

# THE FUSION ALGEBRA OF BIMODULE CATEGORIES

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## Abstract

We establish an algebra-isomorphism between the complexified Grothendieck ring  $\mathcal{F}$  of certain bimodule categories over a modular tensor category and the endomorphism algebra of appropriate morphism spaces of those bimodule categories. This provides a purely categorical proof of a conjecture by Ostrik concerning the structure of  $\mathcal{F}$ .

As a by-product we obtain a concrete expression for the structure constants of the Grothendieck ring of the bimodule category in terms of endomorphisms of the tensor unit of the underlying modular tensor category.

## Introduction

The Grothendieck ring  $K_0(\mathcal{C})$  of a semisimple monoidal category  $\mathcal{C}$  encodes a considerable amount of information about the structure of  $\mathcal{C}$ . If  $\mathcal{C}$  is braided, so that  $K_0(\mathcal{C})$  is commutative, then upon complexification to  $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$  almost all of this information gets lost: what is left is just the number of isomorphism classes of simple objects. In contrast, in the non-braided (but still semisimple) case the complexified Grothendieck ring is no longer necessarily commutative and thus contains as additional information the dimensions of its simple direct summands.

The following statement, which is a refined version of an assertion made in [O] as Claim 5.3, determines these dimensions for a particularly interesting class of categories:

**THEOREM O.**

*Let  $\mathcal{C}$  be a modular tensor category,  $\mathcal{M}$  a semisimple nondegenerate indecomposable module category over  $\mathcal{C}$ , and  $\mathcal{C}_{\mathcal{M}}^*$  the category of module endofunctors of  $\mathcal{M}$ . Then there is an isomorphism of  $\mathbb{C}$ -algebras*

$$\mathcal{F} \cong \bigoplus_{i,j \in I} \text{End}_{\mathbb{C}}(\text{Hom}_{\mathcal{C}_{\mathcal{M}}^*}(\alpha^+(U_i), \alpha^-(U_j))) \quad (1)$$

*between the complexified Grothendieck ring  $\mathcal{F} = K_0(\mathcal{C}_{\mathcal{M}}^*) \otimes_{\mathbb{Z}} \mathbb{C}$  and the endomorphism algebra of the specified space of morphisms in  $\mathcal{C}_{\mathcal{M}}^*$ .*

Here  $I$  is the (finite) set of isomorphism classes of simple objects of  $\mathcal{C}$ ,  $U_i$  and  $U_j$  are representatives of the classes  $i, j \in I$ , and  $\alpha^{\pm}$  are the braided-induction functors from  $\mathcal{C}$  to  $\mathcal{C}_{\mathcal{M}}^*$ . For more details about the concepts appearing in the Theorem see section 1 below, e.g. the tensor functors  $\alpha^{\pm}$  are given in formulas (5) and (6).

In [O] the assertion of Theorem O was formulated with the help of the integers

$$z_{i,j} := \dim_{\mathbb{C}} \text{Hom}_{\mathcal{C}_{\mathcal{M}}^*}(\alpha^+(U_i), \alpha^-(U_j)), \quad (2)$$

in terms of which it states that  $\mathcal{F}$  is isomorphic to the direct sum  $\bigoplus_{i,j} \text{Mat}_{z_{i,j}}$  of full matrix algebras of sizes  $z_{i,j}$ ,  $i, j \in I$ . In this form the statement had been established previously for the particular case that the modular tensor category  $\mathcal{C}$  is a category of endomorphisms of a type III factor (Theorem 6.8 of [BEK]), a result which directly motivated the formulation of the statement in [O]. Indeed, in [O] the additional assumption is made that the quantum dimension of any nonzero object of  $\mathcal{C}$  is positive, a property that is automatically fulfilled for the categories arising in the framework of [BEK], but is violated in other categories (e.g. those relevant for the so-called non-unitary minimal Virasoro models) of interest in physical applications.

In this note we derive the statement in the form of Theorem O, where this positivity requirement is replaced by the condition that  $\mathcal{M}$  is nondegenerate. This property, to be explained in detail further below, is satisfied in particular in the situation studied in [BEK]. We present our proof in section 2. As an additional benefit, it provides a concrete expression for the structure constants of the Grothendieck ring of  $\mathcal{C}_{\mathcal{M}}^*$  in terms of certain endomorphisms of the tensor unit of  $\mathcal{C}$ , see formulas (29) and (30). Various ingredients needed in the proof are supplied in section 1. In section 3 we outline the particularities of the case that  $\mathcal{C}$  comes from endomorphisms of a factor [BEK], and describe further relations between Theorem O and structures arising in quantum field theory; this latter part is, necessarily, not self-contained.

# 1 Bimodule categories and Frobenius algebras

We start by collecting some pertinent information about the quantities used in the formulation of Theorem O.

## Modular tensor categories

A *modular tensor category*  $\mathcal{C}$  in the sense of Theorem O is a semisimple  $\mathbb{C}$ -linear abelian ribbon category with simple tensor unit, having a finite number of isomorphism classes of simple objects and obeying a certain nondegeneracy condition.

Let us explain these qualifications in more detail. A *ribbon* (or *tortile*, or *balanced rigid braided*) category is a rigid braided monoidal category with a ribbon twist, i.e.  $\mathcal{C}$  is endowed with a tensor product bifunctor  $\otimes$  from  $\mathcal{C} \times \mathcal{C}$  to  $\mathcal{C}$ , with tensor unit  $\mathbf{1}$ , and there are families of (right-)duality morphisms  $b_U \in \text{Hom}(\mathbf{1}, U \otimes U^\vee)$ ,  $d_U \in \text{Hom}(U^\vee \otimes U, \mathbf{1})$ , of braiding isomorphisms  $c_{U,V} \in \text{Hom}(U \otimes V, V \otimes U)$ , and of twist isomorphisms  $\theta_U \in \text{End}(U)$  ( $U, V \in \text{Obj}(\mathcal{C})$ ) satisfying relations analogous to ribbons in three-space [JS]. (Details can e.g. be found in section 2.1 of [FRS1]; the category of ribbons indeed enjoys a universal property for ribbon categories, see e.g. chapter XIV.5.1 of [K].) A ribbon category is in particular *sovereign*, i.e. besides the right duality there is also a left duality, with evaluation and coevaluation morphisms  $\tilde{b}_U \in \text{Hom}(\mathbf{1}, {}^\vee U \otimes U)$  and  $\tilde{d}_U \in \text{Hom}(U \otimes {}^\vee U, \mathbf{1})$ , such that the two duality functors coincide, i.e.  ${}^\vee U = U^\vee$  and  ${}^\vee f = f^\vee \in \text{Hom}(V^\vee, U^\vee)$  for all objects  $U$  of  $\mathcal{C}$  and all morphisms  $f \in \text{Hom}(U, V)$ ,  $U, V \in \text{Obj}(\mathcal{C})$ .

