FIVEBRANE STRUCTURES

HISHAM SATI, URS SCHREIBER AND JIM STASHEFF

Abstract. We study the cohomological physics of fivebranes in type II and heterotic string theory. We give an interpretation of the one-loop term in type IIA, which involves the first and second Pontrjagin classes of spacetime, in terms of obstructions to having bundles with certain structure groups. Using a generalization of the Green-Schwarz anomaly cancelation in heterotic string theory which demands the target space to have a *String structure*, we observe that the "magnetic dual" version of the anomaly cancelation condition can be read as a higher analog of String structure, which we call *Fivebrane structure*. This involves lifts of orthogonal and unitary structures through higher connected covers which are not just 3- but even 7-connected. We discuss the topological obstructions to the existence of Fivebrane structures. The dual version of the anomaly cancelation points to a relation of String and Fivebrane structures under electric-magnetic duality.

hisham.sati@yale.edu, schreiber@math.uni-hamburg.de, jds@math.upenn.edu.

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1. Introduction

Cohomological physics began with Gauss in 1833, if not sooner (cf. Kirchoff's laws). The cohomology referred to in Gauss' work was that of differential forms, div, grad, curl and especially Stokes Theorem (the de Rham complex). Gauss explicitly defined the linking number of two circles imbedded in 3-space by an integral defined in terms of the electromagnetic effect of a current circulating in one of the circles

Although Maxwell's equations were for a long time expressed only in coordinate dependent form, they were recast in a particularly attractive way in terms of differential forms on Minkowski space. More subtle differential geometry and implicitly characteristic classes occurred visibly in Dirac's magnetic monopole [23], which lived in a $U(1)$ bundle over $\mathbb{R}^3 - 0$. The magnetic charge was given by the first Chern number; for magnetic charge 1, the monopole lived in the Hopf bundle, introduced that same year 1931 by Hopf [45], though it seems to have taken some decades for that coincidence to be recognized [42]. Thus were characteristic classes (and by implication the cohomology of Lie algebras and of Lie groups) introduced into physics.

In the case of electromagnetic theory, the Lie group was just $U(1)$, but more general Lie groups were involved in Yang-Mills theory. There was a linguistic barrier between physicists and mathematicians (see Yang's reminiscences [76]), which was breached when it was realized that the physicist's gauge potential and field strength were, respectively, the mathematicians connection and curvature from differential geometry.

A major development occurred when Dirac's theory of the electron required – in modern language– lifting from the special orthogonal group $\mathrm{SO}(n)$ to the spinor group $\mathrm{Spin}(n)$, which corresponds to "killing" the first homotopy group $\pi_1(\mathrm{SO}(n))$ of $\mathrm{SO}(n)$.

k		0	1	2	3	4	5	6	7	
$\pi_k(\mathrm{O}(n))$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	fundamental groups
$\mathrm{O}(n)\langle k\rangle$	$\mathrm{O}(n)$	$\mathrm{SO}(n)$	$\mathrm{Spin}(n)$		$\mathrm{String}(n)$				$\mathrm{Fivebrane}(n)$	higher connected covers
	kill π_0	kill π_1		kill π_3			kill π_7			

FIGURE 1. **Homotopy groups of** $\mathrm{O}(n)$ **and its higher connected covers** for $n > k+1$.

Much later it was found that in string theory a further step is needed, namely lifting the spinor group to the String group $\mathrm{String}(n)$, giving rise to String structures. Such structures interpreted in terms of the vanishing condition of the worldsheet anomaly of a superstring were first identified by Killingback [49] and shortly afterwards amplified by Witten [73] in the context of the index theory of Dirac operators on loop space. A nice review and differential geometric description can be found in [54]. In all these articles, the String structure is regarded as a lift of an $L\mathrm{Spin}(n)$-bundle over the free loop space LX through the Kac-Moody central extension $\widehat{L\mathrm{Spin}}(n)$-bundle. It was in [66] where it was pointed out that this lift can also be interpreted as a lift of the original $\mathrm{Spin}(n)$-bundle down on target space X to a principal bundle for the topological group called $\mathrm{String}(n)$. Then it was realized in [**?**] (see also [43]) that this topological group is in fact the realization of the nerve of a *smooth* (as opposed to just topological) albeit categorified group: the String 2-group. This paves the way for a differential geometric treatment of String structures on the target space X.

It was known early on that 2-bundles [9, 7] (aka “crossed module bundle gerbes” [47]) with structure 2-group the String 2-group have the same classification as that of the String-bundles considered by Stolz-Teichner (compare [47] and [**?**]), and hence, rationally [21], as that of the bundles on loop space originally considered by Killingback and by Witten. Detailed proofs of this have recently appeared [6] [8]. Note that the String condition is closely related to the condition on vanishing of the integral seventh Stiefel-Whitney class, $W_7 = 0$, observed in [22] and studied in connection to generalized cohomology in [50]. In particular, the string condition implies the vanishing of W_7 as $W_7 = Sq^3(\frac{1}{2}p_1)$, where Sq^3 is the Steendrod square that raises the cohomology degree by three.

The word *string* being well established for maps from 1-dimensional manifolds, higher dimensional analogs are referred to as *branes* (originally, membranes). The “surface” formed by an evolving string is called the worldsheet and, analogously, the higher-dimensional volumes of evolving branes are referred to as the worldvolumes. The most studied types of branes in string theory are the famous D-branes, which couple to the Ramond-Ramond (RR) fields. We will not be dealing with D-branes, and hence with RR fields, in this paper. In addition to D-branes, there are the Neveu-Schwarz (NS) branes that couple to the NSNS fields. The fundamental string couples to the B-field B_2, whose curvature is H_3 ($= dB_2+$ non-exact, locally). The Hodge dual of H_3 in ten dimensions is H_7, which can be viewed as the curvature of a degree six ‘potential’ B_6, to which couples an extended object, called the NS 5-brane. Thus, the 5-brane is the *magnetic dual* to the string (in the sense familiar in String theory, which mathematicians can find nicely described in [34] in terms of differential characters, which are just another way of talking about higher line bundles with connections). It is to be expected that anomaly freedom of the spinors on the fivebrane's worldvolume require the target to carry an analog of a spin structure even more strict than a String structure. There is a known formula for the dual to the Green-Schwarz anomaly. This formula is discussed in section 3. The same formula can be deduced from anomaly freedom of the worldvolume theory of the super 5-brane [51, 27].

We observe that this known formula can be read as saying that target space X needs to admit a lift of the structure group of TX from $\mathrm{SO}(n)$ through $\mathrm{Spin}(n)$, through $\mathrm{String}(n)$ and then further to the 7-connected cover of $\mathrm{SO}(n)$ which we dub Fivebrane(n). We define a *Fivebrane structure*, which is obtained by killing all up to and including the *seventh* homotopy group of the orthogonal group. This is entirely analogous to the process of killing the third homotopy group in going from the Spin group to the String group. Generally, one could wonder what the even higher connected covers of $\mathrm{SO}(n)$ would correspond to in terms of higher brane physics. While the notions of higher lifts are of course known in homotopy theory, we here discuss the topological structure associated with the physics of fivebranes, much the same as in the known String case.

Definition 1. *An n-dimensional manifold X has a* fivebrane structure *if the classifying map $X \to (BO)(n)$ of the tangent bundle TX lifts to B*Fivebrane $:= (B\mathrm{O})\langle 9\rangle$.

$$\begin{tikzcd} & \mathrm{BO}(n)\langle 9\rangle \arrow[d] \\ M \arrow[ur, dotted, "\hat{f}"] \arrow[r, "f"'] & \mathrm{BO}(n) \end{tikzcd} .$$

One point we make is that the fivebrane – as opposed to branes of other dimensions– is distinguished in 10-dimensional spacetime since it does indeed lead to a structure that naturally generalizes the string structure, due to the existence of the field H_7, the dual field to H_3. The importance of H_7 will be highlighted in section 3. Notice that the existence of this lift implies lifts through all the lower connected covers, which says that for X to have a Fivebrane structure it must also have an orientation, a Spin structure and a String structure.

Our aim is to

- understand the **topological nature** of such *fivebrane structures*, i.e. identify the relevant bundles and the corresponding characteristic classes;
- understand their **differential geometric nature** in terms of characteristic *forms* of such bundles (e.g. generalized connections) analogous to what is done for String theory.

This paper will focus more on the first, while a sequel [61] will emphasize the second. We shall show the following:

Theorem 1. *The obstruction to lifting a String structure on X to a Fivebrane structure on X is the fractional second Pontrjagin class $\frac{1}{6}p_2(TX)$.*

This is our Proposition 1.

- Here the fractional coefficient 1/6 is the crucial subtle point. It is explained in 4.4.2.
- Inequivalent Fivebrane structures on X are classified by a quotient of $H^7(X,\mathbb{Z})$. This is proposition 3.

But this is not the full story yet. String structures are in general carried not just by a manifold X, but by a complex vector bundle $E \to X$ (e.g. the gauge bundle on the target space of the heterotic string): this vector bundle has String structure if its second Chern class [1] cancels the fractional first Pontrjagin class of the Spin bundle X:

$$\begin{aligned} E \to X \text{ has Spin structure} \quad &\Longleftrightarrow \quad \begin{cases} E \text{ is orientable} \\ \text{and} \\ w_2(E) = 0 \end{cases} \\ E \to X \text{ has String structure} \quad &\Longleftrightarrow \quad \begin{cases} E \text{ has Spin structure} \\ \text{and} \\ \frac{1}{2}p_1(TX) = \mathrm{ch}_2(E) \end{cases} . \end{aligned}$$

If the second Chern character of E vanishes, this reduces to saying that X itself has String structure. (The fractional first Pontrjagin class is reviewed in section 4.4.1.)

[1]We review characteristic classes and characters in Appendix A.

This situation generalizes. A Fivebrane structure should be assigned, more generally, to a vector bundle $E \to X$. The dual Green-Schwarz mechanism indicates that the condition on E to have a Fivebrane structure is essentially that the fourth Chern class of E cancels the second Pontrjagin class of X, but there are corrections by decomposable classes (because $\mathrm{ch}_2 = c_2 +$ decomposable classes):

$$E \to X \text{ has Fivebrane structure} \iff \begin{cases} E \text{ has String structure} \\ \text{and} \\ \frac{1}{6}p_2(TX) = \mathrm{ch}_4(E) + \text{decomposable classes} \end{cases} .$$

higher spin-like structure		**defining condition**	
Spin structure on manifold	X	orientation and	$w_2(TX) = 0$
Spin structure on gauge bundle	$E \to X$	orientation and	$w_2(E) = 0$
String structure on manifold	X	Spin structure and	$\frac{1}{2}p_1(TX) = 0$
String structure on gauge bundle	$E \to X$	Spin structure and	$\frac{1}{2}p_1(TX) = \mathrm{ch}_2(E)$
Fivebrane structure on manifold	X	String structure and	$\frac{1}{6}p_2(TX) = 0$
Fivebrane structure on gauge bundle	$E \to X$	String structure and	$\frac{1}{6}p_2(TX) = 8\mathrm{ch}_4(E) + \text{decomposables}$

Table 1. **Higher Spin-like structures** on manifolds and on "gauge bundles" over a manifold X. Here "gauge bundle" technicaly just means: complex vector bundle. The first four entries are well established; we are concerned here with the last two entries.

We will study fivebrane structures arising from anomaly polynomials in two theories: in type IIA string theory and in heterotic string theory. In the first, we will have a fivebrane structure for the tangent bundle of the space itself, since there is no gauge bundle. We address this in full detail. In the second theory, we have, in addition, a gauge bundle. The corresponding condition will involve both the tangent bundle and this gauge bundle. We are still able to give a description when the two separate conditions are imposed: that *separately* the tangent bundle and the gauge bundle admit a fivebrane structure. In the general case we encounter a difficulty, namely that the factors – the second Pontrjagin classes– come with different coefficients. We explain this difficulty and provide (partial) solutions in section 4.6 and towards the end of section 4.7. We will be concerned in this paper primarily with the cohomological aspects of Fivebrane structures rather than the more subtle aspects related to K-theory which we hope to address elsewhere.

The presence of the decomposable cohomology classes (products of nontrivial cohomology classes) appearing on the right can be deduced from the dual Green-Schwarz formula. Notice that a decomposable characteristic class necessarily suspends to a trivial group cocycle. So despite the appearance of those decomposable classes, Fivebrane structures are essentially controlled just by the 7th Lie algebra cohomology of $\mathfrak{so}(n)$, just as String structures are controlled by the third Lie algebra cohomology (compare [60]). We shall discuss in [61] that the decomposable characteristic classes in the dual Green-Schwarz formula affect the differential class (the connection) but not the integral class of the smooth Chern-Simons 7-bundle whose non-triviality obstructs the existence of Fivebrane structures. The description in terms of anomalies will be given in section 3.1.

The notion of and notation for connected covers is explained towards the end of section 4.1.

