arxiv:0710.5467
Hamburger Beiträge zur Mathematik Nr. 280
ZMP-HH/07-10

Gerbes and Lie Groups

Christoph Schweigert and Konrad Waldorf

Organisationseinheit Mathematik
Universität Hamburg
Bundesstraße 55
D-20146, Germany

Abstract

Compact Lie groups do not only carry the structure of a Riemannian manifold, but also canonical families of bundle gerbes. We discuss the construction of these bundle gerbes and their relation to loop groups. We present several algebraic structures for bundle gerbes with connection such as Jandl structures, gerbe modules and gerbe bimodules, and indicate their applications to Wess-Zumino terms in two-dimensional field theories.

Introduction

Compact Lie groups do not only come with a canonical metric (the Killing form), but also with a canonical family of bundle gerbes. These bundle gerbes are geometric objects made of finite dimensional manifolds and maps between those, and provide a way of understanding structure over the infinite dimensional loop group.

As a motivation, consider a central extension of the loop group of a compact connected and simply-connected Lie group G,

$$0 \longrightarrow \mathbb{C}^{\times} \longrightarrow \widehat{LG} \longrightarrow LG \longrightarrow 0.$$

Such extensions are classified by $\mathrm{H}^2(LG, \mathbb{Z})$. By transgression, this cohomology group is in turn isomorphic to the cohomology of G,

$$\mathrm{H}^3(G, \mathbb{Z}) \cong \mathrm{H}^2(LG, \mathbb{Z}).$$

While the cohomology group $\mathrm{H}^2(LG, \mathbb{Z})$ classifies line bundles over LG by their Chern class, $\mathrm{H}^3(G, \mathbb{Z})$ classifies bundle gerbes over G. In this way, every bundle gerbe over the finite dimensional manifold G gives rise to a line bundle over the infinite dimensional manifold LG.

This contribution is organized as follows. In Section 1 we describe bundle gerbes on general manifolds and their classification. In Section 2 we explain how bundle gerbes can be equipped with a connection which allows to define surface holonomies: from this point of view, bundle gerbes generalize the holonomy of principal bundles around curves. In case that the base manifold is a compact Lie group G, we construct examples of bundle gerbes over G in Section 3. Then we explain in Section 4 how a bundle gerbe gives rise to a line bundle over the loop space. In Section 5 we come to additional structures for bundle gerbes like bundle gerbe modules, bimodules and Jandl structures. Finally, in Section 6 we outline the applications of bundle gerbes on Lie groups to two-dimensional conformal field theory and string theory, which are closely related to the surface holonomy from Section 2. In these theories one also recovers the loop space as the space of configurations.

1 Bundle Gerbes

Let M be a smooth manifold. We shall briefly review the classification of complex line bundles over M. For this purpose, let us choose a good open cover $\mathfrak{V} = \{V_i\}_{i \in I}$ of M, i.e. every finite intersection of open sets V_i is contractible. In particular, every line bundle L admits local non-zero sections which determine smooth transition functions

$$g_{ij} : V_i \cap V_j \to \mathbb{C}^\times. \tag{1.1}$$

On three-fold intersections $V_i \cap V_j \cap V_k$, these transition functions satisfy the cocycle condition

$$g_{ik} = g_{ij} \cdot g_{jk}. \tag{1.2}$$

It is fair to call this equality a cocycle condition, since it means that $\delta g = 0$ for the element $g := (g_{ij}) \in \check{C}^1(\mathfrak{V}, \mathbb{C}^\times_M)$ in the Čech cohomology of the sheaf of smooth $\mathbb{C}^\times$-valued functions on M with respect to the cover $\mathfrak{V}$. Since we have chosen the cover $\mathfrak{V}$ to be good, there is a canonical isomorphism

$\check{\mathrm{H}}^1(\mathfrak{V}, \mathbb{C}^\times_M) \cong \mathrm{H}^2(M, \mathbb{Z})$ using the exponential sequence. The image of the class $[g]$ in $\mathrm{H}^2(M, \mathbb{Z})$ is independent of the choice of the sections, and is called the (first) Chern class $c_1(L)$ of the line bundle L. This defines an isomorphism

$$c_1 : \mathrm{Pic}(M) \to \mathrm{H}^2(M, \mathbb{Z}) \tag{1.3}$$

from the group $\mathrm{Pic}(M)$ of isomorphism classes of line bundles to the cohomology group $\mathrm{H}^2(M, \mathbb{Z})$, providing a geometric realization of this group.

A bundle gerbe is a geometric object which realizes the cohomology group $\mathrm{H}^3(M, \mathbb{Z})$ in a similar way. To prepare its definition, we fix the following notation. For a surjective submersion $\pi : Y \to M$ we denote the k-fold fibre product of Y over M by

$$Y^{[k]} = \{(y_1, ..., y_k) \in Y^k | \pi(y_1) = ... = \pi(y_k)\}. \tag{1.4}$$

This is again a smooth manifold, having canonical projections $\pi_{i_1,...,i_\ell} : Y^{[k]} \to Y^{[\ell]}$ on the respective factors.

Definition 1.1 ([Mur96])**.** *A* bundle gerbe $\mathcal{G}$ *over a manifold M is a triple (π, L, μ) consisting of a surjective submersion $\pi : Y \to M$, a line bundle L over $Y^{[2]}$ and an isomorphism*

$$\mu : \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L \tag{1.5}$$

of line bundles over $Y^{[3]}$, such that μ is associative in the sense that the diagram

$$\begin{tikzcd} \pi_{12}^*L \otimes \pi_{23}^*L \otimes \pi_{34}^*L \arrow[r, "\pi_{123}^*\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\pi_{234}^*\mu"'] & \pi_{13}^*L \otimes \pi_{34}^*L \arrow[d, "\pi_{134}^*\mu"] \\ \pi_{12}^*L \otimes \pi_{24}^*L \arrow[r, "\pi_{124}^*\mu"'] & \pi_{14}^*L \end{tikzcd} \tag{1.6}$$

of isomorphisms of line bundles over $Y^{[4]}$ is commutative.

Let us now describe how bundle gerbes realize the cohomology group $\mathrm{H}^3(M, \mathbb{Z})$. Let us again choose a good open cover $\mathfrak{V} = \{V_i\}_{i \in I}$ of M which admits sections $s_i : V_i \to Y$ into the manifold Y of the surjective submersion of a bundle gerbe $\mathcal{G} = (\pi, L, \mu)$. If we denote by $M_\mathfrak{V}$ the disjoint union of all the open sets V_i, the sections s_i patch together to a smooth map $s : M_\mathfrak{V} \to Y$ sending a point $x \in V_i$ to $s_i(x) \in Y$. Note that there are induced maps $M_\mathfrak{V}^{[k]} \to Y^{[k]}$ on fibre products (all denoted by s in order to simplify the notation) where $M_{\mathfrak{V}}^{[k]}$ is the disjoint union of all k-fold intersections of open sets V_i. Now we pull back the line bundle L along s to a line bundle over $M_{\mathfrak{V}}^{[2]}$, and the isomorphism μ to an isomorphism of line bundles over $M_{\mathfrak{V}}^{[3]}$. For a choice $\sigma_{ij} : V_i \cap V_j \to s^*L$ of local sections into the pullback line bundle, we obtain smooth functions

$$g_{ijk} : V_i \cap V_j \cap V_k \to \mathbb{C}^\times \tag{1.7}$$

by

$$s^*\mu(\sigma_{ij} \otimes \sigma_{jk}) = g_{ijk} \cdot \sigma_{ik}. \tag{1.8}$$

The associativity condition (1.6) leads to the equality

$$g_{ijk} \cdot g_{ik\ell} = g_{ij\ell} \cdot g_{jk\ell} \tag{1.9}$$

of functions on four-fold intersections $V_i \cap V_j \cap V_k \cap V_\ell$. In other words, the element $g = (g_{ijk}) \in \check{C}^2(\mathfrak{V}, \mathbb{C}_M^\times)$ is a cocycle and defines a class in the Čech cohomology group $\check{\mathrm{H}}^2(\mathfrak{V}, \mathbb{C}_M^\times)$. Its image in the cohomology group $\mathrm{H}^3(M, \mathbb{Z})$ is called the *Dixmier-Douady class* $\mathrm{dd}(\mathcal{G})$ of the bundle gerbe $\mathcal{G}$; it is analogous to the Chern class of a line bundle.

To obtain a classification result for isomorphism classes of bundle gerbes, we first have to define morphisms between bundle gerbes. To simplify the notation, we work with the convention that we do not write pullbacks along canonical maps, like in (1.10) and (1.11) below.

Definition 1.2. *A* morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ *between two bundle gerbes* $\mathcal{G}_1 = (\pi_1, L_1, \mu_1)$ *and* $\mathcal{G}_2 = (\pi_2, L_2, \mu_2)$ *is a pair* $\mathcal{A} = (A, \alpha)$ *consisting of a vector bundle* A *over the fibre product* $Z := Y_1 \times_M Y_2$ *(whose surjective submersion to* M *is denoted by* ζ*) and an isomorphism*

$$\alpha : L_1 \otimes \zeta_2^* A \to \zeta_1^* A \otimes L_2 \tag{1.10}$$

of vector bundles over $Z^{[2]}$ *such that the diagram*

$$\begin{tikzcd}
\zeta_{12}^*L_1 \otimes \zeta_{23}^*L_1 \otimes \zeta_3^*A \arrow[rr, "\mu_1\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\zeta_{23}^*\alpha"'] & & \zeta_{13}^*L_1 \otimes \zeta_3^*A \arrow[dd, "\zeta_{13}^*\alpha"] \\
\zeta_{12}^*L_1 \otimes \zeta_2^*A \otimes \zeta_{23}^*L_2 \arrow[d, "\zeta_{12}^*\alpha\otimes\mathrm{id}"'] & & \\
\zeta_1^*A \otimes \zeta_{12}^*L_2 \otimes \zeta_{23}^*L_2 \arrow[rr, "\mathrm{id}\otimes\mu_2"'] & & \zeta_1^*A \otimes \zeta_{13}^*L_2
\end{tikzcd} \tag{1.11}$$

of isomorphisms of vector bundles over $Z^{[3]}$ *is commutative.*

The definition of the composition of two morphisms $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_2 \to \mathcal{G}_3$ is quite involved, and we omit its discussion for the purposes of this article, see [Ste00, Wal07] for more details. Bundle gerbes and their morphisms fit into the structure of a 2-category rather than the one of a category. This becomes obvious when comparing two morphisms $\mathcal{A}$ and $\mathcal{A}'$ between the same bundle gerbes: since $\mathcal{A}$ and $\mathcal{A}'$ consist themselves of vector bundles, the natural way to compare them is a morphism of vector bundles.

