CONFORMAL STRUCTURES WITH $\mathrm{G}_{2(2)}$-AMBIENT METRICS

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Abstract. We present conformal structures in signature $(3, 2)$ for which the holonomy of the Fefferman-Graham ambient metric is equal to the non-compact exceptional group $\mathrm{G}_{2(2)}$.

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1. Introduction

In order to describe invariantly the geometry of a conformal structure on a smooth manifold there are two tools. On the one hand, conformal geometry in signature (p, q) can be described as a parabolic geometry of type $(B, \mathfrak{so}(p+1, q+1))$, where B is the stabiliser in $\mathrm{O}(p+1, q+1)$ of a null line. The main ingredient of this description is the normal conformal Cartan connection defined on an B-bundle with values in the Lie algebra $\mathfrak{so}(p+1, q+1)$. On the other hand, there is the construction of the ambient metric by C. Fefferman and C.R. Graham [6, 7] generalising the situation in the flat model. To both constructions a holonomy group is associated: the conformal holonomy of the normal conformal Cartan connection and the holonomy of the ambient metric. One can show that the conformal holonomy is always contained in the ambient holonomy (see Corollary 1) and that both are equal if the conformal class contains an Einstein metric [14, 12].

In this paper we will study the ambient metrics of conformal classes introduced in [15, 16] for which the normal conformal Cartan connection reduces to a Cartan connection with values in the Lie algebra of the non-compact exceptional Lie group $\mathrm{G}_{2(2)} \subset \mathrm{SO}(4, 3)$. In this way it defines a parabolic geometry of type $(P, \mathfrak{g}_{2(2)})$ where P is the parabolic subgroup given by the stabiliser in $\mathrm{G}_{2(2)}$ of a null line. This situation is exceptional in the sense that a reduction of a Cartan connection to a subalgebra $\mathfrak{g} \subsetneq \mathfrak{so}(p+1, q+1)$ imposes very strong algebraic restrictions to $\mathfrak{g}$ and the parabolic subalgebra, as recently shown in [5]. For conformal geometry, only two cases arise: the one of $\mathfrak{g}_{2(2)}$ described in [15], and the one of $\mathfrak{so}(4, 3) \subset \mathfrak{so}(4, 4)$ described in [4]. In [16] a remarkable feature of the $\mathrm{G}_{2(2)}$-conformal structures was noticed: Some of them have a truncated ambient metric, i.e. the ambient metric can be explicitly calculated. Furthermore, examples of such conformal structures were given that do not contain an Einstein metric. In this note we will show that for some of these non-Einstein examples, in fact for a 7-parameter family and a 5-parameter family of conformal classes, the ambient metrics have holonomy exactly $\mathrm{G}_{2(2)}$. In this way we obtain a 7-parameter family and a 5-parameter family of $\mathrm{G}_{2(2)}$-metrics

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on a neighbourhood of the 5-manifold we started with. The idea of the proof is to show that the ambient metric admits exactly one parallel spinor field. We use the inclusion of the conformal holonomy in the ambient holonomy (Corollary 1), and the fact that a metric in signature $(4,3)$ that admits two linearly independent parallel spinors also admits a parallel vector field (Lemma 1). As a final step we calculate explicitly the line of parallel spinors and the parallel three-form defining the $\mathrm{G}_{2(2)}$-structure. The $\mathrm{G}_{2(2)}$-metrics constructed in this way have the remarkable feature that they are highly degenerate in the sense that the rank of the curvature operator on 2-forms is very small, in fact ≤ 4. This means that higher derivatives of the curvature are needed in order to generate the 14-dimensional holonomy algebra.

2. Conformal versus ambient holonomy

A Cartan geometry of type $(B, \mathfrak{g})$ is given by the following data: a Lie group B with Lie-algebra contained in $\mathfrak{g}$ and a $\mathfrak{g}$-valued Cartan connection on a B-principle fibre bundle called the Cartan bundle. A Cartan connection defines an invariant absolute parallelism and hence, gives no horizontal distribution in $T\mathcal{P}$ as it is the case for usual connections in principle fibre bundles. Nevertheless, it is possible to define holonomy with respect to ω in terms of the development map (see [17]).

The geometry of a smooth manifold equipped with a conformal class of metrics with signature (p,q) can be described as a Cartan geometry of type $(B, \mathfrak{so}(p+1,q+1))$ where B is the Lie subgroup with Lie algebra given by the isotropy algebra in $\mathfrak{so}(p+1,q+1)$ of a null line. Furthermore a uniquely defined *normal conformal Cartan connection* exists. Its holonomy is called *(normal conformal) holonomy*.

Related to the conformal holonomy is the following notion of reduction [1]: Assume that $H \subset \mathrm{SO}(p+1,q+1)$ acts transitively on the Möbius sphere $\mathrm{SO}(p+1,q+1)/B$. H defines a *Cartan reduction of conformal geometry* if there is a Cartan geometry given by H, $H \cap B$ and a Cartan connection that pulls back to the normal conformal Cartan connection. Such a reduction to H exists if and only if the normal conformal holonomy is contained in H. An example for this situation was given in [15] with a conformal class of signature $(3,2)$ and $H := \mathrm{G}_{2(2)} \subset SO(4,3)$ being the non-compact form of G_2.

Note that if the Cartan connection reduces to $G \subsetneq \mathrm{SO}(p+1,q+1)$, then its holonomy is also reduced to this group. The converse is not true: The conformal holonomy might be equal to $G \subsetneq \mathrm{SO}(p+1,q+1)$ without the existence of a reduction of the Cartan connection over the same manifold. This is the case, for example, for the reduction of the conformal holonomy to $\mathrm{SU}(1,n+1) \subset \mathrm{SO}(2,2n+2)$ which corresponds to the conformal geometry of a Fefferman space over a stricty pseudo-convex CR-manifold. Other examples of this situation are conformal structures that contain an Einstein metric, or which split into conformal Einstein structures with certain relations between the Einstein constants [2], or Lorentzian conformal structures for which the conformal holonomy admits an invariant null plane [12].

