

SOME REMARKS ON DEFECTS AND T-DUALITY

Gor Sarkissian, Christoph Schweigert *

Organisationseinheit Mathematik, Universität Hamburg
Bereich Algebra und Zahlentheorie
Bundesstraße 55, D-20 146 Hamburg

October 2008

Abstract

The equations of motion for a conformal field theory in the presence of defect lines can be derived from an action that includes contributions from branes. For T-dual toroidal compactifications, they imply a direct relation between Poincaré line bundles and the action of T-duality on boundary conditions. We also exhibit a class of diagonal defects that induce a shift of the B-field. We finally study T-dualities for S^1 -fibrations in the example of the Wess-Zumino-Witten model on $SU(2)$ and lens spaces. Using standard techniques from D-branes, we derive from algebraic data in rational conformal field theories geometric structures familiar from Fourier-Mukai transformations.

* Email addresses:
sarkissian@math.uni-hamburg.de, schweigert@math.uni-hamburg.de

1 Introduction

Defect lines in two-dimensional quantum field theories are oriented lines separating different quantum field theories. Such defect lines have a rich behaviour:

- They can carry defect fields changing the type of the defect, very much in the way the type of a boundary condition can change at the insertion of a boundary field.
- Several defect lines can merge and split at vertices, leading to defect junctions.
- Defect lines with the same conformal field theory on both sides can start and end at disorder fields.

Defects and dualities of lattice models have been studied extensively (see e.g. [13] for a discussion and references). In fact, the simplest realization of a defect line in the Ising model is a line with the property that all bonds crossing this line have antiferromagnetic coupling. In conformal field theories describing the continuum limit of such theories, defects have only been studied recently (see e.g. [2, 18, 31]).

A particularly important subclass of defects are topological defects: they are characterized by the fact that correlation functions do not depend on the precise position of the defect line: They do not change under small continuous changes of position of the defect line. This topological nature implies that there is a well-defined notion of fusion for such defects. It has been shown [15, 16] that the fusion algebra of topological defects contains structural information about symmetries and Kramers-Wannier dualities of the theory. All defects considered in this paper are topological, except possibly the one implementing a shift on the B-field.

Much information about topological defects is by now available in concrete classes of models, and in fact the fusion algebra of topological defects in rational conformal field theories can be computed explicitly in the TFT approach to RCFT correlators [18, 19, 38]. Recently, there has been some progress for the Lagrangian description of topological defects in conformal sigma-models, see e.g. [20, 32–34] for the case of sigma models on group manifolds. In particular, a geometric description of Wess-Zumino terms for sigma models with non-trivial B-field background in has been established [21, 34]. The appropriate framework

for theories with non-trivial background B-field are hermitian bundle gerbes with connection [22]. In such a situation, defects are described by bibranes [21]: submanifolds of the Cartesian product of the target spaces, equipped with a gerbe bimodule, i.e. a specific one-morphism of bundle gerbes. Defect junctions are described [34] by so-called “inter-bibranes”, which contain as part of the data a specific two-morphism of bundle gerbes. From such a Lagrangian description, equations of motion can be derived [34].

The present letter is a case study of defects in different exactly solvable CFT backgrounds which admit a geometric description as tori or toroidal fibrations. For such backgrounds we use both the defect equations of motion and standard techniques from D-branes to make contact to structures encountered in geometric approaches to conformal field theory.

We explain the main tool used in Section 2 and 3: suppose that a defect S separates the worldsheet Σ into two disjoint components Σ_1 and Σ_2 . On Σ_i the theory is described by a target space M_i . In such a situation, the sigma model action is defined for pairs of maps $X : \Sigma_1 \rightarrow M$ and $Y : \Sigma_2 \rightarrow \tilde{M}$. The defect equations (9) for all cases considered in this paper have schematically the form $\partial X = f_{\mathcal{P}}(\partial Y, \bar{\partial} Y)$ and $\bar{\partial} X = \bar{f}_{\mathcal{P}}(\partial Y, \bar{\partial} Y)$. Given a boundary condition of the form $d(\partial X, \bar{\partial} X) = 0$, the defect equation defines a new boundary condition $d_{\mathcal{P}}(\partial Y, \bar{\partial} Y) = 0$, where $d_{\mathcal{P}} = d \circ (f_{\mathcal{P}}, \bar{f}_{\mathcal{P}})$. A notion of fusion between defects and a boundary can be expected in the case of topological defects, since the latter can be moved to the boundary without changing the correlator. For general defects, the correlator depends on the relative position of the defect lines and a notion of fusion is not obvious.

This reasoning provides in particular a direct relation between T-duality of tori and Poincaré line bundles on the product of T-dual tori. Applying a related reasoning to diagonal bibranes shows how the action of a defect can result in a shift of the B-field. Such a phenomenon is known from monodromies in the moduli space of a Calabi-Yau compactification. In fact, the action of a defect on a boundary condition can be expected [21] to be of “Fourier-Mukai” type. This should be compared to Orlov’s theorem [29] which provides a strong relation between fully faithful functors on the derived categories of coherent sheaves on smooth projective varieties and Fourier-Mukai functors. Inducing shifts of the B-field in terms of defects can be seen as a physical realization of such a relation.