Denoting by  $I$  the finite set of labels for the isomorphism classes of simple objects of  $\mathcal{C}$  and by  $U_i$  representatives for those classes, the nondegeneracy condition on  $\mathcal{C}$  is that the  $I \times I$ -matrix  $s$  with entries

$$\begin{aligned} s_{i,j} &:= \text{tr}(c_{U_i, U_j} c_{U_j, U_i}) \\ &\equiv (d_{U_j} \otimes \tilde{d}_{U_i}) \circ [id_{U_j}^\vee \otimes (c_{U_i, U_j} \circ c_{U_j, U_i}) \otimes id_{U_i}^\vee] \circ (\tilde{b}_{U_j} \otimes b_{U_i}) \in \text{End}(\mathbf{1}), \quad i, j \in I, \end{aligned} \quad (3)$$

is invertible. We will identify  $\text{End}(\mathbf{1}) = \mathbb{C} id_{\mathbf{1}}$  with  $\mathbb{C}$ , so that  $id_{\mathbf{1}} = 1 \in \mathbb{C}$  and also the morphisms  $s_{i,j}$  are just complex numbers, and we agree on  $I \ni 0$  and  $U_0 = \mathbf{1}$ . In a sovereign category one has  $s_{i,j} = s_{j,i}$ . The (quantum) dimension of an object  $U$  of a sovereign category is the trace over its identity endomorphism,  $\dim(U) = \text{tr}(id_U) := d_U \circ (id_U \otimes id_{U^\vee}) \circ \tilde{b}_U$ , in particular  $\dim(U_i) = s_{0,i}$ .

## Module and bimodule categories

The notion of a module category is a categorification of the one of a module over a ring. That is, an abelian category  $\mathcal{M}$  is a (right) *module category* over a monoidal category  $\mathcal{C}$  iff there exists an exact bifunctor

$$\boxtimes : \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{M} \quad (4)$$

together with functorial associativity and unit isomorphisms  $(M \boxtimes U) \boxtimes V \cong M \boxtimes (U \otimes V)$  and  $M \boxtimes \mathbf{1} \cong \mathbf{1}$  for  $M \in \text{Obj}(\mathcal{M})$ ,  $U, V \in \text{Obj}(\mathcal{C})$ , which satisfy appropriate pentagon and triangle coherence identities. The latter involve also the corresponding coherence isomorphisms of  $\mathcal{C}$  and are analogous to the identities that by definition of  $\otimes$  are obeyed by the coherence isomorphisms of  $\mathcal{C}$  alone. (Any tensor category  $\mathcal{C}$  is a module category over itself, much like as a ring is a module over itself.) A module category is called *indecomposable* iff it is not equivalent to a direct sum of two nontrivial module categories.

A *module functor* between two module categories  $\mathcal{M}_1$  and  $\mathcal{M}_2$  over the same monoidal category  $\mathcal{C}$  is a functor  $F: \mathcal{M}_1 \rightarrow \mathcal{M}_2$  together with functorial morphisms  $F(M \boxtimes_1 U) \rightarrow F(M) \boxtimes_2 U$  for  $M \in \text{Obj}(\mathcal{M}_1)$  and  $U \in \text{Obj}(\mathcal{C})$ , obeying again appropriate pentagon and triangle identities. For details see section 2.3 of [O]. The composition of two module functors is again a module functor. Thus the category  $\mathcal{C}_{\mathcal{M}}^* := \text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$  of module endofunctors of a right module category  $\mathcal{M}$  over a monoidal category  $\mathcal{C}$  is monoidal, and the action of these endofunctors on  $\mathcal{M}$  turns  $\mathcal{M}$  into a left module category over  $\mathcal{C}_{\mathcal{M}}^*$ .

If  $\mathcal{C}$  has a braiding  $c$  then two functors  $\alpha^{\pm}$  from  $\mathcal{C}$  to  $\mathcal{C}_{\mathcal{M}}^*$  are of particular interest. They are defined by assigning to  $U \in \text{Obj}(\mathcal{C})$  the endofunctors

$$\alpha^+(U) = \alpha^-(U) := \_ \boxtimes U \quad (5)$$

of  $\mathcal{M}$ . In order that  $\alpha^{\pm}$  are indeed functors to  $\mathcal{C}_{\mathcal{M}}^*$ , we have to make  $\alpha^{\pm}(U)$  into module functors, i.e. to specify morphisms  $\alpha^{\pm}(U)(M \boxtimes V) \rightarrow (\alpha^{\pm}(U)(M)) \boxtimes V$ , i.e.

$$M \boxtimes (V \otimes U) \cong (M \boxtimes V) \boxtimes U \longrightarrow (M \boxtimes U) \boxtimes V \cong M \boxtimes (U \otimes V) \quad (6)$$

for  $M \in \text{Obj}(\mathcal{M})$  and  $U, V \in \text{Obj}(\mathcal{C})$ ; this is achieved by using the morphisms<sup>1</sup>  $id_M \otimes c_{V,U}$  for  $\alpha^+$  and  $id_M \otimes c_{U,V}^{-1}$  for  $\alpha^-$ , respectively.

The functors  $\alpha^{\pm}$  defined this way are monoidal; we call them *braided-induction* functors. These functors, originally referred to as *alpha induction* functors, were implicitly introduced [LR] and heavily used [X, BE1, BEK, BE2] in the context of subfactors, and were interpreted as monoidal functors with values in a category of module endofunctors in [O].

Further, by setting

$$(U, X, V) \mapsto \beta^{\epsilon\epsilon'}(U, V) := \alpha^{\epsilon}(U) \circ X \circ \alpha^{\epsilon'}(V) \quad (7)$$

for  $U, V \in \text{Obj}(\mathcal{C})$  and  $X \in \text{Obj}(\mathcal{C}_{\mathcal{M}}^*)$ , for any choice of signs  $\epsilon, \epsilon' \in \{\pm\}$  one obtains a functor  $\beta^{\epsilon\epsilon'}: \mathcal{C} \times \mathcal{C}_{\mathcal{M}}^* \times \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{M}}^*$ ; it can be complemented by two sets of associativity constraints (for the left and right action of  $\mathcal{C}$ ), both obeying separately a pentagon constraint and an additional mixed constraint that expresses the commutativity of the left and right action of  $\mathcal{C}$ . This turns  $\mathcal{C}_{\mathcal{M}}^*$  into a *bimodule category* over  $\mathcal{C}$ .

