ON THE ROSENBERG-ZELINSKY SEQUENCE
IN ABELIAN MONOIDAL CATEGORIES

Till Barmeier [a,b],   Jürgen Fuchs [c],   Ingo Runkel [b],   Christoph Schweigert [a*]

[a] *Organisationseinheit Mathematik, Universität Hamburg*
Schwerpunkt Algebra und Zahlentheorie
Bundesstraße 55, D – 20 146 Hamburg

[b] *Department of Mathematics, King's College London*
Strand, London WC2R 2LS, United Kingdom

[c] *Teoretisk fysik, Karlstads Universitet*
Universitetsgatan 5, S – 651 88 Karlstad

December 2007

Abstract

We consider Frobenius algebras and their bimodules in certain abelian monoidal categories. In particular we study the Picard group of the category of bimodules over a Frobenius algebra, i.e. the group of isomorphism classes of invertible bimodules. The Rosenberg-Zelinsky sequence describes a homomorphism from the group of algebra automorphisms to the Picard group, which however is typically not surjective. We investigate under which conditions there exists a Morita equivalent Frobenius algebra for which the corresponding homomorphism is surjective. One motivation for our considerations is the orbifold construction in conformal field theory.

[*] Email addresses:
barmeier@math.uni-hamburg.de, jfuchs@fuchs.tekn.kau.se, ingo.runkel@kcl.ac.uk, schweigert@math.uni-hamburg.de

1 Introduction

In the study of associative algebras it is often advantageous to collect algebras into a category whose morphisms are not algebra homomorphisms, but instead bimodules. One motivation for this is provided by the following observation. Let $\Bbbk$ be a field and consider finite-dimensional unital associative $\Bbbk$-algebras. The condition on a $\Bbbk$-linear map to be an algebra morphism is obviously not linear. As a consequence the category of algebras and algebra homomorphisms has the unpleasant feature of not being additive.

On the other hand, instead of an algebra homomorphism $\varphi\colon A \to B$ one can equivalently consider the B-A-bimodule B_φ which as a $\Bbbk$-vector space coincides with B and whose left action is given by the multiplication of B while the right action is application of φ composed with multiplication in B. This is consistent with composition in the sense that given another algebra homomorphism $\psi\colon B \to C$ there is an isomorphism $C_\psi \otimes_B B_\varphi \cong C_{\psi\circ\varphi}$ of C-A-bimodules. It is then natural not to restrict one's attention to such special bimodules, but to allow all B-A-bimodules as morphisms from A to B [Be, sect. 5.7]. Of course, as bimodules come with their own morphisms, one then actually deals with the structure of a bicategory. The advantage is that the 1-morphism category $A \to B$, i.e. the category of B-A-bimodules, is additive and even abelian.

Taking bimodules as morphisms has further interesting consequences. First of all, the concept of isomorphy of two algebras A and B is now replaced by Morita equivalence, which requires the existence of an invertible A-B-bimodule. Indeed, in applications involving associative algebras one often finds that not only isomorphic but also Morita equivalent algebras can be used for a given purpose. The classical example is the equivalence of the category of left (or right) modules over Morita equivalent algebras. Another illustration is the Morita equivalence between invariant subalgebras and crossed products, see e.g. [Ri]. Examples in the realm of mathematical physics include the observations that matrix theories on Morita equivalent noncommutative tori are physically equivalent [Sc], and that Morita equivalent symmetric special Frobenius algebras in modular tensor categories describe equivalent rational conformal field theories [FFRS1, FFRS3].

As a second consequence, instead of the automorphism group $\mathrm{Aut}(A)$ one now deals with the invertible A-bimodules. The isomorphism classes of these particular bimodules form the Picard group $\mathrm{Pic}(A\text{-Bimod})$ of A-bimodules. While Morita equivalent algebras may have different automorphism groups, the corresponding Picard groups are isomorphic. One finds that for any algebra A the groups $\mathrm{Aut}(A)$ and $\mathrm{Pic}(A\text{-Bimod})$ are related by the exact sequence

$$0 \longrightarrow \mathrm{Inn}(A) \longrightarrow \mathrm{Aut}(A) \xrightarrow{\Psi_A} \mathrm{Pic}(A\text{-Bimod})\,, \tag{1.1}$$

which is a variant of the Rosenberg-Zelinsky [RZ, KO] sequence. Here $\mathrm{Inn}(A)$ denotes the inner automorphisms of A, and the group homomorphism Ψ_A is given by assigning to an automorphism ω of A the bimodule A_ω obtained from A by twisting the right action of A on itself by ω. In other words, $\mathrm{Pic}(A\text{-Bimod})$ is the home for the obstruction to a Skolem-Noether theorem.

It should be noticed that the group homomorphism Ψ_A in (1.1) is not necessarily a surjection. But for practical purposes in concrete applications it can be of interest to have an explicit realisa-

tion of the Picard group in terms of automorphisms of the algebra available. This leads naturally to the following questions:

- Does there exist another algebra A', Morita equivalent to A, such that the group homomorphism $\Psi_{A'}\colon \mathrm{Aut}(A') \to \mathrm{Pic}(A'\text{-Bimod})$ in (1.1) is surjective?

- And, once such an algebra A' has been constructed: Does this surjection admit a section, i.e. can the group $\mathrm{Pic}(A\text{-Bimod})$ be identified with a subgroup of the automorphism group of the Morita equivalent algebra A'?

We will investigate these questions in a more general setting, namely we consider algebras in $\Bbbk$-linear monoidal categories more general than the one of $\Bbbk$-vector spaces. Like many other results valid for vector spaces, also the sequence (1.1) continues to hold in this setting, see [VZ, prop. 3.14] and [FRS3, prop. 7].

We start in section 2 by collecting some aspects of algebras and Morita equivalence in monoidal categories and review the definition of invertible objects and of the Picard category. Section 3 collects information about fixed algebras under some subgroup of algebra automorphisms. In section 4 we answer the questions raised above for the special case that the algebra A is the tensor unit of the monoidal category $\mathcal{D}$ under consideration. As recalled in section 2, the categorical dimension provides a character on the Picard group with values in $\Bbbk^\times$. The main result of section 4, Proposition 4.3, supplies, for any finite subgroup H of the Picard group on which this character is trivial, an algebra A' that is Morita equivalent to the tensor unit such that the elements of H can be identified with automorphisms of A. Theorem 4.12, in turn, gives a characterisation of group homomorphisms $H \to \mathrm{Aut}(A)$ in terms of cochains on H. In this case the subgroup H is not only required to have trivial character, but in addition a three-cocycle on $\mathrm{Pic}(A\text{-Bimod})$ must be trivial when restricted to H. The relevant three-cocycle is obtained from the associativity constraint of $\mathcal{D}$, see eq. (4.22) below. We also compute the fixed algebra under the corresponding subgroup of automorphisms. In section 5 these results are generalised to algebras not necessarily Morita equivalent to the tensor unit, providing an affirmative answer to the above questions also in the general case. However, similar to the $A = \mathbf{1}$ case, one needs to restrict oneself to a finite subgroup H of $\mathrm{Pic}(A\text{-Bimod})$ such that the corresponding invertible bimodules have categorical dimension equal to 1 in A-Bimod and for which the associativity constraint of A-Bimod is trivial. This is stated in Theorem 5.6, which is the main result of this paper.

Let us also briefly mention a motivation of our considerations which comes from conformal field theory. A consistent rational conformal field theory (on oriented surfaces with possibly non-empty boundary) gives rise to a Morita class of algebras in a modular tensor category [FRS1], and choosing a particular representative A in this Morita class corresponds to selecting a particular 'boundary condition' for the CFT. It turns out that the Picard group $\mathrm{Pic}(A\text{-Bimod})$ – rather than just the automorphism group of A – describes symmetries of this CFT [FFRS3]. For practical purposes it can be useful to choose the representative A of the Morita class such that a given subgroup H of the symmetries is realised as automorphisms of A. Theorem 5.6 provides us with

conditions for when such a representative exists. Finally, the fixed algebra under this subgroup of automorphisms is related to the CFT obtained by 'orbifolding' the original CFT by the symmetry H.

2 Algebras in monoidal categories

In this section we collect information about a few basic structures that will be needed below. Let $\mathcal{D}$ be an abelian category enriched over the category $\mathcal{V}ect_{\Bbbk}$ of finite-dimensional vector spaces over a field $\Bbbk$. An object X of $\mathcal{D}$ is called simple iff it has no proper subobjects. An endomorphism of a simple object X is either zero or an isomorphism (Schur's lemma), and hence the endomorphism space $\mathrm{Hom}(X, X)$ is a finite-dimensional division algebra over $\Bbbk$. An object X of $\mathcal{D}$ is called absolutely simple iff $\mathrm{Hom}(X, X) = \Bbbk \, \mathrm{id}_X$. If $\Bbbk$ is algebraically closed, then every simple object is absolutely simple; the converse holds e.g. if $\mathcal{D}$ is semisimple.

When $\mathcal{D}$ is monoidal, then without loss of generality we assume it to be strict. More specifically, for the rest of this paper we make the following assumption.

Convention 2.1. $(\mathcal{D}, \otimes, \mathbf{1})$ is an abelian strict monoidal category with simple and absolutely simple tensor unit $\mathbf{1}$, and enriched over $\mathcal{V}ect_{\Bbbk}$ for a field $\Bbbk$ of characteristic zero.

In particular, $\mathrm{Hom}(\mathbf{1}, \mathbf{1}) = \Bbbk \, \mathrm{id}_{\mathbf{1}}$, which we identify with $\Bbbk$.

Definition 2.2. A *right duality* on $\mathcal{D}$ assigns to each object X of $\mathcal{D}$ an object $X^{\vee}$, called the right dual object of X, and morphisms $b_X \in \mathrm{Hom}(\mathbf{1}, X \otimes X^{\vee})$ and $d_X \in \mathrm{Hom}(X^{\vee} \otimes X, \mathbf{1})$ such that

$$(\mathrm{id}_X \otimes d_X) \circ (b_X \otimes \mathrm{id}_X) = \mathrm{id}_X \qquad \text{and} \qquad (d_X \otimes \mathrm{id}_{X^{\vee}}) \circ (\mathrm{id}_{X^{\vee}} \otimes b_X) = \mathrm{id}_{X^{\vee}} \,. \tag{2.1}$$

A *left duality* on $\mathcal{D}$ assigns to each object X of $\mathcal{D}$ a left dual object $^{\vee}X$ together with morphisms $\tilde{b}_X \in \mathrm{Hom}(\mathbf{1}, {}^{\vee}X \otimes X)$ and $\tilde{d}_X \in \mathrm{Hom}(X \otimes {}^{\vee}X, \mathbf{1})$ such that

$$(\tilde{d}_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \tilde{b}_X) = \mathrm{id}_X \qquad \text{and} \qquad (\mathrm{id}_{{}^{\vee}X} \otimes \tilde{d}_X) \circ (\tilde{b}_X \otimes \mathrm{id}_{{}^{\vee}X}) = \mathrm{id}_{{}^{\vee}X} \,. \tag{2.2}$$

Note that $\mathbf{1}^{\vee} \xrightarrow{\cong} \mathbf{1}^{\vee} \otimes \mathbf{1} \xrightarrow{d_{\mathbf{1}}} \mathbf{1}$ is nonzero; since by assumption $\mathbf{1}$ is simple, we thus have $\mathbf{1}^{\vee} \cong \mathbf{1}$. In the same way one sees that $^{\vee}\mathbf{1} \cong \mathbf{1}$. Further, given a right duality, the right dual morphism to a morphism $f \in \mathrm{Hom}(X, Y)$ is the morphism

$$f^{\vee} := (d_Y \otimes \mathrm{id}_{X^{\vee}}) \circ (\mathrm{id}_{Y^{\vee}} \otimes f \otimes \mathrm{id}_{X^{\vee}}) \circ (\mathrm{id}_{Y^{\vee}} \otimes b_X) \ \in \mathrm{Hom}(Y^{\vee}, X^{\vee}) \,. \tag{2.3}$$

Left dual morphisms are defined analogously. Hereby each duality furnishes a functor from $\mathcal{D}$ to $\mathcal{D}^{\mathrm{op}}$. Further, the objects $(X \otimes Y)^\vee$ and $Y^\vee \otimes X^\vee$ are isomorphic.

Definition 2.3. A *sovereign*[1] category is a monoidal category that is equipped with a left and a right duality which coincide as functors, i.e. $X^\vee = {}^\vee X$ for every object X and $f^\vee = {}^\vee f$ for every morphism f.

In a sovereign category the *left* and *right traces* of an endomorphism $f \in \mathrm{Hom}(X, X)$ are the scalars (remember that we identify $\mathrm{End}(\mathbf{1})$ with $\Bbbk$)

$$\mathrm{tr}_\mathrm{l}(f) := d_X \circ (\mathrm{id}_{X^\vee} \otimes f) \circ \tilde{b}_X \qquad \text{and} \qquad \mathrm{tr}_\mathrm{r}(f) := \tilde{d}_X \circ (f \otimes \mathrm{id}_{X^\vee}) \circ b_X , \tag{2.4}$$

respectively, and the left and right *dimensions* of an object X are the scalars

$$\dim_\mathrm{l}(X) := \mathrm{tr}_\mathrm{l}(\mathrm{id}_X) , \qquad \dim_\mathrm{r}(X) := \mathrm{tr}_\mathrm{r}(\mathrm{id}_X) . \tag{2.5}$$

Both traces are cyclic, and dimensions are constant on isomorphism classes, multiplicative under the tensor product and additive under direct sums. Further, one has $\mathrm{tr}_\mathrm{l}(f) = \mathrm{tr}_\mathrm{r}(f^\vee)$, and using the fact that in a sovereign category each object X is isomorphic to its double dual $X^{\vee\vee}$ it follows that the right dimension of the dual object equals the left dimension of the object itself,

$$\dim_\mathrm{l}(X) = \dim_\mathrm{r}(X^\vee) , \tag{2.6}$$

and vice versa. In particular, any object that is isomorphic to its dual, $X \cong X^\vee$, has equal left and right dimension, which we then denote by $\dim(X)$. The tensor unit $\mathbf{1}$ is isomorphic to its dual and has dimension $\dim(\mathbf{1}) = 1$.

