, and let X be an A - B -bimodule, Y a B - C -bimodule, and Z an A - C -bimodule. Then

- (i) $\text{Hom}_{A|C}(X \otimes_B Y, Z) \cong \text{Hom}_{B|C}(Y, X^\vee \otimes_A Z)$
and $\text{Hom}_{A|C}(X \otimes_B Y, Z) \cong \text{Hom}_{A|B}(X, Z \otimes_C Y^\vee)$.
- (ii) $\text{Hom}_{A|C}(Z, X \otimes_B Y) \cong \text{Hom}_{B|C}(X^\vee \otimes_A Z, Y)$

and $\text{Hom}_{A|C}(Z, X \otimes_B Y) \cong \text{Hom}_{A|B}(Z \otimes_C Y^\vee, X)$.

Proof:

We denote by e and r the embedding and restriction morphisms for tensor products, as introduced above (2.7). Then the first isomorphism in (i) is given by

$$\text{Hom}_{A|C}(X \otimes_B Y, Z) \ni f \mapsto r_{X^\vee \otimes Z} \circ (id_{X^\vee} \otimes (f \circ r_{X \otimes Y})) \circ (\tilde{b}_X \otimes id_Y). \quad (3.3)$$

To check that (3.3) is indeed an isomorphism one verifies that the morphism $\text{Hom}_{B|C}(Y, X^\vee \otimes_A Z) \ni g \mapsto (\tilde{d}_X \otimes id_Z) \circ (id_X \otimes (e_{X^\vee \otimes Z} \circ g)) \circ e_{X \otimes Y}$ is its inverse. The projectors $e \circ r$ resulting in the composition can be left out because all relevant morphisms are intertwiners of bimodules.

The other three isomorphisms are similar. \checkmark

Lemma 3.4:

Let X be an A - B -bimodule and Y be a B - A -bimodule such that $X \otimes_B Y \cong A$. Then

- (i) X and Y are simple bimodules.
- (ii) $Y \cong X^\vee$ as B - A -bimodule, and $X \cong Y^\vee$ as A - B -bimodule.
- (iii) $Y \otimes_A X \cong B$.

Proof:

(i) can be seen by the same reasoning as in the proof of lemma 3.2, if we use that A is simple and hence $\text{End}_{A|A}(A) \cong \mathbb{C}$.

(ii) and (iii) are consequences of lemma 3.3.

For obtaining (ii) note the isomorphisms $\mathbb{C} \cong \text{Hom}_{A|A}(X \otimes_B Y, A) \cong \text{Hom}_{B|A}(Y, X^\vee)$, which follow by using that $X^\vee \otimes_A A \cong X^\vee$. Since by (i) X and Y are simple, their duals X^\vee and Y^\vee are simple as well. Thus $\text{Hom}_{B|A}(Y, X^\vee) \cong \mathbb{C}$ implies that the simple bimodules Y and X^\vee are isomorphic. $X \cong Y^\vee$ is seen analogously.

For (iii), consider the isomorphisms

$$X \otimes_B (Y \otimes_A X) \cong (X \otimes_B Y) \otimes_A X \cong A \otimes_A X \cong X. \quad (3.4)$$

Since X is simple, $Y \otimes_A X$ must be simple (again by the same reasoning as in the proof of lemma 3.2, applied to the two bimodules X and $Y \otimes_A X$). Further, by (ii) we have $Y \cong X^\vee$. But then

$$\text{Hom}_{B|B}(Y \otimes_A X, B) \cong \text{Hom}_{B|B}(X^\vee \otimes_A X, B) \cong \text{Hom}_{A|B}(X, X) \cong \mathbb{C} \quad (3.5)$$

by lemma 3.3. As both $Y \otimes_A X$ and B are simple, this shows that they are isomorphic. \checkmark

As an immediate consequence of lemma 3.4(i), all group-like bimodules are simple and hence isomorphic to one of the X_μ for $\mu \in \mathcal{K}_{AA}$. We denote by

$$\mathcal{G}_A \subseteq \mathcal{K}_{AA} \quad (3.6)$$

the subset labelling group-like bimodules. Also, from lemma 3.4 (iii) it follows that $X \otimes_A X^\vee \cong A$ iff $X^\vee \otimes_A X \cong A$, so that in definition 3.1 equivalently one could use the condition $X^\vee \otimes_A X \cong A$ to characterise group-like defects.

Lemma 3.5:

The tensor product $X_g \otimes_A X_h$ of two group-like A - A -bimodules X_g, X_h is again group-like.

Proof:

Let $Y = X_g \otimes_A X_h$. Then $Y \otimes_A Y^\vee \cong X_g \otimes_A X_h \otimes_A X_h^\vee \otimes_A X_g^\vee \cong A$. ✓

This observation can be used to define a group structure on \mathcal{G}_A with the product gh of two elements $g, h \in \mathcal{G}_A$ defined by the condition $X_g \otimes_A X_h \cong X_{gh}$. The unit is given by $X_e = X_0 = A$ and the inverse g^{-1} is defined via $X_{g^{-1}} \cong (X_g)^\vee$. That this is a left and right inverse follows from definition 3.1 and from lemma 3.4 (iii). In fact, \mathcal{G}_A is nothing but the Picard group of the tensor category of A - A -bimodules (see definition 2.5 and remark 2.6 of [III]),

$$\mathcal{G}_A = \text{Pic}(\mathcal{C}_{A|A}). \quad (3.7)$$

Because $\mathcal{C}_{A|A}$ is not, in general, braided, the Picard group of $\mathcal{C}_{A|A}$ can be nonabelian, in contrast to $\text{Pic}(\mathcal{C})$. An example for a nonabelian bimodule Picard group is found in the critical three-states Potts model, see section 6.3.

Taking the trace in the defining condition $X \otimes_A X^\vee \cong A$ of a group-like bimodule X_g , we see that

$$(\dim_A(X_g))^2 = 1, \quad (3.8)$$

where $\dim_A(X) = \dim(X)/\dim(A)$ is the quantum dimension in $\mathcal{C}_{A|A}$. The assignment

$$\varepsilon : \mathcal{G}_A \rightarrow \{\pm 1\} \cong \mathbb{Z}_2, \quad \varepsilon(g) := \dim(X_g)/\dim(A) \quad (3.9)$$

defines a group character on \mathcal{G}_A . The following result shows that if tensoring with X_g leaves any simple bimodule fixed, then we are guaranteed that $\varepsilon(g) = 1$.

Proposition 3.6:

Let A and B be simple symmetric special Frobenius algebras in \mathcal{C} . If for $g \in \mathcal{G}_A$ there exists a simple A - B -bimodule Y or a simple B - A -bimodule Y' such that $X_g \otimes_A Y \cong Y$ or $Y' \otimes_A X_g \cong Y'$, respectively, then $\varepsilon(g) = 1$.

Proof:

Suppose there exists a simple A - B -bimodule Y such that $X_g \otimes_A Y \cong Y$. Taking the trace we obtain $\dim(X_g) \dim(Y)/\dim(A) = \dim(Y)$. By lemma 2.6 $\dim(Y) \neq 0$, hence $\dim(X_g) = \dim(A)$, i.e. $\varepsilon(g) = 1$. The reasoning in the case of Y' is analogous. ✓

Remark 3.7:

(i) As the proof of proposition 3.6 shows, in the first case only the left A -module structure of Y is needed, and in the second case only the right A -module structure of Y' . The algebra B is introduced in the statement merely because later on we want to use the result in the discussion of bimodule stabilisers.

(ii) To find examples for $\varepsilon(g) = -1$, consider the simplest case, i.e. $A = \mathbf{1}$. We are then looking for a chiral CFT which has simple currents of quantum dimension -1 . It turns out that certain

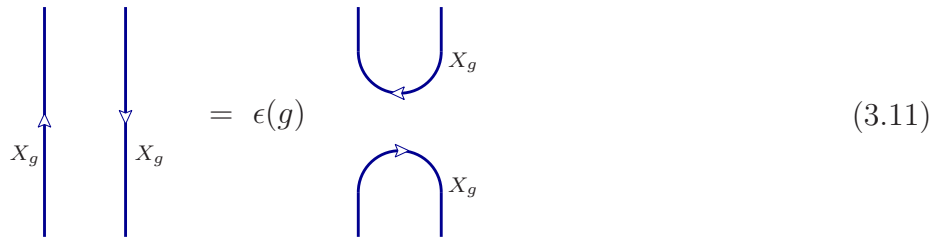
non-unitary Virasoro minimal models in the series $M(p, q)$, $p, q \geq 2$ have this property.⁷ Recall that the quantum dimension $\dim(U)$ of an object U is the trace of the identity morphism id_U . Another notion of dimension is provided by the Perron-Frobenius dimension (see e.g. [42]), which is always positive. Here we are interested in the factors that appear when inserting or removing small circular defects, and these are given in terms of traces of identity morphisms, i.e. by quantum dimensions. For a simple object U_k , the quantum dimension can be computed from the fusing and braiding matrices and the twist as in equation (2.45) of [I]:

$$\dim(U_k) = \theta_k (\mathbb{F}_{00}^{(k \bar{k} k) k} \mathbb{R}^{-(\bar{k} k) 0})^{-1}. \quad (3.10)$$

For Virasoro minimal models, this simplifies to $\dim(U_k) = \mathbb{F}_{00}^{(k k k) k}$. Inserting the F-matrices⁸ of [45], one finds, for instance, that in the minimal model $M(3,5)$ the field with Kac labels $(2, 1)$ has quantum dimension -1 .

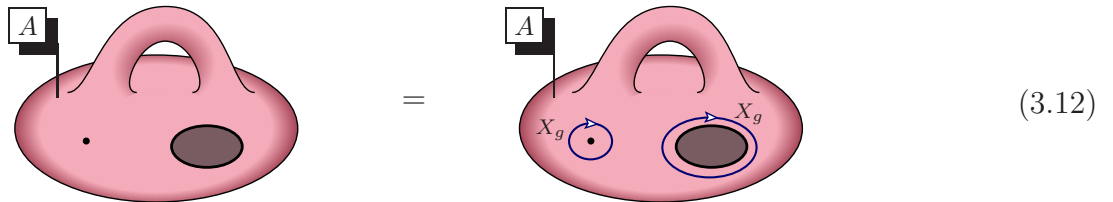
(iii) Note that once negative quantum dimensions occur, there can also be objects of dimension zero even if all simple objects have non-zero quantum dimension. For $M(3,5)$, for example, the object $U = (1, 1) \oplus (2, 1)$ has $\dim(U) = 0$. Recall from lemma 2.6 that a simple A - B -bimodule necessarily has nonzero dimension. In particular, the object U introduced above cannot carry the structure of a simple A - B -bimodule, irrespective of which simple symmetric special Frobenius algebras A and B are considered.

If we apply the identity (2.27) to the case of a group-like defect $X_\sigma = X_\rho = X_g$, we get the relation



$$\begin{array}{c} \uparrow \\ X_g \end{array} \quad \begin{array}{c} \downarrow \\ X_g \end{array} = \epsilon(g) \begin{array}{c} \text{U-shape} \\ X_g \end{array} \quad \begin{array}{c} \text{U-shape} \\ X_g \end{array} \quad (3.11)$$

since only $\mu = 0$ can contribute to the summation coming from (2.27). Using this identity it is easy to see what happens when a group-like defect X_g is inflated in a world sheet X in phase $\text{CFT}(A)$. For example, figure (2.35) then simplifies to



$$\text{Surface with defect } A = \text{Surface with defects } X_g \quad (3.12)$$

Note that here an even number of factors $\epsilon(g)$ occurs. In general, by the identity (3.11) we

⁷ For these models it has not yet been proven that the representation category of the vertex algebra is modular. However, the fusing and braiding matrices can be extracted from the monodromies of the conformal four-point blocks, which in turn are found by solving differential equations resulting from null vectors [45].

⁸ The relevant formulas have been gathered e.g. in formula (A.6) of [47], which in the present notation gives the F-matrix element $\mathbb{F}_{PQ}^{(JKL)I}$.

have the following result for taking a group-like defect X_g past a bulk field.

$$\begin{array}{c} \uparrow X_g \\ | \\ \phi \\ | \end{array} \equiv \begin{array}{c} \uparrow X_g \\ \curvearrowright \\ \phi \\ \curvearrowleft \\ \uparrow X_g \end{array} = \varepsilon(g) \begin{array}{c} X_g \\ \circlearrowleft \\ \phi \\ \circlearrowright \\ X_g \\ | \\ \uparrow \end{array} \quad (3.13)$$

A similar identity holds for taking a group-like defect past a ‘hole’ in the world sheet X , i.e. a connected boundary component. Applying also the moves (2.36) for each handle of X , we find, for an orientable world sheet in phase $\text{CFT}(A)$ of genus h with m bulk field insertions labelled by morphisms ϕ_1, \dots, ϕ_m and b boundary components with boundary conditions M_1, M_2, \dots, M_b ,

$$\begin{aligned} C(X[\phi_1, \dots, \phi_m; M_1, \dots, M_b]) \\ = \varepsilon(g)^{m+b} C(X[D_g(\phi_1), \dots, D_g(\phi_m); X_g \otimes_A M_1, \dots, X_g \otimes_A M_b]), \end{aligned} \quad (3.14)$$

where $D_g \equiv D_{X_g}$ is the map defined in (2.30). In the presence of boundary fields one obtains a similar identity, in which there is in addition an action of the X_g -defect (as in (2.33)) on each boundary field.

We have thus arrived at our first notable conclusion:

Group-like defects give rise to symmetries of CFTs on oriented world sheets.

Further, recall from proposition 2.8 that two non-isomorphic simple defects never act in the same way on all bulk fields. This implies that defects labelled by non-isomorphic group-like bimodules in fact describe *distinct* symmetries.

3.2 Defects generating order-disorder dualities

In this section we consider phase changing defects between $\text{CFT}(A)$ and $\text{CFT}(B)$, which includes $A = B$ as a special case. Similarly as in (3.13) we may want to take an arbitrary B - A -defect Y past a bulk field of $\text{CFT}(A)$. But when doing so we now generically obtain a sum over disorder fields in $\text{CFT}(B)$:

$$\begin{array}{c} \uparrow Y \\ \curvearrowright \\ \phi \\ \curvearrowleft \\ \uparrow Y \end{array} \begin{array}{c} \blacksquare A \\ \blacksquare B \end{array} = \sum_{\mu, \alpha} \begin{array}{c} \theta_{\mu\alpha} \\ \leftarrow X_\mu \\ \blacksquare B \\ \uparrow Y \\ \alpha \\ \blacksquare A \end{array} \quad (3.15)$$

Here first (2.27) is applied, and then (2.28). In the sum, α runs over a basis of $\text{Hom}_{A|A}(X_\mu \otimes_A Y, Y)$ and $\theta_{\mu\alpha}$ denotes the resulting disorder field at the end of the B - B -defect X_μ .

Repeating the above procedure with an A - B -defect Y' will in general again result in a sum over disorder fields. In case it is nevertheless possible to transform (3.15) back into an order correlator, i.e. into a correlator involving only bulk fields, but no disorder fields, we have obtained an order/disorder symmetry. This motivates the

Definition 3.8:

A B - A -defect Y is called a *duality defect* iff there exists an A - B -defect Y' such that, for every bulk field of $\text{CFT}(A)$, taking first Y past that bulk field and then Y' past the resulting sum over disorder fields, gives a sum over bulk fields of $\text{CFT}(A)$.

The bimodule underlying a duality defect is called a *duality bimodule*.

Graphically, the requirement in the definition means that for every choice of bulk field label ϕ one has

$$(3.16)$$

That is, on the right hand side we have a sum over bulk fields only, rather than in addition over defect fields. The dashed box indicates that after taking the two defects Y and Y' past the bulk field, they are no longer separated, but linked by a morphism in $\text{Hom}_{A|A}(Y' \otimes_B Y, Y' \otimes_B Y)$.

We write

$$\mathcal{D}_{BA} := \{ \mu \in \mathcal{K}_{BA} \mid X_\mu \text{ is a duality bimodule} \}. \quad (3.17)$$

Note that in definition 3.8 we do not demand that a duality defect is simple, and indeed non-simple duality defects can appear. However, all duality defects can be obtained as appropriate superpositions of simple duality defects. This will be shown in lemma 3.10 below. Before, however, we give a more convenient characterisation of duality defects, since in practise it would be tedious to check property (3.16) directly. This characterisation only uses the fusion algebra of the defect lines:

Theorem 3.9:

A B - A -bimodule Y is a duality bimodule if and only if $Y^\vee \otimes_B Y$ is a direct sum of group-like bimodules,

$$Y^\vee \otimes_B Y \cong \bigoplus_{g \in \mathcal{G}_A} n_g X_g \quad (3.18)$$

for suitable $n_g \in \mathbb{Z}_{\geq 0}$.

Proof:

“ \Leftarrow ”: Suppose that $Y^\vee \otimes_B Y \cong \bigoplus_g n_g X_g$. The following transformations show that Y is a duality

defect, with the A - B -defect Y' in definition 3.8 given by Y^\vee :

$$= \sum_{g, \alpha} \dots = \sum_{g, \alpha} \epsilon(g) \dots \quad (3.19)$$

In the first step the bimodule $Y^\vee \otimes_B Y$ is decomposed into a direct sum of simple A - A -bimodules, similarly to (2.27) (all prefactors have been absorbed into the definition of the morphisms α and $\tilde{\alpha}$). By assumption, only group-like bimodules appear in the decomposition. The resulting group-like defect can then be moved past the bulk field by using (3.13); this is done in step two. Comparing the right hand side of (3.19) with (3.16), we conclude that Y is a duality defect.

“ \Rightarrow ”: Suppose now that the B - A -defect Y is a duality defect, i.e. that there exists an A - B -defect Y' such that for any choice of bulk field label ϕ in $\text{CFT}(A)$ we have

$$= \sum_{\sigma} \dots \quad (3.20)$$

where the sum is over elements $\sigma \in \text{Hom}_{A|A}(Y' \otimes_B Y, Y' \otimes_B Y)$ and the ϕ_σ label again bulk fields of $\text{CFT}(A)$. Then in particular, the following identity must hold for the two-point correlator of one bulk field and one defect field on the Riemann sphere:

$$= \sum_{\sigma} \dots \quad (3.21)$$

for all choices of simple objects i, j and simple bimodules ρ_1, ρ_2, ν, μ , and for all choices of basis labels $\alpha, \beta, \varepsilon_1, \varepsilon_2, \tau, \gamma$. Here $\varepsilon_{1,2}$ and $\bar{\varepsilon}_{1,2}$ are dual basis elements as in (2.16) and $\tau, \bar{\tau}$ are dual

basis elements as in (2.17). By (2.25), the right hand side of (3.21) is zero if $\mu \neq 0$. For the left hand side we find

l.h.s of (3.21) = $\Gamma(\mu)_{ij\alpha\beta}^{\nu\gamma} \cdot (2\text{-point-blocks})$ (3.22)

Here in the first step one uses the property that $\varepsilon_{1,2}$ and $\bar{\varepsilon}_{1,2}$ are dual to each other, and in the second step that τ and $\bar{\tau}$ are dual, together with the definition (2.39). Thus the identity (3.21) implies that for every simple A - A -bimodule X_ν contained in $Y' \otimes_B Y$, and for every $\mu \in \mathcal{K}_{AA} \setminus \{0\}$, we have

$$\Gamma(\mu)_{ij\alpha\beta}^{\nu\gamma} = 0 \quad \text{for all } \gamma \text{ such that } (\nu\gamma) \in R_2 \text{ and all } (ij\alpha\beta) \in R_1. \quad (3.23)$$

Since by lemma 2.10 the matrix $\Gamma(\mu)$ is non-degenerate, this is only possible if, for every $\mu \neq 0$ and every X_ν contained in $Y' \otimes_B Y$, the index set R_2 does not contain any element of the form $(\nu\gamma)$, i.e. only if $\dim_{\mathbb{C}} \text{Hom}_{A|A}(X_\nu, X_\mu \otimes_A X_\nu) = 0$. In other words, we learn that

$$\text{Hom}_{A|A}(X_\nu, Y' \otimes_B Y) \neq \{0\} \implies \text{Hom}_{A|A}(X_\nu \otimes_A X_\nu^\vee, X_\mu) = \{0\} \text{ for all } \mu \neq 0. \quad (3.24)$$

This is nothing but the statement that all simple sub-bimodules of $Y' \otimes_B Y$ are group-like.