Outlook: differential geometric interpretation. Just as String 2-bundles with connection provide a differential geometric realization of String structure, the higher analog of a Spin bundle with connection, there are analogous higher bundles with connection giving a differential geometric realization of Fivebrane structures. Following the counting pattern these deserve to be called Fivebrane-6-bundles or Fivebrane-5-gerbes with connection, but the reader might want to think of them just as a nonabelian version of differential cocycles with top degree curvature form a 7-form. As will be described in [61], these nonabelian differential cocycles can be obtained from integrating the L_∞-algebra connections of [60]. There the Lie 2-algebra $\mathfrak{string}(n)$ (an L_∞-algebra concentrated in the lowest two degrees) governing String bundles with connection had been accompanied by the Lie 6-algebra $\mathfrak{fivebrane}(n)$ (concentrated in the lowest six degrees) which plays the corresponding role for Fivebrane structures.

In the sense of integration of Lie algebras to Lie groups, the $\mathfrak{string}(n)$ Lie 2-algebra had been integrated in [43, ?] to the String(n) Lie 2-group, the structure Lie 2-group of String 2-bundles. This integration procedure can be thought of as follows: for any L_∞-algebra $\mathfrak{g}$ there is a notion of $\mathfrak{g}$-valued differential forms [60] and there is a generalized smooth space, $S(\mathrm{CE}(\mathfrak{g}))$, which is the *classifying* space for $\mathfrak{g}$-valued forms in that smooth maps from any other smooth space Y into it are in bijection with $\mathfrak{g}$-valued forms on Y:

$$\mathrm{Hom}_{\mathrm{SmoothSpaces}}(Y, S(\mathrm{CE}(\mathfrak{g}))) \simeq \Omega^\bullet(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y))\,. \tag{1}$$

The ∞-groups integrating given L_∞-algebras $\mathfrak{g}$ considered in [38, 43] are obtained essentially by forming the ∞-path groupoid in the sense of Kan complexes of this classifying space, which is nothing but the singular simplicial complex of $S(\mathrm{CE}(\mathfrak{g}))$. This corresponds to a "maximally weakened" integration. Alternatively, one can form the strict globular (instead of simplicial) n-path groupoid Π_n (whose n-morphisms are homotopy classes of n-paths and whose $(k < n)$-morphisms are *thin* homotopy classes of k-paths) which yields the "maximally strictified" n-group integrating the given Lie n-algebra. This way one obtains a strict version of the String Lie 2-group

$$\mathbf{B}\mathrm{String}(n) := \Pi_2\left(S(\mathrm{CE}(\mathfrak{string}(n)))\right) \tag{2}$$

and the strict version of the Fivebrane Lie 6-group

$$\mathbf{B}\mathrm{Fivebrane}(n) := \Pi_6(S(\mathrm{CE}(\mathfrak{fivebrane}(n))))\,. \tag{3}$$

Here on the left the notation $\mathbf{B}G$ indicates a (strict) n-groupoid with a single object corresponding to the (strict) n-group G. The notation is such that taking geometric realization of nerves $|\cdot|$ to produce topological spaces we have

$$|\mathbf{B}G| = B|G|\,, \tag{4}$$

where on the right $|G|$ is a topological group and $B|G|$ its classifying space [8]. The strict realizations of the String and Fivebrane Lie n-groups allow us to employ Ross Street's general theory of descent [69] to formulate String 2- and Fivebrane 6-bundles with connection in terms of nonabelian differential cocycles [62].

2. The context

Before getting to our main discussion, we first indicate the general context in which these questions arise. The word *string* being well established for maps from 1-dimensional manifolds, higher dimensional analogs are referred to as *branes* (originally, membranes). The "surface" formed by an evolving string is called the worldsheet and, analogously, the higher-dimensional volumes of evolving branes are referred to as the worldvolumes. We have an "n-particle", otherwise known as an $(n-1)$-brane, whose worldvolume is an n-dimensional manifold Σ, or rather the image of that n-dimensional manifold under a sufficiently well behaved map

$$\phi : \Sigma \to X \tag{5}$$

into a target space (usually to be thought of as physical spacetime) X. On that spacetime, we have a (generalized, higher) bundle (thought of as an n-bundle or $(n-1)$-gerbe) with (generalized) connection. Henceforth, we will drop the 'generalized' and the n unless crucial. Thinking of an ordinary bundle with connection should be sufficient. This bundle with connection encodes the data specifying the "background field" to which that $(n-1)$-brane "couples". Just as with ordinary connections, connections on an n-bundle can be expressed either in terms of local ($p \leq n$)-forms on the base manifold or in terms of global ($p \leq n$)-forms on the total space [60]. The local representatives of these forms are often referred to as a "background field", though technically speaking the background field is that differential form datum together with the descent/gluing data that makes it a differential cocycle. In particular, we have the 3-form curvature H_3 of the Kalb-Ramond field in string theory, which is, in this sense, the curvature of a "background field", the "B-field", of a string theory. Likewise, we have the 7-form H_7 which is the curvature of a background field for the fivebrane.

2.1. **Σ-models.** We are concerned with the mathematical structure which is supposed to model the physics of *charged n-particles*, usually known as *charged $(n-1)$-branes* or as *quantum field theories of Σ-model type.* Such a Σ-model is specified by choosing

- a "space" X, called the *target space*;
- a "space" (or class of such) Σ, called the *parameter space* or called the *worldvolume*;
- the mapping space $\mathrm{Maps}(\Sigma, X)$ called the *space of fields* or the *configuration space* or sometimes the *moduli space* (the latter is usually a quotient);
- on the target space a *differential n-cocycle* ∇, i.e. a higher generalization of a fiber bundle with connection, called the *background field*;
- a prescription for how to interpret the push-forward of the the pullback $\mathrm{ev}^*\nabla$ along the projection pr_1 onto Σ in the correspondence diagram

$$\begin{tikzcd} & \Sigma \times \mathrm{Maps}(\Sigma, X) \arrow[dl, "pr_1"'] \arrow[dr, "\mathrm{ev}"] & \\ \Sigma & & X \end{tikzcd} \tag{6}$$

called the *path integral* or the *quantization of the Σ-model.*

When the parameter space Σ is n-dimensional, one thinks of this data as encoding the physics of n-fold higher analogs of particles, "n-particles", that propagate on X. The field configuration (5) is thought of as the *trajectory* of such an n-particle in X.

fundamental object	**dimension of worldvolume Σ**	**background field**	**curvature/ field strength**
n-particle	n	n-bundle	$(n+1)$-form
$(n-1)$-brane	n	$(n-1)$-gerbe	$(n+1)$-form

TABLE 2. **The two schools of counting** higher dimensional structures. Here $n = \dim(\Sigma)$ is in $\mathbb{N} = \{1, 2, \cdots\}$. The $n = 1$ case: A particle is a 0-dimensional spatial object moving in time, so that the 'worldvolume' Σ is $(0+1)$-dimensional and hence 1-dimensional in our terminology. The corresponding background field is encoded in a 1-bundle or 0-gerbe with curvature 2-form. The generalization to strings ($n = 2$) and higher extended objects, i.e. the n-particles, works analogously. One can also include the case $n = 0$ corresponding to "instantons", although the corresponding geometric description will be different.

One says that the n-particle *couples* to the background field ∇ or that it is *charged under* the background field. The terminology is entirely motivated from the familiar case of ordinary electromagnetically charged $1-$particles: the electromagnetic background field ∇ which they couple to is modeled by a vector bundle (a line bundle in this case) with connection. For $n = 2$ one speaks of "strings". String theory strictly is the study of those $n = 2$ Σ-models with a special restriction for what the "path integral" is allowed to be. Technically, string theory is required to encode a 2-dimensional superconformal field theory of central charge

15. This condition, however, is of no real relevance for our discussion here, which pertains to all Σ-models which generalize the "spinning 1- particle".

Some of the most interesting ideas concerning such Σ-models have originally been thought by Dan Freed: The interpretation of background fields and of charges as differential cocycles is nicely described and worked out in [34, 46], where the mathematically inclined reader can find rigorous interpretations, in terms of differential cohomology, of the *abelian* kinds of "background fields" and related "anomalies" in string theory with which we are concerned here, which include the Chern-Simons 3- and 7-bundles with connection obstructing the String(n) and Fivebrane(n) lifts, but not these lifts themselves. (For those, one would need *nonabelian* differential cohomology [61]).

Integration as pushforward. The interpretation of quantization and of the path integral as an operation on higher categorical structures has first been explored in [32, 33]. Integration as a push-forward operation plays a promint role in recent developments by Stolz and Teichner and by Hopkins et al. Let us take for instance the simple toy example case where the background field ∇ is a vector bundle (without connection) and where Σ is a point, i.e. $n = 0$. In this case pr_1 is the map from $\mathrm{pt} \times \mathrm{Maps}(\mathrm{pt}, X) = X$ to the point. Integration over the fiber in this case is just integration over X itself, and the ordinary push-forward gives the space of sections of the original vector bundle, as the point is varied over X. That reproduces indeed the desired "quantization over the point" and can, following [32, 33], be regarded as the codimension 1 part of the full path integral for $n = 1$. Stolz and Teichner describe a variation of this which involves push-forward of K-theory classes to the point, which then classifies connected components of all (supersymmetric) 1-dimensional Σ-models. This shows that, while a fully satisfactory mathematical interpretation of the quantization of Σ-models is to date still an open question, a coherent picture, revolving around the correspondence (6), is beginning to emerge. The "higher spin-like structures" on target space X discussed here are believed to ensure the existence of the quantization step in the case that the Σ-model generalizes that describing spinning 1-particles.

The sigma model for the string (i.e. Σ = string worldsheet) has been studied extensively in the literature and provides a consistent model both at the classical and the quantum levels. Due to string/fivebrane duality in ten dimensions [70] [15] [30] [31], one expects that there should be a formulation of string theory via fivebrane sigma models (i.e. Σ = fivebrane worldvolume). While this is expected, we point out that such a program has not been fully completed. However, there are works that point in that direction [27]. At the least for the gravitational parts– i.e. without the gauge bundle– for the fivebrane, there are models in which the anomalies from the worldvolume theory match the expression of the polynomials in the Pontrjagin classes [27] [51]. That is enough for our purposes since our main focus is the cohomological structure resulting from lifts of the tangent bundle of spacetime and, after all, it is not our aim to write down a full quantum fivebrane sigma model action. The paper [51] also highlights some of the difficulties encountered, but also gives partial resolutions.

The reader not further concerned with string theoretic reasoning might proceed to section 4.

2.2. **Background fields.** Independently of how the "background field" ∇ is modeled, it should locally be encoded by differential form data. See table 3

All the relevant background fields that have been considered are locally controlled by some L_∞-algebra $\mathfrak{g}$, and the local differential form data can always be considered as encoding differential forms $A \in \Omega^\bullet(Y, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g}$ [60]. In the case of abelian differential cocycles, these L_∞-algebras are all of the form $b^{n-1}\mathfrak{u}(1)$: the higher dimensional versions of $\mathfrak{u}(1)$.

2.3. **Charges.** Just as an ordinary 1-bundle may be trivialized by a *non-trivial* section, which one may think of as a "twisted 0-bundle", higher n-bundles may be trivialized by "higher sections" which are called "twisted $(n-1)$-bundles". One says the twisted $(n-1)$-bundle is "twisted by" the corresponding n-bundle. A beautiful description of this situation for abelian n-bundles with connection in terms of differential characters

n-particle	background field	global model	local differential form data
(1−)particle	electromagnetic field	line bundle with connection/ Cheeger-Simons differential 2-character/ Deligne 2-cocycle	connection 1-form: $A \in \Omega^1(Y)$ curvature 2-form: $F_2 := dA \in \Omega^2_{\text{closed}}(Y)$
string (2-particle) (1-brane)	Kalb-Ramond field	line 2-bundle with connection/ bundle gerbe with connection ("and curving")/ Cheeger-Simons differential 3-character/ Deligne 3-cocycle	connection 2-form: $B \in \Omega^2(Y)$ curvature 3-form: $H_3 := dB \in \Omega^3_{\text{closed}}(Y)$
membrane (3-particle) (2-brane)	supergravity 3-form field	line 3-bundle with connection/ bundle 2-gerbe with connection ("and curving")/ Cheeger-Simons differential 4-character/ Deligne 4-cocycle	connection 3-form: $C \in \Omega^3(Y)$ curvature 4-form: $G_4 := dC \in \Omega^4_{\text{closed}}(Y)$

Table 3. **Simple (abelian) examples for n-particles and the background fields they couple to**. The background fields are often addressed in terms of the symbols used for their local form data: the Kalb-Ramond field is known as the "B-field" with its "H_3 field strength" . Similarly one speaks of the "C_3-field" and its field strength "G_4", etc. This reflects the historical development, where the local differential form data was discovered first and its global interpretation only much later. (See also the remark on anomalies at the beginning of section 3).

is given in [34, 46]. Twisted nonabelian 1-bundles have been studied in detail under the term "bundle gerbe modules" [12]. Twisted non-abelian 2-bundles have first been considered in [5, 47] under the name "twisted crossed module bundle gerbes". In terms of the L_∞-connections considered in [60] twisted n-bundles with connections are the connections for L_∞-algebras arising as mapping cone L_∞-algebras $(b^{n-1}\mathfrak{u}(1) \to \hat{\mathfrak{g}})$.