## Frobenius algebras

Given a monoidal category  $\mathcal{C}$ , the structure of a module category over  $\mathcal{C}$  on an abelian category  $\mathcal{M}$  is equivalent to a monoidal functor from  $\mathcal{C}$  to the category of endofunctors of  $\mathcal{M}$ . If  $\mathcal{C}$  is semisimple rigid monoidal, with simple tensor unit and with a finite number of isomorphism classes of simple objects, and  $\mathcal{M}$  is indecomposable and semisimple, then it follows [O, Theorem 3.1] that  $\mathcal{M}$  is equivalent to the category  $\mathcal{C}_A$  of (left)  $A$ -modules in  $\mathcal{C}$  for some algebra  $A$  in  $\mathcal{C}$ . By a similar reasoning  $\mathcal{C}_{\mathcal{M}}^*$  is then equivalent to the category  $\mathcal{C}_{A|A}$  of  $A$ -bimodules in  $\mathcal{C}$ . The monoidal product of  $\mathcal{C}_{A|A}$  is the tensor product over  $A$ . The algebra  $A$  such that  $\mathcal{C}_A \simeq \mathcal{M}$  is determined uniquely up to Morita equivalence; it can be constructed as the internal  $\text{End} \underline{\text{End}}(M)$  for any  $M \in \text{Obj}(\mathcal{M})$  [O].

Recall that a (unital, associative) algebra  $A = (A, m, \eta)$  in a (strict) monoidal category  $\mathcal{C}$  consists of an object  $A \in \text{Obj}(\mathcal{C})$  and morphisms  $m \in \text{Hom}(A \otimes A, A)$  and  $\eta \in \text{Hom}(\mathbf{1}, A)$  satisfying  $m \circ (m \otimes id_A) = m \circ (id_A \otimes m)$  and  $m \circ (\eta \otimes id_A) = id_A = m \circ (id_A \otimes \eta)$ . If  $\mathcal{M}$  is indecomposable

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<sup>1</sup> suppressing, for brevity, the mixed associativity morphisms

then the category  $\mathcal{C}_{\mathcal{M}}^*$  has simple tensor unit (it is thus a fusion category in the sense of [ENO]). In terms of the algebra  $A$ , this property means that  $A$  is simple as an object of the category  $\mathcal{C}_{A|A}$  of  $A$ -bimodules in  $\mathcal{C}$ ; such algebras  $A$  are called *simple*.

A left  $A$ -module is a pair  $M = (\dot{M}, \rho)$  consisting of an object  $\dot{M} \in \mathcal{O}bj(\mathcal{C})$  and of a morphism  $\rho \in \text{Hom}(A \otimes \dot{M}, \dot{M})$  that satisfies  $\rho \circ (id_A \otimes \rho) = \rho \circ (m \otimes id_{\dot{M}})$  and  $\rho \circ (\eta \otimes id_{\dot{M}}) = id_{\dot{M}}$ . A right  $A$ -module  $(\dot{M}, \varrho)$ ,  $\varrho \in \text{Hom}(\dot{M}, \dot{M} \otimes A)$ , is defined analogously, and an  $A$ -bimodule is a triple  $X = (\dot{X}, \rho_l, \rho_r)$  such that  $(\dot{X}, \rho_l)$  is a left  $A$ -module,  $(\dot{X}, \rho_r)$  is a right  $A$ -module and the left and right  $A$ -actions commute. The morphism space  $\text{Hom}_A(M, N)$  in  $\mathcal{C}_A$  consists of those morphisms in  $\text{Hom}(\dot{M}, \dot{N})$  which commute with the left  $A$ -action, and an analogous property characterizes the morphism space  $\text{Hom}_{A|A}(X, Y)$  in  $\mathcal{C}_{A|A}$ .

Of interest to us is a particular class of algebras in  $\mathcal{C}$ , the symmetric special Frobenius algebras. A *Frobenius algebra* in a monoidal category  $\mathcal{C}$  is a quintuple  $A = (A, m, \eta, \Delta, \varepsilon)$  such that  $(A, m, \eta)$  is an algebra,  $(A, \Delta, \varepsilon)$  is a coalgebra, and the coproduct  $\Delta$  is a morphism of  $A$ -bimodules.<sup>2</sup> A Frobenius algebra  $A$  in a sovereign monoidal category is *symmetric* iff the morphism  $(d_A \otimes id_A) \circ [id_{A^\vee} \otimes (\Delta \circ \eta \circ \varepsilon \circ m)] \circ (\tilde{b}_A \otimes id_A) \in \text{End}(A)$  (which for any Frobenius algebra  $A$  is an algebra automorphism) equals  $id_A$ . A Frobenius algebra  $A$  is *special* iff  $\Delta$  is a right-inverse of the product  $m$  and the counit  $\varepsilon$  is a left-inverse of the unit  $\eta$ , up to nonzero scalars. For a special Frobenius algebra one has  $\dim(A) \neq 0$ , and one can normalize the counit in such a way that  $m \circ \Delta = id_A$  and  $\eta \circ \varepsilon = \dim(A) id_{\mathbf{1}}$ ; below we assume that this normalization has been chosen.

The structure and representation theory of symmetric special Frobenius algebras have been studied e.g. in [KO, FRS1, FrFRS1, FRS2]. The braided-induction functors  $\alpha^\pm$  which exist when  $\mathcal{C}$  has a braiding can be described in terms of the algebra  $A$  as the functors  $\alpha_A^\pm: \mathcal{C} \rightarrow \mathcal{C}_{A|A} \simeq \mathcal{C}_{\mathcal{M}}^*$  that associate to  $U \in \mathcal{O}bj(\mathcal{C})$  the following  $A$ -bimodules  $\alpha_A^\pm(U)$ : the underlying object is  $A \otimes U$ , the left module structure is the one of an induced left module, and the right module structure is given by the one of an induced right module composed with a braiding  $c_{U,A}^{-1}$  for  $\alpha_A^+$  and  $c_{A,U}$  for  $\alpha_A^-$ , respectively [O, FrFRS1]. In terms of  $A$  the numbers  $z_{ij}$  in (2) are given by  $z_{ij} = z(A)_{ij}$  with

$$z(A)_{i,j} := \dim_{\mathbb{C}} \text{Hom}_{A|A}(\alpha_A^+(U_i), \alpha_A^-(U_j)). \quad (8)$$

Let us also mention that for modular  $\mathcal{C}$  every  $A$ -bimodule can be obtained as a retract of a tensor product  $\alpha_A^+(U) \otimes_A \alpha_A^-(V)$  of a suitable pair of  $\alpha_A^+$ - and  $\alpha_A^-$ -induced bimodules (see the conjecture [O, Claim 5.2] and its proof in [FrFRS2]), and that the matrix  $z(A)$  is a permutation matrix iff the functors  $\alpha_A^\pm$  are monoidal equivalences, i.e. iff  $A$  is Azumaya [FRS2].