An easier way of describing the holonomy of the Cartan connection is via the extension of the Cartan connection to a principle fibre bundle connection in the usual sense. To this end one extends the Cartan bundle to a bundle $\overline{\mathcal{P}}$ with structure group $\mathrm{O}(p+1,q+1)$ via $\overline{\mathcal{P}} = \mathcal{P} \times_B \mathrm{O}(p+1,q+1)$ on which the Cartan connection ω extends to a principle fibre bundle connection $\overline{\omega}$. Then one can show that the holonomy groups of the Cartan connection ω and of $\overline{\omega}$ have the same connected component of the identity [3]. In the following we will only deal with this connected component referring to it as normal conformal holonomy. As $\overline{\omega}$ is a connection in the usual sense, there exists a holonomy reduction to the holonomy group of $\overline{\omega}$. Every Cartan reduction in the above sense gives a holonomy reduction, but there are holonomy reductions that are not given by a Cartan reduction. Examples of this are given by the situation where mentioned above.

By associating the standard representation $\mathbb{R}^{p+1,q+1}$ of $\mathrm{SO}(p+1,q+1)$ to $\overline{P}$ we obtain a vector bundle $\mathcal{T}$ of rank $p+q+2$, called *standard tractor bundle*. $\mathcal{T}$ is equipped with the covariant derivative ∇^ω induced by $\overline{w}$. The holonomy of ∇^ω is the same as the normal conformal holonomy and thus contained in $\mathrm{SO}(p+1,q+1)$. Hence, there is an invariant metric h on $\mathcal{T}$. The triple $(\mathcal{T},\nabla^\omega,h)$ is called *normal conformal standard tractor bundle*. In order to write down ∇^ω explicitly, we fix a metric g in the conformal class $[g]$ inducing a splitting

$$(\mathcal{T},h)\ \simeq \underline{\mathbb{R}}\oplus(TM,g)\oplus\underline{\mathbb{R}},$$

where each $\underline{\mathbb{R}}$ denotes the trivial line bundle $M\times\mathbb{R}$. Both are totally null and orthogonal to TM w.r.t. the bundle metric h. In this splitting ∇^ω is given by the following formula,

$$\nabla^\omega_X\begin{pmatrix}\alpha\\ Y\\ \beta\end{pmatrix} = \begin{pmatrix}X(\alpha)-\mathsf{P}^g(X,Y)\\ \nabla^g_X Y+\alpha X+\beta\mathsf{P}^g(X)^*\\ X(\beta)-g(X,Y)\end{pmatrix}, \tag{1}$$

P^g being the Schouten tensor of the metric g. A parallel section in $\mathcal{T}$ w.r.t. this connection gives a local scale of g to an Einstein metric.

Another tool in conformal geometry is the so-called *Fefferman-Graham ambient metric* (see [6] and [7]). For a conformal class $[g]$ in signature (p,q) on an $n=(p+q)$-dimensional manifold M the *ambient metric* is a metric $\widetilde{g}$ of signature $(p+1,q+1)$ on the product of M with two intervals, $\widetilde{M}:=(-\varepsilon,\varepsilon)\times M\times(1-\delta,1+\delta)$, $\varepsilon>0$, $\delta>0$, that is *compatible with the conformal structure* and, moreover, is *Ricci flat*. In the following we are only interested in the case where M is odd-dimensional. In this case the ambient metric exists and the Ricci-flat condition ensures that it depends uniquely on the conformal class $[g]$. Starting with a formal power series

$$\widetilde{g}=2\,(t\mathrm{d}\rho+\rho\mathrm{d}t)\,\mathrm{d}t+t^2\left(g+\sum_{k=1}^{\infty}\rho^k\mu_k\right) \tag{2}$$

with $\rho\in(-\varepsilon,\varepsilon)$, $t\in(1-\delta,1+\delta)$ and μ_k certain symmetric $(2,0)$ tensors on M, Fefferman and Graham showed that if n is *odd*, the Ricci-flatness of the ambient metric gives equations for $\mu_1,\mu_2,\dots$ that can be solved. However, the μ_k have been determined only for small k or for all k but very special conformal classes. For example, in general one finds that $\mu_1=2\mathsf{P}^g$ and

$$\mu_2=-B^g+\mathsf{tr}(\mathsf{P}^g\otimes\mathsf{P}^g) \tag{3}$$

with B^g being the Bach tensor of g. Furthermore, for an Einstein metric with $\mathsf{P}^g=cg$ we have that $\mu_2=c^2g$ and all other $\mu_i=0$, i.e. the power series in the ambient metric $\tilde{g}_E$ *truncates* at $k=2$. Further calculations of the ambient metric have been carried out for conformal classes that are related to Einstein spaces [8]. However, if the metric g is *not conformally Einstein*, then, except for a few examples [8, 16, 13], no explicit formulae for μ_k, $k>3$ are known.

In order to compare the Levi-Civita connection of the ambient metric with the normal conformal tractor connection we change the coordinate ρ to $u := -\rho t$, i.e. $du = -t\mathrm{d}\rho - \rho \mathrm{d}t$ and the ambient metric takes the form

$$\widetilde{g} \quad = \quad -2\mathrm{d}u\mathrm{d}t + t^2 g - 2ut\mathsf{P}^g + u^2\left(\mu_2 - \frac{u}{t}\mu_3 + \left(\frac{u}{t}\right)^2\mu_4 - \ldots\right).$$

Now we calculate the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ along $\mathcal{Q} = \{u = 0\}$ and obtain that the only non-vanishing terms are

$$\left.\begin{array}{rcl} \widetilde{\nabla}_X \frac{\partial}{\partial u} & = & -\frac{1}{t}\mathsf{P}^g(X)^* \\ \widetilde{\nabla}_X Y & = & \nabla_X Y + t\left(g(X,Y)\frac{\partial}{\partial u} - \mathsf{P}^g(X,Y)\frac{\partial}{\partial t}\right) \\ \widetilde{\nabla}_X \frac{\partial}{\partial t} & = & \frac{1}{t}X \end{array}\right\}, \tag{4}$$

for $X, Y \in \Gamma(TM)$ and ∇ being the Levi-Civita connection of g. Restricting these formulae further to $M = \{t = 1, u = 0\} \subset \mathcal{Q} \subset M$ and comparing it to (1) shows that over M the ambient connection is the same as the normal conformal tractor connection. We arrive at