In a second part of this paper, we study T-dualities in the special case of the group manifold $SU(2)$ and a lens space. In this study, we use the fact that defects

(at least in rational conformal field theories) are uniquely determined by their action on bulk fields. We construct several families of defects by using T-duality and orbifold transformations. For one such family we determine the geometry of the underlying bibrane. We recover structure familiar from Fourier-Mukai transformations.

2 Defect equations

Let us consider a conformal interface between two conformal field theories admitting a sigma model description with target spaces M and \tilde{M} . For simplicity, we consider the situation described in the introduction, where the world sheet, a two-dimensional oriented manifold Σ , is separated by an embedded oriented circle S into two connected components. We take the convention that $\partial\Sigma_1 = S$ and $\partial\Sigma_2 = \bar{S}$ as equalities of oriented manifolds, where \bar{S} is the manifold S with opposite orientation.

In such a situation, the sigma model action is defined for a pair of maps

$$X : \Sigma_1 \rightarrow M \quad \text{and} \quad Y : \Sigma_2 \rightarrow \tilde{M}. \quad (1)$$

On the defect line S itself, one has to impose conditions that relate the two maps. The necessary data are captured by the geometrical structure of a bibrane (for complete definitions see [21]): a bibrane is in particular a submanifold of the Cartesian product of the target spaces, $Q \subset M \times \tilde{M}$. The pair of maps (X, Y) is restricted by the requirement that the combined map

$$\begin{aligned} X_S : S &\rightarrow M \times \tilde{M} \\ s &\mapsto (X(s), Y(s)) \end{aligned} \quad (2)$$

takes its values in the submanifold Q . Locally on each target space, the three-form field strength H on M and \tilde{H} on \tilde{M} can be written as derivatives of two-form connections B and \tilde{B} :

$$H = dB \quad \text{and} \quad \tilde{H} = d\tilde{B}. \quad (3)$$

A second geometric datum of a bibrane is a gerbe bimodule on the bibrane world volume Q . It contains as a particular piece of structure a globally defined two-form ω on Q such that locally the relation

$$\omega = p_1^*B - p_2^*\tilde{B} + dA \quad (4)$$

with a local one-form A holds. Equation (4) implies

$$d\omega = p_1^* H - p_2^* \tilde{H} \quad (5)$$

The Lagrangian then reads in terms of these local data:

$$S = \int_{\Sigma_1} L_1 + \int_{\Sigma_2} L_2 + \int_S X_S^* A \quad (6)$$

with the usual bulk Lagrangians

$$\begin{aligned} L_1 &= G_{ij} \partial_\alpha X^i \partial^\alpha X^j + \epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j \\ L_2 &= \tilde{G}_{ij} \partial_\alpha Y^i \partial^\alpha Y^j + \epsilon^{\alpha\beta} \tilde{B}_{ij} \partial_\alpha Y^i \partial_\beta Y^j . \end{aligned} \quad (7)$$

From now on, we consider the conformal field theory on the complex plane, parametrized by a complex coordinate $z = \tau + i\sigma$. We take Σ_1 to be the left half-plane ($\sigma \leq 0$), and Σ_2 the right half plane ($\sigma \geq 0$) separated by the vertical axis S , parametrized by τ . To write down the defect equations of motion, we assume that the world volume Q of the brane is parametrised by coordinates η^A , $A = 1, \dots, k$ and functions f^i with $i = 1, \dots, \dim M$ and h^j with $j = 1, \dots, \dim \tilde{M}$:

$$X^i = f^i(\eta) \quad (8)$$

$$Y^j = h^j(\eta)$$

Then the defect equations of motion (see also equation (2.22) in [34]) take following form:

$$\partial_A f^i G_{ij} \partial_\sigma X^j - \partial_A h^i \tilde{G}_{ij} \partial_\sigma Y^j + (B_{ij} \partial_A f^i \partial_B f^j - \tilde{B}_{ij} \partial_A h^i \partial_B h^j + F_{AB}) \partial_\tau \eta^B = 0 , \quad (9)$$

where $F = dA$ is the two-form curvature of A .

As a side remark, we mention that a generalization of the arguments of [1,9,25] should allow to deduce that conformal invariance of the defect requires the fields to minimise a modified Dirac-Born-Infeld action

$$S_{DBI} = \int d\eta e^{\phi/2} \sqrt{\det(G_{AB} + \tilde{G}_{AB} + B_{AB} - \tilde{B}_{AB} + F_{AB})} \quad (10)$$

where $G_{AB} = G_{ij} \partial_A f^i \partial_B f^j$, $\tilde{G}_{AB} = \tilde{G}_{ij} \partial_A h^i \partial_B h^j$, $B_{AB} = B_{ij} \partial_A f^i \partial_B f^j$ and $\tilde{B}_{AB} = \tilde{B}_{ij} \partial_A h^i \partial_B h^j$.

