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Connections on non-abelian Gerbes and their Holonomy

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Abstract

We introduce transport 2-functors as a new way to describe connections on gerbes with arbitrary strict structure 2-groups. On the one hand, transport 2-functors provide a manifest notion of parallel transport and holonomy along surfaces. On the other hand, they have a concrete local description in terms of differential forms and smooth functions.

We prove that Breen-Messing gerbes, abelian and non-abelian bundle gerbes with connection, as well as further concepts arise as particular cases of transport 2-functors, for appropriate choices of the structure 2-group. Via such identifications transport 2-functors induce well-defined notions of parallel transport and holonomy for all these gerbes. For abelian bundle gerbes with connection, this induced holonomy coincides with the existing definition. In all other cases, finding an appropriate definition of holonomy is an interesting open problem to which our induced notion offers a systematical solution.

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Introduction

The study of gerbes has a long tradition in geometry and topology. The subject was started in the seventies by Giraud to achieve a geometrical understanding of non-abelian cohomology [Gir71]. In the nineties, Brylinski extended the study of gerbes to their differential geometry with the definition of connections on abelian gerbes [Bry93]. Later Breen and Messing introduced connections on certain non-abelian gerbes [BM05]. Ironically, one of the most interesting consequence of connections, their holonomy, could so far only be treated in the abelian case.

The reason for this may be the lack of a general underlying concept, what a connection on a gerbe is, and around what its holonomy has to be taken. In this article we introduce such a concept. It is based on an alternative description of ordinary connections in ordinary fibre bundles chosen such that the generalization to connections in gerbes is evident. The alternative description of connections in fibre bundles – transport functors – has been introduced by the authors in [SW07]. The relation between transport functors and several classes of fibre bundles with connection has been established in terms of equivalences of categories.

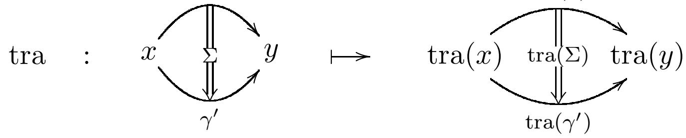
For the purposes of this introduction let us immediately describe the result of the evident generalization to connections in gerbes – transport 2-functors. Their description splits into an algebraical and an analytical part. The algebraical part requires that parallel transport along surfaces has the structure of a 2-functor

$$\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T,$$

hence the name transport 2-functors. These 2-functors are defined on the path 2-groupoid of a smooth manifold M and take values in some „target“ 2-category T. For a moment we may assume T to be the 2-category of categories. An object in the path 2-groupoid is just a point x in M, so that the transport 2-functor attaches a category $\mathrm{tra}(x)$ to each such point. The 1-morphisms between two points x and y are smooth curves connecting x with y, and the transport 2-functor assigns to such a curve γ a functor

$$\mathrm{tra}(\gamma) : \mathrm{tra}(x) \longrightarrow \mathrm{tra}(y).$$

Finally, a 2-morphism in the path 2-groupoid is a smooth homotopy between two curves with fixed endpoints, which sweeps out a disc in X bounded by the two paths. The transport 2-functor assigns to it a natural transformation:



This natural transformation $\mathrm{tra}(\Sigma)$ is the parallel transport along the surface Σ, which is thus manifestly included in the nature of a transport 2-functor.

These assignments of categories to points, functors to paths and natural transformations to discs have to obey the axioms of a 2-functor. For the convenience of the reader, we have included an appendix with the basics about 2-categories and 2-functors. For a transport 2-functor, the axioms practically describe how the functors $\mathrm{tra}(\gamma)$ and the natural transformations $\mathrm{tra}(\Sigma)$ compose when paths or discs are glued together.

The analytic properties of a transport 2-functor demand that the above assignments are smooth in an appropriate sense. It is most natural to discuss smoothness locally: we require that a transport 2-functor is locally trivial. Like for ordinary fibre bundles, a local trivialization is defined with respect to a cover $\mathfrak{V}$ of the base manifold by open sets U_α, and to a typical fibre: this is here a particular „structure“ 2-groupoid Gr together with a 2-functor $i : \mathrm{Gr} \longrightarrow T$ indicating how this structure is realized in the target 2-category of the transport 2-functor. We introduce a local trivialization as a collection of „trivial“ 2-functors $\mathrm{triv}_\alpha : \mathcal{P}_2(U_\alpha) \longrightarrow \mathrm{Gr}$ and of equivalences

$$t_\alpha : \mathrm{tra}|_{U_\alpha} \xrightarrow{\cong} i \circ \mathrm{triv}_\alpha$$

between 2-functors defined on U_α. We show that, like for fibre bundles, local trivializations induce „transition“ transformations

$$g_{\alpha\beta} : i \circ \mathrm{triv}_\alpha \longrightarrow i \circ \mathrm{triv}_\beta$$

by composing an inverse of t_α with t_β. These transition transformations satisfy the usual cocycle conditions only up to morphisms between transformations, so-called modifications. That is, modifications

$$f_{\alpha\beta\gamma} : g_{\beta\gamma} \circ g_{\alpha\beta} \Longrightarrow g_{\alpha\beta} \quad \text{and} \quad \psi_\alpha : \mathrm{id} \Longrightarrow g_{\alpha\alpha}.$$

These modifications again satisfy higher coherence conditions. We call the collection of the 2-functors triv_α, the transformations $g_{\alpha\beta}$ and the modifications $f_{\alpha\beta\gamma}$ and ψ_α the descent data of the transport 2-functor tra extracted from the local trivializations t_α.

It is these descent data on which we impose smoothness conditions. First of all, we require that the 2-functors triv_α are smooth. This makes sense when we also require that the structure 2-groupoid Gr has smooth manifolds of objects, 1-morphisms and 2-morphisms. Thus, in other words, transport 2-functors factor locally through smooth functors to the structure 2-groupoid.

The remaining descent data $g_{\alpha\beta}$, $f_{\alpha\beta\gamma}$ and ψ_α is treated in the following way. We make a crucial observation in abstract 2-category theory: a transformation $g : F \longrightarrow G$ between 2-functors F and G between 2-categories S and T can itself be seen a functor

$$\mathscr{F}(g) : S' \longrightarrow \Lambda T$$

for S' and ΛT appropriate categories constructed out of S and T, respectively. Similarly, modifications $\eta : g \Longrightarrow g'$ between such transformations induce natural transformations $\mathscr{F}(\eta)$ between the functors $\mathscr{F}(g)$ and $\mathscr{F}(g')$.

We apply this abstract consideration to the remaining descent data of a transport 2-functor. The result is a collection of functors

$$\mathscr{F}(g_{\alpha\beta}) : \mathcal{P}_1(U_\alpha \cap U_\beta) \longrightarrow \Lambda T$$

and of natural transformations

$$\mathscr{F}(f_{\alpha\beta\gamma}) : \mathscr{F}(g_{\beta\gamma}) \otimes \mathscr{F}(g_{\alpha\beta}) \Longrightarrow \mathscr{F}(g_{\alpha\gamma}) \quad \text{and} \quad \mathscr{F}(\psi_\alpha) : \mathrm{id} \Longrightarrow \mathscr{F}(g_{\alpha\alpha}).$$

Now, the smoothness condition on these descent data is the requirement that the functors $\mathscr{F}(g_{\alpha\beta})$ are transport functors and that the modifications $\mathscr{F}(f_{\alpha\beta\gamma})$ and $\mathscr{F}(\psi_\alpha)$ are morphisms between transport functors. According to the correspondence between transport functors and fibre bundles with connection established in [SW07], we thus require that certain structures are a smooth fibre bundle with connection, or smooth bundle morphisms that respect the connections.

A detailed development of transport 2-functors is the content of the first part of the present article, including the Sections 1 to 3. In Section 1 we review the path 2-groupoid $\mathcal{P}_2(M)$ of a smooth manifold M and list some features of 2-functors defined on them. We introduce local trivializations of 2-functors and their descent data. This discussion also incorporates transformations between the 2-functors and modifications between those, so that the descent data is naturally arranged in a 2-category $\mathfrak{Des}^2(i)^\infty_M$ associated to the 2-functor $i : \mathrm{Gr} \longrightarrow T$ that realizes the structure 2-groupoid Gr in the target 2-category T.

Section 2 is devoted to the reconstruction of globally defined 2-functors from local descent data. This turns out to be a difficult problem that involves lifts of paths and lifts of homotopies between paths to a Čech-like covering of the path 2-groupoid that combines the path 2-groupoids $\mathcal{P}_2(U_\alpha)$ of the open sets with „jumps“ between those. The result is an equivalence of 2-categories

$$\mathrm{Trans}^2_{\mathrm{Gr}}(M,T) \cong \mathfrak{Des}^2(i)^\infty_M$$

between the 2-category of globally defined transport 2-functors with Gr-structure and the 2-category of locally defined descent data. Section 3 contains a detailed discussion of the smoothness conditions we have imposed on the descent data.

The second part of the present article concerns the relation between transport 2-functors and other gerbes and the impact of our concept for these gerbes. The following observation may illuminate what transport 2-functors have to do with gerbes. The transformation $g_{\alpha\beta}$ which is part of the descent data corresponds by definition to a transport functor

$$\mathscr{F}(g_{\alpha\beta}) : \mathcal{P}_1(U_\alpha \cap U_\beta) \longrightarrow \Lambda T$$

on the two-fold intersection $U_\alpha \cap U_\beta$, whereas transport functors are equivalent to fibre bundles with connection. Hence, transport 2-functors equip the two-fold intersections of an open cover with fibre bundles – one of the significant ingredients of a gerbe, see e.g. [Hit03].

Which particular kind of fibre bundle it is depends on the target 2-category T and the structure 2-groupoid Gr. Mostly, the latter will be of the form $\mathrm{Gr} = \mathcal{B}\mathfrak{G}$: this is the one-object 2-groupoid which is induced from a (strict) Lie 2-group $\mathfrak{G}$. Lie 2-groups play the same role for gerbes as Lie groups do for fibre bundles [SW08]. They can conveniently be understood as crossed modules of ordinary Lie groups: two Lie groups G and H, a Lie group homomorphism $H \longrightarrow G$ and a compatible action of G on H. Several natural examples of crossed modules are available, and via their associated Lie 2-groupoids they give rise to important classes of transport 2-functors. In Section 4 we prove the following list of results that relate some of these classes of transport 2-functors to existing realizations of gerbes with connection:

I.) If $\mathfrak{G}$ is some Lie 2-group, we prove that there is a canonical bijection

$$\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{transport 2-functors} \\ \mathrm{tra} : \mathcal{P}_2(M) \to \mathcal{B}\mathfrak{G} \\ \text{with } \mathcal{B}\mathfrak{G}\text{-structure} \end{array} \right\} \cong \breve{H}^2(M, \mathfrak{G})$$

between transport 2-functors and a set that we identify as the degree two differential non-abelian cohomology of the manifold M with coefficients in the Lie 2-group $\mathfrak{G}$ [BS07]. It purely consists of collections of smooth functions and differential forms with respect to open covers of M in such a way that forgetting the differential forms the usual non-abelian cohomology $H^2(M, \mathfrak{G})$ [Gir71, Bre94, Bar04, Woc08] is reproduced. We show that the set $\breve{H}^2(M, \mathfrak{G})$ also identifies with existing discussions of differential cohomology for particular choices of $\mathfrak{G}$:

(a) The (abelian) Lie 2-group $\mathfrak{G} = \mathcal{B}S^1$ induced from the crossed module $S^1 \longrightarrow 1$. In this case the differential cohomology is the same as the degree two Deligne cohomology [Del91],

$$\breve{H}^2(M, \mathcal{B}\mathcal{B}S^1) = H^2(M, \mathcal{D}(2)).$$

Indeed, Deligne cohomology classifies abelian gerbes with connection [Bry93].

(b) The Lie 2-group $\mathfrak{G} = \mathrm{AUT}(H)$ associated to an ordinary Lie group H and induced by the crossed module $H \longrightarrow \mathrm{Aut}(H)$. In this case we find

$$\breve{H}^2(M, \mathcal{B}\mathrm{AUT}(H)) = \left\{ \begin{array}{c} \text{Equivalence classes of local data} \\ \text{of Breen-Messing } H\text{-gerbes over} \\ M \text{ with (fake-flat) connections} \end{array} \right\}.$$

Breen-Messing gerbes are a realization of non-abelian gerbes on which connections can be defined [BM05]. Our approach infers a new condition on these connections, namely the vanishing of the so-called „fake-curvature“. This condition is not present in [BM05] but arises here from the algebraic properties of a transport 2-functor.

II.) Let $\mathcal{B}S^1$ again be the Lie 2-group from (Ia), but now we consider transport 2-functors whose target is the monoidal category S^1-Tor of manifolds with free and transitive S^1-action, regarded as a 2-category $\mathcal{B}(S^1\text{-Tor})$ with a single object. We show that there is a canonical equivalence of 2-categories

$$\left\{\begin{array}{c}\text{Transport 2-functors}\\ \text{tra}:\mathcal{P}_2(M)\to\mathcal{B}(S^1\text{-Tor})\\ \text{with }\mathcal{B}\mathcal{B}S^1\text{-structure}\end{array}\right\}\cong\left\{\begin{array}{c}S^1\text{-bundle gerbes with}\\ \text{connection over }M\end{array}\right\}.$$

This equivalence arises by realizing that the transport functor $\mathscr{F}(g_{\alpha\beta})$ from the descent data of a transport 2-functor corresponds – in the present situation – to an S^1-bundle with connection over the two-fold intersection of an open cover. After generalizing open covers to surjective submersions, this S^1-bundle, together with the bundle morphisms from the descent data, reproduce exactly Murray's definition [Mur96] of a bundle gerbe.

III.) Let H be a Lie group and let $\text{AUT}(H)$ be the associated Lie 2-group from (Ib). Now we consider transport 2-functors whose target is the monoidal category H-BiTor of smooth manifolds with commuting free and transitive H-actions from the left and from the right, considered as a 2-category $\mathcal{B}(H\text{-BiTor})$. We show that there is a canonical equivalence of 2-categories

$$\left\{\begin{array}{c}\text{Transport 2-functors}\\ \text{tra}:\mathcal{P}_2(M)\to\mathcal{B}(H\text{-BiTor})\\ \text{with }\mathcal{B}\text{AUT}(H)\text{-structure}\end{array}\right\}\cong\left\{\begin{array}{c}\text{Non-abelian }H\text{-bundle gerbes}\\ \text{with connection over }M\end{array}\right\}.$$

Non-abelian bundle gerbes are a generalization of S^1-bundle gerbes introduced by Aschieri, Cantini and Jurco [ACJ05], and the above equivalence arises in the same way as in the abelian case. In particular, we prove that the transport functor $\mathscr{F}(g_{\alpha\beta})$ corresponds to a principal H-bibundle with twisted connection, a key ingredient of a non-abelian bundle gerbe.

Apart from these relations to existing gerbes with connection, transport 2-functors allow to understand further concepts of gerbes and 2-bundles with connection, or to find the correct concepts of connections in cases when only the underlying gerbe is known so far. We indicate how this can be done in the case of 2-vector bundles, in particular string 2-bundles [BBK06, ST04], and principal 2-bundles [Bar04, Woc08].

In the last Section 5 we give a deeper discussion of the notion of parallel transport, which is manifestly included in the concept of a transport 2-functor. Most importantly, we uncover what the holonomy of a transport 2-functor around a surface is. Via the equivalences (Ib) and (III) above, we thereby equip connections on non-abelian gerbes with a well-defined notion of holonomy.

Existing discussions of holonomy of connections in abelian gerbes indicate that such a holonomy should be taken around closed and oriented surfaces. For the non-abelian case we observe a subtlety which also arises in the discussion of ordinary fibre bundles. Namely, while the holonomy of a connection in an S^1-bundle can be taken around a closed and oriented line, a connection in a non-abelian principal bundle requires the choice of a base point. We prove that the holonomy of a non-abelian gerbe around a closed and oriented surface requires the choice of a base point *plus* the choice of a certain loop based at this point. More precisely, the loop has to be chosen together with a contraction which sweeps out the whole surface in a way compatible with the orientation. We show that any closed surfaces admits such choices.

Now suppose that S is a closed and oriented surface, $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ is a transport 2-functor on a smooth manifold M and $\phi : S \longrightarrow M$ is a smooth map. With the choices of a base point $x \in S$, a loop $\tau : x \longrightarrow x$ and a contraction $\Sigma : \tau \Longrightarrow \mathrm{id}_x$, understood as an object, a 1-morphism and a 2-morphism in the path 2-groupoid of S, the holonomy of tra around the surface S is

$$\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma) := \mathrm{tra}(\phi_*\Sigma),$$

where $\phi_* : \mathcal{P}_2(S) \longrightarrow \mathcal{P}_2(M)$ is a 2-functor induced by the smooth map ϕ. The surface holonomy of a transport 2-functor is thus a 2-morphism in its target 2-category T.

We study the dependence of this surface holonomy on the choices of the base point, the loop and the contraction. The first result is that it is independent of the choice of the contraction. The dependence on the base point turns out to be a „conjugation“ of the 2-morphism $\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma)$ by another 2-morphism, very similar to the dependence of the holonomy of a connection in an ordinary fibre bundle on the choice of the base point. Thus, the surface holonomy in general depends on the base point and on the loop, but the dependence can be controlled in a precise way.

Finally, we apply the general concept of the surface holonomy of a transport 2-functor to connections on (non-abelian) gerbes using the equivalences (I), (II) and (III) derived in the first part of the present article. We show that in the abelian cases (Ia) and (II) the dependence on the base point and the loop drops out, and that the surface holonomy $\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma)$ coincides with the usual notion [Gaw88, Mur96] of holonomy of abelian gerbes. In the other cases (Ib) and (III) we obtain new, well-defined quantities associated to connections in non-abelian gerbes and surfaces.

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1 Local Trivializations of 2-Functors

The gerbes we want to consider in this article are certain 2-functors. These 2-functors are defined on the path 2-groupoid of a smooth manifold. We review this 2-groupoid in Section 1.1. Like for fibre bundles, one of the most important properties of our 2-functors is that they are locally trivializable. In Section 1.2 we describe local trivializations for 2-functors on path 2-groupoids. Again, like for fibre bundles, local trivializations of 2-functors admit to extract local data similar to transition functions. This is the content of Sections 1.2 and 1.3. For the basics on 2-categories we refer the reader to Appendix A.

1.1 The Path 2-Groupoid of a smooth Manifold

The basic idea of the path 2-groupoid is very simple: for a smooth manifold X, it is a strict 2-category whose objects are the points of X, whose 1-morphisms are smooth paths in X, and whose 2-morphisms are smooth homotopies between these paths. Its concrete realization needs, however, a more detailed discussion.

For points $x, y \in X$, a *path* $\gamma : x \longrightarrow y$ is a smooth map $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Since the composition $\gamma_2 \circ \gamma_1$ of two paths $\gamma_1 : x \longrightarrow y$ and $\gamma_2 : y \longrightarrow z$ should again be a smooth map we require sitting instants for all paths: a number $0 < \epsilon < \frac{1}{2}$ with $\gamma(t) = \gamma(0)$ for $0 \leq t < \epsilon$ and $\gamma(t) = \gamma(1)$ for $1 - \epsilon < t \leq 1$. The set of these paths is denoted by PX. In order to make the composition associative and to make paths invertible, we need to introduce an equivalence relation on PX.

Definition 1.1. *Two paths $\gamma, \gamma' : x \longrightarrow y$ are called thin homotopy equivalent if there exists a smooth map $h : [0,1]^2 \longrightarrow X$ such that*

(1) h is a homotopy from γ to γ' through paths $x \longrightarrow y$ with sitting instants at γ and γ'.

(2) the differential of h has at most rank 1.

The set of equivalence classes is denoted by P^1X. We remark that any path γ is thin homotopy equivalent to any orientation-preserving reparameterization of γ. The composition of paths induces a well-defined associative composition on P^1X for which the constant paths id_x are identities and the reversed paths γ^{-1} are inverses. These are the axioms of a groupoid $\mathcal{P}_1(X)$ whose set of objects is X and whose set of morphisms is P^1X. This groupoid is called the *path groupoid of* X, see [SW07] for a more detailed discussion.

Remark 1.2. If we drop condition (2) in Definition 1.1, we still obtain a groupoid $\Pi_1(X)$ together with a projection functor $\mathcal{P}_1(X) \longrightarrow \Pi_1(X)$. The groupoid $\Pi_1(X)$ is called the fundamental groupoid of X. Functors $F : \mathcal{P}_1(X) \longrightarrow T$ which factor through $\mathcal{P}_1(X) \longrightarrow \Pi_1(X)$ are called *flat*: they depend only on the homotopy class of the path.

A homotopy h between two paths γ_0 and γ_1 like in Definition 1.1 but without condition (2) on the rank of its differential is called a *bigon* in X and denoted by $\Sigma : \gamma_0 \Longrightarrow \gamma_1$. These bigons form the 2-morphisms of the path 2-groupoid of X. We denote the set of bigons in X by BX. Bigons can be composed in two natural ways. For two bigons $\Sigma : \gamma_1 \Longrightarrow \gamma_2$ and $\Sigma' : \gamma_2 \Longrightarrow \gamma_3$ we have a *vertical composition*

$$\Sigma' \bullet \Sigma : \gamma_1 \Longrightarrow \gamma_3.$$

If two bigons $\Sigma_1 : \gamma_1 \Longrightarrow \gamma_1'$ and $\Sigma_2 : \gamma_2 \Longrightarrow \gamma_2'$ are such that $\gamma_1(1) = \gamma_2(0)$, we have *horizontal composition*

$$\Sigma_2 \circ \Sigma_1 : \gamma_2 \circ \gamma_1 \Longrightarrow \gamma_2' \circ \gamma_1'.$$

Like in the case of paths, we need to define an equivalence relation on BX in order to make the compositions above associative and to make bigons invertible.

Definition 1.3. *Two bigons $\Sigma : \gamma_0 \Longrightarrow \gamma_1$ and $\Sigma' : \gamma_0' \Longrightarrow \gamma_1'$ are called thin homotopy equivalent if there exists a smooth map $h : [0,1]^3 \longrightarrow X$ such that*

(1) h is a homotopy from Σ to Σ' through bigons and has sitting instants at Σ and Σ'.

(2) the induced homotopies $\gamma_0 \Longrightarrow \gamma_0'$ and $\gamma_1 \Longrightarrow \gamma_1'$ are thin.

(3) the differential of h has at most rank 2.

Condition (1) assures that we have defined an equivalence relation on BX, and condition (2) asserts that two thin homotopy equivalent bigons $\Sigma : \gamma_0 \Longrightarrow \gamma_1$ and $\Sigma' : \gamma_0' \Longrightarrow \gamma_1'$ start and end on thin homotopy equivalent paths $\gamma_0 \sim \gamma_0'$ and $\gamma_1 \sim \gamma_1'$. We denote the set of equivalence classes by B^2X. The compositions $\circ$ and $\bullet$ between bigons induce a well-defined composition on B^2X. The *path 2-groupoid* $\mathcal{P}_2(X)$ is now the 2-category whose set of objects is X, whose set of 1-morphisms is P^1X and whose set of 2-morphisms is B^2X, see [SW08] for a more detailed discussion. The path 2-groupoid is strict and all 1-morphisms are strictly invertible.

If we drop condition (3) from Definition 1.3 we still have a strict 2-groupoid, which is denoted by $\Pi_2(X)$ and is called the fundamental 2-groupoid of X. The projection defines a strict 2-functor $\mathcal{P}_2(X) \longrightarrow \Pi_2(X)$.

In this article we describe gerbes as certain (not necessarily strict) 2-functors

$$F : \mathcal{P}_2(M) \longrightarrow T.$$

We call the object $F(x)$ for $x \in M$ the *fibre* of F over x. If T is for instance the 2-category of categories, the fibre over any point is a category. Our 2-functors can be pulled back along smooth maps $f : X \longrightarrow M$: such maps induce strict 2-functors $f_* : \mathcal{P}_2(X) \longrightarrow \mathcal{P}_2(M)$, and we write

$$f^*F := F \circ f_*.$$

Analogously to Remark 1.2 we say that a 2-functor $F : \mathcal{P}_2(M) \longrightarrow T$ is *flat* if it factors through the 2-functor $\mathcal{P}_2(M) \longrightarrow \Pi_2(M)$. See Sections 3.3 and 3.4 for further discussions of flat 2-functors.

1.2 Local Trivializations and Descent Data

Let T be a 2-category, the *target 2-category*. To define local trivializations of a 2-functor $F : \mathcal{P}_2(M) \longrightarrow T$, we fix three attributes:

1. A strict 2-groupoid Gr, the *structure 2-groupoid*. In Section 3 we will require that Gr is a *Lie* 2-groupoid, i.e. it has smooth manifolds of objects, 1-morphisms and 2-morphisms.

2. A 2-functor $i : \mathrm{Gr} \longrightarrow T$ that indicates how the structure 2-groupoid is realized in the target 2-category. In Sections 1 and 2 there will be no further condition on this 2-functor, but in Section 3 we require i to be full and faithful. In all examples we present in Section 4, i will even more be an equivalence of 2-categories.

3. A surjective submersion $\pi : Y \longrightarrow M$, which serves as an open cover of the base manifold M.

Indeed, surjective submersions behave in many aspects like open covers, but generalize them essentially [Mur96]. If M is covered by open sets U_α, the projection from their disjoint union to M defines a surjective submersion $\pi : Y \longrightarrow M$. Notice that for any surjective submersion $\pi : Y \longrightarrow M$ the fibre products $Y^{[k]} := Y \times_M ... \times_M Y$ are again smooth manifolds in such a way that the canonical projections $\pi_{i_1 \dots i_p} : Y^{[k]} \longrightarrow Y^{[k]}$ are smooth maps. In terms of open covers, the k-fold fibre product $Y^{[k]}$ is the disjoint union of all k-fold intersections of the open sets U_α.

Definition 1.4. *A* <u>*π-local i-trivialization*</u> *of a 2-functor*

$$F : \mathcal{P}_2(M) \longrightarrow T$$

is a pair (triv, t) *of a strict 2-functor* $\mathrm{triv} : \mathcal{P}_2(Y) \longrightarrow \mathrm{Gr}$ *and a pseudonatural equivalence*

$$\begin{array}{ccc} \mathcal{P}_1(Y) & \xrightarrow{\pi_*} & \mathcal{P}_1(M) \\ {\scriptstyle \mathrm{triv}}\downarrow & \overset{t}{\swarrow} & \downarrow{\scriptstyle F} \\ \mathrm{Gr} & \xrightarrow[i]{} & T. \end{array}$$

In other words, a 2-functor F is locally trivializable, if its pullback π^*F to the covering space factorizes – up to pseudonatural equivalence– through the fixed Lie 2-groupoid Gr. In terms of an open cover, π^*F is a collection of restrictions $F|_{U_\alpha} : \mathcal{P}_1(U_\alpha) \longrightarrow T$. The

2-functor triv is a collection of „trivial“ strict 2-functors $\mathrm{triv}_\alpha : \mathcal{P}_1(U_\alpha) \longrightarrow \mathrm{Gr}$ such that $i \circ \mathrm{triv}_\alpha \cong F|_{U_\alpha}$.

To abbreviate the notation, we write triv_i instead of $i \circ \mathrm{triv}$ in the following. We define a 2-category $\mathrm{Triv}^2_\pi(i)$ of 2-functors with π-local i-trivialization: an object is a triple (F, triv, t) of a 2-functor $F : \mathcal{P}_2(M) \longrightarrow T$ together with a fixed π-local i-trivialization (triv, t). A 1-morphism

$$(F, \mathrm{triv}, t) \longrightarrow (F', \mathrm{triv}', t')$$

is just a pseudonatural transformation $F \longrightarrow F'$ between the two 2-functors, and a 2-morphism is just a modification between those. In other words, the 2-category $\mathrm{Triv}^2_\pi(i)$ is just a sub-2-category of $\mathrm{Funct}(\mathcal{P}_2(M), T)$, where every object is additionally decorated with a π-local i-trivialization.

Now we define a 2-category $\mathfrak{Des}^2_\pi(i)$ of *descent data*. This 2-category is supposed to be equivalent to $\mathrm{Triv}^2_\pi(i)$ and does yet only contain local data, i.e. structure defined on Y instead of M. This discussion should be considered as being analogous to replacing a globally defined fibre bundle with connection by a collection of transition functions and local 1-forms. We will see in Section 4.1 how the functions and the forms enter.

Definition 1.5. *A descent object is a family* $(\mathrm{triv}, g, \psi, f)$ *consisting of*

1. *a strict 2-functor* $\mathrm{triv} : \mathcal{P}_2(Y) \longrightarrow \mathrm{Gr}$
2. *a pseudonatural equivalence* $g : \pi_1^*\mathrm{triv}_i \longrightarrow \pi_2^*\mathrm{triv}_i$
3. *an invertible modification* $\psi : \mathrm{id}_{\mathrm{triv}_i} \Longrightarrow \Delta^* g$
4. *an invertible modification* $f : \pi_{23}^* g \circ \pi_{12}^* g \Longrightarrow \pi_{13}^* g$

such that the diagrams

$$\begin{array}{ccc} \mathrm{id}_{\pi_2^*\mathrm{triv}_i} \circ g & \xRightarrow{\pi_2^*\psi \circ \mathrm{id}} & \Delta_{22}^* g \circ g \\ & {}_{r}\searrow \quad \swarrow_{\Delta_{122}^* f} & \\ & g & \end{array} \quad , \quad \begin{array}{ccc} g \circ \mathrm{id}_{\pi_1^*\mathrm{triv}_i} & \xRightarrow{\mathrm{id} \circ \pi_1^*\psi} & g \circ \Delta_{11}^* g \\ & {}_{l}\searrow \quad \swarrow_{\Delta_{112}^* f} & \\ & g & \end{array} \tag{1.1}$$

and

$$\begin{array}{ccc} & (\pi_{34}^* g \circ \pi_{23}^* g) \circ \pi_{12}^* g & \\ {}^{a}\swarrow & & \searrow^{\pi_{234}^* f \circ \mathrm{id}} \\ \pi_{34}^* g \circ (\pi_{23}^* g \circ \pi_{12}^* g) & & \pi_{24}^* g \circ \pi_{12}^* g \\ {}_{\mathrm{id} \circ \pi_{123}^* f}\searrow & & \swarrow_{\pi_{124}^* f} \\ \pi_{34}^* g \circ \pi_{13}^* g & \xRightarrow[\mathrm{id} \circ \pi_{134}^* f]{} & \pi_{14}^* g. \end{array} \tag{1.2}$$

are commutative.

In these diagrams, r, l and a are the right and left unifiers and the associator of the 2-category T, $\Delta : Y \longrightarrow Y^{[2]}$ is the diagonal map, and $\Delta_{112}, \Delta_{122} : Y^{[2]} \longrightarrow Y^{[3]}$ are the maps duplicating the first or the second factor, respectively. Let us briefly rephrase the above definition in case that Y is the union of open sets U_α: first there are strict 2-functors $\mathrm{triv}_\alpha : \mathcal{P}_2(U_\alpha) \longrightarrow \mathrm{Gr}$, just like in a local trivialization. To compare the difference between triv_α and triv_β on a two-fold intersection $U_\alpha \cap U_\beta$ there are pseudonatural equivalences $g_{\alpha\beta} : (\mathrm{triv}_\alpha)_i \longrightarrow (\mathrm{triv}_\beta)_i$. If we assume for a moment, that $g_{\alpha\beta}$ was the transition function of some fibre bundle, one would demand that $1 = g_{\alpha\alpha}$ on every U_α and that $g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma}$ on every three-fold intersection $U_\alpha \cap U_\beta \cap U_\gamma$. In the present situation, however, these equalities have been replaced by modifications: the first one by a modification $\psi_\alpha : \mathrm{id}_{(\mathrm{triv}_\alpha)_i} \Longrightarrow g_{\alpha\alpha}$ and the second one by a modification $f_{\alpha\beta\gamma} : g_{\beta\gamma} \circ g_{\alpha\beta} \Longrightarrow g_{\alpha\gamma}$. Finally we have demanded that these modifications satisfy the two coherence conditions (1.1) and (1.2).

Definition 1.6. *Let* $(\mathrm{triv}, g, \psi, f)$ *and* $(\mathrm{triv}', g', \psi', f')$ *be descent objects. A* *descent 1-morphism* $(\mathrm{triv}, g, \psi, f) \longrightarrow (\mathrm{triv}', g', \psi', f')$ *is a pair* (h, ϵ) *of a pseudonatural transformation*

$$h : \mathrm{triv}_i \longrightarrow \mathrm{triv}'_i$$

and an invertible modification

$$\epsilon : \pi_2^* h \circ g \Longrightarrow g' \circ \pi_1^* h$$

such that the diagrams

$$\begin{array}{ccc}
\pi_{23}^* g' \circ (\pi_2^* h \circ \pi_{12}^* g) & \xRightarrow{\;a\;} & (\pi_{23}^* g' \circ \pi_2^* h) \circ \pi_{12}^* g \\
{\scriptstyle \mathrm{id} \circ \pi_{12}^* \epsilon} \Downarrow & & \Downarrow {\scriptstyle \pi_{23}^* \epsilon^{-1} \circ \mathrm{id}} \\
\pi_{23}^* g' \circ (\pi_{12}^* g' \circ \pi_1^* h) & & (\pi_3^* h \circ \pi_{23}^* g) \circ \pi_{12}^* g \\
{\scriptstyle a^{-1}} \Downarrow & & \Downarrow {\scriptstyle a} \\
(\pi_{23}^* g' \circ \pi_{12}^* g') \circ \pi_1^* h & & \pi_3^* h \circ (\pi_{23}^* g \circ \pi_{12}^* g) \\
{\scriptstyle f' \circ \mathrm{id}} \Downarrow & & \Downarrow {\scriptstyle \mathrm{id} \circ f} \\
\pi_{13}^* g' \circ \pi_1^* h & \xRightarrow[\pi_{13}^* \epsilon]{} & \pi_3^* h \circ \pi_{13}^* g.
\end{array} \tag{1.3}$$

and

$$\begin{array}{ccccc}
\mathrm{id}_{\mathrm{triv}'_i} \circ h & \xRightarrow{\;l_h\;} & h & \xRightarrow{\;r_h^{-1}\;} & h \circ \mathrm{id}_{\mathrm{triv}_i} \\
{\scriptstyle \psi' \circ \mathrm{id}_h} \Downarrow & & & & \Downarrow {\scriptstyle \mathrm{id}_h \circ \psi} \\
\Delta^* g' \circ h & & \xRightarrow[\Delta^* \epsilon]{} & & h \circ \Delta^* g.
\end{array} \tag{1.4}$$

are commutative.

We leave it as an exercise to the reader to write out this structure in the case that the surjective submersion comes from an open cover. Finally, we introduce

Definition 1.7. *Let* (h_1, ϵ_1) *and* (h_2, ϵ_2) *be descent 1-morphisms from a descent object* $(\mathrm{triv}, g, \psi, f)$ *to another descent object* $(\mathrm{triv}', g', \psi', f')$. *A descent 2-morphism* $(h_1, \epsilon_1) \Longrightarrow (h_2, \epsilon_2)$ *is a modification*

$$E : h_1 \Longrightarrow h_2$$

such that the diagram

$$\begin{array}{ccc} g' \circ \pi_1^* h_1 & \overset{\epsilon_1}{\Longrightarrow} & \pi_2^* h_1 \circ g \\ {\scriptstyle \mathrm{id} \circ \pi_1^* E} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \pi_2^* E \circ \mathrm{id}} \\ g' \circ \pi_1^* h_2 & \underset{\epsilon_2}{\Longrightarrow} & \pi_2^* h_2 \circ g. \end{array} \tag{1.5}$$

is commutative.

In concrete examples of the target 2-category T these structures have natural interpretations, see Section 4. Descent objects, 1-morphisms and 2-morphisms form a 2-category $\mathfrak{Des}^2_\pi(i)$, called the *descent 2-category*. Let us describe its structure along the lines of Definition A.1.

1. The composition of two descent 1-morphisms

   $$(h_1, \epsilon_1) : (\mathrm{triv}, g, \psi, f) \longrightarrow (\mathrm{triv}', g', \psi', f')$$

   and

   $$(h_2, \epsilon_2) : (\mathrm{triv}', g', \psi', f') \longrightarrow (\mathrm{triv}'', g'', \psi'', f'')$$

   is the pseudonatural transformation $h_2 \circ h_1 : \mathrm{triv}_i \longrightarrow \mathrm{triv}''_i$ and the modification

   $$\begin{array}{l} \pi_2^*(h_2 \circ h_1) \circ g \overset{a}{\Longrightarrow} \pi_2^* h_2 \circ (\pi_2^* h_1 \circ g) \\ \qquad \Big\Downarrow {\scriptstyle \mathrm{id} \circ \epsilon_1} \\ \pi_2^* h_2 \circ (g' \circ \pi_1^* h_1) \overset{a^{-1}}{\Longrightarrow} (\pi_2^* h_2 \circ g') \circ \pi_1^* h_1 \\ \qquad \Big\Downarrow {\scriptstyle \epsilon_2 \circ \mathrm{id}} \\ (g'' \circ \pi_1^* h_2) \circ \pi_1^* h_1 \underset{a}{\Longrightarrow} g'' \circ \pi_1^*(h_2 \circ h_1). \end{array}$$

2. The associators are those of the 2-category $\mathrm{Funct}(\mathcal{P}_2(Y), T)$.

3. The identity descent 1-morphism associated to a descent object $(\mathrm{triv}, g, \psi, f)$ is given by the pseudonatural transformation $\mathrm{id}_{\mathrm{triv}_i}$ and the modification

$$\pi_2^*\mathrm{id}_{\mathrm{triv}_i} \circ g \xRightarrow{r_g} g \xRightarrow{l_g} g \circ \pi_1^*\mathrm{id}_{\mathrm{triv}_i},$$

where r_g and l_g are the right and left unifiers of the 2-category $\mathrm{Funct}(\mathcal{P}_2(Y^{[2]}), T)$.