Next we collect some information about algebra objects in monoidal categories. Recall that a (unital, associative) *algebra* in $\mathcal{D}$ is a triple (A, m, η) consisting of an object A of $\mathcal{D}$ and morphisms $m \in \mathrm{Hom}(A \otimes A, A)$ and $\eta \in \mathrm{Hom}(\mathbf{1}, A)$, such that

$$m \circ (\mathrm{id}_A \otimes m) = m \circ (m \otimes \mathrm{id}_A) \qquad \text{and} \qquad m \circ (\mathrm{id}_A \otimes \eta) = \mathrm{id}_A = m \circ (\eta \otimes \mathrm{id}_A) . \tag{2.7}$$

Dually, a (counital, coassociative) *coalgebra* is a triple (C, Δ, ε) with C an object of $\mathcal{D}$ and morphisms $\Delta \in \mathrm{Hom}(C, C \otimes C)$ and $\varepsilon \in \mathrm{Hom}(C, \mathbf{1})$, such that

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta \qquad \text{and} \qquad (\mathrm{id}_C \otimes \varepsilon) \circ \Delta = \mathrm{id}_C = (\varepsilon \otimes \mathrm{id}_C) \circ \Delta . \tag{2.8}$$

The following concepts are also well known, see e.g. [Mü, FRS1].

Definition 2.4.

(i) A *Frobenius algebra* in $\mathcal{D}$ is a quintuple $(A, m, \eta, \Delta, \varepsilon)$, such that (A, m, η) is an algebra in $\mathcal{D}$, (A, Δ, ε) is a coalgebra and the compatibility relation

$$(\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A) = \Delta \circ m = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta) \tag{2.9}$$

between the algebra and coalgebra structures is satisfied.

---

[1] What we call sovereign is sometimes referred to as *strictly sovereign*, compare [Bi, Br].

(ii) A Frobenius algebra A is called *special* iff $m \circ \Delta = \beta_A \, \mathrm{id}_A$ and $\varepsilon \circ \eta = \beta_\mathbf{1} \, \mathrm{id}_\mathbf{1}$ with $\beta_\mathbf{1}, \beta_A \in \Bbbk^\times$. A is called *normalised special* iff A is special with $\beta_A = 1$.

(iii) If $\mathcal{D}$ is in addition sovereign, an algebra A in $\mathcal{D}$ is called *symmetric* iff the two morphisms

$$\Phi_1 := ((\varepsilon \circ m) \otimes \mathrm{id}_{A^\vee}) \circ (\mathrm{id}_A \otimes b_A) \quad \text{and} \quad \Phi_2 := (\mathrm{id}_{A^\vee} \otimes (\varepsilon \circ m)) \circ (\tilde{b}_A \otimes \mathrm{id}_A) \qquad (2.10)$$

in $\mathrm{Hom}(A, A^\vee)$ are equal.

For (A, m_A, η_A) and (B, m_B, η_B) algebras in $\mathcal{D}$, a morphism $f\colon A \to B$ is called a (unital) morphism of algebras iff $f \circ m_A = m_B \circ (f \otimes f)$ in $\mathrm{Hom}(A \otimes A, B)$ and $f \circ \eta_A = \eta_B$. Similarly one defines (counital) morphisms of coalgebras and morphisms of Frobenius algebras. An algebra S is called a *subalgebra* of A iff there is a monic $i\colon S \to A$ that is a morphism of algebras.

A (unital) *left A-module* is a pair (M, ρ) consisting of an object M in $\mathcal{D}$ and a morphism $\rho \in \mathrm{Hom}(A \otimes M, M)$, such that

$$\rho \circ (\mathrm{id}_A \otimes \rho) = \rho \circ (m \otimes \mathrm{id}_M) \quad \text{and} \quad \rho \circ (\eta \otimes \mathrm{id}_M) = \mathrm{id}_M. \qquad (2.11)$$

Similarly one defines right A-modules. An A-A-*bimodule* (or A-*bimodule* for short) is a triple (M, ρ, ϱ) such that (M, ρ) is a left A-module, (M, ϱ) a right A-module, and the left and right actions of A on M commute. Analogously, A-B-bimodules carry a left action of the algebra A and a commuting right action of the algebra B.

For (M, ρ_M) and (N, ρ_N) left A-modules, a morphism $f \in \mathrm{Hom}(M, N)$ is said to be a morphism of left A-modules (or briefly, a module morphism) iff $f \circ \rho_M = \rho_N \circ (\mathrm{id}_A \otimes f)$. Analogously one defines morphisms of A-B-bimodules. Thereby one obtains a category, with objects the A-B-bimodules and morphisms the A-B-bimodule morphisms. We denote this category by $\mathcal{D}_{A|B}$ and the set of bimodule morphisms from M to N by $\mathrm{Hom}_{A|B}(M, N)$. The Frobenius property (2.9) means that the coproduct Δ is a morphism of A-bimodules.

Definition 2.5. An algebra is called *(absolutely) simple* iff it is (absolutely) simple as a bimodule over itself. Thus A is absolutely simple iff $\mathrm{Hom}_{A|A}(A, A) = \Bbbk \, \mathrm{id}_A$.

Remark 2.6. Since $\mathcal{D}$ is abelian, one can define a tensor product of A-bimodules. This turns the bimodule category $\mathcal{D}_{A|A}$ into a monoidal category. See the appendix for more detail on this and especially on tensor products over special Frobenius algebras.

Remark 2.7. If A is a (not necessarily symmetric) Frobenius algebra in a sovereign category, then the morphisms Φ_1 and Φ_2 in (2.10) are invertible, with inverses

$$\Phi_1^{-1} = (d_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_{A^\vee} \otimes (\Delta \circ \eta)) \quad \text{and} \quad \Phi_2^{-1} = (\mathrm{id}_A \otimes \tilde{d}_A) \circ ((\Delta \circ \eta) \otimes \mathrm{id}_{A^\vee}), \qquad (2.12)$$

respectively. So if A is Frobenius, A and $A^\vee$ are isomorphic, hence the left and right dimension of A are equal. Accordingly we will write $\dim(A)$ for the dimension of a Frobenius algebra in the sequel.

Further one can show (see [FRS1], section 3) that for any symmetric special Frobenius algebra A the relation $\beta_A\,\beta_\mathbf{1} = \dim(A)$ holds. In particular, $\dim(A) \neq 0$. Furthermore, without loss of generality one can assume that the coproduct is normalised such that $\beta_\mathbf{1} = \dim(A)$ and $\beta_A = 1$, i.e. A is normalised special.

Lemma 2.8. *Let (A, m, η) be an algebra with $\dim_\Bbbk \mathrm{Hom}(\mathbf{1}, A) = 1$. Then A is an absolutely simple algebra.*

Proof. By Proposition 4.7 of [FS] one has $\mathrm{Hom}(\mathbf{1}, A) \cong \mathrm{Hom}_A(A, A)$. The result thus follows from $1 \leq \dim_\Bbbk \mathrm{Hom}_{A|A}(A, A) \leq \dim_\Bbbk \mathrm{Hom}_A(A, A)$. $\qquad\square$

Remark 2.9. Obviously the tensor unit $\mathbf{1}$ is a symmetric special Frobenius algebra. One also easily verifies that for any object X in a sovereign category the object $X \otimes X^\vee$ with structural morphisms

$$m := \mathrm{id}_X \otimes d_X \otimes \mathrm{id}_{X^\vee}, \quad \eta := b_X, \quad \Delta := \mathrm{id}_X \otimes \tilde{b}_X \otimes \mathrm{id}_{X^\vee}, \quad \varepsilon := \tilde{d}_X \qquad (2.13)$$
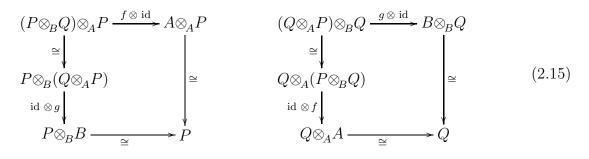
provides an example of a symmetric Frobenius algebra. If the object X has nonzero left and right dimensions, then this algebra is also special, with

$$\beta_{X \otimes X^\vee} = \dim_\mathrm{l}(X), \qquad \beta_\mathbf{1} = \dim_\mathrm{r}(X). \qquad (2.14)$$

The object X is naturally a left module over $X \otimes X^\vee$, with representation morphism $\rho = \mathrm{id}_X \otimes d_X$, while the object $X^\vee$ is a right module over $X \otimes X^\vee$ with $\varrho = d_X \otimes \mathrm{id}_{X^\vee}$.

Next we recall the concept of Morita equivalence of algebras (for details see e.g. [Pa, VZ]).

Definition 2.10. A *Morita context* in $\mathcal{D}$ is a sextuple (A, B, P, Q, f, g), where A and B are algebras in $\mathcal{D}$, $P \equiv {}_AP_B$ is an A-B-bimodule and $Q \equiv {}_BQ_A$ is a B-A-bimodule, such that $f\colon P \otimes_B Q \xrightarrow{\cong} A$ and $g\colon Q \otimes_A P \xrightarrow{\cong} B$ are isomorphisms of A- and B-bimodules, respectively, and the two diagrams

$$
\begin{array}{ccc}
(P\otimes_B Q)\otimes_A P & \xrightarrow{\ f\,\otimes\,\mathrm{id}\ } & A\otimes_A P \\
{\scriptstyle\cong}\Big\downarrow & & \\
P\otimes_B(Q\otimes_A P) & & \Big\downarrow{\scriptstyle\cong} \\
{\scriptstyle\mathrm{id}\,\otimes\,g}\Big\downarrow & & \\
P\otimes_B B & \xrightarrow[\ \cong\ ]{} & P
\end{array}
\qquad
\begin{array}{ccc}
(Q\otimes_A P)\otimes_B Q & \xrightarrow{\ g\,\otimes\,\mathrm{id}\ } & B\otimes_B Q \\
{\scriptstyle\cong}\Big\downarrow & & \\
Q\otimes_A(P\otimes_B Q) & & \Big\downarrow{\scriptstyle\cong} \\
{\scriptstyle\mathrm{id}\,\otimes\,f}\Big\downarrow & & \\
Q\otimes_A A & \xrightarrow[\ \cong\ ]{} & Q
\end{array}
\qquad (2.15)
$$

commute.

If such a Morita context exists, we call the algebras A and B Morita equivalent. In the sequel we will suppress the isomorphisms f and g and write a Morita context as $A \overset{P,Q}{\longleftrightarrow} B$.

Lemma 2.11. *Let $\mathcal{D}$ be in addition sovereign and let U be an object of $\mathcal{D}$ with nonzero left and right dimension. Then the symmetric special Frobenius algebra $U \otimes U^\vee$ is Morita equivalent to the tensor unit, with Morita context $\mathbf{1} \overset{U^\vee,U}{\longleftrightarrow} U \otimes U^\vee$.*

Proof. We only need to show that $U^\vee \otimes_{U \otimes U^\vee} U \cong \mathbf{1}$. Since $U \otimes U^\vee$ is symmetric special Frobenius, the idempotent $P_{U^\vee,U}$ for the tensor product over $U \otimes U^\vee$, as described in appendix A, is well defined. One calculates that $P_{U^\vee,U} = (\dim_l(U))^{-1}\, \tilde{b}_U \circ d_U$. This implies that the tensor unit is indeed isomorphic to the image of $P_{U^\vee,U}$. Finally, commutativity of the diagrams (2.15) follows using the techniques of projectors as presented in the appendix. $\square$

Definition 2.12. An object X in a monoidal category is called *invertible* iff there exists an object X' such that $X \otimes X' \cong \mathbf{1} \cong X' \otimes X$.

If the category $\mathcal{D}$ has small skeleton, then the set of isomorphism classes of invertible objects forms a group under the tensor product. This group is called the *Picard group* $\mathrm{Pic}(\mathcal{D})$ of $\mathcal{D}$.

Lemma 2.13. *Let $\mathcal{D}$ be in addition sovereign.*

(i) *Every invertible object of $\mathcal{D}$ is simple.*

(ii) *An object X in $\mathcal{D}$ is invertible iff $X^\vee$ is invertible.*

(iii) *An object X in $\mathcal{D}$ is invertible iff the morphisms b_X and $\tilde{b}_X$ are invertible.*

(iv) *Every invertible object of $\mathcal{D}$ is absolutely simple.*

Proof. (i) Let $X \otimes X' \cong X' \otimes X \cong \mathbf{1}$. Assume that $e\colon U \to X$ is monic for some object U. Then $\mathrm{id}_{X'} \otimes e\colon X' \otimes U \to X' \otimes X$ is monic. Indeed, if $(\mathrm{id}_{X'} \otimes e) \circ f = (\mathrm{id}_{X'} \otimes e) \circ g$ for some morphisms f and g, then by applying the duality morphism $d_{X'}$ we obtain $e \circ (d_{X'} \otimes \mathrm{id}_U) \circ (\mathrm{id}_{X'^\vee} \otimes f) = e \circ (d_{X'} \otimes \mathrm{id}_U) \circ (\mathrm{id}_{X'^\vee} \otimes g)$. As e is monic this amounts to $(d_{X'} \otimes \mathrm{id}_U) \circ (\mathrm{id}_{X'^\vee} \otimes f) = (d_{X'} \otimes \mathrm{id}_U) \circ (\mathrm{id}_{X'^\vee} \otimes g)$, which by applying $b_{X'}$ and using the duality property of $d_{X'}$ and $b_{X'}$ shows that $f = g$. Thus $\mathrm{id}_{X'} \otimes e\colon X' \otimes U \to X' \otimes X \cong \mathbf{1}$ is monic. As $\mathbf{1}$ is required to be simple, it is thus an isomorphism. Then $\mathrm{id}_X \otimes \mathrm{id}_{X'} \otimes e$ is an isomorphism as well. By assumption there exists an isomorphism $b\colon \mathbf{1} \to X \otimes X'$. With the help of b we can write $e = (b^{-1} \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \mathrm{id}_{X'} \otimes e) \circ (b \otimes \mathrm{id}_U)$. Thus e is a composition of isomorphisms, and hence an isomorphism. In summary, $e\colon U \to X$ being monic implies that e is an isomorphism. Hence X is simple.

(ii) Note that $X^\vee \otimes X'^\vee \cong (X' \otimes X)^\vee \cong \mathbf{1}^\vee \cong \mathbf{1}$, and similarly $X'^\vee \otimes X^\vee \cong \mathbf{1}$.

(iii) Since by part (ii) $X^\vee$ is invertible, so is $X \otimes X^\vee$. By part (i), $X \otimes X^\vee$ is therefore simple and $b_X\colon \mathbf{1} \to X \otimes X^\vee$ is a nonzero morphism between simple objects. By Schur's lemma it is an isomorphism. The argument for $\tilde{b}_X$ proceeds along the same lines, and the converse statement follows by definition.