Definition 1.3. *Let $\mathcal{A} = (A, \alpha)$ and $\mathcal{A}' = (A', \alpha')$ be two morphisms from $\mathcal{G}_1 = (\pi_1, L_1, \mu_1)$ to $\mathcal{G}_2 = (\pi_2, L_2, \mu_2)$. A* 2-morphism

$$\beta : \mathcal{A} \Rightarrow \mathcal{A}' \tag{1.12}$$

is an isomorphism $\beta : A \to A'$ of vector bundles over Z, which is compatible with the isomorphisms α and α' in the sense that the diagram

$$\begin{tikzcd} L_1 \otimes \zeta_2^* A \arrow[r, "\alpha"] \arrow[d, "1\otimes\zeta_2^*\beta"'] & \zeta_1^* A \otimes L_2 \arrow[d, "\zeta_1^*\beta\otimes 1"] \\ L_1 \otimes \zeta_2^* A' \arrow[r, "\alpha'"] & \zeta_1^* A' \otimes L_2 \end{tikzcd} \tag{1.13}$$

of isomorphisms of vector bundles over $Z^{[2]}$ is commutative.

The 2-categorical setup is also the appropriate context to address the question which of the morphisms between two bundle gerbes are invertible.

Proposition 1.4 ([Wal07])**.** *A morphism $\mathcal{A} = (A, \alpha)$ is invertible if and only if the vector bundle A is of rank 1.*

So, the invertible morphisms in the 2-category are the so-called stable isomorphisms from [MS00], Section 3. Let us now return to the relation between bundle gerbes and the cohomology group $\mathrm{H}^3(M, \mathbb{Z})$. Let g and g' be the cocycles extracted from two bundle gerbes $\mathcal{G}$ and $\mathcal{G}'$ over M with respect to the same cover $\mathfrak{V}$ of M and sections $s_i : V_i \to Y$ and $s_i' : V_i \to Y'$. Now let $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ be an isomorphism, $\mathcal{A} = (A, \alpha)$. Let $t : M_{\mathfrak{V}} \to Z := Y \times_M Y'$ be the map sending a point $x \in V_i$ to the pair $(s_i(x), s_i'(x))$. Using t we pull back the line bundle A and choose non-zero sections $\sigma_i : V_i \to A$. Then we obtain smooth functions $h_{ij} : V_i \cap V_j \to \mathbb{C}^\times$ by

$$\alpha(\sigma_i \otimes \zeta_2^* \sigma_{ij}) = h_{ij} \cdot (\zeta_1^* \sigma_{ij} \otimes \sigma_i'). \tag{1.14}$$

The compatibility condition (1.11) between α and the isomorphisms μ and μ' of the bundle gerbes leads to the equation

$$h_{ik} g_{ijk} = g'_{ijk} h_{ij} h_{jk}, \tag{1.15}$$

equivalently, in terms of the Čech coboundary operator, $g = g' \cdot \delta h$. This means that the Dixmier-Douady classes $[g]$ and $[g']$ of the isomorphic bundle gerbes $\mathcal{G}$ and $\mathcal{G}'$ are equal. This is the main ingredient for the following classification result.

Theorem 1.5 ([MS00]). *The Dixmier-Douady class defines a bijection between the set of isomorphism classes of bundle gerbes and the cohomology group* $\mathrm{H}^3(M, \mathbb{Z})$.

In particular, consider $M = G$ a compact, simple, connected and simply-connected Lie group. There is a canonical identification $\mathrm{H}^3(G, \mathbb{Z}) = \mathbb{Z}$, so that we have a canonical sequence of bundle gerbes associated to G. In Section 3, we give a geometric construction of these bundle gerbes.

The 2-category of bundle gerbes admits several additional structures such as pullbacks, tensor products and duals. All these structures are compatible with the Dixmier-Douady class:

$$\mathrm{dd}(\mathcal{G}_1 \otimes \mathcal{G}_2) = \mathrm{dd}(\mathcal{G}_1) + \mathrm{dd}(\mathcal{G}_2) \quad , \quad \mathrm{dd}(\mathcal{G}^*) = -\mathrm{dd}(\mathcal{G}) \tag{1.16}$$

and, for a smooth map $f : X \to M$,

$$\mathrm{dd}(f^*\mathcal{G}) = f^*\mathrm{dd}(\mathcal{G}). \tag{1.17}$$

To close, let us construct a bundle gerbe whose Dixmier-Douady class vanishes, representing the neutral element in $\mathrm{H}^3(M, \mathbb{Z})$. For this purpose, consider the identity $\mathrm{id}_M : M \to M$ as the surjective submersion and the trivial line bundle $M \times \mathbb{C}$ over M. Now, the isomorphism μ can be chosen to be the identity $\mathrm{id}_{\mathbb{C}}$, so that $\mathcal{I} := (\mathrm{id}_M, M \times \mathbb{C}, \mathrm{id}_{\mathbb{C}})$ is a bundle gerbe. It is easy to verify that its Dixmier-Douady class vanishes, $\mathrm{dd}(\mathcal{I}) = 0$.

2 Connections on Bundle Gerbes and Holonomy

Before we discuss more examples of bundle gerbes in Section 3 we introduce several additional structures on bundle gerbes and the appropriate cohomology theory for their classification.

Again, we first review similar additional structures for complex line bundles. One can equip every line bundle with a hermitian metric and a (unitary) connection ∇. Additionally to the transition function $g_{ij} : V_i \cap V_j \to \mathbb{C}^\times$, which now can be determined such that it takes values in $U(1)$, we obtain local connection 1-forms $A_i \in \Omega(V_i)$ by writing the covariant derivatives of the sections $s_i : V_i \to L$ as $\nabla s_i = A_i \otimes s_i$. The Leibniz rule implies the equality

$$A_j - A_i - \mathrm{dlog}(g_{ij}) = 0. \tag{2.1}$$

As we shall see next, the local data (g, A) define a cocycle in the Deligne complex $\mathcal{D}^k_{\mathfrak{V}}(n)$ for $n = 1$. The first cochain groups of this complex are

$$\begin{aligned} \mathcal{D}^0_{\mathfrak{V}}(1) &= \check{C}^0(\mathfrak{V}, U(1)_M) && (2.2)\\ \mathcal{D}^1_{\mathfrak{V}}(1) &= \check{C}^1(\mathfrak{V}, U(1)_M) \oplus \check{C}^0(\mathfrak{V}, \Omega^1_M) && (2.3)\\ \mathcal{D}^2_{\mathfrak{V}}(1) &= \check{C}^2(\mathfrak{V}, U(1)_M) \oplus \check{C}^1(\mathfrak{V}, \Omega^1_M), && (2.4) \end{aligned}$$

and its differential is given by

$$\mathrm{D} : \mathcal{D}^1_{\mathfrak{V}}(1) \to \mathcal{D}^2_{\mathfrak{V}}(1) : (g, A) \mapsto (\delta g, \delta A - \mathrm{dlog}(g)). \tag{2.5}$$

In general, the Deligne complex $\mathcal{D}^k_{\mathfrak{V}}(n)$ is the total complex of the Čech-Deligne double complex, whose rows are Čech cochain groups of the sheaf complex

$$0 \longrightarrow U(1)_M \xrightarrow{\mathrm{dlog}} \Omega^1_M \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n_M, \tag{2.6}$$

truncated in degree n, and the columns are the usual Čech complexes associated to the sheaves $U(1)_M$ and Ω^k_M. The truncation is necessary to describe not just flat objects.

So we can regard the local data (g, A) of a hermitian line bundle with connection as an element of $\mathcal{D}^1_{\mathfrak{V}}(1)$. By equations (1.2) and (2.1) it satisfies $\mathrm{D}(g, A) = 0$, and thus defines a class in the first cohomology group of the Deligne complex, denoted by $\mathrm{H}^1(M, \mathcal{D}(1))$. One can show that this defines a bijection

$$\mathrm{Pic}^\nabla(M) \cong \mathrm{H}^1(M, \mathcal{D}(1)) \tag{2.7}$$

between the group $\mathrm{Pic}^\nabla(M)$ of isomorphism classes of hermitian line bundles with connection and Deligne cohomology [Bry93].

Now we discuss bundle gerbes with similar additional structures.

Definition 2.1. *Let $\mathcal{G} = (\pi, L, \mu)$ be a bundle gerbe over M. It can be equipped successively with the following additional structures.*

(a) A hermitian structure *on $\mathcal{G}$ is a hermitian metric on the line bundle L, such that the isomorphism μ is an isometry.*

(b) A connection *on the hermitian bundle gerbe $\mathcal{G}$ is a connection ∇ on the hermitian line bundle L, such that the isomorphism μ respects the induced connections.*

(c) *A* curving *of a connection* ∇ *on the hermitian bundle gerbe* $\mathcal{G}$ *is a 2-form* $C \in \Omega^2(Y)$*, such that*

$$\pi_2^*C - \pi_1^*C = \mathrm{curv}(\nabla), \tag{2.8}$$

where $\mathrm{curv}(\nabla) \in \Omega^2(Y^{[2]})$ *is the curvature of the connection* ∇ *on* L*.*

Every bundle gerbe admits all of these additional structures [Mur96]. Because the applications of bundle gerbes we discuss later require all this additional structure, we work from now only with hermitian bundle gerbes with connection and curving.

An important feature of those gerbes is that they provide a notion of curvature. To see this, consider the derivative of equation (2.8): since the curvature of the connection ∇ is a closed form, we obtain $\pi_1^*\mathrm{d}C = \pi_2^*\mathrm{d}C$, which means that $\mathrm{d}C$ is the pullback of a 3-form on M,

$$\mathrm{d}C = \pi^*H. \tag{2.9}$$

This 3-form H is uniquely determined and closed; it is called the curvature of the curving of the connection ∇ on the hermitian bundle gerbe $\mathcal{G}$, and denoted by $\mathrm{curv}(C) := H$.