Proposition 1. *Let $(M, [g])$ be conformal manifold with ambient space $(\widetilde{M}, \widetilde{g})$ and normal conformal standard tractor bundle $(\mathcal{T}, \nabla^\omega)$. Then there is a vector bundle isomorphism*

$$\Phi : T\widetilde{M}|_M \to \mathcal{T}$$

which is affine w.r.t. $\widetilde{\nabla}$ and ∇^ω, i.e.

$$\Phi(\widetilde{\nabla}_X U) = \nabla^\omega_X \Phi(U),$$

for $X \in TM$ and $U \in \Gamma(T\widetilde{M}|_M)$. In particular, $Hol_x(\mathcal{T}, \nabla^w) = Hol_x(T\widetilde{M}|_M, \widetilde{\nabla})$.

Clearly, Φ sends $\frac{\partial}{\partial u}$ to $(0, 0, -1) \in \mathcal{T}$, $\frac{\partial}{\partial t}$ to $(1, 0, 0)$, and $X \in TM$ to $(0, X, 0)$. Since the full holonomy of the ambient metric contains the holonomy of the restricted bundle, i.e. $Hol_x(T\widetilde{M}|_M, \widetilde{\nabla}) \subset Hol_x(\widetilde{M}, \widetilde{g})$, we end up with the following result.

Corollary 1. *Let $(M, [g])$ be a conformal manifold with ambient space $(\widetilde{M}, \widetilde{g})$. Then the normal conformal holonomy $Hol^0_x(\mathcal{P}, \omega)$ is contained in the holonomy $Hol_x(\widetilde{M}, \widetilde{g})$ of the ambient space.*

For a conformal class containing an Einstein metric it can be shown that both holonomy groups are equal because in this case the ambient metric truncates after terms of second order in u. In particular, for a Ricci-flat metric, the ambient metric is given as some kind of Brinkmann wave,

$$\widetilde{g} = -2\mathrm{d}u\mathrm{d}t + t^2 g, \tag{5}$$

admitting a parallel null vector field, whereas for an Einstein metric with $\mathsf{P} = cg$ the ambient metric becomes

$$\widetilde{g} = -2\mathrm{d}u\mathrm{d}t + \left(t^2 - 2cut + c^2u^2\right) g.$$

This metric splits into a line and a cone which becomes evident in new coordinates $r = t - cu$ and $s = t + cu$ yielding

$$\widetilde{g} = \frac{1}{2c}(\mathrm{d}r^2 - \mathrm{d}s^2) + r^2 g. \tag{6}$$

Using this truncation of the ambient metric it was proven in [14] for the $c \neq 0$ case and in [12] for the Ricci-flat case that the normal conformal holonomy is equal to the holonomy of the ambient metric.

3. $\mathrm{G}_{2(2)}$-conformal structures with truncated ambient metric

In [15] a conformal structure $[g_F]$ in signature $(3,2)$ was introduced that originated from a first order ODE for two functions y, z of one variable x. We will now describe this construction briefly. Every solution to the first order ODE

$$z' = F(x, y, y', y'', z) \qquad \text{with} \qquad F_{y''y''} \neq 0, .$$

is a curve in the five-dimensional manifold M parametrised by $(x, y, z, p = y', q = y'')$, on which the one-forms

$$\omega^1 = \mathrm{d}z - F(x, y, p, q, z)\mathrm{d}x, \;\; \omega^2 = \mathrm{d}y - p\mathrm{d}x, \;\; \omega^3 = \mathrm{d}p - q\mathrm{d}x \tag{7}$$

vanish. Two triples of three-forms on $\mathbb{R}^5$ are considered to be equivalent, if there is a local diffeomorphism Φ of $\mathbb{R}^5$ and a $GL(3,\mathbb{R})$-valued function $A = (a_{ij})$ on the domain of Φ such that $\Phi^*\hat{\omega}^i = \sum_{j=1}^n a_{ij}\omega^j$. Cartan showed that an equivalence class of a triple of one-forms given by (7) with $F_{qq} \neq 0$ corresponds to a Cartan connection ω on a 14-dimensional principle fibre bundle $\mathcal{P}$ over the five-manifold parametrised by (x, y, z, p, q). This Cartan connection has values in the non-compact exceptional Lie algebra $\mathfrak{g}_{2(2)}$, and $\mathcal{P}$ is the bundle with structure group given by the 9-dimensional parabolic $P := \mathrm{G}_{2(2)} \cap B$, where B is the isotropy group in $\mathrm{SO}(4,3)$ of a null line. The conformal structure on the five-manifold is now constructed as follows: Write the Cartan connection ω as $\omega = (\theta, \Omega)$, where Ω has values in the Lie algebra $\mathfrak{p}$ of P and θ in the five-dimensional complement of $\mathfrak{p}$ in $\mathfrak{g}_{2(2)}$. Write $\theta = (\theta_1, \ldots, \theta_5)$ and $\Omega = (\Omega_1, \ldots, \Omega_9)$ and let $X_1, \ldots, X_5$ and $Y_1, \ldots, Y_9$ be the vector fields on $\mathcal{P}$ dual to θ_i and Ω_μ, respectively. The Y_μ are tangential to the fibres of $\mathcal{P} \to M$. Defining the bilinear form

$$G = 2\theta^1\theta^5 - 2\theta^2\theta^4 + \frac{4}{3}(\theta^3)^2$$

on $\mathcal{P}$ we note that along the fibres G is degenerate and merely scales, i.e.

$$\mathcal{L}_{Y_\mu} G = \lambda_\mu G$$

for some functions λ_μ. Hence, G projects to a conformal class of metrics $[g_F]$ of signature $(+++--)$ on M. This means that the normal conformal Cartan connection for $[g_F]$ reduces (in the Cartan sense) to $\mathrm{G}_{2(2)}$. Hence, the conformal holonomy of $[g_F]$ is contained in this group. Of course, this inclusion might be proper.

Then, in [16], the following remarkable feature of $[g_F]$ was noticed.