By comparing the formalism here with the situation of ordinary electromagnetism, one can regard the twisting n-bundle as encoding the presence of "magnetic charge". This, too, is nicely explained at the beginning of [34]. Accordingly, where an untwisted $(n-1)$-bundle has a curvature n-form H_n which is closed, a twisted $(n-1)$-bundle has a curvature n-form which is "twisted by" the curvature $(n+1)$-form $G_{(n+1)}$ of the twisting n-bundle

$$dH_n = G_{n+1} \,. \tag{7}$$

Indeed, for a twisted $(n-1)$-bundle the curvature H_n is locally no longer the differential of the connection, $dB_{n-1} = H_n$, but receives a contribution from the connection n-form B_n of the twisting n-bundle

$$H_n = dB_{n-1} + B_n \,. \tag{8}$$

The archetypical example is that of ordinary magnetic charge: as Maxwell discovered in the 19th century, in the presence of magnetic charge, which in four dimensions is modelled by a 3-form $H_3 = \star j_1$, the electric field strength 2-form F_2 is no longer closed

$$dF_2 = H_3 \,. \tag{9}$$

When Dirac later discovered at the beginning of the 20th century that H_3 has to have integral periods ("quantization of magentic charge"), the first 2-categorical structure in physics had appeared: the magnetic torsion 2-bundle / bundle-gerbe with deRham class H_3. It seems that this was first explicitly realized in [34].

The next example of this kind received such a great amount of attention that it came to be known as the initiation of the "first superstring revolution": the Green-Schwarz anomaly cancelation mechanism [40]. The interaction of gauge theory with type I supergravity theory leads to the (low energy limit of the) heterotic theory which has a rich mathematical structure. The *Chapline-Manton coupling* [18] amounts essentially to equating the curvature H_3 of the B-field in type I with the Chern-Simons 3-form of the connection on the gauge bundle. More precisely, in terms of the the virtual difference of two Chern-Simons 3-bundles (Chern-Simons 2-gerbes), locally, one has

$$dH_3 = d\mathrm{CS}(\omega) - d\mathrm{CS}(A) \tag{10}$$

for ω and A the local connection 1-forms of a Spin and complex vector bundle, respectively and $\mathrm{CS}(-)$ denoting the corresponding Chern-Simons 3-forms. Thus this leads to nontrivial and rich structures both physically and mathematically. See for instance [34] [11] and [20] for geometric treatments.

As nicely explained by [34], in the "higher gauge theory" given by the effective supergravity target space theory of the heterotic string, the supergravity C-field with curvature 4-form [2] G_4 had to be "trivialized" by the Kalb-Ramond field with curvature 3-form H_3, or conversely the Kalb-Ramond field had to be "twisted" by the supergravity curvature 4-form $dH_3 = G_4$ Moreover, G_4 has to be the curvature of the virtual difference implicit in (10). The Green-Schwarz anomaly cancelation condition can hence be read, equivalently, as saying that

- the supergravity C-field trivializes over the 10-dimensional target of the heterotic string;
- G_4 is the *magnetic 5-brane charge* which the electric heterotic string couples to;
- the Kalb-Ramond field is *twisted* by the supergravity C-field.

This anomaly cancelation has an interesting description in terms of a string structure [49] [21]. This can also be interpreted as a spin structure on the (free) loop space of the ten-dimensional spacetime [73] [64] [66]. This free loop space can be viewed as the configuration space of the string. From a topological point of view, the string structure is equivalent to lifting the structure group of the tangent bundle of spacetime from Spin(10) to its three-connected cover String(10), obtained by killing the first three homotopy groups. The latter is infinite-dimensional but can be captured by some finite-dimensional constructions [66] [**?**].

There is no particular reason to prefer "electric charge" over "magnetic charge": in the presence of a Riemannian structure, the Hodge star dual of an "electric" field strength H_{n+1} may be interpreted as a field strength itself, in which case it is called the "magnetic field strength" $H_{d-n-1} := \star H_{n+1}$. Just as the original field strength H_n coupled to an "electric" n-particle, the dual field strength couples to a "magnetic" $(d-n-2)$-particle. Such electric-magnetic duality is at the heart of what is known as "S-duality" for super Yang-Mills theory, which has recently been argued [48] to be the heart of geometric Langlands duality. It is only for electric 1-particles in $d = 4$ dimensions that their magnetic dual is again a 1-particle. The magnetic dual of the 2-particle in 10 dimensions is the 6-particle. In other words: the magnetic dual of the string is the 5-brane.

Here our starting point is to look at the above situation with the electric string in the presence of magnetic 5-brane charge in the dual formulation, where the magnetic 5-brane couples to a 6-form field with field strength H_7 in the presence of electric string charge, which is then given by an 8-form I_8.

Type I supergravity admits a formulation in terms of the potential B_6 corresponding to the field H_7, which is Hodge dual to H_3 in ten dimensions [17]. There is also a corresponding anomaly cancelation procedure for this dual theory which makes use of a degree seven analog of the Chapline-Manton coupling [37] [58]. This process is also mathematically rich and has been treated in [34] from a K-theoretic point of view and, in fact, in a duality-symmetric fashion, i.e. including both fields on equal footing and at the same time. The

[2] In all of this paragraph, what we mean by the C-field and its corresponding curvature G_4 is their topological part, i.e. the shift of G_4 in the formula $[G_4] - \frac{1}{4}p_1 = a \in H^4(Y^{11}, \mathbb{Z})$ [75], and not the fields themselves –the 'p_1 part' and not the 'G_4 part' of a.

	electric field strength coupled to fundamental electric $(n-1)$-brane		magnetic $(d-n-3)$-brane current		magnetic field strength coupled to fundamental magnetic $(d-n-3)$-brane		electric $(n-1)$-brane current
d	F_{n+1}	$=$	j_B	d	$\star F_{n+1}$	$=$	j_E
	electric KR field coupled to fundamental string		magnetic 5-brane current		magnetic dual KR field coupled to fundamental 5-brane		electric string current
d	H_3	$=$	$\underbrace{\frac{1}{2}p_1(\omega) - \mathrm{ch}_2(A)}_{=I_4}$	d	H_7	$=$	$\underbrace{\frac{1}{48}p_2(\omega) - \mathrm{ch}_4(A) + \text{decomposables}}_{I_8}$

Table 4. The Green-Schwarz formula and its dual version, with its interpretation in terms of electric strings and their magnetic 5-brane duals. The electric current I_4 and the magnetic current I_8 are both fixed such that the anomaly they produce cancels the anomaly from the fermions in the theory (see section 3.1).

magnetic dual discussion of the Green-Schwarz mechanism [37] [58] leads us to consideration of a twisted 6-bundle with field strength $H_7 = \star H_3$, which is twisted by a certain 7-bundle whose field strength eight-form is a sum of two higher characteristic classes plus some mixed terms – see equation (16). This is the formula which we shall refer to as the *dual Green-Schwarz anomaly cancellation condition* and take as the starting point of our discussion.

3. The Dual Green-Schwarz Mechanism and Higher Chern-Simons Forms

Here we recall the string theoretic results which indicate which characteristic classes of a manifold X have to vanish in order for the manifold to qualify as the target for the propagation of a 5-brane. For the main mathematical point that we make in section 4, it is sufficient to note here that there is motivation, from formal high energy physics, for studying the condition (16), below, on characteristic classes of complex vector bundles over X. The reader not further concerned with string theoretic reasoning might just want to note this equation and the observations following it and then proceed to section 4.

3.1. **Anomaly cancelation in string theory.** There are several somewhat different phenomena which are called *anomalies* in physics, but they usually all refer to issues of global topological twists. Physicists are used to develop their concepts in terms of local data and many times implicitly assume that this is sufficient. Generically, the anomaly in our context refers to an inconsistency in the topological assumptions taken for the underlying space or for bundles on that space. For instance, if one accepts that spinors are sections of Spin bundles, then it is obvious that their existence requires the underlying manifold to have a Spin structure. But one way to discover this from the point of view of physics is, as nicely described in [74], to start with a naive action functional for a spinning particle and then to discover that it is ill defined globally unless the target has a Spin structure. Entirely analogous considerations lead to String structures as the "anomaly cancelation conditions" for superstrings, known as the Green-Schwarz anomaly cancelation mechanism.

From the target space perspective, these kinds of anomalies manifest themselves in the fact that the action functional of the theory, supposed to be a function on configuration space, happens to be, in fact, a section of a line bundle. There are (at least) two reasons why this may happen:

- the path integral over the fermionic fields is to be interpreted not as a function over the configuration space of the remaining bosonic fields, but as a section in a Pfaffian line bundle over that space (reviewed in [35]);
- the standard action functional for higher abelian gauge fields in the presence of electric *and* magnetic charges is also in general just a section of a line bundle over configuration space (discussed in [34]).

If the tensor product of these two line bundles, namely of the Pfaffian and Charge line bundles, is a nontrivial bundle with nontrivial connection then the action is *anomalous*. The Green-Schwarz anomaly cancelation mechanism is to introduce electric string and magnetic 5-brane charges in precisely such a way that the line bundle on configuration space thus introduced cancels the nontriviality of the given Pfaffian line bundle due to the fermions in the theory.

The complex **anomaly line bundle** with connection over the space $\mathrm{conf}_{\mathrm{bos}}$ of bosonic fields is the tensor product of a Pfaffian line bundle Pfaff from the fermionic path integral and another line bundle, Charge, due to the presence of electric and magnetic charges.	$\begin{array}{c}\mathrm{Pfaff}\otimes\mathrm{Charge}\\ \downarrow\\ \mathrm{conf}_{\mathrm{bos}}\end{array}$
The **action funcional** e^{-S} is supposed to be a complex function on $\mathrm{conf}_{\mathrm{bos}}$, but is in general, in fact, a section of the anomaly line bundle.	$\begin{tikzcd}\mathrm{Pfaff}\otimes\mathrm{Charge} \arrow[d, shift right] \\ \mathrm{conf}_{\mathrm{bos}} \arrow[u, bend right, "e^{-S}"']\end{tikzcd}$
In order for the starting point of quantization to be well defined one needs **anomaly cancelation**: the anomaly line bundle needs to be trivializable and one needs a choice of trivialization that identifies it with the trivial line bundle with trivial connection.	$\mathrm{Pfaff}\otimes\mathrm{Charge} \xrightarrow{\simeq} (\mathrm{conf}_{\mathrm{bos}}\times\mathbb{C}, \nabla = 0)$
The obstruction to this trivialization is the **anomaly**. The curvature of the connection on Pfaff $\otimes$ Charge is called the **local anomaly**, its holonomy the **global anomaly**.	local anomaly: $\mathrm{curv}(\mathrm{Pfaff}\otimes\mathrm{Charge})$ global anomaly: $\mathrm{Hol}(\mathrm{Pfaff}\otimes\mathrm{Charge})$

TABLE 5. **Anomalies** arising from the fact that the bosonic action functional is, a priori, not a function on the bosonic configuration space of fields $\mathrm{conf}_{\mathrm{bos}}$, but a section of a line bundle over that space.

In the following we describe the anomaly cancelation condition known as the *dual* Green-Schwarz mechanism [37] [58] [34] and related to super 5-branes [51]. It can be obtained from the worldvolume perspective of the super 5-brane again as a generalization of how the Spin-condition for the target of a spinning particle is found. Alternatively, it can be found from the condition that the index of the total Dirac operator (on the fermionic fields called “dilatino”, “gravitino” and “gaugino”) of the effective target space field theory of the heterotic string vanishes.

In String theory, the need for String structures was originally found in terms of anomaly cancelations, either from the target space perspective or from the worldsheet perspective: The effective field theory of the heterotic string on a Spin target space X involves the the (pseudo-)Riemannian metric structure on X

with Levi-Civita SO(10) connection ω (needed for gravity) and a gauge bundle $E \to X$ with connection A. The spinorial field content of the heterotic background theory consists of sections of three different bundles: the Spin bundle S associated to the principal Spin bundle over the Spin manifold X, as well as its tensor products with TX and with the gauge bundle E. Sections of S are states of the the dilatino field, those of $S \otimes TX$ correspond to the gravitino field, and those of $S \otimes E$ correspond to the the gaugino field. There is a Dirac operator associated with all three of these fields, denoted D, D_{TX} and D_E, respectively. The anomaly cancellation condition, which ensures that the action functional for these fields is a well defined function on configuration space, is that a particular linear combination of the indices of these Dirac operators vanishes.

Spin bundle	**name of field**	**symbol for Dirac operator**	**contribution to anomaly**	
S	dilatino	D	$\mathcal{I}_{1/2}(R) := \mathrm{Index}(D)$	$= \hat{A}$
$S \otimes TX$	gravitino	D_{TX}	$\mathcal{I}_{3/2}(R) := \mathrm{Index}(D_{TX})$	$= \frac{1}{8}L(R) + \mathcal{I}_{1/2}(R)$
$S \otimes E$	gaugino	D_E	$\mathcal{I}_{1/2}(R, F) := \mathrm{Index}(D_E)$	$= \mathrm{ch}(E) \wedge \hat{A}$

TABLE 6. **Anomaly contributions** from the three different fermionic fields (sections of spin bundles) of the target space theory. $L(R)$ is the Hirzebruch L-polynomial.