To have a simple example of such additional structures, the bundle gerbe $\mathcal{I} = (\mathrm{id}_M, M\times\mathbb{C}, \mathrm{id}_\mathbb{C})$ with vanishing Dixmier-Douady class becomes a hermitian bundle gerbe with connection by taking the canonical hermitian metric and the trivial flat connection $\nabla := \mathrm{d}$ on the trivial line bundle $M\times\mathbb{C}$. Note that now any 2-form $C \in \Omega^2(M)$ satisfies the condition (2.8) for a curving on I, because ∇ is flat and $\pi_1 = \pi_2 = \mathrm{id}_M$. So we have a canonical hermitian bundle gerbe $\mathcal{I}_C := (\mathrm{id}_M, M\times\mathbb{C}, \mathrm{id}_\mathbb{C}, \mathrm{d}, C)$ with connection and curving for every 2-form $C \in \Omega^2(M)$. The curvature of its curving C is $\mathrm{curv}(C) = \mathrm{d}C$.

Now we extend the cohomological classification from bundle gerbes to hermitian bundle gerbes $\mathcal{G} = (\pi, L, \mu, \nabla, C)$ with connection and curving, using a good open cover $\mathfrak{V}$ of M. Recall that we have extracted a transition function $g_{ijk} : V_i \cap V_j \cap V_k \to \mathbb{C}^\times$ using a choice of sections $s_i : V_i \to Y$ and $\sigma_i : V_i \to s^*L$, defining a representative $g \in \check{C}^2(\mathfrak{V}, \mathbb{C}^\times)$ of the Dixmier-Douady class of the bundle gerbe. Now that L is a hermitian line bundle, we choose the sections σ_i such that g_{ijk} is $U(1)$-valued. Furthermore, by using the connection $s^*\nabla$ on s^*L, we obtain local connection 1-forms $A_{ij} \in \Omega^1(V_i \cap V_j)$. The condition that μ preserves connections implies

$$A_{jk} - A_{ik} + A_{ij} + \mathrm{dlog}(g_{ijk}) = 0. \tag{2.10}$$

Finally, the curving C gives rise to local 2-forms $B_i := s_i^*C \in \Omega^2(V_i)$, and the compatibility (2.8) with the curvature of ∇ implies

$$B_j - B_i - \mathrm{d}A_{ij} = 0. \tag{2.11}$$

Note that the curvature $\mathrm{curv}(C)$ of the curving can be computed from the local data by $H|_{V_i} = \mathrm{d}B_i$. By (2.11), this gives indeed a globally defined 3-form.

In terms of Čech cohomology, we have extracted an element (g, A, B) in the second cochain group of the Deligne complex in degree 2,

$$\mathcal{D}^2_{\mathfrak{V}}(2) = \check{C}^2(\mathfrak{V}, U(1)) \oplus \check{C}^1(\mathfrak{V}, \Omega^1) \oplus \check{C}^0(\mathfrak{V}, \Omega^2). \tag{2.12}$$

The differential is here

$$\mathrm{D} : \mathcal{D}^2_{\mathfrak{V}}(2) \to \mathcal{D}^3_{\mathfrak{V}}(2) : (g, A, B) \mapsto (\delta g, \delta A + \mathrm{dlog}(g), \delta B - \mathrm{d}A), \tag{2.13}$$

so that the cocycle condition (2.8) and equations (2.10) and (2.11) imply the cocycle condition $\mathrm{D}(g, A, B) = 0$. This way, a hermitian bundle gerbe with connection and curving defines a class in the cohomology of the Deligne complex in degree 2, $\mathrm{H}^2(M, \mathcal{D}(2))$. As an exercise, the reader may compute the Deligne class of the canonical hermitian bundle gerbe $\mathcal{I}_C$ with connection and curving associated to any 2-form $C \in \Omega^2(M)$.

Note that both the Deligne cochain groups $\mathcal{D}^1_{\mathfrak{V}}(1)$ from (2.2) and $\mathcal{D}^2_{\mathfrak{V}}(2)$ from (2.12) have projections on the first summand, which commute with the Deligne differential and the Čech coboundary operator, so that we get induced (surjective) group homomorphisms [Bry93]

$$\mathrm{H}^k(M, \mathcal{D}(k)) \to \mathrm{H}^{k+1}(M, \mathbb{Z}). \tag{2.14}$$

This way we obtain the Chern class and the Dixmier-Douady class of a hermitian line bundle with connection and of a hermitian bundle gerbe with connection and curving, respectively, from their Deligne classes. Its surjectivity means that Deligne cohomology refines the ordinary singular cohomology with $\mathbb{Z}$ coefficients.

To achieve a classification result for hermitian bundle gerbes with connection and curving similar to the result (2.7) for hermitian line bundles with connection, we have to adapt the definition of a morphism between bundle gerbes to morphisms between hermitian bundle gerbes with connection and curving.

Definition 2.2. *A morphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ between two hermitian bundle gerbes $\mathcal{G} = (\pi, L, \mu, \nabla, C)$ and $\mathcal{G}' = (\pi', L', \mu', \nabla', C')$ with connection and curving is a morphism $\mathcal{A} = (A, \alpha)$ as in Definition 1.2, together with a connection $\blacktriangledown$ on the vector bundle A, such that*

1. *the isomorphism α respects connections.*

2. the curvature of $\blacktriangledown$ *is related to the curvings by*

$$\frac{1}{n}\mathrm{tr}(\mathrm{curv}(\blacktriangledown)) = C' - C. \tag{2.15}$$

As in Proposition 1.4, a morphism is invertible precisely if the vector bundle is of rank 1. It is again an easy exercise to check, that an isomorphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ of hermitian bundle gerbes with connection and curving with Deligne cocycles (g, A, B) and (g', A', B') respectively, gives rise to a Deligne cochain $(h, W) \in \mathcal{D}^1_{\mathfrak{V}}(2)$ which satisfies

$$(g', A', B') = (g, A, B) + \mathrm{D}(h, W). \tag{2.16}$$

This shows that isomorphism classes of hermitian bundle gerbes with connection and curving have well-defined Deligne classes.

Theorem 2.3 ([MS00])**.** *Isomorphism classes of hermitian bundle gerbes with connection and curving correspond bijectively to the Deligne cohomology group* $\mathrm{H}^2(M, \mathcal{D}(2))$*.*

Particular examples of morphisms are trivializations: a *trivialization* of a hermitian bundle gerbe $\mathcal{G}$ with connection and curving is an isomorphism

$$\mathcal{T} : \mathcal{G} \to \mathcal{I}_\rho \tag{2.17}$$

for some 2-form $\rho \in \Omega^2(M)$. In terms of local data, this isomorphism corresponds to a Deligne cochain $(h, W) \in \mathcal{D}^1_{\mathfrak{V}}(2)$ with

$$(1, 0, \rho) = (g, A, B) + \mathrm{D}(h, W), \tag{2.18}$$

if (g, A, B) is local data of the bundle gerbe $\mathcal{G}$. In particular, the existence of a trivialization implies that the Dixmier-Douady class of $\mathcal{G}$ vanishes. Many assertions about bundle gerbes and their isomorphisms can be proven either in a geometrical way or by computations in Deligne cohomology. As an illustration, we shall prove the following important

Lemma 2.4. *Two trivializations*

$$\mathcal{T}_1 : \mathcal{G} \to \mathcal{I}_{\rho_1} \quad \textit{and} \quad \mathcal{T}_2 : \mathcal{G} \to \mathcal{I}_{\rho_2} \tag{2.19}$$

of the same hermitian bundle gerbe $\mathcal{G}$ *over* M *with connection and curving determine a hermitian line bundle over* M *with connection of curvature* $\rho_2 - \rho_1$*.*

Proof 1 (2-categorical). Using the features of the 2-category of bundle gerbes, we can give a quite conceptual proof: by taking the inverse and composition (which we have not explained in this article, but can be found in [Wal07]), we obtain an isomorphism

$$\mathcal{T}_2 \circ \mathcal{T}_1^{-1} : \mathcal{I}_{\rho_1} \to \mathcal{I}_{\rho_2} \tag{2.20}$$

of trivial bundle gerbes. From the definitions of isomorphisms and trivial bundle gerbes it follows immediately, that $\mathcal{T}_2 \circ \mathcal{T}_1^{-1}$ is a line bundle with curvature $\rho_2 - \rho_1$. □

Proof 2 (geometrical). The two isomorphisms $\mathcal{T}_i = (T_i, \tau_i, \blacktriangledown_i)$ consist of hermitian line bundles T_i over $Z := Y \times_M M \cong Y$, connections $\blacktriangledown_i$ of curvature $\mathrm{curv}(\blacktriangledown_i) = \pi^*\rho_i - C$, and isomorphisms $\tau_i : L \otimes \pi_2^* T_i \to \pi_1^* T_i$ of hermitian line bundles respecting the connections. They can be composed to an isomorphism

$$\tau_2^{-1} \otimes \tau_1^* : \pi_1^*(T_2 \otimes T_1^*) \to \pi_2^*(T_2 \otimes T_1^*) \tag{2.21}$$

of hermitian line bundles with connection over $Y^{[2]}$, which satisfies the obvious cocycle condition over $Y^{[3]}$, due to the commutative diagram (1.11) for the τ_i. This is the condition for the hermitian line bundle $T_2 \otimes T_1^*$ with connection $\blacktriangledown_2 - \blacktriangledown_1$ to descend from Y to M. The descent line bundle has the claimed curvature. □

Proof 3 (cohomological). If the isomorphisms $\mathcal{T}_1$ and $\mathcal{T}_2$ have local data (h_1, W_1) and (h_2, W_2) respectively, both satisfying equation (2.18), their difference satisfies

$$\mathrm{D}(h_2 \cdot h_1^{-1}, W_2 - W_1) = 0, \tag{2.22}$$

which is the cocycle condition for a hermitian line bundle with connection of curvature $\mathrm{d}(W_2 - W_1) = \mathrm{curv}(\blacktriangledown_2) - \mathrm{curv}(\blacktriangledown_1) = \rho_2 - \rho_1$. □

One of the most important aspects of the theory of hermitian bundle gerbes with connection and curving is that they provide a notion of holonomy around surfaces.