4. The right and left unifiers are those of $\mathrm{Funct}(\mathcal{P}_2(Y), T)$.

5. Vertical composition of descent 2-morphisms is the one of modifications.

6. The identity descent 2-morphism associated to a descent 1-morphism (h, ϵ) is the identity modification id_h.

7. Horizontal composition of descent 2-morphisms is the one of modifications.

All axioms for the 2-category $\mathfrak{Des}^2_\pi(i)$ defined like this follow from the axioms of the 2-categories $\mathrm{Funct}(\mathcal{P}_2(Y), T)$ and $\mathrm{Funct}(\mathcal{P}_2(Y^{[2]}), T)$.

We remark that the descent 2-category comes with a strict 2-functor

$$V : \mathfrak{Des}^2_\pi(i) \longrightarrow \mathrm{Funct}(\mathcal{P}_2(Y), T).$$

From a descent object $(\mathrm{triv}, g, \psi, f)$ it keeps only the 2-functor triv and from a descent 1-morphism (h, ϵ) only the pseudonatural transformation h. Thus, in terms of an open cover, the 2-functor V keeps the structure defined on the patches U_α, and forgets the gluing data.

Remark 1.8. Without consequences for the remaining article, let us briefly consider the descent 2-category $\mathfrak{Des}^2_\pi(i)$ in the particular case in which the manifolds M and Y are just points and π is the identity. Let $\mathfrak{C}$ be a tensor category, let Gr be the trivial 2-groupoid (one object, one 1-morphism and one 2-morphism), and let $i : \mathrm{Gr} \longrightarrow \mathcal{B}\mathfrak{C}$ be the canonical 2-functor. Here, $\mathcal{B}\mathfrak{C}$ is the 2-category with one object associated to $\mathfrak{C}$, see Example A.2. Then, a descent object is precisely a one-dimensional special symmetric Frobenius algebra object in $\mathfrak{C}$.

1.3 Descent Data of a 2-Functor

We have so far introduced a 2-category $\mathrm{Triv}^2_\pi(i)$ of 2-functors with π-local i-trivializations and a descent 2-category associated to the surjective submersion π and the 2-functor $i : \mathrm{Gr} \longrightarrow T$. Now we define a 2-functor

$$\mathrm{Ex}_\pi : \mathrm{Triv}^2_\pi(i) \longrightarrow \mathfrak{Des}^2_\pi(i)$$

between these 2-categories. This 2-functor extracts descent data from 2-functors with local trivializations.

Let $F : \mathcal{P}_2(M) \longrightarrow T$ be a 2-functor with a π-local i-trivialization (triv, t). We choose a weak inverse $\bar{t} : \mathrm{triv}_i \longrightarrow \pi^*F$ together with invertible modifications

$$i_t : \bar{t} \circ t \Longrightarrow \mathrm{id}_{\pi^*F} \quad \text{and} \quad j_t : \mathrm{id}_{\mathrm{triv}_i} \Longrightarrow t \circ \bar{t} \tag{1.6}$$

satisfying the identities (A.1). We define a pseudonatural equivalence

$$g : \pi_1^*\mathrm{triv_i} \longrightarrow \pi_2^*\mathrm{triv}_i$$

as the composition $g := \pi_2^*t \circ \pi_1^*\bar{t}$. This composition is well-defined since $\pi_1^*\pi^*F = \pi_2^*\pi^*F$. We obtain $\Delta^*g = t \circ \bar{t}$, so that the definition $\psi := j_t$ yields an invertible modification

$$\psi : \mathrm{id}_{\mathrm{triv}_i} \Longrightarrow \Delta^*g.$$

Finally, we define an invertible modification

$$f : \pi_{23}^*g \circ \pi_{12}^*g \Longrightarrow \pi_{13}^*g$$

as the composition

$$\begin{aligned}(\pi_3^*t \circ \pi_2^*\bar{t}) \circ (\pi_2^*t \circ \pi_1^*\bar{t}) &\Longrightarrow \pi_3^*t \circ ((\pi_2^*\bar{t} \circ \pi_2^*t) \circ \pi_1^*\bar{t}) \\ &\overset{\mathrm{id}\circ(\pi_2^*i_t\circ\mathrm{id})}{\Longrightarrow} \pi_3^*t \circ (\mathrm{id}_{\pi^*F} \circ \pi_1^*\bar{t}) \underset{\mathrm{id}\circ r_{\pi_1^*\bar{t}}}{\Longrightarrow} \pi_3^*t \circ \pi_1^*\bar{t}\end{aligned}$$

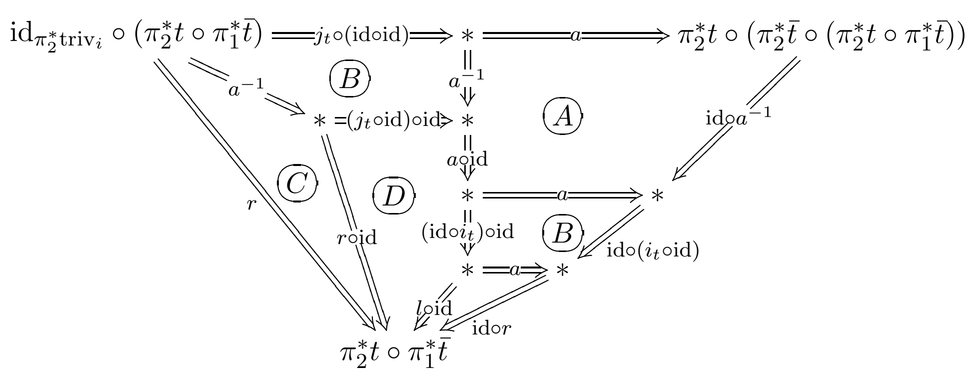
where r is the right unifier of $\mathrm{Funct}(\mathcal{P}_2(Y^{[2]}), T)$, and the first arrow summarizes two obvious occurrences of associators.

Lemma 1.9. *The modifications ψ and f make the diagrams (1.1) and (1.2) commutative, so that*

$$\mathrm{Ex}_\pi(F, \mathrm{triv}, t) := (\mathrm{triv}, g, \psi, f)$$

is a descent object.

Proof. We prove the commutativity of the diagram on the left hand side of (1.1) by patching it together from commutative diagrams:



The six subdiagrams are commutative: A is the Pentagon axiom (C4) of T, B's are the naturality of the associator, C and D are diagrams that follow from the coherence theorem for the 2-category T, and the remaining small triangle is axiom (C2). The commutativity of the second diagram in (1.1) and the one of diagram (1.2) can be shown in the same way. □

Now let $A : F \longrightarrow F'$ be a pseudonatural transformation between two 2-functors with π-local i-trivializations $t : \pi^*F \longrightarrow \mathrm{triv}_i$ and $t' : \pi^*F' \longrightarrow \mathrm{triv}'_i$. Let i_t, j_t and $i_{t'}, j_{t'}$ be the modifications (1.6) we have chosen for the weak inverses $\bar{t}$ and $\bar{t}'$. We define a pseudonatural transformation

$$h : \mathrm{triv}_i \longrightarrow \mathrm{triv}'_i$$

by $h := (t' \circ \pi^*A) \circ \bar{t}$, and an invertible modification ϵ by

$$\begin{array}{c} \pi_2^*h \circ g \Longrightarrow (\pi_2^*t' \circ \pi_2^*\pi^*A) \circ ((\pi_2^*\bar{t} \circ \pi_2^*t) \circ \pi_1^*\bar{t}) \\ \Big\Downarrow {\scriptstyle (\pi_2^* l_{t'}^{-1} \circ \mathrm{id}) \circ (\pi_2^* i_t \circ \mathrm{id})} \\ ((\pi_2^*t' \circ \mathrm{id}) \circ \pi_2^*\pi^*A) \circ (\mathrm{id} \circ \pi_1^*\bar{t}) \\ \Big\Downarrow {\scriptstyle ((\mathrm{id} \circ \pi_1^* i_{t'}^{-1}) \circ \mathrm{id}) \circ \pi_1^* r_t} \\ ((\pi_2^*t' \circ (\pi_1^*\bar{t}' \circ \pi_1^*t')) \circ \pi_1^*\pi^*A) \circ \pi_1^*\bar{t} \Longrightarrow g' \circ \pi_1^*h. \end{array}$$

Here, the unlabelled arrows summarize the definitions of h and g and several obvious occurrences of associators. Arguments similar to those given in the proof of Lemma 1.9 infer

Lemma 1.10. *The modification ϵ makes the diagrams (1.3) and (1.4) commutative, so that $\mathrm{Ex}_\pi(A) := (h, \epsilon)$ is a descent 1-morphism*

$$\mathrm{Ex}_\pi(A) : \mathrm{Ex}_\pi(F) \longrightarrow \mathrm{Ex}_\pi(F').$$

In order to continue the definition of the 2-functor Ex_π we consider a modification $B : A_1 \Longrightarrow A_2$ between pseudonatural transformations $A_1, A_2 : F \longrightarrow F'$ of 2-functors with π-local i-trivializations $t : \pi^*F \longrightarrow \mathrm{triv}_i$ and $t' : \pi^*F' \longrightarrow \mathrm{triv}'_i$. Let $(h_k, \epsilon_k) := \mathrm{Ex}_\pi(A_k)$ be the associated descent 1-morphisms for $k = 1, 2$. We define a modification $E : h_1 \Longrightarrow h_2$ by

$$h_1 = (t' \circ \pi^*A_1) \circ \bar{t} \xrightarrow{(\mathrm{id} \circ \pi^*B) \circ \mathrm{id}} (t' \circ \pi^*A_2) \circ \bar{t} = h_2.$$

Lemma 1.11. *The modification E makes the diagram (1.5) commutative so that $\mathrm{Ex}_\pi(B) := E$ is a descent 2-morphism*

$$\mathrm{Ex}_\pi(B) : \mathrm{Ex}_\pi(A_1) \Longrightarrow \mathrm{Ex}_\pi(A_2).$$

To finish the definition of the 2-functor Ex_π we have to define its compositors and unitors. We consider two composable pseudonatural transformations $A_1 : F \longrightarrow F'$ and $A_2 : F' \longrightarrow F''$ and the extracted descent 1-morphisms $(h_k, \epsilon_k) := \mathrm{Ex}_\pi(A_k)$ for $k = 1, 2$ and $(\tilde{h}, \tilde{\epsilon}) := \mathrm{Ex}_\pi(A_2 \circ A_1)$. The compositor

$$c_{A_1,A_2} : \mathrm{Ex}_\pi(A_2) \circ \mathrm{Ex}_\pi(A_2) \Longrightarrow \mathrm{Ex}_\pi(A_2 \circ A_1)$$

is the modification $h_2 \circ h_1 \Longrightarrow \tilde{h}$ defined by

$$\begin{array}{c} ((t'' \circ \pi^*A_2) \circ \bar{t}') \circ ((t' \circ \pi^*A_1) \circ \bar{t}) \Rightarrow (t'' \circ (\pi^*A_2 \circ ((\bar{t}' \circ t') \circ \pi^*A_1))) \circ \bar{t} \\ \Big\Downarrow {\scriptstyle (\mathrm{id}\circ(\mathrm{id}\circ(i_{t'}\circ\mathrm{id})))\circ\mathrm{id}} \\ (t'' \circ (\pi^*A_2 \circ (\mathrm{id} \circ \pi^*A_1))) \circ \bar{t} \Longrightarrow (t'' \circ \pi^*(A_2 \circ A_1)) \circ \bar{t}, \end{array}$$

which is indeed a descent 2-morphism.

For a 2-functor $F : \mathcal{P}_2(M) \longrightarrow T$ we find $\mathrm{Ex}_\pi(\mathrm{id}_F) = t \circ \bar{t}$. So, the unitor

$$u_F : \mathrm{Ex}_\pi(\mathrm{id}_F) \Longrightarrow \mathrm{id}_{\mathrm{triv}_i}$$

is the modification $u_F := j_t^{-1}$. The identities (A.1) for i_t and j_t show that this modification is a descent 2-morphism. With arguments similar to those given in the proof of Lemma 1.9, we have

Lemma 1.12. *The structure collected above furnishes a 2-functor*

$$\mathrm{Ex}_\pi : \mathrm{Triv}^2_\pi(i) \longrightarrow \mathfrak{Des}^2_\pi(i).$$

We have now described how globally defined 2-functors induce locally defined structure in terms of the 2-functor Ex_π. Going in the other direction is more involved; this is the content of the following section.

2 Reconstruction from Descent Data

In Section 1 we have introduced 2-functors on the path 2-groupoid of a smooth manifold, local trivializations and descent data. We have further described a procedure how to extract descent data from a locally trivialized 2-functor in terms of a 2-functor Ex_π. In this section we prove

Theorem 2.1. *The 2-functor*

$$\mathrm{Ex}_\pi : \mathrm{Triv}^2_\pi(i) \longrightarrow \mathfrak{Des}^2_\pi(i)$$

is an equivalence of 2-categories.

We prove this theorem in a constructive way: we introduce a 2-functor

$$\mathrm{Rec}_\pi : \mathfrak{Des}^2_\pi(i) \longrightarrow \mathrm{Triv}^2_\pi(i)$$

in the opposite direction, which reconstructs a 2-functor from a given descent object, such that Ex_π and Rec_π form a pair of equivalences of 2-categories. The main ingredient is a certain 2-groupoid that we call the *codescent 2-groupoid*. Its definition is the content of Section 2.1. The codescent 2-groupoid joins two important properties. First, it is equivalent to the path 2-groupoid of the underlying manifold M; this is shown in Section 2.2. Secondly, it is „dual“ to the descent 2-category $\mathfrak{Des}^2_\pi(i)$ introduced in the previous section; this duality is worked out in Section 2.3. In Section 2.4 we put the two pieces together and define the 2-functor Rec_π.

2.1 A Covering of the Path 2-Groupoid

In the following we introduce the *codescent 2-groupoid* $\mathcal{P}^\pi_2(M)$ associated to a surjective submersion $\pi : Y \longrightarrow M$. It combines the path 2-groupoid of Y with additional jumps between the fibres. This construction generalizes the one of the groupoid $\mathcal{P}^\pi_1(M)$ from [SW07].

The objects of $\mathcal{P}^\pi_2(M)$ are all points $a \in Y$. There are two „basic“ 1-morphisms:

(1) *Paths*: thin homotopy classes of paths $\gamma : a \longrightarrow a'$ in Y.

(2) *Jumps*: points $\alpha \in Y^{[2]}$ considered as 1-morphisms from $\pi_1(\alpha)$ to $\pi_2(\alpha)$.

The set of 1-morphisms of $\mathcal{P}^\pi_2(M)$ is freely generated from these two basic 1-morphisms, i.e. we have a formal composition $*$ and a formal identity id^*_a (the empty composition) associated to every object $a \in Y$. We introduce six „basic“ 2-morphisms:

(1) Four of *essential* type:

   (a) Thin homotopy classes of bigons $\Sigma : \gamma_1 \Longrightarrow \gamma_2$ in Y going between paths.

   (b) Thin homotopy classes of paths $\Theta : \alpha \longrightarrow \alpha'$ in $Y^{[2]}$ considered as 2-isomorphisms

$$\Theta : \alpha' * \pi_1(\Theta) \Longrightarrow \pi_2(\Theta) * \alpha,$$

   going between 1-morphisms mixed from jumps and paths.

   (c) Points $\Xi \in Y^{[3]}$ considered as 2-isomorphisms

$$\Xi : \pi_{23}(\Xi) * \pi_{12}(\Xi) \Longrightarrow \pi_{13}(\Xi)$$

   going between jumps.

(d) Points $a \in Y$ considered as 2-isomorphisms

$$\Delta_a : \mathrm{id}_a^* \Longrightarrow (a,a)$$

relating the formal identity with the trivial jump.

In (b) to (d) we demand that the 2-morphisms Θ, Ξ and Δ_a come with formal inverses, denoted by Θ^{-1}, Ξ^{-1} and Δ_a^{-1}.

(2) Two of *technical* type:

(a) associators for the formal composition, i.e. 2-isomorphisms

$$a^*_{\beta_1,\beta_2,\beta_3} : (\beta_3 * \beta_2) * \beta_1 \Longrightarrow \beta_3 * (\beta_2 * \beta_1)$$

for β_k either paths or jumps, and unifiers

$$l_\beta : \beta * \mathrm{id}_a^* \Longrightarrow \beta \quad \text{and} \quad r_\beta : \mathrm{id}_b^* * \beta \Longrightarrow \beta.$$

(b) for points $a \in Y$ and composable paths γ_1 and γ_2 2-isomorphisms

$$u_a^* : \mathrm{id}_a \Longrightarrow \mathrm{id}_a^* \quad \text{and} \quad c^*_{\gamma_1,\gamma_2} : \gamma_2 * \gamma_1 \Longrightarrow \gamma_2 \circ \gamma_1$$

expressing that the formal composition restricted to paths compares to the usual composition of paths.

Now we consider the set which is freely generated from these basic 2-morphisms in virtue of a formal horizontal composition $*$ and a formal vertical composition $\circledast$. The formal identity 2-morphisms are denoted by $\mathrm{id}_\beta^\circledast : \beta \Longrightarrow \beta$ for any 1-morphism β. The set of 2-morphisms of the 2-category $\mathcal{P}_2^\pi(M)$ is this set, subject to the following list of identifications:

(I) Identifications of *2-categorical type*. The formal compositions $*$ and $\circledast$, and the 2-isomorphisms of type (2a) form the structure of a 2-category and we impose all identifications required by the axioms (C1) to (C4).

(II) Identifications of *2-functorial type*. We have the structure of a 2-functor

$$\iota : \mathcal{P}_2(Y) \longrightarrow \mathcal{P}_2^\pi(M).$$

This 2-functor regards points, paths and bigons in Y as objects, 1-morphisms of type (1) and 2-morphisms of type (1a), respectively. Its compositors and unitors are the 2-isomorphisms c^* and u^* of type (2b). We impose all identification required by the axioms (F1) to (F4) for this 2-functor.

(III) Identifications of *transformation type*. We have the structure of a pseudonatural transformation

$$\Gamma : \pi_1^*\iota \longrightarrow \pi_2^*\iota$$

between 2-functors defined over $Y^{[2]}$. Its component at a 1-morphism $\Theta : \alpha \longrightarrow \alpha'$ in $\mathcal{P}_1(Y^{[2]})$ is the 2-isomorphism Θ of type (1b). We impose all identifications required by the axioms (T1) and (T2) for this pseudonatural transformation.

(IV) Identification of *modification type*. We have the structure of a modification

$$\pi_{23}^*\Gamma \circ \pi_{12}^*\Gamma \Longrightarrow \pi_{13}^*\Gamma \tag{2.1}$$

between pseudonatural transformations of 2-functors defined over $Y^{[3]}$. Its component at an object $\Xi \in Y^{[3]}$ is the 2-isomorphism Ξ of type (1c). We have the structure of another modification

$$\mathrm{id}_\iota \Longrightarrow \Delta^*\Gamma \tag{2.2}$$

between pseudonatural transformations of 2-functors over Y, whose component at an object $a \in Y$ is the 2-isomorphism Δ_a of type (1d). We impose all identifications required by the commutativity of diagram (A.2) for both modifications.

(V) Identifications of *essential type*:

1. For every point $\Psi \in Y^{[4]}$ we impose the commutativity of the diagram

$$\begin{array}{c}
(\pi_{34}(\Psi) * \pi_{23}(\Psi)) * \pi_{12}(\Psi) \overset{a^*}{\Longrightarrow} \pi_{34}(\Psi) * (\pi_{23}(\Psi) * \pi_{12}(\Psi)) \overset{\mathrm{id}^* * \pi_{123}(\Psi)}{\Longrightarrow} \pi_{34}(\Psi) * \pi_{13}(\Psi) \overset{\pi_{134}(\Psi)}{\Longrightarrow} \pi_{14}(\Psi) \\
(\pi_{34}(\Psi) * \pi_{23}(\Psi)) * \pi_{12}(\Psi) \overset{\pi_{234}(\Psi) * \mathrm{id}^*}{\Longrightarrow} \pi_{24}(\Psi) * \pi_{12}(\Psi) \overset{\pi_{124}(\Psi)}{\Longrightarrow} \pi_{14}(\Psi)
\end{array}$$

   of compositions of jumps.

2. For every point $\alpha \in Y^{[2]}$ we impose the commutativity of the diagrams

$$\begin{array}{c}
\mathrm{id}_b^* * \alpha \overset{b * \mathrm{id}_\alpha^*}{\Longrightarrow} (b,b) * \alpha \overset{(a,b,b)}{\Longrightarrow} \alpha, \qquad \mathrm{id}_b^* * \alpha \overset{r_\alpha^*}{\Longrightarrow} \alpha
\end{array}$$

and

$$\begin{array}{c}
\alpha * \mathrm{id}_a^* \overset{\mathrm{id}_\alpha^* \circ a}{\Longrightarrow} \alpha * (a,a) \overset{(a,a,b)}{\Longrightarrow} \alpha, \qquad \alpha * \mathrm{id}_a^* \overset{l_\alpha^*}{\Longrightarrow} \alpha.
\end{array}$$

According to (I) we have defined a 2-category $\mathcal{P}_2^\pi(M)$. We show in Appendix B that it is actually a 2-groupoid (Lemma B.1). We also have a 2-functor

$$\iota : \mathcal{P}_2(Y) \hookrightarrow \mathcal{P}_2^\pi(M),$$

a pseudonatural transformation Γ and modifications (2.1) and (2.2) claimed by identifications (II), (III) and (IV).

As we shall see next, the codescent 2-groupoid joins two important features: the first relates it to the path 2-groupoid of M and is described in the next subsection. The second relates it to the descent 2-category from Section 1 and is described in Section 2.3.

2.2 Lifts of Paths and Bigons

There is a canonical strict 2-functor

$$p^\pi : \mathcal{P}_2^\pi(M) \longrightarrow \mathcal{P}_2(M)$$

whose composition with the 2-functor ι is equal to the 2-functor $\pi_* : \mathcal{P}_2(Y) \longrightarrow \mathcal{P}_2(M)$ induced from the projection,

$$p^\pi \circ \iota = \pi_*. \tag{2.3}$$

It sends all 1-morphisms and 2-morphisms which are not in the image of ι to identities. In this section we show

Proposition 2.2. *The 2-functor p^π is an equivalence of 2-categories.*

To prove this proposition we introduce an inverse 2-functor

$$s : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^\pi(M).$$

Since the 2-functor p^π is surjective on objects, we call s the *section 2-functor*. To define s, we lift points, paths and bigons in M along the surjective submersion π, and use the jumps and the several 2-morphisms of the codescent 2-groupoid whenever such lifts do not exist.

For preparation we need the following technical lemma whose proof is postponed to Appendix B.

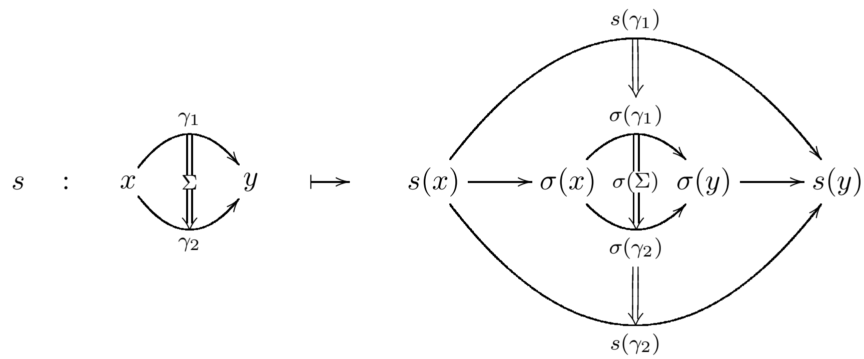
Lemma 2.3. *Let $\gamma : x \longrightarrow y$ be a paths in M, and let $\tilde{x}, \tilde{y} \in Y$ be lifts of the endpoints, i.e. $\pi(\tilde{x}) = x$ and $\pi(\tilde{y}) = y$.*

(a) There exists a 1-morphism $\tilde{\gamma} : \tilde{x} \longrightarrow \tilde{y}$ in $\mathcal{P}_2^\pi(M)$ such that $p^\pi(\tilde{\gamma}) = \gamma$.

(b) Let $\tilde{\gamma} : \tilde{x} \longrightarrow \tilde{y}$ and $\tilde{\gamma}' : \tilde{x} \longrightarrow \tilde{y}$ be two such 1-morphisms. Then, there exists a unique 2-isomorphism $A : \tilde{\gamma} \Longrightarrow \tilde{\gamma}'$ in $\mathcal{P}_2^\pi(M)$ such that $p^\pi(A) = \mathrm{id}_\gamma$.

To construct the 2-functor s we fix choices of an open cover $\{U_i\}_{i\in I}$ of M together with sections $\sigma_i : U_i \longrightarrow Y$, and of lifts $s(p) \in Y$ for all points $p \in M$. We also fix, for every path $\gamma : x \longrightarrow y$ in M, a 1-morphism $s(\gamma) : s(x) \longrightarrow s(y)$ in $\mathcal{P}_2^\pi(M)$. Such lifts exist according to Lemma 2.3 (a). For the identity 1-morphisms id_x we may choose the identity 1-morphisms $\mathrm{id}^*_{s(x)}$. This defines s on objects and 1-morphisms.

Now let $\Sigma : \gamma_1 \Longrightarrow \gamma_2$ be a bigon in M. Its image $\Sigma([0,1]^2) \subset M$ is compact and hence covered by open sets indexed by a finite subset $J \subset I$. We choose a decomposition of Σ in a vertical and horizontal composition of bigons $\{\Sigma_j\}_{j\in J}$ such that $\Sigma_j([0,1]^2) \subset U_j$. Then we define $s(\Sigma)$ to be composed from the 2-morphisms $s(\Sigma_j)$ in the same way as Σ was composed from the Σ_j. It remains to define the 2-functor s on bigons Σ which are contained in one of the open sets U which has a section $\sigma : U \longrightarrow Y$. We define for such a bigon



where the unlabelled 1-morphisms are the obvious jumps, and the unlabelled 2-morphisms are the unique 2-isomorphisms from Lemma 2.3 (b).

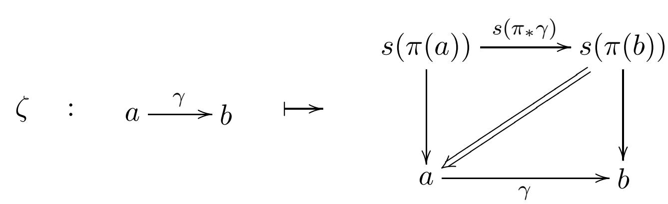
The 2-functor $s : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^\pi(M)$ defined like this is not strict. While its unitor is trivial because we have by definition $s(\mathrm{id}_x) = \mathrm{id}^*_{s(x)}$, its compositor $c_{\gamma_1,\gamma_2} : s(\gamma_2) \circ s(\gamma_1) \Longrightarrow s(\gamma_2 \circ \gamma_1)$ is defined to be the unique 2-isomorphism from Lemma 2.3 (b). All axioms for the 2-functor s follow from the uniqueness of these 2-isomorphisms.

Now we can proceed with the

Proof of Proposition 2.2. By construction we find $p^\pi \circ s = \mathrm{id}_{\mathcal{P}_2(M)}$. It remains to construct a pseudonatural equivalence

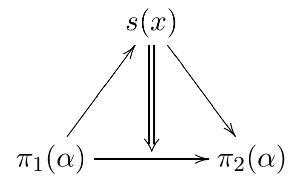
$$\zeta : s \circ p^\pi \longrightarrow \mathrm{id}_{\mathcal{P}_2^\pi(M)}.$$

We define ζ on both basic 1-morphisms. Its component at a path is



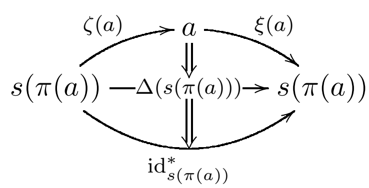
where the unlabelled 1-morphisms are again the obvious jumps, and the 2-isomorphism is the unique one. Notice that if $s(\pi_*\gamma)$ happens to be just a path, this 2-isomorphism is just of type (1b). The component of ζ at a jump is

$$\zeta \quad : \quad \pi_1(\alpha) \xrightarrow{\alpha} \pi_2(\alpha) \quad \longmapsto$$

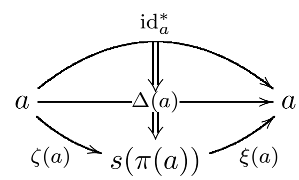


with $x := \pi(\pi_1(\alpha)) = \pi(\pi_2(\alpha))$, this is just one 2-isomorphism of type (1c). For some general 1-morphism, ζ puts the 2-isomorphisms above next to each other; this way axiom (T1) is automatically satisfied. Axiom (T2) follows again from the uniqueness of the 2-morphisms we have used.

In order to show that ζ is invertible we need to find another pseudonatural transformation $\xi : \mathrm{id}_{\mathcal{P}_2^\pi(M)} \longrightarrow s \circ p^\pi$ together with invertible modifications $i_\zeta : \xi \circ \zeta \Longrightarrow \mathrm{id}_{s\circ p^\pi}$ and $j_\zeta : \mathrm{id}_{\mathrm{id}_{\mathcal{P}_2^\pi(M)}} \Longrightarrow \zeta \circ \xi$ that satisfy the zigzag identities. The pseudonatural transformation ξ can be defined in the same way as ζ just by turning the diagrams upside down, using the formal inverses. The modifications i_ζ and j_ζ assign to a point $a \in Y$ the 2-isomorphisms



and



that combine 2-isomorphisms of type (1c) and (1d). The zigzag identities are satisfied due to the uniqueness of 2-isomorphisms we have used. $\square$

Corollary 2.4. *The section 2-functor $s : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^\pi(M)$ is independent (up to pseudonatural equivalence) of all choices, namely the choice of lifts of points and 1-morphisms, the choice of the open cover, and the choice of local sections.*

This follows from the fact that any two weak inverses of a 1-morphism in a 2-category are 2-isomorphic.

2.3 Pairing with Descent Data

In this section we relate the codescent 2-groupoid $\mathcal{P}_2^\pi(M)$ to the descent 2-category $\mathfrak{Des}_\pi^2(i)$ defined in Section 1.2 in terms of a strict 2-functor

$$R : \mathfrak{Des}_\pi^2(i) \longrightarrow \mathrm{Funct}(\mathcal{P}_2^\pi(M), T).$$

This 2-functors expresses that the 2-groupoid $\mathcal{P}_2^\pi(M)$ is „T-dual“ to the descent 2-category; this justifies the notion *co*descent 2-groupoid.

The 2-functor R labels the structure of the codescent 2-groupoid by descent data in a certain way. To start with, let $(\mathrm{triv}, g, \psi, f)$ be a descent object. Its image under R is a 2-functor

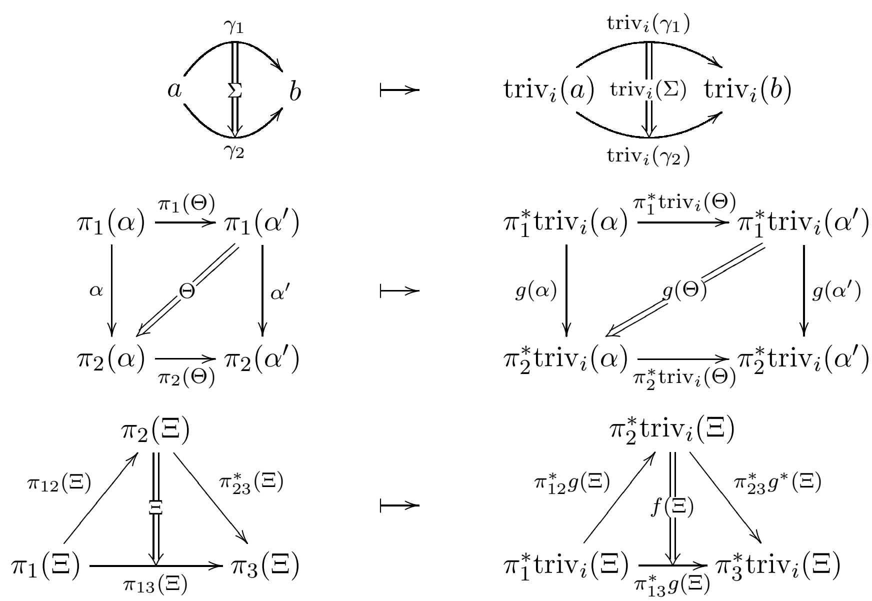
$$R_{(\mathrm{triv},g,\psi,f)} : \mathcal{P}_2^\pi(M) \longrightarrow T$$

which is defined as follows. To an object $a \in Y$ it assigns the object $\mathrm{triv}_i(a)$ in T. On basic 1-morphisms it is defined by the following assignments:

$$a \xrightarrow{\gamma} a' \quad\longmapsto\quad \mathrm{triv}_i(a) \xrightarrow{\mathrm{triv}_i(\gamma)} \mathrm{triv}_i(a')$$

$$\pi_1(\alpha) \xrightarrow{\alpha} \pi_2(\alpha) \quad\longmapsto\quad \pi_1^*\mathrm{triv}_i(\alpha) \xrightarrow{g(\alpha)} \pi_2^*\mathrm{triv}_i(\alpha).$$

To a formal composition of basic 1-morphisms it assigns the composition of the respective images and to the formal identity id_a^* at a point $a \in Y$ it assigns $\mathrm{id}_{\mathrm{triv}_i(a)}$. On the basic 2-morphisms of essential types (1a) to (1d) it is defined by the following assignments:



$$\mathrm{id}_a^* \xRightarrow{\Delta_a} \Delta(a) \quad\longmapsto\quad \mathrm{id}_{\mathrm{triv}_i(a)} \xRightarrow{\psi(a)} \Delta^* g(a).$$

To the basic 2-morphisms of technical type (2a) it assigns associators and unifiers of the 2-category T. To those of type (2b) it assigns unitors and compositors of the 2-functor i, i.e.

$$\mathrm{id}_a \xRightarrow{u_a^*} \mathrm{id}_a^* \quad\longmapsto\quad \mathrm{triv}_i(\mathrm{id}_a) \xRightarrow{u^i_{\mathrm{triv}(a)}} \mathrm{id}_{\mathrm{triv}_i(a)}$$

$$\gamma_2 * \gamma_1 \xRightarrow{c^*_{\gamma_1,\gamma_2}} \gamma_2 \circ \gamma_1 \qquad \longmapsto \qquad \mathrm{triv}_i(\gamma_2) \circ \mathrm{triv}_i(\gamma_1) \xRightarrow{c^i_{\mathrm{triv}(\gamma_1),\mathrm{triv}(\gamma_2)}} \mathrm{triv}_i(\gamma_2 \circ \gamma_1).$$

Finally, some formal horizontal and vertical composition of 2-morphisms is assigned to the composition of the images of the respective basic 2-morphisms, the formal horizontal composition replaced by the horizontal composition $\circ$ of T, and the formal vertical composition replaced by the vertical composition $\bullet$ of T.

By construction, all these assignments are well-defined under the identifications we have declared under the 2-morphisms of $\mathcal{P}_2^\pi(M)$:

- They are well-defined under the identifications (I) due to the axioms of the 2-category T.
- They are well-defined under identifications (II) due to the axioms of the 2-functors triv and i.
- They are well-defined under identifications (III) due to the axioms of the pseudonatural transformation g.
- They are well-defined under identifications (IV) due to the axioms of the modifications ψ and f,
- They are well-defined under the identifications (V) because these are explicitly assumed in the definition of descent objects, see diagrams (1.1) and (1.2).

We have now defined the 2-functor $R_{(\mathrm{triv},g,\psi,f)}$ on descent objects, 1-morphisms and 2-morphisms. Since for all points $a \in Y$

$$R_{(\mathrm{triv},g,\psi,f)}(\mathrm{id}^*_a) = \mathrm{id}_{\mathrm{triv}_i(a)} = \mathrm{id}_{R_{(\mathrm{triv},g,\psi,f)}(a)},$$

it has a trivial unitor. Furthermore,

$$R_{(\mathrm{triv},g,\psi,f)}(\gamma) \circ R_{(\mathrm{triv},g,\psi,f)}(\beta) = R_{(\mathrm{triv},g,\psi,f)}(\gamma * \beta)$$

for all composable 1-morphisms β and γ of any type, so that it also has a trivial compositor. Hence, the 2-functor $R_{(\mathrm{triv},g,\psi,f)}$ is strict, and it is straightforward to see that the remaining axioms (F1) and (F2) are satisfied.