The Dirac-Born-Infeld action (10) is consistent with the gauge invariance

$$B \rightarrow B + d\Lambda \quad (11)$$

$$\tilde{B} \rightarrow \tilde{B} + d\tilde{\Lambda}$$

$$A \rightarrow A - \Lambda + \tilde{\Lambda}$$

of the world sheet action (6). The action (10) should be compared to the “folding trick” [2,3,30,39] which describes defects in terms of D-branes on product $\text{CFT}_1 \times \overline{\text{CFT}}_2$, where $\overline{\text{CFT}}_2$ denotes some “conjugate” conformal field theory, obtained by exchanging in particular left- and right movers. This is compatible with the flipping of the sign of the antisymmetric B-field in the modified Dirac-Born-Infeld action (10).

3 Toroidal backgrounds

We now consider more specifically the case when both target spaces are tori, $M = T$ and $\tilde{M} = \tilde{T}$. To obtain a σ -model with conformal invariance, the gerbes are chosen to be topologically trivial with constant background metric G and constant B-field. They give constant background matrices which have a symmetric and an antisymmetric part, $E = G + B$ and $\tilde{E} = \tilde{G} + \tilde{B}$.

3.1 T-dual tori

We first consider bibranes that fill the whole product space, $Q = T \times \tilde{T}$. Since the gerbes are assumed to be trivial, we have to consider an ordinary line bundle on Q which we take to be topologically trivial, with connection given by a globally defined one-form $A = A_i^X(X, Y)dX^i + A_i^Y(X, Y)dY^i$. The curvature two-form of this line bundle naturally decomposes into four parts F^{XX} , F^{XY} , F^{YX} and F^{YY} . Antisymmetry implies $F^{XY} = -F^{YX}$.

In terms of the background matrices, the defect equations (9) take the following simple form:

$$\begin{aligned} E\partial X - E^T\bar{\partial}X + F^{XX}(\partial + \bar{\partial})X + F^{XY}(\partial + \bar{\partial})Y &= 0 \\ \tilde{E}\partial Y - \tilde{E}^T\bar{\partial}Y - F^{YY}(\partial + \bar{\partial})Y + F^{XY}(\partial + \bar{\partial})X &= 0. \end{aligned} \quad (12)$$

In the case of T-dual tori, $\tilde{E} = E^{-1}$, specific conformal interfaces are provided by Poincaré line bundles. They fill the whole product space and have curvature two-form

$$F^{XX} = F^{YY} = 0 \quad \text{and} \quad F^{XY} = I, \quad (13)$$

where I is unit matrix.

Poincaré line bundles have a well-known relation to T-duality which has become apparent in the context of supersymmetric string backgrounds. Here

Poincaré line bundles enter in the transformation of Ramond-Ramond tensor fields under T-duality [4, 14]: indeed, it is a well-established fact that, in the absence of a Neveu-Schwarz B-field, the associated Fourier-Mukai kernel \mathcal{P} gives rise to the following isomorphism in K-theory [24]

$$T_! = \tilde{p}_! \circ \otimes \mathcal{P} \circ p^! : K^\bullet(T^n \times M) \rightarrow K^{\bullet+n}(\tilde{T}^n \times M) \quad (14)$$

where the tori T and \tilde{T} are T-dual and where we used the projections $\tilde{p} : \tilde{T}^n \times T^n \times M \rightarrow \tilde{T}^n$ and $p : \tilde{T}^n \times T^n \times M \rightarrow T^n$, respectively.

We now turn to study the bibranes given by Poincaré bundles in purely bosonic conformal field theory: in this situation, the defect equations (12) simplify:

$$\begin{aligned} E\partial X - E^T\bar{\partial}X + \partial Y + \bar{\partial}Y &= 0 \\ \tilde{E}\bar{\partial}Y - \tilde{E}^T\bar{\partial}Y + \partial X + \bar{\partial}X &= 0. \end{aligned} \quad (15)$$

Inserting the T-duality condition $\tilde{E} = E^{-1}$ in the second equation and multiplying by E one finds

$$\partial Y - E(E^{-1})^T\bar{\partial}Y + E\partial X + E\bar{\partial}X = 0. \quad (16)$$

Inserting the first defect equation in (15) in equation (16) yields

$$-E(E^{-1})^T\bar{\partial}Y + E\bar{\partial}X + E^T\bar{\partial}X - \bar{\partial}Y = 0. \quad (17)$$

Multiplication with the invertible matrix $E^T(E + E^T)^{-1}$ gives

$$E^T\bar{\partial}X - \bar{\partial}Y = 0. \quad (18)$$

The defect equations are thus equivalent to the two equations

$$E\partial X = -\partial Y \quad E^T\bar{\partial}X = \bar{\partial}Y. \quad (19)$$

Transforming, as indicated in the introduction, Neumann boundary conditions

$$E\partial X - E^T\bar{\partial}X = 0 \quad (20)$$

by such a defect yields Dirichlet boundary conditions $(\partial + \bar{\partial})Y = 0$. This is indeed the expected action of T-duality in toroidal background, interchanging Dirichlet and Neumann boundary conditions.