Definition 2.5 ([CJM02])**.** *Let $\mathcal{G}$ be a hermitian bundle gerbe with connection and curving over M. For a closed oriented surface Σ and a smooth map $\phi : \Sigma \to M$, let*

$$\mathcal{T} : \phi^*\mathcal{G} \to \mathcal{I}_\rho \tag{2.23}$$

be a trivialization of the pullback of $\mathcal{G}$ along ϕ. Then we define

$$\mathrm{hol}_{\mathcal{G}}(\phi) := \exp\left(\int_\Sigma \rho\right) \tag{2.24}$$

to be the holonomy *of $\mathcal{G}$ around $\phi : \Sigma \to M$.*

Note that the Dixmier-Douady class of $\phi^*\mathcal{G}$ vanishes by dimensional reasons, so that the existence of the trivialization $\mathcal{T}$ is guaranteed. However, it is not unique, and different trivializations may have different 2-forms ρ. Now, by Lemma 2.4 we know that the difference $\rho_2 - \rho_1$ between two such 2-forms is the curvature of some line bundle over M, in particular: it is a closed form with integral class. Then, the calculation

$$\exp\left(\int_\Sigma \rho_2\right) = \exp\left(\int_\Sigma \rho_2 - \rho_1\right) \cdot \exp\left(\int_\Sigma \rho_1\right) = \exp\left(\int_\Sigma \rho_1\right) \tag{2.25}$$

shows that the definition $\mathrm{hol}_{\mathcal{G}}(\phi)$ is independent of the choice of the trivialization.

There also exist expressions for the holonomy $\mathrm{hol}_{\mathcal{G}}(\phi)$ in terms of local data of $\mathcal{G}$. They generalize the local formulae for the holonomy of hermitian line bundles with connection. In Section 6 we describe the applications of holonomy of bundle gerbes in conformal field theory.

3 Bundle Gerbes over compact Lie Groups

Now that we have introduced bundle gerbes as geometric objects over arbitrary manifolds, we specialize to manifolds which are Lie groups. We describe in this section, how the Lie group structure allows constructions of examples of bundle gerbes. First constructions of gerbes over different types of compact Lie groups (in realizations different from bundle gerbes) can be found in [Cha98, Bry]. Bundle gerbes (with connection and curving) have been constructed in [GR02, Mei02, GR03].

As already mentioned before, for a compact, simple, connected and simply-connected Lie group G have $\mathrm{H}^3(G,\mathbb{Z}) = \mathbb{Z}$. The (up to isomorphism unique) bundle gerbe over G whose Dixmier-Douady class corresponds to $1 \in \mathbb{Z}$ is called the basic bundle gerbe, and denoted by $\mathcal{G}_0$. The bundle gerbe with Dixmier-Douady class $k \in \mathbb{Z}$ can then be obtained from $\mathcal{G}_0$ or $\mathcal{G}_0^*$ by a k-fold tensor product.

For the purposes of this article, we will restrict ourselves to the construction given in [GR02], by which we obtain the basic bundle gerbe over the special unitary groups $\mathrm{SU}(n)$ and the symplectic groups $\mathrm{Sp}(n)$. First we consider a general compact, simple and simply-connected Lie group G with Lie algebra $\mathfrak{g}$. We choose a maximal torus T with Lie algebra $\mathfrak{t}$ of rank r. We further choose a set of simple roots $\alpha_1, ..., \alpha_r$ and denote the associated positive Weyl chamber by C. Let α_0 the highest root and let

$$\mathfrak{A} := \{\xi \in C \mid \alpha_0(\xi) \leq 1\}. \tag{3.1}$$

be the fundamental alcove. It is bounded by the hyperplanes H_i perpendicular to the roots α_i, and the additional hyperplane H_0 consisting of elements $\xi \in \mathfrak{t}$ with $\alpha_0(\xi) = 1$. So it is an r-dimensional simplex in $\mathfrak{t}$ with vertices $\mu_0 ..., \mu_r$ determined by the condition that $\mu_i \in H_j$ for all $j \neq i$.

For simple and simply connected groups, the fundamental alcove parameterizes conjugacy classes of G in the sense that each conjugacy class contains a unique point $\exp \xi$ with $\xi \in \mathfrak{A}$. This defines a continuous map

$$q : G \to \mathfrak{A}. \tag{3.2}$$

Let $\mathfrak{A}_i$ be the open complement of the face opposite to the vertex μ_i in $\mathfrak{A}$, and consider the open sets $U_i := q^{-1}(\mathfrak{A}_i)$. More generally, for any subset $I \subset \underline{r} = \{0, ..., r\}$ denote by U_I the intersection of all U_i with $i \in I$, and similarly by $\mathfrak{A}_I$ the intersection of all $\mathfrak{A}_i$ with $i \in I$. Of course $U_I = q^{-1}(\mathfrak{A}_I)$. We use the open sets U_i to construct the surjective submersion

$$Y := \bigsqcup_{i \in \underline{r}} U_i \quad \text{and} \quad \pi : Y \to G : (i, x) \mapsto x. \tag{3.3}$$

Note that the k-fold fibre products are disjoint unions of intersections

$$Y^{[k]} = \bigsqcup_{|I|=k} U_I. \tag{3.4}$$

The surjective submersion $\pi : Y \to M$ will serve as the first ingredient of the bundle gerbe we want to construct. To construct the line bundle L over $Y^{[2]}$ we show next that $Y^{[2]}$ projects onto a union of coadjoint orbits.

For any $I \subset \underline{r}$, all group elements $\exp \xi$ with ξ in the open face spanned by the vertices μ_i with $i \in I$ have the same centralizer G_I. For any inclusion $I \subset J$ it follows that $G_J \subset G_I$; for $I = \underline{r}$ we obtain $G_{\underline{r}} = T$. Let S_I be the orbit of $\exp \mathfrak{A}_I \subset T$ under the conjugation with G_I. Consider the set $G \times_{G_I} S_I$ consisting of equivalence classes of pairs $(g, s) \in G \times S_I$ under the equivalence relation $(g, s) \sim (gh, h^{-1}sh)$ for $h \in G_I$. We have the canonical projection $\rho_I : G \times_{G_I} S_I \to G/G_I$ and a smooth map

$$u_I : G \times_{G_I} S_I \to U_I \tag{3.5}$$

which sends a representative (g, s) to $gsg^{-1} \in G$. This is well-defined on equivalence classes, and for $h \in G_I$ and $\xi \in \mathfrak{A}_I$ with $s := h \exp \xi\, h^{-1}$ we find $q(gsg^{-1}) = \xi$ and hence $gsg^{-1} \in U_I$. The map u_I is even a diffeomorphism: for $g \in U_I$ let $\xi := q(g) \in \mathfrak{A}_I$ and $h \in G$ such that $g = h \exp \xi\, h^{-1}$. Then, the inverse sends g to the equivalence class of $(h, \exp \xi)$.

Since G_{ij} fixes the difference $\mu_{ij} := \mu_j - \mu_i$, the quotient G/G_{ij} projects on the coadjoint orbit $\mathcal{O}_{ij}$ of μ_{ij} in $\mathfrak{g}^*$. Now we specialize our construction to the case that μ_{ij} is a weight. This is the case for $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, where the vertices of the fundamental alcove are contained in the weight lattice. For μ_{ij} being a weight there is a canonical line bundle L_{ij} over the coadjoint orbit $\mathcal{O}_{ij}$. Let us recall the construction of the pullback of this line bundle to G/G_{ij}. Let $\chi_{ij} : G_{ij} \to U(1)$ be the character associated to the weight μ_{ij}. Then the line bundle is the bundle associated to the principal G_{ij}-bundle G over G/G_{ij}, namely

$$L_{ij} := G \times_{G_{ij}} \mathbb{C}. \tag{3.6}$$

The line bundles L_{ij} over G/G_{ij} can be pulled back along

$$U_i \cap U_j \xrightarrow{\sim} G \times_{G_{ij}} S_{ij} \longrightarrow G/G_{ij} \tag{3.7}$$

to line bundles over $U_i \cap U_j$, and their disjoint union gives a line bundle L over $Y^{[2]}$.

To close the definition of a bundle gerbe over G, it remains to construct the isomorphism μ of line bundles over $Y^{[3]}$, i.e. we need an isomorphism

$$\mu : \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L \tag{3.8}$$

of line bundles over $Y^{[3]}$. Over each connected component $U_i \cap U_j \cap U_k$, this can be chosen as the pullback of the canonical identification

$$L_{ij} \otimes L_{jk} \cong L_{ik} \tag{3.9}$$

of line bundles over G/G_{ijk}, which comes from the coincidence $\mu_{ik} = \mu_{ij} + \mu_{jk}$ of the weights from which the line bundles are constructed. This identification is obviously associative.

In order to calculate the Dixmier-Douady class of the bundle gerbe we just have constructed, we choose a connection and a curving. This procedure is analogous to the calculation of Chern classes of complex vector bundles using a choice of a hermitian metric and a connection, see [MS76]. Recall that this is based on the fact that the de Rham cohomology class of the curvature of a hermitian line bundle with connection equals the image of its

Chern class under the induced map $\iota^* : \mathrm{H}^2(M,\mathbb{Z}) \to \mathrm{H}^2(M,\mathbb{R})$. The same holds for a bundle gerbe $\mathcal{G} = (\pi, L, \mu, \nabla, C)$:

$$[\mathrm{curv}(C)] = \iota^* \mathrm{dd}(\mathcal{G}), \tag{3.10}$$

which can be proven by a zigzag-argument in the Čech-Deligne double complex [Bry93].

Note that – in a certain normalization of the Killing form [PS86] – the bi-invariant closed 3-form

$$H := \frac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(G) \tag{3.11}$$

is a generator of the cohomology group $\mathrm{H}^3(G,\mathbb{Z})$. Here θ is the left invariant Maurer-Cartan form, a $\mathfrak{g}$-valued 1-form on G; the 3-form H takes at the unit element the value $H_1(X,Y,Z) = \langle X, [Y,Z] \rangle$ on elements $X, Y, Z \in \mathfrak{g}$. Our goal is now to define a connection and a curving with curvature H.

First note that the line bundle L_{ij} from (3.6) inherits a hermitian metric from the standard metric on $\mathbb{C}$, and that the isomorphism μ is an isometry. The line bundle L_{ij} can also be equipped with a connection: we consider the 1-form $A_{ij} := \langle \mu_{ij}, \theta \rangle$ on the total space of the principal G_{ij}-bundle G. It induces a connection on the associated line bundle L_{ij} because μ_{ij} is preserved under the action of G_{ij}. This way the bundle gerbe is a hermitian bundle gerbe with connection.