It remains to show that not only Y' has this property, but also Y^\vee . Let $Y = \bigoplus_r Y_r$ and $Y' = \bigoplus_s Y'_s$ be decompositions of Y and Y' into simple bimodules. The fact that $Y' \otimes_B Y$ is a direct sum of group-like bimodules implies in particular that $Y'_1 \otimes_B Y_r$ is a direct sum of group-like bimodules. Thus there is a $g_r \in \mathcal{G}_A$ such that $X_{g_r} \otimes_A Y'_1 \otimes_B Y_r$ contains A , and hence $X_{g_r} \otimes_A Y'_1 \cong Y_r^\vee$. It follows that for all r, s , $Y_r^\vee \otimes_B Y_s = X_{g_r} \otimes_A Y'_1 \otimes_B Y_s$ is a direct sum of group-like bimodules. Thus also $Y^\vee \otimes_B Y$ is a direct sum of group-like bimodules. \checkmark

Theorem 3.9 can be reformulated as the statement that Y is a duality defect iff $Y^\vee \otimes_B Y$ is in the Picard category of $\mathcal{C}_{A|A}$ (see definition 2.1 of [III]).

Up to now we did not demand the B - A -defect Y to be simple. Suppose that Y decomposes into simple B - A -bimodules X_μ as $Y \cong \bigoplus_{\mu \in \mathcal{K}_{BA}} m_\mu X_\mu$. The statement that Y is a duality defect then amounts to

$$\bigoplus_{\mu, \nu \in \mathcal{K}_{BA}} m_\mu m_\nu X_\mu^\vee \otimes_B X_\nu \cong \bigoplus_{g \in \mathcal{G}_A} n_g X_g. \quad (3.25)$$

This implies in particular that $X_\mu^\vee \otimes_B X_\nu$ is a direct sum of group-like bimodules for any pair of X_μ and X_ν that are sub-bimodules of Y . But then there exists a $g \in \mathcal{G}_A$ such that $X_g \otimes_A X_\mu^\vee \otimes_B X_\nu$ contains the tensor unit A , and hence $X_\mu \otimes_A X_{g^{-1}} \cong X_\nu$. This implies

Lemma 3.10:

Let $\mu_1, \mu_2, \dots, \mu_m \in \mathcal{K}_{BA}$. Then $Y = X_{\mu_1} \oplus \dots \oplus X_{\mu_m}$ is a duality bimodule iff $\mu_1 \in \mathcal{D}_{BA}$ and $\mu_1, \mu_2, \dots, \mu_m$ lie on one and the same orbit of the right action of \mathcal{G}_A , i.e. there exist $g_2, \dots, g_m \in \mathcal{G}_A$ such that $X_{\mu_1} \otimes_A X_{g_k} \cong X_{\mu_k}$ for all $k = 2, 3, \dots, m$.

We have learned that all B - A -duality defects can be obtained as direct sums of the simple duality defects \mathcal{D}_{BA} . Moreover, we are only allowed to superimpose such simple duality defects that lie on one and the same orbit of the right \mathcal{G}_A -action on \mathcal{D}_{BA} . To describe all defect-induced order-disorder dualities it is therefore sufficient to consider simple duality defects only.

In order to better understand which group-like defects appear on the right hand side of (3.18), we introduce the notion of a left and a right stabiliser. This is a straightforward generalisation of the corresponding notion for the braided category \mathcal{C} (in which the left and a right stabilisers coincide, see definition 4.1 (i) of [III]) to the sovereign category $\mathcal{C}_{A|A}$.

Definition 3.11 :

The *left and right stabilisers* $\mathcal{S}^{l,r}(Y)$ of an A - B -bimodule Y are given by

$$\mathcal{S}^l(Y) := \{ g \in \mathcal{G}_A \mid X_g \otimes_A Y \cong Y \} \quad \text{and} \quad \mathcal{S}^r(Y) := \{ h \in \mathcal{G}_B \mid Y \otimes_B X_h \cong Y \}. \quad (3.26)$$

It is straightforward to check that $\mathcal{S}^l(Y) \subseteq \mathcal{G}_A$ and $\mathcal{S}^r(Y) \subseteq \mathcal{G}_B$ are in fact subgroups. With the notion of a stabiliser, for a simple B - A -defect X we can slightly strengthen the condition of theorem 3.9 for X to be a duality defect:

Proposition 3.12 :

Let Y be a simple B - A -defect. Then Y is a duality defect iff

$$Y^\vee \otimes_B Y \cong \bigoplus_{h \in \mathcal{S}^r(Y)} X_h. \quad (3.27)$$

Proof:

First note that $Y \otimes_A X_h$ is again simple, since by lemma 3.3 we have $\text{End}_{B|A}(Y \otimes_A X_h) \cong \text{Hom}_{B|A}(Y, Y \otimes_A X_h \otimes_A X_h^\vee) \cong \text{Hom}_{B|A}(Y, Y)$. Further, the multiplicity n_h of X_h in $Y^\vee \otimes_B Y$ is given by (again using lemma 3.3)

$$n_h = \dim_{\mathbb{C}} \text{Hom}_{A|A}(Y^\vee \otimes_B Y, X_h) = \dim_{\mathbb{C}} \text{Hom}_{B|A}(Y, Y \otimes_A X_h). \quad (3.28)$$

Since Y and $Y \otimes_A X_h$ are simple, it follows that $n_h = 1$ iff $h \in \mathcal{S}^r(Y)$ and $n_h = 0$ otherwise. \checkmark

Let Y be a simple B - A -duality bimodule and let $g, h \in \mathcal{S}^r(Y)$ such that $gh = hg$. Consider the morphism $y_{gh} \in \text{Hom}_{A|A}(X_h \otimes_A X_g, X_g \otimes_A X_h)$ that is given by

$$\begin{array}{c} \text{Diagram 1} \end{array} \equiv \begin{array}{c} \text{Diagram 2} \end{array} := \frac{\dim A}{\dim Y} \begin{array}{c} \text{Diagram 3} \end{array} \quad (3.29)$$

The morphisms in $\text{Hom}_{B|A}(Y \otimes_A X_x, Y)$ and $\text{Hom}_{B|A}(Y, Y \otimes_A X_x)$, for $x = g, h$, that label the junctions in (3.29) are chosen to be dual to each other. Since the morphism spaces are one-dimensional, the resulting morphism y_{gh} is independent of these choices. The normalisation

factor in (3.29) has been introduced in such a way that (3.29) obeys

$$(3.30)$$

This can be seen by the following chain of equalities (we abbreviate $\zeta = \dim(A)/\dim(Y)$).

$$(3.31)$$

In the first step, the intermediate group-like defect X_g is replaced by $X_{g'}$, with the label g' summed over $\mathcal{S}^r(Y)$. This is allowed because for $gh = hg$, $\text{Hom}_{A|A}(X_g \otimes_A X_h, X_h \otimes_A X_{g'})$ is zero unless $g' = g$. In the second step the Y -defect is deformed and the summation over g' performed according to (a reflected version of) the identity (2.27). The X_h -defect can then be omitted, as is easily verified with the help of the TFT formulation in section 4, using also that Y is simple (see equation (4.10)). That the right hand side of (3.31) equals the right hand side of (3.30) can be seen by noting that these morphism spaces are one-dimensional and comparing traces.

A crossing of two group-like defects of the form just considered is implicit in the formulation of the following

Proposition 3.13:

Assume that B - A -duality defects exist, and let Y be a simple B - A -duality defect. Then the torus partition function of $\text{CFT}(B)$ can be expressed in terms of torus amplitudes with defect lines of $\text{CFT}(A)$ as follows. (This is just the way in which the partition function of an orbifold

theory is expressed as a sum over twisted sectors.)

$$\begin{array}{c} \text{Diagram 1: Square with boundary arrows and a box labeled } B \end{array} = \frac{1}{|S^r(Y)|} \sum_{\substack{g, h \in S^r(Y) \\ gh=hg}} \begin{array}{c} \text{Diagram 2: Square with boundary arrows, a box labeled } A, \text{ and a vertical line } g \text{ and horizontal line } h \text{ meeting at a yellow diamond} \end{array} \quad (3.32)$$

Proof:

The statement follows from the equalities

$$\begin{array}{c} \text{Diagram 1: Square with boundary arrows and a box labeled } B \end{array} = \frac{\dim(B)}{\dim(Y)} \begin{array}{c} \text{Diagram 2: Square with boundary arrows, a box labeled } B, \text{ and a box labeled } A \text{ with curved arrows } Y \end{array} \\
 = \frac{\dim(B)}{\dim(Y)} \cdot \left(\frac{\dim(A)}{\dim(Y)} \right)^2 \sum_{g, h} \begin{array}{c} \text{Diagram 3: Square with boundary arrows, boxes } A \text{ and } B, \text{ and lines } g, h, Y \text{ with yellow squares} \end{array} = \frac{\dim(A) \dim(B)}{\dim(Y)^2} \sum_{g, h} \begin{array}{c} \text{Diagram 4: Square with boundary arrows, a box labeled } A, \text{ and lines } g, h \text{ meeting at a yellow diamond} \end{array} \quad (3.33)$$

together with the identity

$$\dim(Y)^2 / \dim(B) = |S^r(Y)| \dim(A), \quad (3.34)$$

which follows from taking the trace of (3.27). That only commuting pairs (g, h) can give a nonzero contribution to the sum is due to the fact that otherwise the relevant coupling space is zero. ✓

Remark 3.14:

In [26] it is pointed out that in the class of models investigated there (minimal models and $\mathfrak{sl}(2)$ WZW models), the models admitting a duality symmetry are precisely those which can be described as their own orbifold, i.e. those which possess an “auto-orbifold” property. Applying proposition 3.13 for the special case $A = B$ shows that indeed an RCFT possessing a duality defect automatically also has the auto-orbifold property. For certain lattice models in the universality classes of the $c < 1$ A-D-E minimal models, the $c = 1$ compactified free boson, or its \mathbb{Z}_2 -orbifolds, a related lattice construction, which also works off criticality and for $A \neq B$, is described in [46].

3.3 Duality defects and Morita equivalence

Suppose that the simple symmetric special Frobenius algebras A and B are related by $X \otimes_B Y \cong A$ via an A - B -bimodule X and a B - A -bimodule Y . Then by lemma 3.4 (iii) also $Y \otimes_A X \cong B$, so that the algebras A and B are Morita equivalent [48, 49]. In this section we show in which sense Morita equivalent algebras lead to equivalent CFTs.

Note that by lemma 3.4(ii), $X \otimes_B Y \cong A$ implies that $Y \otimes_A Y^\vee \cong B$ and $Y^\vee \otimes_B Y \cong A$. Just as in (3.11), let us apply the identity (2.27) to the case $X_\sigma = X_\rho = Y$. This results in

$$\begin{array}{c} \boxed{B} \quad \boxed{A} \quad \boxed{B} \\ \uparrow Y \quad \downarrow Y \\ \text{---} \end{array} = \frac{\dim(B)}{\dim(Y)} \begin{array}{c} \boxed{A} \\ \downarrow Y \\ \boxed{B} \\ \uparrow Y \\ \boxed{A} \end{array} \quad (3.35)$$

Next inflate the defect Y in an oriented connected world sheet X as in (2.35). Whenever two Y -defects run parallel to each other we can make use of the identity (3.35). In this way one obtains a world sheet with only small circular defects labelled by Y , each of which contributes a factor $\dim(Y)/\dim(A)$. Since $Y \otimes_A Y^\vee \cong B$ we have $\dim(Y)/\dim(A) = \dim(B)/\dim(Y)$, so that the factors from circular Y -defects can cancel against the factors in (3.35). Abbreviating

$$\dim(B) / \dim(Y) =: \gamma, \quad (3.36)$$

the net effect is

- a factor of γ^2 for each handle of X ;
 - a factor of γ for each connected boundary component of X ;
 - a factor of γ – to be absorbed into a redefinition of bulk fields – for each bulk insertion of X .
- Altogether we obtain the following relation between correlators of $\text{CFT}(A)$ and $\text{CFT}(B)$ on a connected, oriented world sheet X with genus h , m bulk insertions and b boundary components:

$$\text{Cor}_A(X) = \gamma^{-\chi(X)} \text{Cor}_B(X') \quad \text{with} \quad \gamma = \frac{\dim(B)}{\dim(Y)} = \frac{\dim(Y)}{\dim(A)}, \quad (3.37)$$

where $\chi(X) = 2 - 2h - b$ is the Euler character of X .

The world sheet X' is obtained from X by changing the labelling of bulk fields and boundary conditions. If $\phi \in \text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A)$ labels a bulk field of $\text{CFT}(A)$ on X , then on X' this label is replaced by $\gamma D_Y(\phi) \in \text{Hom}_{B|B}(U \otimes^+ B \otimes^- V, B)$, where $D_Y \equiv D_{0\mu}$ is the map used in (2.31). Using γD_Y instead of D_Y ensures that the identity field of $\text{CFT}(A)$ gets mapped to the identity field of $\text{CFT}(B)$. For a boundary component of X labelled by an A -module M , this label is replaced in X' by the B -module $Y \otimes_A M$. In the presence of boundary fields one must use in addition (2.33).

Via equation (3.37) we define when we consider two CFTs to be equivalent, namely iff there exist isomorphisms between the spaces of bulk fields, boundary fields and boundary conditions of the two CFTs such that the correlator for a world sheet X for the first CFT is equal to the correlator for the corresponding world sheet X' of the second CFT up to an overall constant that only depends on the Euler character of X (which is equal to that of X').

In the case $A = B$ we are dealing with a group-like defect. The discussion above then reduces to the one in section 3.1.

3.4 Equivalence of CFTs on unoriented world sheets

As seen in the previous section, two Morita equivalent simple symmetric special Frobenius algebras result in equivalent CFTs on oriented world sheets. To obtain a CFT that is well defined also on unoriented world sheets we need a Jandl algebra, see [II, def. 2.1]. In this section we present an equivalence relation between simple Jandl algebras, such that two equivalent Jandl algebras yield equivalent CFTs on unoriented surfaces.

Let A and B be two Jandl algebras, and denote by σ_A and σ_B their reversions. Let us for convenience repeat part of the statement of [II, proposition 2.10].

Proposition 3.15 :

Let $X = (\dot{X}, \rho, \tilde{\rho})$ be an A - B -bimodule. Then $X^s = (\dot{X}, \rho^s, \tilde{\rho}^s)$ with

$$\rho^s := \tilde{\rho} \circ c_{B,X} \circ (\sigma_B \otimes id_X) \quad \text{and} \quad \tilde{\rho}^s := \rho \circ c_{X,A} \circ (id_X \otimes \sigma_A) \quad (3.38)$$

is a B - A -bimodule.

Pictorially, the left/right action on X^s is as follows.

Definition 3.16 :

Two simple Jandl algebras A and B are called *Jandl-Morita equivalent* iff there exists a B - A -bimodule Y such that $Y^\vee \otimes_B Y \cong A$ as A - A -bimodules and $Y^s \cong Y^\vee$ as A - B -bimodules.

Remark 3.17 :

- (i) As before, by lemma 3.4(ii) the condition $Y^\vee \otimes_B Y \cong A$ implies that also $Y \otimes_A Y^\vee \cong B$.
- (ii) If Y generates a Jandl-Morita equivalence, one can define an analogue of the Frobenius-Schur indicator (see e.g. definition 3.10 of [33] or equation (2.19) of [I] for a definition). This can be done by picking an isomorphism $g \in \text{Hom}_{A|B}(Y^s, Y^\vee)$ and defining the constant $\nu_Y \in \mathbb{C}$ via (see also [II, section 2.4] and [1, section 11])

Since Y^\vee and Y^s are simple as A - B -bimodules, the morphism space $\text{Hom}_{A|B}(X^s, X^\vee)$ is one-dimensional, so that ν_Y exists and is independent of the choice of g . Also, applying (3.40) twice shows immediately that $\nu_Y \in \{\pm 1\}$.

(iii) By writing out the definitions, one can convince oneself that for simple Jandl algebras, the notion of ‘Jandl-Morita equivalence’ from definition 3.16 is the same as the notion of equivalence of two Jandl algebras in [1, Definition 13]. To this end one turns $Y = M^\vee$, with M the left A -module used in [1], into a B - A -bimodule in the obvious way. The isomorphism $g \in \text{Hom}_A(M, M^\sigma)$ of [1] is related to the isomorphism $f \in \text{Hom}_{A|B}(Y^\vee, Y^s)$ required in definition 3.16 via $f = \theta_M \circ g$.

Let A , B and Y be as in definition 3.16. In order to show that $\text{CFT}(A)$ and $\text{CFT}(B)$ are equivalent for two simple Jandl algebras A and B , we essentially repeat the calculation done in section 3.3. The only new aspect is that the world sheet X can now contain insertions of cross caps. The effect of taking the defect Y past a cross cap is illustrated in the following sequence of deformations:

(3.41)

The dashed line indicates that the local orientation around the defect is thereby reversed.⁹ In step (2) an isomorphism $g \in \text{Hom}_{A|B}(Y^s, Y^\vee)$ is chosen and the identity morphism in the form $g^{-1} \circ g$ is inserted. We also indicate the half-twists of the corresponding ribbon in the 3dTFT representation, see section 3 of [II]. In step (3) we first use (3.35) and then (3.40). In step (4) the remaining section of the Y -defect is dragged through the cross cap, and in step (5) the half-twists are removed and g is cancelled against g^{-1} .

Altogether we arrive at

$$\text{Cor}_A(X) = \gamma^{-\chi(X)} (\nu_Y)^c \text{Cor}_B(X'). \quad (3.42)$$

Here X , X' and γ are as in (3.37), c is the number of cross caps, and $\chi(X) = 2 - 2h - b - c$ is the Euler character of X . Recall that three cross caps can be traded for one cross cap plus one

⁹ For conventions regarding the labelling of defect lines on unoriented surfaces see section 3.8 of [II] and section 3.4 of [IV].

handle, so that the total number of cross caps is only defined modulo two. Since $\nu_Y = \pm 1$, the prefactor in (3.42) is nonetheless well-defined.

Remark 3.18:

A similar sign factor has been found in a geometric approach to WZW theories on unoriented surfaces [50] based on hermitian bundle gerbes with additional structure. Such structures – called Jandl structures in [50] – actually come in pairs whose monodromies on an unoriented surface with c crosscaps differ by a factor of $(-1)^c$.

3.5 Action of duality defects on fields

In the Ising lattice model, order-disorder duality is at the same time a high-low temperature duality. A similar effect occurs for defect-induced order-disorder dualities of CFTs. To exhibit this phenomenon we need to study the behaviour of bulk fields under the duality. As a preparation, we introduce a map ϕ_Y which is a generalisation of the basis independent $6j$ -symbols studied in section 4.1 of [III] to the categories of bimodules.

Definition 3.19:

For Y a simple A - B -bimodule, the map $\phi_Y: \mathcal{S}^l(Y) \times \mathcal{S}^r(Y) \rightarrow \mathbb{C}^\times$ is defined via

for $g \in \mathcal{S}^l(Y)$ and $h \in \mathcal{S}^r(Y)$.

The numbers $\phi_Y(g, h)$ do not depend on the choice of nonzero elements $\alpha \in \text{Hom}(X_g \otimes_A Y, Y)$ and $\beta \in \text{Hom}(Y \otimes_B X_h, Y)$: these two morphism spaces are one-dimensional, and hence choosing different elements changes both sides of (3.43) by the same factor. The map ϕ_Y tells us how to commute two group-like defects attached from opposite sides to a simple A - B -defect:

where $g \in \mathcal{S}^l(Y)$, $h \in \mathcal{S}^r(Y)$.