3.2 Shifts of the B-field

We next consider a different situation: an identical toroidal background on both sides of the defect, $T = \tilde{T}$ and $E = \tilde{E}$, and defects described by bibrane supported on the diagonal $T \subset T \times T$. For a single compactified free boson, this situation has been discussed in [21]. Here, we focus on the new effects related to the presence of a non-trivial B-field on higher dimensional tori.

Our choice of bibrane imposes on all coordinates the equation

$$\partial_\tau(X^i - Y^i) = 0. \quad (21)$$

In the parametrization of the world volume Q of the bibrane by the functions X^i , the defect equations take the following form:

$$E_{ij}\partial X^j - E_{ij}^T\bar{\partial}X^j - E_{ij}\partial Y^j + E_{ij}^T\bar{\partial}Y^j + F_{ij}\partial_\tau X^j = 0 \quad (22)$$

The equations (21) and (22) can be easily seen to imply the two equations

$$\begin{aligned} \partial X &= \partial Y - \frac{1}{2}G^{-1}F(\partial Y + \bar{\partial}Y) \\ \bar{\partial}X &= \bar{\partial}Y + \frac{1}{2}G^{-1}F(\partial Y + \bar{\partial}Y) \end{aligned} \quad (23)$$

Substituting these expressions for ∂X and $\bar{\partial}X$ in Neumann boundary conditions

$$(G + B)\partial X - (G - B)\bar{\partial}X = 0 \quad (24)$$

yields a shift of the B-field:

$$(G + B - F)\partial Y - (G - B + F)\bar{\partial}Y = 0. \quad (25)$$

This shift in the B-field is the simplest analogue of the following geometric fact that plays an important role in superstring compactifications: tensoring a complex of coherent sheaves \mathcal{E}^\bullet on a complex variety X with a line bundle L on X , i.e. $\mathcal{E}^\bullet \rightarrow \mathcal{E}^\bullet \otimes L$, defines an autoequivalence of the bounded derived category of coherent sheaves $\mathcal{D}^b(X) \rightarrow \mathcal{D}^b(X)$ which is just the Fourier-Mukai transform with kernel $\iota_*(L)$, where $\iota : X \rightarrow \Delta \subset X \times X$ is diagonal embedding of X . The shift of the B-field also occurs for Calabi-Yau varieties when one considers monodromies around divisors in moduli space where some even dimensional cycle vanishes [7, 10–12, 27, 28]. In this case the monodromies amount to twisting with the line bundle associated to the divisor.

4 Defects and T-duality on lens space

The construction of the previous section can be applied fibrewise to torus fibration and can be expected to relate pairs of torus fibrations $\pi : E \rightarrow M$ and $\tilde{\pi} : \tilde{E} \rightarrow M$. It should be noted that in this case the fibre product $E \times_M \tilde{E}$ appearing e.g. in [6, 8] can be identified with the subset of elements of the product space $E \times \tilde{E}$ of pairs (e, \tilde{e}) with $\pi(e) = \tilde{\pi}(\tilde{e})$. This submanifold is the world volume Q of the relevant brane.

We exemplify the situation in the case of the T-duality relating conformal sigma-models with a lens space as a target space and the WZW theory based on the compact connected Lie group $SU(2)$ at level k . The relevant lens space is a quotient of the group manifold $SU(2)$ by the right action of a cyclic subgroup \mathbb{Z}_k of a maximal torus of $SU(2)$. In Euler coordinates for $SU(2)$, this corresponds to the identification $\varphi \sim \varphi + \frac{4\pi}{k}$.

We use another parametrisation [23] of the group manifold $SU(2)$ in terms of Pauli matrices,

$$g = e^{i\phi\frac{\sigma_3}{2}} e^{i\theta\frac{\sigma_2}{2}} e^{i(\xi-\phi)\frac{\sigma_3}{2}}, \quad (26)$$

in which the metric takes form

$$ds^2 = (d\xi - (1 - \cos\theta) d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2 \quad (27)$$

with $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and $\xi \in [0, 4\pi]$.

Identifying ξ as the fibre coordinate exhibits the structure of the group manifold $SU(2)$ as an S^1 bundle over S^2 of monopole charge 1, the Hopf bundle. Considering the bundle with charge m amounts to the substitution $\xi \rightarrow \frac{\xi}{m}$. Due to the orbifold description of the lens space $SU(2)/\mathbb{Z}_k$, the latter can be considered as an S^1 -bundle over S^2 with Chern class k and thus admits a parametrization as in (26), but with $\xi \sim \xi + \frac{4\pi}{k}$. It is convenient to reparameterize the lens space bundle coordinate $\tilde{\xi}$ as $\tilde{\xi} = \xi'/k$.