So far we have introduced a 2-functor associated to each descent object. Let us now introduce a pseudonatural transformation

$$R_{(h,\epsilon)} : R_{(\mathrm{triv},g,\psi,f)} \longrightarrow R_{(\mathrm{triv}',g',\psi',f')}$$

associated to any descent 1-morphism

$$(h,\epsilon) : (\mathrm{triv},g,\psi,f) \longrightarrow (\mathrm{triv}',g',\psi',f').$$

Its definition is as straightforward as the one of the 2-functor given before. Its component at an object $a \in Y$ is the 1-morphism

$$h(a) : \mathrm{triv}_i(a) \longrightarrow \mathrm{triv}'_i(a).$$

Its components at basic 1-morphisms are given by the following assignments:

$$a \xrightarrow{\gamma} a' \quad \longmapsto \quad \begin{array}{ccc} \mathrm{triv}_i(a) & \xrightarrow{\mathrm{triv}_i(\gamma)} & \mathrm{triv}_i(a') \\ {\scriptstyle h(a)}\downarrow & \overset{h(\gamma)}{\swarrow} & \downarrow{\scriptstyle h(a')} \\ \mathrm{triv}'_i(a) & \xrightarrow[\mathrm{triv}'_i(\gamma)]{} & \mathrm{triv}'_i(a') \end{array}$$

$$\pi_1(\alpha) \xrightarrow{\alpha} \pi_2(\alpha) \quad \longmapsto \quad \begin{array}{ccc} \pi_1^*\mathrm{triv}_i(\alpha) & \xrightarrow{g(\alpha)} & \pi_2^*\mathrm{triv}_i(\alpha) \\ {\scriptstyle \pi_1^*h(\alpha)}\downarrow & \overset{\epsilon(\alpha)}{\swarrow} & \downarrow{\scriptstyle \pi_2^*h(\alpha)} \\ \pi_1^*\mathrm{triv}'_i(\alpha) & \xrightarrow[g'(\alpha)]{} & \pi_2^*\mathrm{triv}'_i(\alpha) \end{array}$$

For compositions of 1-morphisms, $R_{(h,\epsilon)}$ puts the diagrams for the involved basic 1-morphisms next to each other. For example, to a composition $\gamma * \alpha$ of a jump $\alpha = (x, y)$ with a path $\gamma : y \longrightarrow z$ it assigns the 2-isomorphism

$$h(z) \circ (\mathrm{triv}_i(\gamma) \circ g(\alpha)) \Longrightarrow (\mathrm{triv}'_i(\gamma) \circ g(\alpha)) \circ h(x)$$

which is (up to the obvious associators) obtained by first applying $h(\gamma)$ and then $\epsilon(\alpha)$. This way, axiom (T1) for the pseudonatural transformation $R_{(h,\epsilon)}$, namely the compatibility with the composition of 1-morphisms, is automatically satisfied. It remains to prove

Lemma 2.5. *The assignments $R_{(h,\epsilon)}$ are compatible with the 2-morphisms of the codescent 2-groupoid in the sense of axiom (T2).*

Proof. We check this compatibility separately for each basic 2-morphism. For the essential 2-morphisms it comes from the following properties of the descent 1-morphism (h, ϵ):

- For type (1a) it comes from axiom (T2) for the pseudonatural transformation h.
- For type (1b) it comes from the axiom for the modification ϵ and from axiom (T2) for the pseudonatural transformation h.
- For types (1c) and (1d) it comes from the conditions (1.3) and (1.4) on the descent 1-morphism (h, ϵ).

For the technical 2-morphisms it comes from properties of the 2-category T and the one of the 2-functor i: for type (2a) it is satisfied because the associators and unifiers of T are natural, and for type (2b) it is satisfied because the compositors and unitors of i are natural. $\square$

We have now described a 2-functor associated to each descent object and a pseudonatural transformation associated to each descent 1-morphism. Now let $(\mathrm{triv}, g, \psi, f)$ and $(\mathrm{triv}', g', \psi', f')$ be descent objects and let (h_1, ϵ_1) and (h_2, ϵ_2) be two descent 1-morphisms between these. For a descent 2-morphism

$$E : (h_1, \epsilon_1) \Longrightarrow (h_2, \epsilon_2)$$

we introduce now a modification

$$R_E : R_{(h_1,\epsilon_1)} \Longrightarrow R_{(h_2,\epsilon_2)}.$$

Its component at an object $a \in Y$ is the 2-morphism $E(a) : h_1(a) \Longrightarrow h_2(a)$. The axiom for R_E, the compatibility with 1-morphisms, is satisfied for paths because E is a modification, and for jumps because of the diagram (1.5) in the definition of descent 2-morphisms.

It is now straightforward to see

Proposition 2.6. *The assignments defined above furnish a strict 2-functor*

$$R : \mathfrak{Des}^2_\pi(i) \longrightarrow \mathrm{Funct}(\mathcal{P}_2^\pi(M), T).$$

The 2-functor R represents the descent 2-category in a 2-category of 2-functors; in fact in a faithful way. We recall from Section 1.2 that there is a 2-functor $V : \mathfrak{Des}^2_\pi(i) \longrightarrow \mathrm{Funct}(\mathcal{P}_2(Y), T)$ which is also a representation of the same kind (but not faithful). The relation between these two representations is the simple observation

Lemma 2.7. $R \circ \iota^* = V$.

From this point of view, the codescent 2-groupoid enlarges the path 2-groupoid $\mathcal{P}_2(Y)$ by additional 1-morphisms (the jumps) and additional 2-morphisms in such a way that it carries a faithful representation of the descent 2-category.

2.4 Equivalence Theorem

Now we put the two main aspects of the codescent 2-groupoid together, namely the representation 2-functor R and its equivalence with the path 2-groupoid in terms of the section 2-functor s. The *reconstruction 2-functor* Rec_π is now introduced as the composition

$$\mathfrak{Des}^2_\pi(i) \xrightarrow{R} \mathrm{Funct}(\mathcal{P}_2^\pi(M), T) \xrightarrow{s^*} \mathrm{Funct}(\mathcal{P}_2(M), T) \cdot$$

Here, s^* is the composition with s. According to Corollary 2.4, the reconstruction 2-functor is canonically attached to the surjective submersion $\pi : Y \longrightarrow M$ and the 2-functor $i : \mathrm{Gr} \longrightarrow T$.

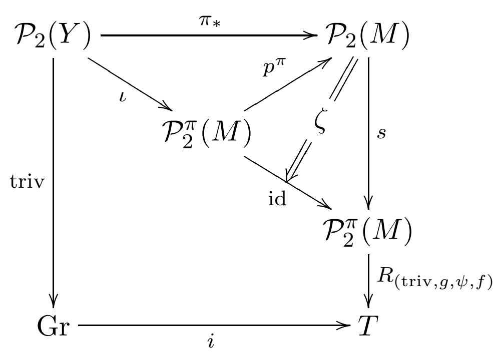
In order to show that the reconstruction ends in the 2-category $\mathrm{Triv}^2_\pi(i)$ instead of just $\mathrm{Funct}(\mathcal{P}_2(M), T)$ it remains to equip, for each descent object $(\mathrm{triv}, g, \psi, f)$, the reconstructed 2-functor

$$F := R_{(\mathrm{triv},g,\psi,f)} \circ s$$

with a π-local i-trivialization (triv, t). Clearly, we take the given 2-functor triv as the first ingredient and are left with the construction of a pseudonatural equivalence

$$t : \pi^*F \longrightarrow \mathrm{triv}_i. \tag{2.4}$$

This equivalence is simply defined by



where ζ is the pseudonatural equivalence from Section 2.2. The triangle on the top of the latter diagram is equation (2.3), and the remaining subdiagram expresses the equation

$$\iota^* R_{(\mathrm{triv},g,\psi,f)} = \mathrm{triv}_i$$

which follows from Lemma 2.7.

We recall that the aim of the present Section 2 was to prove that the extraction of descent data, the 2-functor

$$\mathrm{Ex}_\pi : \mathrm{Triv}^2_\pi(i) \longrightarrow \mathfrak{Des}^2_\pi(i),$$

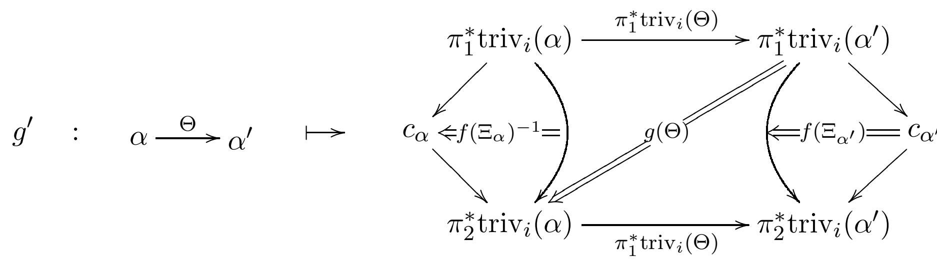
yields an equivalence of 2-categories (Theorem 2.1). We have so far introduced a canonical 2-functor

$$\mathrm{Rec}_\pi : \mathfrak{Des}^2_\pi(i) \longrightarrow \mathrm{Triv}^2_\pi(i)$$

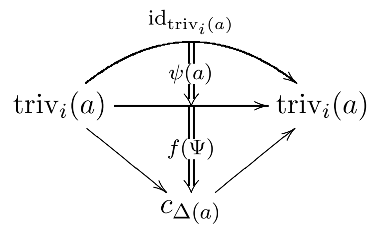
in the opposite direction. To prove Theorem 2.1 it remains to show that the 2-functors Ex_π and Rec_π form a pair of equivalences. This is done in the following two lemmata.

Lemma 2.8. *We have a pseudonatural equivalence* $\mathrm{Ex}_\pi \circ \mathrm{Rec}_\pi \cong \mathrm{id}_{\mathfrak{Des}^2_\pi(i)}$.

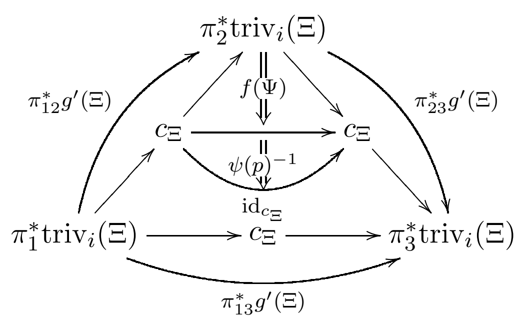
Proof. Given a descent object (triv, g, ψ, f) let us pass to the reconstructed 2-functor and extract its descent data (triv$'$, g', ψ', f'). We find immediately triv$'$ = triv. Furthermore, the pseudonatural transformation g' has the components



where we have introduced an object $c_\alpha := \mathrm{triv}_i(s(p))$ where $p = \pi(\pi_1(\alpha)) = \pi(\pi_2(\alpha))$ and a 2-morphism $\Xi_\alpha := (\pi_1(\alpha), s(p), \pi_2(\alpha))$. It is useful to notice that this means that f is a modification $f : g' \Longrightarrow g$. The modification ψ' has the component



at a point $a \in Y$. Finally, the modification f' has the component



at a point $\Xi \in Y^{[3]}$, where we have introduced the 2-morphism $\Psi := (c_\Xi, \pi_2^*\mathrm{triv}_i(\Xi), c_\Xi)$, and p is again the projection of Ξ to M. Now it is straightforward to construct a descent 1-morphism

$$\rho_{(\mathrm{triv},g,\psi,f)} : (\mathrm{triv}, g', \psi', f') \longrightarrow (\mathrm{triv}, g, \psi, f)$$

which consists of the identity pseudonatural transformation $h := \mathrm{id}_{\mathrm{triv}}$ and of a modification $\epsilon : \pi_2^*h \circ g' \Longrightarrow g \circ \pi_1^*h$ induced from the modification $f : g' \Longrightarrow g$ and the left and right unifiers. This descent 1-morphism is the component of the pseudonatural equivalence $\rho : \mathrm{Ex}_\pi \circ \mathrm{Rec}_\pi \longrightarrow \mathrm{id}$ we have to construct, at the object $(\mathrm{triv}, g, \psi, f)$.

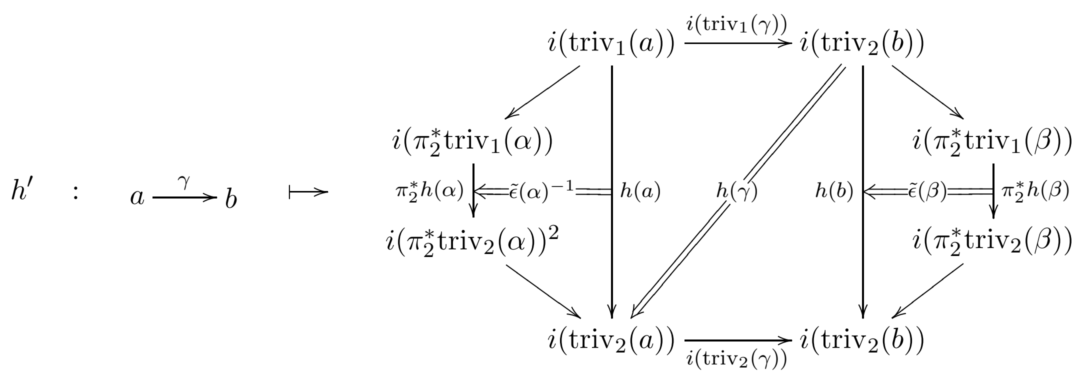
Let us now define the component of ρ at a descent 1-morphism

$$(h, \epsilon) : (\mathrm{triv}_1, g_1, \psi_1, f_1) \longrightarrow (\mathrm{triv}_2, g_2, \psi_2, f_2).$$

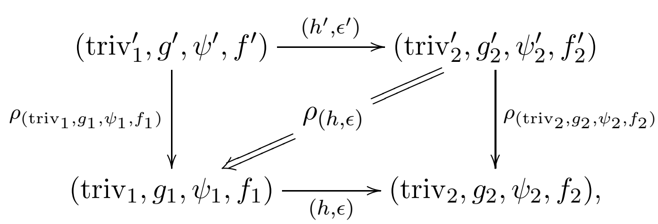
It is useful to introduce a modification $\tilde{\epsilon} : \bar{g}_2 \circ \pi_2^* h \circ g_1 \Longrightarrow \pi_1^* h$ where $\bar{g}_2$ is the pullback of g_2 along the map $Y^{[2]} \longrightarrow Y^{[2]}$ that exchanges the components. It is defined as the following composition of modifications:

$$\bar{g}_2 \circ \pi_2^* h \circ g_1 \xRightarrow{\mathrm{id} \circ \epsilon} \bar{g}_2 \circ g_2 \circ \pi_1^* h \xRightarrow{\Delta_{121}^* f_2 \circ \mathrm{id}} \pi_1^* \Delta^* g \circ \pi_1^* h \xRightarrow[\pi_1^* \psi_2^{-1} \circ \mathrm{id}]{} \pi_1^* \mathrm{id} \circ \pi_1^* h \xRightarrow[l_{\pi_1^* h}]{} \pi_1^* h$$

Now, if we reconstruct and extract local data (h', ϵ'), the pseudonatural transformation h' has the components



with $\alpha := (a, s(\pi(a)))$ and $\beta = (b, s(\pi(b)))$. Like above we observe that $\tilde{\epsilon}$ is hence a modification $\tilde{\epsilon} : h' \Longrightarrow h$. Now, the component $\rho_{(h,\epsilon)}$ we have to define is a descent 2-morphism



this is just a modification $\mathrm{id} \circ h' \Longrightarrow h \circ \mathrm{id}$ since the vertical arrows are the identity pseudonatural transformations. We define $\rho_{(h,\epsilon)}$ from $\tilde{\epsilon}$ and right and left unifiers in the obvious way. It is straightforward to see that this defines indeed a descent 2-morphism.

Finally, we observe that the definitions $\rho_{(\text{triv},g,\psi,f)}$ and $\rho_{(h,\epsilon)}$ furnish a pseudonatural equivalence as required. $\square$

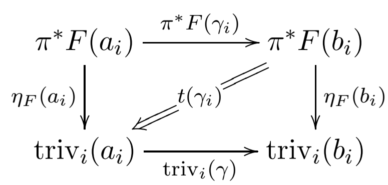
The second part of the proof of Theorem 2.1 is

Lemma 2.9. *There exists a pseudonatural equivalence* $\mathrm{id}_{\mathrm{Triv}^2_\pi(i)} \cong \mathrm{Rec}_\pi \circ \mathrm{Ex}_\pi$.

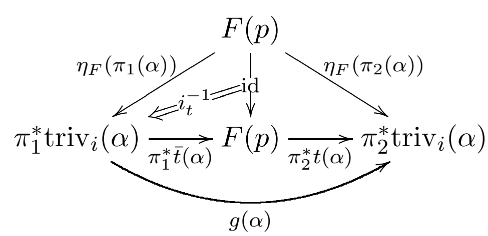
Proof. For a 2-functor $F : \mathcal{P}_2(X) \longrightarrow T$ and a π-local i-trivialization (triv, t), let $(\text{triv}, g, \psi, f)$ be the associated descent data. We find a pseudonatural transformation

$$\eta_F : F \longrightarrow s^*R_{(\text{triv},g,\psi,f)}$$

in the following way. Its component at a point $x \in X$ is the 1-morphism $t(s(x)) : F(x) \longrightarrow \text{triv}_i(s(x))$ in T. To define its component at a path $\gamma : x \longrightarrow y$ we recall that $s(\gamma)$ is a composition of paths $\gamma_i : a_i \longrightarrow b_i$ and jumps α_i, so that we can compose $\eta_F(\gamma)$ from the pieces



and



where $i_t : \bar{t} \circ t \Longrightarrow \text{id}$ is the modification chosen to extract descent data. This defines the pseudonatural transformation η_F associated to a 2-functor F.

Now let $A : F_1 \longrightarrow F_2$ be a pseudonatural transformation between two 2-functors with local trivializations (triv_1, t_1) and (triv_2, t_2). Let (h, ϵ) the associated descent 1-morphism. It is now straightforward to see that

$$\eta_A := i_{t_1}^{-1} : \eta_{F_2} \circ A \Longrightarrow s^*R_{(h,\epsilon)} \circ \eta_{F_1}$$

defines a modification in such a way that both definitions together yield a pseudonatural transformation $\eta : \mathrm{id}_{\mathrm{Triv}^2_\pi(i)} \longrightarrow \mathrm{Rec}_\pi \circ \mathrm{Ex}_\pi$. It is clear that η is even a pseudonatural equivalence. $\square$

We have now derived a correspondence between the globally defined 2-functors and their descent data. This correspondence is important because we can now characterize the transport 2-functors we are aiming at, by imposing conditions on their descent data in a consistent way. This is the subject of the next section.

3 Smoothness Conditions

In the foregoing two sections we have introduced the algebraical setting for locally trivial 2-functors defined on the path 2-groupoid of a smooth manifold. In this section we impose additional smoothness conditions on these 2-functors that yield the appropriate notion of (parallel) transport 2-functors.

In Section 3.1 we review how to decide if a 2-functor on a path 2-groupoid is smooth or not. In Section 3.2 we use this notion of smoothness to characterize smooth descent data among all descent data. Transport 2-functors are defined in Section 3.3 as 2-functors which admit local trivializations with smooth descent data. We discuss several examples of Lie 2-groupoids Gr that correspond to important classes of transport 2-functors. In Section 3.4 we construct an example of a transport 2-functor, the curvature 2-functor associated to any fibre bundle with connection.

3.1 Smooth Functors

Let us start with a review on smooth functors between ordinary categories. The general idea of smooth functors is to consider them internal to smooth manifolds. That is, the sets of objects and morphisms of the involved categories are smooth manifolds, and a smooth functor consists of a smooth map between the objects and a smooth map between the morphisms. Categories internal to smooth manifolds are called *Lie categories*. However, in the situation of a functor

$$F : \mathcal{P}_1(X) \longrightarrow S$$

defined on the path groupoid of a smooth manifold X we encounter the problem that $\mathcal{P}_1(X)$ is not a Lie category: the set P^1X of morphisms of the path groupoid is not a smooth manifold.

One generalization of smooth manifolds which is appropriate here is the „convenient setting“ of diffeological spaces [Sou81]. Diffeological spaces and diffeological maps form a category D^∞ that enlarges the category C^∞ of smooth manifolds by means of a faithful functor

$$C^\infty \longrightarrow D^\infty .$$

This means: any smooth manifold can be regarded as a diffeological space in such a way that a map between two smooth manifolds is smooth if and only if it is diffeological. For an introduction to diffeological spaces we refer the reader to the recent paper [BH08] or to Appendix A.2 of [SW07].

Diffeological spaces admit many constructions that are not possible in the category of smooth manifolds. We need two of them. If X and Y are diffeological spaces, the set $D^\infty(X, Y)$ of diffeological maps from X to Y forms again a diffeological space. In particular, the set of smooth maps between smooth manifolds is a diffeological space. This is relevant for the set PX of paths in a smooth manifold X which is a subset (due to the requirement of sitting instants) of $C^\infty([0, 1], X)$, and hence a diffeological space.

The second construction that we need is taking quotients. That is, if X is a diffeological space and $\sim$ is an equivalence relation on X, the set $X/\sim$ of equivalence classes is again a diffeological space. This is relevant since the set of morphisms of the path groupoid of X is the set $P^1X := PX/\sim$, where $\sim$ is thin homotopy equivalence; thus P^1X is a diffeological space.

Summarizing, the path groupoid $\mathcal{P}_1(X)$ is a category internal to diffeological spaces. We call a functor $F : \mathcal{P}_1(X) \longrightarrow S$ *smooth* if it is also internal to diffeological spaces. Explicitly, a smooth functor consists of a smooth map $F_0 : X \longrightarrow S_0$ on objects, and of a diffeological map $F_1 : P^1X \longrightarrow S_1$ on morphisms. Similarly, a natural transformation $\eta : F \longrightarrow F'$ is called smooth if its components at points $x \in X$ form a smooth map $X \longrightarrow S_1$. The category of smooth functors and smooth natural transformations is denoted by $\mathrm{Funct}^\infty(\mathcal{P}_1(X), S)$.

In order to illuminate that this notion of smooth functors is appropriate for connections in fibre bundles we recall a central result of [SW07] about smooth functors with values in the Lie groupoid $\mathcal{B}G$ associated to a Lie group G. This groupoid has just one object, and G is its set of morphisms. The composition is $g_2 \circ g_1 := g_2 g_1$. Thus, $\mathcal{B}G$ is obviously a Lie groupoid. Associated to the smooth manifold X and the Lie group G is a well-known category $Z^1_X(G)^\infty$ of *G-connections on X* whose objects are 1-forms $A \in \Omega^1(X, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g}$ of G and whose morphisms are smooth functions $g : X \longrightarrow G$ acting as gauge transformations on the 1-forms in the usual way.

Theorem 3.1 (Proposition 4.5 in [SW07])**.** *There is a canonical isomorphism of categories*

$$\mathrm{Funct}^\infty(\mathcal{P}_1(X), \mathcal{B}G) \cong Z^1_X(G)^\infty.$$

Explicitly, the smooth functors $F : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G$ correspond one-to-one to 1-forms $A \in \Omega^1(X, \mathfrak{g})$, and the smooth natural transformations $\eta : F_1 \longrightarrow F_2$ correspond one-to-one to gauge transformations between the associated 1-forms A_1 and A_2. Thus, the notion of diffeological spaces is able to recover well-known differential-geometric structure.

We can go even further. The category $Z^1_X(G)^\infty$ can be seen as the category of local data of *trivial* principal G-bundles with connection, so that the smooth functors correspond to trivial principal G-bundles with connection. This is just the local version of the following global relation:

Theorem 3.2 (Theorem 5.8 in [SW07])**.** *Let X be a smooth manifold. There is a canonical surjective equivalence*

$$\mathrm{Trans}^1_{\mathcal{B}G}(X, G\text{-}\mathrm{Tor}) \cong \mathfrak{Bun}^\nabla_G(X)$$

between the category of transport functors on X with $\mathcal{B}G$-structure and the category of principal G-bundles with connection over X.

Transport functors have been introduced in [SW07] as an alternativ reformulation of fibre bundles with connection, and the latter theorem is one possible manifestation. We omit to give a review on transport functors at this place; for the following discussion it is only important to keep in mind that there is a category of functors $F : \mathcal{P}_1(X) \longrightarrow T$ qualified by „structure groupoids“ Gr, such that for certain choices, e.g. $T = G$-Tor and $\mathrm{Gr} = \mathcal{B}G$ like above, concrete differential-geometric structure is obtained.

Summarizing, diffeological spaces are appropriate to describe differential-geometric structure in category-theoretical terms. We will therefore also use diffeological spaces to define smooth 2-functors.

First we extend the notion of smoothness from functors to 2-functors. The set B^2X of thin homotopy classes of bigons in X is a diffeological space in the same way as the set P^1X explained above. We shall call a strict 2-functor $F : \mathcal{P}_2(X) \longrightarrow S$ with values in a Lie 2-category S *smooth*, if it consists of a smooth map $F_0 : X \longrightarrow S_0$ on objects, of a diffeological map $F_1 : P^1X \longrightarrow S_1$ on 1-morphisms and of a diffeological map $F_2 : B^2X \longrightarrow S_2$ on 2-morphisms. A pseudonatural transformation $\rho : F \longrightarrow F'$ is called smooth if its components $\rho(x)$ at points $x \in X$ and $\rho(\gamma)$ at paths γ in X furnish a smooth map $X \longrightarrow S_1$ and a diffeological map $P^1X \longrightarrow S_2$. A modification $\mathcal{A} : \rho_1 \Longrightarrow \rho_2$ is called smooth if its components $\mathcal{A}(x)$ form a smooth map $X \longrightarrow S_0$. All these form a strict 2-category denoted $\mathrm{Funct}^\infty(\mathcal{P}_2(X), S)$.

We already have evidence that this definition is appropriate: the correspondence of Theorem 3.1 between smooth functors and differential forms extends to 2-functors [SW08] in the following way [SW08]. First, the notion of a Lie group has to be generalized.

Definition 3.3. *A Lie 2-group is a strict monoidal Lie category $(\mathfrak{G}, \boxtimes, \mathbb{1})$ together with a smooth functor $i : \mathfrak{G} \longrightarrow \mathfrak{G}$ such that*

$$X \boxtimes i(X) = \mathbb{1} = i(X) \boxtimes X \quad \text{and} \quad f \boxtimes i(f) = \mathrm{id}_{\mathbb{1}} = i(f) \boxtimes f$$

for all objects X and all morphisms f in $\mathfrak{G}$.

We described in Appendix A, Example A.2, how the strict monoidal category $(\mathfrak{G}, \boxtimes, \mathbb{1})$ defines a strict 2-category $\mathcal{B}\mathfrak{G}$ with a single object. The additional functor i assures that $\mathcal{B}\mathfrak{G}$ is a strict 2-groupoid.

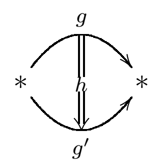
We infer that *every* Lie 2-group can be obtained from a smooth crossed module [BS76], also see [BL04] for a review. These crossed modules are the differential geometric counterpart of the category theoretic definition of a Lie 2-group.

Definition 3.4. *A smooth crossed module is a collection (G, H, t, α) of Lie groups G and H, and of a Lie group homomorphism $t : H \longrightarrow G$ and a smooth map $\alpha : G \times H \longrightarrow H$ which defines a left action of G on H by Lie group homomorphisms such that*

a) $t(\alpha(g, h)) = gt(h)g^{-1}$ *for all $g \in G$ and $h \in H$.*

b) $\alpha(t(h), x) = hxh^{-1}$ *for all $h, x \in H$.*

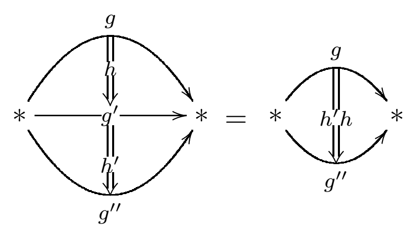
The construction of a Lie 2-group $\mathfrak{G} = \mathfrak{G}(G, H, t, \alpha)$ from a given smooth crossed module (G, H, t, α) can be found in the Appendix of [SW08]. Combining this construction with the one of Lie 2-groupoids out of Lie 2-groups, we obtain a Lie 2-groupoid $\mathcal{B}\mathfrak{G}$ associated to each crossed module $\mathfrak{G} = (G, H, t, \alpha)$. Here it will suffice to describe this resulting Lie 2-groupoid $\mathcal{B}\mathfrak{G}$: it has one object denoted $*$, a 1-morphism is a group element $g \in G$, the identity 1-morphism is the neutral element, and the composition of 1-morphisms is the multiplication, $g_2 \circ g_1 := g_2 g_1$. The 2-morphisms are pairs $(g, h) \in G \times H$, considered as 2-morphisms



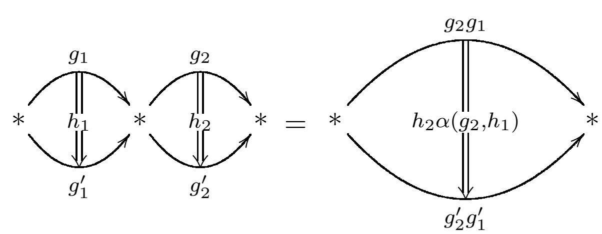
with

$$g' := t(h)g.$$

The vertical composition is



with $g' = t(h)g$ and $g'' = t(h')g' = t(h'h)g$, and the horizontal composition is



All these composition laws are uniquely determined by the crossed module, up to two conventional choices that enter the constructions mentioned above.

Now we are in the position to consider the 2-category $\mathrm{Funct}^{\infty}(\mathcal{P}_2(X), \mathcal{B}\mathfrak{G})$ of smooth 2-functors, smooth pseudonatural transformations and smooth modifications with values in the Lie 2-groupoid $\mathcal{B}\mathfrak{G}$. We have shown [SW08]:

1. Any smooth 2-functor $F : \mathcal{P}_2(X) \longrightarrow \mathcal{B}\mathfrak{G}$ induces a pair of differential forms: a 1-form $A \in \Omega^1(X, \mathfrak{g})$ with values in the Lie algebra of G, and a 2-form $B \in \Omega^2(X, \mathfrak{h})$ with values in the Lie algebra of H.

2. Any smooth pseudonatural transformation $\rho : F \longrightarrow F'$ gives rise to a 1-form $\varphi \in \Omega^1(X, \mathfrak{h})$ and a smooth map $g : X \longrightarrow G$. The identity $\mathrm{id} : F \longrightarrow F$ has $\varphi = 0$ and $g = 1$. If ρ_1 and ρ_2 are composable pseudonatural transformations, the 1-form of their composition $\rho_2 \circ \rho_1$ is $(\alpha_{g_2})_* \circ \varphi_1 + \varphi_2$, and their map is $g_2 g_1 : X \longrightarrow G$.

3. Any smooth modification $\mathcal{A} : \rho \Longrightarrow \rho'$ gives rise to a smooth map $a : X \longrightarrow H$. The identity modification id_ρ has $a = 1$. If two modifications $\mathcal{A}_1$ and $\mathcal{A}_2$ are vertically composable, $\mathcal{A}_2 \bullet \mathcal{A}_1$ has the map $a_2 a_1$. If two modifications $\mathcal{A}_1 : \rho_1 \Longrightarrow \rho_1'$ and $\mathcal{A}_2 : \rho_2 \Longrightarrow \rho_2'$ are horizontally composable, $\mathcal{A}_2 \circ \mathcal{A}_1$ has the map $a_2 \alpha(g_2, a_1)$.

It has been a straightforward but tedious calculation to convert the axioms of 2-functors, pseudonatural transformations and modifications into relations among these forms and functions. The results are the following [SW08]: the axioms of a 2-functor F infer

$$\mathrm{d}A + [A \wedge A] = t_* \circ B. \tag{3.1}$$

The axioms for a pseudonatural transformation $\rho : F \longrightarrow F'$ infer

$$\begin{aligned} A' + t_* \circ \varphi &= \mathrm{Ad}_g(A) - g^*\bar{\theta} && (3.2)\\ B' + \alpha_*(A' \wedge \varphi) + \mathrm{d}\varphi + [\varphi \wedge \varphi] &= (\alpha_g)_* \circ B. && (3.3) \end{aligned}$$

Similar results have been derived in [MP07]. Finally, the axioms for a modification $\mathcal{A} : \rho \Longrightarrow \rho'$ infer

$$g' = (t \circ a) \cdot g \quad \text{and} \quad \varphi' + (r_a^{-1} \circ \alpha_a)_*(A') = \mathrm{Ad}_a(\varphi) - a^*\bar{\theta}. \tag{3.4}$$

This structure made of differential forms and smooth functions naturally forms a strict 2-category $Z^2_X(\mathfrak{G})^\infty$: the objects are pairs (A, B) satisfying (3.1) etc. This 2-category generalizes the category $Z^1_X(G)^\infty$ from above, and has hence to be understood as the *category of* $\mathfrak{G}$*-connections on* X [SW08]. Moreover, the procedure described above furnishes a strict 2-functor

$$\mathcal{D} : \mathrm{Funct}^\infty(\mathcal{P}_2(X), \mathcal{B}\mathfrak{G}) \longrightarrow Z^2_X(\mathfrak{G})^\infty. \tag{3.5}$$

The main result of [SW08] is now

Theorem 3.5 (Theorem 2.20 in [SW08])**.** *The strict 2-functor $\mathcal{D}$ is an isomorphism*

$$\mathrm{Funct}^\infty(\mathcal{P}_2(X), \mathcal{B}\mathfrak{G}) \cong Z^2_X(\mathfrak{G})^\infty,$$

and has a canonical strict inverse 2-functor.

This theorem generalizes Theorem 3.1, and shows that the notion of diffeological spaces is also appropriate to qualify smooth 2-functors.

In the following section we use smooth 2-functors and transport functors to impose smoothness conditions on the descent data of 2-functors. The relation to differential forms will again be important in Section 4.

3.2 Smooth Descent Data

In this section we select a sub-2-category $\mathfrak{Des}^2_\pi(i)^\infty$ of *smooth descent data* in the 2-category $\mathfrak{Des}^2_\pi(i)$ of descent data. Transport 2-functors will then be defined as 2-functors with smooth descent data. Certainly, if $(\mathrm{triv}, g, \psi, f)$ is a descent object, we demand that the strict 2-functor $\mathrm{triv} : \mathcal{P}_2(Y) \longrightarrow \mathrm{Gr}$ has to be smooth in the sense discussed in the previous section. The question what the smoothness condition for the pseudonatural transformation g and the modifications ψ and f is, is more difficult since they take their values not in the Lie 2-category Gr but in the 2-category T which is in most applications not a Lie 2-category.

As anticipated in Section 6.2 of [SW07], the definition of a „transport n-functor“ is supposed to rely on a recursive principle in the sense that it uses the notion of transport $(n-1)$-functors. Accordingly, we will now use transport functors to state the remaining smoothness conditions. Namely, the pseudonatural transformation

$$g : \pi_1^*\mathrm{triv}_i \longrightarrow \pi_2^*\mathrm{triv}_i$$

can be viewed as a certain functor defined on $\mathcal{P}_1(Y^{[2]})$, and the smoothness condition on g will be that it is a transport 1-functor on $Y^{[2]}$. A little motivation might be the observation that g corresponds by Theorem 3.2 to a fibre bundle over $Y^{[2]}$ – one of the well-known ingredients of a bundle gerbe, see Sections 4.2 and 4.3.

Let us first explain in which way a pseudonatural transformation between two 2-functors can be viewed as a functor. We consider 2-functors F and G between 2-categories S and T. Since a pseudonatural transformation $\rho : F \longrightarrow G$ assigns 1-morphisms in T to objects in S and 2-morphisms in T to 1-morphisms in S, the general idea is to construct a category $S_{0,1}$ consisting of objects and 1-morphisms of S and a category ΛT consisting of 1-morphisms and 2-morphisms of T such that ρ yields a functor

$$\mathscr{F}(\rho) : S_{0,1} \longrightarrow \Lambda T.$$

If S is strict, forgetting its 2-morphisms yields immediately the category $S_{0,1}$. The construction of the category ΛT is more involved. If T is strict, its objects are the 1-morphisms of T. A morphism between $f : X_f \longrightarrow Y_f$ and $g : X_g \longrightarrow Y_g$ is a pair of 1-morphisms $x : X_f \longrightarrow X_g$ and $y : Y_f \longrightarrow Y_g$ and a 2-morphism

$$\begin{array}{ccc} X_f & \xrightarrow{x} & X_g \\ {\scriptstyle f}\downarrow & \swarrow\!\!\!\swarrow{\scriptstyle \varphi} & \downarrow{\scriptstyle g} \\ Y_f & \xrightarrow[y]{} & Y_g . \end{array} \tag{3.6}$$

This gives indeed a category ΛT, whose composition is defined by putting the diagrams next to each other. Clearly, any strict 2-functor $f : T' \longrightarrow T$ induces a functor $\Lambda f : \Lambda T' \longrightarrow \Lambda T$. For a more detailed discussion of these constructions we refer the reader to Section 4.2 of [SW08].

Now let $\rho : F \longrightarrow G$ be a pseudonatural transformation between two strict 2-functors from S to T. Sending an object X in S to the 1-morphism $\rho(X)$ and sending a 1-morphism f in S to the 2-morphism $\rho(X)$ now yields a functor

$$\mathscr{F}(\rho) : S_{0,1} \longrightarrow \Lambda T.$$

It respects the composition due to axiom (T1) for ρ and the identities due to Lemma A.9. Moreover, a modification $\mathcal{A} : \rho_1 \Longrightarrow \rho_2$ defines a natural transformation $\mathscr{F}(\mathcal{A}) : \mathscr{F}(\rho_1) \Longrightarrow \mathscr{F}(\rho_2)$, so that the result is a functor

$$\mathscr{F} : \mathrm{Hom}(F, G) \longrightarrow \mathrm{Funct}(S_{0,1}, \Lambda T) \tag{3.7}$$

between the category of pseudonatural transformations between F and G and the category of functors from $S_{0,1}$ to ΛT, for S and T strict 2-categories and F and G strict 2-functors.