## Nondegeneracy

To cover the terms used in the formulation of Theorem O we need to introduce one further notion, the one of nondegeneracy. This is done in the following

DEFINITION.

(i) *An algebra  $A$  in a sovereign tensor category is called nondegenerate iff the morphism*

$$[(\varepsilon_{\mathbf{1}} \circ m) \otimes id_{A^\vee}] \circ (id_A \otimes b_A) \in \text{Hom}(A, A^\vee) \quad (9)$$

*with  $\varepsilon_{\mathbf{1}} := d_A \circ (id_{A^\vee} \otimes m) \circ (\tilde{b}_A \otimes id_A) \in \text{Hom}(A, \mathbf{1})$  is an isomorphism.*

(ii) *A semisimple indecomposable module category  $\mathcal{M}$  over a semisimple sovereign tensor category*

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<sup>2</sup> In the classical case of Frobenius algebras in the category of vector spaces over a field, the Frobenius property can be formulated in several other equivalent ways. The one given here does not require any further structure on  $\mathcal{C}$  beyond monoidality. Also note that neither  $\Delta$  nor the counit  $\varepsilon$  is required to be an algebra morphism.

$\mathcal{C}$  that has simple tensor unit and a finite number of isomorphism classes of simple objects is called nondegenerate iff there exists a nondegenerate algebra  $A$  in  $\mathcal{C}$  such that  $\mathcal{M} \simeq \mathcal{C}_A$ .

It is not difficult to see that the property of an algebra to be nondegenerate is preserved under Morita equivalence. But it is at present not evident to us how to give a definition of nondegeneracy for module categories which does not make direct reference to the corresponding (Morita class of) algebra(s); we plan to come back to this problem elsewhere. That such a formulation must exist is actually the motivation for introducing this terminology. In contrast, for algebras nondegeneracy is not a new concept, owing to

LEMMA 1

*An algebra in a sovereign tensor category is nondegenerate iff it is symmetric special Frobenius.*

Proof.

That a nondegenerate algebra in a sovereign tensor category is symmetric special Frobenius has been shown in lemma 3.12 of [FRS1]. The converse holds because for a symmetric special Frobenius algebra one has  $\varepsilon_{\natural} = \varepsilon$ , and as a consequence  $(d_A \otimes id_A) \circ (id_{A^\vee} \otimes (\Delta \circ \eta)) \in \text{Hom}(A^\vee, A)$  is inverse to the morphism (9).  $\square$

## Fusion rules

We denote by  $[U]$  the isomorphism class of an object  $U$ . If an abelian category  $\mathcal{C}$  is rigid monoidal, then the tensor product bifunctor is exact, so that the Grothendieck group  $K_0(\mathcal{C})$  has a natural ring structure given by  $[U] * [V] := [U \otimes V]$ . A distinguished basis of  $K_0(\mathcal{C})$  is given by the isomorphism classes of simple objects of  $\mathcal{C}$ ; in this basis the structure constants are non-negative integers. If  $\mathcal{M}$  is a module category over  $\mathcal{C}$ , then the Grothendieck group  $K_0(\mathcal{M})$  is naturally a  $K_0(\mathcal{C})$ -module, in fact a based module over the based ring  $K_0(\mathcal{C})$ .

From now on, unless noted otherwise,  $\mathcal{C}$  will stand for a modular tensor category,  $\mathcal{M}$  for a semisimple nondegenerate indecomposable module category over  $\mathcal{C}$ , and  $\mathcal{C}_{\mathcal{M}}^* = \text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$  will be regarded as a bimodule category over  $\mathcal{C}$  via the functor  $\beta^{+-}$ . Then the category  $\mathcal{C}_{\mathcal{M}}^*$  is semisimple  $\mathbb{C}$ -linear abelian rigid monoidal and has finite-dimensional morphism spaces and a finite number of isomorphism classes of simple objects. For simplicity of notation we also tacitly take  $\mathcal{C}$  to be strict monoidal; in the non-strict case the relevant coherence isomorphisms have to be inserted at appropriate places, but all statements about Grothendieck rings remain unaltered.

Analogously as  $K_0(\mathcal{C})$  also the Grothendieck group  $K_0(\mathcal{C}_{\mathcal{M}}^*)$  has a natural ring structure. However,  $\mathcal{C}_{\mathcal{M}}^*$  is not, in general, braided, and hence, unlike  $K_0(\mathcal{C})$ , the ring  $K_0(\mathcal{C}_{\mathcal{M}}^*)$  is in general not commutative. The complexified Grothendieck ring

$$\mathcal{F} := K_0(\mathcal{C}_{\mathcal{M}}^*) \otimes_{\mathbb{Z}} \mathbb{C} \tag{10}$$

of  $\mathcal{C}_{\mathcal{M}}^*$  is a finite-dimensional semisimple associative  $\mathbb{C}$ -algebra, and hence a direct sum of full matrix algebras,  $\mathcal{F} \cong \bigoplus_{p \in P} \text{Mat}_{n_p}$  for some finite index set  $P$ , with  $\text{Mat}_n$  the algebra of  $n \times n$ -matrices with complex entries. We denote the product in  $\mathcal{F}$  by  $*$ . The algebra  $\mathcal{F}$  has a standard basis  $\{e_{p;\alpha\beta}\}$ ,  $\mathcal{F} \cong \bigoplus_{p \in P} \bigoplus_{\alpha,\beta=1}^{n_p} \mathbb{C} e_{p;\alpha\beta}$ , with products  $e_{p;\alpha\beta} * e_{q;\gamma\delta} = \delta_{p,q} \delta_{\beta,\gamma} e_{p;\alpha\delta}$ . The elements  $e_p := \sum_{\alpha=1}^{n_p} e_{p;\alpha\alpha}$  are the primitive idempotents projecting onto the simple summands  $\text{Mat}_{n_p}$  of  $\mathcal{F}$ . Given any other basis  $\{x_{\kappa} \mid \kappa \in K\}$ , there is a basis transformation

$$x_{\kappa} = \sum_{p \in P} \sum_{\alpha,\beta=1}^{n_p} u_{\kappa}^{p;\alpha\beta} e_{p;\alpha\beta}, \quad e_{p;ab} = \sum_{\kappa \in K} (u^{-1})_{p;\alpha\beta}^{\kappa} x_{\kappa}, \tag{11}$$

satisfying  $\sum_{p,\alpha,\beta} (u^{-1})_{p;\alpha\beta}^{\kappa'} u_{\kappa}^{p;\alpha\beta} = \delta_{\kappa,\kappa'}$  and  $\sum_{\kappa} u_{\kappa}^{q;\alpha\beta} (u^{-1})_{p;\gamma\delta}^{\kappa} = \delta_{p,q} \delta_{\alpha,\gamma} \delta_{\beta,\delta}$ , so that the structure constants of  $\mathcal{F}$  in the basis  $\{x_{\kappa}\}$  can be written as

$$N_{\kappa\kappa'}^{\kappa''} = \sum_{p \in P} \sum_{\alpha,\beta,\gamma=1}^{n_p} u_{\kappa}^{p;\alpha\beta} u_{\kappa'}^{p;\beta\gamma} (u^{-1})_{p;\alpha\gamma}^{\kappa''}. \quad (12)$$