In the notation of table 6, the anomaly cancelation formula involves the degree twelve form part of the identity [3] [40] [41]

$$\mathcal{I} := \mathcal{I}_{3/2}(R) - \mathcal{I}_{1/2}(R) + \mathcal{I}_{1/2}(F, R) = 0. \tag{11}$$

Here R and F are the curvature 2-forms of the tangent bundle TX and of the gauge bundle E, respectively. The second of the three terms in the middle is $\mathrm{Index}(D) = \widehat{A}$, the index of the uncoupled Dirac operator given in terms of the A-genus via the index theorem. The first term is $\mathrm{Index}(D_{TM})$, the index of the Dirac operator coupled to the tangent vector bundle, i.e. $S \otimes TM$. The third term is $\mathrm{Index}(D_E)$ is the index of the Dirac operator coupled to the gauge vector bundle, whose curvature is F, i.e. the vector bundle is $\mathrm{Spin}(M) \otimes E$. This is equal to $\mathrm{ch}(E) \wedge \widehat{A}$, by the index theorem.

This fermionic anomaly corresponds to a line bundle with connection on configuration space whose curvature 2-form is the integral of a certain 12-form (see [63] for more detail) over target space X

$$\mathrm{curv}(\mathrm{Pfaff}) = -\int_X I_4 \wedge I_8\,. \tag{12}$$

A similar integral encodes the curvature of the anomaly line bundle due to electric string current j_E (a 4-form) and magnetic current j_B (an 8-form):

$$\mathrm{curv}(\mathrm{Charge}) = \int_X j_E \wedge j_B\,. \tag{13}$$

Anomaly cancellation demands that we identify I_4 (right hand of equation (14) below) and I_8 (right hand of equation (16) below), respectively, as the magnetic currents for the field strengths H_3 and H_7 that appear in the direct and the dual Green-Schwarz anomaly cancelation conditions.

3.2. **Anomalies in terms of H_3 and H_7.** Here we summarize the two pictures we have emphasized.

The standard Green-Schwarz mechanism via H_3. The H-fields appear in two theories of interest to us: Type II and heterotic theories on ten-dimensional X, respectively. In type II, the expression involves the Ramond-Ramond (RR) fields. The direct Green-Schwarz formalism [40] on a ten-dimensional manifold X leads to the appearance of the three-dimensional Chern-Simons term via the Chapline-Manton coupling [18] which makes H_3 no longer closed. Mathematically, this means we assume in that case that in the heterotic theory there is the usual H-field, a priori an $\mathbb{R}$-valued differential form – as presented by supergravity – which gets modified from being closed to

$$\frac{1}{2\pi} dH_3 = \mathrm{ch}_2(A) - \frac{1}{2} p_1(\omega), \tag{14}$$

where A is the gauge connection for an $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$ vector bundle E and ω is the metric connection, so that $p_1(\omega)$ is the first Pontrjagin form of the tangent bundle TX. Recall [49] that the right hand side of (14) is the (the image in real cohomology of the) obstruction to having a String structure on the virtual difference bundle $TM - E$, and that (14) therefore says that this obstruction class has to vanish. Note the contrast of (14) in heterotic string theory to the case of type II string theory on X, where $dH_3 = 0$.

The dual Green-Schwarz mechanism via H_7. The H-field H_3 can be viewed as the dual of a field H_7 where. rationally, this is just Hodge duality $H_7 := \star H_3$ The 'Bianchi identity' of H_7 depends on the specific string theory, i.e. heterotic vs. type II, as was the case for H_3. The reason we say "rationally" is because these fields, like other fields in string theory, can give integral and even torsion elements in cohomology in the quantum theory. In such a case, appropriate notions of Hodge duality will be needed to clarify the relationship between H_3 and H_7. We address this in detail in a separate paper.

Let us now also consider type II string theory on X in the absence of any Ramond-Ramond fields. This theory also has a degree seven dual, H_7, of the H-field H_3. While H_3 in this case is closed, H_7 is not. Instead, from the dimensional reduction from M-theory, H_7 satisfies (cf. [29] and [36] [52])

$$dH_7 = \frac{p_2(\omega) - \left(\frac{1}{2}p_1(\omega)\right)^2}{48}, \tag{15}$$

where p_i are the Pontrjagin classes of the tangent bundle TX. Observe that this only involves the topology of spacetime *without* any gauge bundles.

On the other hand, in the heterotic case, we have a principal bundle with connection, the corresponding modified Bianchi identity will fail by the Chern characters of E. The expression is (see [34])

$$dH_7 = 2\pi \left(\mathrm{ch}_4(A) - \frac{1}{48} p_1(\omega)\mathrm{ch}_2(A) + \frac{1}{64} p_1(\omega)^2 - \frac{1}{48} p_2(\omega) \right). \tag{16}$$

Here ω denotes the Levi-Civita connection on the Spin-lift of the tangent bundle, and A is the connection on the gauge vector bundle E. We interpret this "dual Green-Schwarz mechanism" as saying that H_7 trivializes the obstruction to having a Fivebrane structure on the pair $(TX,\, E)$

Observations: It is useful to see how (16) simplifies in various special cases.
1. If we have a String structure on $TX - E$ coming from String structures on both bundles separately, in that $p_1(TX) = 0 = \mathrm{ch}_2(E)$, then, at the level of cohomology, equation (16) is replaced by

$$\left[\mathrm{ch}_4(E) - \frac{1}{48} p_2(TX) \right] = 0\ . \tag{17}$$

This holds in general when X is 4-connected, in which case the cohomology of X is only in degrees 0, 5 and 10.
2. If in addition to $\mathrm{ch}_2(E) = 0$ we require that c_1^2 and c_2 be equal to zero, then in this case, using

$$\mathrm{ch}_4 = \frac{1}{24}\left(c_1^4 - 4c_1^2 c_2 + 4c_1 c_3 + 2c_2^2 - 4c_4 \right), \tag{18}$$

we have that $\mathrm{ch}_4(E) = \frac{-1}{6} c_4(E)$ so that

$$\left[-\frac{1}{6} c_4(E) - \frac{1}{48} p_2(TX) \right] = 0\ . \tag{19}$$

3. We can write a formula in terms of either the Chern classes or the Pontrjagin classes for both factors in (19), thus giving specialization of the general formula. For this we can consider the complexification of the tangent bundle to the ten-dimensional spacetime. We use

$$p_j(TM) = (-1)^j c_{2j}(TM_{\mathbb{C}}) \tag{20}$$

to write the differential form representative of (19) as

$$dH_7 = \frac{-2\pi}{6}\left(c_4(A) + \frac{1}{8}c_4(\omega)\right), \tag{21}$$

where now it is understood that $c_4(\omega)$ is the fourth Chern class of the complexified tangent bundle with corresponding connection ω, for which, with an abuse of notation, we use the same symbol as for the real connection.

It will follow from our results in section 4 that fivebrane structures for a pair (TM, E) with connections (ω, A) require $\mathrm{ch}_4(A) - \frac{1}{48}p_2(\omega)$ to vanish on top of the string structure.

3.3. **The Chern-Simons forms.** For a principal G-bundle $p : P \to X$, a characteristic class $K_j(F)$, expressed as a polynomial in the $\mathfrak{g}$-valued curvature F of polynomial degree j, is closed and pulled up to the total space is exact

$$K_j(F) = dQ_{2j-1}(A, F), \tag{22}$$

where $Q_{2j-1}(A, F) \in \mathfrak{g}\otimes\Lambda^{2j-1}(M)$ is a ' Chern-Simons form' for $K_j(F)$. This applies to the Chern character as well as the Pontrjagin classes. Thus, we can use this to solve equation (16) on the total space. We denote a specific choice (e.g. by the specific homotopy in the remark below) as $CS_7(A, F)$. Using the expressions

$$\mathrm{ch}_4(F) = dCS_7(A, F) \tag{23}$$

$$\mathrm{ch}_4(R) = dCS_7(\omega, R), \tag{24}$$

equation (21) becomes

$$dH_7 = 2\pi\left[dCS_7(A, F) + \frac{1}{8}dCS_7(\omega, R)\right]. \tag{25}$$

This means that H_7 can be taken to be of the form

$$H_7 = 2\pi\left(CS_7(A, F) + \frac{1}{8}CS_7(\omega, R)\right). \tag{26}$$

We view this setting of H_7 equal to the degree seven Chern-Simons forms as the degree seven analog of the Chapline-Manton coupling.

Remarks. 1. A specific formula for the Chern-Simons form corresponding to the Chern character can be obtained by using the homotopy formula [19] (or in the physics literature [77] [2])

$$CS_{2j-1}(A, F) = \frac{1}{(j-1)!}\left(\frac{i}{2\pi}\right)^j \int_0^1 dt\ \mathrm{Str}(A, F_t^{j-1}), \tag{27}$$

where Str is the symmetrized trace and A_t is a connection that interpolates between connections A_0 and A_1

$$A_t = A_0 + t(A_1 - A_0), \tag{28}$$

with corresponding curvature

$$\begin{aligned} F_t &= dA_t + A_t^2 \\ &= tdA + t^2A^2 \\ &= tF + (t^2 - t)A^2. \end{aligned} \tag{29}$$

For $j = 4$,

$$CS_7(A, F) = \frac{1}{6(2\pi)^4}\int_0^1 dt\ \mathrm{Str}\left(A, F_t^3\right), \tag{30}$$

where F_t is the curvature of the connection $A_t = tA$ that interpolates between the zero connection at $t = 0$ and the connection A at $t = 1$. An analogous formula holds for $CS_7(\omega, R)$.
2. Unlike the Chern character, the Chern-Simons form is not gauge-invariant. Under a transformation $A \to A^g = g^{-1}(A + d)g$,

$$CS_7(A^g, F^g) - CS_7(A, F) = -\frac{3!}{7!}\left(\frac{i}{2\pi}\right)^4 \mathrm{tr}\left[(g^{-1}dg)^7\right] + d\beta_6, \tag{31}$$

where β_6 is a six-form which can be chosen by applying the chosen homotopy operator on the gauge-transformed Chern-Simons form.
3. Alternatively, we see that (19) is obtained from (16) by setting all *decomposable* characteristic forms (all nontrivial wedge products of two characteristic forms) to 0, which is the same as saying that all characteristic classes *suspending to 0* are set to 0. Recall that a characteristic form $P(F_A)$ on a G-bundle $p : P \to X$ is said to suspend to the form $\mu(i^*A)$ on $G \overset{i}{\hookrightarrow} P$ if there is a form $\mathrm{CS}_P(A, F_A)$ on P such that

$$d\mathrm{CS}_P(A, F_A) = p^*P(F_A) \tag{32}$$

and

$$i^*\mathrm{CS}_P(A, F_A) = \mu(i^*A)\,. \tag{33}$$

Put more simply, recall that a form $\omega \in H^*(X)$ on the base of a bundle $p : E \to X$ is said to suspend to the form μ on $F \overset{i}{\hookrightarrow} E$ if there is a form ν on E such that $d\nu = p^*\omega$ and $i^*\nu = \mu$. A decomposabe characteristic form $P(F_A) \wedge P'(F_A)$ necessarily suspends to a 0-form, since

$$p^*(P(F_A) \wedge P'(F_A)) = d\mathrm{CS}_{P\wedge P'}(A, F_A)) := d\left(\mathrm{CS}_P(A, F_A) \wedge P'(F_A)\right) \tag{34}$$

and since

$$i^*\left(\mathrm{CS}_P(A, F_A) \wedge P(F_A)\right) = 0, \tag{35}$$

because $i^*F_A = 0$, and hence $i^*P'(F_A) = 0$.

4. Fivebrane Structures

We recall how String structures are lifts of Spin bundles through the 3-connected cover of $\mathrm{Spin}(n)$ and how this lift is obstructed by the fractional first Pontrjagin class called $\frac{1}{2}p_1$. Then we define *Fivebrane structures* as lifts of the resulting String bundles through the 7-connected cover of $\mathrm{Spin}(n)$. We demonstrate that this lift is obstructed by a degree 8 characteristic class of String bundles, which is a fractional second Pontrjagin class, $\frac{1}{6}p_2$, of the String bundle.

4.1. The homotopy groups of $SO(n)$.

The homotopy groups of the orthogonal group $O(n)$, for n sufficiently large, are

$$\pi_k\left(O(n)\right) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 0, 1 \bmod 8 \\ \mathbb{Z} & \text{for } k = 3, 7 \bmod 8 \\ 0 & \text{otherwise} \end{cases}\,. \tag{36}$$

See also the first two rows of figure 1. The condition on n is best understood by considering the *stable orthogonal group*, also know as the infinite orthogonal group, which is defined as the direct limit of the sequence of inclusions

$$O(1) \subset O(2) \subset \cdots \subset O = \bigcup_{k=0}^{\infty} O(k). \tag{37}$$

That the homotopy groups of $\mathrm{O}(n)$ stabilize, i.e. that for $n > k+1$, one has $\pi_k(\mathrm{O}) = \pi_k(\mathrm{O}(n))$, follows from the fact that the inclusion $O(n) \hookrightarrow O(n+1)$ is $(n-1)$-connected (see below for notation) as we have the fiber bundle

$$O(n) \to O(n+1) \to S^n, \tag{38}$$

i.e. S^n is the homogeneous space $O(n+1)/O(n)$.