(iv) The duality morphisms give an isomorphism $\mathrm{Hom}(X,X) \cong \mathrm{Hom}(X \otimes X^\vee, \mathbf{1})$. From parts (i) and (ii) we know that $X \otimes X^\vee \cong \mathbf{1}$, and so $\mathrm{Hom}(X,X) \cong \mathrm{Hom}(\mathbf{1},\mathbf{1})$. That X is absolutely simple now follows because $\mathbf{1}$ is absolutely simple by assumption. $\square$

Lemma 2.13 implies that for an invertible object X one has

$$\dim_l(X)\, \dim_r(X) = \dim_l(X)\, \dim_l(X^\vee) = \dim_l(X \otimes X^\vee) = \dim_l(\mathbf{1}) = 1\,. \qquad (2.16)$$

8

With the help of this equality one checks that the inverse of b_X is given by $\dim_{\mathrm{l}}(X)\,\tilde{d}_X$,

$$\dim_{\mathrm{l}}(X)\,\tilde{d}_X \circ b_X = \dim_{\mathrm{l}}(X)\dim_{\mathrm{r}}(X)\,\mathrm{id}_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}\,. \tag{2.17}$$

Analogously we have $\dim_{\mathrm{r}}(X)\,d_X \circ \tilde{b}_X = \mathrm{id}_{\mathbf{1}}$; thus in particular the left and right dimensions of an invertible object X are nonzero. Further we have

$$\dim_{\mathrm{l}}(X)\,b_X \circ \tilde{d}_X = \mathrm{id}_{X\otimes X^\vee} \qquad \text{and} \qquad \dim_{\mathrm{r}}(X)\,\tilde{b}_X \circ d_X = \mathrm{id}_{X^\vee \otimes X}\,. \tag{2.18}$$

We denote the object representing an isomorphism class g in $\mathrm{Pic}(\mathcal{D})$ by L_g, i.e. $[L_g] = g \in \mathrm{Pic}(\mathcal{D})$. Then $L_g \otimes L_h \cong L_{gh}$. As the representative of the unit class 1 we take the tensor unit, $L_1 = \mathbf{1}$.

Lemma 2.14. *Let $\mathcal{D}$ be in addition sovereign and H a subgroup of $\mathrm{Pic}(\mathcal{D})$.*

(i) *The mappings $h \mapsto \dim_{\mathrm{l}}(L_h)$ and $h \mapsto \dim_{\mathrm{r}}(L_h)$ are characters on H.*

(ii) *If H is finite, then $\dim_{\mathrm{l}|\mathrm{r}}(\bigoplus_{h\in H} L_h)$ is either 0 or $|H|$. It is equal to $|H|$ iff $\dim_{\mathrm{l}|\mathrm{r}}(L_h) = 1$ for all $h \in H$.*

Proof. Claim (i) follows directly from the multiplicativity of the left and right dimension under the tensor product and from the fact that the dimension only depends on the isomorphism class of an object.

Because of $\dim_{\mathrm{l}|\mathrm{r}}(\bigoplus_{h\in H} L_h) = \sum_{h\in H} \dim_{\mathrm{l}|\mathrm{r}}(L_h)$, part (ii) is a consequence of the orthogonality of characters. $\qquad\square$

Definition 2.15. The *Picard category* $\mathcal{P}ic(\mathcal{D})$ of $\mathcal{D}$ is the full subcategory of $\mathcal{D}$ whose objects are direct sums of invertible objects of $\mathcal{D}$.

3 Fixed algebras

We introduce the notion of a fixed algebra under a group of algebra automorphisms and establish some basic results on fixed algebras.

Definition 3.1. Let (A, m, η) be an algebra in $\mathcal{D}$ and $H \le \mathrm{Aut}(A)$ a group of (unital) automorphisms of A. Then a *fixed algebra* under the action of H is a pair (A^H, j), where A^H is an object of $\mathcal{D}$ and $j\colon A^H \to A$ is a monic with $\alpha \circ j = j$ for all $\alpha \in H$, such that the following universal property is fulfilled: For every object B in $\mathcal{D}$ and morphism $f\colon B \to A$ with $\alpha \circ f = f$ for all $\alpha \in H$, there is a unique morphism $\bar{f}\colon B \to A^H$ such that $j \circ \bar{f} = f$.

The object A^H defined this way is unique up to isomorphism. The following result justifies using the term 'fixed algebra', rather than 'fixed object'.

Lemma 3.2. *Given A, H and (A^H, j) as in definition 3.1, there exists a unique algebra structure on the object A^H such that the inclusion $j\colon A^H \to A$ is a morphism of algebras.*

Proof. For arbitrary $\alpha \in H$ consider the diagrams

$$
\begin{array}{ccc}
A^H \xrightarrow{\ j\ } A \xrightarrow{\ \alpha\ } A & & A^H \xrightarrow{\ j\ } A \xrightarrow{\ \alpha\ } A \\
\big\uparrow {\scriptstyle m\circ(j\otimes j)} & \text{and} & \big\uparrow {\scriptstyle \eta} \\
A^H \otimes A^H & & \mathbf{1}
\end{array}
\tag{3.1}
$$

Since α is a morphism of algebras, we have $\alpha \circ m \circ (j \otimes j) = m \circ ((\alpha \circ j) \otimes (\alpha \circ j)) = m \circ (j \otimes j)$; as this holds for all $\alpha \in H$, the universal property of the fixed algebra yields a unique product morphism $\mu \colon A^H \otimes A^H \to A^H$ such that $j \circ \mu = m \circ (j \otimes j)$. By associativity of A we have

$$
\begin{aligned}
j \circ \mu \circ (\mu \otimes \mathrm{id}_{A^H}) &= m \circ (m \otimes \mathrm{id}_A) \circ (j \otimes j \otimes j) \\
&= m \circ (\mathrm{id}_A \otimes m) \circ (j \otimes j \otimes j) = j \circ \mu \circ (\mathrm{id}_{A^H} \otimes \mu).
\end{aligned}
\tag{3.2}
$$

Since j is monic, this implies associativity of the product morphism μ. Similarly, applying the universal property of (A^H, j) on η gives a morphism $\eta' \colon \mathbf{1} \to A^H$ that has the properties of a unit for the product μ. So (A^H, μ, η') is an associative algebra with unit. $\qquad\square$

We proceed to show that fixed algebras under finite groups of automorphisms always exist in the situation studied here. Let $H \leq \mathrm{Aut}(A)$ be a finite subgroup of the group of algebra automorphisms of A. Set

$$
P = P_H := \frac{1}{|H|} \sum_{\alpha \in H} \alpha \ \in \mathrm{End}(A).
\tag{3.3}
$$

Then $P \circ P = \frac{1}{|H|^2} \sum_{\alpha,\beta \in H} \alpha \circ \beta = \frac{1}{|H|^2} \sum_{\alpha,\beta' \in H} \beta' = \frac{1}{|H|} \sum_{\beta' \in H} \beta' = P$, i.e. P is an idempotent. Analogously one shows that $\alpha \circ P = P$ for every $\alpha \in H$. Further, since $\mathcal{D}$ is abelian, we can write $P = e \circ r$ with e monic and r epi. Denote the image of $P \equiv P_H$ by A_P, so that $e \colon A_P \to A$ and $r \circ e = \mathrm{id}_{A_P}$.

Lemma 3.3. *The pair (A_P, e) satisfies the universal property of the fixed algebra.*

Proof. From $r \circ e = \mathrm{id}_{A_P}$ we see that $\alpha \circ e = \alpha \circ e \circ r \circ e = \alpha \circ P \circ e = P \circ e = e$ for all $\alpha \in H$. For B an object of $\mathcal{D}$ and $f \colon B \to A$ a morphism with $\alpha \circ f = f$ for all $\alpha \in H$, set $\bar{f} := r \circ f$. Then $e \circ \bar{f} = e \circ r \circ f = P \circ f = \frac{1}{|H|} \sum_{\alpha \in H} \alpha \circ f = \frac{1}{|H|} \sum_{\alpha \in H} f = f$. Further, if f' is another morphism satisfying $e \circ f' = f$, then $e \circ f' = e \circ \bar{f}$ and, since e is monic, $\bar{f} = f'$, so $\bar{f}$ is unique. Hence the object A_P satisfies the universal property of the fixed algebra A^H. $\qquad\square$

We would like to express the structural morphisms of the fixed algebra through e and r. To this end we introduce a candidate product m_P and candidate unit η_P on A_P: we set

$$
m_P := r \circ m \circ (e \otimes e) \qquad \text{and} \qquad \eta_P := r \circ \eta.
\tag{3.4}
$$

10

Note that e is a morphism of unital algebras:

$$e \circ m_P = e \circ r \circ m \circ (e \otimes e) = P \circ m \circ (e \otimes e) = \tfrac{1}{|H|} \sum_{\alpha \in H} \alpha \circ m \circ (e \otimes e)$$

$$= \tfrac{1}{|H|} \sum_{\alpha \in H} m \circ ((\alpha \circ e) \otimes (\alpha \circ e)) = \tfrac{1}{|H|} \sum_{\alpha \in H} m \circ (e \otimes e) = m \circ (e \otimes e), \qquad (3.5)$$

$$e \circ \eta_P = e \circ r \circ \eta = P \circ \eta = \tfrac{1}{|H|} \sum_{\alpha \in H} \alpha \circ \eta = \tfrac{1}{|H|} \sum_{\alpha \in H} \eta = \eta.$$

Lemma 3.4. *The algebra (A_P, m_P, η_P) is isomorphic to the algebra structure that A_P inherits as a fixed algebra.*

Proof. An easy calculation shows that m_P is associative and η_P is a unit for m_P; thus (A_P, m_P, η_P) is an algebra. Moreover, since according to lemma 3.2 there is a unique algebra structure on A^H such that the inclusion into A is a morphism of algebras, it follows that A_P and A^H are isomorphic as algebras. $\qquad\square$

In the following discussion the term fixed algebra will always refer to the algebra A_P.

Lemma 3.5. *With $P = \tfrac{1}{|H|} \sum_{\alpha \in H} \alpha = e \circ r$ as in (3.3), we have the following equalities of morphisms:*

$$r \circ m \circ (e \otimes P) = r \circ m \circ (e \otimes \mathrm{id}_A), \qquad r \circ m \circ (P \otimes e) = r \circ m \circ (\mathrm{id}_A \otimes e) \qquad \text{and}$$

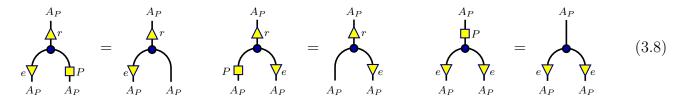$$P \circ m \circ (e \otimes e) = m \circ (e \otimes e). \qquad (3.6)$$

Proof. Indeed, making use of $r \circ \alpha = r$ and $\alpha \circ e = e$ for all $\alpha \in H$, we have

$$r \circ m \circ (e \otimes P) = \frac{1}{|H|} \sum_{\alpha \in H} r \circ m \circ (e \otimes \alpha) = \frac{1}{|H|} \sum_{\alpha \in H} r \circ \alpha \circ m \circ ((\alpha^{-1} \circ e) \otimes \mathrm{id}_A)$$

$$= \frac{1}{|H|} \sum_{\alpha \in H} r \circ m \circ (e \otimes \mathrm{id}_A) = r \circ m \circ (e \otimes \mathrm{id}_A). \qquad (3.7)$$
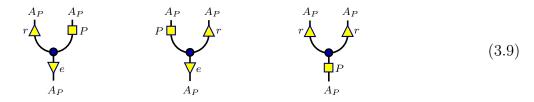
The other two equalities are established analogously. $\qquad\square$

Remark 3.6. With the help of the graphical calculus for morphisms in strict monoidal categories (see [JS, Ka, Ma, BK], and e.g. Appendix A of [FFRS2] for the graphical representation of the structural morphisms of Frobenius algebras in such categories), the equalities in Lemma 3.5 can be visualised as follows:

$$(3.8)$$

If A is a Frobenius algebra, it is understood that $\mathrm{Aut}(A)$ consists of all algebra automorphisms of
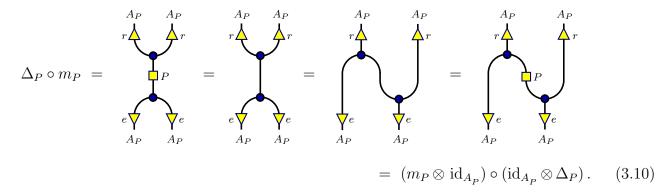
A which are at the same time also coalgebra automorphisms. Then for a Frobenius algebra A the idempotent P can also be omitted in the following situations, which we describe again pictorially:

$$(3.9)$$

Proposition 3.7. *Let A be a Frobenius algebra in $\mathcal{D}$ and $H \leq \mathrm{Aut}(A)$ a finite group of automorphisms of A.*
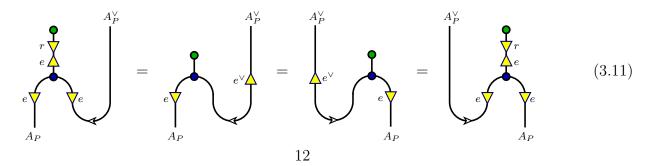
(i) *A_P is a Frobenius algebra, and the embedding $e\colon A_P \to A$ is a morphism of algebras while the restriction $r\colon A \to A_P$ is a morphism of coalgebras.*

(ii) *If the category $\mathcal{D}$ is sovereign and A is symmetric, then A_P is symmetric, too.*

(iii) *If the category $\mathcal{D}$ is sovereign, A is symmetric special and A_P is an absolutely simple algebra and has nonzero left (equivalently right, cf. remark 2.7) dimension, then A_P is special.*

Proof. (i) The algebra structure on A_P has already been defined in (3.4), and according to (3.5) e is a morphism of algebras. Denoting the coproduct on A by Δ and the counit by ε, we further set $\Delta_P := (r \otimes r) \circ \Delta \circ e$ and $\varepsilon_P := \varepsilon \circ e$. Similarly to the calculation in (3.5) one verifies that r is a morphism of coalgebras, and that Δ_P is coassociative and ε_P is a counit. Regarding the Frobenius property, we give a graphical proof of one of the equalities that must be satisfied:

$$= \; (m_P \otimes \mathrm{id}_{A_P}) \circ (\mathrm{id}_{A_P} \otimes \Delta_P). \qquad (3.10)$$

Here it is used that according to remark 3.6 we are allowed to remove and insert idempotents P, and then the Frobenius property of A is invoked. The other half of the Frobenius property is seen analogously.

(ii) The following chain of equalities shows that A_P is symmetric:

$$(3.11)$$

12

Here the notations

$$b_X \;=\; \underset{\ X \qquad\ X^\vee\ }{\bigcup} \qquad \text{and} \qquad \tilde{b}_X \;=\; \underset{\ X^\vee \qquad\ X\ }{\bigcup} \tag{3.12}$$

are used for the duality morphisms b_X and $\tilde{b}_X$, respectively, of an object X. The morphisms d_X and $\tilde{d}_X$ are drawn in a similar way.