For the particular case that  $\{x_{\kappa}\}$  is the distinguished basis of the underlying ring given by the isomorphism classes  $[X_{\kappa}]$  of simple objects of  $\mathcal{C}_{\mathcal{M}}^*$ , it has become customary in various contexts to refer to the structure constants  $N_{\kappa\kappa'}^{\kappa''}$  of  $\mathcal{F}$  (or also to the algebra  $\mathcal{F}$ , or to the Grothendieck ring itself) as the *fusion rules* of the category under study. The formula (12) then constitutes a block-diagonalization of the fusion rules of  $\mathcal{C}_{\mathcal{M}}^*$ . Also note that if  $\mathcal{F}$  is commutative, then (12) reduces to  $N_{\kappa\kappa'}^{\kappa''} = \sum_{p \in P} u_{\kappa}^p u_{\kappa'}^p (u^{-1})_p^{\kappa''}$ , which is sometimes referred to as the Verlinde formula.

In terms of these ingredients, the assertion of Theorem O amounts to the claim that the index set  $P$  and the dimensions  $n_p^2$  of the simple summands of  $\mathcal{F}$  are given by

$$P = \{(i, j) \in I \times I \mid z_{i,j} \neq 0\} \quad \text{and} \quad n_{(i,j)} = z_{i,j} \quad (13)$$

with the integers  $z_{i,j}$  the dimensions defined in formula (2).

We will establish the equalities (13) by finding an explicit expression for the matrix elements  $u_{\kappa}^{p;\alpha\beta} \equiv u_{\kappa}^{(i,j);\alpha\beta}$  of the basis transformation (11) for the case that the basis  $\{x_{\kappa} \mid \kappa \in K\}$  is the distinguished basis of isomorphism classes  $[X_{\kappa}]$  of the simple objects.

## 2 The structure of the algebra $\mathcal{F}$

The considerations above show in particular that a nondegenerate semisimple indecomposable module category  $\mathcal{M}$  over a modular tensor category  $\mathcal{C}$  is equivalent, as a module category, to the category  $\mathcal{C}_A$  for an appropriate simple symmetric special Frobenius algebra  $A$  in  $\mathcal{C}$ . Moreover, the category  $\mathcal{C}_{\mathcal{M}}^*$  of module endofunctors is equivalent to  $\mathcal{C}_{A|A}$  as a monoidal category (and as a bimodule category over  $\mathcal{C}$  and left module category over  $\mathcal{M}$ ). We may thus restate Theorem O as

**THEOREM O'.**

*Let  $\mathcal{C}$  be a modular tensor category and  $A$  a simple symmetric special Frobenius algebra in  $\mathcal{C}$ . Then the complexified Grothendieck ring  $\mathcal{F} = K_0(\mathcal{C}_{A|A}) \otimes_{\mathbb{Z}} \mathbb{C}$  and the endomorphism algebra*

$$\mathcal{E} := \bigoplus_{i,j \in I} \text{End}_{\mathbb{C}}(\text{Hom}_{A|A}(\alpha^+(U_i), \alpha^-(U_j))). \quad (14)$$

*are isomorphic as  $\mathbb{C}$ -algebras.*

We will prove Theorem O' by constructing an isomorphism from  $\mathcal{F}$  to  $\mathcal{E}$ .

### A map $\Phi$ from $\mathcal{F}$ to $\mathcal{E}$

To simplify some of the expressions appearing below, we replace  $j \in I$  in (2) by  $\bar{j}$ , defined as the unique label in  $I$  such that  $U_{\bar{j}} \cong U_j^{\vee}$ , and instead of the spaces  $\text{Hom}_{A|A}(\alpha^+(U), \alpha^-(V^{\vee}))$  we prefer to work with the isomorphic spaces  $\text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A)$ , where for any  $A$ -bimodule  $X = (\dot{X}, \rho_l, \rho_r)$  the  $A$ -bimodules  $U \otimes^+ X$  and  $X \otimes^- V$  are defined as  $U \otimes^+ X := (U \otimes \dot{X}, (id_U \otimes \rho_l) \circ$

$(c_{U,A}^{-1} \otimes id_{\dot{X}}), id_U \otimes \rho_r$ ) and as  $X \otimes^- V := (\dot{X} \otimes V, \rho_l \otimes id_V, (\rho_r \otimes id_V) \circ (id_{\dot{X}} \otimes c_{A,V}^{-1}))$ , respectively. Accordingly instead of with (14) we work with the isomorphic algebra

$$\bigoplus_{i,j \in I} \text{End}_{\mathbb{C}}(\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)) \cong \mathcal{E} \quad (15)$$

which by abuse of notation we still denote by  $\mathcal{E}$ .

For any  $U, V \in \text{Obj}(\mathcal{C})$  and  $X \in \text{Obj}(\mathcal{C}_{A|A})$  we denote by  $D_X^{UV}$  the mapping

$$\begin{aligned} \varphi \in \text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A) \\ \longmapsto D_X^{UV}(\varphi) := (id_A \otimes \tilde{d}_{\dot{X}}) \circ \left[ \left( [id_A \otimes \rho_l \otimes (\varepsilon \circ \varphi)] \circ [(\Delta \circ \eta) \otimes c_{U,\dot{X}} \otimes id_A \otimes id_V] \right) \otimes id_{\dot{X}^\vee} \right] \\ \circ \left[ id_U \otimes \left( [(\rho_r \circ (\rho_l \otimes id_A)) \otimes id_A] \circ [id_A \otimes id_{\dot{X}} \otimes (\Delta \circ \eta)] \right) \otimes id_V \otimes id_{\dot{X}^\vee} \right] \\ \circ \left[ id_U \otimes id_A \otimes \left( [c_{\dot{X},V}^{-1} \otimes id_{\dot{X}^\vee}] \circ [id_V \otimes b_{\dot{X}}] \right) \right] \end{aligned} \quad (16)$$