The following properties of the map ϕ_Y will be important:

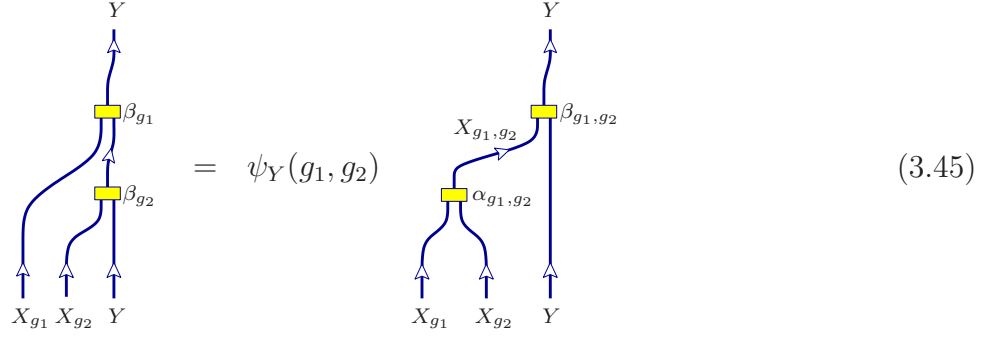
Proposition 3.20 :

Let Y be a simple A - B -bimodule.

- (i) The map $\phi_Y: \mathcal{S}^l(Y) \times \mathcal{S}^r(Y) \rightarrow \mathbb{C}^\times$ is a bihomomorphism.
- (ii) Let Y be in addition a duality bimodule. Then the bihomomorphism ϕ_Y is non-degenerate in the first argument, i.e. if for some $g, g' \in \mathcal{S}^l(Y)$ one has $\phi_Y(g, h) = \phi_Y(g', h)$ for all $h \in \mathcal{S}^r(Y)$, then $g = g'$. In particular, $|\mathcal{S}^r(Y)| \geq |\mathcal{S}^l(Y)|$.

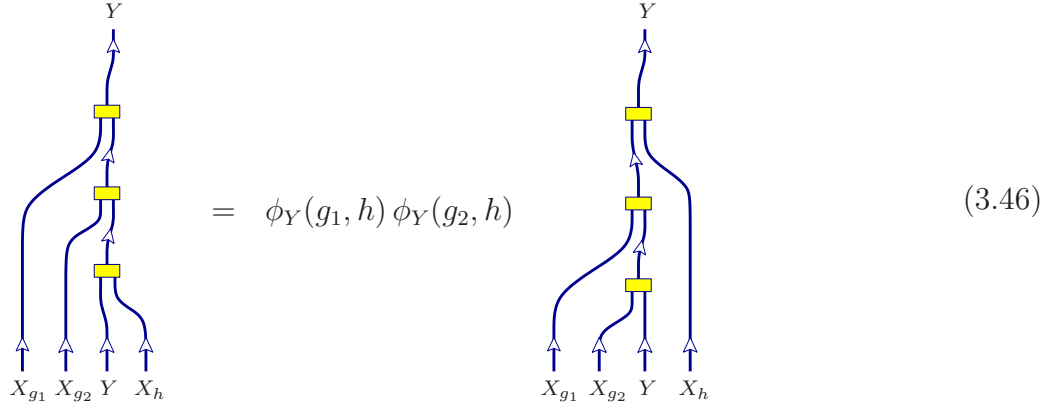
Proof:

- (i) The proof proceeds similar to the one of proposition 4.2 of [III]. Choose basis vectors $\alpha_{g,h} \in \text{Hom}_{A|A}(X_g \otimes_A X_h, X_{gh})$ and $\beta_g \in \text{Hom}_{A|B}(X_g \otimes_A Y, Y)$. For any $g_1, g_2 \in \mathcal{S}^l(Y)$ there are nonzero constants $\psi_Y(g_1, g_2)$ such that (we implicitly use the isomorphisms (2.10))



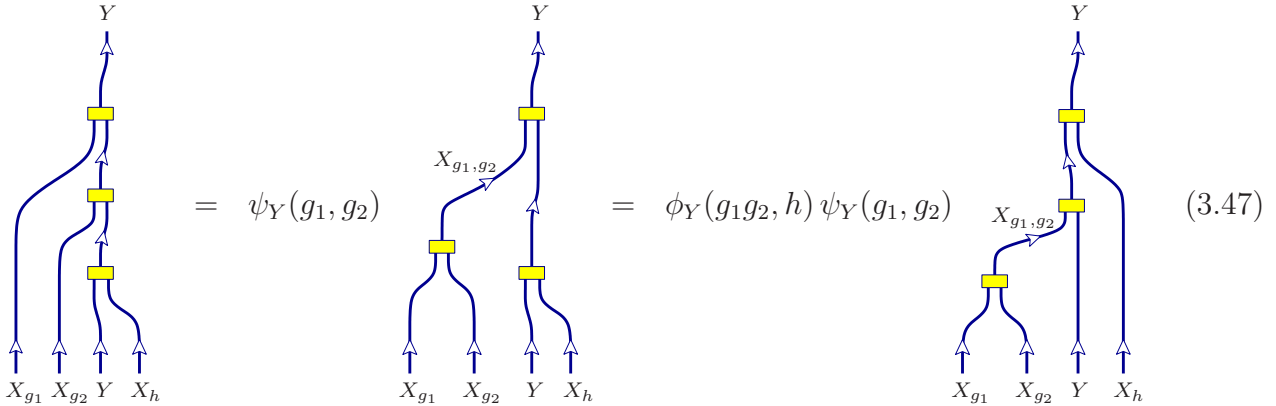
$$(3.45)$$

Then on the one hand one has (we omit the labels for the couplings)



$$(3.46)$$

and on the other hand



$$(3.47)$$

Applying relation (3.45) to the right hand side of (3.47) removes the factor $\psi_Y(g_1, g_2)$ again, and thus comparison with (3.46) yields $\phi_Y(g_1, h) \phi_Y(g_2, h) = \phi_Y(g_1 g_2, h)$. The unit property

$\phi_Y(e, h) = 1$ is immediate. The homomorphism property in the second argument can be checked analogously.

(ii) Consider the equalities

$$= \delta_{g,e} \frac{\dim(Y)}{\dim(A)} id_A . \quad (3.48)$$

The first equality holds because $\text{Hom}_{A|A}(X_g, A)$ is zero unless $g = e$ (i.e., unless $X_g \cong A$), and the second follows by lemma 4.1 below. Next consider the following series of equalities:

$$\stackrel{(3)}{=} \frac{\dim(B)}{\dim(Y)} \sum_{h \in \mathcal{S}^r(Y)} \stackrel{(4)}{=} \frac{\dim(B)}{\dim(Y)} \sum_{h \in \mathcal{S}^r(Y)} \phi_Y(g, h)^{-1} \quad (3.49)$$

Equality (1) is obtained by applying the identity that results from composing both sides of (3.48) with the counit. Step (2) amounts to (2.27). In step (3) the ribbon graph is deformed and the A -ribbons are removed (which is possible owing to the properties of A and the fact that the various intertwiners commute with the action of A); also, it is used that $\dim(X_h) = \dim(B)$ (see proposition 3.6). Finally, (4) uses the definition (3.43) of $\phi_Y(g, h)$ and that the basis elements in $\text{Hom}_{A|B}(Y \otimes_B X_h, Y)$ and $\text{Hom}_{A|B}(Y, Y \otimes_B X_h)$ are dual to each other.

Altogether, the result of (3.49) shows that

$$\sum_{h \in \mathcal{S}^r(Y)} \phi_Y(g, h^{-1}) = \frac{\delta_{g,e} \dim(Y)^2}{\dim(A) \dim(B)} = |\mathcal{S}^r(Y)| \delta_{g,e} , \quad (3.50)$$

where the last equality holds by (3.34). Now $\phi_Y(g, h) = \phi_Y(g', h)$ for all $h \in \mathcal{S}^r(Y)$ implies that $\sum_{h \in \mathcal{S}^r(Y)} \phi_Y(g(g')^{-1}, h) = |\mathcal{S}^r(Y)|$, which by (3.50) is the case only if $g = g'$. \checkmark

Remark 3.21 :

In the remainder of this section we will assume that both the simple B - A -defect Y and the simple A - B -defect Y^\vee are duality defects. This allows us to make stronger statements, but it is not the generic situation. A counter example is provided by a phase-boundary between the tetracritical Ising and the critical three-states Potts model, see section 6 for details. In fact, if both Y and Y^\vee are simple duality defects, then by theorem 3.22 below their stabilisers are abelian. So if a simple duality defect Y has a nonabelian stabiliser, then Y^\vee cannot be a duality defect. This is precisely the situation in the example treated in section 6.

Theorem 3.22 :

Let Y be a simple B - A -bimodule such that both Y and Y^\vee are duality bimodules. Then $\mathcal{S}^l(Y) \subseteq \mathcal{G}_B$ and $\mathcal{S}^r(Y) \subseteq \mathcal{G}_A$ are abelian and are isomorphic as groups.

Proof:

Since Y and Y^\vee are duality bimodules, by proposition 3.20 (ii) we have $|\mathcal{S}^r(Y)| \geq |\mathcal{S}^l(Y)|$ and $|\mathcal{S}^r(Y^\vee)| \geq |\mathcal{S}^l(Y^\vee)|$. Using $\mathcal{S}^r(Y^\vee) = \mathcal{S}^l(Y)$ and $\mathcal{S}^l(Y^\vee) = \mathcal{S}^r(Y)$, this implies $|\mathcal{S}^r(Y)| = |\mathcal{S}^l(Y)|$. Denote by G^* the character group of a group G . Since Y is a duality bimodule, from proposition 3.20 (ii) it follows that the map $\varphi_Y: g \mapsto \phi_Y(g, \cdot)$ is an injective group homomorphism from $\mathcal{S}^l(Y)$ to $\mathcal{S}^r(Y)^*$. Because of $|\mathcal{S}^r(Y)| = |\mathcal{S}^l(Y)|$ this shows that there are at least $|\mathcal{S}^r(Y)|$ different one-dimensional representations of $\mathcal{S}^r(Y)$. Since the number of inequivalent representations of a finite group is equal to the number of its conjugacy classes, this means that every conjugacy class of $\mathcal{S}^r(Y)$ must consist of a single element, i.e. $\mathcal{S}^r(Y)$ is abelian. But a finite abelian group is isomorphic to its character group, and combining this isomorphism with φ_Y we obtain an isomorphism of groups from $\mathcal{S}^l(Y)$ to $\mathcal{S}^r(Y)$. Since $\mathcal{S}^r(Y)$ is abelian, so is $\mathcal{S}^l(Y)$. \checkmark

Let Y be a simple B - A -bimodule such that Y and Y^\vee are duality bimodules. We now investigate what happens to bulk fields when we inflate the duality defect Y in a world sheet. According to (2.31) the bulk fields of $\text{CFT}(A)$ carry a representation $\phi \mapsto D_g(\phi)$ of the group \mathcal{G}_A (and the bulk fields of $\text{CFT}(B)$ a representation of \mathcal{G}_B). In particular, the bulk fields of $\text{CFT}(A)$ furnish a representation of the stabiliser $\mathcal{S}^r(Y)$. Let us define a map $F_{Y,UV}^r$ which assigns to an element $g \in \mathcal{S}^l(Y)$ the subspace of bulk fields with chiral/anti-chiral labels U, V in representation $\phi_Y(g, \cdot)$ of $\mathcal{S}^r(Y)$:

$$\begin{aligned} F_{Y,UV}^r : \mathcal{S}^l(Y) &\longrightarrow \text{set of subspaces of } \text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A) \\ g &\longmapsto \{ v \mid D_h(v) = \phi_Y(g, h)v \text{ for all } h \in \mathcal{S}^r(Y) \}. \end{aligned} \quad (3.51)$$

By theorem 3.22 the stabilisers are abelian, and their irreducible representations are one-dimensional. Together with proposition 3.20 it follows that each irreducible representation of $\mathcal{S}^r(Y)$ is of the form $h \mapsto \phi_Y(g, h)$ for some $g \in \mathcal{S}^l(Y)$. Thus we get the direct sum decomposition

$$\text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A) \cong \bigoplus_{g \in \mathcal{S}^l(Y)} F_{Y,UV}^r(g). \quad (3.52)$$

Analogously we set

$$F_{Y,UV}^l : \mathcal{S}^r(Y) \longrightarrow \text{set of subspaces of } \text{Hom}_{B|B}(U \otimes^+ B \otimes^- V, B) \quad (3.53)$$

$$h \longmapsto \{ v \mid D_g(v) = \phi_Y(g, h)v \text{ for all } g \in \mathcal{S}^l(Y) \}$$

for bulk fields of $\text{CFT}(B)$. The following result shows that in an appropriate basis for the bulk fields, taking a duality defect Y past a bulk field results in a defect field sitting at the end of a single group-like defect, rather than a superposition thereof as one might expect from the relation $Y^\vee \otimes_B Y \cong \bigoplus_{h \in \mathcal{S}^r(Y)} X_h$.

Proposition 3.23:

Let Y be a simple B - A -defect such that both Y and Y^\vee are duality defects, and let $g \in \mathcal{S}^l(Y)$ and $h \in \mathcal{S}^r(Y)$. For $\phi \in F_{Y,UV}^r(g)$ a bulk field of $\text{CFT}(A)$ and $\phi' \in F_{Y,UV}^l(h)$ a bulk field of $\text{CFT}(B)$ we have

and

$$(3.54)$$

Proof:

We establish the first equality in (3.54); the second equality can be seen analogously. Let $u \in \mathcal{S}^l(Y)$ and $\phi \in F_{Y,UV}^r(u)$. Note that

$$(3.55)$$

where we also used that by remark 3.7 (iii), $\varepsilon(g) = 1$. In order for (3.55) to be nonzero, we thus need $\phi_Y(gu, h) = 1$ for all $h \in \mathcal{S}^l$, which by proposition 3.20 (ii) implies $g = u^{-1}$. But then, using also (2.27), we have

$$= \sum_{g \in \mathcal{S}^l(Y)} \frac{\dim(B)}{\dim(Y)} \quad (3.56)$$

Moreover, in the sum on the right hand side, only the term $g = u^{-1}$ can be nonzero. \checkmark

Remark 3.24:

In [26], based on previous results in [51, 52], order-disorder dualities were investigated via the symmetry properties of boundary states. The boundary states were defined to also include sectors twisted by a symmetry of the CFT, and dualities can be found by checking if one can find an invertible transformation on the boundary states that exchanges some periodic sectors with twisted ones. Using defect lines we can recover this relation by considering one-point functions of a bulk field on a disk. These one-point functions give the coefficients of the Ishibashi states in the expansion of a boundary state. Consider a disk in phase $\text{CFT}(A)$ with boundary condition M , and let Y be a B - A -duality defect such that also Y^\vee is a duality defect. Then the manipulations

$$= \frac{\dim(A)}{\dim(Y)} \quad (3.57)$$

show that the coefficient of the periodic Ishibashi state belonging to the bulk field labelled ϕ in the expansion of the boundary state for M is the same as the coefficient of the g -twisted

Ishibashi state belonging to the disorder field labelled θ in the boundary state for $Y \otimes_A M$ (in an appropriate normalisation of twisted Ishibashi states and disorder fields). We also used proposition 3.23 to choose ϕ such that only a single group-like defect contributes on the right hand side of (3.57). Note that the above argument only shows that the relation (3.57) is a necessary condition for the existence of a two-sided duality.

3.6 High-low temperature duality

Consider a bulk field labelled ϕ of $\text{CFT}(A)$ that is invariant under the action of $\mathcal{S}^r(Y)$, i.e. $\phi \in F_{Y,UV}^r(e)$. After acting with the defect Y , it becomes the bulk field $\frac{\dim(B)}{\dim(Y)} D_Y(\phi)$ (in the notation (2.30)) of $\text{CFT}(B)$. Since $\mathcal{S}^l(Y)$ is the left-stabiliser of Y , the field $D_Y(\phi)$, in turn, lies in $F_{Y,UV}^l(e)$. According to (3.34), inflating the dual defect Y^\vee in the world sheet thus takes $\frac{\dim(B)}{\dim(Y)} D_Y(\phi)$ to

$$\frac{\dim(A) \dim(B)}{\dim(Y)^2} D_{Y^\vee} \circ D_Y(\phi) = \frac{1}{|\mathcal{S}^r(Y)|} \sum_{h \in \mathcal{S}^r(Y)} D_h(\phi) = \phi. \quad (3.58)$$

In fact, the map $\varphi \mapsto |\mathcal{S}^r(Y)|^{-1} \sum_{h \in \mathcal{S}^r(Y)} D_h(\varphi)$ is a projector to the subspace of bulk fields invariant under $\mathcal{S}^r(Y)$.

The subspaces $F_{Y,UV}^r(e)$ of the multiplicity spaces of bulk fields are of particular interest, because they are related to the high/low temperature dualities which turn into the order-disorder duality induced by Y at the critical point. To see that, let us study the situation that $\text{CFT}(A)$ is perturbed by the field $\phi \in F_{Y,UV}^r(e)$. Applying the duality induced by Y relates an order correlator of the perturbed $\text{CFT}(A)$ to a disorder correlator in $\text{CFT}(B)$ perturbed by $\phi' := \frac{\dim(B)}{\dim(Y)} D_Y(\phi)$. Schematically,

$$\left\langle (\text{fields in } \text{CFT}(A)) e^{\lambda \int \phi(z) d^2z} \right\rangle = \left\langle (\text{dual fields in } \text{CFT}(B)) e^{\lambda \int \phi'(z) d^2z} \right\rangle. \quad (3.59)$$

The precise form of this relation is obtained by expanding the left hand side of (3.59) in a perturbation series in λ and applying the defect Y at each order. In [14] it was noted that the high-low temperature duality of the Ising model can be found in this way upon choosing ϕ to be the energy field ε .

4 TFT formulation of defect correlators

To prove the rules for manipulating defects laid out in section 2.3, and to establish the non-degeneracy of the defect two-point function on the sphere which is instrumental for our purposes, we employ the formulation of RCFT correlators in terms of three-dimensional topological field theory (3-d TFT) that was developed in [53] and [0–V]. In this approach the chiral CFT is realised by the boundary degrees of freedom of an appropriate 3-d TFT [54, 55]. One can then use the geometry of a three-manifold together with a certain network of Wilson lines to combine left and right moving chiral degrees of freedom in the correct manner.

4.1 TFT derivation of the rules of section 2.3

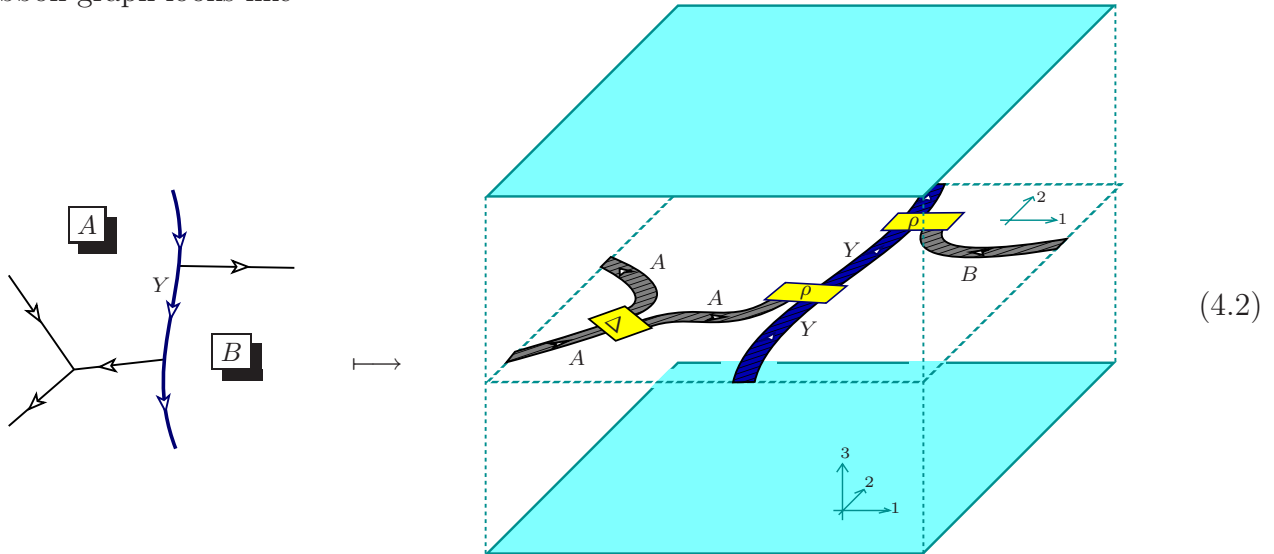
A 3-d TFT can be constructed from any modular tensor category \mathcal{C} [56, 29]. The modular tensor category relevant for the application to CFT is $\mathcal{C} = \mathcal{R}ep(\mathcal{V})$ (or more precisely, an equivalent strict ribbon category), but as already mentioned in remark 2.2, for the calculations in this paper it is irrelevant whether \mathcal{C} can be realised as the representation category of a vertex algebra or not. Given an oriented three-manifold M with embedded ribbon graph, the 3-d TFT assigns to the boundary ∂M a vector space $\mathcal{H}(\partial M)$ and to M itself a vector $Z(M)$ in $\mathcal{H}(\partial M)$. For references and more details on our conventions regarding the 3-d TFT constructed from \mathcal{C} we refer to [I, sect. 2] and [IV, sect. 3.1].