(iii) We have $\varepsilon_P \circ \eta_P = \varepsilon \circ e \circ r \circ \eta = \varepsilon \circ \eta$, which is nonzero by specialness of A. As A_P is associative, m_P is a morphism of A_P-bimodules. The Frobenius property ensures that Δ_P is also a morphism of bimodules. Hence $m_P \circ \Delta_P$ is a morphism of bimodules, and by absolute simplicity of A_P it is a multiple of the identity. Moreover, $m_P \circ \Delta_P$ is not zero: we have

$$\varepsilon_P \circ m_P \circ \Delta_P \circ \eta_P = \varepsilon \circ m \circ (\mathrm{id}_A \otimes P) \circ \Delta \circ \eta \tag{3.13}$$

which, as A is symmetric, is equal to $\mathrm{tr}_{\mathrm{l}}(P) = \dim_{\mathrm{l}}(A_P) \neq 0$. We conclude that $m_P \circ \Delta_P \neq 0$. Hence A_P is special. $\qquad \square$

Remark 3.8. In the above discussion the category $\mathcal{D}$ is assumed to be abelian, but this assumption can be relaxed. Of the properties of an abelian category we only used that the morphism sets are abelian groups, that composition is bilinear, and that the relevant idempotents factorise in a monic and an epi, i.e. that $\mathcal{D}$ is idempotent complete. In addition we assumed that morphisms sets are finite-dimensional $\Bbbk$-vector spaces.

From eq. (3.3) onwards, and in particular in proposition 3.7, it is in addition used that $\mathcal{D}$ is enriched over $\mathcal{V}ect_\Bbbk$. If this is not the case, one can no longer, in general, define an idempotent P through $\frac{1}{|H|} \sum_{\alpha \in H} \alpha$, and there need not exist a coproduct on the fixed algebra A^H, even if there is one on A.

4 Algebras in the Morita class of the tensor unit

Recall that according to our convention 2.1 $(\mathcal{D}, \otimes, \mathbf{1})$ is abelian strict monoidal, with simple and absolutely simple tensor unit and enriched over $\mathcal{V}ect_\Bbbk$ with $\Bbbk$ of characteristic zero. From now on we further assume that $\mathcal{D}$ is skeletally small and sovereign.

We now associate to an algebra (A, m, η) in $\mathcal{D}$ a specific subgroup of its automorphism group – the inner automorphisms – which are defined as follows. The space $\mathrm{Hom}(\mathbf{1}, A)$ becomes a $\Bbbk$-algebra by defining the product as $f * g := m \circ (f \otimes g)$ for $f, g \in \mathrm{Hom}(\mathbf{1}, A)$. The morphism $\eta \in \mathrm{Hom}(\mathbf{1}, A)$ is a unit for this product. We call a morphism f in $\mathrm{Hom}(\mathbf{1}, A)$ invertible iff there exists a morphism $f^- \in \mathrm{Hom}(\mathbf{1}, A)$ such that $f * f^- = \eta = f^- * f$. Now the morphism

$$\omega_f := m \circ (m \otimes f^-) \circ (f \otimes \mathrm{id}_A) \in \mathrm{Hom}(A, A) \tag{4.1}$$

is easily seen to be an algebra automorphism. The automorphisms of this form are called *inner* automorphisms; they form a normal subgroup $\mathrm{Inn}(A) \leq \mathrm{Aut}(A)$ as is seen below.

Definition 4.1. For A an algebra in $\mathcal{D}$ and $\alpha, \beta \in \operatorname{Aut}(A)$, the A-bimodule $_\alpha A_\beta = (A, \rho_\alpha, \varrho_\beta)$ is the bimodule which has A as underlying object and left and right actions of A given by

$$\rho_\alpha := m \circ (\alpha \otimes \operatorname{id}_A) \qquad \text{and} \qquad \varrho_\beta := m \circ (\operatorname{id}_A \otimes \beta), \tag{4.2}$$

respectively. These left and right actions of A are said to be *twisted* by α and β, respectively, and $_\alpha A_\beta$ is called a *twisted bimodule*.

That this indeed defines an A-bimodule structure on the object A is easily checked with the help of the multiplicativity and unitality of α and β. Further, as shown in [VZ, FRS3], the bimodules $_\alpha A_\beta$ are invertible. Denote the isomorphism class of a bimodule X by $[X]$. By setting

$$\Psi_A(\alpha) := [_{\operatorname{id}} A_\alpha] \tag{4.3}$$

one obtains an exact sequence

$$0 \longrightarrow \operatorname{Inn}(A) \longrightarrow \operatorname{Aut}(A) \xrightarrow{\Psi_A} \operatorname{Pic}(\mathcal{D}_{A|A}) \tag{4.4}$$

of groups. In particular one sees that the subgroup $\operatorname{Inn}(A)$ is in fact a normal subgroup, as it is the kernel of the homomorphism Ψ_A. The proof of exactness of this sequence in [VZ, FRS3] is not only valid in braided monoidal categories, but also in the present more general situation.

Let now A and B be Morita equivalent algebras in $\mathcal{D}$, with $A \xleftrightarrow{P,Q} B$ a Morita context $(P \equiv {}_A P_B, Q \equiv {}_B Q_A)$. Then the mapping

$$\begin{aligned} \Pi_{Q,P}: \quad \operatorname{Pic}(\mathcal{D}_{A|A}) \quad &\xrightarrow{\cong} \quad \operatorname{Pic}(\mathcal{D}_{B|B}) \\ [X] \quad &\longmapsto \quad [Q \otimes_A X \otimes_A P] \end{aligned} \tag{4.5}$$

constitutes an isomorphism between the Picard groups $\operatorname{Pic}(\mathcal{D}_{A|A})$ and $\operatorname{Pic}(\mathcal{D}_{B|B})$. In particular, if A is an algebra that is Morita equivalent to the tensor unit $\mathbf{1}$, then we have an isomorphism $\operatorname{Pic}(\mathcal{D}_{A|A}) \cong \operatorname{Pic}(\mathcal{D})$. As Morita equivalent algebras need not have isomorphic automorphism groups, the images of the group homomorphisms $\Psi_A \colon \operatorname{Aut}(A) \to \operatorname{Pic}(\mathcal{D}_{A|A})$ and $\Psi_B \colon \operatorname{Aut}(B) \to \operatorname{Pic}(\mathcal{D}_{B|B})$ will in general be non-isomorphic.

In the following we will consider subgroups of the group $\operatorname{Pic}(\mathcal{D})$. For a subgroup $H \le \operatorname{Pic}(\mathcal{D})$ we put

$$Q \equiv Q(H) := \bigoplus_{h \in H} L_h. \tag{4.6}$$

Remark 4.2. Since the object Q is the direct sum over a whole subgroup of $\operatorname{Pic}(\mathcal{D})$ and $L_g^\vee \cong L_{g^{-1}}$, it follows that $Q \cong Q^\vee$. As a consequence, left and right dimensions of Q are equal, and accordingly in the sequel we use the notation $\dim(Q)$ for both of them.

Proposition 4.3. *Let $H \le \operatorname{Pic}(\mathcal{D})$ be a finite subgroup such that $\dim(Q) \ne 0$ for $Q \equiv Q(H)$. Then the algebra*

$$A \equiv A(H) := Q \otimes Q^\vee \tag{4.7}$$

satisfies $H = \operatorname{im}(\Pi_{Q^\vee, Q} \circ \Psi_A)$, where $\mathbf{1} \xleftrightarrow{Q^\vee, Q} A$ is the Morita context introduced in lemma 2.11.

14

Proof. The isomorphism $\Pi_{Q,Q^\vee}\colon \mathrm{Pic}(\mathcal{D}) \to \mathrm{Pic}(\mathcal{D}_{A|A})$ is given by $[L_g] \mapsto [Q \otimes L_g \otimes Q^\vee]$. For $h \in H$ we want to find automorphisms α_h of A such that ${}_{\mathrm{id}}A_{\alpha_h} \cong Q \otimes L_h \otimes Q^\vee$ as A-bimodules. We first observe the isomorphisms $Q \otimes L_h \cong \bigoplus_{g \in H} L_g \otimes L_h \cong \bigoplus_{g \in H} L_{gh} \cong Q$. We make a (in general non-canonical) choice of isomorphisms $f_h\colon Q \otimes L_h \xrightarrow{\cong} Q$, with the morphism f_1 chosen to be the identity id_Q. Then for each $h \in H$ we define the endomorphism α_h of $Q \otimes Q^\vee$ by

$$\alpha_h \ := \qquad\qquad \tag{4.8}$$

These are algebra morphisms:

$$m \circ (\alpha_h \otimes \alpha_h) \ = \qquad\qquad = $$

$$= \qquad\qquad = \qquad = \ \alpha_h \circ m\,. \tag{4.9}$$

Here we have used that by lemma 2.14(ii) we have $\dim_{l|r}(L_h) = 1$ for $h \in H$. The third equality is then a consequence of $\mathrm{id}_{L_h^\vee \otimes L_h} = \tilde{b}_{L_h} \circ d_{L_h}$, see equation (2.18); in the fourth equality f_h is cancelled against f_h^{-1} by using properties of the duality. (Also, for better readability, here and below we refrain from labelling some of the L_h-lines.)

Further, the morphisms α_h are also unital:

$$\alpha_h \circ \eta \ = \qquad = \qquad = \qquad = \ \eta, \tag{4.10}$$

where again by lemma 2.14 we have $\dim_r(L_h) = 1$. The inverse of α_h is given by

$$\alpha_h^{-1} \ = \qquad\qquad \tag{4.11}$$

15

as is seen in the following calculations:

$$\alpha_h \circ \alpha_h^{-1} \;=\; \text{[diagram]} \;=\; \text{[diagram]} \;=\; \operatorname{id}_{Q\otimes Q^\vee}\,,$$

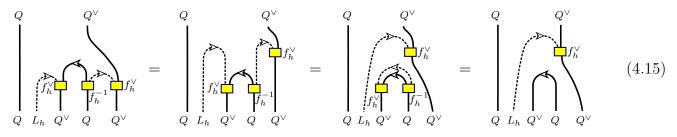$$\alpha_h^{-1} \circ \alpha_h \;=\; \text{[diagram]} \;=\; \text{[diagram]} \;=\; \operatorname{id}_{Q\otimes Q^\vee}\,. \tag{4.12}$$

where in particular (2.18) and $\dim_{\mathrm{l|r}} L_h = 1$ is used.

A bimodule isomorphism $Q \otimes L_h \otimes Q^\vee \to {}_{\mathrm{id}}A_{\alpha_h}$ is now given by

$$F_h := \operatorname{id}_Q \otimes \left((\tilde{d}_{L_h} \otimes \operatorname{id}_{Q^\vee}) \circ (\operatorname{id}_{L_h} \otimes f_h^\vee) \right). \tag{4.13}$$

First we see that F_h is invertible with inverse $F_h^{-1} = \operatorname{id}_Q \otimes \left((\operatorname{id}_{L_h} \otimes f_h^{-\vee}) \circ (b_{L_h} \otimes \operatorname{id}_{Q^\vee}) \right)$, where $f_h^{-\vee}$ stands for the dual of the inverse of f_h. That F_h^{-1} is indeed inverse to F_h is seen as follows:

$$F_h \circ F_h^{-1} \;=\; \text{[diagram]} \;=\; \text{[diagram]} \;=\; \operatorname{id}_{Q\otimes Q^\vee}\,,$$

$$F_h^{-1} \circ F_h \;=\; \text{[diagram]} \;=\; \text{[diagram]} \;=\; \operatorname{id}_{Q\otimes L_h\otimes Q^\vee}\,. \tag{4.14}$$

Moreover, F_h clearly intertwines the left actions of A on $Q \otimes L_h \otimes Q^\vee$ and on ${}_{\mathrm{id}}A_{\alpha_h}$. That it intertwines the right actions as well is verified as follows:

$$\text{[diagram]} \;=\; \text{[diagram]} \;=\; \text{[diagram]} \;=\; \text{[diagram]} \tag{4.15}$$

Here similar steps are performed as in the proof that α_h respects the product of A. We conclude that we have $[Q \otimes L_h \otimes Q^\vee] \in \operatorname{im}(\Psi_A)$ for all $h \in H$, and thus $\Pi_{Q,Q^\vee}(H)$ is a subgroup of $\operatorname{im}(\Psi_A)$.

On the other hand, for $g \notin H$, $Q \otimes L_g \otimes Q^\vee \cong \bigoplus_{h,h' \in H} L_{hgh'}$ is not isomorphic to $Q \otimes Q^\vee$, not even as an object, so that $\Pi_{Q,Q^\vee}(g) \notin \mathrm{im}(\Psi_A)$. Together it follows that $\mathrm{im}(\Pi_{Q,Q^\vee} \circ \Psi_A) = H$. $\qquad\square$

Remark 4.4. Similarly to the calculation that the morphisms α_h in (4.8) are morphisms of algebras, one shows that they also respect the coproduct and the counit of $A(H)$. So in fact we have found automorphisms of Frobenius algebras.

We denote the inclusion morphisms $L_h \to Q = \bigoplus_{g \in H} L_g$ by e_h and the projections $Q \to L_h$ by r_h, such that $r_g \circ e_h = 0$ for $g \neq h$ and $r_g \circ e_g = \mathrm{id}_{L_g}$. Then $e_g \circ r_g = P_g$ is a nonzero idempotent in $\mathrm{End}(Q)$, and we have $\sum_{h \in H} P_h = \mathrm{id}_Q$.

Lemma 4.5. *Let $H \leq \mathrm{Pic}(\mathcal{D})$ be a finite subgroup such that $\dim(Q) \neq 0$ for $Q \equiv Q(H)$. Given $g \in H$ and an automorphism α of $A = Q \otimes Q^\vee$ such that $\Psi_A(\alpha) = [Q \otimes L_g \otimes Q^\vee]$, there exists a unique isomorphism $f_g \in \mathrm{Hom}(Q \otimes L_g, Q)$ such that $r_g \circ f_g \circ (e_\mathbf{1} \otimes \mathrm{id}_{L_g}) = \mathrm{id}_{L_g}$ and $\alpha = \alpha_g$ with α_g as in (4.8).*

Proof. We start by proving existence. Let $\varphi_g \colon Q \otimes L_g \otimes Q^\vee \xrightarrow{\cong}{}_{\mathrm{id}} A_\alpha$ be an isomorphism of bimodules. As a first step we show that $\varphi_g = \mathrm{id}_Q \otimes h$ for some morphism $h \colon L_g \otimes Q^\vee \xrightarrow{\cong} Q^\vee$:

$$\text{(4.16)}$$

Here in the first step we just inserted the dimension of Q, using that it is nonzero, and the second step is the statement that φ_g intertwines the left action of A on $Q \otimes L_g \otimes Q^\vee$ and on ${}_{\mathrm{id}}A_\alpha$. Note that upon setting

$$f_g := \frac{1}{\dim(Q)} \quad \text{(4.17)}$$

this amounts to $\varphi_g = \mathrm{id}_Q \otimes ((\tilde{d}_{L_g} \otimes \mathrm{id}_{Q^\vee}) \circ (\mathrm{id}_{L_g} \otimes f_g^\vee))$, as in proposition 4.3.