In the case that the 2-category T is not strict, the construction of ΛT suffers from the fact that the composition is not longer associative. The situation becomes treatable if one requires the objects X_f, Y_f and X_g, Y_g and the 1-morphisms x and y in (3.6) to be contained the image of a strict 2-category T^{str} under some 2-functor $i : T^{\mathrm{str}} \longrightarrow T$. The result is a

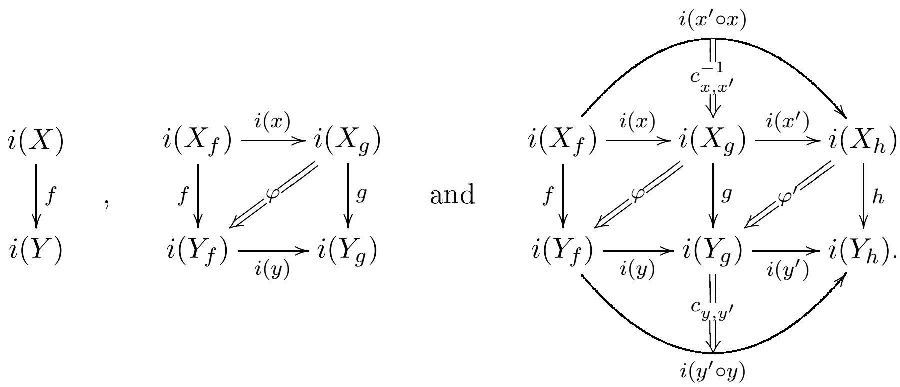


Figure 1: Objects, morphisms and the composition of the category $\Lambda_i T$ (the diagram on the right hand side ignores the associators and the bracketing of 1-morphisms). Here, c is the compositor of the 2-functor i.

category $\Lambda_i T$, in which the associativity of the composition is restored by axiom (F3) on the compositor of the 2-functor i. We omit a more formal definition and refer the reader to Figure 1 for an illustration. For any 2-functor $f : T \longrightarrow T'$, a functor

$$\Lambda F : \Lambda_i T \longrightarrow \Lambda_{F \circ i} T'$$

is induced by applying f to all involved objects, 1-morphisms and 2-morphisms. If S is the strict 2-category from above, we may now consider strict 2-functors F and G from S to T^{str}. Then, the functor 3.7 generalizes straightforwardly to a functor

$$\mathscr{F} : \mathrm{Hom}(i \circ F, i \circ G) \longrightarrow \mathrm{Funct}(S_{0,1}, \Lambda_i T)$$

between the category of pseudonatural transformations between $i \circ F$ and $i \circ G$ and the category of functors from $S_{0,1}$ to $\Lambda_i T$.

The following properties of $\mathscr{F}$ are easy to see. It is natural with respect to strict 2-functors $f : S' \longrightarrow S$ in the sense that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(i \circ F, i \circ G) & \xrightarrow{\mathscr{F}} & \mathrm{Funct}(S_{0,1}, \Lambda_i T) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Hom}(i \circ F \circ f, i \circ G \circ f) & \xrightarrow[\mathscr{F}]{} & \mathrm{Funct}(S'_{0,1}, \Lambda_i T) \end{array} \tag{3.8}$$

is commutative. It also preserves composition: if $F, G, H : S \longrightarrow T^{\mathrm{str}}$ are three strict 2-functors, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(i\circ G, i\circ H) \times \mathrm{Hom}(i\circ F, i\circ G) & \xrightarrow{\mathscr{F}\times\mathscr{F}} & \mathrm{Funct}(S_{0,1}, \Lambda_i T) \times \mathrm{Funct}(S_{0,1}, \Lambda_i T) \\ {\scriptstyle \circ}\downarrow & & \downarrow{\scriptstyle \otimes} \\ \mathrm{Hom}(i \circ H, i \circ F) & \xrightarrow[\mathscr{F}]{} & \mathrm{Funct}(S_{0,1}, \Lambda_i T) \end{array} \tag{3.9}$$

is commutative. Here, the tensor product $\otimes$ has the following meaning. The composition of morphisms in $\Lambda_i T$ was defined by putting the diagrams (3.6) next to each other as shown in Figure 1. But one can also put the diagrams of appropriate morphisms on top of each other, provided that the arrow on the bottom of the upper one coincides with the arrow on the top of the lower one. This is indeed the case for the morphisms in the image of composable pseudonatural transformations under $\mathscr{F} \times \mathscr{F}$, so that the above diagram makes sense. In a more formal context, the tensor product $\otimes$ can be discussed in the formalism of *weak double categories*, but we will not stress this point. However, it will obtain a concrete meaning in Section 4.2 and 4.3.

In what follows the strict 2-category S will be the path 2-groupoid of some smooth manifold, and the strict 2-functors $f : S' \longrightarrow S$ will be induced by smooth maps. Notice that for $S = \mathcal{P}_2(X)$ we obtain $S_{0,1} = \mathcal{P}_1(X)$, the path 1-groupoid of the manifold X.

Now we begin the discussion of smooth descent data of 2-functors $\mathcal{P}_2(M) \longrightarrow T$ with π-local i-trivializations, for $\pi : Y \longrightarrow M$ a surjective submersion and $i : \mathrm{Gr} \longrightarrow T$ a 2-functor. Let $(\mathrm{triv}, g, \psi, f)$ be a descent object in the associated descent 2-category $\mathfrak{Des}^2_\pi(i)$. Now, the strict Lie 2-groupoid Gr plays the role of the strict 2-category T^{str} in the above setting, and the path 2-groupoid $\mathcal{P}_2(Y^{[2]})$ the one of S. The two 2-functors F and G are

$\pi_1^*\mathrm{triv}$ and $\pi_2^*\mathrm{triv}$. Accordingly, the pseudonatural transformation $g : \pi_1^*\mathrm{triv}_i \longrightarrow \pi_2^*\mathrm{triv}_i$ induces a functor

$$\mathscr{F}(g) : \mathcal{P}_1(Y^{[2]}) \longrightarrow \Lambda_i T.$$

Similarly, the modification $\psi : \mathrm{id}_{\mathrm{triv}_i} \longrightarrow \Delta^* g$ induces a natural transformation

$$\mathscr{F}(\psi) : \mathscr{F}(\mathrm{id}_{\mathrm{triv}_i}) \Longrightarrow \Delta^*\mathscr{F}(g),$$

and here we have used the commutativity of diagram (3.8). Finally, the modification f induces a natural transformation

$$\mathscr{F}(f) : \pi_{23}^*\mathscr{F}(g) \otimes \pi_{12}^*\mathscr{F}(g) \Longrightarrow \pi_{13}^*\mathscr{F}(g)$$

where we have again used the commutativity of diagram (3.8) and also the one of (3.9).

We have now converted a descent object into a 2-functor triv, a functor $\mathscr{F}(g)$ and two natural transformations $\mathscr{F}(\psi)$ and $\mathscr{F}(f)$. In order for the functor $\mathscr{F}(g)$ to qualify as a transport functor we need a Lie groupoid and a functor to its target category $\Lambda_i T$. This will be the functor

$$\Lambda i : \Lambda\mathrm{Gr} \longrightarrow \Lambda_i T,$$

and $\Lambda\mathrm{Gr}$ is indeed a Lie groupoid because Gr is a Lie 2-groupoid. Summarizing,

Definition 3.6. *A descent object* $(\mathrm{triv}, g, \psi, f)$ *is called* <u>*smooth*</u> *provided the 2-functor* $\mathrm{triv} : \mathcal{P}_2(Y) \longrightarrow \mathrm{Gr}$ *is smooth, the functor* $\mathscr{F}(g)$ *is a transport functor with* $\Lambda\mathrm{Gr}$*-structure and the natural transformations* ψ *and* f *are morphisms of transport functors.*

In the same way we qualify smooth descent 1-morphisms and descent 2-morphisms. A descent 1-morphism

$$(h, \epsilon) : (\mathrm{triv}, g, \psi, f) \longrightarrow (\mathrm{triv}', g', \psi', f')$$

is converted into a functor

$$\mathscr{F}(h) : \mathcal{P}_1(Y) \longrightarrow \Lambda_i T$$

and a natural transformation

$$\mathscr{F}(\epsilon) : \pi_2^*\mathscr{F}(h) \otimes \mathscr{F}(g) \Longrightarrow \mathscr{F}(g') \otimes \pi_1^*\mathscr{F}(h).$$

We call the descent 1-morphism (h, ϵ) *smooth*, provided the functor $\mathscr{F}(h)$ is a transport functor with $\Lambda\mathrm{Gr}$-structure and the natural transformation $\mathscr{F}(\epsilon)$ is a 1-morphism of transport functors. A descent 2-morphism $E : (h, \epsilon) \Longrightarrow (h', \epsilon')$ is converted into a natural transformation

$$\mathscr{F}(E) : \mathscr{F}(h) \Longrightarrow \mathscr{F}(h'),$$

and we call E *smooth*, provided the natural transformation $\mathscr{F}(E)$ is a 1-morphism of transport functors. Compositions of smooth descent 1-morphisms and smooth descent 2-morphisms are again smooth, so that we obtain a sub-2-category $\mathfrak{Des}_\pi^2(i)^\infty$ of $\mathfrak{Des}_\pi^2(i)$, called the *2-category of smooth descent data*. The following discussion shows that one can consistently characterize globally defined 2-functors by smooth descent data.

Using the equivalence Ex_π from Section 1.3 we obtain a sub-2-category $\mathrm{Triv}^2_\pi(i)^\infty$ of the 2-category $\mathrm{Triv}^2_\pi(i)$ of 2-functors with π-local i-trivialization consisting only of those objects, 1-morphisms and 2-morphisms whose associated descent data is smooth.

Lemma 3.7. *The 2-functors* Ex_π *and* Rec_π *restrict to equivalences of 2-categories,*

$$\mathrm{Triv}^2_\pi(i)^\infty \cong \mathfrak{Des}^2_\pi(i)^\infty.$$

Proof. First of all, it is clear that the restriction of Ex_π to $\mathrm{Triv}^2_\pi(i)^\infty$ is a 2-functor whose image is contained in $\mathfrak{Des}^2_\pi(i)^\infty$. To prove that the image of the restriction of Rec_π is contained in $\mathrm{Triv}^2_\pi(i)^\infty$ we have to show that $\mathrm{Ex}_\pi \circ \mathrm{Rec}_\pi$ restricts to an endo-2-functor of $\mathfrak{Des}^2_\pi(i)^\infty$. Indeed, by Lemma 2.8 this 2-functor is pseudonaturally equivalent to the identity, and going through the proof of this lemma shows that the components of the pseudonatural equivalence ρ we have constructed there are smooth descent 1-morphisms and smooth descent 2-morphisms.

Secondly, the pseudonatural equivalence $\eta : \mathrm{id}_{\mathrm{Triv}^2_\pi(i)} \longrightarrow \mathrm{Rec}_\pi \circ \mathrm{Ex}_\pi$ constructed in the proof of Lemma 2.9 has components $\eta(F)$ in *smooth* pseudonatural transformations and $\eta(A)$ in *smooth* modifications, i.e. those with smooth descent data. Namely, for a functor F with trivialization (π, t, triv) and the canonical trivialization (2.4) of 2-functors in the image of Rec_π, the descent 1-morphism corresponding to the pseudonatural transformation $\eta(F)$ is given by the pseudonatural transformation g of the descent object $(\mathrm{triv}, g, \psi, f)$ corresponding to F and a modification composed from the modifications f and ψ. The descent object is by assumption smooth, and so is $\eta(F)$. The same argument shows that the component $\eta(A)$ of a pseudonatural transformation $A : F \longrightarrow F'$ with smooth descent data is smooth. ☐

We have now generalized Theorem 2.1, the equivalence between 2-functors with local trivialization and descent objects, to the smooth case. This will be an important part of the equivalence between smooth descent data and transport 2-functors that we introduce in the following section.

3.3 Transport 2-Functors

Now we come to the main point of Section 3.

Definition 3.8. *Let M be a smooth manifold,* Gr *a strict Lie 2-groupoid, T a 2-category and $i : \mathrm{Gr} \longrightarrow T$ a 2-functor.*

1. *A transport 2-functor on M with* Gr*-structure is a 2-functor*

$$\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$$

such that there exists a surjective submersion $\pi : Y \longrightarrow M$ *and a* π*-local* i*-trivialization* (triv, t) *whose descent object* $\mathrm{Ex}_\pi(\mathrm{tra}, \mathrm{triv}, t)$ *is smooth.*

2. *A* <u>*transport transformation*</u> *between transport 2-functors* tra *and* tra$'$ *is a pseudonatural transformation*

$$A : \mathrm{tra} \longrightarrow \mathrm{tra}'$$

such that there exists a surjective submersion π *together with* π*-local* i*-trivializations of* tra *and* tra$'$ *for which the descent 1-morphism* $\mathrm{Ex}_\pi(A)$ *is smooth.*

3. *A* <u>*transport modification*</u> *is a modification* $B : A_1 \Longrightarrow A_2$ *such that the descent 2-morphism* $\mathrm{Ex}_\pi(B)$ *is smooth.*

Transport 2-functor $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ with Gr-structure, transport transformations and transport modifications form a 2-category that we denote by $\mathrm{Trans}^2_{\mathrm{Gr}}(M, T)$. We emphasize that in the structure of a transport 2-functor no surjective submersion or open cover is fixed: transport 2-functors are manifest *globally defined* structures.

We want to establish an equivalence between these globally defined transport 2-functors and their smooth descent data. For this purpose we remark that the 2-categories $\mathrm{Triv}^2_\pi(i)^\infty$ of 2-functors with smooth local trivializations and $\mathfrak{Des}^2_\pi(i)^\infty$ of smooth descent data form directed systems with respect to the surjective submersion $\pi : Y \longrightarrow M$ and refinements of those: surjective submersions $\zeta : Y' \longrightarrow Y$ such that $\pi' = \pi \circ \zeta$. Namely, for each such refinement ζ there are canonical 2-functors

$$\mathrm{res}_\zeta : \mathrm{Triv}^2_\pi(i)^\infty \longrightarrow \mathrm{Triv}^2_{\pi'}(i)^\infty \quad \text{and} \quad \mathrm{res}_\zeta : \mathfrak{Des}^2_\pi(i)^\infty \longrightarrow \mathfrak{Des}^2_{\pi'}(i)^\infty.$$

These 2-functors just pullback all the structure along the refinement map $\zeta : Y' \longrightarrow Y$. It is thus clear that they compose strictly for iterated refinements. Now we take the direct limit over all surjective submersions and their refinements. This direct limit is to be taken in the category of 2-categories, in order to make things as easiest as possible.

In general, suppose that $S(\pi)$ are 2-categories, one for each surjective submersion $\pi : Y \longrightarrow X$, and $F(\zeta) : S(\pi) \longrightarrow S(\pi')$ are 2-functors, one for each refinement $\zeta : Y' \longrightarrow Y$, such that $F(\zeta' \circ \zeta) = F(\zeta') \circ F(\zeta)$ for repeated refinements. In this situation, the direct limit is a 2-category

$$S_M := \lim_{\overrightarrow{\pi}} S(\pi)$$

together with 2-functors $G(\pi) : S(\pi) \longrightarrow S_M$ such that

(a) $G(\pi) = G(\pi') \circ F(\zeta)$ for every refinement $\zeta : Y' \longrightarrow Y$ and

(b) the following universal property is satisfied: for any other 2-category S' and 2-functors $G'(\pi) : S(\pi) \longrightarrow S'$ satisfying (a) there exists a unique 2-functor

$$C : S_M \longrightarrow S'$$

such that $G'(\pi) = C \circ G(\pi)$.

In the category of 2-categories, these (co)limits always exist and are uniquely determined up to strict equivalences of 2-categories.

In the present situation, we obtain 2-categories

$$\mathrm{Triv}^2(i)^\infty_M := \lim_{\overrightarrow{\pi}} \mathrm{Triv}^2_\pi(i)^\infty \quad \text{and} \quad \mathfrak{Des}^2(i)^\infty_M := \lim_{\overrightarrow{\pi}} \mathfrak{Des}^2_\pi(i)^\infty.$$

Since the 2-functors Ex_π and Rec_π commute with the 2-functors rec_ζ above, it is easy to deduce from the universal property and Lemma 3.7 that these two 2-categories are equivalent.

Next we want to show that the 2-categories $\mathrm{Triv}^2(i)^\infty_M$ and $\mathrm{Trans}^2_{\mathrm{Gr}}(M, T)$ are equivalent. From the universal property we obtain a unique 2-functor

$$v^\infty : \mathrm{Triv}^2(i)^\infty_M \longrightarrow \mathrm{Trans}^2_{\mathrm{Gr}}(X, T)$$

induced by $(\mathrm{tra}, \pi, \mathrm{triv}, t) \longmapsto \mathrm{tra}$, i.e. by forgetting the chosen trivialization. In order to prove that v^∞ is an equivalence we have to make a slight assumption on the 2-functor i. We call a 2-functor $i : \mathrm{Gr} \longrightarrow T$ *full and faithful*, if it induces an equivalence on Hom-categories. In particular, i is full and faithful if it is an equivalence of 2-categories, which is the case in all examples we are going to discuss.

Lemma 3.9. *Under the assumption that the 2-functor i is full and faithful, the 2-functor v^∞ is an equivalence of 2-categories.*

Proof. It is clear that an inverse functor w^∞ picks a given transport 2-functor and chooses a smooth local trivialization for some surjective submersion $\pi : Y \longrightarrow M$. It follows immediately that $v^\infty \circ w^\infty = \mathrm{id}$. It remains to construct a pseudonatural equivalence $\mathrm{id} \cong w^\infty \circ v^\infty$, i.e. a 1-isomorphism

$$A : (\mathrm{tra}, \pi, \mathrm{triv}, t) \longrightarrow (\mathrm{tra}, \pi', \mathrm{triv}', t')$$

in $\mathrm{Triv}^2(i)^\infty_M$, where the original π-local trivialization (triv, t) has been forgotten and replaced by a new π'-local trivialization (triv', t'). But since the 1-morphisms in $\mathrm{Triv}^2(i)^\infty_M$ are just pseudonatural transformation between the 2-functors ignoring the trivializations, we only have to prove that the identity pseudonatural transformation

$$A := \mathrm{id}_{\mathrm{tra}} : \mathrm{tra} \longrightarrow \mathrm{tra}$$

of a transport 2-functor tra has smooth descent data (h, ϵ) with respect to *any* two trivializations (π, triv, t) and $(\pi', \mathrm{triv}', t')$.

The first step is to choose a refinement $\zeta : Z \longrightarrow Y \times_M Y'$ of the common refinement of the to surjective submersions. One can choose Z such that is has contractible connected components. If $c : Z \times [0, 1] \longrightarrow Z$ is such a contraction, it defines for each point $z \in Z$ a path $c_z : z \longrightarrow z_k$ that moves z to the distinguished point z_k to which the component of Z that contains z is contracted. It further defines for each path $\gamma : z_1 \longrightarrow z_2$ a bigon $c_\gamma : \gamma \Longrightarrow c_{z_2}^{-1} \circ c_{z_1}$. Axiom (T2) for the pseudonatural transformation

$$h := t' \circ \bar{t} : \mathrm{triv}_i \longrightarrow \mathrm{triv}'_i$$

applied to the bigon c_γ yields the commutative diagram

$$\begin{array}{ccc}
h(z_2) \circ \mathrm{triv}_i(\gamma) & \xRightarrow{\;h(\gamma)\;} & \mathrm{triv}'_i(\gamma) \circ h(z_1) \\
{\scriptstyle \mathrm{id} \circ \mathrm{triv}_i(c_\gamma)} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \mathrm{triv}'_i(c_\gamma) \circ \mathrm{id}} \\
h(z_2) \circ \mathrm{triv}_i(c_{z_2}^{-1} \circ c_{z_1}) & \xRightarrow[h(c_{z_2}^{-1} \circ c_{z_1})]{} & \mathrm{triv}'_i(c_{z_2}^{-1} \circ c_{z_1}) \circ h(z_1).
\end{array}$$

Notice that the 1-morphisms $h(z_j) : \mathrm{triv}_i(z_j) \longrightarrow \mathrm{triv}'_i(z_j)$ have by assumption preimages $\kappa_j : \mathrm{triv}(z_j) \longrightarrow \mathrm{triv}'(z_j)$ under i in Gr, and that the 2-morphism $h(c_{z_2}^{-1} \circ c_{z_1})$ also has a preimage Γ in Gr. Thus,

$$h(\gamma) = i\left((\mathrm{triv}'(c_\gamma) \circ \mathrm{id})^{-1} \bullet \Gamma \bullet (\mathrm{id} \circ \mathrm{triv}(c_\gamma))\right).$$

This is nothing but the Wilson line $\mathcal{W}_{z_1,z_2}^{\mathscr{F}(h),\Lambda i}$ of the functor $\mathscr{F}(h)$ and it is smooth since triv and triv$'$ are smooth 2-functors. Hence, by Theorem 3.12 in [SW07], $\mathscr{F}(h)$ is a transport functor with ΛGr-structure.

It remains to prove that the modification $\epsilon : \pi_2^* h \circ g \Longrightarrow g' \circ \pi_1^* h$ induces a morphism $\mathscr{F}(\epsilon)$ of transport functors. This simply follows from the general fact that under the assumption that the functor $i : \mathrm{Gr} \longrightarrow T$ is full, every natural transformation η between transport functors with Gr-structure is a morphism of transport functors. We have not shown this in [SW07] but it can easily be deduced from the naturality conditions on trivializations t and t' and on η, evaluated for paths with a fixed starting point. $\square$

The final consequence of the latter lemma is the following important result on transport 2-functors.

Theorem 3.10. *Let M be a smooth manifold, and let $i : \mathrm{Gr} \longrightarrow T$ be a full and faithful 2-functor. There is a canonical equivalence*

$$\mathrm{Trans}^2_{\mathrm{Gr}}(M, T) \cong \mathfrak{Des}^2(i)^\infty_M$$

between the 2-category of globally defined transport 2-functors on M and the 2-category of smooth descent data.

In the following we introduce several features of transport 2-functors, which make contact between the abstract setting and some more concrete notions.

Operations on Transport 2-Functors. It is straightforward to see that transport 2-functors allow a list of natural operations.

1. *Pullbacks*: Let $f : M \longrightarrow N$ be a smooth map. The pullback $f^*\mathrm{tra}$ of any transport 2-functor on N is a transport 2-functor on M.

2. *Tensor products*: Let $\otimes : T \times T \longrightarrow T$ be a monoidal structure on a 2-category T. For transport 2-functors $\mathrm{tra}_1, \mathrm{tra}_2 : \mathcal{P}_2(M) \longrightarrow T$ with Gr-structure, the pointwise tensor product $\mathrm{tra}_1 \otimes \mathrm{tra}_2 : \mathcal{P}_2(M) \longrightarrow T$ is again a transport 2-functor with Gr-structure, and makes the 2-category $\mathrm{Trans}^2_{\mathrm{Gr}}(M,T)$ a monoidal 2-category.

3. *Change of the target 2-category*: Let T and T' be two target 2-categories equipped with 2-functors $i : \mathrm{Gr} \longrightarrow T$ and $i' : \mathrm{Gr} \longrightarrow T'$, and let $F : T \longrightarrow T'$ be a 2-functor together with a pseudonatural equivalence
$$\rho : F \circ i \longrightarrow i'.$$
If $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ is a transport 2-functor with Gr-structure, $F \circ \mathrm{tra}$ is also a transport 2-functor with Gr-structure. In particular, this is the case for $i' := F \circ i$ and $\rho = \mathrm{id}$.

4. *Change of the structure 2-groupoid*: Let $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ be a transport 2-functor with Gr-structure, for a 2-functor $i : \mathrm{Gr} \longrightarrow T$ which is a composition
$$\mathrm{Gr} \xrightarrow{F} \mathrm{Gr}' \xrightarrow{i'} T$$
in which F is a smooth 2-functor. Then, tra is also a transport 2-functor with Gr'-structure, since for any local i-trivialization (triv, t) of tra we have a local i'-trivialization $(F \circ \mathrm{triv}, t)$. Conversely, if $\mathrm{tra}' : \mathcal{P}_2(M) \longrightarrow T$ is a transport 2-functor with Gr'-structure, it is *not* necessarily a transport 2-functor with Gr-structure.

Structure Lie 2-Groups. As we have described in Section 3.1, a Lie 2-group $\mathfrak{G}$ gives rise to a Lie 2-groupoid $\mathcal{B}\mathfrak{G}$, and hence to important examples of structure 2-groupoids. Transport 2-functors with $\mathcal{B}\mathfrak{G}$-structure play the role of *gerbes with connection*, as Section 4 will prove. The Lie 2-group $\mathfrak{G}$ is the *structure 2-group* of these gerbes. In the following we list important examples of such structure 2-groups.

(a) Let A be an abelian Lie group. A smooth crossed module is defined by $G = \{1\}$ and $H := A$. This fixes the maps to $t(a) := 1$ and $\alpha(1, a) := a$. Notice that axiom b) is only satisfied because A is abelian. The associated Lie 2-group $\mathfrak{G}$ is denoted by $\mathcal{B}A$. Transport 2-functors with $\mathcal{B}\mathcal{B}A$-structure play the role of *abelian gerbes with connection*, see Section 4.2.

(b) Let G be any Lie group. A smooth crossed module is defined by $H := G$, $t = \mathrm{id}$ and $\alpha(g,h) := ghg^{-1}$. The associated Lie 2-group is denoted by $\mathcal{E}G$. This notation is devoted to the fact that the geometric realization of the nerve of the category $\mathcal{E}G$ yields the universal G-bundle EG. Transport 2-functors with $\mathcal{B}\mathcal{E}G$-structure arise as the *curvature of transport 1-functors*, see Section 3.4.

(c) Let H be a connected Lie group, so that the group of Lie group automorphisms of H is again a Lie group $G := \mathrm{Aut}(H)$. The definitions $t(h)(x) := hxh^{-1}$ and $\alpha(\varphi, h) := \varphi(h)$ yield a smooth crossed module whose associated Lie 2-group $\mathfrak{G}$ is denoted by $\mathrm{AUT}(H)$. Transport 2-functors with $\mathcal{B}\mathrm{AUT}(H)$-structure play the role *non-abelian gerbes with connection*, see Section 4.3.

(d) Let

$$1 \longrightarrow N \xrightarrow{t} H \xrightarrow{p} G \longrightarrow 1$$

be an exact sequence of Lie groups denoted by $\mathfrak{N}$. There is a canonical action α of H on N defined by requiring

$$t(\alpha(h,n)) = ht(n)h^{-1}.$$

This defines a smooth crossed module, whose associated Lie 2-group we also denote $\mathfrak{N}$. Transport 2-functors with $\mathcal{B}\mathfrak{N}$-structure correspond to (non-abelian) *lifting gerbes*. They generalize the abelian lifting gerbes [Bry93, Mur96] for central extensions to arbitrary short exact sequences of Lie groups.

Transgression to Loop Spaces. Let us briefly indicate that transport 2-functors on a smooth manifold M induce tautologically structure on the loop space LM. This comes from the fact that there is a canonical diffeological functor

$$\ell : \mathcal{P}_1(LM) \longrightarrow \Lambda\mathcal{P}_2(M)$$

expressing the fact that a point in LM is just a particular path in M, and that a path in LM is just a particular bigon in M [SW08]. The composition of ℓ with

$$\Lambda\mathrm{tra} : \Lambda\mathcal{P}_2(M) \longrightarrow \Lambda_{\mathrm{tra}}T$$

yields a functor

$$\mathrm{Tgr}(\mathrm{tra}) := \Lambda\mathrm{tra} \circ \ell : \mathcal{P}_1(LM) \longrightarrow \Lambda_{\mathrm{tra}}T$$

that we call the *transgression of* tra *to the loop space*. In order to cut the discussion of the functor Tgr(tra) short we make two simplifying assumptions:

1. We assume that there exists a surjective submersion $\pi : Y \longrightarrow M$ for which tra admits smooth local trivializations and for which $L\pi : LY \longrightarrow LM$ is also a surjective submersion.

2. We assume that the target 2-category T is strict, so that ΛT is the target category of the functor $\mathrm{Tgr}(\mathrm{tra})$.

We also restrict the following consideration to the based loop space $\Omega_p M$, for $p \in M$ any point, and identify $\mathrm{Tgr}(\mathrm{tra})$ with its pullback along the embedding $\iota_p : \Omega_p M \longrightarrow LM$.

Proposition 3.11. *Let* $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ *be a transport 2-functor with* Gr*-structure such that the two simplifying assumptions above are satisfied. Then,*

$$\mathrm{Tgr}(\mathrm{tra}) : \mathcal{P}_1(\Omega_p M) \longrightarrow \Lambda T$$

is a transport functor with $\Lambda\mathrm{Gr}$*-structure.*

Proof. Let $t : \pi^*\mathrm{tra} \longrightarrow \mathrm{triv}_i$ be a π-local i-trivialization of tra for π a surjective submersion satisfying the simplifying assumption. A local trivialization $\tilde{t}$ of $\mathrm{Tgr}(\mathrm{tra})$ is given by

$$\begin{array}{ccc}
\mathcal{P}_1(LY) & \xrightarrow{(L\pi)_*} & \mathcal{P}_1(LM) \\
{\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle \ell} \\
\Lambda\mathcal{P}_2(Y) & \xrightarrow{\pi_*} & \Lambda\mathcal{P}_2(M) \\
{\scriptstyle \Lambda\mathrm{triv}}\downarrow & \swarrow\!\!\swarrow_{\Lambda t} & \downarrow{\scriptstyle \Lambda\mathrm{tra}} \\
\Lambda\mathrm{Gr} & \xrightarrow[\Lambda i]{} & \Lambda T
\end{array}$$

in which the upper subdiagram is commutative on the nose. If $g : \pi_1^*\mathrm{triv}_i \longrightarrow \pi_2^*\mathrm{triv}_i$ is the pseudonatural transformation in the smooth descent object $\mathrm{Ex}_\pi(\mathrm{tra}, t, \mathrm{triv})$, and $\tilde{g}$ is the natural transformation in the descent object $\mathrm{Ex}_\pi(\mathrm{Tgr}(\mathrm{tra}), \tilde{t}, \ell^*\Lambda\mathrm{triv})$ associated to the above trivialization, we find

$$\tilde{g} = \ell^*\Lambda g.$$

Since $\mathscr{F}(g)$ is a transport 2-functor with $\Lambda\mathrm{Gr}$-structure, it has smooth Wilson lines [SW07]: for a fixed point $\alpha \in Y^{[2]}$ there exists a smooth natural transformation $g' : \pi_1^*\ell^*\Lambda\mathrm{triv} \longrightarrow \pi_2^*\ell^*\Lambda\mathrm{triv}$ with $g = i(g')$. This shows that $\tilde{g}$ factors through a smooth natural transformation $\ell^*\Lambda g'$, so that $\mathrm{Tgr}(\mathrm{tra})$ is a transport functor. □

With a view to the equivalence of Theorem 3.1 between transport functors and fibre bundles with connection, this means that transport 2-functors on a manifold M naturally induce fibre bundles with connection on the loop space LM. In general, these are so-called groupoid bundles [MM03, SW07] with connection, whose structure groupoid is $\Lambda\mathrm{Gr}$. However, in the abelian case, i.e. $\mathrm{Gr} = \mathcal{BB}A$ for an abelian Lie group A, we find $\Lambda\mathrm{Gr} \cong \mathcal{B}A$ (see Lemma 4.7 below), so that the transgression $\mathrm{Tgr}(\mathrm{tra})$ is a principal A-bundle with connection over $\Omega_p M$.

Curvature Forms. Suppose $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ is a transport 2-functor with $\mathcal{B}\mathfrak{G}$-structure, for $\mathfrak{G}$ some Lie 2-group coming from a smooth crossed module (G, H, t, α). Since such 2-functors play the role of gerbes with connection, one wants to assign a *3-form curvature* to tra. Since we also capture non-abelian gerbes, it is not to be expected that the curvature will be a globally defined 3-form on the base manifold M.

However, since transport 2-functors have a manifest local behaviour, it is easy to produce a locally defined 3-form. Let $\pi : Y \longrightarrow M$ be a surjective submersion, and let (triv, t) be a π-local trivialization associated to which we find a smooth descent object. In particular, we have a smooth 2-functor

$$\mathrm{triv} : \mathcal{P}_2(Y) \longrightarrow \mathcal{B}\mathfrak{G},$$

which corresponds according to Theorem 3.5 to a pair (A, B) of a 1-form $A \in \Omega^1(Y, \mathfrak{g})$ and a 2-form $B \in \Omega^2(Y, \mathfrak{h})$, for $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. The curvature of tra is now defined (see Remark A.12 in [SW08]) to be the 3-form

$$\mathrm{curv}(\mathrm{tra}) = \mathrm{d}B + \alpha_*(A \wedge B) \in \Omega^3(Y, \mathfrak{h}). \tag{3.10}$$

We recall that we proposed to call a 2-functor $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ flat if it factors through the projection $\mathcal{P}_2(M) \longrightarrow \Pi_2(M)$ of thin homotopy classes of bigons to homotopy classes. Now we obtain

Proposition 3.12. *Suppose that the 2-functor $i : \mathcal{B}\mathfrak{G} \longrightarrow T$ is injective on 2-morphisms. A transport 2-functor* $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ *with $\mathcal{B}\mathfrak{G}$-structure is flat if and only if its local curvature 3-form* $\mathrm{curv}(\mathrm{tra}) \in \Omega^3(Y, \mathfrak{h})$ *with respect to any smooth local trivialization vanishes.*

Proof. We proceed in two parts. (a): curv(tra) vanishes if and only if triv is a flat 2-functor, and (b): tra is flat if and only if triv is flat. The claim (a) follows from Lemma A.11 in [SW08]. To see (b) consider two bigons $\Sigma_1 : \gamma \Longrightarrow \gamma'$ and $\Sigma_2 : \gamma \Longrightarrow \gamma'$ in Y which are smoothly homotopic so that they define the same element in $\Pi_2(Y)$. Suppose tra is flat and let $\Sigma := \Sigma_2^{-1} \bullet \Sigma_1$. Axiom (T2) for the trivialization t is then

$$\begin{array}{ccc} t(y) \circ \pi^*\mathrm{tra}(\gamma) & \xRightarrow{\;t(\gamma)\;} & \mathrm{triv}_i(\gamma) \circ t(x) \\ {\scriptstyle \mathrm{id}_{t(y)} \circ \pi^*\mathrm{tra}(\Sigma)} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \mathrm{triv}_i(\Sigma) \circ \mathrm{id}_{t(x)}} \\ t(y) \circ \pi^*\mathrm{tra}(\gamma) & \xRightarrow[\;t(\gamma)\;]{} & \mathrm{triv}_i(\gamma) \circ t(x) \end{array}$$

and since $\pi^*\mathrm{tra}(\Sigma) = \mathrm{id}$ by assumption it follows that $\mathrm{triv}_i(\Sigma) = \mathrm{id}$, i.e. triv is flat. Conversely, assume that triv is flat. The latter diagram shows that tra is then flat on all bigons in the image of π_*. This is actually enough: let $h : [0, 1]^3 \longrightarrow M$ be a smooth homotopy between two bigons Σ_1 and Σ_2 which are not in the image of π_*. Like explained in Appendix A.3 of [SW08] the cube $[0,1]^3$ can be decomposed into small cubes such that h restricts to smooth homotopies between small bigons that bound these cubes. The decomposition can be chosen so small that each of these bigons is contained in the image of π_*, so that tra assigns the same value to the source and the target bigon of each small cube. By 2-functorality of tra, this infers $\mathrm{tra}(\Sigma_1) = \mathrm{tra}(\Sigma_2)$. ☐

We hence see that the two notions of flatness, namely the one given on the level of 2-functors, and the one given on the level of differential forms, coincide. It is, however, clear that the first notion is much more general: it makes sense for structure Lie 2-groupoid Gr which are not of the form $\mathrm{Gr} = \mathcal{B}\mathfrak{G}$, and even for any 2-functor defined on the path 2-groupoid of a smooth manifold M, without putting smoothness conditions on the 2-functor itself.

3.4 An Example: Curvature 2-Functors

If P is a principal G-bundle with connection ω over M, one can compare the parallel transport maps along two paths $\gamma_1, \gamma_2 : x \longrightarrow y$,

$$P_x \underset{\tau_{\gamma_2}}{\overset{\tau_{\gamma_1}}{\rightrightarrows}} P_y,$$

by an automorphism of P_y, namely the holonomy around the loop $\gamma_2 \circ \gamma_1^{-1}$,

$$\tau_{\gamma_2} = \mathrm{Hol}_\nabla(\gamma_2 \circ \gamma_1^{-1}) \circ \tau_{\gamma_1}.$$

If the paths γ_1 and γ_2 are the source and the target of a bigon $\Sigma : \gamma_1 \Longrightarrow \gamma_2$, this holonomy is immediately related to the curvature of ∇. So, a principal G-bundle with connection does not only assign fibres P_x to points $x \in M$ and parallel transport maps τ_γ to paths, it also assigns a curvature-related quantity to bigons Σ.

Under the equivalence between principal G-bundles with connection and transport functors on X with $\mathcal{B}G$-structure (Theorem 3.2), the principal bundle (P,ω) corresponds to the transport functor

$$\mathrm{tra}_P : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$$

that assigns the fibres P_x to points $x \in M$ and the parallel transport maps τ_γ to paths γ. Adding an assignment for bigons is supposed to yields a „curvature 2-functor"

$$K(\mathrm{tra}_P) : \mathcal{P}_2(M) \longrightarrow \widehat{G\text{-Tor}}$$

where $\widehat{G\text{-Tor}}$ is the category G-Tor regarded as a strict 2-category with a unique 2-morphism between each pair of 1-morphisms. The uniqueness of the 2-morphisms expresses the fact that the curvature is already determined by the parallel transport.

The goal of this section is to define a curvature 2-functor associated to any transport functor, and to prove that these are transport 2-functors. This procedure is able to capture the curvature of connections on principal bundles, but is in principle more general.