To define a curving for this bundle gerbe, i.e. a 2-form $C \in \Omega^2(Y)$, we use the fact that the linear retraction of $\mathfrak{A}_i$ to the vertex μ_i lifts to a smooth retraction of U_i to the conjugacy class $\mathcal{C}_{\mu_i}$ [Mei02],

$$r_i : U_i \times [0,1] \to U_i. \tag{3.12}$$

On the conjugacy class $\mathcal{C}_{\mu_i}$, the 3-form H becomes exact, $\iota_i^* H = \mathrm{d}\omega_{\mu_i}$ where $\iota_i : \mathcal{C}_{\mu_i} \to G$ is the inclusion, and

$$\omega_{\mu_i} := \left\langle \iota_i^* \theta \wedge \frac{\mathrm{Ad}^{-1} + \mathrm{id}_{\mathfrak{g}}}{\mathrm{Ad}^{-1} - \mathrm{id}_{\mathfrak{g}}} \iota_i^* \theta \right\rangle \in \Omega^2(\mathcal{C}_{\mu_i}) \tag{3.13}$$

is an invariant 2-form on the conjugacy class [BRS01]. Here the notation is to be understood as follows: at some element $h \in \mathcal{C}_{\mu_i}$, the 2-form ω_{μ_i} is obtained by considering the Maurer-Cartan forms in h and taking the inverse of the adjoint action $(\mathrm{Ad}_h)^{-1} : \mathfrak{g} \to \mathfrak{g}$. The endomorphism $(\mathrm{Ad}_h)^{-1} - \mathrm{id}_{\mathfrak{g}}$ becomes invertible when restricted to the image of $\iota_i^* \theta$ so that the fraction makes sense.

By pullback along r_i and fibre integration, one obtains a 2-form $C_i \in \Omega^2(U_i)$ with

$$H|_{U_i} = \mathrm{d}C_i. \tag{3.14}$$

One can now show that $C_j - C_i = \langle \mu_{ij}, \mathrm{d}\theta \rangle$ on $U_i \cap U_j$, which is the condition on the curving. So, our construction realizes, for $G = \mathrm{SU}(n)$ and $G = \mathrm{Sp}(n)$ a hermitian bundle gerbe with connection and curving with Dixmier-Douady class $1 \in \mathbb{Z}$, the basic bundle gerbe.

For the other compact, simple, connected and simply-connected Lie groups, one can find an integer k_0 for which the vertices of $k_0\mathfrak{A}$ are weights. Using the weights $k_0\mu_{ij}$ in the construction of the line bundles L_{ij}, and the 2-forms $k_0 C_i$ in the definition of the curving, we obtain bundle gerbes with Dixmier-Douady class $k_0 \in \mathbb{Z}$ [Mei02]. The smallest such integer k_0 is tabulated in [Bou68]:

G	SU(n)	Spin(n)	Sp($2n$)	E_6	E_7	E_8	F_4	G_2
k_0	1	2	1	3	12	60	6	2

The construction of the basic bundle gerbes on the groups with $k_0 > 1$ requires more advanced techniques [Mei02, GR03]. Here it becomes in particular important that the definition of a bundle gerbes admits $\pi : Y \to G$ to be a surjective submersion, which is more general than just an open cover of G.

Starting from a bundle gerbe $\mathcal{G}$ on a simply-connected Lie group G, we now describe a method to obtain bundle gerbes on the non simply-connected Lie groups G/Z which are quotients of G by a subgroup Z of the center of G. More generally, let Γ be a finite group acting on a manifold M by diffeomorphisms.

Definition 3.1. *A* Γ-equivariant structure $(\mathcal{A}_\gamma, \varphi_{\gamma_1,\gamma_2})$ *on a bundle gerbe* $\mathcal{G}$ *over* M *consists of isomorphisms*

$$\mathcal{A}_\gamma : \mathcal{G} \to \gamma^*\mathcal{G} \tag{3.15}$$

for each $\gamma \in \Gamma$ *and of 2-isomorphisms*

$$\varphi_{\gamma_1,\gamma_2} : \gamma_1^*\mathcal{A}_{\gamma_2} \circ \mathcal{A}_{\gamma_1} \Longrightarrow \mathcal{A}_{\gamma_2\gamma_1} \tag{3.16}$$

for each pair $\gamma_1, \gamma_2 \in Z$*, such that the diagram*

$$\begin{tikzcd} \gamma_1^*\gamma_2^*\mathcal{A}_{\gamma_3} \circ \gamma_1^*\mathcal{A}_{\gamma_2} \circ \mathcal{A}_{\gamma_1} \arrow[r, Rightarrow, "\mathrm{id}\circ\varphi_{\gamma_1,\gamma_2}"] \arrow[d, Rightarrow, "\gamma_1^*\varphi_{\gamma_2,\gamma_3}\circ\mathrm{id}"'] & \gamma_1^*\gamma_2^*\mathcal{A}_{\gamma_3} \circ \mathcal{A}_{\gamma_2\gamma_1} \arrow[d, Rightarrow, "\varphi_{\gamma_2\gamma_1,\gamma_3}"] \\ \gamma_1^*\mathcal{A}_{\gamma_3\gamma_2} \circ \mathcal{A}_{\gamma_1} \arrow[r, Rightarrow, "\varphi_{\gamma_1,\gamma_3\gamma_2}"'] & \mathcal{A}_{\gamma_3\gamma_2\gamma_1} \end{tikzcd} \tag{3.17}$$

of 2-isomorphisms is commutative.

Not every bundle gerbe $\mathcal{G}$ over M admits a Γ-equivariant structure, and if it does, it may not be unique. To obtain obstructions and classifications we use the cohomological language. For this purpose, we impose the structure of a Γ-module on the Deligne cochain groups $\mathcal{D}^n_{\mathfrak{V}}(k)$ [GSW]. We assume the existence of a good open cover $\mathfrak{V} = \{V_i\}_{i\in I}$ of M which is compatible with the group action in the sense that there is an induced action of Γ on the index set I such that $\gamma(V_i) = V_{\gamma i}$. For example, the open sets U_i we have used in the construction of the basic bundle gerbe satisfy this condition. Then Γ acts by pullback on the cochain groups $\mathcal{D}^n_{\mathfrak{V}}(k)$.

For each Γ-module W, one can build the usual group cohomology complex, consisting of cochain groups $C^p_\Gamma(W) := \mathrm{Map}(\Gamma^{p+1}, W)$ and the usual coboundary operator

$$\mathrm{d} : C^{k-1}_\Gamma(W) \to C^k_\Gamma(W). \tag{3.18}$$

A simple key observation is that the coboundary operator d and the Deligne differential D commute so that we have a double complex with cochain groups $C^p_\Gamma(\mathcal{D}^k_{\mathfrak{V}}(n))$ in degree (p,k). We denote the total cohomology of this double complex by $\mathrm{H}^q_\Gamma(M, \mathcal{D}(n))$. In particular, note that we have a natural group homomorphism

$$\mathrm{eq} : \mathrm{H}^2(M, \mathcal{D}(2)) \to \mathrm{H}^2_\Gamma(M, \mathcal{D}(2)) : \xi \mapsto (\gamma^*\xi - \xi, 0, 0) \tag{3.19}$$

that includes an ordinary Deligne cohomology class into the cohomology of the total complex we just have defined.

Now let $\mathcal{G}$ be a hermitian bundle gerbe with connection and curving with Deligne class ξ, and let $(\mathcal{A}_z, \varphi_{z_1,z_2})$ be a Γ-equivariant structure on $\mathcal{G}$. For local data $a_\gamma \in C^1_\Gamma(\mathcal{D}^1_{\mathfrak{V}}(2))$ of the isomorphism $\mathcal{A}_\gamma$, i.e.

$$\mathrm{D}a_\gamma = \gamma^*\xi - \xi, \tag{3.20}$$

and local data $b_{\gamma_1,\gamma_2} \in C^2_\Gamma(\mathcal{D}^0_{\mathfrak{V}}(2))$ of the 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$, i.e.

$$\mathrm{D}b_{\gamma_1,\gamma_2} = (\mathrm{d}a)_{\gamma_1,\gamma_2}, \tag{3.21}$$

the commutativity of diagram (3.17) imposes the condition $(\mathrm{d}b)_{\gamma_1,\gamma_2,\gamma_3} = 0$. This means for a bundle gerbe with Deligne class ξ that the class $\mathrm{eq}(\xi) \in \mathrm{H}^2_\Gamma(M, \mathcal{D}(2))$ is the obstruction class for $\mathcal{G}$ to admit Γ-equivariant structures. Furthermore, the cohomology group $\mathrm{H}^1_\Gamma(M, \mathcal{D}(2))$ classifies the inequivalent choices.

In the case of bundle gerbes over a compact, simple, connected and simply-connected Lie group G we consider the action of a subgroup Z of the center of G by multiplication. In this case the relevant cohomology groups reduce to the usual group cohomology of the finite group Z, so that there is an obstruction class in $\mathrm{H}^3_{\mathrm{Grp}}(Z, U(1))$, and the possible Z-equivariant structures are classified by $\mathrm{H}^2_{\mathrm{Grp}}(Z, U(1))$. For the bundle gerbes we have constructed above, all obstruction classes against Z-equivariant structures for all subgroups Z of the center of G can be calculated [GR02, GR03].