Similarly the condition that φ_g intertwines the right action of A on $Q \otimes L_g \otimes Q^\vee$ and ${}_{\mathrm{id}}A_\alpha$ means that

$$\text{(4.18)}$$

and this is equivalent to equality of the same pictures with the identity morphisms at the left sides

removed. Applying duality morphisms to both sides of the resulting equality we obtain

$$\text{(4.19)}$$

Next we apply the morphism $(\mathrm{id}_Q \otimes \tilde{d}_{L_g} \otimes \mathrm{id}_{Q^\vee}) \circ (f_g^{-1} \otimes \mathrm{id}_{L_g^\vee} \otimes \mathrm{id}_{Q^\vee})$, leading to

$$\text{(4.20)}$$

Using also that by lemma 2.14 (ii) we have $\dim_{\mathrm{r}}(L_g) = 1$, this shows that $\alpha = \alpha_g$ with α_g as in (4.8). Since L_g is absolutely simple (see lemma 2.13 (iv)), we have $r_g \circ f_g \circ (e_{\mathbf{1}} \otimes \mathrm{id}_{L_g}) = \xi \, \mathrm{id}_{L_g}$ for some $\xi \in \Bbbk$. As r_g is injective and f_g is an isomorphism, $r_g \circ f_g \colon Q \otimes L_g \to L_g$ is nonzero. But this morphism can only be nonvanishing on the direct summand $\mathbf{1}$ of Q, and hence also $r_g \circ f_g \circ (e_{\mathbf{1}} \otimes \mathrm{id}_{L_g}) \neq 0$, i.e. $\xi \in \Bbbk^\times$. Finally note that α does not change if we replace f_g by a nonzero multiple of f_g; hence after suitable rescaling f_g obeys both $\alpha = \alpha_g$ and $r_g \circ f_g \circ (e_{\mathbf{1}} \otimes \mathrm{id}_{L_g}) = \mathrm{id}_{L_g}$, thus proving existence.

To show uniqueness, suppose that there is another isomorphism $f_g' \colon Q \otimes L_g \to Q$ such that $\alpha = \alpha_g'$ (where α_g' is given by (4.8) with f_g' instead of f_g) and $r_g \circ f_g' \circ (e_{\mathbf{1}} \otimes \mathrm{id}_{L_g}) = \mathrm{id}_{L_g}$. Composing both sides of the equality $\alpha_g = \alpha_g'$ with $f_g \otimes \mathrm{id}_{Q^\vee}$ from the right and taking a partial trace over Q of the resulting morphism $Q \otimes L_g \otimes Q^\vee \to Q \otimes Q^\vee$ gives

$$(d_Q \otimes \mathrm{id}_{Q^\vee}) \circ (\mathrm{id}_{Q^\vee} \otimes \alpha_g) \circ (\mathrm{id}_{Q^\vee} \otimes f_g \otimes \mathrm{id}_{Q^\vee}) \circ (\tilde{b}_Q \otimes \mathrm{id}_{L_g} \otimes \mathrm{id}_{Q^\vee})$$
$$= (d_Q \otimes \mathrm{id}_{Q^\vee}) \circ (\mathrm{id}_{Q^\vee} \otimes \alpha_g') \circ (\mathrm{id}_{Q^\vee} \otimes f_g \otimes \mathrm{id}_{Q^\vee}) \circ (\tilde{b}_Q \otimes \mathrm{id}_{L_g} \otimes \mathrm{id}_{Q^\vee})$$

$$\text{(4.21)}$$

Substituting the explicit form of α_g and α_g', and using that $\dim(Q) \neq 0$ and that L_g is absolutely simple, one finds that $f_g = \lambda \, f_g'$ for some $\lambda \in \Bbbk$. The normalisation conditions $r_g \circ f_g \circ (e_{\mathbf{1}} \otimes \mathrm{id}_{L_g}) = \mathrm{id}_{L_g}$ and $r_g \circ f_g' \circ (e_{\mathbf{1}} \otimes \mathrm{id}_{L_g}) = \mathrm{id}_{L_g}$ then force $\lambda = 1$, proving uniqueness. $\qquad\square$

The construction of the automorphisms α_h presented in proposition 4.3 and lemma 4.5 still depends on the choice of isomorphisms f_h or φ_h. As each such automorphism gets mapped to $[Q \otimes L_h \otimes Q^\vee]$, due to exactness of the sequence (4.4) different choices of f_h lead to automorphisms which differ only by inner automorphisms. On the other hand, the mapping $h \mapsto \alpha_h$ need not be a homomorphism of groups for any choice of the isomorphisms f_h.

In the following we will formulate necessary and sufficient conditions that $H \leq \mathrm{Pic}(\mathcal{D})$ must satisfy for the assignment $h \mapsto \alpha_h$ to yield a group homomorphism from H to $\mathrm{Aut}(A)$. Recall that $L_g \otimes L_h \cong L_{gh}$. Thus by lemma 2.13 (iv) the spaces $\mathrm{Hom}(L_g \otimes L_h, L_{gh})$ are one-dimensional (but there is no canonical choice of an isomorphism to the ground field $\Bbbk$). For each pair

$g, h \in \mathrm{Pic}(\mathcal{D})$ we select a basis isomorphism $_g b_h \in \mathrm{Hom}(L_g \otimes L_h, L_{gh})$. We denote their inverses by $^g b^h \in \mathrm{Hom}(L_{gh}, L_g \otimes L_h)$, i.e. $^g b^h \circ {}_g b_h = \mathrm{id}_{L_g \otimes L_h}$ and $_g b_h \circ {}^g b^h = \mathrm{id}_{L_{gh}}$. For $g = 1$ we take $_1 b_g$ and $_g b_1$ to be the identity, which is possible by the assumed strictness of $\mathcal{D}$.

For any triple $g_1, g_2, g_3 \in \mathrm{Pic}(\mathcal{D})$ the collection $\{_g b_h\}$ of morphisms provides us with two bases of the one-dimensional space $\mathrm{Hom}(L_{g_1} \otimes L_{g_2} \otimes L_{g_3}, L_{g_1 g_2 g_3})$, namely with $_{g_1 g_2} b_{g_3} \circ (_{g_1} b_{g_2} \otimes \mathrm{id}_{L_{g_3}})$ as well as $_{g_1} b_{g_2 g_3} \circ (\mathrm{id}_{L_{g_1}} \otimes {}_{g_2} b_{g_3})$. These differ by a nonzero scalar $\psi(g_1, g_2, g_3) \in \Bbbk$:

$$_{g_1 g_2} b_{g_3} \circ (_{g_1} b_{g_2} \otimes \mathrm{id}_{L_{g_3}}) = \psi(g_1, g_2, g_3) \, _{g_1} b_{g_2 g_3} \circ (\mathrm{id}_{L_{g_1}} \otimes {}_{g_2} b_{g_3}) \,. \tag{4.22}$$

The pentagon axiom for the associativity constraints of $\mathcal{D}$ implies that ψ is a three-cocycle on the group $\mathrm{Pic}(\mathcal{D})$ with values in $\Bbbk^\times$ (see e.g. appendix E of [MS], chapter 7.5 of [FK], or [Ya]). Any other choice of bases leads to a cohomologous three-cocycle. Observe that by taking $_1 b_h$ and $_h b_1$ to be the identity on L_h the cocycle ψ is normalised, i.e. satisfies $\psi(g_1, g_2, g_3) = 1$ as soon as one of the g_i equals 1.

Lemma 4.6. *For $g, h, k \in \mathrm{Pic}(\mathcal{D})$, the bases introduced above obey the relation*

$$\tag{4.23}$$

Proof. In pictures:

$$\tag{4.24}$$

where in the second step we inserted the definition of ψ and abbreviated $\psi \equiv \psi(h, h^{-1}k, k^{-1}g)$. $\quad\square$

Definition 4.7. Given the normalised cocycle ψ on $\mathrm{Pic}(\mathcal{D})$, a normalised two-cochain ω on H with values in $\Bbbk^\times$ is called a *trivialisation* of ψ on H iff it satisfies $\mathrm{d}\omega = \psi\big|_H$.

Proposition 4.8. *Given a finite subgroup H of $\mathrm{Pic}(\mathcal{D})$ and a function $\omega \colon H \times H \to \Bbbk^\times$, define $Q := \bigoplus_{h \in H} L_h$ and*

$$
\begin{aligned}
m &\equiv m(H, \omega) := \sum_{g,h \in H} \omega(g, h) \, e_{gh} \circ {}_g b_h \circ (r_g \otimes r_h) && \in \mathrm{Hom}(Q \otimes Q, Q) \,, \\
\eta &\equiv \eta(H, \omega) := e_1 && \in \mathrm{Hom}(\mathbf{1}, Q) \,, \\
\Delta &\equiv \Delta(H, \omega) := |H|^{-1} \sum_{g,h \in H} \omega(g, h)^{-1} \, (e_g \otimes e_h) \circ {}^g b^h \circ r_{gh} && \in \mathrm{Hom}(Q, Q \otimes Q) \,, \\
\varepsilon &\equiv \varepsilon(H, \omega) := |H| \, r_1 && \in \mathrm{Hom}(Q, \mathbf{1}) \,.
\end{aligned}
\tag{4.25}
$$

Then the following statements are equivalent:

(i) *ω is a trivialisation of ψ on H.*

(ii) (Q, m, η) is an associative unital algebra.

(iii) (Q, Δ, ε) is a coassociative counital coalgebra.

Moreover, if any of these equivalent conditions holds, then $Q(H, \omega) \equiv (Q, m, \Delta, \eta, \varepsilon)$ is a special Frobenius algebra with $m \circ \Delta = \mathrm{id}_Q$ and $\varepsilon \circ \eta = |H| \, \mathrm{id}_\mathbf{1}$.

Proof. The equivalence of conditions (i) – (iii) follows by direct computation using only the definitions; we refrain from giving the details. The Frobenius property then follows with the help of lemma 4.6. □

Lemma 4.9. *Let ω be a trivialisation of ψ on $H \leq \mathrm{Pic}(\mathcal{D})$ and let $Q \equiv Q(H, \omega)$ be the special Frobenius algebra defined in proposition 4.8. Then Q is symmetric iff $\dim(Q) \neq 0$.*

Proof. Recall the morphisms Φ_1 and Φ_2 from (2.10). Since $Q = \bigoplus_{h \in H} L_h$, the condition $\Phi_1 = \Phi_2$ is equivalent to $\Phi_1 \circ e_g = \Phi_2 \circ e_g$ for all $g \in H$. By the definition of the multiplication on Q, this amounts to

$$(4.26)$$

which in turn, by applying duality morphisms and composing with the morphisms ${}^g b^{g^{-1}}$, is equivalent to

$$(4.27)$$

The right hand side of (4.27) is evaluated to

$$(4.28)$$

The first step is an application of lemma 4.6, the second is due to our convention that the morphisms ${}^1 b^g$ are chosen to be identity morphisms. So the condition that Q is a symmetric Frobenius algebra is equivalent to the condition $\omega(g, g^{-1}) = \omega(g^{-1}, g) \, \psi(g^{-1}, g, g^{-1}) \, \dim_\mathrm{l}(L_g)$ for all $g \in H$. As ω is a trivialisation of ψ and $\mathrm{d}\omega(g^{-1}, g, g^{-1}) = \omega(g, g^{-1}) / \omega(g^{-1}, g)$, this is equivalent to $\dim_\mathrm{l}(L_g) = 1$ for all $g \in H$. By lemma 2.14 the latter condition holds iff $\dim(Q) \neq 0$. □

Definition 4.10. An *admissible* subgroup of $\mathrm{Pic}(\mathcal{D})$ is a finite subgroup $H \leq \mathrm{Pic}(\mathcal{D})$ such that $\dim_{\mathrm{l}|r}(L_h) = 1$ for all $h \in H$ and such that there exists a trivialisation ω of ψ on H.

Remark 4.11. We see that $Q(H, \omega)$ is a symmetric special Frobenius algebra if and only if H is an admissible subgroup of $\mathrm{Pic}(\mathcal{D})$ and ω is a trivialisation of ψ on H. One can show that every structure of a special Frobenius algebra on the object $Q = \bigoplus_{h \in H} L_h$ is of the type $Q(H, \omega)$ described in proposition 4.8 for a suitable trivialisation ω of ψ (see [FRS2], proposition 3.14). So giving product and coproduct morphisms on Q is equivalent to giving a trivialisation ω of ψ.

Also observe that multiplying a trivialisation ω of ψ with a two-cocycle γ on H gives another trivialisation ω' of ψ. One can show that the Frobenius algebras $Q(H, \omega)$ and $Q(H, \omega')$ are isomorphic as Frobenius algebras if and only if ω and ω' differ by multiplication with an exact two-cocycle. Accordingly, in the sequel we call two trivialisations ω and ω' for ψ equivalent iff $\omega / \omega' = \mathrm{d}\eta$ for some one-cochain η. Thus if ψ is trivialisable on H, then the equivalence classes of trivialisations form a torsor over $H^2(H, \Bbbk^\times)$.

Theorem 4.12. *Let H be an admissible subgroup of $\mathrm{Pic}(\mathcal{D})$, put $Q = Q(H)$ as in (4.6), and let $A = A(H) = Q \otimes Q^\vee$ be the algebra defined in (4.7). Then there is a bijection between*

- *trivialisations ω of ψ on H and*
- *group homomorphisms $\alpha \colon H \to \mathrm{Aut}(A)$ with $\Pi_{Q^\vee, Q} \circ \Psi_A \circ \alpha = \mathrm{id}_H$.*

Proof. Denote by $\mathcal{T}$ the set of all trivialisations ω of ψ on H, and by $\mathcal{H}$ the set of all group homomorphisms $\alpha \colon H \to \mathrm{Aut}(A)$ satisfying $\Pi_{Q^\vee, Q} \circ \Psi_A \circ \alpha = \mathrm{id}_H$. The proof that $\mathcal{T} \cong \mathcal{H}$ as sets is organised in three steps: defining maps $F \colon \mathcal{T} \to \mathcal{H}$ and $G \colon \mathcal{H} \to \mathcal{T}$, and showing that they are each other's inverse.

(i) Let $\omega \in \mathcal{T}$. For each $h \in H$ define

$$
f_h := \sum_{g \in H} \omega(g, h) \quad \begin{array}{c} Q \\ \bigtriangledown_{e_{gh}} \\ \square_{g b_h} \\ r_g \bigtriangleup \\ Q \quad L_h \end{array}
\tag{4.29}
$$

Since ω takes values in $\Bbbk^\times$, these are in fact isomorphisms, with inverse given by

$$
f_h^{-1} = \sum_{g \in H} \omega(g, h)^{-1} \left(e_g \otimes \mathrm{id}_{L_h} \right) \circ {}^g b^h \circ r_{gh} \,.
\tag{4.30}
$$

Define the function $F(\omega)$ from H to $\mathrm{Aut}(A)$ by $F(\omega) \colon h \mapsto \alpha_h$ for α_h given by (4.8) with f_h as in (4.29). We proceed to show that $F(\omega) \in \mathcal{H}$.