To obtain the correlator for an oriented world sheet X , one considers the *connecting manifold* M_X , defined as

$$M_X =: X \times [-1, 1] / \sim, \quad \text{where } (x, t) \sim (x, -t) \text{ for all } x \in \partial X, t \in [-1, 1]. \quad (4.1)$$

This amounts to a ‘fattening’ of the world sheet. Note that $\iota: x \mapsto (x, 0)$ gives an embedding of X into M_X . The relevant ribbon graph – or framed Wilson graph – in M_X is obtained by choosing a dual triangulation¹⁰ with directed edges on X and inserting ribbons in M_X along the images of these edges under the embedding ι . A ribbon has an orientation as a surface and a direction, and it is labelled by an object of \mathcal{C} . The ribbons are to be embedded in $\iota(X)$ such that their orientation is opposite to that of $\iota(X)$ and their direction is opposite to that of the edge of the dual triangulation¹¹. If the edge of the dual triangulation lies in a region of the world sheet in phase $\text{CFT}(A)$, then the corresponding ribbon is labelled by the object A of \mathcal{C} .

Close to defect lines, world sheet boundaries and field insertions, special pieces of ribbon graph have to be inserted. Specifically, close to a defect labelled by an A - B -bimodule Y , the ribbon graph looks like

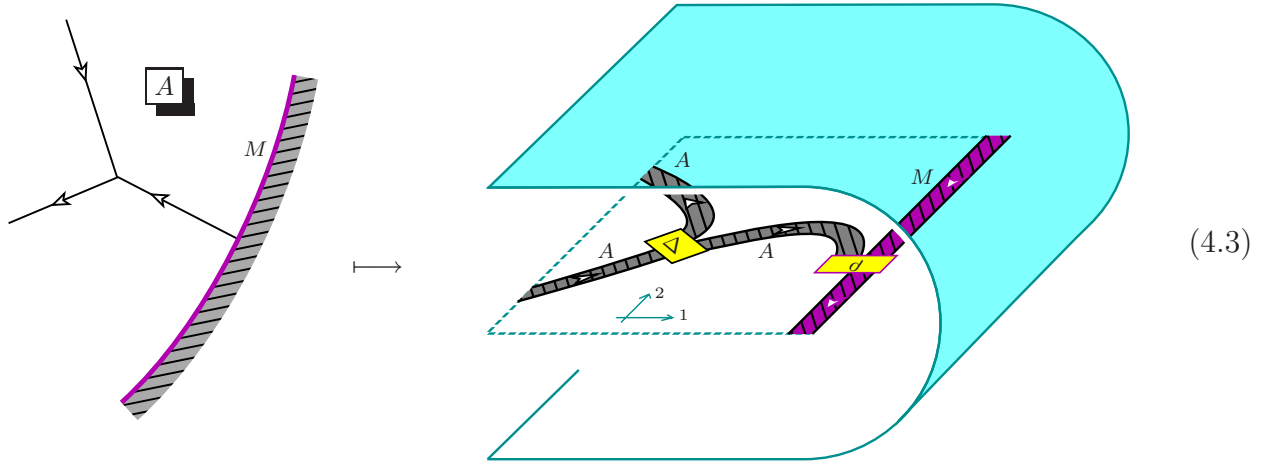


In this picture the orientations and a possible choice of dual triangulation are also shown. All ribbons are showing their ‘black side’, i.e. their surface orientation is opposite to the one

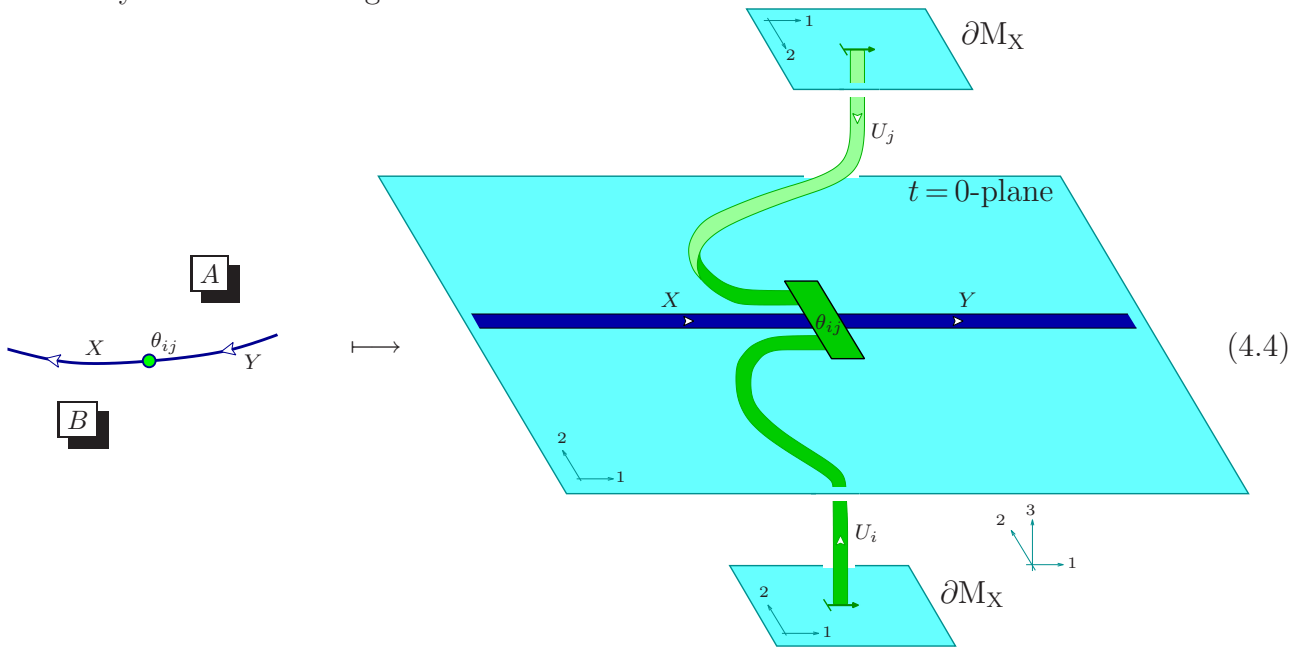
¹⁰ That is, a covering of X with a 2-complex whose vertices are three-valent and whose faces can be arbitrary polygons.

¹¹ That orientation and direction are chosen opposite to one another is just a convention (see section 3.1 of [IV] for the reasoning behind it), and does not have any deeper meaning.

indicated on the embedded world sheet. In a neighbourhood of a world sheet boundary we have



For a defect field insertion in left/right representation $U_i \times U_j$, there are additional ribbons, labelled by the simple objects U_i and U_j , which connect the embedded world sheet $\iota(X)$ to the boundary of the connecting manifold:



The special case of the insertion of a local bulk field is obtained for $X = Y = A$. A more detailed description of the TFT construction is given in appendix A of [V] and in section 3.3 of [IV].

We now derive some of the rules given in section 2.3; the remaining ones can be verified along similar lines. Let us start by explaining why the fusion of defects corresponds to the

tensor product of bimodules, see (2.23). We have the equality

(4.5)

Here we show only the relevant fragments of the complete cobordisms, and it is understood that $Z(\cdot)$ is applied to each side of the equality. We also passed to the ‘blackboard framing’ convention for the ribbons, in which a solid line means that the ribbon is showing its ‘white’ side to the reader. Note that the orientation of the embedded world sheet is reversed with respect to (4.2)–(4.4), which means that, for convenience, we are drawing the cobordism viewed from a different angle, so that we face the white side of the embedded ribbons. The equality in (4.5) follows from the corresponding equality for morphisms, namely $e_{X,Y} \circ r_{X,Y} = P_{X,Y}$, see section 2.1.

Next, the second rule in (2.26) amounts to the following identity for invariants of cobordisms:

$$= \sum_{\mu, \alpha} \quad (4.6)$$

Here the morphisms are the basis morphisms as chosen in (2.18). Equation (4.6) follows from semisimplicity of the bimodule category $\mathcal{C}_{A|C}$; we have

$$\sum_{\mu, \alpha} \bar{\Lambda}_{(\rho\sigma)\mu}^{\alpha} \circ \Lambda_{(\rho\sigma)\mu}^{\alpha} = P_{X_{\rho}, X_{\sigma}}. \quad (4.7)$$

The ribbon graph for the identity (2.27) is similar to (4.6). The corresponding equality for morphisms of \mathcal{C} reads

$$= \sum_{\mu, \gamma} \frac{\dim(X_{\mu})}{\dim(X_{\rho})} \quad (4.8)$$

To obtain this equality, first note that

$$\begin{aligned} L_{(\rho\sigma)\mu}^{\gamma} &= (id_{X_{\mu}} \otimes \tilde{d}_{X_{\sigma}}) \circ (\bar{\Lambda}_{(\mu\sigma)\rho}^{\gamma} \otimes id_{X_{\sigma}^{\vee}}) \circ e_{X_{\rho}, X_{\sigma}^{\vee}} \in \text{Hom}_{A|C}(X_{\rho} \otimes_B X_{\sigma}^{\vee}, X_{\mu}) \quad \text{and} \\ \bar{L}_{(\rho\sigma)\mu}^{\gamma} &= r_{X_{\rho}, X_{\sigma}^{\vee}} \circ (\Lambda_{(\mu\sigma)\rho}^{\gamma} \otimes id_{X_{\sigma}^{\vee}}) \circ (id_{X_{\mu}} \otimes b_{X_{\sigma}}) \in \text{Hom}_{A|C}(X_{\mu}, X_{\rho} \otimes_B X_{\sigma}^{\vee}) \end{aligned} \quad (4.9)$$

provide bases of the two morphism spaces, respectively. Since the bimodule X_μ is simple, we have $L_{(\rho\sigma)\mu}^\gamma \circ \bar{L}_{(\rho\sigma)\mu}^\varepsilon = c_{\gamma\varepsilon} \text{id}_{X_\mu}$ for some constants $c_{\gamma\varepsilon}$ (which can also depend on ρ, σ, μ). To determine the value of the constant, one takes the trace of both sides and uses the defining properties of the morphisms Λ and $\bar{\Lambda}$ introduced in (2.18). This results in $c_{\gamma\varepsilon} \dim(X_\mu) = \delta_{\gamma,\varepsilon} \dim(X_\rho)$. Thus

$$L_{(\rho\sigma)\mu}^\gamma \circ \bar{L}_{(\rho\sigma)\mu}^\varepsilon = \delta_{\gamma,\varepsilon} \frac{\dim(X_\rho)}{\dim(X_\mu)} \text{id}_{X_\mu}. \quad (4.10)$$

Next, by semisimplicity of $\mathcal{C}_{A|C}$ there are numbers $C_{\mu,\gamma\varepsilon}$ such that

$$\text{id}_{X_\rho \otimes_B X_\sigma^\vee} = \sum_{\mu \in \mathcal{K}_{AC}} \sum_{\gamma,\varepsilon} C_{\mu,\gamma\varepsilon} \bar{L}_{(\rho\sigma)\mu}^\varepsilon \circ L_{(\rho\sigma)\mu}^\gamma. \quad (4.11)$$

These constants can be determined by composing both sides with $\bar{L}_{(\rho\sigma)\nu}^\beta$ from the right, which yields

$$\bar{L}_{(\rho\sigma)\nu}^\beta = \sum_{\varepsilon} C_{\nu,\beta\varepsilon} \frac{\dim(X_\rho)}{\dim(X_\nu)} \bar{L}_{(\rho\sigma)\nu}^\varepsilon. \quad (4.12)$$

Since the morphisms $\bar{L}_{(\rho\sigma)\nu}^\beta$ form a basis, this forces $C_{\nu,\beta\varepsilon} = \delta_{\beta,\varepsilon} \dim(X_\nu)/\dim(X_\rho)$. This shows that the constants in (4.11) indeed have the value used in (4.8).

Next consider the equality (2.28) for wrapping a defect line around a bulk insertion. In terms of cobordisms, it amounts to the following defining equality for $D_{\mu\nu\alpha}(\phi)$:

(4.13)

This enables us to deduce the explicit form of the linear map $D_{\mu\nu\alpha}$ that we introduced in (2.29):

$$D_{\mu\nu\alpha}(\phi) = \text{Diagram} \tag{4.14}$$

As a final rule we establish the one for inserting a little defect loop as in equation (2.34). The corresponding identity for invariants of cobordisms is

$$= \frac{\dim(A)}{\dim(Y)} \text{Diagram} \tag{4.15}$$

That this is indeed valid is a consequence of the following result (note that it requires the algebra A to be simple):

Lemma 4.1 :

For any left A -module M over a simple symmetric special Frobenius algebra A one has

$$= \frac{\dim(M)}{\dim(A)} id_A . \tag{4.16}$$

Proof:

Denote the morphism on the left hand side of (4.16) by f . Using that A is symmetric Frobenius, one verifies that $f \in \text{Hom}_{A|A}(A, A)$. Since A is simple, this space is one-dimensional, so that $f = \lambda id_A$ for some $\lambda \in \mathbb{C}$. Composing with unit and counit determines this coefficient to be $\lambda = \dim(M)/\dim(A)$. \checkmark

4.2 Non-degeneracy of defect two-point correlators

In this section we prove a non-degeneracy result for the two-point correlator for disorder fields on S^2 , i.e. for

$$C := \tag{4.17}$$

and derive a number of consequences. In (4.17), X_μ is a simple B - B -defect, X_ν a simple A - A -defect, X_τ and X_σ are simple B - A -defects, and γ, δ label basis elements in the relevant morphism spaces. As in (2.39) we can write the correlator (4.17) as a multiple of a product of two-point blocks:

$$C = \Gamma(\mu, \nu)_{ij\alpha\beta}^{\tau\sigma\gamma\delta} \beta[i, \bar{i}](z, w) \beta[j, \bar{j}](z^*, w^*) . \tag{4.18}$$

The coefficients $\Gamma(\mu, \nu)_{ij\alpha\beta}^{\tau\sigma\gamma\delta}$ can be computed by applying the TFT construction for defect correlators outlined in section 4.1 to the correlator (4.18). This results in the following ribbon invariant in S^3 :

$$\Gamma(\mu, \nu)_{ij\alpha\beta}^{\tau\sigma\gamma\delta} = \frac{1}{S_{00}^2} = \frac{1}{S_{00}^2} \tag{4.19}$$

Let R_1 denote the set of labels $(ij\alpha\beta)$ and R_2 the set of labels $(\tau\sigma\gamma\delta)$. By definition, their cardinalities are

$$\begin{aligned} |R_1| &= \sum_{i,j \in \mathcal{I}} \dim_{\mathbb{C}} \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, X_\nu) \cdot \dim_{\mathbb{C}} \text{Hom}_{A|A}(U_i \otimes^+ X_\mu \otimes^- U_j, B), \\ |R_2| &= \sum_{\sigma, \tau \in \mathcal{K}_{BA}} \dim_{\mathbb{C}} \text{Hom}_{B|A}(X_\sigma \otimes_A X_\nu, X_\tau) \cdot \dim_{\mathbb{C}} \text{Hom}_{B|A}(X_\tau, X_\mu \otimes_B X_\sigma). \end{aligned} \quad (4.20)$$

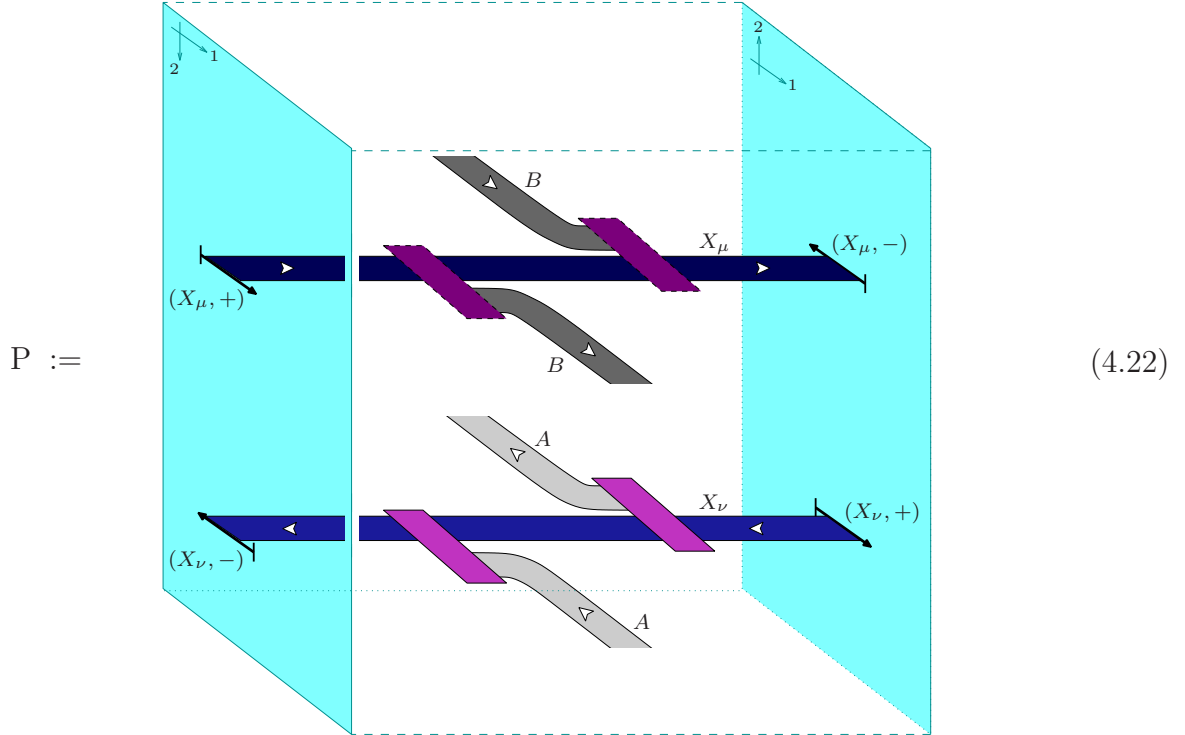
Re-expressing $|R_1|$ via (2.15) and performing the sum over τ in $|R_2|$ we can alternatively write

$$|R_1| = \sum_{i,j \in \mathcal{I}} Z(A)_{ij}^{A|X_\nu} Z(B)_{ij}^{X_\mu|B}, \quad |R_2| = \sum_{\sigma \in \mathcal{K}_{BA}} \dim_{\mathbb{C}} \text{Hom}_{B|A}(X_\sigma \otimes_A X_\nu, X_\mu \otimes_B X_\sigma). \quad (4.21)$$

Theorem 4.2:

Let A and B be simple symmetric special Frobenius algebras and let $\mu \in \mathcal{K}_{BB}$ and $\nu \in \mathcal{K}_{AA}$. Then the label sets R_1 and R_2 either obey $|R_1| = |R_2| = 0$, or else the $|R_1| \times |R_2|$ -matrix $\Gamma(\mu, \nu)$ with entries given by (4.19) is non-degenerate.