Below the stable range, i.e. for $n \leq k+1$, the description of the homotopy groups of $\mathrm{SO}(n)$ becomes incomplete because one is essentially looking at the homotopy groups of spheres, which are not completely known. For example, the homotopy groups of $\mathrm{SO}(3)$ are the same as the homotopy groups of the sphere S^3 which are known only in specific degrees (but nonetheless in a range sufficient for most practical purposes). But in the applications of all these considerations to string theory which we have in mind, the base manifold X from which one wishes to consider homotopy classes of maps into $B(\mathrm{SO}(n)\langle k\rangle)$ is $(n=10)$-dimensional. Therefore in this application figure 1 is fully applicable. One obtains a sequence of topological groups from $\mathrm{O}(n)$ by successively passing to its k-connected covers. This process is known as the "killing of homotopy groups". See the third row of figure 1.

The description of the unitary group is analogous but much simpler due to the fact that Bott periodicity in this case has period two instead of period eight. In the stable range, i.e. for $i < 2n$, the inclusion $U(n)$ into $U(n+1)$ induces an isomorphism between the homotopy groups $\pi_i(U(n))$ and $\pi_i(U(n+1))$. The infinite unitary group U is defined in an analogous way as the orthogonal group above. The Bott periodicity theorem implies that the homotopy groups of U are particularly simple: $\pi_i(U)$ is trivial if i is even and isomorphic to $\mathbb{Z}$ if i is odd.

The standard notation for the k-connected cover of a space X for which all the homotopy groups vanish up to *and including* π_k is $X\langle k+1\rangle$. Thus the groups $U\langle 2k\rangle$ and $O\langle 2k\rangle$ denote the connected covers of the unitary and orthogonal group, respectively, having the first potentially nontrivial homotopy group in dimension $2k$. For example, $O\langle 3\rangle$ refers to the orthogonal group having first homotopy group in dimension three, which means that the first three homotopy groups are killed (starting with π_0) and therefore is 2-connected. The result of killing the first two homotopy groups of $O(n)$ are very familiar: these are the groups $SO(n)$ and $\mathrm{Spin}(n)$, respectively. The group $SO(n)$ is the connected component of the group $O(n)$ and $\mathrm{Spin}(n)$ is the simply-connected cover of $SO(n)$. Less familiar but by now well known is the result of killing the next nontrivial homotopy group beyond that, π_3: this is the String group $\mathrm{String}(n)$.

It is noteworthy that $O(n)$, $SO(n)$ and $\mathrm{Spin}(n)$ are not just topological groups, but of course carry the structure of a Lie group. On the other hand, there is no known model for $\mathrm{String}(n)$ as a group with a manifold structure. Notice that since all *compact* Lie groups G have $\pi_3(G) = \mathbb{Z}$ or several copies of $\mathbb{Z}$, $\mathrm{String}(n)$ has no chance of being a compact Lie group. The known models for $\mathrm{String}(n)$ are "huge" topological spaces. Notice however that [**?**] give a Lie model for $\mathrm{String}(n)$ when regarded not as a mere group, but as a Lie 2-group (a 2-group whose space of objects and of morphisms is a Fréchet) manifold.

Caveat: notation for higher connected covers. A comment on the notation used for the connected covers of the corresponding classifying spaces is in order. In another approach, the connected covers of the groups are defined not directly but via the corresponding classifying space

$$G\langle n\rangle := \Omega BG\langle n\rangle, \tag{39}$$

where Ω denotes the based loop space. However, for our purposes we are also interested in the classifying space itself. To avoid confusion (hopefully), we will use the notation

$$B(G\langle n\rangle) = (BG)\langle n+1\rangle. \tag{40}$$

Note the shift in n compared to (39). For example, $(BO)\langle 4\rangle = B(O\langle 3\rangle)$ refers to the classifying space having first homotopy group in dimension four, which means that the first four homotopy groups are killed, and therefore is 3-connected.

4.2. **String structures revisited.** Recall how the group $\mathrm{String}(n)$ can be constructed from $\mathrm{Spin}(n)$ via the sequence

$$1 \to K(\mathbb{Z},2) \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to 1, \tag{41}$$

where $\mathrm{Spin}(n)$ is the simply connected cover of $\mathrm{SO}(n)$ and, in fact, the first nonzero homotopy group of the Lie group $Spin(n)$ is $\pi_3 = \mathbb{Z}$. It follows from the Hurewicz theorem that the first nonzero cohomology group occurs at the same degree, i.e. $H^3(\mathrm{Spin}(n), \mathbb{Z}) = \mathbb{Z}$. This singles out the (homotopy class of a) map

$$f : \mathrm{Spin}(n) \to K(\mathbb{Z}, 3) \simeq BK(\mathbb{Z}, 2) \tag{42}$$

that generates $H^3(\mathrm{Spin}(n), \mathbb{Z})$ under the identification $H^3(X, \mathbb{Z}) \simeq [X, K(\mathbb{Z}, 3)]$, the set of homotopy classes of maps $X \to K(\mathbb{Z}, 3)$. This map classifies a principal $K(\mathbb{Z}, 2)$-bundle over $\mathrm{Spin}(n)$

$$\begin{tikzcd}
\mathrm{String}(n) := f^* EK(\mathbb{Z}, 2) \arrow[r] \arrow[d] & EK(\mathbb{Z}, 2) \arrow[d] \\
\mathrm{Spin}(n) \arrow[r, "f"] & K(\mathbb{Z}, 3) \simeq BK(\mathbb{Z}, 2)
\end{tikzcd} \tag{43}$$

and this bundle is the extension (41). Applying the classifying functor on (41) leads to a weakly exact (homotopy exact) sequence

$$K(\mathbb{Z}, 3) \to B\mathrm{String}(n) \to B\mathrm{Spin}(n). \tag{44}$$

The corresponding complex analog is

$$K(\mathbb{Z}, 3) \to (BU)\langle 6\rangle \to BSU \tag{45}$$

and on into $K(\mathbb{Z}, 4)$ obtained by mapping from BSU by the Chern class c_2. The structures are summarized in the following diagram

$$\begin{tikzcd}
(BU)\langle 6\rangle \arrow[rr] \arrow[d] & & B\mathrm{String} = (BO)\langle 8\rangle \arrow[d] & \\
BSU = (BU)\langle 4\rangle \arrow[rr] \arrow[d] & & B\mathrm{Spin} = (BO)\langle 4\rangle \arrow[r, "\frac{1}{2}p_1"] \arrow[d] & K(\mathbb{Z}, 4) \\
BU \arrow[rr] \arrow[dr] & & BSO = (BO)\langle 2\rangle \arrow[r, "w_2"] \arrow[dd] & K(\mathbb{Z}_2, 2) \\
& K(\mathbb{Z}, 2) \arrow[urr, "\mathrm{mod}\, 2"'] & & \\
& & BO = (BO)\langle 1\rangle \arrow[r, "w_1"] & K(\mathbb{Z}_2, 1).
\end{tikzcd} \tag{46}$$

Remarks. 1. $(BU)\langle 6\rangle$ and $(BO)\langle 8\rangle$ are the homotopy fibers of the respective maps to $K(\mathbb{Z}, 4)$.
2. The map from BSU to BSpin is an isomorphism on fourth cohomology H^4. This comes from the isomorphism

$$\mathrm{Spin}(3) \cong SU(2) \cong S^3. \tag{47}$$

3. The composite map from BSU to $K(\mathbb{Z}, 4)$ in the second row is given by the second Chern class c_2.

4. The composite map from BU to $K(\mathbb{Z}_2, 2)$ in the third row is given by c_1 followed by reduction mod 2. That is, we have the following commutative diagram

$$\begin{tikzcd} BU \arrow[r, "c_1"] \arrow[d] & K(\mathbb{Z},2) \arrow[d, "\text{mod } 2"] \\ BU \arrow[r, "w_2"] & K(\mathbb{Z}_2,2). \end{tikzcd} \tag{48}$$

4.3. **Cohomology of the connected covers.**

4.3.1. *In characteristic 0.* In this section we work in characteristic zero and we (briefly) address torsion in (4.3.2). The definition of String(n) as the extension of Spin(n) in (41) induced by the map (42) allows us to compute the cohomology of String(n). It is easier to compute first the cohomology of BString(n) from the homotopy fibration sequence

$$K(\mathbb{Z},3) \to B\mathrm{String}(n) \to B\mathrm{Spin}(n), \tag{49}$$

since in characteristic 0 we have (all coefficients are rational)

$$H^*(K(\mathbb{Z},3)) \simeq H^*(S^3) \tag{50}$$

and we can apply the long exact Gysin sequence

$$\cdots \longrightarrow H^n(E) \xrightarrow{\pi_*} H^{n-k}(M) \xrightarrow{e\wedge} H^{n+1}(M) \xrightarrow{\pi^*} H^{n+1}(E) \longrightarrow \cdots \tag{51}$$

where $e\wedge$ is the wedge product of a differential form with the Euler class e. The result is

$$H^*(B\mathrm{String}) \simeq P[p_2, p_3, \ldots], \tag{52}$$

from which it readily follows that $H^*(\mathrm{String})$ is an exterior algebra on generators x_i of degree $4i-1$ starting with $i=2$.

A very close analog of the above can be carried over starting with BU. The result is

$$H^*((BU)\langle 2k\rangle) \simeq P\left[c_k, c_{k+1}, \cdots\right]/(c_k) = P\left[c_{k+1}, \cdots\right]. \tag{53}$$

Except for the change in indexing, the proof is the same as for $B\mathrm{String} = B\mathrm{Spin}\langle 4\rangle$.

Remarks. 1. The rational cohomology of $(BU)\langle k\rangle$ can be calculated in several ways, for example using the Gysin sequence, as we saw above, or by induction on k using the Leray-Serre spectral sequence. While the first method is more straightforward, we include the second one in Appendix B because it is likely to be needed for applications in future work where torsion is important.
2. In the geometric description of $K(\mathbb{Z},2)$ as $PU(\mathcal{H})$, the projective unitary group on an infinite-dimensional Hilbert space $\mathcal{H}$, the H-field H_3 occurs as the canonical degree three class x_1 of $PU(\mathcal{H})$ bundles. In the operator algebra language this is called the Dixmier-Douady class [24]. From a physical point of view if X is ten-dimensional, it is natural to ask whether a similar interpretation of the dual field H_7 can be given. From a mathematical point of view, it is natural to ask whether a similar construction to the degree three case can be carried out with the first non-trivial homotopy group π_7 of String.

4.3.2. *Over the integers and the integers mod p.* **1.** The cohomology for all connected covers for the classifying spaces of the orthogonal and the unitary groups are also known. At $p=2$ this is a result of Stong [68]. For general p, this is a result of W. Singer [65], in terms of generators and relations, and Giambalvo [39].
2. $H_*\left((BU)\langle 6\rangle;\mathbb{Z}\right)$ is calculated as a ring in [4]. It is torsion-free and concentrated in even degrees. From this the cohomology can be read via

$$H^* = \hom\left(H_*((BU)\langle 6\rangle;\mathbb{Z}), -\right), \tag{54}$$

as a map from rings to sets. This is related in an interesting way to the what are known as cubical structures on the additive group [4].

4.4. **Fractional Pontrjagin classes.** When our manifold X has extra structure, such that it admits a lift of the structure group of its tangent bundle to a higher connected cover of $O(n)$, we can refine the Pontrjagin classes of X to *fractional* Pontrjagin classes. For X a d-dimensional orientable manifold, its kth Pontrjagin class p_k is taken to be the Pontrjagin class of the tangent bundle TX, regarded as an associated $SO(d)$-bundle. So it is given by a map

$$X \xrightarrow{f} BSO(d) \xrightarrow{p_k} K(\mathbb{Z}, 4k) \; . \tag{55}$$

4.4.1. *Spin structures and the first Pontrjagin class.* Saying that a d-dimensional manifold X is *spin* means we can lift the classifying map f of its tangent bundle TX to a map $\hat{f} : X \to B\mathrm{Spin}(d)$. Since Spin is 2-connected and $\pi_3(\mathrm{Spin}) = \mathbb{Z}$, it follows that $H^4(B\mathrm{Spin}, \mathbb{Z}) = \mathbb{Z}$. If we denote the generator of this fourth cohomology group by ω, the situation looks as follows:

$$\begin{tikzcd} & B\mathrm{Spin}(d) \arrow[r, "\omega"] \arrow[d, "\pi"] & K(\mathbb{Z},4) \\ X \arrow[ur, "\hat{f}"] \arrow[r, "f"] & B\mathrm{SO}(d) \arrow[r, "p_1"] & K(\mathbb{Z},4) \end{tikzcd} \cdot \tag{56}$$

Since, by assumption, ω is a generator, it must be true that $\pi^* p_1$ is an integral multiple of ω. One finds that ω can be chosen so that

$$\pi^* p_1 = 2\omega \,, \tag{57}$$

i.e.

$$\begin{tikzcd} & B\mathrm{Spin}(d) \arrow[r, "\omega"] \arrow[d, "\pi"] & K(\mathbb{Z},4) \arrow[d, "\cdot 2"] \\ X \arrow[ur, "\hat{f}"] \arrow[r, "f"] & B\mathrm{SO}(d) \arrow[r, "p_1"] & K(\mathbb{Z},4) \end{tikzcd} \cdot \tag{58}$$

This motivates the notation

$$\omega := \frac{1}{2} p_1 \tag{59}$$

for the generator of $H^4(B\mathrm{Spin}, \mathbb{Z})$. Accordingly, the pullback

$$\frac{1}{2} p_1(X) := \hat{f}^* \frac{1}{2} p_1 : \; X \xrightarrow{\hat{f}} B\mathrm{Spin} \xrightarrow{\frac{1}{2}p_1} K(\mathbb{Z}, 4) \tag{60}$$

is "half the Pontrjagin class" of the spin-manifold X. Notice that for $\frac{1}{2}p_1(X)$ to be zero, the vanishing of $p_1(X)$ is a necessary but not a sufficient condition: $\frac{1}{2}p_1(X)$ might be non-vanishing but 2-torsion.