One checks that  $D_X^{UV}(\varphi)$ , which by construction is a morphism in  $\text{Hom}(U \otimes A \otimes V, A)$ , is actually again in the subspace  $\text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A)$ , so that  $D_X^{UV}$  is an endomorphism of the vector space  $\text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A)$ . Further, we consider the linear map

$$\Phi : \mathcal{F} = K_0(\mathcal{C}_{A|A}) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \mathcal{E} \quad (17)$$

defined by

$$\Phi([X]) := D_X = \bigoplus_{i,j \in I} D_X^{U_i U_j}. \quad (18)$$

It must be admitted that the expression (16) for the map  $D_X^{UV}$  is not exceedingly transparent. However, with the help of the graphical notation for monoidal categories as described e.g. in [JS, Ma, K], it can easily be visualized. Indeed, using

$$id_U = \begin{array}{c} U \\ | \\ U \end{array} \quad f = \begin{array}{c} V \\ | \\ \boxed{f} \\ | \\ U \end{array} \quad g \circ f = \begin{array}{c} W \\ | \\ \boxed{g} \\ | \\ V \\ | \\ \boxed{f} \\ | \\ U \end{array} \quad f \otimes f' = \begin{array}{cc} V & V' \\ | & | \\ \boxed{f} & \boxed{f'} \\ | & | \\ U & U' \end{array} \quad (19)$$

for identity morphisms, general morphisms  $f \in \text{Hom}(U, V)$ , and for composition and tensor product of morphisms of  $\mathcal{C}$ ,

$$\begin{array}{cc} c_{U,V} = \begin{array}{cc} V & U \\ | & | \\ \diagdown & / \\ | & | \\ \diagup & \diagdown \\ U & V \end{array} & c_{U,V}^{-1} = \begin{array}{cc} U & V \\ | & | \\ \diagdown & / \\ | & | \\ \diagup & \diagdown \\ V & U \end{array} & b_U = \begin{array}{c} U & U^\vee \\ | & | \\ \diagdown & / \\ | & | \\ \diagup & \diagdown \end{array} & \tilde{b}_U = \begin{array}{c} U^\vee & U \\ | & | \\ \diagdown & / \\ | & | \\ \diagup & \diagdown \end{array} \\ d_U = \begin{array}{c} U^\vee & U \\ | & | \\ \diagdown & / \\ | & | \\ \diagup & \diagdown \end{array} & \tilde{d}_U = \begin{array}{c} U & U^\vee \\ | & | \\ \diagdown & / \\ | & | \\ \diagup & \diagdown \end{array} \end{array} \quad (20)$$

for braiding and duality morphisms of  $\mathcal{C}$ , as well as



braiding and the fact that the left and right  $A$ -actions on the bimodule  $Y$  commute, one obtains

$$D_X \circ D_Y(\varphi) = \text{Diagram} \tag{24}$$

The assertion then follows immediately by the fact [KO, FrFRS2] that the morphism

$$(\rho_r^X \otimes \rho_l^Y) \circ (id_{\dot{X}} \otimes (\Delta \circ \eta) \otimes id_{\dot{Y}}) = \text{Diagram} \tag{25}$$

is the idempotent corresponding to the epimorphism that restricts  $X \otimes Y$  to  $X \otimes_A Y$ . To show also the first of the equalities (23), first note that for  $X = A$  the left and right  $A$ -actions are just given by the product  $m$  of  $A$ . One can then use the various defining properties of  $A$  and the fact that  $\varphi \in \text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A)$  intertwines the left action of  $A$  to arrive at

$$D_A^{UV}(\varphi) = \text{Diagram} \quad (26)$$

By again using the associativity, symmetry and Frobenius property of  $A$ , together with the fact that  $\varphi$  is a bimodule intertwiner, one can reduce the right hand side to  $\varphi$ , except for an additional  $A$ -loop being attached to the  $A$ -line. This  $A$ -loop, in turn, is removed by invoking specialness of  $A$ . Thus  $D_A^{UV}$  acts as the identity on  $\text{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A)$ , which establishes the claim.  $\square$

### $\Phi$ is an isomorphism

Maps  $D_X$  of the type appearing in (18) have already been considered in [FrFRS2], where their properties were studied using the relation [T] between modular tensor categories and three-dimensional topological field theory. Indeed, these maps are special cases of certain linear maps  $\text{Hom}_{B|B}(U \otimes^+ B \otimes^- V, B) \rightarrow \text{Hom}_{A|A}(U \otimes^+ Y \otimes^- V, A)$  defined for a pair  $A, B$  of symmetric special Frobenius algebras and an  $A$ -bimodule  $Y$ , see equations (2.29) and (4.14) of [FrFRS2].

#### LEMMA 3

*The algebra morphism  $\Phi$  defined in (18) is an isomorphism.*

Proof.

It has been shown in Proposition 2.8 of [FrFRS2] that if the equality  $D_X(\varphi) = D_Y(\varphi)$  holds for all  $\varphi \in \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$  and  $i, j \in I$ , then  $X$  and  $Y$  are isomorphic bimodules. Hence  $\Phi$  is injective.

That  $\Phi$  is surjective thus follows from the fact that

$$\dim_{\mathbb{C}}(\mathcal{F}) = \text{tr}(z(A)^t z(A)) \equiv \sum_{i,j \in I} (z(A)_{i,j})^2 = \dim_{\mathbb{C}}(\mathcal{E}). \quad (27)$$

Here the first equality has been established in Remark 5.19(ii) of [FRS1], while the last equality is satisfied because, by definition of the numbers  $z(A)_{i,j}$ ,  $\dim_{\mathbb{C}}(\text{End}_{\mathbb{C}}(\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A))) = (z(A)_{i,j})^2$ .  $\square$

This completes the proof of Theorem O', and thus also of Theorem O.  $\square$

## Block-diagonalization

Using the fact that for simple  $A$  the mapping  $f \mapsto \varepsilon \circ f \circ \eta$  furnishes a canonical isomorphism  $\text{End}_{A|A}(A) \rightarrow \text{End}(\mathbf{1}) = \mathbb{C}$ , one sees that with respect to the bases of  $\mathcal{F}$  and  $\mathcal{E}$  considered in section 1 the linear map  $\Phi$  (18) is given by the matrix  $d$  with entries

$$d_{\kappa}^{ij;\alpha\beta} := \varepsilon \circ D_{X_{\kappa}}^{U_i U_j}(h_{\beta}^{(ij)}) \circ \bar{h}_{\alpha}^{(ij)} \circ \eta \in \text{End}(\mathbf{1}) = \mathbb{C}, \quad (28)$$

where  $\kappa \in K$ ,  $i, j \in I$  and  $h_{\beta}^{(ij)} \in \Gamma_{ij}$ ,  $\bar{h}_{\alpha}^{(ij)} \in \bar{\Gamma}_{ij}$ , with  $\Gamma_{ij}$  a basis of  $\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$  and  $\bar{\Gamma}_{ij}$  a dual basis of  $\text{Hom}_{A|A}(A, U_i \otimes^+ A \otimes^- U_j) \cong \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)^*$ . Since  $\Phi$  is an isomorphism,  $d$  is invertible.