The idea of the proof is to relate the assertion (which is a statement about conformal blocks on surfaces of genus zero) to properties of conformal blocks on surfaces of genus one. It relies on a number of ingredients. The first is a certain projector P on the vector space $\mathcal{H}(\mathbb{T}_{X_\mu, X_\nu})$, with $\mathbb{T}_{X_\mu, X_\nu}$ the extended surface consisting of a 2-torus with two marked points with labels $(X_\mu, +)$ and $(X_\nu, -)$ (here the signs (\cdot, \pm) refer to different orientations of the core of a ribbon, see the conventions in [I, sect. 2.4] and [IV, sect. 3.1]),



This is just a slight generalisation of the projector given in (5.15) of [V], with the A -ribbons running from left to right replaced by an X_μ - and an X_ν -ribbon, respectively. That (4.22) indeed defines a projector is seen in the same way as in [V].

Let $\text{Im}(P) \subseteq \text{End}(\mathcal{H}(\mathbb{T}_{X_\mu, X_\nu}))$ be the image of the projector P . We will introduce two bases for the vector space $\text{Im}(P)$ and show that the matrix transforming one basis into the other is given by $\Gamma(\mu, \nu)$, thus establishing that $\Gamma(\mu, \nu)$ is nondegenerate.

For $\alpha \in \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, X_\nu)$ and $\beta \in \text{Hom}_{A|A}(U_{\bar{i}} \otimes^+ X_\mu \otimes^- U_{\bar{j}}, B)$, consider the vectors $M_{ij, \alpha\beta}^{\mu\nu} \in \mathcal{H}(\mathbb{T}_{X_\mu, X_\nu})$ given by the invariant

$M_{ij, \alpha\beta}^{\mu\nu} :=$ (4.23)

The three-manifold in this figure is a solid torus in “wedge presentation”. The boundary of the solid torus is the vertical face containing the two marked points, which is the 2-torus $\mathbb{T}_{X_\mu, X_\nu}$. The other two vertical faces are to be identified, as are the horizontal faces at the top and bottom. We refer to section 5.1 of [V] for more details on the wedge presentation of three-manifolds.

Lemma 4.3:

The invariants $M_{ij, \alpha\beta}^{\mu\nu}$, with $(ij\alpha\beta) \in R_1$, form a basis of $\text{Im}(P)$.

Proof:

The invariants (4.23) constitute a generalisation of the ones displayed in (5.10) of [V]. The proof works in the same way as the one of lemma 5.2 (ii) of [V]. ✓

To find the second basis, consider the cobordisms $K_{\sigma\tau\gamma\delta}^{\mu\nu}: \emptyset \rightarrow \mathbb{T}_{X_\mu, X_\nu}$ and $\overline{K}_{\sigma\tau\gamma\delta}^{\mu\nu}: \mathbb{T}_{X_\mu, X_\nu} \rightarrow \emptyset$

given by

$$\begin{aligned}
 K_{\sigma\tau\gamma\delta}^{\mu\nu} &:= \text{[Diagram 1: A 3D box with a cyan plane. A blue ribbon graph with strands labeled } X_\nu, X_\sigma, X_\tau, X_\mu \text{ and yellow bands labeled } \bar{\gamma}, \delta \text{ is shown.}] \\
 \bar{K}_{\sigma\tau\gamma\delta}^{\mu\nu} &:= \text{[Diagram 2: A 3D box with a cyan plane. A blue ribbon graph with strands labeled } X_\mu, X_\tau, X_\nu \text{ and yellow bands labeled } \gamma, \delta \text{ is shown.}]
 \end{aligned}
 \tag{4.24}$$

Here γ runs over a basis of the space $\text{Hom}_{A|B}(X_\sigma, X_\tau \otimes_A X_\nu)$ and $\bar{\gamma}$ over the corresponding dual basis of $\text{Hom}_{A|B}(X_\tau \otimes_A X_\nu, X_\sigma)$; the meaning of δ and $\bar{\delta}$ is analogous. Again, we implicitly use the isomorphisms (2.10). To establish that (4.24) are dual bases, the following result is helpful:

Lemma 4.4:

Let A be a simple symmetric special Frobenius algebra and X_μ a simple A - A -bimodule. Then

$$\begin{aligned}
 &\text{[Diagram: A box labeled } S^2 \times I \text{ containing a blue ribbon graph with a loop and a strand labeled } X_\mu \text{ connecting two cyan ovals.]} \\
 &= \frac{\delta_{\mu,0}}{\dim(A)} \\
 &\text{[Diagram: A box labeled } S^2 \times I \text{ containing a blue ribbon graph with a single strand labeled } X_\mu \text{ connecting two cyan ovals.]}
 \end{aligned}
 \tag{4.25}$$

Proof:

Let us start from the ribbon graph invariant on the left hand side of (4.25), to be denoted by Q . By the same reasoning as in the proof of lemma 5.2 in [I], we see that $Q \in \text{End}(\mathcal{H}(S^2; X_\mu))$ is an idempotent. Next consider the equalities

$$\begin{aligned}
 Q &= \sum_{k \in \mathcal{I}} \sum_{\alpha} \text{[Diagram: A box labeled } S^2 \times I \text{ with a blue ribbon graph. A green strand labeled } k \text{ connects two yellow triangles labeled } \alpha \text{ and } \bar{\alpha} \text{ on a blue strand labeled } X_\mu \text{ connecting two cyan ovals.]} \\
 &= \sum_{\alpha} \text{[Diagram: A box labeled } S^2 \times I \text{ with a blue ribbon graph. A red dashed box highlights a loop structure involving yellow triangles } \alpha \text{ and } \bar{\alpha} \text{ on a blue strand labeled } X_\mu \text{ connecting two cyan ovals.]} \\
 &= 0 \text{ if } \mu \neq 0.
 \end{aligned}
 \tag{4.26}$$

Here in the first step the bimodule X_μ is regarded as an object of \mathcal{C} and decomposed into simple objects of \mathcal{C} . The second step follows because $\mathcal{H}(S^2; U_k)$ has nonzero dimension only for $U_k = \mathbf{1}$, so that the sum over k restricts to $k=0$. We also deformed the A -ribbon and used that A is symmetric. Finally, the morphism in the dashed box constitutes an element in $\text{Hom}_{A|A}(A, X_\mu)$, which is non-zero only for $\mu=0$. If $\mu=0$, i.e. if $X_\mu = A$, then the morphism from A to A on the left hand side of (4.25) is just the projector on the left center $C_l(A)$ of A , see section 2.4 of [30] for details and references. Since A is simple we have $\dim_{\mathbb{C}} \text{Hom}(\mathbf{1}, C_l(A)) = 1$ (this follows from the case $U = V = \mathbf{1}$ of proposition 2.36 of [30]). The unit morphism η of A lies in the left center, so that the left and the right hand side of (4.25) must be proportional. The constant is determined by demanding the right hand side to be an idempotent. \checkmark

Lemma 4.5 :

(i) $K_{\sigma\tau\gamma\delta}^{\mu\nu} \in \text{Im}(\text{P})$.

(ii) $\bar{K}_{\sigma'\tau'\gamma'\delta'}^{\mu\nu} \circ K_{\sigma\tau\gamma\delta}^{\mu\nu} = \delta_{\sigma,\sigma'} \delta_{\tau,\tau'} \delta_{\gamma,\gamma'} \delta_{\delta,\delta'}$.

In particular, the vectors $\{K_{\sigma\tau\gamma\delta}^{\mu\nu}\}_{\sigma,\tau,\gamma,\delta}$ are linearly independent.

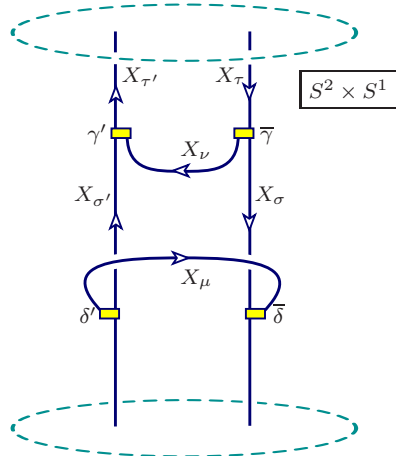
(iii) The vectors $\{K_{\sigma\tau\gamma\delta}^{\mu\nu}\}_{\sigma,\tau,\gamma,\delta}$ span $\text{Im}(\text{P})$.

Proof:

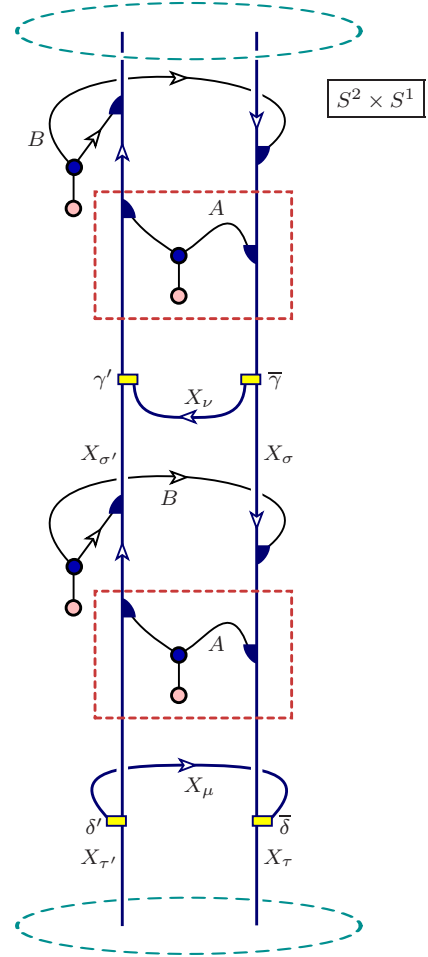
(i) can be seen by using the property that $\bar{\gamma}$ and $\bar{\delta}$ are bimodule intertwiners – this allows one to remove the A - and B -ribbons in the composition $\text{P} \circ K_{\sigma\tau\gamma\delta}^{\mu\nu}$ by moves similar to the ones used in the proof of lemma 5.2(iii) of [V].

For (ii) note that composing $\overline{K}_{\sigma'\tau'\gamma'\delta'}^{\mu\nu} \circ K_{\sigma\tau\gamma\delta}^{\mu\nu}$ yields the following ribbon graph in $S^2 \times S^1$:

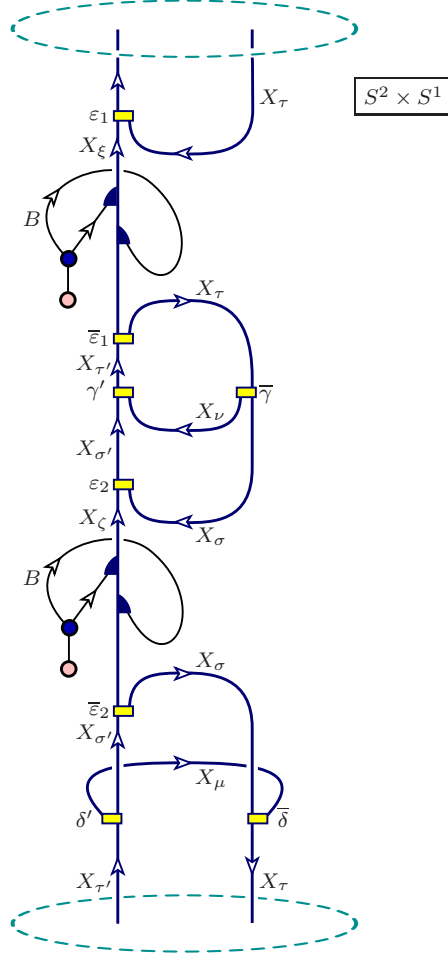
$$\overline{K}_{\sigma'\tau'\gamma'\delta'}^{\mu\nu} \circ K_{\sigma\tau\gamma\delta}^{\mu\nu} \stackrel{(1)}{=} \quad (1)$$



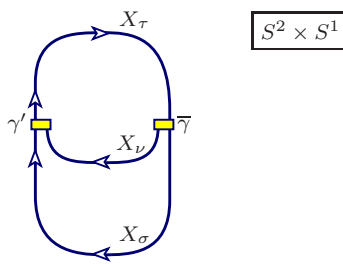
$$\stackrel{(2)}{=} \quad (2)$$



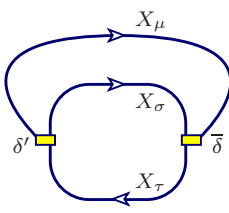
$$\underline{(3)} \quad \sum_{\zeta, \xi \in \mathcal{K}_{BB}} \sum_{\varepsilon_1, \varepsilon_2} \frac{\dim(X_\zeta)}{\dim(X_{\sigma'})} \frac{\dim(X_\xi)}{\dim(X_{\tau'})}$$



$$\underline{(4)} \quad \frac{\delta_{\sigma, \sigma'} \delta_{\tau, \tau'}}{\dim(X_\sigma) \dim(X_\tau)}$$



$$\underline{(5)} \quad \delta_{\sigma, \sigma'} \delta_{\tau, \tau'} \delta_{\gamma, \gamma'} \delta_{\delta, \delta'}$$



(4.27)

The manipulations leading to the individual equalities are as follows: (1) consists in composing the two cobordisms (4.24). For (2) one first wraps the X_μ -ribbon around the ‘horizontal’ S^2 so

that it runs behind the vertical ribbons, and then uses the presence of the X_μ - and X_ν -ribbons to insert the A - and B -ribbons (this equality is more easily checked in the opposite direction, using the presence of the two bimodule ribbons to remove A and B). In (3) the identity (4.8) is applied to the morphisms in the two dashed boxes. In (4) lemma 4.4 is applied to restrict the sums to $\zeta = \xi = 0$. Finally, in (5) one uses that γ' and $\bar{\gamma}$, as well as δ' and $\bar{\delta}$, label dual bases. To show (iii), we start from the ribbon invariants

$$b_{nm\tau s, \alpha\beta\varepsilon\varphi} := \text{Diagram} \tag{4.28}$$

which form a basis of $\mathcal{H}(T_{X_\mu, X_\nu})$. Here n, m, s, t label simple objects of \mathcal{C} and $\alpha, \beta, \varepsilon, \sigma$ label basis elements in the appropriate morphism spaces. Applying the idempotent P and deforming the resulting ribbon graph suitably gives

$$P \circ b_{nm\tau s, \alpha\beta\varepsilon\varphi} = \text{Diagram} \tag{4.29}$$

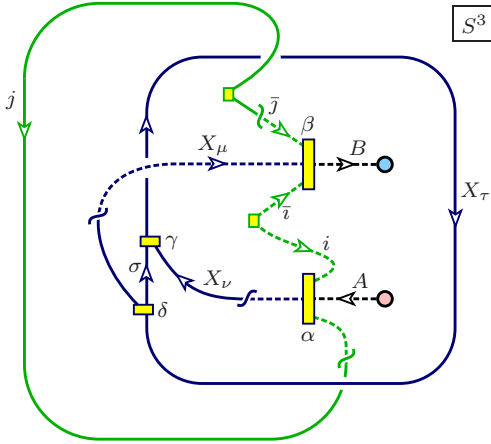
Inside the dashed boxes we deal with the induced B - A -bimodules $B \otimes U_s \otimes A$ and $B \otimes U_t \otimes A$, which we can decompose in simple B - A -bimodules X_σ and X_τ , respectively. In this decomposition, the morphisms inside the dashed circles give elements in $\text{Hom}_{B|A}(X_\tau, X_\sigma \otimes_A X_\nu)$ and $\text{Hom}_{B|A}(X_\mu \otimes_B X_\sigma, X_\tau)$, respectively. It follows that the vector (4.29) can be written as a linear combination of the elements $K_{\sigma\tau\gamma\delta}^{\mu\nu}$ in (4.24). Thus indeed $\text{Im}(P)$ is spanned by the vectors $K_{\sigma\tau\gamma\delta}^{\mu\nu}$. \checkmark

Proof of theorem 4.2:

By lemmas 4.3 and 4.5 we know that $M_{ij,\alpha\beta}^{\mu\nu}$ and $K_{\sigma\tau\gamma\delta}^{\mu\nu}$ are bases of $\text{Im}(P)$. Thus we can write

$$M_{ij,\alpha\beta}^{\mu\nu} = \sum_{\sigma',\tau',\gamma',\delta'} \Lambda(\mu,\nu)_{ij\alpha\beta}^{\sigma'\tau'\gamma'\delta'} K_{\sigma'\tau'\gamma'\delta'}^{\mu\nu} \quad (4.30)$$

for some nondegenerate matrix $\Lambda(\mu,\nu)$. Composing both sides with $\overline{K}_{\sigma\tau\gamma\delta}^{\mu\nu}$ from the left and using lemma 4.5 (ii) yields

$$\Lambda(\mu,\nu)_{ij\alpha\beta}^{\sigma\tau\gamma\delta} = \overline{K}_{\sigma\tau\gamma\delta}^{\mu\nu} \circ M_{ij,\alpha\beta}^{\mu\nu} =$$


$$(4.31)$$

The ribbon graph on the right hand side can be deformed into the one shown in (4.19), so that in fact $\Gamma(\mu,\nu) = \Lambda(\mu,\nu)/S_{00}^2$. \checkmark

Lemma 2.10 is a consequence of theorem 4.2, obtained by specialising to $\nu=0$. Since for $\Gamma(\mu,\nu)$ to be nondegenerate the cardinalities (4.21) of the index sets R_1 and R_2 must be equal, another immediate consequence is the following

Corollary 4.6:

For A, B simple symmetric Frobenius algebras, X_ν a simple A - A -bimodule and X_μ a simple B - B -bimodule, we have

$$\begin{aligned} \sum_{i,j \in \mathcal{I}} \dim_{\mathbb{C}} \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, X_\nu) \dim_{\mathbb{C}} \text{Hom}_{B|B}(U_i \otimes^+ X_\mu \otimes^- U_j, B) \\ = \sum_{\tau \in \mathcal{K}_{BA}} \dim_{\mathbb{C}} \text{Hom}_{B|A}(X_\tau \otimes_A X_\nu, X_\mu \otimes_B X_\tau). \end{aligned} \quad (4.32)$$

Recall that the space $\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, X_\nu)$ labels disorder fields which turn a defect A into a defect X_ν and which are in left/right representation (i, j) . Thinking of the CFT as the scaling limit of a lattice model, one would expect that for every defect X_ν there is at least one disorder field on which that defect can start, i.e. at least one pair (i, j) for which $\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, X_\nu)$ is nonzero. In representation theoretic terms, this is established by the following

Proposition 4.7:

Every simple A - A -bimodule of a simple symmetric special Frobenius algebra A in a modular tensor category \mathcal{C} is a sub-bimodule of $U \otimes^+ A \otimes^- V$ for suitable simple objects U, V of \mathcal{C} .

Proof:

It suffices to show that $\dim_{\mathbb{C}} \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, X_\nu)$ is nonzero for a suitable choice of $(i, j) \in \mathcal{I} \times \mathcal{I}$. Suppose the contrary. Then the left hand side of (4.32), evaluated for $A = B$ and $\mu = \nu$, is zero. The sum on the right hand side, on the other hand, contains the contribution $\dim_{\mathbb{C}} \text{Hom}_{A|A}(A \otimes_A X_\mu, X_\mu \otimes_A A)$ for $\tau = 0$, which is equal to one. This is a contradiction; hence $\dim_{\mathbb{C}} \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, X_\nu)$ cannot be zero for all i, j . \checkmark

Remark 4.8:

(i) By taking suitable direct sums, proposition 4.7 implies in particular that every A - A -bimodule is a sub-bimodule of $U \otimes^+ A \otimes^- V$ for suitable objects U and V .