4.4.2. *String structures and the second Pontrjagin class.* The same kind of reasoning continues to apply as we keep killing homotopy groups of $O(d)$. We say that X admits a *string* structure or that X is string if the classifying map f for TX lifts to $B\mathrm{String}(d)$. Now $B\mathrm{String}$ is 7-connected and $H^8(B\mathrm{String}, \mathbb{Z}) \simeq \mathbb{Z}$. Let ν denote the corresponding generator. The pullback $\pi^* p_2$ of the second Pontrjagin class has to be an integer multiple of this generator. In the next section (see Proposition 1 section 4.5), we will show that the integer multiple is 6:

(61)
$$\begin{tikzcd} & B\mathrm{String}(d) \arrow[r, "\nu"] \arrow[d, "\pi"] & K(\mathbb{Z},8) \arrow[d, "\cdot 6"] \\ X \arrow[ur, "\hat{f}"] \arrow[r, "f"] & B\mathrm{SO}(d) \arrow[r, "p_2"] & K(\mathbb{Z},8) \end{tikzcd} .$$

Therefore we should give the generator ν the name $p_2/6$ and define

(62)
$$\frac{1}{6}p_2(X) := \hat{f}^* \frac{1}{6}p_2,$$

the fractional second Pontrjagin class of the String manifold X. Later (see equation (93)) we will make use of the spin characteristic classes, which better describe spin bundles than do Pontrjagin classes.

4.5. **Fivebrane Structures.** The point of view we take is that in the same way that H_3 was part of the obstruction to lifting a Spin bundle to a String bundle, we would like to interpret H_7 as another obstruction. Further, H_7 serves as a higher obstruction in the sense that it makes sense to talk about it once the 'lower' obstructions vanish. The next task is to make this more precise.

The first nonzero homotopy group of the topological group String(n) is $\pi_7 \simeq \mathbb{Z}$. Then, again, the Hurewicz theorem implies that the first nonzero cohomology group occurs in degree 7. As $H^7(X,\mathbb{Z}) = [X, K(\mathbb{Z},7)]$ and $K(\mathbb{Z},7) = BK(\mathbb{Z},6)$, it follows that $K(\mathbb{Z},6)$ is a fiber in a nontrivial fibration sequence with String(n) as the base. From the structure of the homotopy groups (36), the extension will be $O\langle 8\rangle$. Thus (compare also Def. 1)

Definition 2. *The extension* $1 \longrightarrow K(\mathbb{Z},6) \longrightarrow O\langle 8\rangle \longrightarrow \mathrm{String} \longrightarrow 1$*, classified by the canonical map* $\mathrm{String} \to K(\mathbb{Z},7)$*, which replaces the sequence (41), we call the* Fivebrane*-extension* $\mathrm{Fivebrane} := O\langle 8\rangle$.

Remarks. 1. The fact that String occurs as the base of the sequence in Def. 2 is compatible with the interpretation of the classes of H_3 and H_7 as corresponding to the fibrations being pulled back from $K(\mathbb{Z},3)$ and $K(\mathbb{Z},7)$ respectively.
2. When the space is Spin, the first Pontrjagin class p_1 is divisible by 2, and the obstruction to lifting Spin to String is $\frac{1}{2}p_1$. When the space is String, the second Pontrjagin class p_2 is divisible by 6, and the obstruction to lifting *String* to Fivebrane is $\frac{1}{6}p_2$.

Proposition 1. *The obstruction to lifting a String bundle to an* $O\langle 8\rangle$ *bundle is given by* $\frac{1}{6}p_2$.

Proof. The classifying spaces of the above sequences induce a map from $B\mathrm{String} \to B\mathrm{Spin}$. The composite map $B\mathrm{String} \to B\mathrm{Spin} \to BSO$ will map p_2 to a multiple of the generator of $H^8(B\mathrm{String},\mathbb{Z})$. This multiple is obtained by noticing that the map from BString to $(BU)\langle 7\rangle$ is an isomophism on π_8 and that the fourth Chern class c_4 restricts to 6 times the generator of π_8 on the complex side.

The complex vector bundle on S^{2k} that represents the generator of $\pi_{2k}(BU)$ is the k-fold (external) tensor product of $(1-L)$, where 1 is the tautological (trivial) line bundle and L is the Hopf line bundle on S^2. All that needs to be done is find the value of c_k on this bundle. Using the multiplicative properties of the Chern character, this is just $(k-1)!$ (see e.g. [56], Theorem 5.1). Thus for $k=4$, the answer is 6 as claimed. ☐

It follows from connectivity that there is a commutative pullback diagram $(BO)\langle 9\rangle \to (BU)\langle 10\rangle$ sitting over $(BO)\langle 8\rangle \to (BU)\langle 8\rangle$. The map $(BU)\langle 10\rangle \to (BU)\langle 8\rangle$ is the fibration corresponding to $\frac{1}{6}c_4$, which restricts to $\frac{1}{6}p_2$ in $(BO)\langle 8\rangle$. The map $(BO)\langle 8\rangle \to (BU)\langle 8\rangle$ is an isomorphism on π_8 and lifts to $(BO)\langle 9\rangle \to$

$(BU)\langle 10\rangle$, which is also an isomorphism on π_8. Thus we have the following diagram, where BFivebrane is as in Def. 1,

$$\begin{tikzcd}
B\mathrm{Fivebrane} = (BO)\langle 9\rangle \arrow[r] \arrow[d] & (BU)\langle 10\rangle \arrow[d] & \\
B\mathrm{String} = (BO)\langle 8\rangle \arrow[r] \arrow[d] & (BU)\langle 8\rangle \arrow[r, "\frac{1}{6}c_4"] \arrow[d] & K(\mathbb{Z},8) \\
B\mathrm{Spin} = (BO)\langle 4\rangle \arrow[r] \arrow[d] & BSU = (BU)\langle 4\rangle \arrow[r, "c_2"] \arrow[d] & K(\mathbb{Z},4) \\
BSO = (BO)\langle 2\rangle \arrow[r] \arrow[d] & BU & \\
BO = (BO)\langle 1\rangle & &
\end{tikzcd} \tag{63}$$

Remarks. 1. In order for a bundle to be lifted from String to $O\langle 8\rangle$, it has already to be Spin. Thus the first Pontrjagin classes in expressions (15) and (16) are set to zero and then dH_7 in both cases is some linear combination of the second Pontrjagin classes, corresponding to the (lifts of the) tangent and the gauge bundles, TX and E (cf. [49]).
2. That the first column in (63) is real and the second column is complex follows the complexification map $KO \to KU$ on bundles, since we are interested in the divisibility properties of the Pontrjagin classes, and those are considered as the pullback of the complexification, i.e. the Chern classes.
3. The map from BO to BU is an isomorphism on π_{8k} and is multiplication by 2 on π_{8k+4}. In particular, the map is an isomorphism on π_8 and is multiplication by 2 on π_4 (see e.g. [57]).
4. In the second row, we could have written $(BU)\langle 6\rangle$ in the second entry. The reason for having $(BU)\langle 8\rangle$ instead is because we would then have the map from the real to the complex side, $(BO)\langle 8\rangle \to (BU)\langle 8\rangle$ to be an isomorphism on π_8. In contrast, the map $B\mathrm{Spin} \to BSU$ is not an isomorphism on π_4 but is in fact given by multiplication by 2 (by part 3 above) – see [57] for instance.
5. In the first row, we wrote $(BU)\langle 10\rangle$ instead of $(BU)\langle 9\rangle$ because there are no homotopy groups of odd degree in BU, i.e. killing the homotopy in degree eight automatically gets us to degree ten.
6. In both cases, BString and BFivebrane, we are killing a $\mathbb{Z}$ in the homotopy groups.
7. The map from $(BO)\langle 8\rangle$ to $(BU)\langle 8\rangle$ in the second row is an isomorphism in degree eight because the generator of π_8 for both spaces is v_1^4, where v_1 is the Bott generator whose degree is two.
8. The map from BSpin to BSU in the third row is given by multiplication by two.

Real vs. Complex. In diagram (46) we had the complex spaces in the first column and the real spaces in the second column. In diagram (63) we had instead the real spaces in the first column and the complex spaces in the second column. We describe this further here. Consider the composite map from BSU to itself factoring through $B\mathrm{Spin}$

$$BSU \xrightarrow{\cong \text{ in deg } 4} B\mathrm{Spin} \xrightarrow{\times 2} BSU. \tag{64}$$

The composition is given by multiplication by 2 in degree 4 and acts on vector bundles via

$$V \mapsto V \oplus \overline{V} = 2V, \tag{65}$$

so that $c_2(2V) = 2c_2(V)$. Note that since we have SU bundles then $c_1(V) = 0$.

Next going from BU to BSO we have

$$\mathbb{Z} \times BU \xrightarrow[\text{forget}]{\times 2,\ \cong \text{ in deg } 0} \mathbb{Z} \times BSO \xrightarrow[\text{complexify}]{\cong \text{ deg } 0} \mathbb{Z} \times BU \longrightarrow \mathbb{Z} \times BSO. \tag{66}$$

The first map amounts to forgetting the complex structure and the second to complexifying. The map from second to the fourth term is $V \mapsto 2V$, and that from the first to the third is $V \mapsto V \oplus \overline{V}$.

4.6. **Congruence. Remarks.**

1. At the integral level, there is a very crucial difference between p_2 being zero and $\frac{1}{6}p_2$ being zero, due to the possible existence of very important 2- and 3-torsion. In other words, $\frac{1}{6}p_2 = 0$ certainly implies that p_2 is zero, but the converse is not necessarily true – and in fact in most interesting cases it is not true. This gives us the important conclusion that: *Unlike the String case, both two- and three-torsion are important for $(BO)\langle 9\rangle$ structures.*

2. Note that $\pi_* O$ has only 2-torsion so that in the stable range there is no 4-torsion, so there is no difference between $\frac{1}{2}p_1$ and $\frac{1}{4}p_1$ as obstructions. The latter is the shift in the quantization condition of the field strength G_4 in M-theory [75] (cf. footnote 2).

3. Likewise, in the fivebrane case – again assuming the stable range– there is no difference, as far as the type of torsion is concerned, between the obstructions $\frac{1}{6}p_2$ and $\frac{1}{48}p_2$. This is because they are both of the form $\frac{1}{2^n 3^m}p_2$, i.e. involve 2− and 3-torsion. This leads to the following conclusion: *In the stable range (hence no 8-torsion) for the tangent bundle, our definition of the fivebrane structure captures the physical condition for a fivebrane structure, i.e. that $\frac{1}{6}p_2$ is essentially the same as $\frac{1}{48}p_2$ in the sense that has just been explained.*

4. In both case we can avoid subtleties of division by 2 and 3, by *inverting* those two primes. Thus if we invert 2 in the string case, starting with a ring R (e.g. $\mathbb{Z}$) we can work with the ring $R[\frac{1}{2}]$, and if we invert $6 = 2 \times 3$ in the fivebrane case, we can work with the ring $R[\frac{1}{6}]$.

The division by 8. We have seen that the obstruction to the fivebrane structure is given by $\frac{1}{6}p_2$, while the expression appearing in the anomaly is $\frac{1}{48}p_2$. Here we give a description of this further division by 8, which, by the above remark, does not generate any new torsion. In particular this means that the situation in the fivebrane case is better than that in the string case, where there was a crucial difference between p_1 vanishing and $\frac{1}{2}p_1$ vanishing, namely that coming from 2-torsion.