Proposition 2. *There exist functions F such that the ambient metric of a $g_F \in [g_F]$ truncates after terms of second order, i.e.*

$$\widetilde{g}_F = -2\mathrm{d}t\mathrm{d}u + t^2 g_F - 2ut\mathsf{P} + u^2\beta, \tag{8}$$

with P the Schouten tensor of g_F and $\beta = \mu_2$ defined as in Eq. (3)

Examples of such F's given in [16] include $F = F(q)$ and $F = q^2 + \sum_{i=0}^6 a_i p^i + bz$. The proof is based on the form and the uniqueness of the ambient metric in odd dimensions proved in [7] and the observation, that the metric (8) is Ricci-flat.

This concise form of the ambient metric makes it possible to study the relation between the conformal holonomy of these structures and the holonomy of the ambient metric. This is done by distinguishing situations where the conformal class contains an Einstein metric or does not contain an Einstein metric. Also in [16] several examples of such conformal structures depending on the function F in (7) with $F_{qq} \neq 0$ were considered. On the one hand it was shown that for $F = F(q)$ the conformal class given by F contains a Ricci flat metric. We have seen that for a conformal class that contains a Ricci flat metric, the ambient metric is a certain Brinkmann metric, and that the holonomy of the ambient metric is the same as the holonomy of the conformal Cartan connection. Based on the result in [16] we obtain:

Proposition 3. *Let $[g_F]$ be a conformal class where $F = F(q)$ with $F_{qq} \neq 0$. Then $[g_F]$ contains a Ricci flat metric, the ambient metric is given by Equations (5), the holonomy of the ambient metric is equal to the conformal holonomy and contained in the eight-dimensional stabiliser in $\mathrm{G}_{2(2)}$ of a null vector.*

This shows that the ambient metric of conformal classes $g_{F(q)}$ are $\mathrm{G}_{2(2)}$- metrics that admit a parallel null vector field, and thus can be considered as $\mathrm{G}_{2(2)}$-Brinkmann waves.

Furthermore, in [16] a conformal structure $[g_F]$ in signature $(3,2)$ was introduced that still has an ambient metric in the truncated form (8) but does *not* contain an Einstein metric. This is defined by

$$F = q^2 + \sum_{i=0}^{6} a_i p^i + bz. \tag{9}$$

Explicitly,

$$g_F = 2\theta^1\theta^5 - 2\theta^2\theta^4 + (\theta^3)^2, \tag{10}$$

where the co-frames θ_i are defined as

$$\begin{aligned}
\theta^1 &= \mathrm{d}y - p\mathrm{d}x \\
\theta^2 &= \mathrm{d}z - F\mathrm{d}x - 2q(\mathrm{d}p - q\mathrm{d}x) \\
\theta^3 &= -\tfrac{2^{4/3}}{\sqrt{3}}(\mathrm{d}p - q\mathrm{d}x) \\
\theta^4 &= 2^{-1/3}\mathrm{d}x \\
15(2)^{1/3}\theta^5 &= \left(9\left(a_2 + 3a_3p + 6a_4p^2 + 10a_5p^3 + 15a_6p^4\right) + 2b^2\right)(\mathrm{d}y - p\mathrm{d}x) + \\
&\quad 10b(\mathrm{d}p - q\mathrm{d}x) - 30\mathrm{d}q + \\
&\quad 15(a_1 + 2a_2p + 3a_3p^2 + 4a_4p^3 + 5a_5p^4 + 6a_6p^5 + 2bq)\mathrm{d}x.
\end{aligned}$$

Proposition 4. *If $a_4 + 5a_5p + 15a_6p^2 \neq 0$, then the conformal class $[g_F]$ corresponding to $F = q^2 + \sum_{i=0}^{6} a_i p^i + bz$ does not contains an Einstein metric.*

Proof. First, observe that a metric which is Einstein also has vanishing Cotton tensor. Hence, a metric which is conformal to an Einstein metric is also conformally Cotton. But it is known (see e.g. [9]) that the latter implies the existence of a vector field T such that

$$C(T) := C + W(T,.,.,.) \equiv 0. \tag{11}$$

Assuming $a_4 + 5a_5p + 15a_6p^2 \neq 0$, we solve this equation for $\tau = g_F(T,.) = \tau_i\theta^i$. Using the coframe $\theta^1, \ldots, \theta^5$ we get that the $\{112\}$ component of $C(T)$ is $C(T)_{112} =$

$-\frac{9}{10}(a_4+5a_5p+15a_6p^2)\tau_2$. Hence, $\tau_2 = 0$. Using this we obtain $C(T)_{214} = -\frac{9}{10}(a_4 + 5a_5p + 15a_6p^2)\tau_5$, and thus $\tau_5 = 0$. After imposing this condition $C(T)_{314} = -\frac{3^{3/2}}{10}(a_4 + 5a_5p + 15a_6p^2)$. This means that with our assumptions about F, the metric g_F cannot be conformally Cotton. □

Remark 1. Observe the remarkable fact that for any F as in (9) the Riemann tensor of $\widetilde{g}_F$ considered as an endomorphism of $\Lambda^2 T^*\widetilde{M}$ has rank ≤ 4. In some cases it can be even more degenerate. Hence, in order to obtain the 14 dimensional group $G_{2(2)}$ as holonomy group also derivatives of the curvature have to contribute to the holonomy algebra.