To the S^1 -bundle description of the lens space and the group manifold, we can apply the standard geometric T-duality construction [6] for torus fibrations. It involves a correspondence space $E \times_M \tilde{E}$, where in our case $E := SU(2)$, \tilde{E} is the lens space and the base manifold is $M := S^2$. It leads to the following relations for the first Chern classes of the S^1 -bundles on M and the three-forms H and \tilde{H} on E and \tilde{E} , respectively:

$$F = c_1(E) = \tilde{\pi}_*\tilde{H} \quad \text{and} \quad \tilde{F} = c_1(\tilde{E}) = \pi_*H, \quad (28)$$

where π_* is integration on the S^1 -fibre. It is observed in [6] that the pullbacks π^*F and $\tilde{\pi}^*\tilde{F}$ are exact on E and \tilde{E} respectively, and therefore can be written as

$$\pi^*F = dA \quad \text{and} \quad \tilde{\pi}^*\tilde{F} = d\tilde{A}, \quad (29)$$

where $A \in \Omega^1(E)$ and $\tilde{A} \in \Omega^1(\tilde{E})$ are global one-forms on E and \tilde{E} , respectively, which are assumed to be normalized such that

$$\pi_*A = 1 = \tilde{\pi}_*\tilde{A} \quad (30)$$

It is shown in Section 3 of [6] that there exists a three-form Ω on the base manifold M that obeys the two relations

$$H = A \wedge \pi^*\tilde{F} - \pi^*\Omega \quad \text{and} \quad \tilde{H} = \tilde{\pi}^*F \wedge \tilde{A} - \tilde{\pi}^*\Omega. \quad (31)$$

One then introduces a two-form ω on the correspondence space $E \times_M \tilde{E}$ by

$$\omega := \tilde{p}^*\tilde{A} \wedge p^*A \quad (32)$$

where p and \tilde{p} are the projections $E \times_M \tilde{E} \rightarrow E$ and $E \times_M \tilde{E} \rightarrow \tilde{E}$ respectively. They obey the relation $\pi p = \tilde{\pi} \tilde{p}$.

It follows from (31) and (29) and commutativity $p^*\pi^* = \tilde{p}^*\tilde{\pi}^*$, that

$$d\omega = -\tilde{p}^*\tilde{H} + p^*H \quad (33)$$

This two-form also enters [6] in the following isomorphism of twisted cohomologies

$$T_* : p_* \circ e^\omega \circ \tilde{p}^* : H^\bullet(\tilde{E}, \tilde{H}) \rightarrow H^{\bullet+1}(E, H). \quad (34)$$

Let us comment on the important role of the equation (33) in the isomorphism (34). It is shown in [6] that thanks to this equation H -twisted cohomologies mapped to \tilde{H} -twisted cohomologies. On the other hand, this equation coincides with equation (5), which was derived in [21] from the requirement of a well-defined worldsheet action. This coincidence can be seen as additional evidence for the relation between defects and kernels of Fourier-Mukai transforms we propose in this paper.

In the case when $E = \text{SU}(2)$ and \tilde{E} is a lens space, this yields

$$\begin{aligned} H &= \frac{k}{16\pi^2} \sin \theta \, d\phi \, d\theta \, d\xi \\ F &= \frac{1}{4\pi} \sin \theta \, d\phi \, d\theta \\ A &= \frac{1}{4\pi} (d\xi - (1 - \cos \theta) d\phi) \end{aligned} \quad (35)$$

and

$$\begin{aligned}\tilde{H} &= \frac{1}{16\pi^2} \sin \theta \, d\phi \, d\theta \, d\xi' \\ \tilde{A} &= \frac{1}{4\pi} (d\xi' - k(1 - \cos \theta) \, d\phi) \\ \tilde{F} &= \frac{k}{4\pi} \sin \theta \, d\phi \, d\theta\end{aligned}\tag{36}$$

and thus suppressing the projectors p and \tilde{p} for brevity in calculations in explicit coordinates

$$\tilde{A} \wedge A = \frac{1}{16\pi^2} (da \wedge d\xi + (1 - \cos \theta) d\phi \wedge da)\tag{37}$$

where a is defined by the equation

$$\tilde{\xi} = \frac{\xi'}{k} = \xi + \frac{a}{k}.\tag{38}$$

4.1 Defect operators on bulk fields

In this section, we describe defects by their action on bulk fields. In the case of rational conformal field theories, it is known (see Proposition 2.8 of [16]) that this action characterizes a defect uniquely.