(ii) It is instructive to reformulate proposition 4.7 in terms of α -induction. Recall that, given an algebra A in a braided tensor category \mathcal{C} , there are two tensor functors α_A^\pm from \mathcal{C} to the category of A - A -bimodules, called α -induction, with the right A -action on $\alpha_A^\pm(U)$ involving the braiding, see e.g. section 5.1 of [28]. Comparing the definitions (see equations (2.18) of [IV] and (2.33) of [30]) one finds

$$\alpha_A^+(U) = A \otimes^+ U \quad \text{and} \quad \alpha_A^-(U) = A \otimes^- U. \quad (4.33)$$

This leads to the isomorphisms

$$\alpha_A^+(U) \otimes_A \alpha_A^-(V) \cong (A \otimes^+ U) \otimes_A (A \otimes^- V) \cong U \otimes^+ A \otimes^- V, \quad (4.34)$$

where the last isomorphism is again easily seen by writing out the definitions. Proposition 4.7 then implies that for simple A , every A - A -bimodule is a submodule of the tensor product $\alpha_A^+(U) \otimes_A \alpha_A^-(V)$ of two α -induced bimodules. This result has been obtained previously in the framework of subfactor theory in [27, Theorem 5.10] and has been conjectured in the category theoretic framework in Claim 2 of [28].

When analysing the fusion rules of defect lines, it is useful to know that defects labelled by α -induced bimodules always lie in the center of the fusion algebra. This is established in the following

Lemma 4.9:

Let A be a symmetric special Frobenius algebra in an idempotent complete ribbon category \mathcal{C} . For every object U of \mathcal{C} and for every A - A -bimodule Y we have, for $\nu \in \{\pm\}$, an isomorphism

$$\alpha_A^\nu(U) \otimes_A Y \cong Y \otimes_A \alpha_A^\nu(U) \quad (4.35)$$

of A - A -bimodules.

Proof:

An isomorphism can be constructed using the braiding of \mathcal{C} . For example, for $\nu = +$ consider

the two morphisms

$$f := \quad \text{and} \quad g := \quad (4.36)$$

One verifies that $f \in \text{Hom}_{A|A}(\alpha_A^+(U) \otimes_A Y, Y \otimes_A \alpha_A^+(U))$ and that $g \in \text{Hom}_{A|A}(Y \otimes_A \alpha_A^+(U), \alpha_A^+(U) \otimes_A Y)$. It is furthermore straightforward to check that g is a left and right inverse to f . ✓

5 Defects for simple current theories

A class of symmetric special Frobenius algebras that play an important role in applications and are at the same time particularly tractable are Schellekens algebras. By definition, a *Schellekens algebra* is a symmetric special Frobenius algebra which is simple as a left-module over itself and all of whose simple subobjects are invertible objects of \mathcal{C} , or, in other words [57, 58], simple currents.

The structure and representation theory of Schellekens algebras has been developed in [III]. As an object of \mathcal{C} , a Schellekens algebra is of the form

$$A \cong \bigoplus_{h \in H} L_h \quad (5.1)$$

with H a subgroup of the group $\text{Pic}(\mathcal{C})$ of isomorphism classes of invertible objects (simple currents) of \mathcal{C} . The subgroup $H = H(A)$ is called the *support* of A . Since \mathcal{C} is braided, its Picard group $\text{Pic}(\mathcal{C})$ is abelian and, as a consequence, the support H is abelian as well. Indeed, the simple objects appearing in A are even further restricted:

Lemma 5.1:

For any Schellekens algebra $A \cong \bigoplus_{h \in H} L_h$ in an additive ribbon category \mathcal{C} we have:

- (i) $\dim(L_h) = 1$ (rather than -1) for all $h \in H$.
- (ii) The twist $\theta_h := \theta_{U_h}$ obeys $(\theta_h)^{N_h} = \text{id}_{L_h}$ for N_h the order of L_h .
(Invertible objects obeying this relation are said to be in the ‘effective center’ of \mathcal{C} .)

Proof:

(i) Since by definition A is simple as a left module over itself, it is in particular simple as an algebra, i.e. is a simple A - A -bimodule. Also, it follows from the defining properties of a symmetric special Frobenius algebra that $\dim(A) \neq 0$, see section 2.1. On the other hand, $\dim(\cdot)$ is a group homomorphism from H to \mathbb{Z}_2 . Thus $\sum_{h \in H} \dim(L_h) = \dim(A) \neq 0$ implies that $\dim(\cdot)$ is trivial on H .

(ii) is Proposition 3.14 of [III]. ✓

One and the same object of the form $A \cong \bigoplus_{h \in H} L_h$ can, in general, be endowed with several non-isomorphic structures of a symmetric special Frobenius algebra. A convenient concept for classifying non-isomorphic algebra structures is the Kreuzer-Schellekens bihomomorphism:

Definition 5.2:

Given a subgroup H of $\text{Pic}(\mathcal{C})$ that is contained in the effective center, a *Kreuzer-Schellekens bihomomorphism* (or *KSB*, for short) on H is a bihomomorphism

$$\Xi : H \times H \rightarrow \mathbb{C}^\times \tag{5.2}$$

which on the diagonal coincides with the quadratic form on H that is given by the twist, i.e. $\theta_{L_g} = \Xi(g, g) \text{id}_{L_g}$ for all $g \in H$.

Let us fix monomorphisms $e_g: L_g \rightarrow A$ as auxiliary data. One can show (proposition 3.20 of [III]) that for any Schellekens algebra A with multiplication m the equation

$$m \circ c_{L_g, L_h} \circ (e_g \otimes e_h) =: \Xi_A(h, g) m \circ (e_g \otimes e_h) \tag{5.3}$$

defines a KSB Ξ_A on the support H of A . This KSB is independent of the choice of morphisms e_h and characterises a Schellekens algebra up to isomorphism. Conversely, every KSB appears in the description of some Schellekens algebra. In brief, the KSB plays the role for the classification of Schellekens algebras that is played by alternating bihomomorphisms for the classification of twisted group algebras of abelian groups.

The main goal of this section is to extract as much information as possible about the bimodules of a Schellekens algebra A from the support $H(A)$ and the KSB Ξ_A . As it turns out, to this end we need to know the automorphism group of A ; this group is determined in the next subsection.

5.1 Automorphisms of Schellekens algebras

The endomorphisms of A as an object of \mathcal{C} are of the form

$$t_\psi := \bigoplus_{h \in H} \psi(h) \text{id}_{L_h}, \tag{5.4}$$

where $\psi: H \rightarrow \mathbb{C}$ is an arbitrary function. Obviously,

$$t_{\psi_1} \circ t_{\psi_2} = t_{\psi_1 \cdot \psi_2}, \tag{5.5}$$

where the product of functions on H is defined by pointwise multiplication.

Lemma 5.3:

The endomorphism t_ψ of A is an algebra automorphism if and only if ψ is a character of H . This identifies canonically the character group of H with $\text{Aut}(A)$.

Proof:

The statement follows from the fact that for a Schellekens algebra the function μ defined by $m \circ (e_g \otimes e_h) = \mu(g, h) e_{gh}$ is nonzero for all $g, h \in H$. \checkmark

For any simple object of \mathcal{C} we set

$$\chi_U(h) := \frac{s_{U, L_h}}{s_{U, \mathbf{1}}}. \quad (5.6)$$

As shown in [III] (see definition 3.24 and proposition 3.26 of [III]), this furnishes a character χ_U on H , and it is just the exponentiated negative *monodromy charge* of U ,

$$\chi_U(h) = \frac{\theta_{L_h \otimes U}}{\theta_{L_h} \theta_U}. \quad (5.7)$$

Furthermore,

$$c_{U, L_g} = \chi_U(g) c_{L_g, U}^{-1} \quad \text{and} \quad \chi_{U^\vee}(g) = \chi_U(g^{-1}). \quad (5.8)$$

For brevity, we denote the algebra automorphism of A corresponding to the character χ_U by t_U , i.e. write $t_U := t_{\chi_U}$.

These considerations immediately imply the following result (which justifies the term monodromy charge).

Lemma 5.4:

Let A be a Schellekens algebra and U be a simple object of \mathcal{C} . Then

- (i) $c_{U, A} = c_{A, U}^{-1} \circ (id_U \otimes t_U)$.
- (ii) $c_{A, U} = c_{U, A}^{-1} \circ (t_U \otimes id_U)$.

Proof:

For (i) note that expanding the braiding morphisms in direct sums, (5.8) implies that

$$c_{U, A} = \bigoplus_{h \in H} c_{U, L_h} = \bigoplus_{h \in H} \chi_U(h) c_{L_h, U}^{-1} = c_{U, A}^{-1} \circ (t_U \otimes id_U). \quad (5.9)$$

(ii) is seen similarly. \checkmark

We close this subsection with the

Lemma 5.5:

Let \mathcal{C} be a ribbon category, A a Schellekens algebra in \mathcal{C} , and $t_\psi \in \text{Aut}(A)$. Then

$$m \circ (id_A \otimes t_\psi) \circ \Delta = \delta_{\psi, e} id_A = m \circ (t_\psi \otimes id_A) \circ \Delta. \quad (5.10)$$

Proof:

First note that $\text{tr}(t_\psi) = \sum_{h \in H} \psi(h) \dim(L_h)$. Now according to lemma 5.1 we have $\dim(L_h) = 1$,

so that $\text{tr}(t_\psi) = \sum_{h \in H} \psi(h) = |H| \delta_{\psi, e}$. Together with the fact that A is a symmetric special Frobenius algebra it follows that

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} = \frac{1}{\dim(A)} \begin{array}{c} \text{Diagram 3} \end{array} = \frac{\text{tr}(t_\psi)}{\dim(A)} \begin{array}{c} \text{Diagram 4} \end{array} \quad (5.11)$$

Because of $\dim(A) = |H|$ this implies the first equality in (5.10). The second equality can be seen in the same way. \checkmark

5.2 Bimodules of Schellekens algebras

To enter the discussion of bimodules, we observe that both the left and the right action of A on an A - A -bimodule can be twisted by the action of automorphisms of A (see section 9 of [1]):

Definition 5.6:

Let \mathcal{C} be a tensor category and A an algebra in \mathcal{C} . For $X = (\dot{X}, \rho_l, \rho_r)$ an A - A -bimodule and $t, t' \in \text{Aut}(A)$ we denote by ${}_t X_{t'}$ the A - A -bimodule

$$(\dot{X}, \rho_l \circ (t \otimes id_{\dot{X}}), \rho_r \circ (id_{\dot{X}} \otimes t')). \quad (5.12)$$

In the case of a Schellekens algebra A with support H we write ${}_\varphi X_\psi := {}_{t_\varphi} X_{t_\psi}$ for $\varphi, \psi \in H^*$, i.e.

$${}_\varphi X_\psi = (\dot{X}, \rho_l \circ (t_\varphi \otimes id_{\dot{X}}), \rho_r \circ (id_{\dot{X}} \otimes t_\psi)). \quad (5.13)$$

We also use the abbreviation $X_\psi \equiv id X_\psi$.

Next we assume that \mathcal{C} is modular. We can then show that every bimodule over a Schellekens algebra can be recovered as a submodule of a twisted α -induced bimodule:

Proposition 5.7:

Let A be a Schellekens algebra in modular tensor category \mathcal{C} . Then every simple A - A -bimodule is a submodule of $\alpha_A^+(U)_\psi$ for some simple object U of \mathcal{C} and some automorphism $t_\psi \in \text{Aut}(A)$.

Proof:

Let X be a simple A - A -bimodule. Since a Schellekens algebra is in particular simple, by proposition 4.7 there are simple objects U and V of \mathcal{C} such that X is a sub-bimodule of $U \otimes^+ A \otimes^- V$. We claim that for a Schellekens algebra one has

$$U \otimes^+ A \otimes^- V \cong \alpha_A^+(U \otimes V)_\psi \quad (5.14)$$

as A - A -bimodules, with $\psi \in H^*$ the character for which t_ψ is the automorphism (5.4), $t_\psi = t_V^{-1}$. An isomorphism is provided by

$$f = c_{U,A} \otimes id_V. \quad (5.15)$$

To see that f is an isomorphism of bimodules, note that

$$\alpha_A^+(U \otimes V)_\psi = \text{diagram 2} = \text{diagram 3} = \alpha_A^+(U \otimes V)_\psi \quad (5.16)$$

where in the second step lemma 5.8 (i) is used. Thus X is also a sub-bimodule of $\alpha_A^+(U \otimes V)_\psi$. If $U \otimes V$ is not simple, then this bimodule is a direct sum of bimodules of the form $\alpha_A^+(U_i)_\psi$ with U_i simple, and X is contained in at least one of the summands. \checkmark

This result implies immediately that any simple A - A -bimodule, considered as an object of \mathcal{C} , is a direct sum of simple objects of \mathcal{C} whose classes in $K_0(\mathcal{C})$ are on the same H -orbit. Thus the problem to classify simple A - A -bimodules of a Schellekens algebra decomposes into separate problems for each H -orbit.

The situation simplifies when one considers H -orbits of simple objects in \mathcal{C} on whose class in $K_0(\mathcal{C})$ the support H of A acts freely by fusion. Objects with this property have been termed non-fixed points in the physics literature. We call such a simple object H -torsorial, and the orbit of its class in $K_0(\mathcal{C})$ a torsorial H -orbit. Note that invertible objects are in particular H -torsorial for any subgroup H of the Picard group $\text{Pic}(\mathcal{C})$.

As shown in section 5 of [III], the algebraic structure that controls the decomposition of the A - A -bimodule $A \otimes U \otimes A$ is a twisted group algebra. If U is H -torsorial, then the underlying group is given by

$$\mathcal{S}_{AA}(U) = \{(h, h^{-1}) \mid h \in H\} \cong H. \quad (5.17)$$

The commutator two-cocycle describing the twist is, according to proposition 5.4 of [III],

$$\begin{aligned} \varepsilon(g, g^{-1}, h, h^{-1}) &= \phi_U(1, 1) \beta(h, g^{-1}) \Xi_A(h, g) \Xi_A(g^{-1}, h^{-1}) \\ &= \beta(h, g^{-1}) \Xi_A(h, g) \Xi_A(g, h) = 1, \end{aligned} \quad (5.18)$$

with β defined by $c_{L_g, L_h} = \beta(g, h) c_{L_h, L_g}^{-1}$. Thus the group algebra is in fact untwisted.

We conclude that there are exactly $|H|$ simple A - A -bimodules that are direct sums of simple elements on the H -orbit of a given torsorial simple object U . On the other hand, proposition 5.9 below provides $|H|$ mutually non-isomorphic A - A -bimodules on this orbit, namely the bimodules $\alpha_A^+(U)_\varphi$ for $\varphi \in H^*$. Furthermore, applying Frobenius reciprocity in the form

$$\text{Hom}_A(\text{Ind}_A(U), \text{Ind}_A(U)) \cong \text{Hom}_{\mathcal{C}}(U, A \otimes U) = \bigoplus_{g \in H} \text{Hom}(U, L_g \otimes U) = \text{Hom}(U, U) \quad (5.19)$$

shows that the bimodules $\alpha_A^\pm(U)$ induced from H -torsorial simple objects U are simple as left A -modules and thus, a fortiori, simple as A - A -bimodules. Thus the $\alpha_A^+(U)_\varphi$ represent the $|H|$ isomorphism classes of simple A - A -bimodules that can be constructed using objects on the H -orbit of U . Note that for this argument it is not required that the category is modular.

Proposition 5.8 :

For A an algebra in a braided tensor category \mathcal{C} , twists by automorphisms of A can be shifted from the left to the right action of A on α -induced bimodules, and vice versa:

$$\varphi^{-1}\psi_1\alpha_A^\pm(U)_{\psi_2} \cong \psi_1\alpha_A^\pm(U)_{\varphi\psi_2} \quad (5.20)$$

for any object U of \mathcal{C} .

Proof:

The claim follows by verifying, similarly as in the proof of lemma 6 (i) of [1], that $t_\varphi \otimes id_U$ furnishes an isomorphism between the two bimodules. ✓

Proposition 5.9 :

Let A be a Schellekens algebra in an additive ribbon category \mathcal{C} and U a simple object of \mathcal{C} .

- (i) One can work with one type of α -induction only: $\psi_1\alpha_A^+(U)_{\psi_2} \cong \psi_1\alpha_A^-(U)_{\chi_U\psi_2}$.
- (ii) If U is H -torsorial, then one has $\alpha_A^+(U)_\varphi \cong \alpha_A^+(U)_\psi$ as bimodules if and only if $\varphi = \psi$.

Proof:

- (i) follows immediately from lemma 5.4.
- (ii) Because of proposition 5.8 it suffices to establish the claim for $\varphi \equiv 1$. By the definition of α -induction (see formula (2.11), remark 4.8 (ii) and definition 5.6), we have $\alpha_A^+(U)_\psi = A_\psi \otimes^+ U$. Suppose now there exists an isomorphism $f \in \text{Hom}_{A|A}(A \otimes^+ U, A_\psi \otimes^+ U)$, and consider its partial trace

$$g := (id_A \otimes \tilde{d}_U) \circ (f \otimes id_{U^\vee}) \circ (id_A \otimes b_U) \in \text{Hom}(A, A). \quad (5.21)$$

Since U is H -torsorial, f can be written as $f = \sum_{h \in H} f_h id_{L_h} \otimes id_U$, with all of the coefficients $f_h \in \mathbb{C}$ being nonzero. Substituting this decomposition of f gives $g = \dim(U) \sum_{h \in H} f_h id_{L_h}$. Since U is simple, $\dim(U) \neq 0$, and since the f_h are nonzero, we have $g \neq 0$. It is also easy to check that in fact $g \in \text{Hom}_{A|A}(A, A_\psi)$. Since A and A_ψ are simple and g is nonzero, g is an isomorphism. Lemma 6 (iv) of [1] now implies that t_ψ is an inner automorphism of A , $t_\psi \in \text{Inn}(A)$ (for the definition of the notion of inner automorphism, see section 8 of [1]). However, for a Schellekens algebra one has $\text{Inn}(A) = \{id_A\}$, and thus $t_\psi = id_A$.

The converse direction of the claim is trivial. ✓

By a calculation generalising (5.19) one shows that the bimodules $\psi_1\alpha_A^+(U \otimes L_h)_{\psi_2}$ and $\psi_1\alpha_A^+(U)_{\psi'_2}$ are isomorphic as left A -modules for any $h \in H$ and for every choice of $\psi_2, \psi'_2 \in H^*$. The following result establishes equivalences of A - A -bimodules between bimodules of this type.

Proposition 5.10:

Let A be a Schellekens algebra in an additive ribbon category, let H be the support of A , and let U be an H -torsorial simple object. Then for any $\varphi, \psi \in H^*$ we have the isomorphism

$$\psi \alpha_A^+(L_h \otimes U)_\varphi \cong \psi \alpha_A^+(U)_{\Xi_A(\cdot, h)\varphi} \quad (5.22)$$

of A - A -bimodules.

Proof:

First note that, owing to the fact that α -induction is a tensor functor, the assertion can be reduced to the case $U = \mathbf{1}$:

$$\begin{aligned} \psi \alpha_A^+(L_h \otimes U)_\varphi &\cong \psi \alpha_A^+(L_h) \otimes_A \alpha_A^+(U)_\varphi \\ &\cong \psi \alpha_A^+(\mathbf{1})_{\Xi_A(\cdot, h)} \otimes_A \alpha_A^+(U)_\varphi \\ &\cong \psi \Xi_A(\cdot, h)^{-1} \alpha_A^+(\mathbf{1}) \otimes_A \alpha_A^+(U)_\varphi \cong \psi \alpha_A^+(U)_{\Xi_A(\cdot, h)\varphi}. \end{aligned} \quad (5.23)$$

To establish the isomorphism for $U = \mathbf{1}$, we first show that the two morphisms

$$\begin{aligned} s_h &:= (id_A \otimes r_h) \circ \Delta \in \text{Hom}(A, A \otimes L_h) \quad \text{and} \\ f_h &:= \dim(A) m \circ (id_A \otimes e_h) \in \text{Hom}(A \otimes L_h, A), \end{aligned} \quad (5.24)$$

with e_h and r_h embedding and restriction morphisms for L_h as a retract of A , are each other's inverse. That s_h is left-inverse to f_h follows by noting that after applying the Frobenius property to $s_h \circ f_h$ one can insert an idempotent p_1 in the intermediate A -line, and then using that

$$p_1 \equiv r_1 \circ e_1 = \frac{1}{\dim(A)} \eta \circ \varepsilon, \quad (5.25)$$

which, in turn, follows by using that $\text{Hom}(\mathbf{1}, A) \cong \mathbb{C}$ and composing with the counit ε . That s_h is also a right-inverse follows by noting that $f_h \circ s_h$ is $\dim(A)$ times the left hand side of (5.11) with t_ψ replaced by $p_h = r_h \circ e_h$, and using also $\text{tr}(p_h) = 1$.