Abbreviate $\alpha \equiv F(\omega)$. That $\Pi_{Q^\vee, Q} \circ \Psi_A \circ \alpha = \mathrm{id}_H$ follows from the proof of proposition 4.3. To see

that $\alpha(g)\circ\alpha(h)=\alpha(gh)$, first rewrite $\alpha(g)\circ\alpha(h)$ using (4.29) and (4.30):

$$\alpha(g)\circ\alpha(h) \;=\; \sum_{k,l,m,n}\frac{\omega(k,g)\,\omega(l,h)}{\omega(m,g)\,\omega(n,h)}\;\;[\text{diagram}]\;\;=\;\sum_{k,m}\frac{\omega(k,g)\,\omega(kg,h)}{\omega(m,g)\,\omega(mg,h)}\;\;[\text{diagram}]$$

$$=\;\sum_{k,m}\xi_{km}\;\;[\text{diagram}]\;\;=\;\sum_{k,m}\xi_{km}\;\;[\text{diagram}] \tag{4.31}$$

with

$$\xi_{km}\;=\;\frac{\omega(k,g)\,\omega(kg,h)\,\psi(k,g,h)}{\omega(m,g)\,\omega(mg,h)\,\psi(m,g,h)}\;. \tag{4.32}$$

Here the second step uses that there are no nonzero morphisms $L_n\to L_{mg}$ unless $n=mg$; by the same argument we conclude that $l=kg$. In the third step one applies relation (4.22). Now the condition $\alpha(g)\circ\alpha(h)=\alpha(gh)$ is equivalent to

$$\frac{\omega(k,g)\,\omega(kg,h)\,\psi(k,g,h)}{\omega(m,g)\,\omega(mg,h)\,\psi(m,g,h)}=\frac{\omega(k,gh)}{\omega(m,gh)}\quad\text{for all }\;m,k\in H\,, \tag{4.33}$$

which in turn can be rewritten as

$$\frac{\mathrm{d}\omega(m,g,h)}{\mathrm{d}\omega(k,g,h)}=\frac{\psi(m,g,h)}{\psi(k,g,h)}\quad\text{for all }\;m,k\in H\,. \tag{4.34}$$

The last condition is satisfied because by assumption $\mathrm{d}\omega=\psi|_H$. So indeed we have $F(\omega)\in\mathcal{H}$.

(ii) Given $\alpha\in\mathcal{H}$, for each $h\in H$ the automorphism $\alpha(h)$ satisfies the conditions of lemma 4.5. As a consequence we obtain a unique isomorphism $f_h\colon Q\otimes L_h\to Q$ such that $\alpha(h)=\alpha_h$ and $r_h\circ f_h\circ(e_{\mathbf{1}}\otimes\mathrm{id}_{L_h})=\mathrm{id}_{L_h}$. Define a function $\omega\colon H\times H\to\Bbbk$ via

$$\omega(g,h)\;\;[\text{diagram}]\;\;=\;\;[\text{diagram}] \tag{4.35}$$

Then define the map G from $\mathcal{H}$ to functions $H\times H\to\Bbbk$ by $G(\alpha):=\omega$, with ω obtained as in (4.35). We will show that $G(\alpha)\in\mathcal{T}$.

Given $\alpha \in \mathcal{H}$, abbreviate $\omega \equiv G(\alpha)$. First note that ω takes values in $\Bbbk^\times$, as f_h is an isomorphism. Next, the normalisation condition $r_h \circ f_h \circ (e_{\mathbf{1}} \otimes \mathrm{id}_{L_h}) = \mathrm{id}_{L_h}$ implies $\omega(1, h) = 1$ for all $h \in H$. Since α is a group homomorphism we have $\alpha(1) = \mathrm{id}_{Q \otimes Q^\vee}$. By the uniqueness result of lemma 4.5 this implies that $f_1 = \mathrm{id}_Q$, and so $\omega(g, 1) = 1$ for all $g \in H$. Altogether it follows that ω is a normalised two-cochain with values in $\Bbbk^\times$. By following again the steps (4.31) to (4.34) one shows that, since α is a group homomorphism, ω must satisfy (4.34). Setting $k = 1$ and using that ω and ψ are normalised finally demonstrates that $\mathrm{d}\omega = \psi\big|_H$. Thus indeed $G(\alpha) \in \mathcal{T}$.

(iii) That $F(G(\alpha)) = \alpha$ is immediate by construction, and that $G(F(\omega)) = \omega$ follows from the uniqueness result of lemma 4.5. $\qquad\square$

We have seen that if H is an admissible subgroup of $\mathrm{Pic}(\mathcal{D})$ and ω a trivialisation of ψ, then $Q(H, \omega)$ is a symmetric special Frobenius algebra. Using the product and coproduct morphisms of $Q(H, \omega)$, the automorphisms α_h induced by ω as described in theorem 4.12 can be written as

$$\alpha_h \;=\; |H| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.36)$$

Note that in this picture the circle on the left stands for the coproduct of Q, while the circle on the right stands for the dual of the product.

Given a trivialisation ω of ψ, theorem 4.12 allows us to realise H as a subgroup of $\mathrm{Aut}(A)$. In particular the fixed algebra under the action of H is well defined; we denote it by A^H. We already know from proposition 3.7 that A^H is a symmetric Frobenius algebra. It will turn out that it is isomorphic to $Q(H, \omega)$. In particular, by remark 4.11 any equivalent choice of a trivialisation ω' of ψ will give an isomorphic fixed algebra.

Theorem 4.13. *Let H be an admissible subgroup of $\mathrm{Pic}(\mathcal{D})$ with trivialisation ω, put $A = A(H)$ as in (4.7), and embed $H \to \mathrm{Aut}(A)$ as in theorem 4.12. Then the fixed algebra A^H is well defined and it is isomorphic to $Q(H, \omega)$.*

Proof. Let $Q \equiv Q(H, \omega)$. By lemma 2.14 we have $\dim(Q) = |H|$. Identify H with its image in $\mathrm{Aut}(A)$ via the embedding $H \to \mathrm{Aut}(A)$ determined by ω as in theorem 4.12. Now define morphisms $i \colon Q \to A$ and $s \colon A \to Q$ by

$$i := (m \otimes \mathrm{id}_{Q^\vee}) \circ (\mathrm{id}_Q \otimes b_Q)\,, \qquad s := (\mathrm{id}_Q \otimes \tilde{d}_Q) \circ (\Delta \otimes \mathrm{id}_{Q^\vee})\,. \qquad (4.37)$$
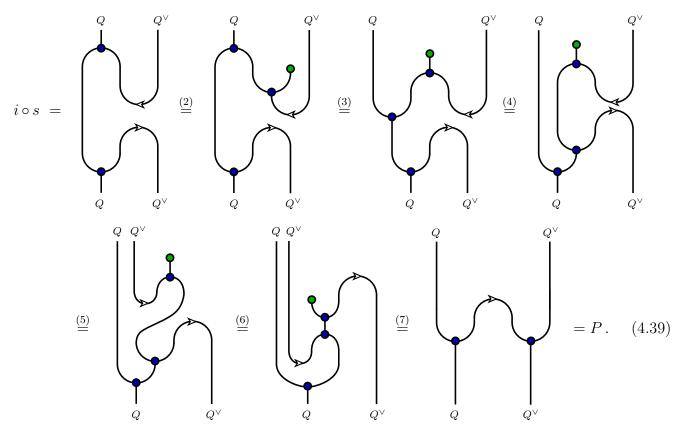
We have $s \circ i = \mathrm{id}_Q$, implying that i is monic and Q is a retract of A. We claim that i is the inclusion morphism of the fixed algebra. Recall from (3.3) the definition $P = |H|^{-1} \sum_{h \in H} \alpha_h$ of the idempotent corresponding to the fixed algebra object. In the case under consideration, P
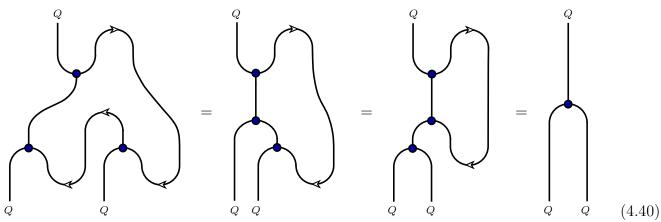
takes the form

$$P \;=\; \text{(diagram)} \tag{4.38}$$

as follows from (4.36) together with $\mathrm{id}_Q = \sum_{h \in H} P_h$. We calculate that $i \circ s = P$:

$$i \circ s \;=\; \text{(diagram)} \;\overset{(2)}{=}\; \text{(diagram)} \;\overset{(3)}{=}\; \text{(diagram)} \;\overset{(4)}{=}\; \text{(diagram)}$$

$$\overset{(5)}{=}\; \text{(diagram)} \;\overset{(6)}{=}\; \text{(diagram)} \;\overset{(7)}{=}\; \text{(diagram)} \;=\; P . \tag{4.39}$$

Here in the second step a counit morphism is introduced and in the third step the Frobenius property is applied. The next step uses coassociativity, while the fifth step follows because $Q(H, \omega)$ is symmetric. Then one uses the Frobenius property and duality. So Q satisfies the universal property of the image of P and hence the universal property of the fixed algebra. We now calculate the product morphism that the fixed algebra inherits from $Q \otimes Q^\vee$, starting from (3.4):

$$\text{(diagram)} \;=\; \text{(diagram)} \;=\; \text{(diagram)} \;=\; \text{(diagram)} \tag{4.40}$$

24

The first step uses duality, the second one holds by associativity of m. In the last step one uses that $s \circ i = \mathrm{id}_Q$. The inherited unit morphism is given by

$$ (4.41) $$

which due to $\dim_{\Bbbk} \mathrm{Hom}(\mathbf{1}, Q) = 1$ is equal to $\zeta \eta$ for some $\zeta \in \Bbbk$. Applying ε to both these morphisms and using that $\dim(Q) = |H|$, we see that $\zeta = 1$. Similarly one shows that the coproduct and counit morphisms that Q inherits as a fixed algebra equal those defined in proposition 4.8. So $Q(H, \omega)$ is isomorphic to the fixed algebra A^H as a Frobenius algebra. □

Remark 4.14. The algebra structure on the object $Q(H, \omega)$ is a kind of twisted group algebra of the group H which is not twisted by a closed two-cochain, but rather by a trivialisation of the associator of $\mathcal{D}$. Algebras of this type have appeared in applications in conformal field theory [FRS2].

5 Algebras in general Morita classes

In this section we solve the problem discussed in the previous section for algebras that are not Morita equivalent to the tensor unit. Throughout this section we will assume the following.

Convention 5.1. $(\mathcal{C}, \otimes, \mathbf{1})$ has the properties listed in convention 2.1 and is in addition skeletally small and sovereign. $(A, m, \eta, \Delta, \varepsilon)$ is a simple and absolutely simple symmetric normalised special Frobenius algebra in $\mathcal{C}$, and $H \leq \mathrm{Pic}(\mathcal{C}_{A|A})$ is a finite subgroup.

Recall from remark 2.7 that the conditions above imply $\dim(A) \neq 0$. In the sequel we will find a symmetric special Frobenius algebra $A' = A'(H)$ and a Morita context $A \xleftrightarrow{P, P'} A'$ in $\mathcal{C}$, such that H is a subgroup of $\mathrm{im}(\Pi_{P,P'} \circ \Psi_{A'})$, where $\Pi_{P,P'}$ is the isomorphism introduced in (4.5). This generalises the results of proposition 4.3.

We will apply some of the results of the previous section to the strictification $\mathcal{D}$ of the category $\mathcal{C}_{A|A}$. Note that $\mathcal{D}$ has the properties stated in convention 2.1 and is in addition sovereign, as can be seen by straightforward calculations which are parallel to those of [FS] for the category $\mathcal{C}_A$ of left A-modules. By applying the inverse equivalence functor $\mathcal{D} \xrightarrow{\simeq} \mathcal{C}_{A|A}$ this will then yield a symmetric special Frobenius algebra in $\mathcal{C}_{A|A}$ that has the desired properties. Note that the graphical representations of morphisms used below are meant to represent morphisms in $\mathcal{C}$. Pertinent facts about the structure of the category $\mathcal{C}_{A|A}$ are collected in appendix A; in the sequel we will freely use the terminology presented there. It is worth emphasising that for establishing various of the results below, it is essential that A is not just an algebra in $\mathcal{C}$, but even a simple and absolutely simple symmetric special Frobenius algebra.

As a first step we study how concepts like algebras and modules over algebras can be transported from $\mathcal{C}_{A|A}$ to $\mathcal{C}$. If X is an object of $\mathcal{C}_{A|A}$, i.e. an A-bimodule in $\mathcal{C}$, we denote the corresponding object of $\mathcal{C}$ by $\dot{X}$.
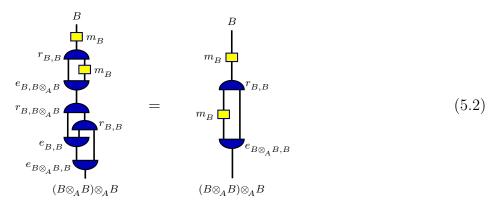
Proposition 5.2.