The bulk partition function for the rational conformal field theory associated to a lens space is

$$Z(q) = \sum_{j=0}^{k/2} \sum_{n \in \mathbb{Z}} \chi_j^{SU(2)}(q) \chi_{jn}^{PF}(\bar{q}) \psi_{-n}^{U(1)}(\bar{q}).\tag{39}$$

To derive conformal defects between $SU(2)_k$ and the lens space $SU(2)/\mathbb{Z}_k$ we need the following endomorphisms of a direct sum of Fock spaces for left movers and right movers, respectively:

$$P_{r\pm}^{U(1)} = \exp \left[\pm \sum_{n=1}^{\infty} \frac{\alpha_{-n}^0 \alpha_n^1}{n} \right] \sum_{l \in \mathbb{Z}} \left| \frac{r+2kl}{\sqrt{2k}} \right\rangle_0 \otimes {}_1 \langle \pm \frac{r+2kl}{\sqrt{2k}} | \tag{40}$$

$$\bar{P}_{r'\pm}^{U(1)} = \exp \left[\pm \sum_{n=1}^{\infty} \frac{\tilde{\alpha}_{-n}^0 \tilde{\alpha}_n^1}{n} \right] \sum_{l' \in \mathbb{Z}} \overline{\left| \pm \frac{r'+2kl'}{\sqrt{2k}} \right\rangle_0} \otimes {}_1 \langle \overline{\frac{r'+2kl'}{\sqrt{2k}}} |, \tag{41}$$

where the subscripts 0 and 1 distinguish free boson theories on the two sides of the defect. The bra- and ket-states are highest weight states in Fock spaces. They obey the following conservation equations for the $U(1)$ -currents

$$J_0^3 \pm J_1^3 = 0, \quad \bar{J}_0^3 \pm \bar{J}_1^3 = 0,\tag{42}$$

where e.g. the first equation is a short hand for the intertwining property

$$P_{\pm}^{U(1)} J_1^3 = \mp J_0^3 P_{\pm}^{U(1)}.$$

Similarly, we consider for the parafermion theories $\mathcal{A}_0^{PF(k)} \times \mathcal{A}_1^{PF(k)}$ the following two operators

$$P_{[j,n]}^{PF} = \sum_N |j, n, N\rangle_0 \otimes {}_1\langle j, n, N|, \quad (43)$$

$$\bar{P}_{[j,n]}^{PF} = \sum_M \overline{|j, n, M\rangle_0} \otimes {}_1\overline{\langle j, n, M|}, \quad (44)$$

where the sums over M and N are over orthonormal bases of the parafermion state spaces. Here $j \in \{0, \frac{1}{2}, 1, \dots, \frac{k}{2}\}$ and $n \in \mathbb{Z}/2k\mathbb{Z}$ satisfy the constraint $2j + n = 0 \pmod{2}$. The pairs (j, n) and $(k/2 - j, k + n)$ have to be identified.

Our starting point are symmetry preserving defects in the $SU(2)$ -theory. The corresponding operators on bulk fields can be expressed in terms of the modular matrix S of $SU(2)$ and the identity operators on irreducible highest weight modules of the corresponding untwisted affine Lie algebra,

$$P_j^{SU(2)} = \sum_N |j, N\rangle_0 \otimes {}_1\langle j, N|, \quad (45)$$

$$\bar{P}_j^{SU(2)} = \sum_M \overline{|j, M\rangle_0} \otimes {}_1\overline{\langle j, M|}, \quad (46)$$

where the sums over M and N are over orthonormal bases of the $SU(2)$ state spaces. These endomorphisms preserve, of course, all $SU(2)$ symmetries,

$$J_0^a + J_1^a = 0, \quad (47)$$

$$\bar{J}_0^a + \bar{J}_1^a = 0, \quad (a = 1, 2, 3). \quad (48)$$

The action of a symmetry preserving defect on bulk fields is given in terms of these endomorphisms by [31]:

$$X_a = \sum_j \frac{S_{aj}}{S_{0j}} P_j^{SU(2)} \bar{P}_j^{SU(2)}. \quad (49)$$

Since in the situation at hand no field identification fixed points occur, we can apply the procedure described in [17, 26, 37] to derive a new family of defects separating $SU(2)_k$ and the lens space $SU(2)/\mathbb{Z}_k$. Performing a T-duality in (49) yields

$$Y_a^{AB} = \sum_j \sum_n \frac{S_{aj}}{S_{0j}} P_j^{SU(2)} \bar{P}_{j,n}^{PF} \bar{P}_{n-}^{U(1)}. \quad (50)$$

The defects (50) preserve all left moving currents, but only the right moving current corresponding to the maximal torus,

$$J_0^a + J_1^a = 0, \quad (a = 1, 2, 3) \quad (51)$$

$$\bar{J}_0^3 - \bar{J}_1^3 = 0. \quad (52)$$

As a consequence of these equations, the defects (50) transform A-type branes on $SU(2)_k$ to B-type brane on $SU(2)/\mathbb{Z}_k$.