Now  $\Phi(x_{\kappa})$  acts on the elements of a basis  $\bigcup_{i,j \in I} \Gamma_{ij}$  of  $\bigoplus_{i,j \in I} \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$  as  $h_{\alpha}^{(ij)} \mapsto \sum_{\beta} d_{\kappa}^{ij;\beta\alpha} h_{\beta}^{(ij)}$ , while the elements  $\Phi(e_{(k,l);\gamma\delta})$  are by definition the matrix units of  $\mathcal{E}$  with respect to the basis  $\{h_{\alpha}^{(ij)}\}$ , mapping  $h_{\alpha}^{(ij)}$  to  $\delta_{ik}\delta_{jl}\delta_{\alpha\delta} h_{\gamma}^{(ij)}$ , and hence we have  $\Phi(x_{\kappa}) = \sum_{i,j,\alpha,\beta} d_{\kappa}^{ij;\alpha\beta} \Phi(e_{(i,j);\alpha\beta})$ . Thus  $d$  is precisely the basis transformation matrix  $u$  appearing in formula (11). As a consequence, according to (12) the fusion rules of  $\mathcal{C}_{A|A}$  can be written as

$$N_{\kappa\kappa'}^{\kappa''} = \sum_{i,j \in I} \sum_{\alpha,\beta,\gamma=1}^{z_{i,\bar{j}}} d_{\kappa}^{ij;\alpha\beta} d_{\kappa'}^{ij;\beta\gamma} (d^{-1})_{ij;\alpha\gamma}^{\kappa''}. \quad (29)$$

Pictorially, the entries of the block-diagonalization matrix are given by

$$d_{\kappa}^{ij;\alpha\beta} = \text{Diagram 1} = \text{Diagram 2} \quad (30)$$

Here the second equality is presented to facilitate comparison with formulas in [FrFRS2]; it follows by repeated use of functoriality of the braiding, the various properties of  $A$  (symmetric, special, Frobenius) and  $X$  ( $A$ -bimodule) and of  $h_{\beta}^{(ij)}$ ,  $h_{\bar{\alpha}}^{(ij)}$  (bimodule morphisms), and by invoking the isomorphism  $\text{End}(\mathbf{1}) \cong \text{End}_{A|A}(A)$ .

## 3 Quantum field theory

A main motivation for the investigations in [O] came from work in low-dimensional quantum field theory, and that area also constitutes an important arena for applications of results like Theorem O. This section describes some of the connections with quantum field theory.

## Subfactors

As already mentioned, braided induction first appeared in the study of subfactors, or more precisely, of a new construction of subfactors that was inspired by examples from quantum field theory, see e.g. [LR, X, I]. In that context it is defined in terms of the so-called statistics operators (describing the braiding) and of the canonical and dual canonical endomorphisms of the subfactor.

In a series of papers starting with [BE1, BEK], various aspects of braided induction were studied in this setting (for reviews see e.g. [BE2] or section 2 of [EP]). The results include in particular the subfactor version of Theorem O, describing the fusion rules of  $N'$ - $N'$ -morphisms for a finite index subfactor  $N \subseteq N'$ , which was established as Theorem 6.8 of [BEK]. Indeed, the formulation of Theorem O in [O] was directly motivated by this result, as were several other conjectures, all of which have meanwhile been proven in a purely categorical setting as well.

The proof of Theorem 6.8 in [BEK] is based on techniques and results from the theory of subfactors. As a consequence, the statement of the theorem itself is somewhat weaker than the one of Theorem O. More specifically, in the subfactor context the modular tensor category  $\mathcal{C}$  arises as a category  $\mathcal{E}nd(N)$  of endomorphisms of a type III factor  $N$  (see e.g. [Mü2, Def./Prop. 2.5]). This in turn implies that  $\mathcal{C}$  as well as  $\mathcal{M} = \mathcal{H}om(N, N')$  and  $\mathcal{C}_{\mathcal{M}}^* = \mathcal{E}nd(N')$  are  $*$ -categories. A  $*$ -category  $\mathcal{C}$  is a  $\mathbb{C}$ -linear category endowed with a family  $\{*_U, V: \text{Hom}(U, V) \rightarrow \text{Hom}(V, U) \mid U, V \in \text{Obj}(\mathcal{C})\}$  of maps which are antilinear, involutive and contravariant, as well as, in case  $\mathcal{C}$  is a monoidal category, monoidal. In a  $*$ -category the dimension of any object is positive. This is the reason why positivity of the dimension was included in the original form [O] of Theorem O; as already stressed, this condition is not essential.

Let us also point out that in the subfactor setting algebras arise in the guise of so-called  $Q$ -systems as introduced in [Lo]. Indeed [EP], a  $Q$ -system is the same as a symmetric special Frobenius  $*$ -algebra, where the  $*$ -property means that  $\Delta = *_A, A(m)$  and  $\varepsilon = *_1, A(\eta)$ . The  $Q$ -system associated to  $N \subseteq N'$  is constructed from the embedding morphism  $N \hookrightarrow N'$  and its two-sided adjoint. Given a separable type III factor  $N$  there is a bijection between finite index subfactors  $N \subseteq N'$  and  $Q$ -systems in  $\mathcal{E}nd(N)$  (see e.g. [Mü1]).

## Conformal field theory

One of the basic ideas in the subfactor studies was to introduce the matrix (2) and to establish that it enjoys various remarkable properties, like commuting with certain matrices  $S$  and  $T$  that generate a representation of the modular group  $\text{SL}(2, \mathbb{Z})$ . These properties are, in fact, familiar from the matrix describing the so-called torus partition function in two-dimensional rational conformal field theory (RCFT), see e.g. [DMS]. One of the basic ingredients of RCFT is the ‘category of chiral sectors’, which provides basic input data for the field content of a given model of RCFT. This category  $\mathcal{C}$  is a modular tensor category. In the formalization of RCFT through conformal nets of subfactors  $\mathcal{C}$  is given by the endomorphism category  $\mathcal{E}nd(N)$  of the relevant factor; in another formalization it arises as the representation category of a conformal vertex algebra [Le].

However, the properties of the matrix  $z$  verified in the subfactor studies are by far not the only requirements that a valid torus partition function of RCFT must fulfill. (In fact, one knows of many examples of non-negative integral matrices in the commutant of the action of  $\text{SL}(2, \mathbb{Z})$  that do not describe a valid partition function, see e.g. section 4 of [FSS] and [G].) It has been shown that a necessary [FjFRS2] and sufficient [FRS1, FjFRS1] condition for *all* constraints on the torus partition function, as well as on all other correlation functions of an RCFT, to be satisfied is that there exists a simple symmetric special Frobenius algebra in  $\mathcal{C}$ , through the representation theory of which the field content of the theory can be understood. This algebra is determined only up to

Morita equivalence. A Morita invariant formulation leads precisely to the definition (2) for  $z$ .