What we would like to describe is a space, which we will call $\mathcal{F}$, that sits in the fibration

$$\mathcal{F} \longrightarrow (BO)\langle 9\rangle \xrightarrow{\times 8} (BO)\langle 9\rangle \tag{67}$$

where the second map takes $\frac{p_2}{48}$ to $\frac{p_2}{6}$. The question is then: what is $\mathcal{F}$? As a warm-up, consider a degree two class, in which case we have

$$F \longrightarrow K(\mathbb{Z}, 2) \xrightarrow{\times n} K(\mathbb{Z}, 2), \tag{68}$$

and the answer is $F = K(\mathbb{Z}_n, 1)$. We then seek a lift

$$
\begin{tikzcd}
 & K(\mathbb{Z},2) \arrow[d, "\times n"] & \\
X \arrow[ur, dotted, "?"] \arrow[r] & K(\mathbb{Z},2) \arrow[r, "\alpha_n"] & K(\mathbb{Z}_n,2),
\end{tikzcd}
\tag{69}
$$

which exists if and only if the pullback of the map α_n to X is zero, where α_n is 'reduction mod 2'. For a String structure on a space Y we have the following diagram

$$
\begin{tikzcd}
Y \arrow[r, "x"] \arrow[dr] & \mathcal{F} = \mathrm{String}^{\mathcal{F}} \arrow[r] \arrow[d] & K(\mathbb{Z},8) \arrow[d, "\times 8"] & \\
 & (BO)\langle 8\rangle \arrow[r, "\frac{1}{6}p_2"'] & K(\mathbb{Z},8) \arrow[r] & K(\mathbb{Z}_8,8),
\end{tikzcd}
\tag{70}
$$

where x is our class $\frac{1}{48}p_2$ which naturally lives not in $(BO)\langle 8\rangle$ but rather in the desired space $\mathcal{F}$. The above diagram specifies $\mathcal{F}$. The modification of Proposition 1 is then

Proposition 2. *The class $\frac{1}{48}p_2$ is the obstruction to lifting a String$^{\mathcal{F}}$ bundle, defined by diagram (70), to a Fivebrane bundle.*

We conclude this section with a caveat. The formula that arise from the anomaly cancelation, e.g. (16), involve different coefficients for each of the second Pontrjagin classes of the two bundles. Had we had the same factor or a just minus sign, then we could have simply applied the formulae in the discussion at the end of section 4.7. However, we have relative factors of 48, so how can one make sense of such factors? For instance, in the case when the string condition applies to all bundles, is $p_2(E) + \frac{1}{n}p_2(F)$ the second Pontrjagin class of some combination of the bundles E and F? It does not make sense to talk about $E + \frac{1}{n}F$, but one can talk about $nE + F$. But then, we would have the issue of torsion, especially that $n = 48$ includes both the important 2- and 3-torsion. We can thus say that, away from such torsion, the Fivebrane class of $48TM - E$ is equal to 48 times the Fivebrane class of TM minus the Fivebrane class of E. In the case when $H^8(X, \mathbb{Z})$ has 2- or 3-torsion we cannot draw such a conclusion.

4.7. **Inequivalent Fivebrane Structures: The Fivebrane Class.** In this section we consider the set of inequivalent Fivebrane structures on our manifold or on a pair consisting of a manifold with a gauge bundle, In order to do so, let us first recall the situation for String structures on manifolds. The complex version of the fibration

$$K(\mathbb{Z}, 3) \to B\mathrm{String} \to B\mathrm{Spin} \tag{71}$$

is

$$K(\mathbb{Z}, 3) \to (BU)\langle 6\rangle \to BSU, \tag{72}$$

where the space $(BU)\langle 6\rangle$ is the 5-connected cover of BU. A choice of String structure on a space X is a lift $\hat{f}$ of the classifying map f in the diagram

(73)
$$\begin{tikzcd} & (BO)\langle 8\rangle \arrow[d] & \\ X \arrow[ur, dotted, "\hat{f}"] \arrow[r, "f"] & (BO)\langle 4\rangle \arrow[r, "\frac{1}{2}p_1"] & K(\mathbb{Z},4). \end{tikzcd}$$

A choice of $(BU)\langle 6\rangle$ structure on a space X is a lift in the diagram

(74)
$$\begin{tikzcd} & B(U)\langle 6\rangle \arrow[d] & \\ X \arrow[ur, dotted, "\hat{f}"] \arrow[r, "f"] & BSU \arrow[r, "c_2"] & K(\mathbb{Z},4). \end{tikzcd}$$

The appearance of the factor $\frac{1}{2}$ in (73) vs. (79) is a reflection of the fact mentioned earlier that the map from BSpin to BSU is given by multiplication by two. The fibration sequences

(75)
$$\begin{array}{c} K(\mathbb{Z},3) \longrightarrow (BO)\langle 8\rangle \longrightarrow (BO)\langle 4\rangle \\ K(\mathbb{Z},3) \longrightarrow (BU)\langle 6\rangle \longrightarrow BSU \end{array}$$

show that two String or $(BU)\langle 6\rangle$ structures differ by a map to $K(\mathbb{Z},3)$ for a fixed Spin or BSU structure, respectively. Therefore, the set of lifts, i.e. the set of String structures for a fixed Spin structure in the real case, or the set of $(BU)\langle 6\rangle$ structures for a fixed BSU structure in the complex case, is a torsor [3] for a quotient of the third integral cohomology group $H^3(X;\mathbb{Z})$. The elements of the torsor are the string classes, corresponding to the NS degree three H-field in string theory.

Now we are ready to study the set of Fivebrane structures. The complex version of the fibration

(76)
$$K(\mathbb{Z},7) \to B\text{Fivebrane} \to B\text{String},$$

obtained by applying the classifying functor on the sequence in Def. 2, is

(77)
$$K(\mathbb{Z},7) \to (BU)\langle 10\rangle \to (BU)\langle 8\rangle,$$

where the space $(BU)\langle 8\rangle$ is the 7-connected cover of BU. A choice of Fivebrane structure on a space X is a lift in the diagram

(78)
$$\begin{tikzcd} & (BO)\langle 9\rangle \arrow[d] & \\ X \arrow[ur, dotted, "\hat{f}"] \arrow[r, "f"] & (BO)\langle 8\rangle \arrow[r, "\frac{1}{2}p_1"] & K(\mathbb{Z},8). \end{tikzcd}$$

[3] Here a torsor is a set with a group action which is free and transitive. So as a set it is the same as the group. The group acts but there is no canonical identification of the set with the group.

A choice of $(BU)\langle 9\rangle$ structure on a space X is a lift in the diagram

(79)
$$\begin{tikzcd} & (BU)\langle 10\rangle \arrow[d] & \\ X \arrow[ur, dotted, "\hat{f}"] \arrow[r, "f"] & (BU)\langle 8\rangle \arrow[r, "\frac{1}{6}c_4"] & K(\mathbb{Z},8). \end{tikzcd}$$

The fibration sequences (76) (77) show that two Fivebrane or $(BU)\langle 9\rangle$ structures differ by a map to $K(\mathbb{Z},7)$ for a fixed String or $(BU)\langle 7\rangle$ structure, respectively.

Remarks.

1. A degree seven class $X \to K(Z,7)$ that corresponds to the fivebrane structure cannot be specified. Consider the diagram

(80)
$$\begin{tikzcd} & K(\mathbb{Z},7) \arrow[d] \\ & (BO)\langle 9\rangle \arrow[d] \\ X \arrow[uur, dotted, "g"] \arrow[ur, dotted, "\hat{f}"] \arrow[r, "f"] & (BO)\langle 8\rangle \end{tikzcd}$$

and note that f and g do not determine $\hat{f}$ since $BO\langle 9\rangle \neq BO\langle 8\rangle \times K(\mathbb{Z},7)$. This means that there is no degree seven class that picks $\hat{f}$.

2. For a given string structure, the set of compatible fivebrane structures has a transitive action by degree seven cohomology. We have

Proposition 3. *The set of lifts, i.e. the set of Fivebrane structures for a fixed String structure in the real case, or the set of $(BU)\langle 9\rangle$ structures for a fixed $(BU)\langle 7\rangle$ structure in the complex case, is a torsor for a quotient (to be described below) of the seventh integral cohomology group $H^7(X;\mathbb{Z})$.*

We call the elements of the latter the *fivebrane classes*, corresponding to the (dual) NS degree seven H-field in string theory. We now explain the quotient in the proposition. Whether or not we have a free action depends on whether the map

(81)
$$[X,(\Omega(BO))\langle 8\rangle] \to [X, K(Z,7)]$$

coming from the diagram

$$
\begin{tikzcd}
 & (\Omega(BO))\langle 8\rangle \arrow[d] \\
 & K(\mathbb{Z},7) \arrow[d] \\
 & (BO)\langle 9\rangle \arrow[d] \\
X \arrow[r, "f"] \arrow[uur, dotted, "g"] \arrow[ur, dotted, "\hat{f}"] & BO\langle 8\rangle
\end{tikzcd}
\tag{82}
$$

is the zero map. Note that (81) is just the map of homotopy classes induced by the top arrow in (82). However, the intention is to consider *two* lifitngs $\hat{f}_i$ for $i = 1, 2$ which therefore differ by a map g as in (80). Since the maps of homotopy classes give long exact sequences for any three of the target spaces, then if (81) were the zero map then the next one would be onto. Failure to be onto is measured by the failure of $[X, (BO)\langle 8\rangle] \to [X, K(\mathbb{Z}, 7)]$ to be zero. The set of lifts is a torsor over the quotient

$$H^7(X, \mathbb{Z})/\,[X, \Omega(BO)\langle 8\rangle]\,. \tag{83}$$

3. The fibration (76) is a map of infinite loop spaces so we can think of it in terms of a map

$$K(\mathbb{Z}, 7) \times B\text{Fivebrane} \to B\text{Fivebrane}, \tag{84}$$

realizing an action of $K(\mathbb{Z}, 7)$ on the space of inequivalent fivebrane structures.
4. At the level of spectra, which are objects whose homotopy groups represent generalized cohomology theories [1], we have

$$\Sigma^7 H\mathbb{Z} \times ko\langle 9\rangle \to ko\langle 9\rangle, \tag{85}$$

where $ko\langle 9\rangle$ is the *connective* version of real K-theory KO, i.e. the version with no negative degrees, of the K-theory corresponding to $BO\langle 9\rangle$.
5. Addition of vector bundles is encoded in a product

$$BSO \times BSO \to BSO, \tag{86}$$

which has a lift to

$$B\text{Fivebrane} \times B\text{Fivebrane} \to B\text{Fivebrane}. \tag{87}$$

This gives us a way of adding Fivebrane bundles; for example, in our setting of heterotic string theory, we have the virtual difference of the tangent bundle and the gauge bundle. In fact lifts such as (87) work for even higher connected covers as well.
6. The bundles we have are the tangent bundle and the gauge vector bundle E, with the total bundle being the K-theoretic virtual difference. Hence it is natural to ask about the fivebrane structure of sums and differences of bundles. There is a distinction here according to whether the bundles are spin or not. For the two possible gauge bundles, the $E_8 \times E_8$ bundle is Spin, while the $\text{Spin}(32)/\mathbb{Z}_2$ bundle is not, as its second Stiefel-Whitney class is non-vanishing. The (integral) Pontrjagin classes for two bundles E_1 and E_2 satisfy [53]

$$p_k(E_1 \oplus E_2) = \sum_{i+j=k} p_i(E_1) \cup p_i(E_2) \text{ mod elements of order 2.} \tag{88}$$

Thus, applying to our case,

$$p_2(TM \oplus E) = p_1(TM) \cup p_1(E) + p_2(TM) + p_2(E) \text{ mod } 2 - \text{torsion.} \tag{89}$$

But we are assuming $p_1(TM)$ to be zero – recall that in general, we have to have a String structure on a bundle to talk about Fivebrane structure as we have to have a Spin structure on a bundle to talk about String structure. Hence we get:

$$p_2(TM \oplus E) = p_2(TM) + p_2(E) \text{ mod } 2 - \text{torsion}, \tag{90}$$

so

$$2[p_2(TM \oplus E) - p_2(TM) - p_2(E)] = 0. \tag{91}$$

We have worked out the case of a direct sum, but the case of virtual difference should be analogous. Indeed, since $V \oplus (-V)$ is equivalent to a trivial bundle, $p_1(E_1 - E_2) = p_1(E_1) - p_1(E_2)$ and $p_2(E_1 - E_2) = p_2(E_1) - p_2(E_2)$, and hence we get

$$2[p_2(TM - E) - p_2(TM) + p_2(E)] = 0. \tag{92}$$

We can actually give a better description by using special characteristic classes more adapted for covers of the tangent bundle and, in addition, use integral coefficients.

Note that, making use of the fact that the bundles are not just real but also spin,

$$H^*(B\mathrm{Spin};\mathbb{Z}) = \mathbb{Z}[Q_1, Q_2, \cdots] \oplus \gamma, \tag{93}$$

with γ a 2-torsion factor, i.e. $2\gamma = 0$ [72], concentrated in degrees *not* congruent to 0 mod 4. The two degrees relevant to our discussion are

$$\begin{aligned} H^4(B\mathrm{Spin};\mathbb{Z}) &\cong \mathbb{Z} \text{ with generator } Q_1 \\ H^8(B\mathrm{Spin};\mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z} \text{ with generators } Q_1^2, Q_2, \end{aligned} \tag{94}$$

where Q_1 and Q_2 are determined by their relation to the Pontrjagin classes

$$\begin{aligned} p_1 &= 2Q_1 \\ p_2 &= Q_1^2 + 2Q_2. \end{aligned} \tag{95}$$

Then the spin generators are given in terms of the Pontrjagin classes by $Q_1 = \frac{1}{2}p_1$ and $Q_2 = \frac{1}{2}p_2 - \frac{1}{2}\left(\frac{1}{2}p_1\right)^2$. Note that $Q_2 = \frac{1}{2}p_2$ holds when Q_1 vanishes. Their importance in the study of anomalies was emphasized in [59]. For two Spin bundles E and F, such as the spinor bundle and the $E_8 \times E_8$ bundle

$$\begin{aligned} Q_2(E \oplus F) &= \sum_{i+j=2} Q_i(E) \cup Q_j(F) \\ &= Q_2(E) + Q_2(F), \end{aligned} \tag{96}$$

where again the condition $Q_1(TM) = 0$ is used.