Proposition 5. *For $F = q^2 + a_3p^3 + a_2p^2 + a_1p + a_0 + bz$ with $a_3 \neq 0$ the metric g_F is not conformally Einstein but conformally Cotton. Furthermore, the Riemann tensor of the ambient metric $\tilde{g}_F$ acting on 2-forms has in general rank 2.*

Proof. First one shows that if $a_3 = 0$ then g_F is conformal to an Einstein metric. As in the previous proof we evaluate the conformal Cotton condition (11). Here, under the assumption $a_3 \neq 0$, we also obtain that $\tau_2 = \tau_5 = 0$. In fact, as the most general τ solving Equation (11) we get

$$\tau = \frac{2^{4/3}b\lambda}{\sqrt{3}}\theta^1 + \lambda\theta^3 - \frac{2^{4/3}b}{3}\theta^4,$$

where λ is an arbitrary function. The metric g_F is conformally Cotton if and only if the form τ is closed. One checks that this is equivalent to $\lambda = 0$. In fact, the metric $\mathrm{e}^{-4bx/3}g_F$ is the unique (up to constant rescaling) Cotton flat metric in this $[g_F]$. After this condition has been imposed we calculate

$$E := \mathsf{P} - \nabla\tau + \tau^2,$$

which has to be a multiple of the metric if g_F is conformally Einstein. Since the tensor E is given by

$$E = -\frac{9a_3}{10 \cdot 2^{2/3}}\theta^1\theta^4 - \frac{9a_2 + 27a_3p + 2b^2}{45 \cdot 2^{1/3}}(\theta^4)^2,$$

we get a contradiction to a_3 being nonzero. Determining the rank of the curvature operator is a straightforward calculation with Mathematica. □

We can summarise the results of this section in

Theorem 1. *Let F be given by $F = q^2 + \sum_{i=0}^{6} a_ip^i + bz$ with at least one of a_3, a_4, a_5, a_6 not equal to zero. Then the conformal class $[g_F]$ does not contain an Einstein metric. If furthermore, $a_4 = a_5 = a_6 = 0$, then $[g_F]$ contains a Cotton flat metric.*

4. Ambient metrics with holonomy $G_{2(2)}$

For those conformal classes introduced in the previous section that are not conformally Einstein the relation between the holonomy of the ambient metric and the conformal holonomy is more involved than in the conformally Einstein case. We will now show that the ambient metric has holonomy exactly $G_{2(2)}$. The strategy is to show that the ambient manifold admits exactly one parallel spinor which is not null. Before verifying the existence of this parallel spinor, we will recall some facts about spin representations in signature $(4,3)$ and describe the situation in the presence of two parallel spinors.

On the complex spinor module $\Delta^{\mathbb{C}}_{4,3}$ for the Clifford algebra $\mathrm{Cl}(4,3)$ there is a $\mathrm{Spin}(4,3)$-invariant hermitian product $\langle .,. \rangle$ of signature $(4,4)$ defined by

$$\langle \varphi, \psi \rangle := -\mathrm{i}\,(e_4 \cdot e_5 \cdot e_6 \cdot \varphi, \psi)\,, \tag{12}$$

where $(.,.)$ is the standard hermitian scalar product on $\mathbb{C}^8$, $X \cdot \varphi$ denotes the Clifford multiplication, and e_4, e_5, e_6 are orthogonal time-like unit vectors squaring to the identity. Then $\langle .,. \rangle$ satisfies

$$\langle X \cdot \varphi, \psi \rangle = \langle \varphi, X \cdot \psi \rangle. \tag{13}$$

This implies that for two spinors φ and ψ and any vector X the number $\langle X \cdot \varphi, \psi \rangle + \langle X \cdot \psi, \varphi \rangle$ is real because

$$\overline{\langle X \cdot \varphi, \psi \rangle} + \overline{\langle X \cdot \psi, \varphi \rangle} = \overline{\langle \varphi, X \cdot \psi \rangle} + \overline{\langle \psi, X \cdot \varphi \rangle} = \langle X \cdot \varphi, \psi \rangle + \langle X \cdot \psi, \varphi \rangle.$$

Recall that in dimension 7 and signature $(4,3)$, the spin representation is real, meaning that the irreducible complex spinor module $\Delta^{\mathbb{C}}_{4,3} = \mathbb{C}^{4,4}$ is the complexification of an irreducible real module $\Delta_{4,3}$. Furthermore, the invariant hermitian scalar product restricts to a real scalar product $\langle .,. \rangle$ of the same signature $(4,4)$, i.e. $\Delta_{4,3} = \mathbb{R}^{4,4}$. Equation (13) then reads as

$$\langle X \cdot \varphi, \psi \rangle = \langle X \cdot \psi, \varphi \rangle. \tag{14}$$

$\langle .,. \rangle$ gives a metric $\langle .,. \rangle$ on the spin bundle that is parallel w.r.t. the lift of the Levi-Civita connection ∇. It also satisfies the relation

$$Y\left(\langle X \cdot \varphi, \psi \rangle\right) = \langle \nabla_Y X \cdot \varphi, \psi \rangle + \langle X \cdot \nabla_Y \varphi, \psi \rangle + \langle X \cdot \varphi, \nabla_Y \psi \rangle \tag{15}$$

for two spinor field φ and ψ, and two vector fields X and Y. In the proof of the following lemma we will use this real representation and the real spinor bundle.

Lemma 1. *If a 7-dimensional spin manifold with metric g of signature $(4,3)$ admits two linearly independent spinor fields, it also admits a parallel vector field.*

Proof. Let ψ_1 and ψ_2 be two linearly independent parallel spinors. We may assume that $\langle \psi_1, \psi_2 \rangle = 0$. We associate to them a vector field V via transposing the Clifford multiplication, i.e.

$$g(V, X) \;=\; \langle X \cdot \psi_1, \psi_2 \rangle$$

for all $X \in TM$. Because of Equation (15), V is a parallel vector field, but it might happen that V is everywhere zero. We will show that this contradicts ψ_1 and ψ_2 being linearly independent. Assume that

$$g(V, X) \;=\; \langle X \cdot \psi_1, \psi_2 \rangle = 0 \tag{16}$$

for all $X \in TM$. At each tangent space $T_pM = \mathbb{R}^{4,3}$, for $i = 1, 2$ consider the linear maps

$$\Phi_i : \mathbb{R}^{4,3} \ni X \mapsto X \cdot \psi_i \in \mathbb{R}^{4,4}.$$

If $G \subset \mathrm{Spin}(4,3)$ denotes the isotropy group of both spinors, these maps are G-equivariant. Hence, their images V_i and their kernels N_i are invariant under the corresponding representations of G. Furthermore, recall that the subspaces N_i of $\mathbb{R}^{4,3}$ are totally null or trivial. Using the transitive action of $\mathrm{Spin}(4,3)$ on spheres and the light cone in $\mathbb{R}^{4,4}$, one can show [10] that ψ_i is null if and only if N_i is three dimensional, and that ψ_i is not null if and only if N_i is trivial.