(i) *Let (B, m_B, η_B) be an algebra in $\mathcal{C}_{A|A}$. Then $(\dot{B}, m_B \circ r_{B,B}, \eta_B \circ \eta)$ is an algebra in $\mathcal{C}$.*

(ii) *If $(C, \Delta_C, \varepsilon_C)$ is a coalgebra in $\mathcal{C}_{A|A}$, then $(\dot{C}, e_{C,C} \circ \Delta_C, \varepsilon \circ \varepsilon_C)$ is a coalgebra in $\mathcal{C}$.*

(iii) *A morphism $\gamma \colon B \to B'$ of algebras in $\mathcal{C}_{A|A}$ is also a morphism of algebras in $\mathcal{C}$.*

(iv) *Let (B, m_B, η_B) be an algebra in $\mathcal{C}_{A|A}$ and (M, ρ) be a left B-module in $\mathcal{C}_{A|A}$. Then $(\dot{M}, \rho \circ r_{B,M})$ is a left $\dot{B}$-module in $\mathcal{C}$. Similarly right B-modules and B-bimodules in $\mathcal{C}_{A|A}$ can be transported to $\mathcal{C}$. Further, if $f \colon (M, \rho_M) \to (N, \rho_N)$ is a morphism of left B-modules in $\mathcal{C}_{A|A}$, then f is also a morphism of left $\dot{B}$-modules in $\mathcal{C}$, and an analogous statement holds for morphisms of right- and bimodules.*

(v) *If $(B, m_B, \Delta_B, \eta_B, \varepsilon_B)$ is a Frobenius algebra in $\mathcal{C}_{A|A}$, then $(\dot{B}, m \circ r_{B,B}, e_{B,B} \circ \Delta, \eta_B \circ \eta, \varepsilon \circ \varepsilon_B)$ is a Frobenius algebra in $\mathcal{C}$. If B is special in $\mathcal{C}_{A|A}$, then $\dot{B}$ is special in $\mathcal{C}$. If B is symmetric in $\mathcal{C}_{A|A}$, then $\dot{B}$ is symmetric in $\mathcal{C}$.*

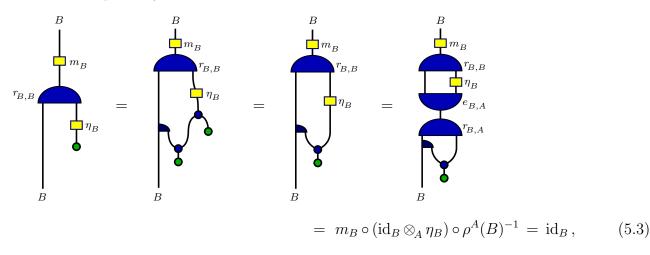*Proof.* (i) That m_B is an associative product for B in $\mathcal{C}_{A|A}$ means that

$$m_B \circ (\mathrm{id}_B \otimes_A m_B) \circ \alpha_{B,B,B} = m_B \circ (m_B \otimes_A \mathrm{id}_B)\,, \tag{5.1}$$

where $\alpha_{B,B,B}$ is the associator defined in (A.4). After inserting the definitions of the tensor product of morphisms and the associator $\alpha_{B,B,B}$ this reads

$$\tag{5.2}$$

After composing both sides of this equality with the morphism $r_{B \otimes_A B, B} \circ (r_{B,B} \otimes \mathrm{id}_B)$ the resulting idempotents $P_{B, B \otimes_A B}$, $P_{B \otimes_A B, B}$ and $P_{B,B}$ can be dropped from the left hand side, and $P_{B \otimes_A B, B}$ from the right hand side. We see that $m_B \circ r_{B,B}$ is indeed an associative product for $\dot{B}$ in $\mathcal{C}$. Next

consider the morphism η_B; we have

$$= m_B \circ (\mathrm{id}_B \otimes_A \eta_B) \circ \rho^A(B)^{-1} = \mathrm{id}_B, \qquad (5.3)$$

with ρ^A the unit constraint as given by (A.7) in the appendix. Here in the first step the idempotent $P_{B,B}$ is introduced and then moved downwards, and likewise in the third step. The fifth step is the unit property of η_B in $\mathcal{C}_{A|A}$. So we see that $\eta_B \circ \eta$ is indeed a right unit for $\dot{B}$ in $\mathcal{C}$, similarly one shows that it is also a left unit.

(ii) is proved analogously to the preceding statement.

(iii) Let m_B and $m_{B'}$ denote the products of B and B' in $\mathcal{C}_{A|A}$. Then

$$\gamma \circ m_B \circ r_{B,B} = m_{B'} \circ (\gamma \otimes_A \gamma) \circ r_{B,B} = m_{B'} \circ r_{B',B'} \circ (\gamma \otimes \gamma) \circ P_{B,B}$$
$$= m_{B'} \circ r_{B',B'} \circ P_{B',B'} \circ (\gamma \otimes \gamma) = m_{B'} \circ r_{B',B'} \circ (\gamma \otimes \gamma), \qquad (5.4)$$

where the third equality uses that γ is a morphism in $\mathcal{C}_{A|A}$. Further we have $\gamma \circ \eta_B \circ \eta = \eta_{B'} \circ \eta$, as γ respects the unit η_B of B in $\mathcal{C}_{A|A}$. So γ is also a morphism of algebras in $\mathcal{C}$.

(iv) The statement that M is a left B-module in $\mathcal{C}_{A|A}$ reads

$$\rho \circ (\mathrm{id}_B \otimes_A \rho) \circ \alpha_{B,B,M} = \rho \circ (m_B \otimes_A \mathrm{id}_M), \qquad (5.5)$$

which is an equality in $\mathrm{Hom}_{A|A}((B \otimes_A B) \otimes_A M, M)$. Similarly to the proof in i), one shows that this indeed implies that $(M, \rho \circ r_{B,M})$ is a left $\dot{B}$-module in $\mathcal{C}$.

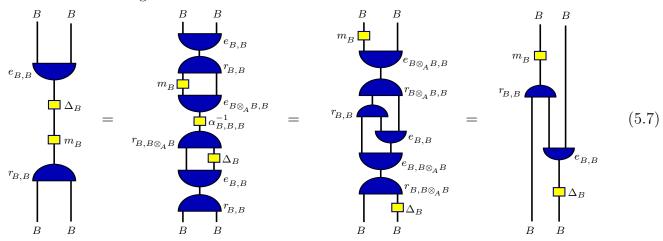Now for any morphism $f: (M, \rho_M) \to (N, \rho_N)$ we have

$$f \circ \rho_M \circ r_{B,M} = \rho_N \circ (\mathrm{id}_B \otimes_A f) \circ r_{B,M} = \rho_N \circ r_{B,N} \circ (\mathrm{id}_B \otimes f) \circ P_{B,M}$$
$$= \rho_N \circ r_{B,N} \circ P_{B,N} \circ (\mathrm{id}_B \otimes f) = \rho_N \circ r_{B,N} \circ (\mathrm{id}_B \otimes f), \qquad (5.6)$$

showing that f is a morphism of left $\dot{B}$-modules in $\mathcal{C}$. Similarly one verifies the conditions for right- and bimodules.

(v) To see that the product and coproduct morphisms for $\dot{B}$ satisfy the Frobenius property in $\mathcal{C}$

consider the following calculation:

$$(5.7)$$

The first equality is the assertion that B is a Frobenius algebra in $\mathcal{C}_{A|A}$, the second one implements the definition of $\alpha^{-1}_{B,B,B}$. In the last step the resulting idempotents are moved up or down, upon which they can be dropped. A parallel argument establishes the second identity.

Similarly one checks that specialness and symmetry of $\dot{B}$ are transported to $\mathcal{C}$ as well. $\square$
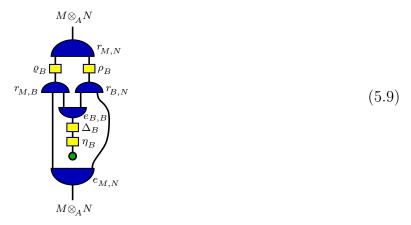
To apply the results of the previous section to the algebra A we also need to deal with Morita equivalence in $\mathcal{C}_{A|A}$. We start with the following observation.

Lemma 5.3. *Let B be a symmetric special Frobenius algebra in $\mathcal{C}_{A|A}$, and C and D be algebras in $\mathcal{C}_{A|A}$ and let $({}_C M_B, \rho_C, \varrho_B)$ be a C-B-bimodule and $({}_B N_D, \rho_B, \varrho_D)$ a B-D-bimodule. Then the tensor product $M \otimes_B N$ in $\mathcal{C}_{A|A}$ is isomorphic, as a $\dot{C}$-$\dot{D}$-bimodule in $\mathcal{C}$, to the tensor product $\dot{M} \otimes_{\dot{B}} \dot{N}$ over the algebra $\dot{B}$ in $\mathcal{C}$.*

Proof. Since B is symmetric special Frobenius in $\mathcal{C}_{A|A}$, the idempotent $P^B_{M,N}$ corresponding to the tensor product of M and N over B is well defined in $\mathcal{C}_{A|A}$. Explicitly it reads
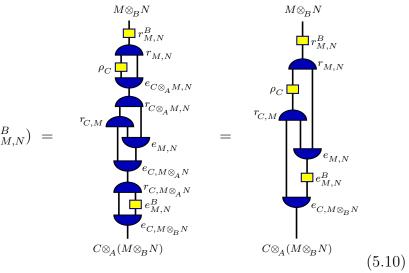
$$(\varrho_B \otimes_A \rho_B) \circ \alpha^{-1}_{M,B,B \otimes_A N} \circ (\mathrm{id}_M \otimes_A \alpha_{B,B,N}) \circ (\mathrm{id}_M \otimes_A (\Delta_B \circ \eta_B) \otimes_A \mathrm{id}_N)) \circ (\mathrm{id}_M \otimes_A \lambda^A(N)^{-1}) . \quad (5.8)$$

One calculates that this equals the morphism given by the following morphism in $\mathcal{C}$:

$$(5.9)$$

Composing with the morphisms $e_{M,N}$ and $r_{M,N}$ for the tensor product over A, we see that $P^{\dot{B}}_{\dot{M},\dot{N}} = e_{M,N} \circ P^B_{M,N} \circ r_{M,N} = e_{M,N} \circ e^B_{M,N} \circ r^B_{M,N} \circ r_{M,N}$. This furnishes a different decomposition

28

of $P_{\dot{M},\dot{N}}^{\dot{B}}$ into a monic and an epi, hence there is an isomorphism $f\colon M\otimes_B N \xrightarrow{\cong} \dot{M}\otimes_{\dot{B}}\dot{N}$ of the images of $P_{M,N}^B$ and $P_{\dot{M},\dot{N}}^{\dot{B}}$, such that $f\circ r_{M,N}^B\circ r_{M,N}=r_{\dot{M},\dot{N}}^{\dot{B}}$ and $e_{M,N}\circ e_{M,N}^B=e_{\dot{M},\dot{N}}^{\dot{B}}\circ f$. Now the left action of C on $M\otimes_B N$ is given by

$$
r_{M,N}^B\circ(\rho_C\otimes_A\mathrm{id}_N)\circ\alpha_{C,M,N}^{-1}\circ(\mathrm{id}_C\otimes_A e_{M,N}^B)\;=
$$

$$
\tag{5.10}
$$

Compose this morphism from the right with $r_{C,M\otimes_B N}$ and drop the resulting idempotent to get the transported left action of $\dot{C}$. Now composing with f from the left and replacing $f\circ r_{M,N}^B\circ r_{M,N}$ by $r_{\dot{M},\dot{N}}^{\dot{B}}$ and $e_{M,N}\circ e_{M,N}^B$ by $e_{\dot{M},\dot{N}}^{\dot{B}}\circ f$ shows that f also intertwines the left actions of $\dot{C}$. Similarly one shows that f is also an isomorphism of $\dot{D}$-right modules. $\qquad\square$

Corollary 5.4. *Assume that B and C are symmetric special Frobenius algebras in $\mathcal{C}_{A|A}$, and that $B\xrightarrow{P,P'}C$ is a Morita context in $\mathcal{C}_{A|A}$. Then $\dot{B}\xrightarrow{\dot{P},\dot{P}'}\dot{C}$ is a Morita context in $\mathcal{C}$.*

Proof. Follows from lemma 5.3 above. The commutativity of the diagrams (2.15) in $\mathcal{C}$ follows from the commutativity of their counterparts in $\mathcal{C}_{A|A}$. $\qquad\square$

We are now in a position to generalise the results of the previous section to the case of algebras in arbitrary Morita classes.

Proposition 5.5. *Let $H\le\mathrm{Pic}(\mathcal{C}_{A|A})$ be a finite subgroup of the Picard group of $\mathcal{C}_{A|A}$ and assume that $\dim_A(\bigoplus_{h\in H}L_h)\ne 0$ for representatives L_h of H in $\mathcal{C}_{A|A}$. Then there exists an algebra A' in $\mathcal{C}$ and a Morita context $A\xrightarrow{P,P'}A'$ in $\mathcal{C}$ such that $\Pi_{P',P}$ maps H into the image of $\Psi_{A'}$ in $\mathrm{Pic}(\mathcal{C}_{A'|A'})$. In other words, for any $h\in H$ there is an algebra automorphism β_h of A' such that the twisted bimodule ${}_{\mathrm{id}}A'_{\beta_h}$ is isomorphic to $(P'\otimes_A L_h)\otimes_A P$.*

Proof. Applying proposition 4.3 to the tensor unit of $\mathcal{D}$ yields, by the equivalence $\mathcal{D}\simeq\mathcal{C}_{A|A}$, a symmetric special Frobenius algebra B in $\mathcal{C}_{A|A}$ and a Morita context $A\xrightarrow{Q,Q'}B$ in $\mathcal{C}_{A|A}$, such that there are automorphisms β_h of B and B-bimodule isomorphisms $F_h\colon (Q'\otimes_A L_h)\otimes_A Q\xrightarrow{\cong}{}_{\mathrm{id}}B_{\beta_h}$ for all $h\in H$. By proposition 5.2 and corollary 5.4 this gives rise to a Morita context $\dot{A}\xrightarrow{P,P'}\dot{B}$ in $\mathcal{C}$, where $P=\dot{Q}$ and $P'=\dot{Q}'$. The morphisms F_h remain isomorphisms of bimodules when transported to $\mathcal{C}$, see proposition 5.2.

It follows that $(P'\otimes_A L_h)\otimes_A P\cong{}_{\mathrm{id}}\dot{B}_{\beta_h}$ as $\dot{B}$-bimodules in $\mathcal{C}$. $\qquad\square$

29

Since the algebra $\dot{B}$ might have a larger automorphism group in $\mathcal{C}$, its image under $\Psi_{\dot{B}}$ might be larger in $\mathrm{Pic}(\mathcal{C}_{\dot{B}|\dot{B}})$. So we cannot conclude that $H \cong \mathrm{im}(\Psi_{\dot{B}})$ in this case. But as we have $\mathrm{Aut}_{A|A}(B) \le \mathrm{Aut}_{\mathcal{C}}(\dot{B})$ as subgroups, we can still generalise theorem 4.12. This furnishes the main result of this paper:

Theorem 5.6. *Let $\mathcal{C}$ be a skeletally small sovereign abelian monoidal category with simple and absolutely simple tensor unit that is enriched over $\mathcal{V}\mathrm{ect}_{\Bbbk}$, with $\Bbbk$ a field of characteristic zero. Let A be a simple and absolutely simple symmetric special Frobenius algebra in $\mathcal{C}$, and let ψ be a normalised three-cocycle describing the associator of the Picard category of $\mathcal{C}_{A|A}$. Let H be an admissible subgroup of $\mathrm{Pic}(\mathcal{C}_{A|A})$ (cf. definition 4.10).*

Then there exist a symmetric special Frobenius algebra A' in $\mathcal{C}$ and a Morita context $A \xleftrightarrow{P,P'} A'$ such that for each trivialisation ω of ψ on H (cf. definition 4.7) the following holds.