We start with a given transport functor $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow T$ with $\mathcal{B}G$-structure for some Lie group G and some functor $i : \mathcal{B}G \longrightarrow T$. We recall from [SW07] that this means that there exists a surjective submersion $\pi : Y \longrightarrow M$, a functor $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathcal{B}G$ and a natural equivalence

$$t : \pi^*\mathrm{tra} \longrightarrow \mathrm{triv}_i$$

such that its descent data is smooth: the functor triv is smooth, and the natural transformation $g : \pi_1^*\mathrm{triv}_i \longrightarrow \pi_2^*\mathrm{triv}_i$ factors through a smooth natural transformation $\tilde{g} : \pi_1^*\mathrm{triv} \longrightarrow \pi_2^*\mathrm{triv}$, i.e. $g(\alpha) = i(\tilde{g}(\alpha))$ for every $\alpha \in Y^{[2]}$.

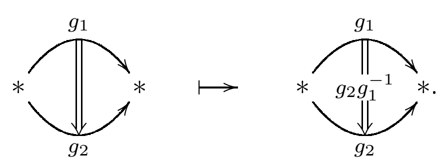
The curvature 2-functor associated to tra is the strict 2-functor

$$K(\mathrm{tra}) : \mathcal{P}_2(M) \longrightarrow \widehat{T}$$

which does on objects and 1-morphisms the same as tra and is on 2-morphisms determined by the fact that $\widehat{T}$ has a only one 2-morphism between each pair of 1-morphisms. In the same way, we obtain a strict 2-functor

$$K(i) : \widehat{\mathcal{B}G} \longrightarrow \widehat{T}$$

which sends the unique 2-morphisms on the left hand side to the unique ones on the right. We observe that the Lie 2-groupoids $\widehat{\mathcal{B}G}$ and $\mathcal{BE}G$ are canonically isomorphic under the assignment



Now we can check that

Lemma 3.13. *The curvature 2-functor* $K(\mathrm{tra})$ *is a transport 2-functor with* $\mathcal{BE}G$*-structure.*

Proof. We construct a local trivialization of $K(\mathrm{tra})$ starting with a local trivialization (triv, t) of tra with respect to some surjective submersion $\pi : Y \longrightarrow M$. Let $\mathrm{dtriv} : \mathcal{P}_2(Y) \longrightarrow \mathcal{BE}G$ be the derivative 2-functor associated to triv [SW08]: on objects and 1-morphisms it is given by triv, and it sends every bigon $\Sigma : \gamma_1 \Longrightarrow \gamma_2$ in Y to the unique 2-morphism in $\mathcal{BE}G$ between the images of γ_1 and γ_2 under triv. A pseudonatural equivalence

$$K(t) : \pi^*K(\mathrm{tra}) \longrightarrow K(i) \circ \mathrm{dtriv}$$

is defined as follows. Its component at a point $a \in Y$ is the 1-morphism

$$K(t)(a) := t(a) : \mathrm{tra}(\pi(a)) \longrightarrow i(*)$$

in T. Its component $t(\gamma)$ at a path $\gamma : a \longrightarrow b$ is the unique 2-morphism in $\widehat{T}$. Notice that since t is a natural transformation, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{tra}(\pi(a)) & \xrightarrow{\mathrm{tra}(\pi(\gamma))} & \mathrm{tra}(\pi(b)) \\ {\scriptstyle t(a)}\downarrow & & \downarrow{\scriptstyle t(b)} \\ i(*) & \xrightarrow[\mathrm{triv}_i(\gamma)]{} & i(*) \end{array}$$

meaning that $t(\gamma) = \mathrm{id}$. This defines the pseudonatural transformation t as required.

Now we assume that the descent data (triv, g_t) associated to the local trivialization (triv, t) is smooth, and show that then also the descent object $(\mathrm{dtriv}, g_{K(t)}, \psi, f)$ is smooth. As observed in [SW08], the derivative 2-functor dtriv is smooth if and only if triv is smooth. To extract the remaining descent data according to the procedure described in Section 1.3, we have to choose a weak inverse $\overline{K(t)}$ of the trivialization $t(K)$. It is clear that for t^{-1} the natural transformation inverse to t, $\overline{K(t)} := K(t^{-1})$ is even a strict inverse. This means that the 2-isomorphisms i_t and j_t are identities, and in turn, the modifications ψ and f are identities. The only non-trivial descent datum is the pseudonatural transformation

$$g_{K(t)} : \pi_1^*\mathrm{dtriv}_{K(i)} \longrightarrow \pi_2^*\mathrm{dtriv}_{K(i)}.$$

Its component at a point $\alpha \in Y^{[2]}$ is given by $g_{K(t)}(\alpha) := g_t(\alpha)$, and its component at some path $\Theta : \alpha \longrightarrow \alpha'$ is again the identity.

The last step is to show that

$$\mathscr{F}(g_{K(t)}) : \mathcal{P}_1(Y^{[2]}) \longrightarrow \Lambda_{K(i)}\widehat{T}$$

is a transport functor with $\Lambda\mathcal{BE}G$-structure. To do so we have to find a local trivialization with smooth descent data. This is here particulary simple: the functor $\mathscr{F}(g_{K(t)})$ is *globally* trivial in the sense that it factors through the functor

$$\Lambda K(i) : \Lambda\mathcal{BE}G \longrightarrow \Lambda_{K(i)}\widehat{T}.$$

To see this we use the smoothness condition on the natural transformation g_t, namely that it factors through a smooth natural transformation $\tilde{g}_t$. We obtain a smooth pseudonatural transformation $\tilde{g}_{K(t)} : \pi_1^*\mathrm{dtriv} \longrightarrow \pi_2^*\mathrm{dtriv}$ such that $g_{K(t)} = K(i)(\tilde{g}_{K(t)})$. This finally gives us

$$\mathscr{F}(g_{K(t)}) = \Lambda K(i) \circ \mathscr{F}(\tilde{g}_{K(t)})$$

meaning that $\mathscr{F}(g_{K(t)})$ is a transport functor with $\Lambda\mathcal{BE}G$-structure. □

We have now obtained a first example of a transport 2-functor. In terms of gerbes, it is a non-abelian gerbe with structure 2-group $\mathcal{BE}G$, and is hence neither equivalent to an abelian or non-abelian bundle gerbe nor to a Breen-Messing gerbe. In the remainder of this section we collect some properties of curvature 2-functors.

Since the value of the curvature 2-functor $K(\mathrm{tra})$ on bigons does not depend on the bigon itself but only on its source and target path, it is in particular independent of the thin homotopy classes of the bigon. Hence,

Proposition 3.14. *The curvature 2-functor $K(\mathrm{tra})$ associated to any transport functor is flat.*

This proposition gains a very nice interpretation when we relate the curvature of a connection ω in a principal G-bundle $p : P \longrightarrow M$ to the curvature 2-functor $K(\mathrm{tra}_P)$ associated to the corresponding transport functor tra_P. We identify the curvature of ω with a 2-form $\mathrm{curv}(\omega) \in \Omega^2(P, \mathfrak{g})$.

Lemma 3.15. *The curvature 2-functor $K(\mathrm{tra}_P) : \mathcal{P}_2(M) \longrightarrow \widehat{G\text{-}\mathrm{Tor}}$ has a canonical smooth p-local trivializations (p, t, triv). If $B \in \Omega^2(P, \mathfrak{g})$ is the 2-form associated to* triv *by Theorem 3.5,*

$$B = \mathrm{curv}(\omega).$$

Proof. As described in detail in Section 5.1 of [SW07], tra_P admits local trivializations with respect to the surjective submersion $p : P \longrightarrow M$ and with smooth descent data (triv', g) such that the connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ of the bundle P corresponds to the smooth functor $\mathrm{triv}' : \mathcal{P}_1(P) \longrightarrow \mathcal{B}G$ under the bijection of Theorem 3.2. Then, by Lemma 3.5 in [SW08], the 2-form B' associated to dtriv' is given by

$$B' = \mathrm{d}\omega + [\omega \wedge \omega].$$

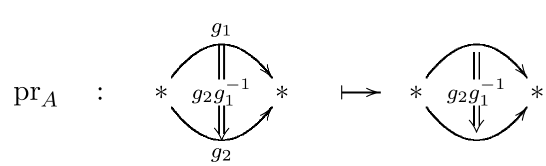
This is indeed the curvature of the connection ω. □

The announced interpretation of Proposition 3.14 is now as follows: using Lemma 3.15 one can now calculate the 3-form curvature (3.10) $\mathrm{curv}(K(\mathrm{tra}_P))$ of the curvature 2-functor of tra_P. The calculation involves the second Bianchi identity for the connection ω on the principal G-bundle P, and the result is

$$\mathrm{curv}(K(\mathrm{tra}_P)) = 0,$$

which is according to Proposition 3.12 an independent proof of Proposition 3.14. In other words, Proposition 3.14 is equivalent to the second Bianchi identity for connections on fibre bundles.

In case that G is an abelian Lie group A the situation is simplified by the fact that there exists a canonical smooth 2-functor $\mathrm{pr}_A : \mathcal{BE}A \longrightarrow \mathcal{BB}A$ given by



The composition of pr_A with $K(i)$ yields a 2-functor $\mathcal{B}\mathcal{B}A \longrightarrow \widehat{T}$. We leave it to the reader to prove the following lemma.

Lemma 3.16. *If* $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow T$ *is a transport functor with* $\mathcal{B}A$*-structure, the curvature 2-functor* $K(\mathrm{tra})$ *is a globally trivial transport 2-functor with* $\mathcal{B}\mathcal{B}A$*-structure.*

As a consequence, if P is a principal A-bundle over M with connection ω, its curvature $\mathrm{curv}(\omega) \in \Omega^2(M, \mathfrak{a})$ is precisely the 2-form which corresponds to $K(\mathrm{tra}_P)$ under the bijection of Theorem 3.5.

4 Relation to Gerbes with Connection

We have now developed the general theory of transport 2-functors. In this section, we reduce it to special cases by picking particular target 2-categories T, structure 2-groups $\mathfrak{G}$ and appropriate 2-functors

$$i : \mathcal{B}\mathfrak{G} \longrightarrow T.$$

We claim that *every* reasonable concept of a „2-bundle with connection“ can be obtained like this. We provide proofs of this claim for differential cocycles arising from Breen-Messing gerbes [BM05] in Section 4.1, for abelian bundle gerbes [Mur96] in Section 4.2 and for non-abelian bundle gerbes [ACJ05] in Section 4.3. Section 4.4 contains an outlook on further relations between transport 2-functors and 2-bundles with connection, in particular string 2-bundles.

4.1 Differential non-abelian Cohomology

Let $\mathfrak{G}$ be a Lie 2-group. In this section we consider transport 2-functors

$$\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow \mathcal{B}\mathfrak{G} \tag{4.1}$$

with $\mathcal{B}\mathfrak{G}$-structure, for $i := \mathrm{id}_{\mathcal{B}\mathfrak{G}} : \mathcal{B}\mathfrak{G} \longrightarrow \mathcal{B}\mathfrak{G}$ the identity 2-functor. Notice that such transport 2-functors can be produced from a transport 2-functor $\widetilde{\mathrm{tra}}$ with $\mathcal{B}\mathfrak{G}$-structure and target 2-category T, whenever the 2-functor $\widetilde{i} : \mathcal{B}\mathfrak{G} \longrightarrow T$ is an equivalence of 2-categories. This is the case in all examples that appear in this article. Then, for $F : T \longrightarrow \mathcal{B}\mathfrak{G}$ a weak inverse to $\widetilde{i}$, the 2-functor $F \circ \widetilde{\mathrm{tra}}$ is a transport 2-functor (4.1) according to Section 3.3.

In this section we prove that the descent 2-category $\mathfrak{Des}^2_\pi(\mathrm{id}_{\mathcal{B}\mathfrak{G}})^\infty$ can be replaced by a 2-category of *degree two differential* $\mathfrak{G}$*-cocycles*, whenever the surjective submersion $\pi : Y \longrightarrow M$ is *two-contractible*: both Y and the two-fold fibre product $Y^{[2]}$ have contractible connected components. For example, any good open cover of M defines such a two-contractible surjective submersion. The differential cocycles we want to substitute for the descent data are very concrete objects: they consist solely of ordinary smooth functions and differential forms defined on Y an fibre products of Y. One might thus consider differential cocycles as „local data“ of transport 2-functors. Degree two differential $\mathfrak{G}$-cocycles have first been considered in [BS07]. We will here retrieve their definition in a systematical way. In order to convert descent data into such smooth functions and differential forms, we use the 2-functor $\mathcal{D}$ from (3.5),

$$\mathcal{D} : \mathrm{Funct}^{\infty}(\mathcal{P}_2(X), \mathcal{B}\mathfrak{G}) \longrightarrow Z^2_X(\mathfrak{G})^{\infty},$$

which is an isomorphism of 2-categories, see Theorem 3.5.

Let us start with a smooth descent object $(\mathrm{triv}, g_0, \psi_0, f_0)$. It contains a smooth 2-functor $\mathrm{triv} : \mathcal{P}_2(Y) \longrightarrow \mathcal{B}\mathfrak{G}$ and a pseudonatural transformation

$$g_0 : \pi_1^*\mathrm{triv}_i \longrightarrow \pi_2^*\mathrm{triv}_i$$

whose associated functor $\mathscr{F}(g_0)$ is a transport functor over the contractible space $Y^{[2]}$. By Corollary 3.13 in [SW07] we can hence assume that g_0 is equivalent to a *smooth* pseudonatural transformation $g_\infty : \pi_1^*\mathrm{triv} \longrightarrow \pi_2^*\mathrm{triv}$. Similarly, the modifications ψ_0 and f_0 induce *smooth* modifications ψ_∞ and f_∞. Now we apply the 2-functor $\mathcal{D}$ to all this structure and obtain

(a) an object $(A, B) := \mathcal{D}(\mathrm{triv})$ in $Z^2_Y(\mathfrak{G})^\infty$, i.e. differential forms $A \in \Omega^1(Y, \mathfrak{g})$ and $B \in \Omega^2(Y, \mathfrak{h})$ satisfying relation (3.1).

(b) a 1-morphism

$$(g, \varphi) := \mathcal{D}(g_\infty) : \pi_1^*(A, B) \longrightarrow \pi_2^*(A, B)$$

in $Z^2_{Y^{[2]}}(\mathfrak{G})^\infty$, i.e. a smooth function $g : Y^{[2]} \longrightarrow G$ and a 1-form $\varphi \in \Omega^1(Y^{[2]}, \mathfrak{h})$ satisfying the relations (3.2) and (3.3).

(c) a 2-morphism

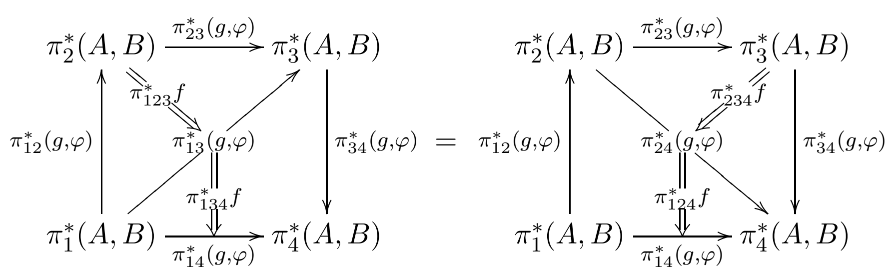
$$f := \mathcal{D}(f_\infty) : \pi_{23}^*(g, \varphi) \circ \pi_{12}^*(g, \varphi) \Longrightarrow \pi_{13}^*(g, \varphi)$$

in $Z^2_{Y^{[3]}}(\mathfrak{G})^\infty$ and a 2-morphism

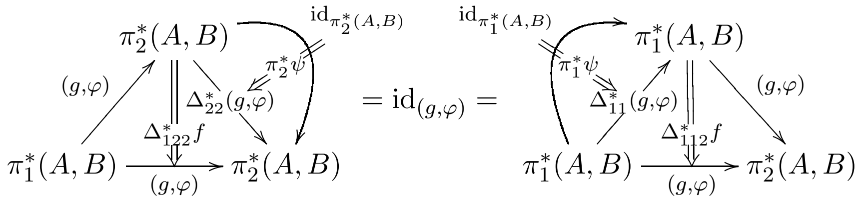
$$\psi := \mathcal{D}(\psi_\infty) : \mathrm{id}_{(A,B)} \Longrightarrow \Delta^*(g, \varphi)$$

in $Z^2_Y(\mathfrak{G})^\infty$; these are smooth functions $f : Y^{[3]} \longrightarrow H$ and $\psi : Y \longrightarrow H$ satisfying relations (3.4).

Furthermore, the two conditions (1.1) and (1.2) on descent objects translate into corresponding conditions, which are, expressed by pasting diagrams



and



The collection (a), (b), (c) satisfying these two relations is called a *differential $\mathfrak{G}$-cocycle in degree two*. Notice that the diagrams above still involve the composition laws of the 2-categories $Z^2_{Y^{[4]}}(\mathfrak{G})^\infty$ and $Z^2_{Y^{[2]}}(\mathfrak{G})^\infty$, respectively. We will write out all relations in a second step on the next page.

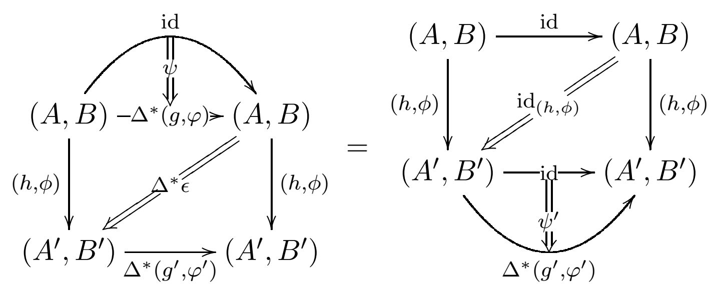
First we proceed similarly with a descent 1-morphism. The result is a *1-morphism between differential $\mathfrak{G}$-cocycles in degree two*: a 1-morphism

$$(h,\phi) : (A,B) \longrightarrow (A',B')$$

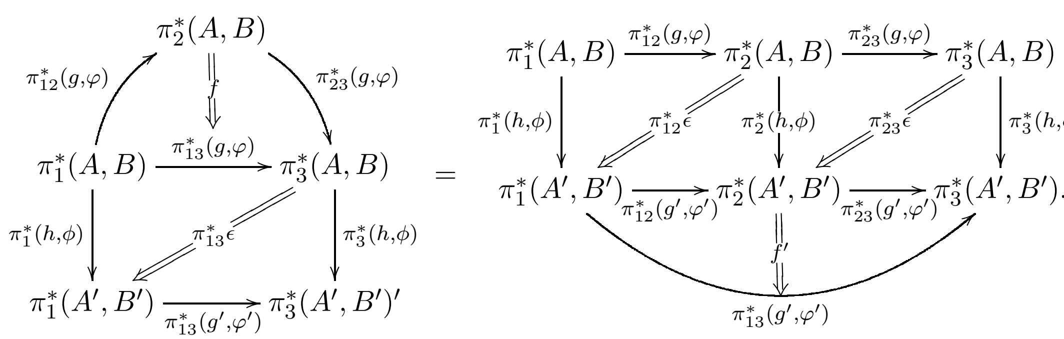
in $Z^2_Y(\mathfrak{G})^\infty$, i.e. a smooth function $h : Y \longrightarrow G$ and a 1-form $\phi \in \Omega^1(Y,\mathfrak{h})$ satisfying relations (3.2) and (3.3), and a 2-morphism

$$\epsilon : \pi_2^*(h,\phi) \circ (g,\varphi) \implies (g',\varphi') \circ \pi_1^*(h,\phi)$$

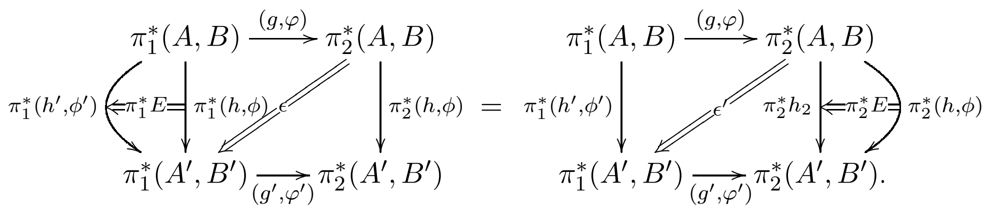
in $Z^2_{Y^{[2]}}(\mathfrak{G})^\infty$, i.e. a smooth function $\epsilon : Y^{[2]} \longrightarrow H$ satisfying (3.4). Conditions (1.3) and (1.4) for descent 1-morphisms result in the identities



and



Finally, a descent 2-morphism induces a 2-morphism $E : (h, \phi) \Longrightarrow (h', \phi')$ in $Z^2_Y(\mathfrak{G})^\infty$, i.e. a smooth function $E : Y \longrightarrow H$ that satisfies (3.4), and condition (1.5) infers



It is clear that differential cocycles together with their 1-morphisms and 2-morphisms form a 2-category, which we denote by $Z^2_\pi(\mathfrak{G})^\infty$, the *2-category of degree two differential $\mathfrak{G}$-cocycles*. It is also clear that the 2-functor $\mathcal{D}$ induces a strict 2-functor between the descent 2-category and this 2-category. Since $\mathcal{D}$ is strictly invertible by Theorem 3.5, we have even more

Proposition 4.1. *Let $\mathfrak{G}$ be a Lie 2-group and let $\pi : Y \longrightarrow M$ be a two-contractible surjective submersion. Then, the 2-functor $\mathcal{D}$ induces an isomorphism of 2-categories*

$$\mathfrak{Des}^2_\pi(\mathrm{id}_{\mathcal{B}\mathfrak{G}})^\infty \cong Z^2_\pi(\mathfrak{G})^\infty.$$

The 2-category $Z^2_\pi(\mathfrak{G})^\infty$ of degree two differential $\mathfrak{G}$-cocycles can, however, be considered for an arbitrary surjective submersion. As mentioned above, it plays the role of local data of transport 2-functors. To make this more transparent, let us now write out differential cocycles in terms of smooth functions and differential forms which are implicit in the categories $Z^2_{Y^{[k]}}(\mathfrak{G})^\infty$ appearing above. Let us additionally assume that the surjective submersion π comes from an open cover $\mathfrak{V}$ of M, in which case we write $Z^2_{\mathfrak{V}}(\mathfrak{G})^\infty$.

A differential $\mathfrak{G}$-cocycle in degree two $((A, B), (g, \varphi), \psi, f)$ has the following smooth functions and differential forms:

(a) On every open set V_i,

$$\psi_i : V_i \longrightarrow H \quad , \quad A_i \in \Omega^1(V_i, \mathfrak{g}) \quad \text{and} \quad B_i \in \Omega^2(V_i, \mathfrak{h}).$$

(b) On every two-fold intersection $V_i \cap V_j$,

$$g_{ij} : V_i \cap V_j \longrightarrow G \quad \text{and} \quad \varphi_{ij} \in \Omega^1(V_i \cap V_j, \mathfrak{h}).$$

(c) On every three-fold intersection $V_i \cap V_j \cap V_k$,

$$f_{ijk} : V_i \cap V_j \cap V_k \longrightarrow H.$$

The cocycle conditions are the following:

1. Over every open set V_i,

$$\begin{aligned} \mathrm{d}A_i + [A_i \wedge A_i] &= t_*(B_i) \\ g_{ii} &= t(\psi_i) \\ \varphi_{ii} &= -(r_{\psi_i}^{-1} \circ \alpha_{\psi_i})_*(A_i) - \psi_i^*\bar{\theta}. \end{aligned} \tag{4.2}$$

2. Over every two-fold intersection $V_i \cap V_j$,

$$\begin{aligned} A_j &= \mathrm{Ad}_{g_{ij}}(A_i) - g_{ij}^*\bar{\theta} - t_*(\varphi_{ij}) \\ B_j &= (\alpha_{g_{ij}})_*(B_i) - \alpha_*(A_j \wedge \varphi_{ij}) - \mathrm{d}\varphi_{ij} - [\varphi_{ij} \wedge \varphi_{ij}] \\ 1 &= f_{ijj}\psi_j = f_{iij}\, \alpha_{g_{ij}}(\psi_i). \end{aligned}$$

3. Over every three-fold intersection $V_i \cap V_j \cap V_k$,

$$\begin{aligned} g_{ik} &= t(f_{ijk})g_{jk}g_{ij} \\ \mathrm{Ad}_{f_{ijk}}(\varphi_{ik}) &= (\alpha_{g_{jk}})_*(\varphi_{ij}) + \varphi_{jk} + (r_{f_{ijk}}^{-1} \circ \alpha_{f_{ijk}})_*(A_k) + f_{ijk}^*\bar{\theta}. \end{aligned}$$

4. Over every four-fold intersection $V_i \cap V_j \cap V_k \cap V_l$,

$$f_{ikl}\alpha(g_{kl}, f_{ijk}) = f_{ijl}f_{jkl}.$$

Additionally, the curvature of the differential cocycle is according to (3.10) given by

$$H_i := \mathrm{d}B_i + \alpha_*(A_i \wedge B_i) \in \Omega^3(V_i, \mathfrak{h}).$$

We remark that particular examples of differential cocycles, namely those with $\psi_i = 1$, $\varphi_{ij} = 0$ and $f_{ijk} = 1$ have been considered in [MP07] as *categorical connections* on ordinary principle G-bundles whose classifying cocycle is given by g_{ij}.

A 1-morphism $((h, \epsilon), \phi)$ between differential cocycles has the following structure:

(a) On every open set V_i,

$$h_i : V_i \longrightarrow G \quad \text{and} \quad \phi_i \in \Omega^1(V, \mathfrak{h}).$$

(b) On every two-fold intersection $V_i \cap V_j$,

$$\epsilon_{ij} : V_i \cap V_j \longrightarrow H.$$

The following conditions have to be satisfied:

1. Over every open set V_i,

$$\begin{aligned} B'_i &= (\alpha_{h_i})_*(B_i) - \alpha_*(A'_i \wedge \phi_i) - \mathrm{d}\phi_i - [\phi_i \wedge \phi_i] \\ A'_i &= \mathrm{Ad}_{h_i}(A_i) - t_*(\phi_i) - h_i^*\bar{\theta} \\ \psi'_i &= \epsilon_{ii}\alpha(h_i, \psi_i). \end{aligned} \tag{4.3}$$

2. Over every two-fold intersection $V_i \cap V_j$,

$$\begin{aligned} g'_{ij} &= t(\epsilon_{ij})h_j g_{ij} h_i^{-1} \\ \varphi'_{ij} &= \mathrm{Ad}_{\epsilon_{ij}}((\alpha_{h_j})_*(\varphi_{ij}) + \phi_j) - (\alpha_{g'_{ij}})_*(\phi_i) - (r_{\epsilon_{ij}}^{-1} \circ \alpha_{\epsilon_{ij}})_*(A'_{ij}) - \epsilon_{ij}^*\bar{\theta} \end{aligned}$$

3. Over every three-fold intersection $V_i \cap V_j \cap V_k$,

$$f'_{ijk} = \epsilon_{ik}\alpha(h_k, f_{ijk})\alpha(g'_{ik}, \epsilon_{ij}^{-1})\epsilon_{jk}^{-1}.$$

Finally, a 2-morphism E between differential cocycles has, for any open set V_i, a smooth function $E_i : V_i \longrightarrow H$ such that on every open set V_i

$$\begin{aligned} h'_i &= t(E_i)h_i \\ \phi'_i &= \mathrm{Ad}_{E_i}(\phi_i) - (r_{E_i}^{-1} \circ \alpha_{E_i})_*(A') - E_i^*\bar{\theta} \end{aligned}$$

and, on every 2-fold intersection $V_i \cap V_j$,

$$\epsilon'_{ij} = \alpha(g'_{ij}, E_i)\epsilon_{ij}E_j^{-1}.$$

Remark 4.2. The structure and the relations listed above are *direct consequences* of the structure and axioms of 2-functors, pseudonatural transformations and modifications; neither choices nor additional assumptions had to be made.

Summarizing, we may have started with a transport 2-functor $\widetilde{\mathrm{tra}} : \mathcal{P}_2(M) \longrightarrow T$ with $\mathcal{B}\mathfrak{G}$-structure, and $\widetilde{i} : \mathcal{B}\mathfrak{G} \longrightarrow T$ an equivalence of 2-categories. With the choice of a weak inverse 2-functor $F : T \longrightarrow \mathcal{B}\mathfrak{G}$, we have formed the associated 2-functor $F \circ \widetilde{\mathrm{tra}} : \mathcal{P}_2(M) \longrightarrow \mathcal{B}\mathfrak{G}$ with $\mathcal{B}\mathfrak{G}$-structure. For $\mathfrak{V}$ a good open cover, and $\pi : Y \longrightarrow M$ the associated 2-contractible surjective submersion, it defines a smooth descent object in $\mathfrak{Des}^2_\pi(\mathrm{id}_{\mathcal{B}\mathfrak{G}})^\infty$, and in turn, via the 2-functor $\mathcal{D}$, a degree two differential $\mathfrak{G}$-cocycle in $Z^2_{\mathfrak{V}}(\mathfrak{G})^\infty$. This differential cocycle consists of smooth functions and differential forms, yielding local data for the transport 2-functor tra.

In order to relate differential $\mathfrak{G}$-cocycles to the cohomology of the underlying manifold M we consider the set of isomorphism classes of differential $\mathfrak{G}$-cocycles, i.e. the set of objects in $Z^2_{\mathfrak{V}}(\mathfrak{G})^\infty$ subject to the equivalence relation according to which two elements are equivalent if and only if there exists a 1-morphism between them. We remark that every differential $\mathfrak{G}$-cocycle is equivalent to a differential $\mathfrak{G}$-cocycle with trivial „normalization function" ψ_i, i.e. $\psi_i = 1$ for all i. We denote the set of equivalence classes of differential $\mathfrak{G}$-cocycles by $\check{H}^2(\mathfrak{V}, \mathfrak{G})$.

We make the following observation. If one drops all differential forms from the above data and only keeps the smooth functions, the set $\check{H}^2(\mathfrak{V}, \mathfrak{G})$ coincides with the non-abelian cohomology $H^2(\mathfrak{V}, \mathfrak{G})$, as it appears for instance in [Gir71, Bre94, Bar04, Woc08]. This justifies the following

Definition 4.3. *The set $\check{H}^2(\mathfrak{V}, \mathfrak{G})$ of isomorphism classes of degree two differential $\mathfrak{G}$-cocycles is called the degree two differential non-abelian cohomology of the cover $\mathfrak{V}$ with values in the Lie 2-group $\mathfrak{G}$. The direct limit*

$$\check{H}^2(M, \mathfrak{G}) := \varinjlim_{\mathfrak{V}} \check{H}^2(\mathfrak{V}, \mathfrak{G})$$

is called the degree two differential non-abelian cohomology of M with values in $\mathfrak{G}$.

Combining Proposition 4.1 with Theorem 3.10 we obtain

Theorem 4.4. *Let $i : \mathcal{B}\mathfrak{G} \longrightarrow T$ be an equivalence of 2-categories. Then, isomorphism classes of transport 2-functors* $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ *with $\mathcal{B}\mathfrak{G}$-structure are in bijection with the differential non-abelian cohomology $\check{H}^2(M, \mathfrak{G})$.*

Let us specify two particular examples of differential non-abelian cohomology which have been treated in the literature:

1. The Lie 2-group $\mathfrak{G} = \mathcal{B}S^1$. We leave it as an easy exercise to the reader to check that our differential non-abelian cohomology is precisely degree two Deligne cohomology,

   $$\check{H}^2(M, \mathcal{B}S^1) = H^2(M, \mathcal{D}(2)).$$

   Deligne cohomology [Bry93] is one of the well-known local description of abelian gerbes with connection, which hence appears as a particular case of local data for transport 2-functors.

2. The Lie 2-group $\mathfrak{G} = \mathrm{AUT}(H)$ for H some ordinary Lie group H. We also leave it to the reader to check our differential cocycles corresponds precisely to the local description of connections in non-abelian gerbes given by Breen and Messing [BM05] (see Remark 4.5 below). Furthermore, the existence of 1-morphisms between differential cocycles corresponds precisely to the equivalence relation used in [BM05]. Summarizing, we have an equality

   $$\check{H}^2(M, \mathrm{AUT}(H)) = \left\{ \begin{array}{c} \text{Equivalence classes of local data} \\ \text{of Breen-Messing } H\text{-gerbes} \\ \text{with connection over } M \end{array} \right\} .$$

   Hence, also Breen-Messing gerbes with connection appear as a particular case of transport 2-functors.

Remark 4.5. This remark concerns the condition (4.2) between the 1-form A and the 2-form B which are part of our differential $\mathfrak{G}$-cocycles. It is present neither in the Breen-Messing gerbes [BM05] nor in the approach by Aschieri, Cantini and Jurco [ACJ05] using non-abelian bundle gerbes [ACJ05], which is discussed in Section 4.3. Breen and Messing call the local 2-form

$$t_*(B_i) - \mathrm{d}A_i - [A_i \wedge A_i]$$

which is here zero by (4.2), the *fake curvature* of the gerbe. In this terminology, transport 2-functors only cover Breen-Messing gerbes with vanishing fake curvature.

The crucial point is here that neither for the Breen-Messing gerbes nor for the non-abelian bundle gerbes reasonable notions of holonomy or parallel transport are known, while transport 2-functors have such notions, as we will demonstrate in Section 5. And indeed, equation (4.2) comes from an important consistency condition on this parallel transport, namely from the target-source matching condition for the transport 2-functor, which makes it possible to decompose parallel transport in pieces. So we understand equation (4.2) as an integrability condition which has necessarily to be satisfied if parallel transport is supposed to work. This is affirmed by Martins-Picken categorical connections [MP07], for which parallel transport plays an important role and where equation (4.2) is also present.

4.2 Abelian Bundle Gerbes with Connection

In this section we consider the target 2-category $T = \mathcal{B}(S^1\text{-Tor})$, the monoidal category of S^1-torsors viewed as a 2-category with a single object like in Example A.2. Associated to this 2-category is the 2-functor $i_{S^1} : \mathcal{BB}S^1 \longrightarrow \mathcal{B}(S^1\text{-Tor})$ that sends the single 1-morphism of $\mathcal{BB}S^1$ to the circle – viewed as an S^1-torsor over itself. Now we consider transport 2-functors

$$\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow \mathcal{B}(S^1\text{-Tor})$$

with $\mathcal{BB}S^1$-structure. For any surjective submersion $\pi : Y \longrightarrow M$ we relate the associated descent 2-category $\mathfrak{Des}^2_\pi(i_{S^1})^\infty$ to a 2-category $\mathfrak{BGrb}^\nabla(\pi)$ of S^1-bundle gerbes with connection over M. Let us recall the definition of these bundle gerbes following [Mur96, MS00].

1. A bundle gerbe with connection (B, L, ω, μ) is a 2-form $B \in \Omega^2(Y)$, a circle bundle L with connection ω over $Y^{[2]}$ of curvature $\mathrm{curv}(\omega) = \pi_1^*B - \pi_2^*B$, and an associative isomorphism

$$\mu : \pi_{23}^*L \otimes \pi_{12}^*L \longrightarrow \pi_{13}^*L$$

   of circle bundles over $Y^{[3]}$ that respects connections.

2. A bundle gerbe 1-morphism $(B, L, \omega, \mu) \longrightarrow (B', L', \omega', \mu')$, also known as *stable isomorphism*, is a circle bundle A with connection ς over Y of curvature $\mathrm{curv}(\varsigma) = B - B'$ together with an isomorphism

$$\alpha : \pi_2^*A \otimes L \longrightarrow L' \otimes \pi_1^*A$$

of circle bundles that respects the connections, such that the diagram

$$\begin{array}{ccc} \pi_3^*A\otimes\pi_{23}^*L\otimes\pi_{12}^*L & \xrightarrow{\mathrm{id}\otimes\mu} & \pi_3^*A\otimes\pi_{13}^*L \\ \Big\downarrow{\scriptstyle \pi_{23}^*\alpha\otimes\mathrm{id}} & & \\ \pi_{23}^*L'\otimes\pi_2^*A\otimes L & & \Big\downarrow{\scriptstyle\pi_{13}^*\alpha} \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi_{12}^*\alpha} & & \\ \pi_{23}^*L'\otimes\pi_{12}^*L'\otimes\pi_1^*A & \xrightarrow[\mu'\otimes\mathrm{id}]{} & \pi_{13}^*L'\otimes\pi_1^*A \end{array} \tag{4.4}$$

of isomorphisms of circle bundles over $Y^{[3]}$ is commutative.

3. A bundle gerbe 2-morphism $(A,\varsigma,\alpha) \Longrightarrow (A',\varsigma',\alpha')$ is an isomorphism $\varphi : A \longrightarrow A'$ of circle bundles over Y that respects the connections, such that the diagram

$$\begin{array}{ccc} \pi_2^*A\otimes L & \xrightarrow{\alpha} & L'\otimes\pi_1^*A \\ \Big\downarrow{\scriptstyle \pi_2^*\varphi\otimes\mathrm{id}_L} & & \Big\downarrow{\scriptstyle \mathrm{id}_{L'}\otimes\pi_1^*\varphi} \\ \pi_2^*A'\otimes L & \xrightarrow[\alpha']{} & L'\otimes\pi_1^*A' \end{array} \tag{4.5}$$

of isomorphisms of circle bundles over $Y^{[2]}$ is commutative.