First consider the case that ψ_1 is not null. Because of

$$2\langle X \cdot \psi_1, Y \cdot \psi_1 \rangle = -g(X, Y)\langle \psi_1, \psi_1 \rangle, \tag{17}$$

$V_1 \subset \mathbb{R}^{4,4}$ is non-degenerate. Equation (16) shows that $\mathbb{R}\psi_2 = V_1^{\perp}$ implying $\langle\psi_2,\psi_2\rangle = \langle\psi_1,\psi_1\rangle$. Hence, V_2 is non degenerate as well and we get a G-invariant, non degenerate, 6-dimensional subspace $V_1 \cap V_2 \subset \mathbb{R}^{4,4}$. The pre-image under either isomorphism Φ_i is a 6-dimensional, non-degenerate, G-invariant subspace of $\mathbb{R}^{4,3}$ with a G-invariant time-like or space-like line as orthogonal complement. But this leads to a parallel time-like or space-like vector field on M.

Now, assume that ψ_1 is null. In this case V_1 is 4-dimensional and Equation (17) shows that V_1 is totally null. This means that $V_1^{\perp} = V_1$, and thus, relation (16) then shows that $\psi_2 \in V_1$. Hence, ψ_2 has to be null as well. Since N_1 and N_2 are three-dimensional, ψ_1 and ψ_2 are pure spinors. As they are linearly independent, this implies that $N_1 \neq N_2$ (see for example [11]). Hence, there is a null vector $Z \in \mathbb{R}^{4,3}$ such that $Z \in N_1$ but $Z \notin N_2$. The relation

$$\langle Z\cdot\psi_2, X\cdot\psi_1\rangle = -\langle Z\cdot\psi_2, X\cdot Z\cdot\psi_1\rangle - g(X,Z)\langle\psi_1,\psi_2\rangle = 0$$

shows that $0 \neq Z\cdot\psi_2 \in V_1^{\perp} = V_1$. To Z we can find another null vector $Z^* \in N_2$ with $g(Z,Z^*) = 1$ and $Z^* \notin N_1$. Using Z^* we can write $\psi_2 \in V_1^{\perp} = V_1$ as

$$\psi_2 = Z^*\cdot\psi_1 + X\cdot\psi_1$$

with X orthogonal to Z. But this implies that

$$Z\cdot\psi_2 = -\langle Z,Z^*\rangle\psi_1 = -\psi_1,$$

and thus $\psi_1 \in V_1$. But this is a contradiction. Indeed, if $\psi_1 = X\cdot\psi_1 \in V_1$, then

$$\psi_1 = X^2\cdot\psi_1 = -g(X,X)\psi_1$$

and therefore $g(X,X) = -1$. On the other hand, for $Y \in N_1$ we get that

$$-g(X,Y)\psi_1 = Y\cdot X\cdot\psi_1 = Y\cdot\psi_1 = 0,$$

i.e. $X \in N_1^{\perp}$. But $N_1^{\perp}$ contains only space-like and null vectors. □

Note that this lemma is also true in the Riemannian signature following immediately from the Berger list.

Returning to our original situation, a parallel vector field of the ambient metric implies that the ambient holonomy admits an invariant vector, and by Corollary 1 the same is true for the conformal holonomy. Therefore, the normal conformal tractor connection admits a parallel section that, on the other hand, locally corresponds to an Einstein scale. We get

Proposition 6. *Let $[g]$ be a conformal structure in signature $(3,2)$ such that the ambient metric admits two linearly independent parallel spinors. Then g is locally conformally Einstein.*

Now we return to the conformal structures and the ambient metric of the previous section. First we have to find a suitable co-frame for the ambient manifold. Recall from [16] that the ambient metric for g_F was given as

$$\begin{aligned}
&\tilde{g}_F = t^2 g_F \ - 2\ \mathrm{d}t\mathrm{d}u\ - \\
&2\ tu\ [\ \tfrac{1}{20}(-2a_2 + 4b^2 + 3a_3p + 6a_4p^2 - 20a_5p^3 - 120a_6p^4)\mathrm{d}x^2 - \\
&\tfrac{9}{20}(a_3 - 10a_5p^2 - 40a_6p^3)\mathrm{d}x\mathrm{d}y - \tfrac{9}{10}(a_4 + 5a_5p + 15a_6p^2)\mathrm{d}y^2\ ]\ + \\
&u^2\ [\ \tfrac{3}{20(2)^{2/3}}(a_4 - 10a_5p + 60a_6p^2)\mathrm{d}x^2 + \tfrac{9}{4(2)^{2/3}}(a_5 - 12a_6p)\mathrm{d}x\mathrm{d}y + \tfrac{81}{4(2)^{2/3}}a_6\mathrm{d}y^2\ ].
\end{aligned}$$

Note that this ambient metrics has no u^2 terms for the conformal classes $[g_F]$, if $a_4 = a_5 = a_6 = 0$. This means that for such F it truncates at the same order as the ambient metric of a conformal class with an Einstein metric, although it does not contain an Einstein metric if $a_3 \neq 0$.

In order to absorb the terms in the ambient metric coming from the terms of first and second order in u, we introduce the following co-frame on M:

$$
\begin{aligned}
\eta^1 &= \theta^1\\
\eta^2 &= t\theta^2 + \\
&+\Big(\tfrac{1}{5\cdot 2^{\frac{1}{3}}}\left(2b^2-a_2-3a_3p-6a_4p^2-10a_5p^3-15a_6p^4\right)-\tfrac{3u}{8t}\left(\tfrac{a_4}{5}+a_5p+3a_6p^2\right)\Big)u\theta^4\\
\eta^3 &= \theta^3\\
\eta^4 &= \theta^4\\
\eta^5 &= \theta^5+\frac{9}{2}\left(\frac{a_4}{5}+a_5p+3a_6p^2+\frac{9}{4\cdot 2^{\frac{2}{3}}}\frac{u}{t}\right)u\theta^1+\\
&\quad +\frac{9}{2\cdot 2^{2/3}}\left(\frac{a_3}{5}+\frac{4a_4}{5}p+2a_5p^2+4a_6p^3\right)u\theta_4+\frac{9}{8\cdot 2^{1/3}}\left(a_5+6a_6p\right)\frac{u^2}{t}\theta^4
\end{aligned}
$$