(i) *There is an injective homomorphism $\alpha_{\omega}\colon H \to \mathrm{Aut}(A')$ such that $\Pi_{P,P'} \circ \Psi_{A'} \circ \alpha_{\omega} = \mathrm{id}_H$. The assignment $\omega \mapsto \alpha_{\omega}$ is injective.*

(ii) *The fixed algebra of $\mathrm{im}(\alpha_{\omega}) \le \mathrm{Aut}(A')$ is isomorphic to $\dot{Q}(H,\omega)$, where $\dot{Q}(H,\omega)$ is the algebra $Q(H,\omega)$ in $\mathcal{C}_{A|A}$ as described in proposition 4.8, transported to $\mathcal{C}$.*

Proof. (i) Denote again by $\mathcal{D}$ the strictification of the bimodule category $\mathcal{C}_{A|A}$. By propositions 4.3 and 4.8 and theorem 4.12 we find a symmetric special Frobenius algebra B in $\mathcal{C}_{A|A}$ and a Morita context $A \xleftrightarrow{Q,Q'} B$ in $\mathcal{C}_{A|A}$ such that there is a homomorphism $\phi_{\omega}\colon H \to \mathrm{Aut}_{A|A}(B)$ with $\Pi_{Q,Q'} \circ \Psi_B\colon \mathrm{Aut}_{A|A}(B) \to \mathrm{Pic}(\mathcal{C}_{A|A})$ as one-sided inverse. According to proposition 5.2, $\dot{B}$ is a symmetric special Frobenius algebra in $\mathcal{C}$ and $A \xleftrightarrow{P,P'} \dot{B}$ is a Morita context in $\mathcal{C}$ with $P = \dot{Q}$ and $P' = \dot{Q}'$. Further, we can extend Ψ_B to $\mathrm{Aut}_{\mathcal{C}}(\dot{B})$ by putting $\Psi_{\dot{B}}(\gamma) = [_{\mathrm{id}}\dot{B}_{\gamma}]$ for $\gamma \in \mathrm{Aut}_{\mathcal{C}}(\dot{B})$. Since we have $\mathrm{Aut}_{A|A}(B) \subseteq \mathrm{Aut}_{\mathcal{C}}(\dot{B})$ as a subgroup, ϕ_{ω} then gives a homomorphism $\alpha_{\omega}\colon H \to \mathrm{Aut}_{\mathcal{C}}(\dot{B})$ that has $\Pi_{P,P'} \circ \Psi_{\dot{B}}$ as one-sided inverse. As was seen in theorem 4.12, the assignment $\omega \mapsto \phi_{\omega}$ is a bijection, and hence the assignment $\omega \mapsto \alpha_{\omega}$ is still injective.

(ii) Put $\beta_h := \alpha_{\omega}(h)$. From theorem 4.13 we know that the algebra $Q(H,\omega)$ is isomorphic to the fixed algebra B^H in $\mathcal{C}_{A|A}$. It comes together with an algebra morphism $i\colon Q(H,\omega) \to B$ and a coalgebra morphism $s\colon B \to Q(H,\omega)$ such that $s \circ i = \mathrm{id}_{Q(H,\omega)}$ and $i \circ s = |H|^{-1} \sum_{h \in H} \beta_h$. Now let $f \in \mathrm{Hom}_{\mathcal{C}}(X,\dot{B})$ be a morphism obeying $\beta_h \circ f = f$ for all $h \in H$. Then for $\bar{f} := s \circ f\colon X \to \dot{Q}(H,\omega)$ we find $i \circ \bar{f} = i \circ s \circ f = \frac{1}{|H|} \sum_{h \in H} \beta_h \circ f = f$, i.e. $\dot{Q}(H,\omega)$ satisfies the universal property of the fixed algebra in $\mathcal{C}$. So $\dot{B}^H \cong \dot{Q}(H,\omega)$ as objects in $\mathcal{C}$. By proposition 5.2, i is still a morphism of Frobenius algebras when transported to $\mathcal{C}$. It follows that $\dot{B}^H \cong \dot{Q}(H,\omega)$ as Frobenius algebras in $\mathcal{C}$. $\qquad\square$
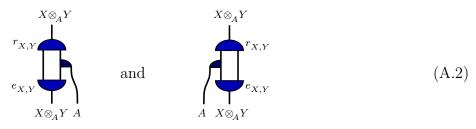
A   Appendix

Here we collect some facts about the category of bimodules over an algebra in a monoidal category. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a an abelian sovereign strict monoidal category, enriched over $\mathcal{V}ect_{\Bbbk}$ with $\Bbbk$ a field of characteristic zero, and with simple and absolutely simple tensor unit. Let A be an algebra in $\mathcal{C}$.

The tensor product $X \otimes_A Y$ of two A-bimodules $X \equiv (X, \rho_X, \varrho_X)$ and $Y \equiv (Y, \rho_Y, \varrho_Y)$ is defined to be the cokernel of the morphism $(\varrho_X \otimes \mathrm{id}_Y - \mathrm{id}_X \otimes \rho_Y) \in \mathrm{Hom}(X \otimes A \otimes Y, X \otimes Y)$. In the following we will only deal with tensor products over symmetric special Frobenius algebras. In this case the notion of tensor product can equivalently be described as follows. Let $(A, m, \eta, \Delta, \varepsilon)$ be a symmetric normalised special Frobenius algebra. Consider the morphism

$$P_{X,Y} \;:=\; \raisebox{-1.2em}{\includegraphics{placeholder}} \quad \in \; \mathrm{Hom}_{A|A}(X \otimes Y, X \otimes Y). \tag{A.1}$$

Since A is symmetric special Frobenius, $P_{X,Y}$ is an idempotent. Writing $P_{X,Y} = e_{X,Y} \circ r_{X,Y}$ as a composition of a monic $e_{X,Y}$ and an epi $r_{X,Y}$, one can check that the morphism $r_{X,Y}$ satisfies the universal property of the cokernel of $(\varrho_X \otimes \mathrm{id}_Y - \mathrm{id}_X \otimes \rho_Y)$.

The object $X \otimes_A Y$ is in the bimodule category $\mathcal{C}_{A|A}$ again. Indeed, right and left actions of A on $X \otimes_A Y$ can be defined by

$$\raisebox{-2em}{\includegraphics{placeholder}} \qquad \text{and} \qquad \raisebox{-2em}{\includegraphics{placeholder}} \tag{A.2}$$

The tensor product over A of two morphisms $f \in \mathrm{Hom}_{A|A}(X, X')$ and $g \in \mathrm{Hom}_{A|A}(Y, Y')$ is defined as
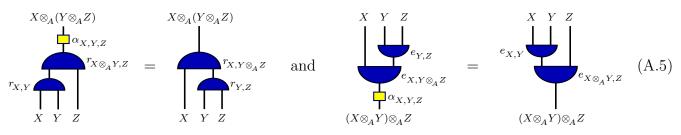
$$f \otimes_A g := r_{X',Y'} \circ (f \otimes g) \circ e_{X,Y}. \tag{A.3}$$

So in particular the morphisms $e_{X,Y}$, $r_{X,Y}$ and $f \otimes_A g$ are morphisms of A-bimodules.

For any three A-bimodules X, Y, Z one defines

$$\alpha_{X,Y,Z} \;:=\; \raisebox{-3em}{\includegraphics{placeholder}} \quad \in \; \mathrm{Hom}_{A|A}\big((X \otimes_A Y) \otimes_A Z, X \otimes_A (Y \otimes_A Z)\big), \tag{A.4}$$

which is a morphism of A-bimodules. These morphisms are in fact isomorphisms and have the following properties:
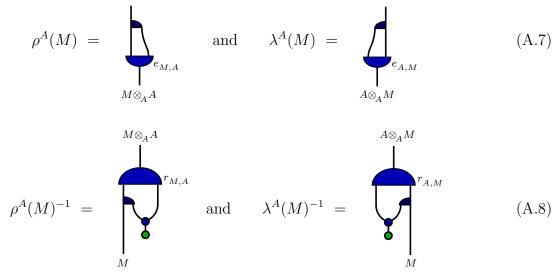

$$\tag{A.5}$$

This can be seen by writing out $\alpha_{X,Y,Z}$ and letting the occurring idempotents disappear using the properties of A as a symmetric special Frobenius algebra.
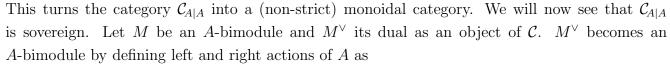
An easy, albeit lengthy, calculation then shows that the isomorphisms $\alpha_{X,Y,Z}$ obey the pentagon condition for the associativity constraints in a monoidal category, i.e. one has
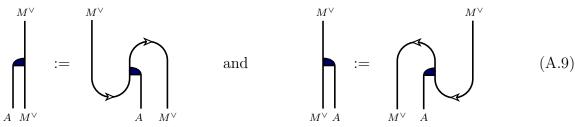
$$(\mathrm{id}_U \otimes_A \alpha_{V,W,X}) \circ \alpha_{U,V \otimes_A W,X} \circ (\alpha_{U,V,W} \otimes_A \mathrm{id}_X) = \alpha_{U,V,W \otimes_A X} \circ \alpha_{U \otimes_A V,W,X} \tag{A.6}$$

for any quadruple U, V, W, X of A-bimodules. For an A-bimodule M, unit constraints are given by

$$\rho^A(M) \;=\; \qquad \text{and} \qquad \lambda^A(M) \;=\; \tag{A.7}$$

with inverses

$$\rho^A(M)^{-1} \;=\; \qquad \text{and} \qquad \lambda^A(M)^{-1} \;=\; \tag{A.8}$$

This turns the category $\mathcal{C}_{A|A}$ into a (non-strict) monoidal category. We will now see that $\mathcal{C}_{A|A}$ is sovereign. Let M be an A-bimodule and $M^\vee$ its dual as an object of $\mathcal{C}$. $M^\vee$ becomes an A-bimodule by defining left and right actions of A as

$$:= \qquad \text{and} \qquad := \tag{A.9}$$

The structural morphisms of left and right dualities for $\mathcal{C}_{A|A}$ are given by

$$
b_M^A \;=\; \text{} \;, \qquad\qquad d_M^A \;=\; \text{}
$$

$$
\tilde{d}_M^A \;=\; \text{} \;, \qquad\qquad \tilde{b}_M^A \;=\; \text{} \tag{A.10}
$$

One checks that this indeed furnishes dualities on $\mathcal{C}_{A|A}$, and that they coincide with those of $\mathcal{C}$ not only on objects, but also on morphisms. Thus if $\mathcal{C}$ is sovereign, then so $\mathcal{C}_{A|A}$. One also easily verifies that $\mathcal{C}_{A|A}$ is abelian and that its morphism groups are $\Bbbk$-vector spaces with $\mathrm{Hom}_{A|A}(M,N) \subset \mathrm{Hom}_{\mathcal{C}}(M,N)$ as subspaces, and hence $\dim_{\Bbbk} \mathrm{Hom}_{A|A}(M,N) \leq \dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(M,N)$. For M an A-bimodule, we denote its left and right dimension as an object of $\mathcal{C}_{A|A}$ by $\dim_{\mathrm{l}A}(M)$ and $\dim_{\mathrm{r}A}(M)$, respectively. If A is in addition an absolutely simple algebra, one has

$$
\dim_{\mathrm{r}}(M) = \dim_{\mathrm{r}}(A)\,\dim_{\mathrm{r}A}(M) \qquad \text{and} \qquad \dim_{\mathrm{l}}(M) = \dim_{\mathrm{l}}(A)\,\dim_{\mathrm{l}A}(M)\,. \tag{A.11}
$$

References

[BK]   B. Bakalov and A.A. Kirillov, Jr., *Lectures on Tensor Categories and Modular Functors*, American Mathematical Society, Providence 2001
(see http://www.math.sunysb.edu/~kirillov/tensor/tensor.html)

[Be]   S. Bénabou, *Introduction to bicategories*, Springer Lecture Notes in Mathematics 106 (1967), 1–75

[Bi]   J. Bichon, *Cosovereign Hopf algebras*, J. Pure Appl. Alg. 157 (2001), 121–133 [math.QA/9902030]

[Br]   A. Bruguières, *Double braidings, twists, and tangle invariants*, J. Pure Appl. Alg. 204 (2006), 170–194 [math.QA/0407217]

[FFRS1] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Kramers-Wannier duality from conformal defects*, Phys. Rev. Lett. 93 (2004), 070601-1–4 [cond-mat/0404051]

[FFRS2] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Correspondences of ribbon categories*, Adv. in Math. 199 (2006), 192–329 [math.CT/0309465]

[FFRS3] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Duality and defects in rational conformal field theory*, Nucl. Phys. B 763 (2007), 354–430 `[hep-th/0607247]`

[FK] J. Fröhlich and T. Kerler, *Quantum Groups, Quantum Categories and Quantum Field Theory*, Springer Lecture Notes in Mathematics 1542 (Springer Verlag, Berlin 1993)

[FRS1] J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators I: Partition functions*, Nucl. Phys. B 646 (2002), 353–497 `[hep-th/0204148]`

[FRS2] J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators III: Simple currents*, Nucl. Phys. B 694 (2004), 277–353 `[hep-th/0403157]`

[FRS3] J. Fuchs, I. Runkel and C. Schweigert, *Ribbon categories and (unoriented) CFT: Frobenius algebras, automorphisms, reversions*, Contemp. Math. 431 (2007), 203–224 `[math.CT/0511590]`

[FS] J. Fuchs and C. Schweigert, *Category theory for conformal boundary conditions*, Fields Inst. Commun. 39 (2003), 25–71 `[math.CT/0106050v4]`

[JS] A. Joyal and R. Street, *The geometry of tensor calculus, I*, Adv. in Math. 88 (1991), 55–112

[Ka] C. Kassel, *Quantum Groups* (Springer Verlag, New York 1995)

[KO] M.-A. Knus and M. Ojanguren, *Théorie de la Descente et Algèbres d'Azumaya*, Springer Lecture Notes in Mathematics 389 (Springer Verlag, Berlin 1974)

[Ma] S. Majid, *Foundations of Quantum Group Theory* (Cambridge University Press, Cambridge 1995)

[MS] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Commun. Math. Phys. 123 (1989), 177–254

[Mü] M. Müger, *From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories*, J. Pure Appl. Alg. 180 (2003), 81–157 `[math.CT/0111204]`

[Pa] B. Pareigis, *Non-additive ring and module theory III: Morita theorems*, Publ. Math. Debrecen 25 (1978), 177–186

[Ri] M.A. Rieffel, *Morita equivalence for C^*-algebras and W^*-algebras*, J. Pure Appl. Alg. 5 (1974), 51–96

[RZ] A. Rosenberg and D. Zelinsky, *Automorphisms of separable algebras*, Pacific J. Math. 11 (1961), 1109–1118

[Sc] A.S. Schwarz, *Morita equivalence and duality*, Nucl. Phys. B 534 (1998), 720–738 `[hep-th/9805034]`

[VZ] F. Van Oystaeyen and Y.H. Zhang, *The Brauer group of a braided monoidal category*, J. Algebra 202 (1998), 96–128

[Ya] S. Yamagami, *Group symmetry in tensor categories and duality for orbifolds*, J. Pure Appl. Alg. 167 (2002), 83–128