Let us now describe how a choice $(\mathcal{A}_z, \varphi_{z_1,z_2})$ of a Γ-equivariant structure on a given bundle gerbe $\mathcal{G} = (\pi, L, \mu, \nabla, C)$ with connection and curving over M defines a quotient bundle gerbe $\overline{\mathcal{G}} = (\overline{\pi}, \overline{L}, \overline{\mu}, \overline{\nabla}, \overline{C})$ over $\overline{M} := M/\Gamma$. Following [GR02], we set $\overline{Y} := Y$ and $\overline{\pi} := p \circ \pi : \overline{Y} \to \overline{M}$, where $p : M \to \overline{M}$ is the projection to the quotient. Note that the fibre products are

$$\overline{Y}^{[2]} = \bigsqcup_{\gamma \in \Gamma} Z^{\gamma} \quad \text{and} \quad \overline{Y}^{[3]} = \bigsqcup_{\gamma_1, \gamma_2 \in \Gamma} Z^{\gamma_1, \gamma_2} \tag{3.22}$$

for the smooth manifolds

$$Z^{\gamma} := Y_{\gamma} \times_M Y \quad \text{and} \quad Z^{\gamma_1, \gamma_2} = Y_{\gamma_1 \gamma_2} \times_M Z^{\gamma_1}, \tag{3.23}$$

where $Y_\gamma := Y$ as manifolds but with projection $\gamma^{-1} \circ \pi$ instead of π. The manifolds $\overline{Y}^{[2]}$ and $\overline{Y}^{[3]}$ have again projections $\overline{\pi}_i := \pi_i$ and $\overline{\pi}_{ij} := \pi_{ij}$ to $\overline{Y}$ and to $\overline{Y}^{[2]}$ respectively. The curving of the quotient bundle gerbe will be $\overline{C} := C \in \Omega^2(\overline{Y})$, and the line bundle $\overline{L} \to \overline{Y}^{[2]}$ will be $L|_{Z^\gamma} := A_\gamma$. Axiom (G1) for the curvature of L is

$$\mathrm{curv}(\overline{L})|_{Z^\gamma} = \mathrm{curv}(A_\gamma) = \pi_2^* C - \pi_1^* C = \overline{\pi}_2^* \overline{C} - \overline{\pi}_1^* \overline{C}, \tag{3.24}$$

and hence satisfied. The isomorphism $\overline{\mu}$ over $\overline{Y}^{[3]}$ is defined by

$$\overline{\mu}|_{Z^{\gamma_1,\gamma_2}} := \varphi_{Z^{\gamma_1,\gamma_2}} \tag{3.25}$$

and its associativity by means of axiom (G2) is nothing but the condition on the isomorphisms $\varphi_{\gamma_1,\gamma_2}$ from Definition 3.1. So we have defined a bundle gerbe $\overline{\mathcal{G}}$ over $\overline{M}$ with connection and curving.

Applying this procedure to the bundle gerbes over the simply connected Lie groups and their equivariant structures, one obtains examples of bundle gerbes over all compact simple connected and simply-connected Lie groups.

4 Structure on Loop Spaces from Bundle Gerbes

We describe a construction of a line bundle over the loop space LM of a manifold M from a given hermitian bundle gerbe over M with connection and curving. This construction is adapted from the one in [Bry93].

For preparation, we have to describe the set of isomorphisms between two fixed hermitian bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ with connection and curving. We denote the set of isomorphism classes of morphisms $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ by $\mathrm{Iso}(\mathcal{G}_1, \mathcal{G}_2)$. The following lemma can easily be shown using the cohomological description.

Lemma 4.1 ([SSW07])**.** *The group* $\mathrm{Pic}_0^{\nabla}(M)$ *of isomorphism classes of flat line bundles with connection over* M *acts freely and transitively on the set* $\mathrm{Iso}(\mathcal{G}_1, \mathcal{G}_2)$.

Now let $LM := C^{\infty}(S^1, M)$ be the free loop space of M, equipped with a smooth manifold structure as described in [Bry93]. Let $\mathcal{G} = (\pi, L, \mu, \nabla, C)$ be a hermitian bundle gerbe on M with connection and curving. The total space of the line bundle we are going to define is, as a set,

$$\mathcal{L} := \bigsqcup_{\gamma \in LM} \mathrm{Iso}(\gamma^*\mathcal{G}, \mathcal{I}_0). \tag{4.1}$$

It comes with the evident projection to LM, and by Lemma 4.1 every fibre is a torsor over the group

$$\mathrm{Pic}_0^{\nabla}(S^1) \cong \mathrm{Hom}(\pi_1(S^1), U(1)) \cong U(1). \tag{4.2}$$

Note that the canonical projection $p : \mathcal{L} \to LM$ admits local sections: for a contractible open subset $U \subset M$ we have an isomorphism $\mathcal{T} : \mathcal{G}|_U \to \mathcal{I}_\rho$ which provides a section $\sigma : LU \to \mathcal{L} : \gamma \mapsto (\gamma, \gamma^*\mathcal{T})$.

Proposition 4.2. *There is a unique differentiable structure on* $\mathcal{L}$*, such that the projection* p *and the sections* σ *are smooth, and* $\mathcal{L}$ *becomes a principal* $U(1)$*-bundle over* LM*.*

Proof. Since every gerbe over S^1 is trivializable, none of the fibres $p^{-1}(\gamma)$ is empty. Hence each fibre is a $U(1)$-torsor. For the same reason, the image $\gamma(S^1)$ of any loop γ has an open neighbourhood $U \subset M$ (cf. the proof of Proposition 6.2.1 in [Bry93]), such that $\mathcal{G}|_U$ admits a trivialization $\mathcal{T} : \mathcal{G}|_U \to \mathcal{I}_\rho$. The corresponding section σ identifies $p^{-1}(LU)$ with $LU \times U(1)$, and thus defines a topology and a differentiable structure on each preimage $p^{-1}(LU)$. Let a topology on $\mathcal{L}$ be generated by all open subsets of all the fibres $p^{-1}(LU)$. Now, for two intersecting subsets U_1 and U_2 and trivializations $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, let N be the line bundle over $U_1 \cap U_2$ from Lemma 2.4. The transition map $LU_1 \times U(1) \to LU_2 \times U(1)$ is then given by $(\gamma, z) \mapsto (\gamma, z \cdot \mathrm{hol}_{\gamma^*N}(S^1)^{-1})$, and hence differentiable with respect to the loop $\gamma : S^1 \to U_1 \cap U_2$. $\square$

Instead of a principal $U(1)$-bundle, we will often and equivalently consider $\mathcal{L}$ as a hermitian line bundle. The construction just discussed also applies to the case when the bundle gerbe $\mathcal{G}$ is hermitian and has a connection. A curving defines a connection ∇ on the line bundle $\mathcal{L}$, whose curvature is

$$\mathrm{curv}(\nabla) = \int_{S^1} \mathrm{ev}^* H, \tag{4.3}$$

where H is the curvature of the gerbe $\mathcal{G}$, and $\mathrm{ev} : LM \times S^1 \to M$ is the evaluation map. The hermitian line bundle $\mathcal{L}$ with connection is functorial in the following sense:

Proposition 4.3. *For a hermitian bundle gerbe $\mathcal{G}$ over M with connection and curving, denote the associated hermitian line bundle with connection by $\mathcal{L}_{\mathcal{G}}$.*

i) Any isomorphism of gerbes $\mathcal{G} \to \mathcal{G}'$ induces an isomorphism $\mathcal{L}_{\mathcal{G}'} \to \mathcal{L}_{\mathcal{G}}$ of line bundles.

ii) For a smooth map $f : X \to M$ denote the induced map on loop spaces by $Lf : LX \to LM$. Then, the line bundles $(Lf)^*\mathcal{L}_{\mathcal{G}}$ and $\mathcal{L}_{f^*\mathcal{G}}$ are canonically isomorphic.

iii) For the dual gerbe $\mathcal{G}^*$ we obtain a canonical isomorphism $\mathcal{L}_{\mathcal{G}}^* \cong \mathcal{L}_{\mathcal{G}^*}$.

Let us also have a view on the cohomological counterpart of the construction of a line bundle over LM from a bundle gerbe over M. For this purpose, one has to extend the usual fibre integration of differential forms,

$$\int_{S_1} : \Omega^{k+1}(X \times S^1) \to \Omega^k(X), \tag{4.4}$$

to the Deligne cohomology groups. Here, X can be any smooth and possibly infinite dimensional manifold. Such extensions have been described in various ways [Gaw88, Bry93, GT01]. Then, for $X = LM$, the concatenation of this extension with the pullback along the evaluation map

$$\mathrm{ev} : LM \times S^1 \to M \tag{4.5}$$

gives a group homomorphism

$$\int_{S^1} \circ\, \mathrm{ev}^* : \mathrm{H}^3(M, \mathcal{D}(3)) \to \mathrm{H}^2(LM, \mathcal{D}(2)). \tag{4.6}$$

One can show that the image of the Deligne class $[(g, A, B)]$ of a hermitian bundle gerbe $\mathcal{G}$ with connection and curving under this group homomorphism gives exactly the Deligne class of the line bundle $\mathcal{L}$ with the connection ∇ we have constructed above in a direct geometric way.

The construction of the line bundle $\mathcal{L}$ can in particular be applied to the bundle gerbes over compact Lie groups from Section 3. In this case, we obtain a sequence

$$1 \longrightarrow U(1) \longrightarrow \mathcal{L} \xrightarrow{p} LG \longrightarrow 1 \tag{4.7}$$

of smooth maps, where $U(1)$ is mapped to the fibre of $\mathcal{L}$ over the loop which is constantly $1 \in G$. Now it is a natural question, whether one can equip the total space $\mathcal{L}$ with a group structure, such that the sequence (4.7) is an exact sequence of groups. This would provide a geometric construction of central extensions of loop groups. For simply-connected groups there exist definitions of group structures on $\mathcal{L}$ [Bry93], while in the general case the geometric construction of loop group extensions from bundle gerbes is still an open problem.

5 Algebraic Structures for Gerbes

There are several additional structures for bundle gerbes, some of which we introduce in this section. We describe the particular case of those additional structures on bundle gerbes over compact connected Lie groups.

5.1 Bundle Gerbe Modules

Bundle gerbe modules, also known as twisted vector bundles, have been introduced in [BCM$^+$02] in order to realize twisted K-theory geometrically. They are also the appropriate structure to extend the definition of holonomy to surfaces with boundary [CJM02].

Definition 5.1. *Let $\mathcal{G}$ be a bundle gerbe over M. A $\mathcal{G}$-*module *is a 1-morphism $\mathcal{E} : \mathcal{G} \to \mathcal{I}_\omega$ for some 2-form $\omega \in \Omega^2(M)$. The 2-form ω is called the curvature of the gerbe module.*

Let us compare this definition with the original definition of bundle gerbe modules in [BCM$^+$02]. A left $\mathcal{G}$-module $\mathcal{E} : \mathcal{G} \to \mathcal{I}_\omega$ consists of a vector bundle E over Y and of an isomorphism $\epsilon : L \otimes \pi_2^* E \to \pi_1^* E$ of vector bundles over $Y^{[2]}$ which satisfies

$$\pi_{13}^* \epsilon \circ (\mu \otimes \mathrm{id}) = \pi_{23}^* \epsilon \circ \pi_{12}^* \epsilon. \tag{5.1}$$

The similarity with an action of L on E justifies the notion gerbe module. The curvature of E is restricted by (2.15) to

$$\frac{1}{n}\mathrm{tr}(\mathrm{curv}(E)) = \pi^*\omega - C \tag{5.2}$$

with n the rank of E.