Then we write the ambient metric as

$$\widetilde{g}_F = -2\mathrm{d}t\mathrm{d}u + 2\eta^1\eta^5 - 2\eta^2\eta^4 + (\eta^3)^2.$$

Then, for the calculation of the parallel spinor we use the orthonormal basis

$$
\begin{aligned}
&\xi^0 = \frac{1}{\sqrt{2}}(\mathrm{d}t-\mathrm{d}u),\quad \xi^1 = \frac{1}{\sqrt{2}}(\eta^1+\eta^5),\quad \xi^2 = \frac{1}{\sqrt{2}}(\eta^2-\eta^4),\quad \xi^3 = \eta^3\\
&\xi^4 = \frac{1}{\sqrt{2}}(\eta^2+\eta^4),\quad \xi^5 = \frac{1}{\sqrt{2}}(\eta^1-\eta^5),\quad \xi^6 = \frac{1}{\sqrt{2}}(\mathrm{d}t+\mathrm{d}u),
\end{aligned}
$$

in which $\widetilde{g}_F$ reads as

$$\widetilde{g}_F = (\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^4)^2 - (\xi^5)^2 - (\xi^6)^2,$$

and the following representation of the Clifford algebra $\mathrm{Cl}(4,3)$ via the Gamma-matrices

$$
\gamma_0 = \mathrm{i}\begin{pmatrix} \sigma_1 & 0 & 0 & 0\\ 0 & \sigma_1 & 0 & 0\\ 0 & 0 & \sigma_1 & 0\\ 0 & 0 & 0 & \sigma_1 \end{pmatrix},\qquad
\gamma_1 = -\mathrm{i}\begin{pmatrix} \sigma_2 & 0 & 0 & 0\\ 0 & \sigma_2 & 0 & 0\\ 0 & 0 & \sigma_2 & 0\\ 0 & 0 & 0 & \sigma_2 \end{pmatrix},
$$

$$
\gamma_2 = -\mathrm{i}\begin{pmatrix} 0 & \sigma_3 & 0 & 0\\ \sigma_3 & 0 & 0 & 0\\ 0 & 0 & 0 & \sigma_3\\ 0 & 0 & \sigma_3 & 0 \end{pmatrix},\qquad
\gamma_3 = \begin{pmatrix} 0 & \sigma_3 & 0 & 0\\ -\sigma_3 & 0 & 0 & 0\\ 0 & 0 & 0 & \sigma_3\\ 0 & 0 & -\sigma_3 & 0 \end{pmatrix},
$$

$$
\gamma_4 = \begin{pmatrix} 0 & 0 & -\sigma_3 & 0\\ 0 & 0 & 0 & \sigma_3\\ -\sigma_3 & 0 & 0 & 0\\ 0 & \sigma_3 & 0 & 0 \end{pmatrix},\qquad
\gamma_5 = \mathrm{i}\begin{pmatrix} 0 & 0 & -\sigma_3 & 0\\ 0 & 0 & 0 & \sigma_3\\ \sigma_3 & 0 & 0 & 0\\ 0 & -\sigma_3 & 0 & 0 \end{pmatrix},
$$

$$
\gamma_6 = \begin{pmatrix} \sigma_3 & 0 & 0 & 0\\ 0 & -\sigma_3 & 0 & 0\\ 0 & 0 & -\sigma_3 & 0\\ 0 & 0 & 0 & \sigma_3 \end{pmatrix}.
$$

The σ's are the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we have to solve the parallel spinor equations

$$0 = \widetilde{\nabla}\psi = d\psi - \frac{1}{4}\sum_{k,l=0}^{6} \widetilde{\omega}^{kl}\gamma_k\gamma_l\psi. \tag{18}$$

Here $\widetilde{\omega}^{kl}$ are the Levi-Civita connection 1-forms for the ambient metric $\widetilde{g}_F$ in the orthonormal co-frame ξ^i. I.e., $\widetilde{\omega}^{ij}$ are determined by $\widetilde{\omega}^{ij} = -\widetilde{\omega}^{ji}$, $d\xi^i + \widetilde{\omega}^i{}_j \wedge \xi^j = 0$, and $\widetilde{\omega}^{ij} = \widetilde{\omega}^i{}_k g^{kj}$.

Proposition 7. *Let $F = q^2 + \sum_{i=0}^{6} a_i p^i + bz$ with $b = 0$. The non-null spinor*

$$\psi_1 = \left(\mathrm{i}(7+2\sqrt{6}), 5, -5, -\mathrm{i}(7+2\sqrt{6}), -\mathrm{i}(7+2\sqrt{6}), 5, 5, -\mathrm{i}(7+2\sqrt{6})\right).$$

is a solution of the parallel spinor equation (18).

Proof. To see that ψ_1 is parallel, one checks that if $b = 0$ then ψ_1 is annihilated by $\sum_{k,l=0}^{6} \widetilde{\omega}^{kl}\gamma_k\gamma_l$. Obviously, $d\psi$ is also zero. A direct calculation using (12) shows that $\|\psi_1\|^2 = -16(1+\sqrt{6})(6+\sqrt{6}) \neq 0$. □

Corollary 2. *Let $F = q^2 + \sum_{i=0}^{6} a_i p^i + bz$ with $b = 0$. If at least one of a_3, a_4, a_5, or a_6 is not zero ψ_1 spans the space of parallel spinors for $\widetilde{g}_F$. In particular, the ambient holonomy of the conformal class $[g_F]$ defined by F has holonomy $G_{2(2)}$.*

Proof. If the ambient metric admits two linearly independent spinors, by Proposition 6 we know that there exists an Einstein metric in the conformal class. But this was not possible if at least one of a_4, a_5, or a_6 is not zero, or if $a_3 \neq 0$ and all the other a_i's zero (see Propositions 4 and 5 of the previous section). □

Note the fact that the parallel spinor written w.r.t. the frame $\xi^0, \dots, \xi^6$ has constant coefficients. This shows that this frame is a section of the holonomy bundle of $(\widetilde{M}, \widetilde{g})$, i.e. obtained by parallel displacements along a path.