A third family of defects is obtained by summing over the images of (49) under the action of \mathbb{Z}_k , with a prefactor determined by the Cardy condition:

$$Y_a^{AA} = \sqrt{k} \sum_j \frac{S_{aj}}{S_{0j}} P_j^{SU(2)} \left(\bar{P}_{j,0}^{PF} \bar{P}_{0+}^{U(1)} + \bar{P}_{j,k}^{PF} \bar{P}_{k+}^{U(1)} \right). \quad (53)$$

The defects (53) satisfy the conservation equations

$$J_0^a + J_1^a = 0, \quad (a = 1, 2, 3) \quad (54)$$

$$\bar{J}_0^3 + \bar{J}_1^3 = 0, . \quad (55)$$

and transform A-type branes on $SU(2)_k$ to A-type branes on the lens space $SU(2)/\mathbb{Z}_k$.

Performing a T-duality on the defects (53), one derives another family of defects on $SU(2)_k$ that map an A-type brane on $SU(2)_k$ to a B-type brane on $SU(2)_k$:

$$X_a^{AB} = \sqrt{k} \sum_j \frac{S_{aj}}{S_{0j}} P_j^{SU(2)} \left(\bar{P}_{j,0}^{PF} \bar{P}_{0-}^{U(1)} + \bar{P}_{j,k}^{PF} \bar{P}_{k-}^{U(1)} \right). \quad (56)$$

The defects (56) satisfy the conservation equations

$$J_0^a + J_1^a = 0, \quad (a = 1, 2, 3) \quad (57)$$

$$\bar{J}_0^3 - \bar{J}_1^3 = 0, . \quad (58)$$

Summing over images and performing T-duality in the left moving sector of (53) yields a fifth family of defects that map B-type branes on $SU(2)_k$ to A-type branes on $SU(2)/\mathbb{Z}_k$:

$$Y_a^{BA} = k \sum_j \frac{S_{aj}}{S_{0j}} \left(P_{j,0}^{PF} P_{0-}^{U(1)} + P_{j,k}^{PF} P_{k-}^{U(1)} \right) \left(\bar{P}_{j,0}^{PF} \bar{P}_{0+}^{U(1)} + \bar{P}_{j,k}^{PF} \bar{P}_{k+}^{U(1)} \right). \quad (59)$$

The defects (59) satisfy the conservation equations:

$$J_0^3 - J_1^3 = 0 \quad (60)$$

$$\bar{J}_0^3 + \bar{J}_1^3 = 0, . \quad (61)$$

4.2 Geometry of defects

We finally determine the geometry of the family of defects (50) relating $SU(2)$ and the lens space $SU(2)/\mathbb{Z}_k$. To this end, we parametrize bulk fields in terms of Euler angles $\vec{\theta}$ using the representation function $\mathcal{D}_{mm'}^j$ of the spin j representation:

$$|\vec{\theta}\rangle := \sum_{j,m,m'} \sqrt{2j+1} \mathcal{D}_{mm'}^j(\vec{\theta}) |j, m, m'\rangle. \quad (62)$$

We are thus interested in the overlap $\langle \vec{\theta}_0 | Y_a^{AB} | \vec{\theta}_1 \rangle$ as a function of two sets of Euler angles. As in the calculation in [5], the definition of the lens spaces as right quotients implies that only terms of the defect operator (50) with $n = 0, k$ contribute to the overlap; in the large k limit also the term with $n = k$ can be ignored. Therefore, we arrive in the limit of large level k at the function

$$\langle \vec{\theta}_0 | Y_a^{AB} | \vec{\theta}_1 \rangle \sim \sum_j \frac{k}{\pi} \sin[(2j+1)\hat{\psi}] \mathcal{D}_{00}^j(g_0^{-1}(\vec{\theta}_0)g_1(\vec{\theta}_1)), \quad (63)$$

where the angle $\hat{\psi}$ is given in terms of a by $\hat{\psi} := \frac{(2a+1)\pi}{k+2}$.

To proceed, we express [26] the Wigner D-functions in terms of Legendre polynomials as $\mathcal{D}_{00}^j = P_j(\cos \theta)$. We find for the sum appearing on the right hand side of equation (63)

$$\sum_j \frac{e^{(2j+1)i\hat{\psi}} - e^{-(2j+1)i\hat{\psi}}}{2i} P_j(\cos \theta) = \frac{e^{i\hat{\psi}}}{2i} \sum_j e^{(2j)i\hat{\psi}} P_j(\cos \theta) - \frac{e^{-i\hat{\psi}}}{2i} \sum_j e^{-(2j)i\hat{\psi}} P_j(\cos \theta) \quad (64)$$

which allows us to use the generating function for Legendre polynomials

$$\sum_n t^n P_n(x) = \frac{1}{\sqrt{1-2tx+t^2}}. \quad (65)$$

to simplify equation (63). We evaluate the sum on the right hand side of equation (63) to

$$\frac{e^{i\hat{\psi}}}{2i} \frac{1}{\sqrt{e^{2i\hat{\psi}} (e^{-2i\hat{\psi}} - 2 \cos \theta + e^{2i\hat{\psi}})}} + \text{c.c.} = -\frac{1}{2\sqrt{2}} \frac{1}{\sqrt{\cos \theta - \cos \hat{\psi}}} + \text{c.c.}, \quad (66)$$

and thus the overlap to

$$\langle \vec{\theta}_0 | Y_a^{AB} | \vec{\theta}_1 \rangle \sim \frac{\Theta(\cos \delta - \cos 2\hat{\psi})}{\sqrt{\cos \delta - \cos 2\hat{\psi}}}. \quad (67)$$