4.8. **Overview.**

- We considered the notion of a "Fivebrane structure" on a manifold, which generalizes that of a String structure: as a String structure is defined to be a lift of the tangent bundle of a Spin-manifold to the 3-connected cover of Spin, a Fivebrane structure is the further lift to a 7-connected cover. We showed that Fivebrane structures exist when a fractional second Pontrjagin class vanishes. This holds for the tangent bundle as well as gauge bundles. When considering direct sums or K-theoretic virtual bundles, as in the case of heterotic string theory where we have both TX and E, the obstructions are given by the sums and differences of the individual fivebrane structures, respectively.
- In analogy to how String structures appear in terms of the vanishing of a worldsheet anomaly for the superstring, Fivebrane structures are related to the vanishing of an anomaly in the 6-dimensional worldvolume theory of the fivebranes. In M-theory and type IIA string theory, this is the anomaly of the M-theory fivebrane [25] [26] [55] [10] [16], while in heterotic string theory this is the worldvolume anomaly of the heterotic fivebrane [51]. Note that the electric-magnetic duality between Strings and 5-branes in ten dimensions relates String structures and Fivebrane structures:

- We notice that string theory suggests that String structures and Fivebrane structures are related by a duality which generalizes Hodge duality. Apart from the well-known fact (see, for instance, [28] for a survey) that NS 5-branes are magnetic duals to strings, we notice that there is a known formula for the Hodge dual of the 3-form curvature which appears in the Green-Schwarz mechanism which relates to Fivebrane structures as the former relates to String structures. We leave the detailed study of this duality for a separate treatment.

Appendix A. Recollection of characteristic classes

We recall elements of the theory of characteristic classes.

A.1. **Universal characteristic classes.** For any topological group G, there is a *classifying space* BG and a universal principal G-bundle

$$EG \to BG \tag{97}$$

such that equivalence classes of (numerable) G-bundles $E \to B$ are in 1-1 correspondence with homotopy classes of maps $B \to BG$.

For G a Lie group, elements of the cohomology $H^\bullet(BG)$ (with any coefficients) are called the *universal characteristic classes for G-bundles*. Given a G-bundle $E \to B$, its characteristic classes are the pull-backs of the universal characteristic classes via the classifying map $B \to BG$. It is a classical theorem that, for G compact connected, the image in real cohomology, $H^\bullet(BG, \mathbb{R})$, of this cohomology ring is finitely freely generated in even degree and is isomorphic to the the ring of invariant polynomials on $\mathfrak{g} = \mathrm{Lie}(G)$. Moreover, the real cohomology ring of G itself $H^\bullet(G, \mathbb{R}) \simeq H^\bullet(\mathfrak{g}, \mathbb{R})$ is isomorphic to the Lie algebra cohomology ring of $\mathfrak{g}$, which is generated in odd degree. The relation between $H^*(G, \mathbb{R})$ and $H^*(BG, \mathbb{R})$ is as follows:

- $H^*(G)$ is isomorphic to an exterior algebra $\Lambda(x_1, \cdots, x_n)$, where the x_i are all of odd degree.
- $H^*(BG)$ is isomorphic to a polynomial algebra $P[y_1, \cdots, y_n]$ where

$$\text{degree } y_i = \text{degree } x_i + 1. \tag{98}$$

The isomorphism of the respective vector spaces of generators can in fact be expressed invariantly: Let $\mathcal{P}H^*(G)$ denote the subspace of *primitive* elements, i.e. those $x \in H^*(G)$ such that

$$m^*(x) = 1 \otimes x + x \otimes 1 \tag{99}$$

where $m : G \times G \to G$ is the group multiplication. Let $\mathcal{Q}H^*(BG)$ denote the *quotient space of indecomposables*, i.e. $H^+(BG)/H^+(BG) \cdot H^+(BG)$ where H^+ denotes the subalgebra of elements of positive degree. The isomorphism we have described in terms of generators is in fact

$$\tau : \mathcal{P}H^*(G) \to \mathcal{Q}H^{*+1}(BG). \tag{100}$$

The isomorphism says that the two vector spaces are *transgressively related*. Unfortunately the name *transgression* is sometimes applied in algebraic topology to τ (cf. H. Cartan, Borel, Serre) and sometimes in differential geometry to its inverse (cf. Chern and disciples). In algebraic topology, the map $\sigma : H^*(BG) \to H^{*-1}(G)$ is always defined whereas τ is defined only on primitive elements.

An element c in $H^n(BG, \mathbb{Z})$ can be identified with a (homotopy class of a) map f from BG to the Eilenberg-MacLane space $K(\mathbb{Z}, n)$ (and similarly for $\mathbb{R}$), then the characteristic class f^*c of a G-bundle E corresponding to the characteristic class c of G is given simply by the composite map

$$B \xrightarrow{f} BG \xrightarrow{c} K(\mathbb{Z}, n)\ . \tag{101}$$

A.2. **The cohomology of BSO and BU.** We are particularly concerned in this paper with $H^*(BSO)$ and $H^*(BU)$. See [53]. In characteristic 0,

$$H^*(BSO) \simeq P[y_4, y_8, \cdots], \tag{102}$$

where now the indexing is by the degree of the (Pontrjagin) class. In any characteristic,

$$H^*(BU) \simeq P[c_1, c_2, \cdots], \tag{103}$$

where we have used the traditonal notation for the Chern classes c_i. The basic definitions and properties of Chern and Pontrjagin classes are recalled next.

A.3. **Chern-Weil homomorphism.** Given a G-bundle $P \to X$ with connection A, the Chern-Weil homomorphism

$$\mathrm{inv}(\mathfrak{g}) \to H^\bullet(X, \mathbb{R}) \tag{104}$$

sends each degree n invariant polynomial $k \in \mathrm{inv}(\mathfrak{g})$ on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ to the differential form

$$k \mapsto k(F_A) := k(F_A \wedge \cdots \wedge F_A) \tag{105}$$

obtained by wedging n copies of the curvature 2-form and evaluating its Lie algebra value in k. This $k(F_A)$ is a closed form. The corresponding class $[k(F_A)]$ in deRham cohomology is the characteristic class of P corresponding to k. This class is independent of the choice of connection on P.

$$\begin{tikzcd}
\mathrm{inv}(\mathfrak{g}) \arrow[r] \arrow[rr, bend right, "\text{Chern-Weil homomorphism}"] & \Omega^\bullet(X) \arrow[r] & H^\bullet(X, \mathbb{R}) \\
k \arrow[r, mapsto] & k(F_A) \arrow[r, mapsto] & {[k(F_A)]}
\end{tikzcd}$$

invariant polynomial	characteristic form	characteristic class

TABLE 7. **The Chern-Weil homomorphism** sends, for each G-bundle $P \to X$, any degree n invariant polynomial on $\mathfrak{g} = \mathrm{Lie}(G)$ to the deRham class of the differential form $k(F_A) = k(F_A \wedge \cdots \wedge F_A)$ obtained by inserting the curvature 2-form of any connection on P into k.

A.4. **Polynomials and classes for matrix Lie algebras.** Given a matrix Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n)$, the trace and the determinant operation on matrices provide families of $\mathfrak{g}$-invariant polynomials.

- The assignment

$$X \mapsto \mathrm{ch}(X) := \mathrm{Tr}(\exp(X)) \tag{106}$$

  defines the invariant polynomials $\mathrm{ch}_k(X)$ as

$$\mathrm{ch}(X) := \sum_{k=0}^{\infty} t^k \mathrm{ch}_k(X, \cdots, X)\,. \tag{107}$$

  The expression $\mathrm{ch}(X)$ is the total Chern-character.
- Let $\mathfrak{g} \subset \mathfrak{gl}(c, \mathbb{C})$ be a Lie algebra of *complex* matrices. Then the assignment

$$X \mapsto c(X) := \det(t + iX) \tag{108}$$

defines the invariant polynomials c_k as

$$c(X) = \sum_{k=0}^{n} t^{n-k} c_k(X, \cdots, X)\,. \tag{109}$$

These are the Chern Polynomials. The corresponding class $[c_k(F_A)]$ is the kth Chern class.

- Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ be a Lie algebra of *real* matrices. Then the assignment

$$X \mapsto c(X) := \det(t - X) \tag{110}$$

defines the invariant polynomials $p_{k/2}$ as

$$c(X) = \sum_{k=0}^{n} t^{n-k} p_{k/2}(X, \cdots, X)\,. \tag{111}$$

These are the Pontrjagin polynomials. The corresponding classes $[p_{k/2}(F_A)]$ are the Pontrjagin classes.

- The restrictions of the c_k from $\mathfrak{gl}(n, \mathbb{C})$ to $\mathfrak{gl}(n, \mathbb{R})$ satisfy

$$i^k c_k(X, \cdots, X) = p_{k/2}(X, \cdots, X)\,. \tag{112}$$

- When $\mathfrak{g} \subset \mathfrak{sl}(n)$ all elements are traceless and various cancellations occur. In particular for $\mathfrak{g} = \mathfrak{so}(n)$ the Pontrjagin classes have relatively simple relation to the Chern characters.

Lie algebra	$\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$	$\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$	$\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$
invariant polynomial	ch_k	c_k	$\mathrm{p}_{k/2}$
definition	$\mathrm{ch}(X)$ $= \mathrm{tr}(\exp(itX))$ $= \sum_k t^k \mathrm{ch}_k(X, \cdots, X)$	$\mathrm{c}(X)$ $= \det(t + iX)$ $= \sum_k t^{n-k} \mathrm{c}_k(X, \cdots, X)$	$\det(t - X)$ $= \sum_k t^{n-k} p_{k/2}(X, \cdots, X)$
characteristic class	Chern character	Chern class	Pontrjagin class

TABLE 8. **Characteristic classes for matrix Lie algebras** obtained from the trace and the determinant.

APPENDIX B. THE LERAY-SERRE SPECTRAL SEQUENCE

Suppose that $H^*\left((BU)\langle 2k\rangle; \mathbb{Q}\right) = P\left[c_k, c_{k+1}, \cdots\right]$. Corresponding to the fibration

$$\begin{tikzcd} K(\mathbb{Z}, 2k-1) \arrow[r] & (BU)\langle 2k+2\rangle \arrow[d] & \\ & (BU)\langle 2k\rangle \arrow[r] & K(\mathbb{Z}, 2k) \end{tikzcd} \tag{113}$$

we have

$$H^*\left((BU)\langle 2k\rangle; H^*\left(K(\mathbb{Z}, 2k-1); \mathbb{Q}\right)\right) \Longrightarrow H^*\left((BU)\langle 2k+2\rangle\right). \tag{114}$$

Here the shift in 2 in the degree of the Eilenberg-MacLane space comes from Bott periodicity. The cohomology $H^*(K(\mathbb{Z}, 2k-1); \mathbb{Q})$ is an exterior algebra $\Lambda(e_{2k-1})$ on a generator of degree $2k-1$, e_{2k-1}, which satisfies $e_{2k-1}^2 = 0$. The differential in the sequence takes this generator to the k-th Chern class c_k. Modding out by this class gives (53). In fact, the spectral sequence collapses from here and there is no extension problem for the E_∞ term.

Since G is simply connected, the E_2 term of this spectral sequence is in bidegree (p, q)

$$E_2^{p,q} = H^p(G) \otimes H^q(K(\mathbb{Z}, 2) \tag{115}$$

The cohomology of $K(\mathbb{Z}, 2)$ is a polynomial algebra on one generator y in degree 2: $H(K(\mathbb{Z}, 2)) = \mathbb{R}[y]$, and the differential $d_2 \colon E_2^{p,q} \to E_2^{p+2,q-1}$ raises total degree by 1.

$$E_2^{p,q} = \begin{cases} H^p(G) \otimes \mathbb{R}y^k \text{ if } q = 2k \text{ is even} \\ 0 \text{ if } q \text{ is odd.} \end{cases} \tag{116}$$

Moreover, since we are dealing with real coefficients, the cohomology of G is an exterior algebra $H(G) = \Lambda(x_3, x_5, \ldots)$. So we are computing the cohomology of a complex which looks like

$$\Lambda V \otimes \mathbb{R}[y], \tag{117}$$

where E is a graded vector space concentrated in odd degrees and x is of degree 2. Clearly the differential d_2 is zero so that $E_3 = E_2$. The new differential $d_3 \colon E_3^{p,q} \to E_3^{p+3,q-2}$ sends y to $x_3 = H$, the generator of $H^3(G)$. Since d_3 is a derivation with respect to the algebra structure of $E_3^{p,q}$. It follows that

$$d_3(z \otimes y^k) = kHz \otimes y^{k-1} \tag{118}$$

where $z \in H(G)$.

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Hisham Sati
Department of Mathematics
Yale University
10 Hillhouse Avenue
New Haven,
CT 06511

Urs Schreiber
Fachbereich Mathematik
Schwerpunkt Algebra und Zahlentheorie
Universität Hamburg
Bundesstraße 55
D–20146 Hamburg

Jim Stasheff
Department of Mathematics
University of Pennsylvania
David Rittenhouse Lab.
209 South 33rd Street
Philadelphia, PA 19104-6395