In terms of local data, a rank-n bundle gerbe module $\mathcal{E}\colon \mathcal{G} \to \mathcal{I}_\omega$ is described by a collection (G_{ij}, Π_i) of smooth functions $G_{ij} : U_i \cap U_j \to U(n)$ and $\mathfrak{u}(n)$-valued 1-forms $\Pi_i \in \Omega^1(U_i) \otimes \mathfrak{u}(n)$ which relate the local data of the bundle gerbes $\mathcal{G}$ and $\mathcal{I}_\omega$ in the following way:

$$\begin{array}{rcll} 1 & = & g_{ijk} \cdot G_{ik}\, G_{jk}^{-1}\, G_{ij}^{-1} & \text{on } \; U_i \cap U_j \cap U_k \,, \\ 0 & = & A_{ij} + \Pi_j - G_{ij}^{-1}\Pi_i\, G_{ij} - G_{ij}^{-1}\, \mathrm{d}G_{ij} & \text{on } \; U_i \cap U_j \,, \\ \omega & = & B_i + \dfrac{1}{n}\,\mathrm{tr}(\mathrm{d}\Pi_i) & \text{on } \; U_i \,. \end{array} \tag{5.3}$$

Note that the derivative of the last equality gives

$$\mathrm{d}\omega = \mathrm{d}B_i = \mathrm{curv}(\mathcal{G}). \tag{5.4}$$

Also note that if the bundle gerbe $\mathcal{G}$ is itself trivial, i.e. has local data $(1, 0, C|_{U_i})$ for a globally defined 2-form $C \in \Omega^2(M)$, then (G_{ij}, Π_i) are the local data of a rank-n vector bundle over M with curvature of trace $n\,(\omega - C)$. This explains the terminology "twisted" vector bundle in the non-trivial case.

According to (5.4), a necessary condition for the existence of a bundle gerbe module is that the curvature is an exact form. However, this is not the case in many situations, for example for the bundle gerbes on compact Lie groups we have constructed in Section 3, whose curvature is the canonical 3-form H. For this reason, one often considers a pair $(Q, \mathcal{E})$ of a submanifold $Q \subset M$ together with a gerbe module $\mathcal{E} : \mathcal{G}|_Q \to \mathcal{I}_\omega$ for the restriction of the gerbe to this submanifold. In conformal field theory, the pair $(Q, \mathcal{G})$ is also called a D-brane.

In particular we can consider this situation for the bundle gerbes over Lie groups G constructed in Section 3. In this case, the important submanifolds are conjugacy classes $Q = \mathcal{C}_\lambda$, and we already know that the curvature $\mathrm{curv}(\mathcal{G}) = H$ becomes exact when restricted to a conjugacy class, $H|_{\mathcal{C}_\lambda} = \mathrm{d}\omega_\lambda$. So the necessary condition (5.4) is satisfied. One can furthermore show [Gaw05] that precisely for integrable weights λ there exists a $\mathcal{G}|_{\mathcal{C}_\lambda}$-module with curvature ω_λ. This is the appropriate description of "flux stabilization of D-branes" in string theory [BDS00].

5.2 Bundle Gerbe Bimodules

Bundle gerbe bimodules generalize bundle gerbe modules for one bundle gerbe $\mathcal{G}$ to a structure for two bundle gerbes.

Definition 5.2 ([FSW]). *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be bundle gerbes over M. A $\mathcal{G}_1$-$\mathcal{G}_2$-* bimodule *is a morphism*

$$\mathcal{D} : \mathcal{G}_1 \to \mathcal{G}_2 \otimes \mathcal{I}_{\varpi} \tag{5.5}$$

for some 2-form $\varpi \in \Omega^2(M)$. The 2-form ϖ is called the curvature of the bimodule.

This definition is related to the one of a gerbe module in the sense that – using the appropriate notion of duality for bundle gerbes [Wal07] – a $\mathcal{G}_1$-$\mathcal{G}_2$-bimodule is the same as a $(\mathcal{G}_1 \otimes \mathcal{G}_2^*)$-module.

We drop the discussion of the local data of a gerbe bimodule. However, it is clear that there is – analogously to equation (5.4) – a necessary condition on the curvature of the two bundle gerbes,

$$H_1 = H_2 + \mathrm{d}\varpi. \tag{5.6}$$

It is again useful to consider bimodules for restrictions of the bundle gerbes to a submanifold $Q \subset M$. In particular, if M is the direct product of two manifolds M_1 and M_2, each equipped with a bundle gerbe $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, one considers $p_1^*\mathcal{G}_1|_Q$-$p_2^*\mathcal{G}_2|_Q$-bimodules, where $p_i : M \to M_i$ is the projection on the ith factor. This is the setup to define holonomies around surfaces with defect lines [FSW].

Examples for such bimodules are again provided by compact Lie groups and the basic bundle gerbes thereon [FSW]. The study of such examples leads to the following relevant submanifolds Q of $G \times G$, so-called biconjugacy classes

$$\mathcal{B}_{h_1,h_2} := \left\{ (x_1 h_1 x_2^{-1}, x_1 h_2 x_2^{-1}) \in G \times G \mid x_1, x_2 \in G \right\} \tag{5.7}$$

for any pair $(h_1, h_2) \in G \times G$.

Biconjugacy classes inherit from the diagonal left and diagonal right actions of G on $G \times G$ two commuting actions of G. One observes that the smooth map

$$\tilde{\mu} : G \times G \to G : (g_1, g_2) \mapsto g_1 g_2^{-1} \tag{5.8}$$

intertwines the diagonal left and diagonal right action of G on $G \times G$ and the adjoint and trivial actions of G on itself, respectively. It now follows that a biconjugacy class in $G \times G$ is the preimage of a conjugacy class in G under the projection $\tilde{\mu}$:

$$\mathcal{B}_{h_1,h_2} = \tilde{\mu}^{-1}(\mathcal{C}_{h_1 h_2^{-1}}) = \left\{ (g_1, g_2) \in G \times G \mid g_1 g_2^{-1} \in \mathcal{C}_{h_1 h_2^{-1}} \right\}. \tag{5.9}$$

We introduce the two-form

$$\varpi_{h_1,h_2} := \tilde{\mu}^*\omega_{h_1h_2^{-1}} - \frac{k}{2}\langle p_1^*\theta \wedge p_2^*\theta\rangle \tag{5.10}$$

on $\mathcal{B}_{h_1,h_2}$, where both summands are restricted to the submanifold $\mathcal{B}_{h_1,h_2}$ of $G \times G$. From the intertwining properties of $\tilde{\mu}$ and the bi-invariance of ω it follows that the two-form ϖ is also bi-invariant. One can show that it satisfies

$$p_1^*H = p_2^*H + \mathrm{d}\varpi_{h_1,h_2} \tag{5.11}$$

on a biconjugacy class $\mathcal{B}_{h_1,h_2}$, which is the necessary condition (5.6) [FSW].

5.3 Jandl Structures

Another structure we want to introduce is a Jandl structure on a bundle gerbe $\mathcal{G}$. Jandl structures extend the definition of holonomy we gave in Section 2 to unoriented, and in particular unorientable surfaces.

Definition 5.3 ([SSW07])**.** *A* Jandl structure $\mathcal{J}$ *on a bundle gerbe* $\mathcal{G}$ *over* M *is an involution* k *of* M *together with an isomorphism*

$$\mathcal{A} : k^*\mathcal{G} \to \mathcal{G}^* \tag{5.12}$$

and a 2-morphism

$$\varphi : k^*\mathcal{A} \Rightarrow \mathcal{A}^* \tag{5.13}$$

which satisfies the equivariance condition

$$k^*\varphi = \varphi^*. \tag{5.14}$$

To give an impression of the details of a Jandl structure, recall that an isomorphism such as $\mathcal{A} = (A, \alpha)$ consists of a line bundle A over the space Z which is build up from the two surjective submersions of the bundle gerbes $k^*\mathcal{G}$ and $\mathcal{G}^*$. In this particular situation, there is a canonical lift $\tilde{k}$ of the involution k into the space Z, and it is in fact easy to work out that the 2-morphism φ defines a $\tilde{k}$-equivariant structure on the line bundle A, which is compatible with the isomorphism α. Summarizing, a Jandl structure $\mathcal{J}$ on $\mathcal{G}$ is an isomorphism

$$\mathcal{A} : k^*\mathcal{G} \to \mathcal{G}^* \tag{5.15}$$

whose line bundle A is equivariant with respect to the involution $\tilde{k}$ on Z [SSW07].

Recall that we introduced equivariant structures on bundle gerbes in Section 3 in order to produce bundle gerbes over quotients of a manifold M

by a discrete group Z. One can combine equivariant structures and Jandl structures to Z-equivariant Jandl structures, leading to the mathematically appropriate description of so-called orientifolds in string theory [GSW]. The idea behind this combination is, that a bundle gerbe $\mathcal{G}$ over M with Γ-equivariant Jandl structure defines a bundle gerbe $\overline{\mathcal{G}}$ with Jandl structure over the quotient M/Z. For a cohomological description, one modifies the action of Z on the Deligne cohomology group to an action of the semidirect product $\Gamma := \mathbb{Z}_2 \ltimes Z$ [GSW].

For bundle gerbes over compact Lie groups, where Z is a subgroup of the center of G, the relevant involutions are given by

$$k_z : G \to G : g \mapsto (zg)^{-1} \tag{5.16}$$

for any z in the center. Again, using the basic bundle gerbes constructed in Section 3, one can classify all equivariant Jandl structures over all these bundle gerbes [GSW].