Next we observe that by combining (5.3) with the Frobenius property it follows that

The diagrammatic equation (5.26) shows the equality of two expressions. The left side is a morphism from $L_g \otimes L_{g'}$ to $A \otimes L_h$ represented by two strands from L_g and $L_{g'}$ that cross and then merge into a single strand A , which then splits into two strands L_h and A . This is equal to $\Xi_A(g', gh^{-1})^{-1}$ times a similar morphism where the strands from L_g and $L_{g'}$ cross and merge into a single strand A , which then splits into two strands L_h and A . This is equal to $\Xi_A(g', gh^{-1})^{-1}$ times a morphism where the strands from L_g and $L_{g'}$ merge into a single strand A , which then splits into two strands L_h and A . This is equal to $\Xi_A(g', h)$ times a morphism where the strands from L_g and $L_{g'}$ merge into a single strand A , which then splits into two strands L_h and A .

$$\text{Diagram 1} = \Xi_A(g', gh^{-1})^{-1} \text{Diagram 2} = \Xi_A(g', gh^{-1})^{-1} \text{Diagram 3} = \Xi_A(g', h) \text{Diagram 4} \quad (5.26)$$

where the factor on the right hand side arises as the product $\Xi_A(g', gh^{-1})^{-1} \Xi_A(g', g)$. This shows that s_h is an isomorphism of bimodules from $\psi A_{\Xi_A(\cdot, h)\varphi} = \psi \alpha_A^+(\mathbf{1})_{\Xi_A(\cdot, h)\varphi}$ to $\psi \alpha_A^+(L_h)_\varphi$. \checkmark

Another special situation of interest in which we can make more specific statements are Schellekens algebras that are braided commutative, i.e. obey $m \circ c_{A,A} = m$. This class is also of particular practical interest, since it describes extensions of the chiral algebra and is therefore used to implement projections on conformal field theories, e.g. [59] in string theory the GSO-projection and the alignment of fermionic boundary conditions in various sectors. In this case the following results hold also for simple objects which are not H -torsorial.

Theorem 5.11 :

Let A be a commutative Schellekens algebra in a modular tensor category \mathcal{C} , and let $M_\kappa, \kappa \in \mathcal{J}_A$, be representatives of the isomorphism classes of simple left A -modules. Then the isomorphism classes of simple A - A -bimodules can be labelled by pairs consisting of an element of \mathcal{J}_A and an algebra automorphism:

$$\mathcal{K}_{AA} = \{(\kappa, \psi) \mid \kappa \in \mathcal{J}_A, \psi \in \text{Aut}(A)\}. \quad (5.27)$$

A representative $M_{\kappa, \psi}$ of (κ, ψ) is given by the simple left module M_κ with right action



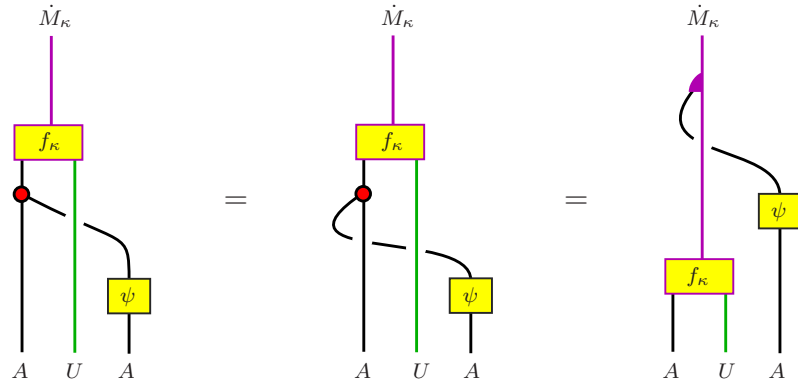
$$(5.28)$$

In particular, all simple A - A -bimodules are already simple as left A -modules, and the equality $|\mathcal{K}_{AA}| = \dim(A) |\mathcal{J}_A|$ holds.

Proof:

First note that in order to verify that $M_{\kappa, \psi}$ is indeed a bimodule one uses commutativity of A . All $M_{\kappa, \psi}$ are simple as bimodules because they are already simple as left modules. Let now X be a simple A - A -bimodule. Then by proposition 5.7, X is a sub-bimodule of $\alpha_A^+(U)_\psi$ for some $U \in \text{Obj}(\mathcal{C})$ and some $\psi \in \text{Aut}(A)$; we denote by $e \in \text{Hom}(X, \alpha_A^+(U)_\psi)$ the corresponding embedding morphism.

Let further $f_\kappa \in \text{Hom}_A(\text{Ind}_A(U), M_\kappa)$ be a morphism of A -modules. Because of



$$(5.29)$$

f_κ is also a morphism of A - A -bimodules from $\alpha_A^+(U)_\psi$ to $M_{\kappa, \psi}$; here it is also used that A is commutative. Since e is nonzero and since $\text{Ind}_A(U)$ can be written as a direct sum of simple modules, there exists an f_κ such that $e \circ f_\kappa$ is nonzero. Thus $e \circ f_\kappa \in \text{Hom}(X, M_{\kappa, \psi})$ is a nonzero morphism of bimodules. Since both X and $M_{\kappa, \psi}$ are simple, it is thus an isomorphism. Hence every simple A - A -bimodule is isomorphic to one of the $M_{\kappa, \psi}$.

Suppose now that for some choice of $\kappa, \mu \in \mathcal{K}_{AA}$ and of $\psi, \varphi \in \text{Aut}(A)$ there is an isomorphism $f \in \text{Hom}_{A|A}(M_{\kappa, \psi}, M_{\mu, \varphi})$. Then f is in particular an isomorphism of A -left modules, so that

$\kappa = \mu$. Furthermore,

(the last step uses lemma 5.5), showing that $\psi = \varphi$.
 Thus the $M_{\kappa, \psi}$ are pairwise non-isomorphic. ✓

Remark 5.12:

The statement does not apply to non-commutative algebras. For example, denote by U_l the simple object given by the integrable highest weight representation of $A_1^{(1)}$ with highest weight l at level k . In the theory described by the D-type modular invariant of the WZW model based on $A_1^{(1)}$ at level $k = 2 \pmod 4$, the object $U_{k/2}$ carries two non-isomorphic structures $U_{k/2}^+$ and $U_{k/2}^-$ of (simple) left module, and there is a simple bimodule structure defined on $U_{k/2}^+ \oplus U_{k/2}^-$, which as a left module decomposes as $U_{k/2}^+ \oplus U_{k/2}^-$. Also the sum rule $|\mathcal{K}_{AA}| = \dim(A) |\mathcal{J}_A|$ does not hold any longer.

As a generalisation of lemma 6 (ii) of [1] the following statement will be useful,

Lemma 5.13:

Let A be an algebra in a braided tensor category \mathcal{C} . Then for $U, V \in \text{Obj}(\mathcal{C})$ and $\psi, \varphi \in \text{Aut}(A)$ we have

$$\alpha_A^+(U)_\psi \otimes_A \alpha_A^+(V)_\varphi \cong \alpha_A^+(U \otimes V)_{\psi\varphi} \tag{5.31}$$

as A - A -bimodules.

Proof:

First we use proposition 5.8 to consider instead of $\alpha_A^+(U)_\psi$ the isomorphic bimodule ${}_{\psi^{-1}}\alpha_A^+(U)$. Then we use that, since α -induction is a tensor functor, we have $\alpha_A^+(U) \otimes_A \alpha_A^+(V) \cong \alpha_A^+(U \otimes V)$, and thereby automatically also ${}_{\psi^{-1}}\alpha_A^+(U) \otimes_A \alpha_A^+(V)_\varphi \cong {}_{\psi^{-1}}\alpha_A^+(U \otimes V)_\varphi$. The claim then follows by invoking once more proposition 5.8. \checkmark

5.3 The generic symmetry group

For any full local conformal field theory described by a Schellekens algebra A , the results of the previous subsection allow us to determine a subgroup of the symmetry group of that conformal field theory: those given by group-like defects that are induced from invertible objects of \mathcal{C} . We call this group the *generic group of symmetries* and denote it by Sy_{gen} . It can be described as follows.

Proposition 5.14:

The generic group of symmetries of a conformal field theory corresponding to a Schellekens algebra A with support H and KSB Ξ_A is

$$\text{Sy}_{\text{gen}} = H^* \times_H \text{Pic}(\mathcal{C}). \quad (5.32)$$

Here H acts on $\text{Pic}(\mathcal{C})$ by (left) multiplication, and via $\psi^h(\cdot) = \psi(\cdot) \Xi_A(\cdot, h)$ for $h \in H$ from the right on H^* .

Proof:

The assertion is a special case of proposition 5.16 below, to which we turn after the following remarks.

Remark 5.15:

(i) As a quotient of an abelian group, Sy_{gen} is abelian. In other words: non-abelian symmetries come from resolved fixed points. This is illustrated in the next section with the example of the Potts model.

(ii) It is instructive to consider the special example of a Schellekens algebra with support $H \cong \mathbb{Z}_2$. Two cases must be distinguished: the nontrivial invertible element U_h can have twist $\theta_h = \pm id_{U_h}$. If $\theta_h = id_{U_h}$, then $\Xi_A(h, h) = 1$ and the KSB is trivial. As a consequence, H acts trivially on H^* and we get

$$\text{Sy}_{\text{gen}} \cong H^* \times_H \text{Pic}(\mathcal{C}) \cong H^* \times \text{Pic}(\mathcal{C}) / H. \quad (5.33)$$

Note that this is an abelian group of the same order as $\text{Pic}(\mathcal{C})$, but the two groups are, in general, not isomorphic. For $\theta_h = -id_{U_h}$, we have $\Xi_A(h, h) = -1$ and the KSB is nontrivial. As a consequence, the action of H on H^* removes that factor, so that

$$\text{Sy}_{\text{gen}} \cong H^* \times_H \text{Pic}(\mathcal{C}) \cong \text{Pic}(\mathcal{C}). \quad (5.34)$$

This should not come as a surprise, since in this case A is an Azumaya algebra, which implies that $\text{Pic}(\mathcal{C}_{A|A}) \cong \text{Pic}(\mathcal{C})$.

(iii) The assertion in proposition 5.14, which was announced in [60] with a sketch of the proof, has been refined in theorem 4.5 of [61], where also the associator on the subcategory whose objects correspond to elements of Sy_{gen} is discussed.

Proposition 5.16 :

Let \mathcal{C} be a semisimple ribbon category and let A be a Schellekens algebra in \mathcal{C} with support H . If all simple objects of \mathcal{C} are H -torsorial, then there is a ring isomorphism

$$K_0(\mathcal{C}_{A|A}) \cong \mathbb{Z}H^* \otimes_{\mathbb{Z}H} K_0(\mathcal{C}) \tag{5.35}$$

which preserves the distinguished bases.

Proof:

We will construct a surjective ring homomorphism $f: \mathbb{Z}H^* \otimes_{\mathbb{Z}} K_0(\mathcal{C}) \rightarrow K_0(\mathcal{C}_{A|A})$ and show that it descends to a ring isomorphism $\tilde{f}: \mathbb{Z}H^* \otimes_{\mathbb{Z}H} K_0(\mathcal{C}) \rightarrow K_0(\mathcal{C}_{A|A})$. For X an object of \mathcal{C} (respectively, of $\mathcal{C}_{A|A}$), denote by $[X]$ its class in $K_0(\mathcal{C})$ (respectively, in $K_0(\mathcal{C}_{A|A})$). It is enough to define f on pairs $(\psi, [U])$ with U a simple object of \mathcal{C} . We set $f(\psi, [U]) := [\alpha^+(U)_\psi]$. As observed in section 5.2 (see the arguments before proposition 5.8), every simple A - A -bimodule is isomorphic to a bimodule of the form $\alpha^+(U)_\psi$, so f is surjective. That f is a ring homomorphism follows by direct calculation using lemma 5.13. Define a right action of H on H^* by $\psi^h(\cdot) := \psi(\cdot) \Xi_A(\cdot, h)$ for $h \in H$, and a left action on $K_0(\mathcal{C})$ by ${}^h[U] := [L_h \otimes U]$. It is then an immediate consequence of proposition 5.10 that $f(\psi^h, [U]) = f(\psi, {}^h[U])$ for all $h \in H$. Thus f indeed gives rise to a well-defined surjective ring homomorphism $\tilde{f}: \mathbb{Z}H^* \otimes_{\mathbb{Z}H} K_0(\mathcal{C}) \rightarrow K_0(\mathcal{C}_{A|A})$. It remains to show that \tilde{f} is injective. Let us denote the image of $(\psi, [U])$ in $\mathbb{Z}H^* \otimes_{\mathbb{Z}H} K_0(\mathcal{C})$ by $\{\psi, [U]\}$. Suppose that $\tilde{f}(\varphi, [U]) = \tilde{f}(\psi, [V])$ for simple objects U and V of \mathcal{C} . Then by definition $\alpha^+(U)_\varphi \cong \alpha^+(V)_\psi$; this is only possible if U and V lie on the same H -orbit. We can thus use the action of H to find ψ' such that $\{\psi', [U]\} = \{\psi, [V]\}$. By proposition 5.9 (ii), the resulting equality $\tilde{f}(\varphi, [U]) = \tilde{f}(\psi', [U])$ then implies $\varphi = \psi'$ and thus $\{\varphi, [U]\} = \{\psi', [U]\} = \{\psi, [V]\}$. Hence \tilde{f} is injective. ✓

Let us add a few comments on the situation when the symmetry group $\text{Pic}(\mathcal{C}_{A|A})$ contains non-generic elements, which is a prerequisite for having a nonabelian symmetry group. A non-generic element must appear as the class of a proper subobject of a twisted α -induced bimodule $\alpha_A^+(U)_\psi$ for some simple object U of \mathcal{C} . Now $\dim_{A|A}(\alpha^+(U)_\psi) = \dim(U)$, and if $\alpha^+(U)_\psi$ is a direct sum of N_U simple bimodules, then [III] N_U must be a divisor of $|H|$ and each of the simple sub-bimodules S has the same dimension $\dim_{A|A}(S) = \dim(U)/N_U$. Since a group-like bimodule has dimension 1, this implies that non-generic symmetries can only come from non- H -torsorial simple objects of \mathcal{C} with low integral dimension. In all known classes of models such objects are rare (see e.g. [62, 63]). Moreover, in many models that come in series, e.g. labelled by a “level”, such as minimal or WZW models, the dimension of non- H -torsorial objects grows with the level. In all such cases non-abelian symmetries will be a low-level phenomenon. Also, of course, independently of whether there are non- H -torsorial objects or not, the symmetry is abelian whenever the algebra A is Azumaya, since then \mathcal{C} and $\mathcal{C}_{A|A}$ are equivalent tensor categories.

6 Tetracritical Ising and three-states Potts model

In this section we investigate topological defects and phase boundaries for the tetracritical Ising and for the critical three-states Potts model. For both models the chiral symmetry algebra contains the Virasoro vertex algebra of central charge $c=4/5$, which is rational. Thus we only need to require preservation of the Virasoro symmetry, and the analysis below gives all topological defects and phase boundaries consistent with conformal symmetry.

6.1 Chiral data of the minimal model M(5,6)

The M(5,6) minimal model has central charge $c=4/5$. The first of the following two tables gives the lowest conformal weight of the representation corresponding to a given entry of the Kac table. The second table shows our choice of representatives and the names we will use for these representations.

4	3	$\frac{13}{8}$	$\frac{2}{3}$	$\frac{1}{8}$	0
3	$\frac{7}{5}$	$\frac{21}{40}$	$\frac{1}{15}$	$\frac{1}{40}$	$\frac{2}{5}$
3	$\frac{2}{5}$	$\frac{1}{40}$	$\frac{1}{15}$	$\frac{21}{40}$	$\frac{7}{5}$
1	0	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{13}{8}$	3
	1	2	3	4	5

4					
3	$\hat{\mathbf{1}}$	\hat{u}	\hat{f}	\hat{v}	\hat{w}
2					
1	$\mathbf{1}$	u	f	v	w
	1	2	3	4	5

Denote by $\mathcal{C}_{5,6}$ the representation category of the vertex algebra for the M(5,6) model. The index set \mathcal{I} of representative simple objects in $\mathcal{C}_{5,6}$ will be taken in the order

$$\mathcal{I} = \{\mathbf{1}, u, f, v, w, \hat{\mathbf{1}}, \hat{u}, \hat{f}, \hat{v}, \hat{w}\}. \quad (6.1)$$

All objects of $\mathcal{C}_{5,6}$ are isomorphic to their duals, so that $\bar{k}=k$ for all $k \in \mathcal{I}$. The fusion rules of $\mathcal{C}_{5,6}$ are of the form $\mathfrak{su}(2)_4 \times (\text{Lee-Yang})$. In more detail, the fusion ring of $\{\mathbf{1}, u, f, v, w\}$ is that of $\mathfrak{su}(2)_4$, i.e. $u \cdot u = 1 + f$, $u \cdot f = u + v$, etc. Multiplication with hatted fields is determined by the rules

$$\hat{\mathbf{1}} \cdot x = \hat{x} \quad \text{for } x \in \{\mathbf{1}, u, f, v, w\} \quad \text{and} \quad \hat{\mathbf{1}} \cdot \hat{\mathbf{1}} = \mathbf{1} + \hat{\mathbf{1}} \quad (6.2)$$

together with associativity and commutativity.

The modular S -matrix is given by (see e.g. section 10.6 of [64])

$$S = \begin{pmatrix} \zeta M & \xi M \\ \xi M & -\zeta M \end{pmatrix}, \quad \text{where } \zeta = \sqrt{\frac{1}{2}(1 - \frac{1}{\sqrt{5}})}, \quad \xi = \sqrt{\frac{1}{2}(1 + \frac{1}{\sqrt{5}})} \quad (6.3)$$

and M is the matrix

$$M = \begin{pmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{2} & \frac{1}{\sqrt{3}} & \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2\sqrt{3}} & -\frac{1}{2} & \frac{1}{\sqrt{3}} & -\frac{1}{2} & \frac{1}{2\sqrt{3}} \end{pmatrix}. \quad (6.4)$$

Note that $\xi/\zeta = \frac{1}{2}(1+\sqrt{5})$. Rows and columns of the matrix (6.3) are ordered according to (6.1). The invariant $s_{i,j}$ of the Hopf link is the ratio

$$s_{i,j} = S_{i,j}/S_{0,0} \quad (6.5)$$

and thus in particular the quantum dimension of the simple objects are

k	$\mathbf{1}$	u	f	v	w	$\hat{\mathbf{1}}$	\hat{u}	\hat{f}	\hat{v}	\hat{w}
$\dim(U_k)$	1	$\sqrt{3}$	2	$\sqrt{3}$	1	$\frac{\xi}{\zeta}$	$\sqrt{3}\frac{\xi}{\zeta}$	$2\frac{\xi}{\zeta}$	$\sqrt{3}\frac{\xi}{\zeta}$	$\frac{\xi}{\zeta}$

(6.6)

The entries of the fusion matrices F are not really needed in the sequel. For the sake of concreteness, we will nonetheless use the following entry, obtained in the conventions of (A.6) in [47]:

$$F_{f\mathbf{1}}^{(w w f) f} = \frac{81}{26}. \quad (6.7)$$

6.2 The tetracritical Ising model

The tetracritical Ising model is the A-series modular invariant. It is described by the Morita class of the symmetric special Frobenius algebra $A = \mathbf{1}$. Using the ribbon invariant (5.30) of [I] one finds that the torus partition function is diagonal, as it should be ($k = \bar{k}$ since all representations are self-conjugate):

$$Z(A)_{ij} = \delta_{ij}. \quad (6.8)$$

Further, $A = \mathbf{1}$ implies that the simple objects U_k , $k \in \mathcal{I}$, are also representatives of all simple A -left modules as well as A -bimodules, and thus

$$\mathcal{J}_A = \mathcal{I}, \quad \mathcal{K}_{AA} = \mathcal{I}. \quad (6.9)$$

Accordingly, the fusion algebra of A - A -defects is just the fusion algebra of $\mathcal{C}_{5,6}$. We see that there are two group-like A - A -defects $X_{\mathbf{1}}$ and X_w , which form a group isomorphic to \mathbb{Z}_2 , and that there are no A - A -duality defects that are not already group-like.