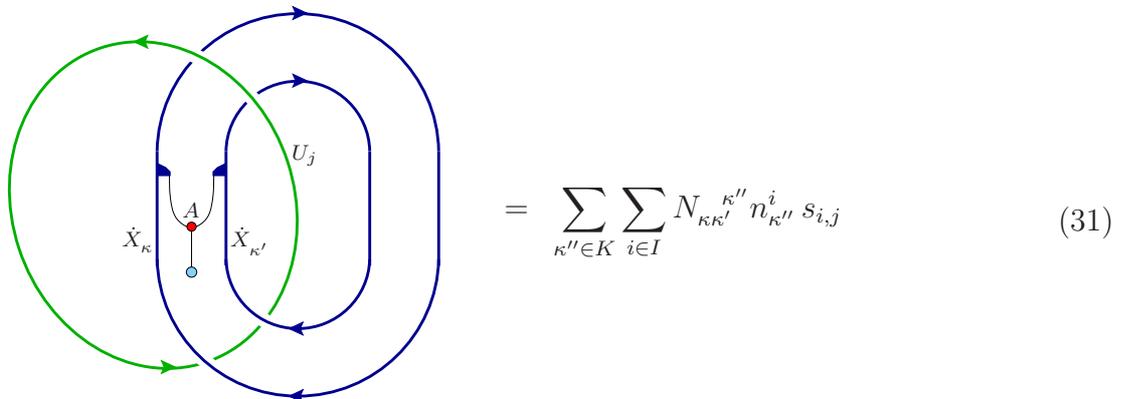
In the approach to RCFT via Frobenius algebras  $A$  in modular tensor categories (for a brief survey see e.g. [FRS2, SFR]), the role of the bimodule category  $\mathcal{C}_{A|A}$  is to specify the allowed types of topological defect lines. More generally, for any pair  $A, B$  of symmetric special Frobenius algebras the defect lines described by  $\mathcal{C}_{A|B}$  constitute one-dimensional phase boundaries on the two-dimensional world sheet, and their properties can be used to extract symmetries and order-disorder dualities of the RCFT. (The phase boundaries are referred to as *topological* defect lines because, as it turns out [FrFRS2], correlation functions remain invariant under continuous variation of their location.) The results from [FrFRS2] that were referred to and used above arose from the study of such defect lines in RCFT.

Fusion rules for topological defect lines in RCFT were first studied in [PZ1, PZ2], where it was e.g. verified that the formula (12) yields non-negative integers when one inserts for the entries of the basis transformation matrix  $u$  explicit numerical values that can be extracted from known data for the  $\mathfrak{sl}(2)$  Wess-Zumino-Witten models, a particular class of RCFT models.

The objects of the category  $\mathcal{C}_A$  of  $A$ -modules correspond to conformal boundary conditions of the RCFT. That  $\mathcal{C}_A$  is a module category over  $\mathcal{C}_{A|A}$  corresponds in the RCFT context to the fact that one can fuse a topological defect line with a boundary condition, thereby obtaining another boundary conditions of the theory [FrFRS2].

It is also worth mentioning that the matrix  $z(A)$  naturally arises not only in the discussion of the bimodule category  $\mathcal{C}_{A|A}$ , but also of the module category  $\mathcal{C}_A$ . In particular, similarly as  $\dim_{\mathbb{C}}(\mathcal{F}) = \text{tr}(z(A)^t z(A))$ , one finds that the number of isomorphism classes of simple  $A$ -modules equals  $\text{tr}(z(A))$  [FRS1, Theorem 5.18]. A crucial ingredient of the proof of this relation in [FRS1]<sup>3</sup> is the matrix  $S^A$  that implements a modular S-transformation on the conformal one-point blocks on the torus. The rows of this matrix are naturally labeled by the isomorphism classes of simple  $A$ -modules, while its columns correspond to a basis of the subspace of so-called *local* morphisms [FRS1, Def. 5.5] in  $\bigoplus_{i \in I} \text{Hom}(A \otimes U_i, U_i)$ . The dimension of the latter space is easily seen to be  $\text{tr}(z(A))$ , and the result then follows by observing [FRS1, Eq. (5.113)] that the matrix  $S^A$  is invertible and hence in particular square. (The notion of local morphisms is also closely related to certain interesting endofunctors of  $\mathcal{C}$ , compare section 3.1 of [FrFRS1].)

In view of this connection with  $\mathcal{C}_A$  it not surprising that the structure constants  $N_{\kappa\kappa'}^{\kappa''}$  of the bimodule fusion rules can be related to the matrix  $S^A$  as well. To see that this is the case, one can start from the fact that (25) is the idempotent that corresponds to forming the tensor product over  $A$ , which allows one to derive that



$$= \sum_{\kappa'' \in K} \sum_{i \in I} N_{\kappa\kappa'}^{\kappa''} n_{\kappa''}^i s_{i,j} \quad (31)$$

<sup>3</sup> In the arguments in [FRS1] the three-dimensional topological field theory associated [T, K] to  $\mathcal{C}$  is used. But they can in fact be formulated entirely in categorical language, a crucial ingredient being semisimplicity of  $\mathcal{C}$ .

with  $n_\kappa^i = \dim_{\mathbb{C}} \text{Hom}(U_i, \dot{X}_\kappa)$ . By a calculation repeating the steps in formula (5.131) of [FRS1], one then arrives for any  $j \in I$  at the sum rule

$$\sum_{\kappa'' \in K} n_j^{\kappa''} N_{\kappa\kappa'}^{\kappa''} = \sum_{i \in I} \sum_{\gamma} \sum_{\mu, \nu \in J} m_{\kappa\nu}^{\mu} m_{\kappa'}^{\nu} S_{\mu, i\gamma}^A \frac{S_{i, \bar{j}}}{S_{i, 0}} (S^A)_{i\gamma, \nu}^{-1}, \quad (32)$$

where the  $\gamma$ -summation extends over a basis of the local morphisms in  $\text{Hom}(A \otimes U_i, U_i)$  and the  $\mu$ - and  $\nu$ -summations over a set  $J$  of representatives of simple left  $A$ -modules, while the integers  $m_\kappa^\mu = \dim_{\mathbb{C}} \text{Hom}_A(M_\mu, X_\kappa)$  count the multiplicity of the simple left  $A$ -module  $M_\mu$  in the simple  $A$ -bimodule  $X_\kappa$ , regarded as a (not necessarily simple) left  $A$ -module.

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