6 Applications to Conformal Field Theory

Let us explain the relation between conformal field theory and Lie theory, which arises in the study of non-linear sigma models on a Lie group G. Such a model can be defined by amplitudes $\mathcal{A}(\phi)$ for some path integral, where ϕ is a map from a closed complex curve Σ – the world sheet – into the target space G of the model. In [Wit84], Witten gives the following definition for $G = SU(2)$. Σ is the boundary of a three dimensional manifold B, and because the homotopy groups $\pi_i(SU(2))$ vanish for $i = 1, 2$, every map $\phi : \Sigma \to M$ can be extended into the interior B to a map $\Phi : B \to G$. Witten showed that – due to the integrality of the canonical 3-form H –

$$\mathcal{A}(\phi) := \exp\left(S_{\mathrm{kin}}(\phi) + \int_B \Phi^* H\right) \tag{6.1}$$

neither depends on the choice of B nor on the choice of the extension Φ, so that one obtains a well-defined amplitude. Here $S_{\mathrm{kin}}(\phi)$ is a kinetic term, and with a certain relative normalization of the two terms in (6.1) this model is called the Wess-Zumino-Witten model on G at level k. For non-simply-connected Lie groups, the extension Φ of ϕ to B does not exist in general. In these cases, the second summand of (6.1) has to be generalized.

Proposition 6.1. *Let $\mathcal{G}$ be a hermitian bundle gerbe over G with connection and curving of curvature H. For a three-dimensional oriented manifold B*

with boundary and a map $\Phi : B \to G$, we have

$$\mathrm{hol}_{\mathcal{G}}(\Phi|_{\partial B}) = \exp\left(\int_B \Phi^* H\right). \tag{6.2}$$

Proof. Remember that for any trivialization $\mathcal{T} : \Phi^*\mathcal{G}|_{\partial B} \to \mathcal{I}_\rho$ we have $\Phi^* H|_{\partial B} = \mathrm{d}\rho$. The rest follows by Stokes' Theorem. □

This way we reproduce the amplitude of the coupling term of the Wess-Zumino-Witten model by

$$\mathcal{A}(\phi) = \exp\left(S_{\mathrm{kin}}(\phi)\right) \cdot \mathrm{hol}_{\mathcal{G}}(\phi). \tag{6.3}$$

Notice that using bundle gerbes we did not impose any condition on the topology of the target space G. For compact connected and simply-connected Lie groups, it reproduces Witten's original definition. However, for general target spaces there may be bundle gerbes with same curvature, which are not isomorphic. This occurs for instance for the Lie group $\mathrm{Spin}(4n)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Here, the theory of bundle gerbes over Lie groups has brought new insights into the Lagrangian description of Wess-Zumino-Witten models.

In conformal field theory many applications require to consider surfaces with boundary. For those, we are not able to apply Definition 2.5 of the holonomy of a bundle gerbe $\mathcal{G}$: for a change of the chosen trivialization, a boundary term emerges which has to be compensated to achieve a holonomy independent of the choice of the trivialization. The compensating term is provided by the choice of a symmetric D-brane $(\mathcal{C}_\lambda, \mathcal{E}_\lambda)$: a conjugacy class $\mathcal{C}_\lambda$ for an integrable weight λ, together with a $\mathcal{G}|_{\mathcal{C}_\lambda}$-module $\mathcal{E}_\lambda$ of curvature ω_λ [Gaw05].

Another class of conformal field theories involves unoriented world sheets Σ. Again, Definition 2.5 has to be generalized, since it involves the integral of a differential form over Σ. It has been shown in [SSW07] that the choice of a Jandl structure on the bundle gerbe $\mathcal{G}$ makes the holonomy again well-defined. The precise classification of Jandl structures leads to a complete classification of unoriented Wess-Zumino-Witten models [SSW07, GSW].

Let us also indicate the relevance of the line bundle $\mathcal{L}$ over the loop group LG we have constructed in Section 4. The Hilbert space of holomorphic sections in $\mathcal{L}$ (completed with respect to its hermitian metric) serves as the space of states for the quantized theory [GR02]. The choice of additional structures like bundle gerbe modules or Jandl structures, has implications on this space. For example, a Jandl structure on the bundle gerbe $\mathcal{G}$ implies by Proposition 4.3 an isomorphism $\varphi : Lk^*\mathcal{L} \to \mathcal{L}^*$ which satisfies $Lk^*\varphi = \varphi$.

7 Open Questions

We conclude this contribution with the discussion of some lines for further research. One obvious direction is to extend the results explained here to gauged Wess-Zumino-Witten models, so-called coset theories. Those models having fixed points under the action of the group implementing field identifications for which the so-called untwisted stabilizer is strictly smaller than the stabilizer [FRS04] should be particularly interesting: in this case, simple gerbe modules and bimodules of rank strictly bigger than one appear naturally. A precise understanding of such theories requires the notion of an H-equivariant gerbe (bi-)module on the ambient group G.

Another subtle issue is the generalization of our results to non-compact Lie groups; this is partially due to the fact that much less is known about these theories in algebraic approaches.

The following two points seem to be conceptually appealing questions: our initial motivating question in this paper was about central extensions of Lie groups. While hermitian bundle gerbes naturally account for a line bundle $\mathcal{L}$ on loop space, more specific structure on the gerbe is needed to obtain a group structure on the line bundle $\mathcal{L}$. Similarly, one wishes to find structure on a gerbe module E that endows the associated bundles $\mathcal{E}$ over loop- and interval spaces with a natural $\mathcal{L}$-module structure.

Finally, we point out that gerbe bimodules have a natural operation of fusion which is very much in spirit of a convolution of correspondences. Imposing additional properties on gerbe bimodules over a compact Lie group (that in physical applications ensure the existence of enough conserved quantities) one should be able to single out interesting subcategories of gerbe bimodules that are semi-simple tensor categories. If G is simply connected, their fusion ring can be expected to be the corresponding Verlinde algebra; for non-simply connected groups, we expect interesting cousins of the Verlinde algebra.

References

[BCM+02] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray and D. Stevenson, *Twisted K-Theory and K-Theory of Bundle Gerbes*, Commun. Math. Phys. **228**(1), 17–49 (2002), http://arxiv.org/abs/hep-th/0106194.

[BDS00] C. Bachas, M. Douglas and C. Schweigert, *Flux Stabilization of D-branes*, JHEP **0005**(048) (2000), http://arxiv.org/abs/hep-th/0003037v2.

[Bou68] N. Bourbaki, *Éléments de Mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV-VI*, Hermann, 1968.

[BRS01] P. Bordalo, S. Ribault and C. Schweigert, *Flux Stabilization in compact Groups*, JHEP **0110**(036) (2001).

[Bry] J.-L. Brylinski, *Gerbes on complex reductive Lie Groups*, http://arxiv.org/abs/math/0002158.

[Bry93] J.-L. Brylinski, *Loop spaces, Characteristic Classes and Geometric Quantization*, volume 107 of *Progress in Mathematics*, Birkhäuser, 1993.

[Cha98] D. S. Chatterjee, *On the Construction of Abelian Gerbes*, PhD thesis, Cambridge Univ., 1998.

[CJM02] A. L. Carey, S. Johnson and M. K. Murray, *Holonomy on D-Branes*, J. Geom. Phys. **52**(2), 186–216 (2002), http://arxiv.org/abs/hep-th/0204199.

[FRS04] J. Fuchs, I. Runkel and C. Schweigert, *TFT Construction of RCFT Correlators III: Simple Currents*, Nucl. Phys. B **694**, 277–353 (2004), http://arxiv.org/abs/hep-th/0403157.

[FSW] J. Fuchs, C. Schweigert and K. Waldorf, *Bi-Branes: Target Space Geometry for World Sheet topological Defects*, http://arxiv.org/abs/hep-th/0703145, submitted.

[Gaw88] K. Gawędzki, Topological Actions in two-dimensional Quantum Field Theories, in *Non-perturbative Quantum Field Theory*, edited by G. Hooft, A. Jaffe, G. Mack, K. Mitter and R. Stora, pages 101–142, Plenum Press, 1988.

[Gaw05] K. Gawędzki, *Abelian and non-Abelian Branes in WZW Models and Gerbes*, Commun. Math. Phys. **258**, 23–73 (2005), http://arxiv.org/abs/hep-th/0406072.

[GR02] K. Gawędzki and N. Reis, *WZW Branes and Gerbes*, Rev. Math. Phys. **14**(12), 1281–1334 (2002), http://arxiv.org/abs/hep-th/0205233.

[GR03] K. Gawędzki and N. Reis, *Basic Gerbe over non simply connected compact Groups*, J. Geom. Phys. **50**(1–4), 28–55 (2003), http://arxiv.org/abs/math.dg/0307010.

[GSW] K. Gawędzki, R. R. Suszek and K. Waldorf, *WZW Orientifolds and Finite Group Cohomology*, http://arxiv.org/abs/hep-th/0701071, submitted.

[GT01] K. Gomi and Y. Terashima, *Higher-dimensional parallel Transports*, Math. Research Letters **8**, 25–33 (2001).

[Mei02] E. Meinrenken, *The Basic Gerbe over a compact simple Lie Group*, Enseign. Math., II. Sér. **49**(3–4), 307–333 (2002), http://arxiv.org/abs/math/0209194.

[MS76] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals of Mathematical Studies, Princton University Press, 1976.

[MS00] M. K. Murray and D. Stevenson, *Bundle Gerbes: Stable Isomorphism and local Theory*, J. Lond. Math. Soc. **62**, 925–937 (2000), http://arxiv.org/abs/math/9908135.

[Mur96] M. K. Murray, *Bundle Gerbes*, J. Lond. Math. Soc. **54**, 403–416 (1996), http://arxiv.org/abs/dg-ga/9407015.

[PS86] A. Pressley and G. Segal, *Loop Groups*, Oxford Univ. Press, 1986.

[SSW07] U. Schreiber, C. Schweigert and K. Waldorf, *Unoriented WZW Models and Holonomy of Bundle Gerbes*, Commun. Math. Phys. **274**(1), 31–64 (2007), http://arxiv.org/abs/hep-th/0512283.

[Ste00] D. Stevenson, *The Geometry of Bundle Gerbes*, PhD thesis, University of Adelaide, 2000, math.DG/0004117.

[Wal07] K. Waldorf, *More Morphisms between Bundle Gerbes*, Theory Appl. Categories **18**(9), 240–273 (2007), http://arxiv.org/abs/math.CT/0702652.

[Wit84] E. Witten, *Nonabelian Bosonization in two Dimensions*, Commun. Math. Phys. **92**, 455–472 (1984).