What we have described here is a simplified version of the full 2-category $\mathfrak{BGrb}^{\nabla}(M)$ of S^1-bundle gerbes with connection over M, in which every bundle gerbe has an individual surjective submersion, see [Ste00, Wal07]. We obtain the full 2-category back as the direct limit

$$\mathfrak{BGrb}^{\nabla}(M) := \varinjlim_{\pi} \mathfrak{BGrb}^{\nabla}(\pi).$$

We return later to this direct limit. In the following we show first

Theorem 4.6. *For any surjective submersion $\pi : Y \longrightarrow M$ there is a canonical surjective equivalence of 2-categories*

$$\mathfrak{Des}^2_{\pi}(i_{S^1})^{\infty} \cong \mathfrak{BGrb}^{\nabla}(\pi).$$

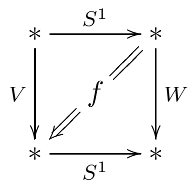
A 2-functor $\mathfrak{Des}^2_{\pi}(i_{S^1})^{\infty} \longrightarrow \mathfrak{BGrb}^{\nabla}(\pi)$ realizing the claimed equivalence is defined in the following way. For a descent object $(\mathrm{triv}, g, \psi, f)$, the smooth 2-functor $\mathrm{triv} : \mathcal{P}_2(Y) \longrightarrow \mathcal{B}\mathcal{B}S^1$ defines by Theorem 3.5 a 2-form $B \in \Omega^2(Y)$, this is the first ingredient of the bundle gerbe. The pseudonatural transformation g yields a transport functor

$$\mathscr{F}(g) : \mathcal{P}_1(Y^{[2]}) \longrightarrow \Lambda_{i_{S^1}}\mathcal{B}(S^1\text{-Tor})$$

with $\Lambda\mathcal{B}\mathcal{B}S^1$-structure. Let us translate this functor into familiar language. First of all, we have evidently $\Lambda\mathcal{B}\mathcal{B}S^1 = \mathcal{B}S^1$. Second, there is a canonical equivalence of categories

$$\Lambda_{i_{S^1}}\mathcal{B}(S^1\text{-Tor}) \cong S^1\text{-Tor}. \tag{4.6}$$

This comes from the fact that an object is in both categories an S^1-torsor. A morphism between S^1-torsors V and W in $\Lambda_{i_{S^1}}\mathcal{B}(S^1\text{-Tor})$ is by definition a 2-morphism



in $\mathcal{B}(S^1\text{-Tor})$, and this is in turn an S^1-equivariant map

$$f : W \otimes S^1 \longrightarrow S^1 \otimes V.$$

It can be identified canonically with an S^1-equivariant map $f^{-1} : V \longrightarrow W$, i.e. a morphism in S^1-Tor. It is straightforward to see that (4.6) is even a monoidal equivalence. In combination with Theorem 3.2 we have

Lemma 4.7. *For X a smooth manifold, there is a canonical surjective equivalence of monoidal categories*

$$\mathfrak{Bun}^{\nabla}_{S^1}(X) \cong \mathrm{Trans}^1_{\Lambda\mathcal{B}\mathcal{B}S^1}(X, \Lambda_{i_{S^1}}\mathcal{B}(S^1\text{-Tor}))$$

between circle bundles with connection and transport functors with $\Lambda\mathcal{B}\mathcal{B}S^1$-structure.

Despite of the heavy notation, this lemma allows us to transform all the remaining descent data into geometrical data. First, the transport functor $\mathscr{F}(g)$ is a circle bundle L with connection ω over $Y^{[2]}$. This circle bundle will be the second ingredient of the bundle gerbe.

Lemma 4.8. *The curvature of the connection ∇ on the circle bundle L satisfies*

$$\mathrm{curv}(\omega) = \pi_1^*B - \pi_2^*B.$$

Proof. Let U_α be open sets covering $Y^{[2]}$, and let $(\widetilde{\mathrm{triv}}, \tilde{t})$ be a local i_{S^1}-trivialization of the transport functor $\mathscr{F}(g)$ consisting of smooth functors $\widetilde{\mathrm{triv}}_\alpha : \mathcal{P}_1(U_\alpha) \longrightarrow \mathcal{B}S^1$ and natural transformations

$$\tilde{t}_\alpha : \mathscr{F}(g)|_{U_\alpha} \longrightarrow (\widetilde{\mathrm{triv}}_\alpha)_{i_{S^1}}.$$

We observe that the functors $\widetilde{\mathrm{triv}}_\alpha$ and the natural transformation $\tilde{t}_\alpha$ lie in the image of the functor $\mathscr{F}$, such that there exist smooth pseudonatural transformations $\rho_\alpha : \pi_1^*\mathrm{triv}|_{U_\alpha} \longrightarrow \pi_2^*\mathrm{triv}|_{U_\alpha}$ and modifications $t_\alpha : g|_{U_\alpha} \Longrightarrow \rho_\alpha$ with

$$\widetilde{\mathrm{triv}}_\alpha = \mathscr{F}(\rho_\alpha) \quad \text{and} \quad \tilde{t}_\alpha = \mathscr{F}(t_\alpha).$$

As found in [SW08] and reviewed in Section 3.1 of the present article, associated to the smooth pseudonatural transformation ρ_α is a 1-form $\varphi_\alpha \in \Omega^1(U_\alpha)$, and equation (3.3) infers in the present situation

$$\pi_1^*B - \pi_2^*B = \mathrm{d}\varphi_\alpha.$$

It remains to trace back the relation between φ_α and the curvature of the connection ω on circle bundle L. Namely, if A_α is the 1-form corresponding to the smooth functor $\widetilde{\mathrm{triv}}_\alpha$, we have

$$A_\alpha = \varphi_\alpha \quad \text{and} \quad \mathrm{d}A_\alpha = \mathrm{curv}(\omega).$$

This shows the claim. □

Second, the modification $f : \pi_{23}^*g \circ \pi_{12}^*g \Longrightarrow \pi_{13}^*g$ induces an isomorphism

$$\mathscr{F}(f) : \pi_{23}^*\mathscr{F}(g) \otimes \pi_{12}^*\mathscr{F}(g) \longrightarrow \pi_{13}^*\mathscr{F}(g)$$

of transport functors; again by Lemma 4.7 this defines an isomorphism

$$\mu : \pi_{12}^*L \otimes \pi_{23}^*L \longrightarrow \pi_{13}^*L$$

of circle bundles with connection, which is the last ingredient of the bundle gerbe. The pentagon identity (1.2) infers the associativity condition on μ. This shows that (B, L, ∇, μ) is a bundle gerbe with connection. We remark that the descent datum ψ has been forgotten.

Using Lemma 4.7 in the same way as just demonstrated it is easy to assign bundle gerbe 1-morphisms to descent 1-morphisms and bundle gerbe 2-morphisms to descent 2-morphisms. Here the conditions (1.3) and (1.5) on the descent 1-morphisms translate one-to-one to the commutative diagrams (4.4) and (4.5). Most naturally, the composition law of morphisms between bundle gerbes (which we have not carried out above) is precisely reproduced by the composition laws of the descent 2-category $\mathfrak{Des}^2_\pi(i_{S^1})^\infty$.

It is evident that the 2-functor we just have defined is an equivalence of 2-categories, since all manipulations we have made are equivalences according to Lemma 4.7 and Theorem 3.5. We only remark that the descent datum ψ can be reproduced in a canonical way from a given bundle gerbe using the existence of dual circle bundles, see Lemma 1 in [Wal07].

Summarizing, bundle gerbes with connection are precisely the descent objects of transport 2-functors with $\mathcal{B}\mathcal{B}S^1$-structure and values in $\mathcal{B}(S^1\text{-Tor})$. This equivalence clearly commutes with the refinement of surjective submersions. Hence, as a consequence of Theorem 3.10 we have

Corollary 4.9. *We have an equivalence*

$$\mathrm{Trans}^2_{\mathcal{B}\mathcal{B}S^1}(M, \mathcal{B}(S^1\text{-Tor})) \cong \mathfrak{BGrb}^\nabla(M)$$

between the 2-category of transport 2-functors and the 2-category of bundle gerbes with connection over M.

In the next section we proceed similarly for non-abelian bundle gerbes.

4.3 Non-abelian Bundle Gerbes with Connection

The first problem one encounters when trying to generalize S^1-bundle gerbes to non-abelian H-bundle gerbes is that the category of H-torsors is not monoidal. This problem can be solved using H-bitorsors [BM05]. More difficult is to say what connections on such non-abelian bundle gerbes are. In [ACJ05] a suitable definition was presented involving twisted connections on bibundles.

We show here that just as abelian S^1-bundle gerbes with connection are nothing but descent objects for $i : \mathcal{B}\mathcal{B}S^1 \longrightarrow \mathcal{B}(S^1\text{-Tor})$, the non-abelian H-bundle gerbes with connection from [ACJ05] are nothing but descent objects for a 2-functor

$$i : \mathcal{B}\mathrm{AUT}(H) \longrightarrow \mathcal{B}(H\text{-BiTor}). \tag{4.7}$$

In particular the curious twist on the connections on the bibundles finds a natural interpretation as one component of a pseudonatural transformation.

4.3.1 Bibundles with twisted Connections

The first thing we have to do is to generalize the equivalence between circle bundles with connection and certain transport functors obtained in Lemma 4.7 to principal H-bibundles with twisted connections. For this purpose, let us carry out the details of the category of such bibundles, which are implicit in [ACJ05].

A *principal H-bibundle over X* is a bundle $P \longrightarrow X$ that is both a left and a right principal H-bundle such that the two actions commute with each other. Morphisms between two principal H-bibundles are smooth fibrewise bi-equivariant bundle maps.

We will denote the left and right actions by an element $h \in H$ on a bibundle P by l_h and r_h, respectively. We remark that measuring the difference between the left and the right action in the sense of $l_h(p) = r_{g(h)}(p)$ furnishes a smooth map

$$g : P \longrightarrow \mathrm{Aut}(H). \tag{4.8}$$

In the following we denote by $\mathfrak{aut}(H)$ the Lie algebra of $\mathrm{Aut}(H)$. Like in the construction of the Lie 2-group $\mathrm{AUT}(H)$ in Section 3.3 we denote by $t : H \longrightarrow \mathrm{Aut}(H)$ the assignment of inner automorphisms and by $\alpha : \mathrm{Aut}(H) \times H \longrightarrow H$ the evaluation.

Definition 4.10 ([ACJ05]). *Let $p : P \longrightarrow X$ be a principal H-bibundle, and let $A \in \Omega^1(X, \mathfrak{aut}(H))$ be a 1-form on the base space. An <u>A-twisted (right) connection on P</u> is a 1-form $\phi \in \Omega^1(P, \mathfrak{h})$ satisfying*

$$\phi_{\rho h}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\rho h)\right) = \mathrm{Ad}_h^{-1}\left(\phi_\rho\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)\right) - (r_h \circ \alpha_h)_* \circ (p^*A) + \theta_h\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right) \tag{4.9}$$

for all smooth curves $\rho : [0,1] \longrightarrow P$ and $h : [0,1] \longrightarrow H$. A morphism $f : P \longrightarrow P'$ respects A-twisted connections ϕ on P and ϕ' on P' if $f^*\phi' = \phi$.

We write $\mathfrak{Bibun}_H^{\nabla}(X, A)$ for the category of principal H-bibundles with A-twisted connection over X, and $\mathfrak{Bibun}_H^{\nabla}(X)$ for the union of these categories over all 1-forms A.

Remark 4.11. For $A = 0$ an A-twisted right connection on P is the same as an ordinary connection on P regarded as a right principal bundle. One can give an analogous definition of a twisted left connection. Then, a twisted right connection gives rise to a twisted left connection, for a different twist, and vice versa. This is discussed in detail in [ACJ05], but will be a manifest consequence of the reformulation which we give later.

Lemma 4.12. *Let $A \in \Omega^1(X, \mathfrak{aut}(H))$ be a 1-form and let $p : P \longrightarrow X$ be a principal H-bibundle. For any A-twisted connection ϕ on P there exists a unique 1-form $A_\phi \in \Omega^1(X, \mathfrak{aut}(H))$ satisfying*

$$p^*A_\phi = \mathrm{Ad}_g(p^*A) - g^*\bar{\theta} - t_* \circ \phi,$$

where g is the map from (4.8).

A twisted connection in a principal bibundle P gives rise to parallel transport maps

$$\tau_\gamma : P_x \longrightarrow P_y$$

between the fibres of P over points x, y associated to any path $\gamma : x \longrightarrow y$. It is obtained in the same way as in an ordinary principal bundle but using equation (4.9) instead of the usual one. As a result of the twist, the maps τ_γ are not bi-equivariant; they satisfy

$$\tau_\gamma(l_{F_\phi(\gamma)(h)}(p)) = l_{h^{-1}}(\tau_\gamma(p)) \quad \text{and} \quad \tau_\gamma(r_h(p)) = r_{F(\gamma)^{-1}(h^{-1})}\tau_\gamma(p) \tag{4.10}$$

where $F, F_\phi : PX \longrightarrow \mathrm{Aut}(H)$ come from the functors associated to the 1-forms A and A_ϕ by Theorem 3.1. These complicated relations have a very easy interpretation, as we will see in the next section.

Finally, an A-twisted connection ϕ on a principal H-bundle P has a curvature: this is the 2-form

$$\mathrm{curv}(\phi) := \mathrm{d}\phi + [\phi \wedge \phi] + \alpha_*(A \wedge \phi) \in \Omega^2(P, \mathfrak{h}).$$

As usual in the non-abelian case, this 2-form will in general not induce a globally defined 2-form on the base manifold.

For two principal H-bibundles P and P' over X one can fibrewise take the tensor product of P and P' yielding a new principal H-bibundle $P \times_H P$ over X. If the two bibundles are equipped with twisted connections, the bibundle $P \times_H P'$ inherits a twisted connection only if the two twists satisfy an appropriate matching condition. Suppose the principal H-bibundle P is equipped with an A-twisted connection ϕ, and P' is equipped with an A'-twisted connection ϕ', and suppose that the matching condition

$$A'_{\phi'} = A \tag{4.11}$$

is satisfied. Then, the tensor product bibundle $P \times_H P'$ carries an A'-twisted connection $\phi_{\text{tot}} \in \Omega^1(P \times_H P, \mathfrak{h})$ characterized uniquely by the condition that

$$\text{pr}^*\phi_{\text{tot}} = (g \circ p')_* \circ p^*\phi + p'^*\phi' ,$$

where $\text{pr} : P \times_X P' \longrightarrow P \times_H P'$ is the projection to the tensor product and p and p' are the projections to the two factors. This tensor product, which is defined only between appropriate pairs of bibundles with twisted connections, turns $\mathfrak{Bibun}_H^{\nabla}(X)$ into a „monoidoidal" category.

A better point of view is to see it as a 2-category: the objects are the twists, i.e. 1-forms $A \in \Omega^1(X, \mathfrak{aut}(H))$, a 1-morphism $A \longrightarrow A'$ is a principal H-bibundle P with A'-twisted connection ϕ such that $A'_{\phi'} = A$, and a 2-morphism $(P, \phi) \Longrightarrow (P', \phi')$ is just a morphism of principal H-bibundles that respects the A'-twisted connections.

4.3.2 Transport Functors of twisted Connections in Bibundles

We are now going to identify the category $\mathfrak{Bibun}_H^{\nabla}(X)$ of principal H-bibundles with twisted connections over X with a (subcategory of a) category of transport functors.

For preparation, we write H-BiTor for the category whose objects are smooth manifolds with commuting smooth left and right H-actions, both free and transitive, and whose morphisms are smooth bi-equivariant maps. Using the product over H this is naturally a (non-strict) monoidal category. As usual we write $\mathcal{B}(H\text{-BiTor})$ for the corresponding one-object (non-strict) 2-category. The announced 2-functor (4.7),

$$i : \mathcal{B}\text{AUT}(H) \longrightarrow \mathcal{B}(H\text{-BiTor}),$$

is now defined as follows. It sends a 1-morphism $\varphi \in \text{Aut}(H)$ to the H-bitorsor ${}_{\varphi}H$ which is the group H on which an element h acts from the right by multiplication and from the left by multiplication with $\varphi(h)$. The compositors of i are given by the canonical identifications

$$c_{g_1,g_2} : {}_{g_1}H \times_H {}_{g_2}H \longrightarrow {}_{g_2g_1}H,$$

and the unitor is the identity. The 2-functor i further sends a 2-morphism $h : \varphi_1 \Longrightarrow \varphi_2$ to the bi-equivariant map

$${}_{\varphi_1}H \longrightarrow {}_{\varphi_2}H : x \longmapsto hx.$$

While the bi-equivariance with respect to the right action is obvious, the one with respect to the left action follows from the condition $\varphi_2(x) = h\varphi_1(x)h^{-1}$ we have for the 2-morphisms in $\text{AUT}(H)$ for all $x \in H$.

Remark 4.13. The 2-functor i is an equivalence of 2-categories, and exhibits $\mathcal{B}(H\text{-BiTor})$ as a framed bicategory in the sense of [Shu07].

As described in Section 3.2, the 2-functor i admits the construction of a category $\Lambda_i\mathcal{B}(H\text{-BiTor})$ and of a functor

$$\Lambda i : \Lambda\mathcal{B}\mathrm{AUT}(H) \longrightarrow \Lambda_i\mathcal{B}(H\text{-BiTor}).$$

We can now prove the announced generalization of Lemma 4.7 to the non-abelian case.

Proposition 4.14. *There exists a canonical functor*

$$\mathfrak{Bibun}_H^{\nabla}(X) \longrightarrow \mathrm{Trans}^1_{\Lambda\mathcal{B}\mathrm{AUT}(H)}(X, \Lambda_i\mathcal{B}(H\text{-BiTor}))$$

which is surjective and faithful.

Proof. Given a principal H-bibundle P with A-twisted connection, we define the associated transport functor by

$$\mathrm{tra}_P \quad : \quad x \xrightarrow{\gamma} y \quad \longmapsto \quad \begin{array}{ccc} i(*) & \xrightarrow{i(F(\gamma))} & i(*) \\ {\scriptstyle P_x}\big\downarrow & \overset{\tau_\gamma^{-1}}{\swarrow\!\!\!\!\swarrow} & \big\downarrow{\scriptstyle P_y} \\ i(*) & \xrightarrow[i(F_\phi(\gamma))]{} & i(*). \end{array}$$

Here $F, F_\phi : PX \longrightarrow \mathrm{Aut}(H)$ are the maps defined by A and A_ϕ that we have already used in the previous section. The definition contains the claim that the parallel transport map τ_γ gives a bi-equivariant map

$$\tau_\gamma^{-1} : P_y \times_H {}_{F(\gamma)}H \longrightarrow {}_{F_\phi(\gamma)}H \times_H P_x;$$

it is indeed easy to check that this is precisely the meaning of equations (4.10). A morphism $f : P \longrightarrow P'$ between bibundles with A-twisted connections induces a natural transformation $\eta_f : \mathrm{tra}_P \longrightarrow \mathrm{tra}_{P'}$ between the associated functors, whose component at a point x is the bi-equivariant map $f_x : P_x \Longrightarrow P'_x$. This is a particular morphism in $\Lambda_i\mathcal{B}(H\text{-BiTor})$ for which the horizontal 1-morphisms are identities. Here it becomes clear that the assignments

$$(P, \phi) \longmapsto \mathrm{tra}_P \quad \text{and} \quad f \longmapsto \eta_f$$

define a functor which is faithful but not full.

It remains to check that the functor tra_P is a transport 2-functor. We leave it as an exercise for the reader to construct a local trivialization (t, triv) of tra_P with smooth descent data. Hint: use an ordinary local trivialization of the bibundle P and follow the proof of Proposition 5.2 in [SW07]. □

The two categories appearing in the last proposition have both the feature that they have tensor products between appropriate objects. Concerning the bibundles with twisted connections, we have described this in terms of the matching condition (4.11) on the twists. Concerning the category of transport functors, this tensor product is inherited from the one on $\Lambda_i\mathcal{B}(H\text{-BiTor})$, which has been discussed in Section 3.2.

Lemma 4.15. *The matching condition (4.11) corresponds to the required condition for tensor products in* $\Lambda_i\mathcal{B}(H\text{-BiTor})$ *under the functor from Proposition 4.14. Furthermore, the functor respects tensor products whenever they are well-defined.*

Proof. Suppose that the matching condition $A'_{\phi'} = A$ holds, so that principal H-bibundles P and P' with connections ϕ and ϕ have a tensor product. It follows that the map $F_{\phi'}$ which labels the horizontal 1-morphisms at the bottom of the images of $\mathrm{tra}_{P'}$ is equal to the map F which labels the ones at the top of the images of tra_P; this is the required condition for the existence of the tensor product $\mathrm{tra}_{P'} \otimes \mathrm{tra}_P$. That the tensor products are respected follows from the definition of the twisted connection ϕ_{tot} on the tensor product bibundle. □

An alternative formulation of Lemma 4.15 would be that the functor from Proposition 4.14 respects the monoidoidal structures, or, that it is a double functor between (weak) double categories.

4.3.3 Non-Abelian Bundle Gerbes as Transport 2-Functors

We claim that the relation between non-abelian H-bundle gerbes with connection and transport 2-functors with $\mathcal{B}\mathrm{AUT}(H)$-structure is a straightforward generalization of the abelian case, see Theorem 4.6. Here, a *non-abelian H-bundle gerbe with connection* and surjective submersion $\pi : Y \longrightarrow M$ is a 2-form $B \in \Omega^2(Y, \mathfrak{h})$, a principal H-bibundle $p : P \longrightarrow Y^{[2]}$ with twisted connection ϕ such that

$$\mathrm{curv}(\phi) = (\pi_1 \circ p)^*B - (\alpha_g)_* \circ (\pi_2 \circ p)^*B, \tag{4.12}$$

and an associative morphism

$$\mu : \pi_{23}^*P \;\times_H\; \pi_{12}^*P \longrightarrow \pi_{13}^*P$$

of bibundles over $Y^{[3]}$ that respects the twisted connections [ACJ05]. In (4.12), g is the smooth map (4.8) and $\alpha : \mathrm{Aut}(H) \times H \longrightarrow H$ is the evaluation. The definitions of bundle gerbe 1-morphisms and bundle gerbe 2-morphisms generalize analogously to the non-abelian case.

Theorem 4.16. *There is a canonical surjective and faithful 2-functor*

$$H\text{-}\mathfrak{BGrb}^{\nabla}(\pi) \longrightarrow \mathfrak{Des}^2_\pi(i)^\infty.$$

Proof. All relations concerning the bimodules are analogous to those in the abelian case, when generalizing Lemma 4.7 to Proposition 4.14. Relation (4.12) for the 2-form B can be proven in the same way as in the proof of Lemma 4.8, but now using the full version of equation (3.3). The comments concerning the descent datum ψ also remain valid. □

The last result induces with Theorem 3.10

Corollary 4.17. *Let M be a smooth manifold. There exists a canonical 2-functor*

$$H\text{-}\mathfrak{BGrb}^{\nabla}(M) \longrightarrow \mathrm{Trans}^2_{\mathcal{B}\mathrm{AUT}(H)}(M, \mathcal{B}(H\text{-}\mathrm{BiTor})).$$

Let us close with a few remarks on non-abelian bundle gerbes.

1. The fact that the functor from Proposition 4.14 from bibundles to transport functors is not full means that the bibundle theory developed in [ACJ05] oversees a whole class of morphisms. As a consequence, one could consider a more general version of non-abelian bundle gerbes involving such morphisms over $Y^{[3]}$.

2. A non-abelian S^1-bundle gerbe is *not* the same as an abelian S^1-bundle gerbe: for the non-abelian bundle gerbes also the automorphisms are important, and $\mathrm{Aut}(S^1) \cong \mathbb{Z}_2$. For transport 2-functors this is even more obvious: the Lie 2-groups $\mathcal{B}\mathcal{B}S^1$ and $\mathcal{B}\mathrm{AUT}(S^1)$ are not equivalent.

3. The non-abelian bundle gerbes we have considered here are „fake-flat". See Remark 4.5 why this has to be.

4.4 Outlook: Connections on 2-Vector Bundles and more

Additionally to the equivalence between transport functors and principal G-bundles with connection (Theorem 3.1), [SW07] also contains an analogous equivalence for vector bundles with connection. It has an immediate generalization to 2-vector bundles with many applications, on which we shall give a brief outlook.

4.4.1 Models for 2-Vector Spaces

We fix some 2-category 2Vect standing for a 2-category of 2-vector spaces. Given a 2-group $\mathfrak{G}$, a representation of $\mathfrak{G}$ on such a 2-vector space is a 2-functor

$$\rho : \mathcal{B}\mathfrak{G} \longrightarrow 2\mathrm{Vect}.$$

A 2-vector bundle with connection and structure 2-group $\mathfrak{G}$ is nothing but a transport 2-functor $\mathrm{tra} : \mathcal{P}_2(X) \longrightarrow 2\mathrm{Vect}$ with $\mathcal{B}\mathfrak{G}$-structure. Important classes of 2-vector bundles are line 2-bundles and string bundles.

Depending on the precise application there is some flexibility in what one may want to understand under a 2-vector space. Usually 2-vector spaces are abelian module categories over a given monoidal category. For k a field, two important classes of examples are the following. First, let $\widehat{k}$ be the discrete monoidal category over k. Then, 2Vect is 2-category of module categories over $\widehat{k}$. This is equivalent to the 2-category of categories internal to k-vector spaces. These *Baez-Crans 2-vector spaces* [BC04] are appropriate for the discussion of Lie 2-algebras.

The second model for 2Vect is the 2-category of module categories over the monoidal category $\mathrm{Vect}(k)$ of k-vector spaces,

$$2\mathrm{Vect} := \mathrm{Vect}(k)\text{-Mod}.$$

In its totality this is rather unwieldy, but it contains two important sub-2-categories: the 2-category $\mathrm{KV}(k)$ of *Kapranov-Voevodsky 2-vector spaces* [KV94] and the 2-category $\mathrm{Bimod}(k)$, whose objects are k-algebras, whose 1-morphisms are bimodules over these algebras and whose 2-morphisms are bimodule homomorphisms [Shu07]. Indeed, there is a canonical inclusion 2-functor

$$\iota : \mathrm{Bimod}(k) \hookrightarrow \mathrm{Vect}(k)\text{-Mod}$$

that sends a k-algebra A to the category A-Mod of ordinary (say, right) A-modules. This is a module category over $\mathrm{Vect}(k)$ by tensoring a right module from the left by a vector space. A 1-morphism, an A-B-bimodule N, is sent to the functor that tensores a right A-module from the right by N, yielding a right B-module. A bimodule morphism induces evidently a natural transformation of these functors.

If one restricts the 2-functor ι to the full sub-2-category formed by those algebras that are direct sums $A = k^{\oplus n}$ of the ground field algebra, the 2-vector spaces in the image of ι are of the form $\mathrm{Vect}(k)^n$, i.e. tuples of vector spaces. The 1-morphisms in the image are $(m \times n)$-matrices whose entries are k-vector spaces. These form the 2-category of Kapranov-Voevodsky 2-vector spaces [KV94].

4.4.2 The canonical Representation of a 2-Group

Every automorphism 2-group $\mathrm{AUT}(H)$ of a Lie group H has a canonical representation on 2-vector spaces, namely

$$\mathcal{B}\mathrm{AUT}(H) \xrightarrow{A} \mathrm{Bimod}(k) \xrightarrow{\iota} \mathrm{Vect}(k)\text{-Mod}, \tag{4.13}$$

where the 2-functor A is defined similar as the one we have used for the non-abelian bundle gerbes in (4.7). It sends the single object to k regarded as a k-algebra, it sends a 1-morphism $\varphi \in \mathrm{Aut}(H)$ to the bimodule ${}_{\varphi}k$ in the notation of Section 4.3.2, and it sends a 2-morphism $(\varphi, h) : \varphi \implies c_h \circ \varphi$ to the multiplication with h from the left.

Now let $\mathfrak{G}$ be any smooth Lie 2-group corresponding to a smooth crossed module (G, H, t, α). We have a canonical 2-functor

$$\mathcal{B}\mathfrak{G} \longrightarrow \mathcal{B}\mathrm{AUT}(H) \quad : \quad \ast \underset{g'}{\overset{g}{\Downarrow h}} \ast \;\longmapsto\; \ast \underset{\alpha_{g'}}{\overset{\alpha_g}{\Downarrow h}} \ast \tag{4.14}$$

whose composition with (4.13) gives a representation of $\mathfrak{G}$, that we call the *canonical k-representation*.

Example 4.18. A very simple but useful example is the standard $\mathbb{C}$-representation of $\mathcal{B}\mathbb{C}^\times$. In this case the composition (4.13) is the 2-functor

$$\rho \quad : \quad \mathcal{B}\mathcal{B}\mathbb{C}^\times \longrightarrow \mathrm{Vect}(\mathbb{C})\text{-Mod} \quad : \quad \ast \underset{\mathrm{id}}{\overset{\mathrm{id}}{\Downarrow z}} \ast \;\longmapsto\; \mathrm{Vect}(\mathbb{C}) \underset{-\otimes\mathbb{C}}{\overset{-\otimes\mathbb{C}}{\Downarrow -\cdot z}} \mathrm{Vect}(\mathbb{C})$$

for all $z \in \mathbb{C}^\times$. Notice that $\mathrm{Vect}(\mathbb{C})$ is the canonical 1-dimensional 2-vector space over $\mathbb{C}$ in the same sense in that $\mathbb{C}$ is the canonical 1-dimensional complex 1-vector space. Therefore, transport 2-functors

$$\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow \mathrm{Vect}(\mathbb{C})\text{-Mod}$$

with $\mathcal{B}\mathcal{B}\mathbb{C}^\times$-structure deserve to be addressed as *line 2-bundles with connection*. Let us make two remarks:

1. Going through the discussion of abelian bundle gerbes with connection in Section 4.2 it is easy to see that line 2-bundles with connection are equivalent to bundle gerbes with connection defined via line bundles instead of circle bundles.

2. The fibre $\mathrm{tra}(x)$ of a line 2-bundle tra at a point x is an algebra which is Morita equivalent to the ground field $\mathbb{C}$. These are exactly the finite rank operators on a separable Hilbert space. Thus, line 2-bundles with connection are a form of bundles of finite rank operators with connection, this is the point of view taken in [BCM$^+$02].

The canonical 2-functor $A : \mathcal{B}\mathrm{AUT}(H) \longrightarrow \mathrm{Bimod}(k)$ we have used above can be deformed to a 2-functor A^ρ using an ordinary representation $\rho : \mathcal{B}H \longrightarrow \mathrm{Vect}(k)$ of H. It sends the object of $\mathcal{B}\mathrm{AUT}(H)$ to the algebra $A^\rho(\ast)$ which is the vector space generated from all the linear maps $\rho(h)$. A 1-morphism $\varphi \in \mathrm{Aut}(H)$ is again sent to the bimodule ${}_\varphi A^\rho(\ast)$, and the 2-morphisms as before to left multiplications. The original 2-functor is reproduced $A = A^{\mathrm{triv}_k}$ from the trivial representation of H on k.

Example 4.19. For G a compact simple and simply-connected Lie group, we consider the level k central extension $H_k := \hat{\Omega}_k G$ of the group of based loops in G. For a positive energy representation $\rho : \mathcal{B}\hat{\Omega}_k G \longrightarrow \mathrm{Vect}(k)$ the algebra $A^\rho(*)$ turns out to be a von Neumann-algebra while the bimodules ${}_\varphi A^\rho(*)$ are Hilbert bimodules. In this infinite-dimensional case we have to make the composition of 1-morphisms more precise: here we take not the algebraic tensor product of these Hilbert bimodules but the Connes fusion tensor product [ST04]. Connes fusion product still respects the composition: for A a von Neumann algebra and ${}_\varphi A$ the bimodule structure on it induced from twisting the left action by an algebra automorphism φ, we have

$$ {}_\varphi A \otimes {}_{\varphi'} A \simeq {}_{\varphi' \circ \varphi} A $$

under the Connes fusion tensor product. Now let $\mathfrak{G} = \mathrm{String}_k(G)$ be the string 2-group defined from the crossed module $\hat{\Omega}_k G \longrightarrow P_0 G$ of Fréchet Lie groups [BCSS07]. Together with the projection 2-functor (4.14) we obtain an induced representation

$$ i : \mathcal{B}\mathrm{String}_k(G) \longrightarrow \mathrm{Bimod}_{\mathrm{CF}}(k) $$

The fibres of a transport 2-functor

$$ \mathrm{tra} : \mathcal{P}_2(M) \longrightarrow \mathrm{Bimod}_{\mathrm{CF}}(k) \tag{4.15} $$

with $\mathcal{B}\mathrm{String}_k(G)$-structure are hence von Neumann algebras, and its parallel transport along a path is a Hilbert bimodule for these fibres. In conjunction with the result [BS08, BBK06] that $\mathrm{String}_k(G)$-2-bundles have the same classification as ordinary fibre bundles whose structure group is the topological String group, this says that transport 2-functors (4.15) have to be addressed as *String 2-bundles with connection*, already appearing in [ST04].

4.4.3 More: Twisted Vector Bundles

Vector bundles over M twisted by a class $\xi \in H^3(M, \mathbb{Z})$ are the same thing as gerbe modules for a bundle gerbe $\mathcal{G}$ whose Dixmier-Douady class is ξ [BCM+02]. These modules are in turn nothing else but certain (generalized) 1-morphisms in the 2-category of bundle gerbes $\mathfrak{BGrb}(M)$ [Wal07]. The same is true for connections on twisted vector bundles. More precisely, a twisted vector bundle with connection is the same as a 1-morphism

$$ \mathcal{E} : \mathcal{G} \longrightarrow \mathcal{I}_\rho $$

from the bundle gerbe $\mathcal{G}$ with connection to the trivial bundle gerbe $\mathcal{I}$ equipped with the connection 2-form $\rho \in \Omega^2(M)$.

Now let

$$ \mathrm{tra} : \mathcal{P}_2(M) \longrightarrow \mathrm{Vect}(\mathbb{C})\text{-Mod} $$

be a transport 2-functor which plays the role of the bundle gerbe $\mathcal{G}$, but we allow an arbitrary structure 2-group $\mathfrak{G}$ and any representation $\rho : \mathcal{B}\mathfrak{G} \longrightarrow \mathrm{Vect}(\mathbb{C})\text{-Mod}$. Let tra^∞ :

$\mathcal{P}_2(M) \longrightarrow \mathcal{B}\mathfrak{G}$ be a smooth 2-functor which plays the role of the trivial bundle gerbe. We shall now consider transport transformations

$$A : \mathrm{tra} \longrightarrow \mathrm{tra}_\rho^\infty .$$

Let $\pi : Y \longrightarrow M$ be a surjective submersion for which tra admits a local trivialization with smooth descent data $(\mathrm{triv}, g, \psi, f)$. The descent data of tra^∞ is of course $(\pi^*\mathrm{tra}^\infty, \mathrm{id}, \mathrm{id}, \mathrm{id})$. Now the transport transformation A has the following descent data: the first part is a pseudonatural transformation $h : \mathrm{triv} \longrightarrow \pi^*\mathrm{tra}^\infty$ whose associated functor $\mathscr{F}(h) : \mathcal{P}_1(Y) \longrightarrow \Lambda_\rho(\mathrm{Vect}(\mathbb{C})\text{-Mod})$ is a transport functor with $\Lambda\mathcal{B}\mathfrak{G}$-structure. The second part is a modification $\epsilon : \pi_2^* h \circ g \Longrightarrow \mathrm{id} \circ \pi_1^* h$ whose associated natural transformation

$$\mathscr{F}(\epsilon) : \pi_2^*\mathscr{F}(h) \otimes \mathscr{F}(g) \longrightarrow \pi_1^*\mathscr{F}(h)$$

is a morphism of transport functors over $Y^{[2]}$. According to conditions (1.3) and (1.4) on descent 1-morphisms, it fits into the commutative diagram

$$\begin{array}{ccc} \pi_3^*\mathscr{F}(h) \otimes \pi_{23}^*\mathscr{F}(g) \otimes \pi_{12}^*\mathscr{F}(g) & \xrightarrow{\pi_{23}^*\mathscr{F}(\epsilon)\otimes \mathrm{id}} & \pi_2^*\mathscr{F}(h) \otimes \pi_{12}^*\mathscr{F}(g) \\ {\scriptstyle \mathrm{id}\otimes\mathscr{F}(f)}\downarrow & & \downarrow{\scriptstyle \pi_{12}^*\mathscr{F}(\epsilon)} \\ \pi_3^*\mathscr{F}(h) \otimes \pi_{13}^*\mathscr{F}(g) & \xrightarrow[\pi_{13}^*\mathscr{F}(\epsilon)]{} & \pi_1^*\mathscr{F}(h) \end{array} \tag{4.16}$$

of morphisms of transport functors over $Y^{[3]}$ and satisfies $\Delta^*\mathscr{F}(\epsilon) \circ \mathscr{F}(\psi) = \mathrm{id}$. The transport functor

$$\mathscr{F}(h) : \mathcal{P}_1(Y) \longrightarrow \Lambda_\rho(\mathrm{Vect}(\mathbb{C})\text{-Mod})$$

together with the natural transformation $\mathscr{F}(\epsilon)$ is the general version of a vector bundle with connection twisted by a transport 2-functor tra. According to Sections 4.1 and 4.3, the twists can thus be Breen-Messing gerbes or non-abelian bundle gerbes with connection.