The bulk fields of $\text{CFT}(A)$ are elements of $\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A) = \text{Hom}(U_i \otimes U_j, \mathbf{1})$, which has dimension δ_{ij} . Consider the bulk field with left/right representation index i labelled by the basis morphism $\phi_i := \lambda_{(ii)_0}$ (compare formula (2.29) of [I]). The action (2.30) on bulk fields amounts to multiplication with a ratio of s -matrix elements (not assuming $k = \bar{k}$ for the moment),

$$D_\nu(\phi_i) \cong \frac{S_{i,\bar{\nu}}}{S_{i,0}} \phi_i \quad (6.10)$$

From the explicit form of the s -matrix (6.5) we find that the group-like A - A -defect D_w acts on morphisms labelling A -bulk fields as

$$\begin{aligned} D_w(\phi_i) &= \phi_i & \text{for } i \in \{\mathbf{1}, f, w, \hat{\mathbf{1}}, \hat{f}, \hat{w}\}, \\ D_w(\phi_i) &= -\phi_i & \text{for } i \in \{u, v, \hat{u}, \hat{v}\}. \end{aligned} \quad (6.11)$$

6.3 The three-state Potts model

The Potts model is the D-series modular invariant. It is described by the Morita class of the symmetric special Frobenius algebra $B = \mathbf{1} \oplus U_w$. For concreteness, let us fix $m_{ww}^{\mathbf{1}} = 1$ in the expansion in [I, eq. (3.7)] for the multiplication on B . The automorphism group of B is

$$\text{Aut}(B) = \{e, \omega\} \cong \mathbb{Z}_2 \quad (6.12)$$

with

$$e = id_A \quad \text{and} \quad \omega = id_{\mathbf{1}} \oplus (-id_{U_w}). \quad (6.13)$$

To determine the simple B -modules we use the method presented in section 4.2 of [III]. This tells us that the induced modules

$$M_x = \text{Ind}_B(U_x) \quad \text{with} \quad x \in \{\mathbf{1}, u, \hat{\mathbf{1}}, \hat{u}\} \quad (6.14)$$

are simple and that, owing to $\dim_{\mathbb{C}}(\text{Hom}_B(\text{Ind}_B(U_x), \text{Ind}_B(U_x))) = \dim_{\mathbb{C}}(\text{Hom}(A \otimes U_x, U_x)) = 2$ for $x \in \{f, \hat{f}\}$, we have

$$\text{Ind}_B(U_f) \cong M_e \oplus M_\omega \quad \text{and} \quad \text{Ind}_B(U_{\hat{f}}) \cong M_{\hat{e}} \oplus M_{\hat{\omega}} \quad (6.15)$$

for some simple B -modules $M_e, M_\omega, M_{\hat{e}}$ and $M_{\hat{\omega}}$. (While we use the same symbols e and ω that label elements of the group $\text{Aut}(B)$ to distinguish the simple modules, it should be kept in mind that this labelling is not canonical and that the two modules only form a torsor over $\text{Aut}(B)$.) These modules can be described as follows. Let $M_e = (U_f, \rho_f)$, where $\rho_f \in \text{Hom}(A \otimes U, U)$ is of the form

$$\rho_f = \begin{array}{c} U_f \\ | \\ \text{---} \\ | \\ B \end{array} \oplus \mu \begin{array}{c} U_f \\ | \\ \square \\ / \quad \backslash \\ U_w \quad U_f \\ | \quad | \\ B \quad U_f \end{array} \quad (6.16)$$

for some $\mu \in \mathbb{C}$. In order for (U, ρ_f) to be a B -module, ρ_f must obey in particular

$$\begin{array}{c} M_e \\ | \\ \square \\ / \quad \backslash \\ B \quad B \\ | \quad | \\ U_w \quad U_w \end{array} = \begin{array}{c} M_e \\ | \\ \square \\ / \quad \backslash \\ B \quad B \\ | \quad | \\ U_w \quad U_w \end{array} \implies \mu^2 \begin{array}{c} U_f \\ | \\ \square \\ / \quad \backslash \\ U_w \quad U_w \end{array} = \begin{array}{c} U_f \\ | \\ \square \\ / \quad \backslash \\ U_w \quad U_w \end{array} \quad (6.17)$$

Using the F-matrix element (6.7) we see that $\mu^2 = \frac{26}{81}$. Let us choose $\mu = +\sqrt{26}/9$. One verifies that (U_f, ρ_f) is indeed a B -module.

For $\psi \in \text{Aut}(B)$ and $M = (\dot{M}, \rho)$ a B -module, denote by ${}_\psi M$ the B -module ${}_\psi M = (\dot{M}, \rho \circ (\psi \otimes \text{id}_{\dot{M}}))$. Then we can choose

$$M_\omega = {}_\omega(M_e), \quad M_{\hat{e}} = M_e \otimes \hat{\mathbf{1}}, \quad M_{\hat{\omega}} = {}_\omega(M_e) \otimes \hat{\mathbf{1}} \quad (6.18)$$

with ω as defined in (6.13). The label set of simple B -modules is then

$$\mathcal{J}_B = \{\mathbf{1}, u, e, \omega, \hat{\mathbf{1}}, \hat{u}, \hat{e}, \hat{\omega}\}. \quad (6.19)$$

According to theorem 5.11, the isomorphism classes of simple B - B -bimodules are labelled by pairs

$$\mathcal{K}_{BB} = \{(\kappa, \psi) \mid \kappa \in \mathcal{J}_B, \psi \in \text{Aut}(B)\}. \quad (6.20)$$

Thus there are 16 isomorphism classes of simple bimodules. Their dimensions are

$$\begin{array}{c|cccccccc} x & \mathbf{1} & u & e & \omega & \hat{\mathbf{1}} & \hat{u} & \hat{e} & \hat{\omega} \\ \hline \dim_B(M_{x,\psi}) & 1 & \sqrt{3} & 1 & 1 & \frac{\xi}{\zeta} & \sqrt{3} \frac{\xi}{\zeta} & \frac{\xi}{\zeta} & \frac{\xi}{\zeta} \end{array} \quad (6.21)$$

It follows that there are six group-like bimodules, namely

$$\mathcal{G}_B = \{(\mathbf{1}, e), (\mathbf{1}, \omega), (e, e), (e, \omega), (\omega, e), (\omega, \omega)\}. \quad (6.22)$$

The fusion algebra of defects in the Potts model (or in the related $\mathfrak{su}(2)_4$ WZW model) has also been considered in [43, 12]. To see how it can be obtained in the present framework, we start with the bimodule $\hat{B} = M_{\hat{\mathbf{1}}, e} = \alpha_B^+(\hat{\mathbf{1}})$. From lemmas 4.9 and 5.13 we see that

$$\hat{B} \otimes_B X \cong X \otimes_B \hat{B} \quad \text{and} \quad \hat{B} \otimes_B \hat{B} \cong B \oplus \hat{B} \quad (6.23)$$

for any B - B -bimodule X . Further, with (6.14) and (6.18) it is straightforward to check that

$$M_{x,\psi} \otimes_B \hat{B} \cong M_{\hat{x},\psi} \quad \text{for all } x \in \{1, u, e, \omega\} \text{ and all } \psi \in \text{Aut}(B). \quad (6.24)$$

It is therefore enough to understand the tensor products of $M_{x,\psi}$ for $x \in \{1, u, e, \omega\}$. Let us work in the Grothendieck ring of the tensor category of B -bimodules to simplify notation. We denote the isomorphism class of a bimodule X by $[X]$ and abbreviate $[M_{x,\psi}] =: (x, \psi)$. Lemma 5.13 then tells us that

$$[\alpha_B^+(U_x)_\phi] \cdot [\alpha_B^+(U_y)_\psi] = [\alpha_B^+(U_x \otimes U_y)_{\phi\psi}] \quad (6.25)$$

for $x, y \in \{1, u, f\}$ and $\phi, \psi \in \{e, \omega\}$. Using $(\mathbf{1}, \phi) = [B_\phi]$, $(u, \phi) = [\alpha^+(U_u)_\phi]$ and noting further the decomposition $[\alpha_B^+(U_f)_\phi] = (e, \phi) + (\omega, \phi)$ which follows from (6.15), we find

$$\begin{aligned} (\mathbf{1}, \phi) \cdot (\mathbf{1}, \psi) &= (\mathbf{1}, \phi\psi), & (\mathbf{1}, \phi) \cdot (u, \psi) &= (u, \phi\psi), \\ (u, \phi) \cdot (u, \psi) &= (\mathbf{1}, \phi\psi) + (e, \phi\psi) + (\omega, \phi\psi), \\ ((e, \phi) + (\omega, \phi)) \cdot (u, \psi) &= 2(u, \phi\psi). \end{aligned} \quad (6.26)$$

It follows in particular that

$$(\mathbf{1}, \phi) \cdot (u, \psi) = (e, \phi) \cdot (u, \psi) = (\omega, \phi) \cdot (u, \psi) = (u, \phi\psi). \quad (6.27)$$

Let us now have a closer look at the group-like bimodules. (6.27) tells us that (u, ψ) is a fixed point for the group-like bimodules labelled by $(\mathbf{1}, e)$, (e, e) , (ω, e) . Together with the third equality in (6.26) it follows that these form the stabiliser of (u, ψ) and thus a subgroup of \mathcal{G}_B . A group with three elements is isomorphic to \mathbb{Z}_3 , and so

$$(e, e) \cdot (e, e) = (\omega, e), \quad (\omega, e) \cdot (\omega, e) = (e, e), \quad (e, e) \cdot (\omega, e) = (\omega, e) \cdot (e, e) = (\mathbf{1}, e). \quad (6.28)$$

For the multiplication by $(\mathbf{1}, \phi)$ we find by direct computation that

$$(\mathbf{1}, \phi) \cdot (\alpha, \psi) = (\alpha\phi, \phi\psi) \quad \text{and} \quad (\alpha, \psi) \cdot (\mathbf{1}, \phi) = (\alpha, \phi\psi) \quad (6.29)$$

for $\alpha, \phi, \psi \in \{e, \omega\}$. This shows that the fusion of bimodules is nonabelian. Since there are just two non-isomorphic groups of order 6, the cyclic group, which is abelian, and the dihedral group, which is nonabelian and isomorphic to the symmetric group S_3 , it follows that the Picard group of B -bimodules is isomorphic to S_3 and thus indeed coincides with the symmetry group one expects for the three-state Potts model. In the sequel we will work out this group structure more explicitly (also note again that the identification of bimodule labels with group elements is not canonical). We first derive the equalities (6.29). For the first product we need to show that $M_{\mathbf{1}, \phi} \otimes_B M_{\alpha, \psi} \cong M_{\alpha\phi, \phi\psi}$. To this end we write the projector $P \in \text{End}_{B|B}(M_{\mathbf{1}, \phi} \otimes_B M_{\alpha, \psi})$, whose image defines the tensor product, in the form $P = e \circ r$ such that $r \circ e = \text{id}_{M_{\alpha\phi, \phi\psi}}$. This is done in the following equation (for convenience we also indicate the left and right action of B):

$$(6.30)$$

To make the connection with (6.29) we also use that $\text{Aut}(B)$ is commutative and that $\psi = \psi^{-1}$

for all $\psi \in \text{Aut}(B)$. The corresponding calculation for the second product in (6.29) is

$$(6.31)$$

where we also use that the algebra B is commutative.

Finally we can express $(\alpha, \psi) = (\alpha, e) \cdot (\mathbf{1}, \psi) = (\mathbf{1}, \psi) \cdot (\alpha\psi, e)$. Thus all products are determined in terms of the ones already computed.

We proceed to display the S_3 group structure of the fusion. We use the cycle notation for elements of S_n , i.e. $(abc \cdots d)$ stands for the permutation $a \mapsto b, b \mapsto c, \dots, d \mapsto a$, and consider the following assignment of permutations to elements of \mathcal{G}_B :

$$(6.32)$$

$$\begin{aligned} id &\mapsto (\mathbf{1}, e), & (123) &\mapsto (e, e), & (132) &\mapsto (\omega, e), \\ (12) &\mapsto (\mathbf{1}, \omega), & (23) &\mapsto (\omega, \omega), & (13) &\mapsto (e, \omega). \end{aligned}$$

One verifies that this is a group isomorphism $S_3 \xrightarrow{\cong} \mathcal{G}_B$. For example, $(23) \cdot (13) = (123)$, as well as $(\omega, \omega) \cdot (e, \omega) = (\omega, e) \cdot (\mathbf{1}, \omega) \cdot (\mathbf{1}, \omega) \cdot (\omega, e) = (e, e)$.

Inspecting once more the fusion rules (6.26) we see that (u, e) and (u, ω) are the only simple duality defects that are not already group-like, i.e.

$$(6.33)$$

$$\mathcal{D}_{BB} = \mathcal{G}_B \cup \{(u, e), (u, \omega)\}.$$

Note that e.g. $(\mathbf{1}, \omega) \cdot (u, e) = (u, \omega)$, so that the two new duality defects form one orbit under the left/right \mathcal{G}_B -action.

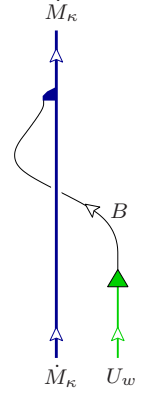
6.4 Phase changing defects

Topological defects that separate a tetracritical Ising phase $\text{CFT}(A)$ of a world sheet from a three-states Potts phase $\text{CFT}(B)$ are described by B - A -bimodules (or equivalently, by A - B -bimodules). Because of $A = \mathbf{1}$ these are nothing but left B -modules. The isomorphism classes of simple B - A -bimodules can thus be labelled as

$$(6.34)$$

$$\mathcal{K}_{BA} = \mathcal{J}_B = \{\mathbf{1}, u, e, \omega, \hat{\mathbf{1}}, \hat{u}, \hat{e}, \hat{\omega}\}.$$

Let us compute the $\mathcal{G}_B \times \mathcal{G}_A$ -action on \mathcal{K}_{BA} to see how many orbits, i.e. phase-changing defects not linked by a symmetry, there are. For the right action we get $M_\kappa \otimes U_w \cong M_\kappa$ as B - A -bimodules, for all $\kappa \in \mathcal{K}_{BA}$. Since B is commutative, a possible isomorphism is



$$\in \text{Hom}_{BA}(M_\kappa \otimes U_w, M_\kappa) . \quad (6.35)$$

Thus \mathcal{G}_A acts trivially. To find the left \mathcal{G}_B -action we can use the fusion rules of B - B -bimodules and forget the right B -action. One obtains four \mathcal{G}_B -orbits:

$$\{\mathbf{1}, e, \omega\}, \quad \{\hat{\mathbf{1}}, \hat{e}, \hat{\omega}\}, \quad \{u\}, \quad \{\hat{u}\} . \quad (6.36)$$

Next we check whether any of the phase changing defects are duality defects. It is enough to do this for a representative of every orbit. The tensor product of two ‘hatted’ bimodules is never a sum of group-like bimodules, so the only candidates for duality bimodules are the orbits of the B - A -bimodules labelled by $\mathbf{1}$ and u , i.e. B and $\text{Ind}_B(U_u)$. For B we obtain

$$B^\vee \otimes_B B \cong \mathbf{1} \oplus U_w \quad \text{and} \quad B \otimes B^\vee \cong B \oplus B_\omega , \quad (6.37)$$

where B_ω denotes the bimodule ${}_{id}B_\omega$ as given in definition 5.6. To see the last equivalence, note that $B^\vee \cong B$ as right B -module and that

$$\Delta \in \text{Hom}_{B|B}(B, B \otimes B) \quad \text{and} \quad (id_B \otimes \omega) \circ \Delta \in \text{Hom}_{B|B}(B_\omega, B \otimes B) \quad (6.38)$$

are monomorphisms of B and B_ω into $B \otimes B$, respectively (note that Δ is a monomorphism, since B is simple and Δ is not zero). Evaluating the quantum dimension on both sides of $B \otimes_A B^\vee \cong B \oplus B_\omega$ shows that these are all simple submodules. Thus B is both a B - A -duality defect and an A - B -duality defect.

For $\text{Ind}_B(U_u)$ the analogous calculation yields

$$\text{Ind}_B(U_u)^\vee \otimes_B \text{Ind}_B(U_u) \cong B \otimes U_u^\vee \otimes U_u = \mathbf{1} \oplus 2U_f \oplus U_w \quad (6.39)$$

and

$$\begin{aligned} \text{Ind}_B(U_u) \otimes \text{Ind}_B(U_u)^\vee &\cong \alpha_B^+(U_u \otimes U_u^\vee) \oplus \alpha_B^+(U_u \otimes U_u^\vee)_\omega \\ &\cong B \oplus M_{e,e} \oplus M_{w,e} \oplus B_\omega \oplus M_{e,\omega} \oplus M_{w,\omega} . \end{aligned} \quad (6.40)$$

The first equivalence follows with the help of the embedding morphisms

$$\in \text{Hom}_{B|B}(\alpha_B^+(U_u \otimes U_u^\vee)_\phi, \text{Ind}_B(U_u) \otimes \text{Ind}_B(U_u)^\vee) \quad (6.41)$$

for $\phi \in \{e, \omega\}$.

We see that $\text{Ind}_B(U_u) \otimes \text{Ind}_B(U_u)^\vee$ decomposes into a direct sum of group-like B - B -bimodules, whereas $\text{Ind}_B(U_u)^\vee \otimes_B \text{Ind}_B(U_u)$ is not a direct sum of only group-like A - A -bimodules. Thus by theorem 3.9, $\text{Ind}_B(U_u)$ is an example of a topological defect that is not itself a B - A -duality defect even though $\text{Ind}_B(U_u)^\vee$ is an A - B -duality defect.

Given a simple B - A -bimodule X_μ , $\mu \in \mathcal{K}_{BA}$, we obtain a simple A - B -bimodule by taking its dual $(X_\mu)^\vee$. Let us choose these as representatives for the isomorphism classes of simple A - B -bimodules, so that $\mathcal{K}_{AB} = \mathcal{K}_{BA}$. The above calculation then shows that the simple A - B - and B - A -duality defects are

$$\mathcal{D}_{AB} = \{1, e, \omega, u\} \quad \text{and} \quad \mathcal{D}_{BA} = \{1, e, \omega\}, \quad (6.42)$$

respectively. According to proposition 3.13, there exists thus one way more to obtain the torus partition function of the tetra-critical Ising model from the Potts model than vice versa.

In more detail, each of the subgroups of S_3 is conjugate to either $\{id\}$, $\{id, (12)\} \cong \mathbb{Z}_2$, $A_3 \cong \mathbb{Z}_3$, or S^3 . The above results imply that when applied to the subgroups $\{id\}$ or A_3 , the orbifold-like expression (3.32) gives again the partition function of the Potts model, while applying the same prescription to the subgroups \mathbb{Z}_2 or S_3 results in the partition function of the tetra-critical Ising model. In the reverse direction, we can only use the \mathbb{Z}_2 symmetry of the tetra-critical Ising model to obtain the Potts model.

An analogous lattice construction, relating e.g. the A_5 height model and the D_4 height model via a ‘non-critical orbifold’ was given in [46].

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