Here Θ is the Heavyside step function and δ is the second Euler angle of the product element $g_0^{-1}(\vec{\theta}_0)g_1(\vec{\theta}_1)$. Thus equation (67) implies [35,36] that the ‘‘difference’’ $g_0^{-1}(\vec{\theta}_0)g_1(\vec{\theta}_1)$ takes its values in a subset consisting of products of an element in a fixed conjugacy class C with an element $L \in U(1)$:

$$g_0^{-1}(\vec{\theta}_0)g_1(\vec{\theta}_1) \in CL. \quad (68)$$

We next determine the two-form ω satisfying equation $(H_1 - H_2)|_{\text{bibrane}} = d\omega$ that is part of the bibrane-data. Its value in the element $xfx^{-1}L$ with f a fixed element of the conjugacy class C and $x \in G$ and $L \in U(1)$ arbitrary can be derived from the Polyakov-Wiegmann identity

$$\omega^{WZ}(gh) = \omega^{WZ}(g) + \omega^{WZ}(h) - d\text{Tr}(g^{-1}dg dhh^{-1})$$

for the Wess-Zumino three-form $\omega^{WZ}(g) = \frac{1}{3}\text{Tr}(g^{-1}dg)^3$ as follows: we compute for $g_0^{-1}g_1 \in CL$ the difference

$$\begin{aligned} \omega^{WZ}(g_0) - \omega^{WZ}(g_1) &= \omega^{WZ}(g_0) - \omega^{WZ}(g_0CL) \\ &= \omega^{WZ}(g_0) - [\omega^{WZ}(g_0) + \omega^{WZ}(CL) - d\text{Tr}(g_0^{-1}dg_0 d(CL)(CL)^{-1})] \\ &= -\omega^{WZ}(C) + d\text{Tr}(C^{-1}dC dLL^{-1}) + d\text{Tr}(g_0^{-1}dg_0 d(CL)(CL)^{-1}). \end{aligned}$$

As a consequence, the two-form

$$\omega := \frac{k}{8\pi^2}\text{Tr}(C^{-1}dC dLL^{-1} + g_0^{-1}dg_0 d(CL)(CL)^{-1}) - \omega^f(x) \quad , \quad (69)$$

where the two form

$$\omega^f(x) = \frac{k}{8\pi^2}\text{Tr}(x^{-1}dxfx^{-1} dx f^{-1}) \quad (70)$$

obeys $d\omega^f(C) = \frac{k}{8\pi^2}\omega^{WZ}(C)$, has the desired property $\frac{k}{8\pi^2}\omega^{WZ}(g_0) - \frac{k}{8\pi^2}\omega^{WZ}(g_1) = d\omega$. The coefficient fixed by the requirement $\int_{SU(2)} \frac{k}{8\pi^2}\omega^{WZ}(g) = k$ to make contact with the geometrical consideration.

Asymptotically, for large k , the situation simplifies in the case when $f \approx e$, and the bibrane worldvolume, i.e. the correspondence space, consists of all pairs of the form (g_0, g_0L) , with $g_0 \in SU(2)$ and $L \in U(1)$. The corresponding two-form takes the form

$$\omega = \frac{k}{8\pi^2}\text{Tr}(g_0^{-1}dg_0 dLL^{-1}) \quad (71)$$

In this case, the defect acts as an isomorphism on bulk fields, and we thus expect a relation to T-duality. Indeed, we find in the parametrization (26)

$$(g^{-1}dg)_{11} = -(g^{-1}dg)_{22} = i\frac{d\xi}{2} - id\phi\frac{1 - \cos\theta}{2}. \quad (72)$$

Writing $L = e^{i\frac{a\sigma_3}{2k}}$, we see that the two-form (71) coincides with the two-form (37) from the geometric approach. This nicely demonstrates how geometric structure familiar from Fourier-Mukai transformations is encoded in the algebraic data describing defects.

Acknowledgements

We are grateful to Jürgen Fuchs and Ingo Runkel for useful discussions and comments on the paper.

G.S. would like to thank King’s College London for hospitality. Both authors received partial support from the Collaborative Research Centre 676 “Particles, Strings and the Early Universe - the Structure of Matter and Space-Time”.

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