Depending on the choice of the representation ρ, our twisted vector bundles can be translated into more familiar language. Let us demonstrate this in the case of Example 4.18, in which the twist is a line 2-bundle with connection, i.e. a transport 2-functor

$$\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow \mathrm{Vect}(\mathbb{C})\text{-Mod}$$

with $\mathcal{B}\mathcal{B}\mathbb{C}^\times$-structure. In order to obtain the usual twisted vector bundles, we restrict the target 2-category to $\mathcal{B}\mathrm{Vect}(\mathbb{C})$, the monoidal category of complex vector spaces considered as a 2-category. The following Lie category Gl is appropriate: its objects are the natural numbers $\mathbb{N}$, and it has only morphisms between equal numbers, namely all matrices $\mathrm{Gl}_n(\mathbb{C})$. The composition is the product of matrices. The Lie category Gl is strictly monoidal: the tensor product of two objects $m, n \in \mathbb{N}$ is the product $nm \in \mathbb{N}$, and the one of two matrices $A \in \mathrm{Gl}(m)$ and $B \in \mathrm{Gl}(n)$ is the ordinary tensor product $A \otimes B \in \mathrm{Gl}(m \times n)$. In fact, Gl carries a second monoidal structure coming from the sum of natural numbers and the direct sum of matrices, so that Gl is actually a *bipermutative category*, see Example 3.1 of [BDR04].

Notice that we have a canonical inclusion functor $\iota : \mathcal{B}\mathbb{C}^{\times} \hookrightarrow \mathrm{Gl}$, which induces another inclusion

$$\iota_* : \mathrm{Trans}^2_{\mathcal{B}\mathcal{B}\mathbb{C}^{\times}}(M, \mathcal{B}\mathrm{Vect}(\mathbb{C})) \longrightarrow \mathrm{Trans}^2_{\mathcal{B}\mathrm{Gl}}(M, \mathcal{B}\mathrm{Vect}(\mathbb{C}))$$

of line 2-bundles with connection into more general vector 2-bundles with connection. Here we have used the representation

$$\rho : \mathcal{B}\mathrm{Gl} \longrightarrow \mathcal{B}\mathrm{Vect}$$

obtained as a generalization of Example 4.18 from $\mathbb{C}^{\times} = \mathrm{Gl}_1(\mathbb{C})$ to $\mathrm{Gl}_n(\mathbb{C})$ for all $n \in \mathbb{N}$. The composition $\rho \circ \iota$ reproduces the representation of Example 4.18.

Using the above inclusion, the given transport 2-functor tra induces a transport 2-functor $\iota_* \circ \mathrm{tra} : \mathcal{P}_2(M) \longrightarrow \mathcal{B}\mathrm{Vect}(\mathbb{C})$ with $\mathcal{B}$Gl-structure, and one can study transport transformations

$$A : \mathrm{tra} \longrightarrow \mathrm{tra}^{\infty}_{\rho}$$

in that greater 2-category $\mathrm{Trans}^2_{\mathcal{B}\mathrm{Gl}}(M, \mathcal{B}\mathrm{Vect}(\mathbb{C}))$. Along the lines of the general procedure described above, we have transport functors $\mathscr{F}(g)$ and $\mathscr{F}(h)$ coming from the descent data of tra and A, respectively. In the present particular situation, the first one takes values in the category $\Lambda_{\rho\circ\iota}\mathcal{B}\mathrm{Vect}_1(\mathbb{C})$ whose objects are one-dimensional complex vector spaces and whose morphisms from V to W are invertible linear maps $f : W \otimes \mathbb{C} \longrightarrow \mathbb{C} \otimes V$. Similar to Lemma 4.7, this category is equivalent to the category $\mathrm{Vect}_1(\mathbb{C})$ of one dimensional complex vector spaces itself. Thus, the transport functor $\mathscr{F}(g)$ with $\mathcal{B}\mathbb{C}^{\times}$-structure is a complex line bundle L with connection over $Y^{[2]}$. The second transport functor, $\mathscr{F}(h)$, takes values in the category $\Lambda_{\rho\circ\iota}\mathcal{B}\mathrm{Vect}(\mathbb{C})$. This category is equivalent to the category $\mathrm{Vect}(\mathbb{C})$ itself. It has $\Lambda_{\iota}\mathcal{B}$Gl-structure, which is equivalent to Gl. Thus, $\mathscr{F}(h)$ is a transport functor with values in $\mathrm{Vect}(\mathbb{C})$ and Gl-structure. It thus corresponds to a finite rank vector bundle E over Y with connection.

Since all identifications we have made so far a functorial, the morphisms $\mathscr{F}(f)$ and $\mathscr{F}(\epsilon)$ of transport functors induce morphisms of vector bundles that preserve the connections, namely an associative morphism

$$\mu : \pi_{23}^* L \otimes \pi_{12}^* L \longrightarrow \pi_{13}^* L$$

of line bundles over $Y^{[2]}$, and a morphism

$$\varrho : \pi_2^* E \otimes L \longrightarrow \pi_1^* E$$

of vector bundles over Y which satisfies a compatibility condition corresponding to (4.16). This reproduces the definition of a twisted vector bundle with connection [BCM+02]. We remark that the 2-form ρ that corresponds to the smooth 2-functor tra_ρ^∞ which was the target of the transport transformation A we have considered, is related to the curvature of the connection on the vector bundle E: it requires that

$$\mathrm{curv}(E) = I_n \cdot (\mathrm{curv}(L) - \pi^*\rho),$$

where I_n is the identity matrix and n is the rank of E. This condition can be derived similar to Lemma 4.8.

5 Holonomy of Transport 2-Functors

From the viewpoint of a transport 2-functor, parallel transport and holonomy are basically evaluation on paths or bigons.

5.1 Parallel Transport along Paths and Bigons

Let $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ be a transport 2-functor with $\mathcal{B}\mathfrak{G}$-structure on M. Its fibres over points $x, y \in M$ are objects $\mathrm{tra}(x)$ and $\mathrm{tra}(y)$ in T, and we say that its *parallel transport along a path* $\gamma : x \longrightarrow y$ is given by the 1-morphism

$$\mathrm{tra}(\gamma) : \mathrm{tra}(x) \longrightarrow \mathrm{tra}(y)$$

in T, and its *parallel transport along a bigon* $\Sigma : \gamma \Longrightarrow \gamma'$ is given by the 2-morphism

$$\mathrm{tra}(\Sigma) : \mathrm{tra}(\gamma) \Longrightarrow \mathrm{tra}(\gamma')$$

in T.

The rules how these 1-morphisms and 2-morphisms behave under the composition of paths and bigons are precisely the axioms of the 2-functor tra. We make some examples. If $\gamma_1 : x \longrightarrow y$ and $\gamma_2 : y \longrightarrow z$ are composable paths, the separate parallel transports along the two paths are related to the one along their composition by the compositor

$$c_{\gamma_1,\gamma_2} : \mathrm{tra}(\gamma_2) \circ \mathrm{tra}(\gamma_1) \Longrightarrow \mathrm{tra}(\gamma_2 \circ \gamma_1). \tag{5.1}$$

If id_x is the constant path at x, the parallel transport along id_x is related to the identity at the fibre $\mathrm{tra}(x)$ by the unitor

$$u_x : \mathrm{tra}(\mathrm{id}_x) \Longrightarrow \mathrm{id}_{\mathrm{tra}(x)}.$$

The parallel transports along vertically composable bigons $\Sigma : \gamma_1 \Longrightarrow \gamma_2$ and $\Sigma' : \gamma_2 \Longrightarrow \gamma_3$ obey for example axiom (F1), namely

$$\mathrm{tra}(\Sigma' \bullet \Sigma) = \mathrm{tra}(\Sigma') \bullet \mathrm{tra}(\Sigma).$$

The complete list of gluing axioms is precisely the list of axioms of a 2-functor, see Definition A.5.

In the previous Section 4 we have identified differential cocycles, abelian bundle gerbes and non-abelian bundle gerbes with connection with smooth descent data of particular transport 2-functors. Reconstructing the transport 2-functor from such descent data like described in Section 2, and evaluating this 2-functor on paths and bigons, yields a well-defined notion of parallel transport for these gerbes.

Let us start with a smooth descent object $(\mathrm{triv}, g, \psi, f)$ in the descent 2-category $\mathfrak{Des}^2_\pi(i)^\infty$ associated to some surjective submersion $\pi : Y \longrightarrow M$ and some 2-functor $i : \mathcal{B}\mathfrak{G} \longrightarrow T$. Suppose we want to compute the parallel transport of the reconstructed transport 2-functor

$$\mathrm{tra} := s^* R_{(\mathrm{triv}, g, \psi, f)} : \mathcal{P}_2(M) \longrightarrow T \tag{5.2}$$

along some path $\gamma : x \longrightarrow y$. Applying the section 2-functor s to γ we obtain a 1-morphism $s(\gamma) : s(x) \longrightarrow s(y)$ in the codescent 2-groupoid $\mathcal{P}^2_\pi(M)$. In general this 1-morphism is a composition of paths γ_ℓ in Y and jumps α_ℓ in the fibres:

$$s(\gamma) \;=\; s(x) \xrightarrow{\alpha_1} p_1 \xrightarrow{\gamma_1} p_2 \xrightarrow{\alpha_2} \cdots \longrightarrow p_n \xrightarrow{\gamma_m} s(y).$$

Then we have to compute the pairing between $s(\gamma)$ and the descent object $(\mathrm{triv}, g, \psi, f)$. The pairing procedure prescribes the piecewise evaluation of triv_i on the paths γ_ℓ and of g on the jumps α_ℓ. This yields composable 1-morphisms in T, whose composition is $\mathrm{tra}(\gamma)$.

Example 5.1. Let us give the following three examples for parallel transport along a path.

1. *Differential cocycle*. We represent the Lie 2-group $\mathfrak{G}$ as a crossed module (G, H, t, α). The target 2-category is now $T = \mathcal{B}\mathfrak{G}$, has only one object and the 1-morphisms are group elements $g \in G$. Thus, the parallel transport will be a group element $\mathrm{tra}(\gamma) \in G$.

   The differential cocycle is given by a tuple $(B, A, \varphi, \psi, g, f)$ of which A is a 1-form $A \in \Omega^1(Y, \mathfrak{g})$ and g is a smooth function $g : Y^{[2]} \longrightarrow G$. Parsing through the relation between the differential cocycle and the associated descent object, we obtain for γ_ℓ one of the paths one finds in $s(\gamma)$,

   $$\mathrm{triv}_i(\gamma_\ell) = \mathcal{P}\exp\left(\int_{\gamma_\ell} A\right) \in G \tag{5.3}$$

   where the path-ordered exponential $\mathcal{P}\exp$ stands for the solution of a differential equation governed by A. The evaluation of g at one of the jumps α_ℓ is just $g(\alpha_\ell) \in G$.

   Then, the parallel transport $\mathrm{tra}(\gamma) \in G$ is the product of the $\mathrm{triv}_i(\gamma_\ell)$ and the $g(\alpha_\ell)$, taken in the same order as the pieces appear in $s(\gamma)$.

2. *Abelian bundle gerbe.* Here the target 2-category is $T = \mathcal{B}(S^1\text{-Tor})$, so that the parallel transport will be an S^1-torsor $\text{tra}(\gamma)$.

   The abelian bundle gerbe is given by a tuple (L, ∇, μ, B), of which L is a circle bundle over $Y^{[2]}$. Since the structure 2-group is $\mathcal{B}\mathcal{B}S^1$ it is clear that the 2-functor $\text{triv} : \mathcal{P}_2(Y) \longrightarrow \mathcal{B}\mathcal{B}S^1$ is constant on the paths γ_ℓ, so that $\text{triv}_i(\gamma_\ell) = S^1$. Further, the pseudonatural transformation g corresponds to the circle bundle L, so that the pairing between a jump α_ℓ and g yields the fibre L_{α_ℓ} of L over the point $\alpha_\ell \in Y^{[2]}$.

   Then, the parallel transport $\text{tra}(\gamma)$ is the tensor product of S^1 viewed as a torsor over itself and the S^1-torsors L_{α_ℓ}.

3. *Non-abelian bundle gerbe.* Here the target 2-category is $T = \mathcal{B}(H\text{-BiTor})$, so that the parallel transport will be an H-bitorsor $\text{tra}(\gamma)$.

   The non-abelian bundle gerbe is given by a tuple $(E, \varphi, \nabla, \mu, A, B)$, of which E is a principal H-bibundle over $Y^{[2]}$ with φ-twisted connection ∇. The 2-functor $\text{triv} : \mathcal{P}_2(Y) \longrightarrow \mathcal{B}\text{AUT}(H)$ assigns to the paths γ_ℓ the automorphisms (5.3) so that $\text{triv}_i(\gamma_\ell) = {}_{\text{triv}(\gamma_\ell)}H$. Further, the pseudonatural transformation g corresponds to the bibundle E, so that the pairing between a jump α_ℓ and g yields the fibre E_{α_ℓ} of E over the point $\alpha_\ell \in Y^{[2]}$.

   Then, the parallel transport $\text{tra}(\gamma)$ is the tensor product of the H-bitorsors ${}_{\text{triv}(\gamma_\ell)}H$ and the H-bitorsors E_{α_ℓ}.

Let us now compute the parallel transport of transport 2-functor tra (5.2) that we have reconstructed from given a descent object $(\text{triv}, g, \psi, f)$, around a bigon $\Sigma : \gamma_1 \Longrightarrow \gamma_2$. According to the prescription, we use again the section 2-functor s and obtain some 2-isomorphism $s(\Sigma) : s(\gamma_1) \Longrightarrow s(\gamma_2)$. In general, this 2-morphism $s(\Sigma)$ can be a huge vertical and horizontal composition of 2-morphisms of $\mathcal{P}_2^\pi(M)$ of any kind. The pairing between $s(\Sigma)$ and the descent object $(\text{triv}, g, \psi, f)$ evaluates according to the prescription of Section 2.3 the 2-functor triv on bigons, g on the 2-morphisms of type (1b), f on those of type (1c) and ψ on those of type (1d). The result is a 2-morphism $\text{tra}(\Sigma) : \text{tra}(\gamma_1) \Longrightarrow \text{tra}(\gamma_2)$.

Example 5.2. Let us again go through our three examples.

1. *Differential cocycle.* The parallel transport along Σ is a group $\text{tra}(\Sigma) \in H$ that satisfies the equation
   $$\text{tra}(\gamma_2) = t(\text{tra}(\Sigma)) \cdot \text{tra}(\gamma_1),$$
   where $\text{tra}(\gamma_1), \text{tra}(\gamma_2) \in G$ are the parallel transports along the source path and the target path, and $t : H \longrightarrow G$ is the Lie group homomorphism from the crossed module $\mathfrak{G}$.

2. *Abelian bundle gerbe.* The parallel transport along Σ is an equivariant map
   $$\text{tra}(\Sigma) : \text{tra}(\gamma_1) \longrightarrow \text{tra}(\gamma_2)$$

between the S^1-torsors $\mathrm{tra}(\gamma_1)$ and $\mathrm{tra}(\gamma_2)$.

3. *Non-abelian bundle gerbe.* The parallel transport along Σ is a bi-equivariant map

$$\mathrm{tra}(\Sigma) : \mathrm{tra}(\gamma_1) \longrightarrow \mathrm{tra}(\gamma_2)$$

between the H-bitorsors $\mathrm{tra}(\gamma_1)$ and $\mathrm{tra}(\gamma_2)$.

In the next section we concentrate in certain bigons that parameterize surfaces; the parallel transport along these bigons will be called the holonomy of the transport 2-functor tra.

5.2 Holonomy around Surfaces

Usually, holonomy is understood as the parallel transport along closed paths. In particular „holonomy around a closed line“ is not a well-defined expression since it depends on the choice of a base point and of an orientation. In other words, one has to represent the closed line as the image of a closed path.

In the same way one cannot expect that „holonomy around a closed surface“ is well-defined. We infer that one first has to represent the closed surface as the image of a „closed bigon“ that generalizes a closed path. Possible generalizations are:

(a) Bigons $\Sigma : \gamma \Longrightarrow \gamma$ from some path $\gamma : x \longrightarrow y$ to itself.

(b) More particular, bigons $\Sigma : \tau \Longrightarrow \tau$ from some loop $\tau : x \longrightarrow x$ to itself.

(c) Even more particular, bigons $\Sigma : \mathrm{id}_x \Longrightarrow \mathrm{id}_x$.

The evaluation of a transport 2-functor $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ on such bigons gives indeed rise to interesting structure: in case (a) one obtains a 2-groupoid whose objects are the points in the base manifold and whose 1-morphisms are the images $\mathrm{tra}(\gamma)$ of all paths $\gamma : x \longrightarrow y$. In case (b) one obtains a (probably weak) Lie 2-group attached to each point x, whose objects are the images $\mathrm{tra}(\tau)$ of all loops located at x. In case (c) one obtains an ordinary group attached to each point, whose elements are the images $\mathrm{tra}(\Sigma)$ of all bigons $\Sigma : \mathrm{id}_x \Longrightarrow \mathrm{id}_x$. These groups are actually abelian: this follows from the same kind of Eckman-Hilton argument which proves that the second homotopy group of a space is abelian.

We can thus associate a holonomy 2-groupoid, a holonomy 2-group or a holonomy group to a transport 2-functor. The investigation of these structures for particular examples of transport 2-functors could be an interesting and difficult problem. In the remainder of this article we shall, however, return to the problem of defining the „holonomy around a closed surface“ by representing the given surface as the image of a particular bigon. This problem is mainly motivated by the applications of gerbes with connection in conformal field theory, where these surface holonomies contribute terms to certain action functionals, see e.g. [Gaw88].

It is clear that only surfaces of particular topology can be represented by bigons from the above list. We should hence take a different class of bigons into account. These bigons have the form

$$\Sigma : \tau \Longrightarrow \mathrm{id}_x,$$

starting at a loop $\tau : x \longrightarrow x$ and ending at the identity path at x.

Definition 5.3. *If S is a closed and oriented surface, we call a bigon $\Sigma : \tau \Longrightarrow \mathrm{id}_x$ in S a fundamental bigon for S, if its map $\Sigma : [0,1]^2 \longrightarrow S$ is orientation-preserving, surjective, and – restricted to the interior $(0,1)^2$ – injective.*

It is easy to see that any closed oriented surface has a fundamental bigon. First, the surface can be represented by a fundamental polygon, which has an even number of pairwise identified edges. Let x be a vertex of this polygon, and let $\tau : x \longrightarrow x$ be a parameterization of the boundary, oriented in the way induced from the orientation of S. If the surface is of genus n, τ has the form

$$\tau = \alpha_{2n}^{-1} \circ \alpha_{2n-1}^{-1} \circ \alpha_{2n} \circ \alpha_{2n-1} \circ \ldots \circ \alpha_2^{-1} \circ \alpha_1^{-1} \circ \alpha_2 \circ \alpha_1$$

for paths $\alpha_i : x \longrightarrow x$ that parameterize the edges of the polygon. Now, a fundamental bigon $\Sigma : \tau \Longrightarrow \mathrm{id}_x$ is given by the linear contraction of the polygon to the point x.

Definition 5.4. *Let S be a closed and oriented surface and let $\phi : S \longrightarrow M$ be a smooth map. For a transport 2-functor $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ and a fundamental bigon Σ for S we call the 2-morphism*

$$\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma) := \mathrm{tra}(\phi_*\Sigma) : \mathrm{tra}(\phi_*\tau) \Longrightarrow \mathrm{tra}(\mathrm{id}_{\phi(x)})$$

in T the holonomy of tra around S.

In general, the holonomy around a surface depends on the choice of the fundamental bigon. In the following we want to specify this dependence in more detail.

Lemma 5.5. *Let S be a closed and oriented surface with fundamental bigon*

$$\Sigma : \tau \Longrightarrow \mathrm{id}_x,$$

let $\phi : S \longrightarrow M$ be a smooth map and let $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ be a transport 2-functor.

(a) If $\Sigma' : \tau \Longrightarrow \mathrm{id}_x$ is another fundamental bigon for S with the same loop τ,

$$\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma) = \mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma'),$$

i.e. the holonomy is – for fixed base point x and fixed loop τ – independent of the choice of the fundamental bigon.

(b) If $\gamma : x \longrightarrow y$ is a path, $\tau_\gamma := \gamma \circ \tau \circ \gamma^{-1}$ is loop based at y and $\Sigma^\gamma := \mathrm{id}_\gamma \circ \Sigma \circ \mathrm{id}_{\gamma^{-1}}$ is a fundamental bigon $\Sigma^\gamma : \tau_\gamma \Longrightarrow \mathrm{id}_y$ for S. Then,

$$\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma^\gamma) = \mathrm{id}_{\mathrm{tra}(\phi_*\gamma)} \circ \mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma) \circ \mathrm{id}_{\mathrm{tra}(\phi_*\gamma^{-1})},$$

i.e. the holonomy becomes conjugated when the base point is moved.

(c) Suppose τ has the form $\tau = \gamma_2 \circ \alpha^{-1} \circ \gamma_1 \circ \alpha \circ \gamma_0$ for $\alpha : a \longrightarrow b$ some path, for instance when τ is like in (5.2) and α is one of the α_i. Let $\alpha' : a \longrightarrow b$ be another path and let $\Delta : \alpha' \Longrightarrow \alpha$ be a bigon whose map $\Delta : [0,1]^2 \longrightarrow S$ is injective restricted to the interior. Then, $\tau' := \gamma_2 \circ \alpha'^{-1} \circ \gamma_1 \circ \alpha' \circ \gamma_0$ is another loop based at x, and $\Sigma' := \Sigma \bullet (\mathrm{id}_{\gamma_2} \circ \Delta^\# \circ \mathrm{id}_{\gamma_1} \circ \Delta \circ \mathrm{id}_{\gamma_0})$ is another fundamental bigon, and

$$\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma') = \mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma) \bullet (\mathrm{id}_{\gamma_2} \circ \Delta^\# \circ \mathrm{id}_{\gamma_1} \circ \Delta \circ \mathrm{id}_{\gamma_0}),$$

where $\Delta^\# : \alpha'^{-1} \Longrightarrow \alpha^{-1}$ is the „horizontally inverted“ bigon given by $\Delta^\#(s,t) := \Delta(s, 1-t)$.

Proof. The first assertion follows from the fact that the two fundamental bigons are homotopy equivalent, and thus, since S is a manifold of dimension two, even thin homotopy equivalent. The second and the third assertion follow from the 2-functorality of tra. □

Summarizing, the holonomy of a transport 2-functor around a closed and oriented surface S depends on the choice of a base point $x \in S$ and on the choice of a loop τ based at x. In the remainder of this section we discuss this dependence for differential $\mathfrak{G}$-cocycles and abelian bundle gerbes.

Holonomy of differential cocycles. Let $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow \mathcal{B}\mathfrak{G}$ be a transport 2-functor with $\mathcal{B}\mathfrak{G}$-structure corresponding to a degree two differential $\mathfrak{G}$-cocycle as discussed in Section 4.1. As always, the Lie 2-group $\mathfrak{G}$ is represented by a smooth crossed module (G, H, t, α). According to Examples 5.1 and 5.2, the holonomy of this differential cocycle around a surface S with fundamental bigon $\Sigma : \tau \Longrightarrow \mathrm{id}_x$ is a group element $\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma) \in H$ such that

$$t(\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma)) = \mathrm{tra}(\tau)^{-1}.$$

If the base point is moved along a path γ like investigated in Lemma 5.5 (b), it is changed by the action of $\mathrm{tra}(\gamma) \in G$,

$$\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma^\gamma) = \alpha(\mathrm{tra}(\gamma), \mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma)).$$

This follows from the horizontal composition rule of the 2-groupoid $\mathcal{B}\mathfrak{G}$, see Section 3.1.

If the loop is changed by a bigon as described in Lemma 5.5 (c) we find

$$\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma') = \mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma) \cdot \alpha(g_2 g^{-1}, h^{-1}\alpha(g_1, h)), \tag{5.4}$$

where $h := \mathrm{tra}(\Delta)$, $g := \mathrm{tra}(\alpha')$, $g_2 := \mathrm{tra}(\gamma_2 \circ \alpha'^{-1})$ and $g_1 := \mathrm{tra}(\gamma_1)$. Here we have used $\mathrm{tra}(\Delta^{\#}) = \alpha(g^{-1}, h^{-1})$, which follows from the axioms of the 2-functor tra. In order to handle the formula (5.4) let us introduce the following notation. We write $[G, H] \subset H$ for the Lie subgroup of H which is generated by all elements of the form $h^{-1}\alpha(g, h)$, for $h \in H$ and $g \in G$. The group $[G, H]$ generalizes the commutator subgroup $[H, H]$ of H, see Example 5.7 below. The axioms of the crossed module (G, H, t, α) infer

Lemma 5.6. *The subgroup $[G, H]$ of H is invariant under automorphisms $\alpha_g : H \longrightarrow H$ for all $g \in G$. In particular, it is invariant under conjugation and hence a normal subgroup of H.*

Thus, the image of $\mathrm{Hol}_{\mathrm{tra}}(\phi, \Sigma) \in H$ in the quotient $H/[G, H]$ is independent of the choice of the loop τ.

Example 5.7. Let us specify to two examples of Lie 2-groups $\mathfrak{G}$:

(a) In the case of the 2-group $\mathcal{B}A$ for A an ordinary abelian Lie group, the holonomy is an element in A, and since here α is the trivial action and $[1, A]$ is the trivial group, the holonomy is independent of both under the choice of the base point and under changes of the loop.

(b) Let G be an ordinary Lie group and let $\mathcal{E}G$ the associated 2-group of inner automorphisms, see Section 3.3. Since α here is the conjugation action of G on itself, the holonomy becomes conjugated when moving the base point, just like in the case of ordinary holonomy. Further, the subgroup $[G, H]$ we have considered above is here just the ordinary commutator subgroup $[G, G]$, so that the image of the holonomy in $G/[G, G]$ is independent of the choice of the loop.

Holonomy of abelian bundle gerbes. Let $\mathcal{G}$ be an abelian bundle gerbe with connection over M, and let $\mathrm{tra}_{\mathcal{G}} : \mathcal{P}_2(M) \longrightarrow \mathcal{B}(S^1\text{-Tor})$ be the associated transport 2-functor. According to Examples 5.1 and 5.2, the holonomy of $\mathrm{tra}_{\mathcal{G}}$ around a surface S with fundamental bigon $\Sigma : \tau \Longrightarrow \mathrm{id}_x$ is a S^1-equivariant map

$$\mathrm{Hol}_{\mathrm{tra}_{\mathcal{G}}}(\phi, \Sigma) : \mathrm{tra}_{\mathcal{G}}(\tau) \longrightarrow S^1$$

which one can uniquely identify with an element $\mathrm{Hol}_{\mathrm{tra}_{\mathcal{G}}}(\phi, \Sigma) \in S^1$. By Lemma 5.5 it is clear that it is independent of the choice of the fundamental bigon, of the choice of the base point and of the choice of the loop τ. We can thus compare it with the holonomy of the abelian bundle gerbe $\mathcal{G}$, which has been defined in [Mur96], see also [GR02, CJM02].

Proposition 5.8. *Let $\mathcal{G}$ be an abelian bundle gerbe with connection over M, and let $\mathrm{tra}_{\mathcal{G}} : \mathcal{P}_2(M) \longrightarrow \mathcal{B}(S^1\text{-Tor})$ be the associated transport 2-functor. Then, the holonomies of $\mathcal{G}$ and $\mathrm{tra}_{\mathcal{G}}$ coincide, i.e.*

$$\mathrm{Hol}_{\mathcal{G}}(\phi, S) = \mathrm{Hol}_{\mathrm{tra}_{\mathcal{G}}}(\phi, \Sigma)$$

for S an oriented closed surface, $\Sigma : \tau \Longrightarrow \mathrm{id}_x$ a fundamental for S and $\phi : S \longrightarrow M$ a smooth map.

Proof. To see this, we have to recall how the holonomy of the bundle gerbe $\mathcal{G}$ with connection is defined. The pullback $\phi^*\mathcal{G}$ is by dimensional reasons isomorphic to a trivial bundle gerbe, which has a connection solely given by a 2-form $\rho \in \Omega^2(S)$. Then,

$$\mathrm{Hol}_{\mathcal{G}}(\phi, S) = \exp\left(\mathrm{i}\int_S \rho\right). \tag{5.5}$$

Now, since Corollary 4.9 infers an equivalence between 2-categories of bundle gerbes and of transport 2-functors, also the pullback $\phi^*\mathrm{tra}_{\mathcal{G}}$ is equivalent to a trivial transport 2-functor tra_ρ^∞, where ρ is the same 2-form as above. We infer that in the present case of transport 2-functors with $\mathcal{B}\mathcal{B}S^1$-structure equivalent transport 2-functors have the same holonomies. The holonomy of the trivial transport 2-functor tra_ρ^∞ is according to [SW08]

$$\mathrm{tra}_\rho^\infty(\Sigma) = \exp\left(\mathrm{i}\int_{[0,1]} \mathcal{A}_\Sigma\right) = \exp\left(\mathrm{i}\int_{[0,1]^2} \Sigma^*\rho\right),$$

where $\mathcal{A}_\Sigma \in \Omega^1([0,1])$ is the 1-form from equation (2.25) in [SW08] reduced to the present abelian case. Since Σ is a regular and orientation-preserving parameterization of the surface S, the last expression coincides with (5.5). □

A Basic 2-Category Theory

We introduce notions and facts that we need in this article. For a more complete introduction to 2-categories, see, e.g. [Lei98].

Definition A.1. *A (small) 2-category consists of a set of objects, for each pair (X, Y) of objects a set of 1-morphisms denoted $f : X \longrightarrow Y$ and for each pair (f, g) of 1-morphisms $f, g : X \longrightarrow Y$ a set of 2-morphisms denoted $\varphi : f \Longrightarrow g$, together with the following structure:*

1. *For every pair (f, g) of 1-morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, a 1-morphism $g \circ f : X \longrightarrow Y$, called the composition of f and g.*

2. *For every triple (f, g, h) of 1-morphisms $f : W \longrightarrow X$, $g : X \longrightarrow Y$ and $h : Y \longrightarrow Z$, a 2-morphism*

$$a_{f,g,h} : (h \circ g) \circ f \Longrightarrow h \circ (g \circ f)$$

called the associator of f, g and h.

3. *For every object X, a 1-morphism $\mathrm{id}_X : X \longrightarrow X$, called the identity 1-morphism of X.*

4. *For every 1-morphism $f : X \longrightarrow Y$, 2-morphisms $l_f : f \circ \mathrm{id}_X \Longrightarrow f$ and $r_f : \mathrm{id}_Y \circ f \Longrightarrow f$, called the left and the right unifier.*

5. *For every pair (φ, ψ) of 2-morphisms $\varphi : f \Longrightarrow g$ and $\psi : g \Longrightarrow h$, a 2-morphism $\psi \bullet \varphi : f \Longrightarrow h$, called the vertical composition of φ and ψ.*

6. *For every 1-morphism f, a 2-morphism $\mathrm{id}_f : f \Longrightarrow f$, called the identity 2-morphism of f.*

7. *For every triple (X, Y, Z) of objects, 1-morphisms $f, f' : X \longrightarrow Y$ and $g, g' : Y \longrightarrow Z$, and every pair (φ, ψ) of 2-morphisms $\varphi : f \Longrightarrow f'$ and $\psi : g \Longrightarrow g'$, a 2-morphism $\psi \circ \varphi : g \circ f \Longrightarrow g' \circ f'$, called the horizontal composition of φ and ψ.*

This structure has to satisfy the following list of axioms:

(C1) The vertical composition of 2-morphisms is associative,

$$(\phi \bullet \varphi) \bullet \psi = \phi \bullet (\varphi \bullet \psi)$$

whenever these compositions are well-defined, while the horizontal composition is compatible with the associator in the sense that the diagram

$$\begin{array}{ccc}
(h \circ g) \circ f & \xRightarrow{(\psi\circ\varphi)\circ\phi} & (h' \circ g') \circ f' \\
{\scriptstyle a_{f,g,h}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle a_{f',g',h'}} \\
h \circ (g \circ f) & \xRightarrow[\psi\circ(\varphi\circ\phi)]{} & h' \circ (g' \circ f')
\end{array}$$

is commutative.

(C2) The identity 2-morphisms are units with respect to vertical composition,

$$\varphi \bullet \mathrm{id}_f = \mathrm{id}_g \bullet \varphi$$

for every 2-morphism $\varphi : f \Longrightarrow g$, while the identity 1-morphisms are compatible with the unifiers and the associator in the sense that the diagram

$$\begin{array}{ccc}
(g \circ \mathrm{id}_Y) \circ f & \xRightarrow{a_{f,\mathrm{id}_Y,g}} & g \circ (\mathrm{id}_Y \circ f) \\
& {\scriptstyle l_g \circ \mathrm{id}_f} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_g \circ r_f} & \\
& g \circ f &
\end{array}$$

is commutative. Horizontal composition preserves the identity 2-morphisms in the sense that

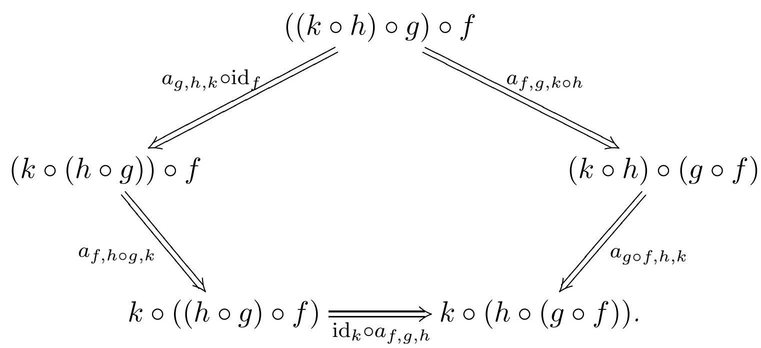
$$\mathrm{id}_g \circ \mathrm{id}_f = \mathrm{id}_{g\circ f}.$$

(C3) Horizontal and vertical compositions are compatible in the sense that

$$(\psi_1 \bullet \psi_2) \circ (\varphi_1 \bullet \varphi_2) = (\psi_1 \circ \varphi_1) \bullet (\psi_2 \circ \varphi_2)$$

whenever these compositions are well-defined.

(C4) All associators and unifiers are invertible 2-morphisms and natural in f, g and h, and the associator satisfies the pentagon axiom



In (C4) we have called a 2-morphism $\varphi : f \implies g$ *invertible* or 2-*iso*morphism, if there exists a 2-morphism $\psi : g \implies f$ such that $\psi \bullet \varphi = \mathrm{id}_f$ and $\varphi \bullet \psi = \mathrm{id}_g$. The axioms imply a *coherence theorem*: all diagrams of 2-morphisms whose arrows are labelled by associators, right or left unifiers, and identity 2-morphisms, are commutative. A 2-category is called *strict*, if

$$(h \circ g) \circ f = h \circ (g \circ f) \quad \text{and} \quad a_{f,g,h} = \mathrm{id}_{h\circ g\circ f}$$

for all triples (f, g, h) of composable 1-morphisms, and if

$$f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f \quad \text{and} \quad r_f = l_f = \mathrm{id}_f$$

for all 1-morphisms f. Strict 2-categories allow us to draw pasting diagrams, since multiple compositions of 1-morphisms are well-defined without putting brackets. Pasting diagrams are often more instructive than commutative diagrams of 2-morphisms. Notice that for a *strict* 2-category

- axiom (C1) claims that both vertical and horizontal composition are associative,
- axiom (C2) claims that the 2-morphisms id_f are identities with respect to the vertical composition and preserved by the horizontal composition,

- axiom (C3) is as before,
- while axiom (C4) can be dropped.

For an explicit discussion of the strict case the reader is referred our Appendix A.1 in [SW08].

Example A.2. Let $\mathfrak{C}$ be a monoidal category, i.e. a category equipped with a functor $\otimes : \mathfrak{C}\times\mathfrak{C} \longrightarrow \mathfrak{C}$, a distinguished object $\mathbb{1}$ in $\mathfrak{C}$, a natural transformation α with components

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z),$$

and natural transformations ρ and λ with components

$$\rho_X : \mathbb{1} \otimes X \longrightarrow X \quad \text{and} \quad \lambda_X : X \otimes \mathbb{1} \longrightarrow X$$

which are subject to the usual coherence conditions, see, e.g. [ML97]. The monoidal category $\mathfrak{C}$ defines a 2-category $\mathcal{B}\mathfrak{C}$ in the following way: it has a single object, the 1-morphisms are the objects of $\mathfrak{C}$ and the 2-morphisms between two 1-morphisms X and Y are the morphisms $f : X \longrightarrow Y$ in $\mathfrak{C}$. The composition of 1-morphisms and the horizontal composition is the tensor product $\otimes$, and the associator $a_{X,Y,Z}$ is given by the component $\alpha_{Z,Y,X}$. The identity 1-morphism is the tensor unit $\mathbb{1}$, and the unifiers are given by the natural transformations ρ and λ. The vertical composition and the identity are just the ones of $\mathfrak{C}$. It is straightforward to check that axioms (C1) to (C4) are either satisfies due to the axioms of the category $\mathfrak{C}$, the functor $\otimes$, or the natural transformations α, ρ and λ, or due to the coherence axioms. The 2-category $\mathcal{B}\mathfrak{C}$ is strict if and only if the monoidal category $\mathfrak{C}$ is strict.

In any 2-category, a 1-morphism $f : X \longrightarrow Y$ is called *invertible* or 1-*iso*morphism, if there exists another 1-morphism $g : Y \longrightarrow X$ together with natural 2-isomorphisms $i : g \circ f \Longrightarrow \mathrm{id}_X$ and $j : \mathrm{id}_Y \Longrightarrow f \circ g$ such that the diagrams

$$\begin{array}{ccc} (f\circ g)\circ f & \xRightarrow{j^{-1}\circ \mathrm{id}_f} & \mathrm{id}_Y \circ f \\ {\scriptstyle a_{f,g,f}}\Downarrow & & \\ f\circ(g\circ f) & & \Downarrow{\scriptstyle r_f} \\ {\scriptstyle \mathrm{id}_f\circ i}\Downarrow & & \\ f\circ \mathrm{id}_X & \xRightarrow[l_f]{} & f \end{array} \quad \text{and} \quad \begin{array}{ccc} (g\circ f)\circ g & \xRightarrow{i\circ \mathrm{id}_f} & \mathrm{id}_X \circ g \\ {\scriptstyle a_{g,f,g}}\Downarrow & & \\ g\circ(f\circ g) & & \Downarrow{\scriptstyle r_g} \\ {\scriptstyle \mathrm{id}_g\circ j^{-1}}\Downarrow & & \\ g\circ \mathrm{id}_Y & \xRightarrow[l_g]{} & g \end{array} \tag{A.1}$$

are commutative. Let us remark that neither in the strict nor in the general case the inverse 1-morphism g is uniquely determined. We call a choice of g a *weak inverse* of f.

Remark A.3. Often a 2-category is called bicategory, while a strict 2-category is called 2-category. Invertible 1-morphisms are often called adjoint equivalences.

Definition A.4. *A (strict) 2-category in which every 1-morphism and every 2-morphism is invertible, is called (strict) 2-groupoid.*

The following definition generalizes the one of a functor between categories.

Definition A.5. *Let S and T be two 2-categories. A 2-functor $F : S \longrightarrow T$ assigns*

1. *an object $F(X)$ in T to each object X in S,*
2. *a 1-morphism $F(f) : F(X) \longrightarrow F(Y)$ in T to each 1-morphism $f : X \longrightarrow Y$ in S, and*
3. *a 2-morphism $F(\varphi) : F(f) \Longrightarrow F(g)$ in T to each 2-morphism $\varphi : f \Longrightarrow g$ in S.*

Furthermore, it has

(a) *a 2-isomorphism $u_X : F(\mathrm{id}_X) \Longrightarrow \mathrm{id}_{F(X)}$ in T for each object X in S, and*

(b) *a 2-isomorphism $c_{f,g} : F(g) \circ F(f) \Longrightarrow F(g \circ f)$ in T for each pair of composable 1-morphisms f and g in S.*

Four axioms have to be satisfied:

(F1) The vertical composition is respected in the sense that

$$F(\psi \bullet \varphi) = F(\psi) \bullet F(\varphi) \quad \text{and} \quad F(\mathrm{id}_f) = \mathrm{id}_{F(f)}$$

for all composable 2-morphisms φ and ψ, and any 1-morphism f.

(F2) The horizontal composition is respected in the sense that the diagram

$$\begin{array}{ccc} F(g) \circ F(f) & \xRightarrow{F(\psi) \circ F(\varphi)} & F(g') \circ F(f') \\ {\scriptstyle c_{f,g}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle c_{f',g'}} \\ F(g \circ h) & \xRightarrow[F(\psi \circ \varphi)]{} & F(g' \circ f') \end{array}$$

is commutative for all horizontally composable 2-morphisms φ and ψ.

(F3) The compositor $c_{f,g}$ is compatible with the associators of S and T in the sense that the diagram

$$\begin{array}{ccc}
(F(h)\circ F(g))\circ F(f) & \xRightarrow{a_{F(f),F(g),F(h)}} & F(h)\circ (F(g)\circ F(f)) \\
{\scriptstyle c_{g,h}\circ \mathrm{id}_{F(f)}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mathrm{id}_{F(h)}\circ c_{f,g}} \\
F(h\circ g)\circ F(f) & & F(h)\circ F(g\circ f) \\
{\scriptstyle c_{f,h\circ g}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle c_{g\circ f,h}} \\
F((h\circ g)\circ f) & \xRightarrow[F(a_{f,g,h})]{} & F(h\circ (g\circ f))
\end{array}$$

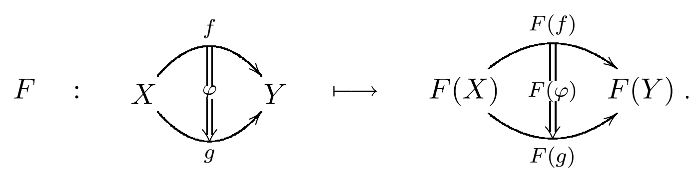
is commutative for all composable 1-morphisms f, g and h.

(F4) Compositor and unitor are natural and compatible with the unifiers of S and T in the sense that the diagrams

$$\begin{array}{ccc}
F(f)\circ F(\mathrm{id}_X) & \xRightarrow{c_{\mathrm{id}_X,f}} & F(f\circ \mathrm{id}_X) \\
{\scriptstyle \mathrm{id}_{F(f)}\circ u_X}\Big\Downarrow & & \Big\Downarrow{\scriptstyle F(l_f)} \\
F(f)\circ \mathrm{id}_{F(X)} & \xRightarrow[l_{F(f)}]{} & F(f)
\end{array}
\quad \textit{and} \quad
\begin{array}{ccc}
F(\mathrm{id}_Y)\circ F(f) & \xRightarrow{c_{f,\mathrm{id}_Y}} & F(\mathrm{id}_Y\circ f) \\
{\scriptstyle u_Y\circ \mathrm{id}_{F(f)}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle F(r_f)} \\
\mathrm{id}_{F(Y)}\circ F(f) & \xRightarrow[r_{F(f)}]{} & F(f)
\end{array}$$

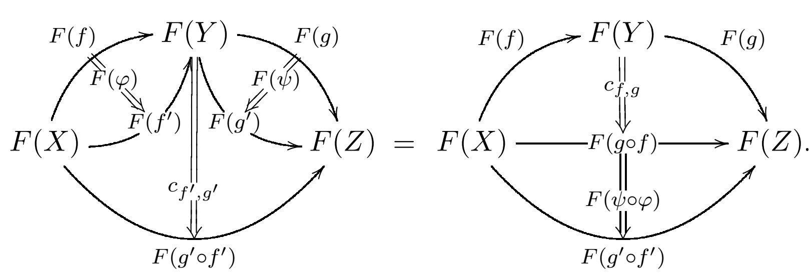
are commutative for every 1-morphism f.

Sometimes we represent a 2-functor $F : S \longrightarrow T$ diagrammatically as an assignment

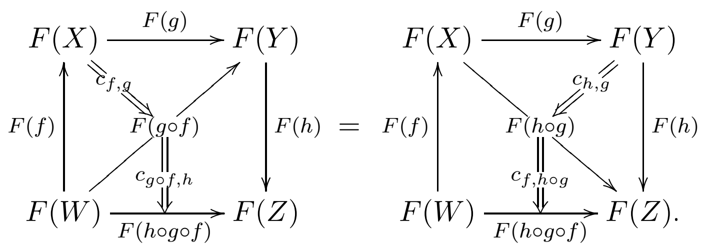


In case that the 2-category T is strict, and the axioms (F2) to (F4) can be expressed by pasting diagrams in the following way:

- Axioms (F2) is equivalent to the equality



- Axiom (F3) is equivalent to the tetrahedron identity



- Axiom (F4) is equivalent to the equalities

$$c_{\mathrm{id}_X,f} = \mathrm{id}_{F(f)} \circ u_X \quad \text{and} \quad c_{f,\mathrm{id}_Y} = u_Y \circ \mathrm{id}_{F(f)}.$$

A 2-functor $F : S \longrightarrow T$ is called *strict*, if

$$F(g) \circ F(f) = F(g \circ f) \quad \text{and} \quad c_{f,g} = \mathrm{id}_{F(g\circ f)}$$

for all composable 1-morphisms f and g, and if

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \text{and} \quad u_X = \mathrm{id}_{\mathrm{id}_{F(X)}}$$

for all objects X in S. In case of strict 2-functors between strict 2-categories only axioms (F1) and (F2) remain, claiming that both compositions are respected. The following definition generalizes a natural transformation between two functors.

Definition A.6. *Let F_1 and F_2 be two 2-functors from S to T. A pseudonatural transformation $\rho : F_1 \longrightarrow F_2$ assigns*

1. *a 1-morphism $\rho(X) : F_1(X) \longrightarrow F_2(X)$ in T to each object X in S, and*
2. *a 2-isomorphism $\rho(f) : \rho(Y) \circ F_1(f) \Longrightarrow F_2(f) \circ \rho(X)$ in T to each 1-morphism $f : X \longrightarrow Y$ in S,*

such that two axioms are satisfied:

(T1) The composition of 1-morphisms in S is respected in the sense that the diagram

$$\begin{array}{ccc}
(\rho(Z) \circ F_1(g)) \circ F_1(f) & \xRightarrow{a_{F_1(f),F_1(g),\rho(Z)}} & \rho(Z) \circ (F_1(g) \circ F_1(f)) \\
\Big\Downarrow {\scriptstyle \rho(g)\circ \mathrm{id}_{F_1(f)}} & & \Big\Downarrow {\scriptstyle \mathrm{id}_{\rho(Z)}\circ (c_1)_{f,g}} \\
(F_2(g) \circ \rho(Y)) \circ F_1(f) & & \rho(Z) \circ F_1(g \circ f) \\
\Big\Downarrow {\scriptstyle a_{F_1(f),\rho(Y),F_2(g)}} & & \Big\Downarrow {\scriptstyle \rho(g\circ f)} \\
F_2(g) \circ (\rho(Y) \circ F_1(f)) & & F_2(g \circ f) \circ \rho(X) \\
\Big\Downarrow {\scriptstyle \mathrm{id}_{F_2(g)}\circ \rho(f)} & & \Big\Downarrow {\scriptstyle (c_2)^{-1}_{f,g}\circ \mathrm{id}_{\rho(X)}} \\
F_2(g) \circ (F_2(f) \circ \rho(X)) & \xRightarrow[a^{-1}_{\rho(X),F_2(f),F_2(g)}]{} & (F_2(g) \circ F_2(f)) \circ \rho(X)
\end{array}$$

is commutative for all composable 1-morphisms f and g. Here, a is the associator of the 2-category T and c_1 and c_2 are the compositors of the 2-functors F_1 and F_2, respectively.

(T2) It is natural in the sense that the diagram

$$\begin{array}{ccc} \rho(Y) \circ F_1(f) & \xRightarrow{\rho(f)} & F_2(f) \circ \rho(X) \\ {\scriptstyle \mathrm{id}_{\rho(Y)} \circ F_1(\varphi)} \Downarrow & & \Downarrow {\scriptstyle F_2(\varphi) \circ \mathrm{id}_{\rho(X)}} \\ \rho(Y) \circ F_1(g) & \xRightarrow[\rho(g)]{} & F_2(g) \circ \rho(X) \end{array}$$

is commutative for all 2-morphisms $\varphi : f \implies g$.

If one considers a version of pseudonatural transformations where the 2-morphisms $\rho(f)$ do not have to be invertible, there is a third axiom related to the value of ρ at the identity 1-morphism id_X of an object X in S. In our setup this axiom follows:

Lemma A.7. *Let $\rho : F_1 \longrightarrow F_2$ be a pseudonatural transformation between 2-functors with unitors u^1 and u^2, respectively, Then, the diagram*

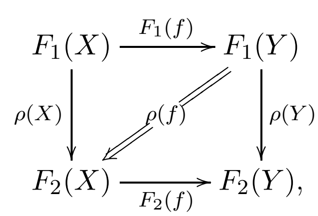
$$\begin{array}{ccccc} \rho(X) \circ F_1(\mathrm{id}_X) & & \xRightarrow{\rho(\mathrm{id}_X)} & & F_2(\mathrm{id}_X) \circ \rho(X) \\ {\scriptstyle \mathrm{id}_{\rho(X)} \circ u^1_X} \Downarrow & & & & \Downarrow {\scriptstyle u^2_X \circ \mathrm{id}_{\rho(X)}} \\ \rho(X) \circ \mathrm{id}_{F_1(X)} & \xRightarrow[l_{\rho(X)}]{} & \rho(X) & \xRightarrow[r^{-1}_{\rho(X)}]{} & \mathrm{id}_{F_2(X)} \circ \rho(X) \end{array}$$

is commutative.

Proof. One applies axiom (T1) to 1-morphisms $f = g = \mathrm{id}_X$. Then one uses axiom (T2) for ρ, axiom (F4) for both 2-functors, axiom (C2) for T, and the invertibility of the 2-morphism $\rho(g)$ and of the 1-morphism $F_2(\mathrm{id}_X)$. $\square$

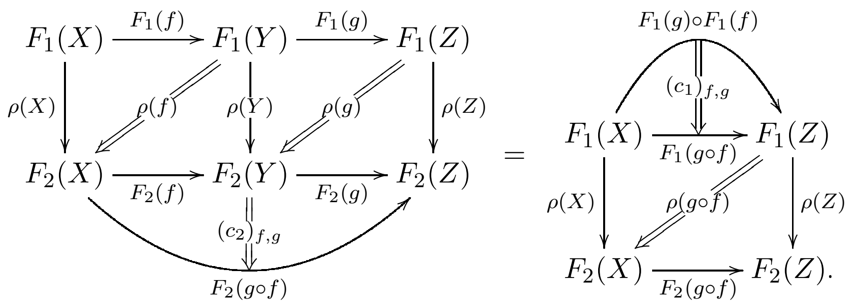
Sometimes we represent a pseudonatural transformation $\rho : F_1 \longrightarrow F_2$ diagrammatically by

$\rho \quad : \quad X \xrightarrow{f} Y \quad \longmapsto$

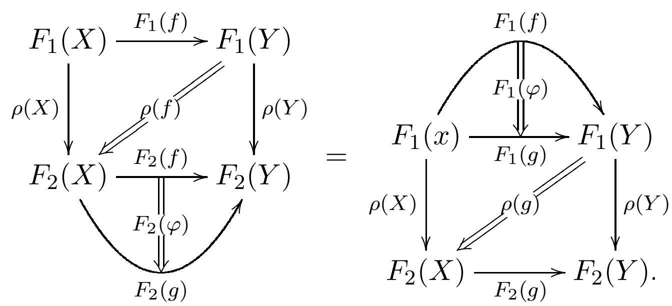


and the axioms can be expressed by pasting diagrams in the following way:

- Axiom (T1) is equivalent to



- Axiom (T2) is equivalent to



Still for the case that the 2-category T is strict, Lemma A.9 implies

$$\rho(\mathrm{id}_X) = ((u_X^2)^{-1} \circ \mathrm{id}_{\rho(X)}) \circ (\mathrm{id}_{\rho(X)} \circ u_X^1).$$

If also the 2-functors F_1 and F_2 are strict, we obtain $\rho(\mathrm{id}_X) = \mathrm{id}_{\rho(X)}$.

We need one more definition for situations where two pseudonatural transformations are present.

Definition A.8. *Let $F_1, F_2 : S \longrightarrow T$ be two 2-functors and let $\rho_1, \rho_2 : F_1 \longrightarrow F_2$ be pseudonatural transformations. A* *<u>modification</u>* *$\mathcal{A} : \rho_1 \Longrightarrow \rho_2$ assigns a 2-morphism*

$$\mathcal{A}(X) : \rho_1(X) \Longrightarrow \rho_2(X)$$

in T to any object X in S such that the diagram

$$\begin{array}{ccc} \rho_1(Y) \circ F_1(f) & \xrightarrow{\rho_1(f)} & F_2(f) \circ \rho_1(X) \\ {\scriptstyle \mathcal{A}(Y) \circ \mathrm{id}_{F_1(f)}} \Downarrow & & \Downarrow {\scriptstyle \mathrm{id}_{F_2(f)} \circ \mathcal{A}(X)} \\ \rho_2(Y) \circ F_1(f) & \xrightarrow[\rho_2(f)]{} & F_2(f) \circ \rho_2(X) \end{array} \tag{A.2}$$

is commutative for every 1-morphism f.

In the case that T is a strict 2-category, the latter diagram is equivalent to a pasting diagram, see Definition A.4 in [SW08].

As one might expect, 2-Functors, pseudonatural transformations and modifications fit again into the structure of a 2-category:

Lemma A.9. *Let S and T be 2-categories. The set of all 2-functors $F : S \longrightarrow T$, the set of all pseudonatural transformations $\rho : F_1 \longrightarrow F_2$ between these 2-functors and the set of all modifications $\mathcal{A} : \rho_1 \Longrightarrow \rho_2$ between those form a 2-category* $\mathrm{Funct}(S, T)$.

Let us describe the structure of this 2-category:

1. The composition of two pseudonatural transformations $\rho_1 : F_1 \longrightarrow F_2$ and $\rho_2 : F_2 \longrightarrow F_3$ is defined by the 1-morphism

$$(\rho_2 \circ \rho_1)(X) := \rho_2(X) \circ \rho_1(X)$$

   and the 2-morphism $(\rho_2 \circ \rho_1)(f)$ which is the following composite:

$$\begin{array}{l}
(\rho_2 \circ \rho_1)(Y) \circ F_1(f) \\
\Big\Downarrow a_{F_1(f),\rho_1(Y),\rho_2(Y)} \\
\rho_2(Y) \circ (\rho_1(Y) \circ F_1(f)) \xRightarrow{\mathrm{id}_{\rho_2(Y)} \circ \rho_1(f)} \rho_2(Y) \circ (F_2(f) \circ \rho_1(X)) \\
\Big\Downarrow a^{-1}_{\rho_1(X),F_2(f),\rho_2(Y)} \\
(\rho_2(Y) \circ F_2(f)) \circ \rho_1(X) \xRightarrow[\rho_2(f) \circ \mathrm{id}_{\rho_1(X)}]{} (F_3(f) \circ \rho_2(X)) \circ \rho_1(X) \\
\Big\Downarrow a_{\rho_1(X),\rho_2(X),F_3(f)} \\
F_3(f) \circ (\rho_2 \circ \rho_1)(X).
\end{array}$$

2. The associator for the above composition of pseudonatural transformations is the modification defined by

$$a_{\rho_1,\rho_2,\rho_3}(X) := a_{\rho_1(X),\rho_2(X),\rho_3(X)},$$

   where a on the right hand side is the associator of T.

3. The identity pseudonatural transformation $\mathrm{id}_F : F \longrightarrow F$ associated to a 2-functor F is defined by $\mathrm{id}_F(X) := \mathrm{id}_{F(X)}$, and $\mathrm{id}_F(f)$ is the composite

$$\mathrm{id}_{F(Y)} \circ F(f) \xRightarrow{r_{F(f)}} F(f) \xRightarrow{l^{-1}_{F(f)}} F(f) \circ \mathrm{id}_{F(X)}.$$

4. The right and left unifiers are the modifications defined by

$$r_\rho(X) := r_{\rho(X)} \quad \text{and} \quad l_\rho(X) := l_{\rho(X)}.$$

5. The vertical composition of two modifications $\mathcal{A} : \rho_1 \Longrightarrow \rho_2$ and $\mathcal{A}' : \rho_2 \Longrightarrow \rho_3$ is defined by
$$(\mathcal{A}' \bullet \mathcal{A})(X) := \mathcal{A}'(X) \bullet \mathcal{A}(X).$$

6. The identity modification associated to a pseudonatural transformation $\rho : F_1 \longrightarrow F_2$ is defined by $\mathrm{id}_\rho(X) := \mathrm{id}_{\rho(X)}$.

7. The horizontal composition of two modifications $\mathcal{A} : \rho_1 \Longrightarrow \rho_2$ and $\mathcal{A}' : \rho_1' \Longrightarrow \rho_2'$ is defined by
$$(\mathcal{A}' \circ \mathcal{A})(X) := \mathcal{A}'(X) \circ \mathcal{A}(X).$$

We leave it to the reader to verify that the axioms of a 2-category are satisfied. From 2. and 4. of the above list it is clear that the 2-category $\mathrm{Funct}(S,T)$ is strict if and only if T is strict. In this case, the composition of pseudonatural transformations introduced in 1. can be depicted as in (A.1) of [SW08].

Another consequence of Lemma A.9 is that we know what invertibility means in the 2-category $\mathrm{Funct}(S,T)$: a 2-isomorphism in the 2-category $\mathrm{Funct}(S,T)$ is called *invertible modification*, and a 1-isomorphism is called *pseudonatural equivalence*. This leads to the following

Definition A.10. *Let S and T be 2-categories.*

- *A 2-functor $F : S \longrightarrow T$ is called an equivalence of 2-categories, if there exists a 2-functor $G : T \longrightarrow S$ together with pseudonatural equivalences $\rho_S : G \circ F \longrightarrow \mathrm{id}_S$ and $\rho_T : F \circ G \longrightarrow \mathrm{id}_T$.*

- *If the 2-categories S and T and the 2-functor F are strict, and G can be chosen strict, F is called a strict equivalence.*

- *If additionally the pseudonatural equivalences ρ_S and ρ_T are identities, F is called an isomorphism of 2-categories.*

B Lifts to the Codescent 2-Groupoid

Here we deliver the proofs of two properties of the codescent 2-groupoid $\mathcal{P}_2^\pi(M)$ we have introduced in Section 2.1.

Lemma B.1. *The category $\mathcal{P}_2^\pi(M)$ is a 2-groupoid.*

Proof. All 2-morphisms except those of type (1a) are invertible by definition. But for a 2-morphism of type (1a), a bigon $\Sigma : \gamma \Longrightarrow \gamma'$, we have
$$\Sigma^{-1} \circledast \Sigma \overset{\text{(II)}}{=} \Sigma^{-1} \bullet \Sigma = \mathrm{id}_\gamma \overset{\text{(II)}}{=} \mathrm{id}_\gamma^{\circledast},$$

and analogously $\Sigma \circledast \Sigma^{-1} = \mathrm{id}_{\gamma'}^{\circledast}$. Here we have used identification (II); more precisely axiom (F1) of the 2-functor $\iota : \mathcal{P}_2(Y) \longrightarrow \mathcal{P}_2^{\pi}(M)$. To see that a path $\gamma : a \longrightarrow b$ is invertible, we claim that γ^{-1} is a weak inverse. It is easy to construct the 2-isomorphisms i_γ and j_γ using the 2-isomorphisms of type (2b). The required identities (A.1) for these 2-isomorphisms are then satisfied due to identification (II). To see that a jump $\alpha \in Y^{[2]}$ with $\alpha = (x, y)$ is invertible, we claim that $\bar{\alpha} := (y, x)$ is a weak inverse. The 2-isomorphisms i_α and j_α can be constructed from 2-isomorphisms of types (1c) and (1d). The identities (A.1) are satisfied du to identifications (V1) and (V2). □

The proof of the „Lifting" Lemma 2.3 requires some preparation.

Lemma B.2. *Let $p \in M$ be a point and $a, b \in Y$ with $\pi(a) = \pi(b) = p$. Let $\alpha : a \longrightarrow b$ and $\beta : a \longrightarrow b$ be 1-morphisms in $\mathcal{P}_2^{\pi}(M)$ which are compositions of jumps.*

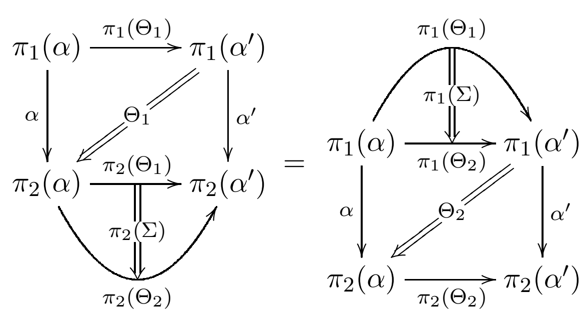
(a) There exists a 2-isomorphism $\Xi : \alpha \Longrightarrow \beta$ with $p^{\pi}(\Xi) = \mathrm{id}_{\mathrm{id}_p}$.

(b) Any 2-isomorphism $\Xi : \alpha \Longrightarrow \beta$ with $p^{\pi}(\Xi) = \mathrm{id}_{\mathrm{id}_p}$ can be represented by a composition of 2-morphisms of type (1c).

(c) The 2-isomorphism from (a) is unique.

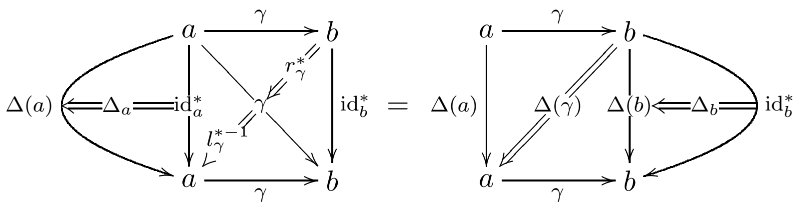
Proof. It is easy to construct the 2-isomorphism from (a) using only 2-isomorphisms of type (1c) and their inverses. To show (b) let $\Xi : \alpha \Longrightarrow \beta$ be a 2-isomorphism with $p^{\pi}(\Xi) = \mathrm{id}_{\mathrm{id}_p}$, represented by a composition of 2-morphisms of any type. In the following we draw pasting diagrams to demonstrate that all 2-morphisms of types (1a), (1b) and (1d) can subsequently be killed.

To prepare some machinery notice that identification (III) imposes axiom (T2) for the pseudonatural transformation Γ, which is, for any bigon $\Sigma : \Theta_1 \Longrightarrow \Theta_2$ in $Y^{[2]}$, the identity:



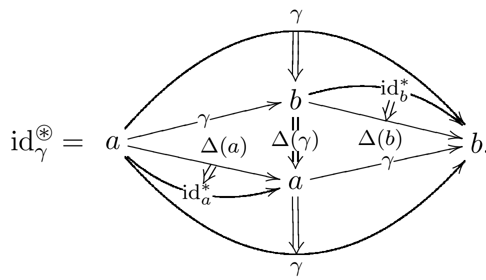
(B.1)

In the same way, identification (IV) imposes the axiom for the modification $\mathrm{id}_\iota \Longrightarrow \Delta^*\Gamma$, which is, for any path $\gamma : a \longrightarrow b$ in Y, the identity



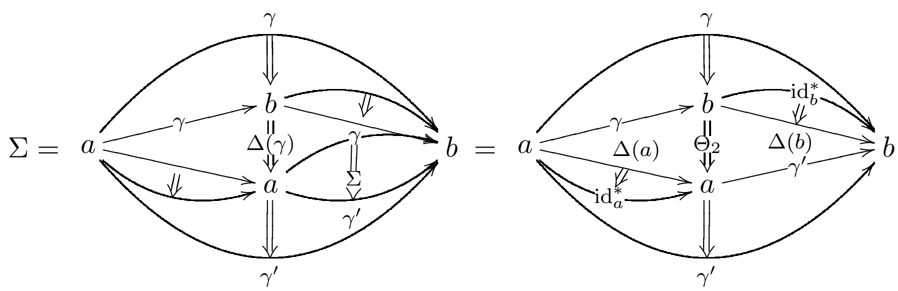
(B.2)

Using (B.2) we can write the identity 2-morphism associated to the path γ in a very fancy way, namely



(B.3)

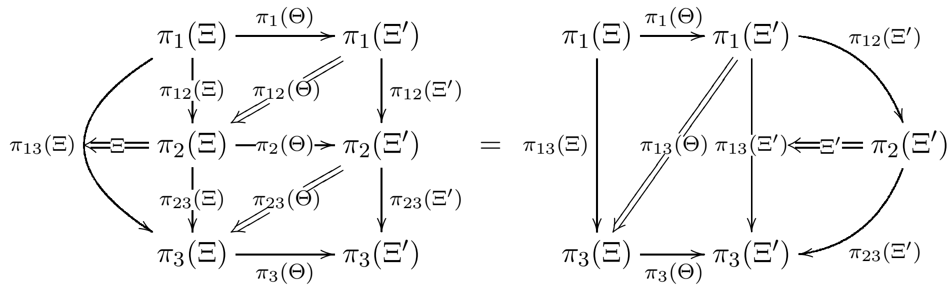
Now suppose that $\Sigma : \gamma \Longrightarrow \gamma'$ is some 2-morphism of type (1a) that we want to kill. We write $\Sigma = \Sigma \circledast \mathrm{id}^{\circledast}_\gamma$ and use (B.3). Using the naturality of the 2-morphism l^*_γ claimed by identification (I) we have



where the second identity is obtained from (B.1) by taking $\Theta_1 := \Delta(\gamma)$ and $\Theta_2 := (\gamma, \gamma')$ which is only possible because we have assumed that $p^\pi(\Sigma) = \mathrm{id}_{\mathrm{id}_p}$. We can thus kill every 2-morphism of type (1a).

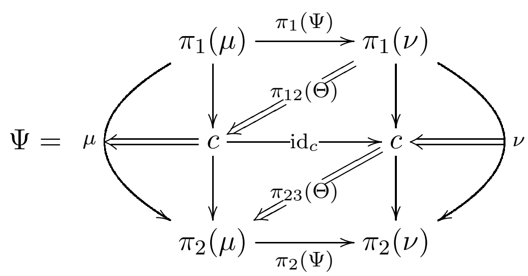
Suppose now that $\Psi : \mu \Longrightarrow \nu$ is a 2-morphism of type (1b). To kill it we need identification (IV), namely the axiom for the modification $\pi^*_{23}\Gamma \circ \pi^*_{12}\Gamma \Longrightarrow \pi^*_{13}\Gamma$. For any path

$\Theta : \Xi \longrightarrow \Xi'$ in $Y^{[3]}$, the corresponding pasting diagram is



(B.4)

Here we suppress writing the associators and the bracketing of the 1-morphisms. Using this identity we have

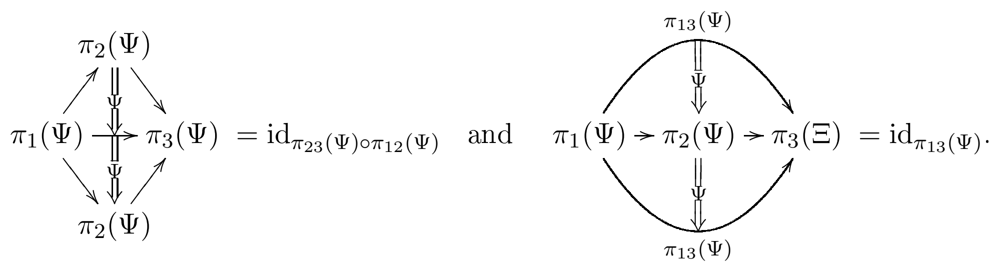


(B.5)

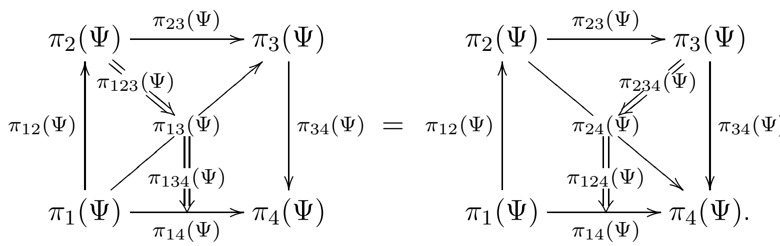
for $c \in Y$ an arbitrary fixed point with $\pi(c) = p$ and $\Theta := (\pi_1(\Psi), \mathrm{id}_c, \pi_2(\Psi))$ which is only possible because $p^\pi(\Psi) = \mathrm{id}_{\mathrm{id}_p}$.

We can so far assume that the 2-morphism $\Xi : \alpha \Longrightarrow \beta$ we started with contains no 2-morphism of type (1a) and by (B.5) only those 2-morphism $\Theta = (\gamma_1, \gamma_2)$ for which γ_1 or γ_2 is the identity path of the point c. If both γ_1 and γ_2 are identity paths, we can replace Θ according to (B.2) by two 2-morphisms of type (1d). It is now a combinatorial task to kill all 2-morphisms which start or end on paths, in particular all 2-morphisms of type (2b). Then one kills all 2-morphisms of types (1d) and the remaining unifiers l_β^* and r_β^*. Finally, all associators a^* can be killed using their naturality with respect to 2-morphisms of type (1c).

To prove (c) we assume that $\Xi' : \alpha \Longrightarrow \beta$ is any 2-isomorphism with $p^\pi(\Xi) = \mathrm{id}_{\mathrm{id}_p}$. By (b) we can assume that Ξ' is composed of 2-isomorphisms of type (1c). First we remark that Ξ and Ξ' induce triangulations of the disc D^2. If we assume that the triangulations induced by Ξ and Ξ' coincide, we already have $\Xi = \Xi'$, since orientations and labels of the edges and of the triangles of Ξ are uniquely determined. If the triangulations do not coincide, we infer that two triangulations of the Riemann surface D^2 can be transformed into each other via the so-called *fusion* and *bubble* moves, see [FHK94, FRS02]. It remains to show that our identifications among the 2-morphisms imply these two moves. The bubble move follows from the fact that the 2-morphisms of type (1c) are invertible:



The fusion move follows from identification (V1), which is in pasting diagrams for a point $\Psi \in Y^{[4]}$



Analogous identities for inverses $\bar{\Psi}$ and mixtures of Ψ and $\bar{\Psi}$ can also be deduced. □

Now let $\gamma : x \longrightarrow y$ be a path in M, and let $\tilde{x}, \tilde{y} \in Y$ be lifts of the endpoints, i.e. $\pi(\tilde{x}) = x$ and $\pi(\tilde{y}) = y$. We are ready to prove Lemma 2.3 from Section 2.2, namely

(a) There exists a 1-morphism $\tilde{\gamma} : \tilde{x} \longrightarrow \tilde{y}$ in $\mathcal{P}_2^\pi(M)$ such that $p^\pi(\tilde{\gamma}) = \gamma$.

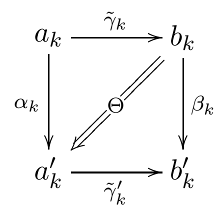
(b) Let $\tilde{\gamma} : \tilde{x} \longrightarrow \tilde{y}$ and $\tilde{\gamma}' : \tilde{x} \longrightarrow \tilde{y}$ be two such 1-morphisms. Then, there exists a unique 2-isomorphism $A : \tilde{\gamma} \Longrightarrow \tilde{\gamma}'$ in $\mathcal{P}_2^\pi(M)$ such that $p^\pi(A) = \mathrm{id}_\gamma$.

The assertion (a) is proven in Lemma 2.15 of [SW07]. To prove (b), we compare the two lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ of γ in the following way. Let $P \subset M$ be the set of points over whose fibre either $\tilde{\gamma}_1$ or $\tilde{\gamma}_2$ has a jump. The set P is finite and ordered by the orientation of the path γ, so that we may put $P = \{p_0, ..., p_n\}$ with $p_0 = x$ and $p_n = y$. Now we write

$$\gamma = \gamma_n \circ ... \circ \gamma_1$$

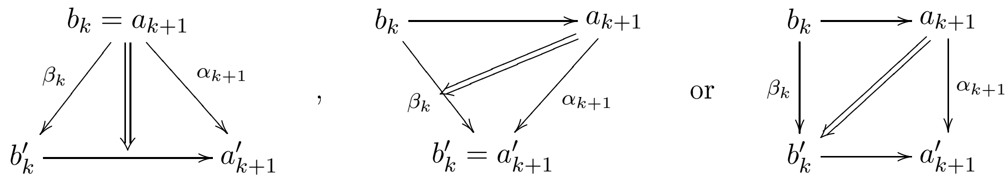
for paths $\gamma_k : p_{k-1} \longrightarrow p_k$. We also write $\tilde{\gamma}$ as a composition of lifts $\tilde{\gamma}_k : a_k \longrightarrow b_k$ of γ_k and (possibly multiple) jumps $b_k \longrightarrow \alpha_{k+1}$ located over the points p_k; analogously for $\tilde{\gamma}'$. This defines jumps $\alpha_k := (a_k, a'_k)$ and $\beta_k := (b_k, b'_k)$. Now, over the paths γ_k, we take

2-isomorphisms



(B.6)

with $\Theta := (\tilde{\gamma}_k, \tilde{\gamma}'_k)$. Over the points p_k we need 2-isomorphisms of the form



(B.7)

the first whenever $\tilde{\gamma}'$ has jumps over p_k and $\tilde{\gamma}$ has not, the second whenever $\tilde{\gamma}$ has jumps and $\tilde{\gamma}'$ has not, and the third whenever both lifts have jumps. By Lemma B.2 these 2-isomorphisms exist and are unique. Then, all of the four diagrams above can be put next to each other; this defines a 2-isomorphism $\tilde{\gamma} \Longrightarrow \tilde{\gamma}'$. We call the 2-morphism constructed like this the canonical 2-morphism.

It remains to show that every 2-morphism $A : \tilde{\gamma} \Longrightarrow \tilde{\gamma}'$ with $p^\pi(A) = \mathrm{id}_\gamma$ is equal to this canonical 2-morphism. First, we kill all bigons contained in A by the argument given in the proof of Lemma B.2. We consider two cases:

1. A contains no paths except those contained in $\tilde{\gamma}$ or $\tilde{\gamma}'$. In this case A is already equal to the canonical 2-morphism. Namely, each of the pieces $\tilde{\gamma}_k$ or $\tilde{\gamma}'_k$ can only be target or source of a 2-morphism of type (1b). These are now necessarily the pieces (B.6). It remains to consider the 2-morphisms between the jumps. But these are by Lemma B.2 equal to the pieces (B.7). This shows that A is the canonical 2-morphism.

2. There exists a path γ_0 in $\mathcal{P}_2^\pi(M)$ which is target or source of some 2-morphism contained in A but not contained in $\tilde{\gamma}$ or $\tilde{\gamma}'$. In this case there exists a 1-morphism $\tilde{\gamma}_o : \tilde{x} \longrightarrow \tilde{y}$ together with 2-morphisms $A_1 : \tilde{\gamma} \Longrightarrow \tilde{\gamma}_0$ and $A_2 : \tilde{\gamma}_0 \Longrightarrow \tilde{\gamma}'$ such that $A = A_2 \bullet A_1$. By iteration, we can decompose A in a vertical composition of 2-morphisms which fall into case 1, i.e. into a vertical composition of canonical 2-morphisms.

It remains to conclude with the observation that the vertical composition $A_2 \bullet A_1$ of two canonical 2-morphisms is again canonical. □

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