Remark 2. We remark that the validity of the above result can also be checked by a direct integration of equation (18). Indeed, if ψ is a solution of this equation, it must satisfy the integrability conditions

$$\widetilde{\mathcal{R}}_{ij}{}^{kl}\gamma_k\gamma_l\psi = 0,$$

where $\widetilde{\mathcal{R}}_{ij}{}^{kl}$ are the components of the Riemann tensor of $\widetilde{g}_F$ in the orthonormal co-frame ξ^i. Assuming the parameter b in $\widetilde{g}_F$ is zero and that one of a_4, a_5, or a_6 is non zero, we find that there is a two-dimensional space of spinor fields satisfying this algebraic condition. This space is spanned by ψ_1 and

$$\psi_2 \quad = \quad \left(1, \mathrm{i} - \frac{2\mathrm{i}\alpha u}{\beta t + \alpha u}, \mathrm{i} - \frac{12\mathrm{i}\alpha u}{\beta t + \alpha u}, -1, 1, \mathrm{i} - \frac{12\mathrm{i}\alpha u}{\beta t + \alpha u}, \mathrm{i} - \frac{12\mathrm{i}\alpha u}{\beta t + \alpha u}, 1\right)$$

where $\alpha = 6+\sqrt{6}$ and $\beta = 6(1+\sqrt{6})$. Hence, every parallel spinor ψ has to lie in the span of ψ_1 and ψ_2, i.e. $\psi = f_1\psi_1 + f_2\psi_2$ with smooth functions f_1 and f_2 satsifying the PDE $0 = \widetilde{\nabla}\psi$. Setting $\widetilde{\nabla}\psi = \chi_0\xi^0 + \dots + \ \chi_6\xi^6$ with spinor valued functions χ_i, we solve $\chi_6 \equiv 0$ obtaining $f_2 \equiv 0$. Then on checks that ψ_1 itself is parallel, therefore f_1 is a constant. Hence, the space of parallel spinors is one-dimensional spanned by $\psi = \psi_1$.

Note also that if $b = 0$ but the a_i's are arbitrary, the spinors ψ_1 and ψ_2 are always in the solution space of the integrability condition, which might in general have dimension bigger than two.

Now, we will give a formula for the parallel three-form φ that defines the $\mathrm{G}_{2(2)}$ structure. φ is related to the spinor ψ by the following relation (see for example [10]): First one defines a skew $(2,1)$-tensor A^ψ depending on ψ via

$$X \cdot Y \cdot \psi + \widetilde{g}(X,Y)\psi = A^\psi(X,Y) \cdot \psi$$

and obtains φ by dualising it

$$\varphi(X,Y,Z) := \widetilde{g}(X, A^\psi(Y,Z)).$$

Calulating this with Mathematica we get that φ is a constant multiple of

$$5\left(\xi^{012} - \xi^{045} + \xi^{146} - \xi^{256}\right) + 7\left(\xi^{014} - \xi^{025} + \xi^{126} - \xi^{456}\right) + 2c\left(\xi^{036} + \xi^{135} + \xi^{234}\right),$$

where

$$c = \frac{(12 + 7\sqrt{6})(73 + 28\sqrt{6})}{847 + 342\sqrt{6}}$$

and $\xi^{ijk} := \xi^i \wedge \xi^j \wedge \xi^k$. A direct calculation verifies that φ and its Hodge dual are closed.

Corollary 2 confirms our belief that for conformal classes with truncated ambient metrics the conformal holonomy should be equal to the ambient metric holonomy. However, in principle, for the remaining cases with $b \neq 0$ the ambient metric holonomy could be bigger than the conformal holonomy. So far we focussed on the case $b = 0$, because here the formulae turned out to be simple enough in order to determine explicitly the solution to the spinor field equation. For $b \neq 0$, the equations are much more complicated and we were not able to find a parallel spinor.

However, some statements can be made for a conformal class given by $F = q^2 + a_3p^3 + a_2p^2 + a_1p + a_0 + bz$ with $a_3 \neq 0$.

We recall that in this case g_F is *conformally* Cotton for every $b \in \mathbb{R}$ (see Proposition 5), and that if $b = 0$ it is Cotton *flat*. The coincidence of the two facts, (1) that when $b = 0$ the metric g_F is Cotton flat and (2) that in this case it is possible to find a parallel spinor explicitly, suggests that it is the Cotton flatness of g_F that simplifies the integration of the parallel spinor equations. Thus in *all* cases when g_F is conformally Cotton, one should start solving the parallel spinor equations only after rescalling the metric to the Cotton flat form. It turns out that the application of this idea to the $F = q^2 + a_3p^3 + a_2p^2 + a_1p + a_0 + bz$, $b \neq 0$, $a_3 \neq 0$ case enormously simplifies the calculations and leads to the explicit formula for the parallel spinor.

Indeed, from the proof of Proposition 5 we know that the Cotton scale is $\sigma(x) = \mathrm{e}^{-2bx/3}$ with the Cotton flat metric $\sigma^2 g_F$. In this scale, using the coframe $\sigma\theta^i$ of the metric $\sigma^2 g_F$, we consider a spinor ψ with components $\psi = (s_1, \ldots, s_8)$ given by

$$\begin{aligned}
s_1 = -s_8 &= \mathrm{i}\mathrm{e}^{\frac{-bx}{3}}\left(2^{5/6}3c_1 + \left(2^{1/3} + 4bc_3\right)\mathrm{e}^{\frac{2bx}{3}}\right)\\
s_2 = s_7 &= \mathrm{e}^{\frac{-bx}{3}}2^{1/6}c_1\left(2^{2/3}3 - 2^{1/6}c_1\mathrm{e}^{\frac{2bx}{3}}\right)\\
s_3 = -s_6 &= \mathrm{e}^{\frac{-bx}{3}}\left(-2^{5/6}3c_1 + c_2\mathrm{e}^{\frac{2bx}{3}}\right),
\end{aligned}$$

where the constants c_1, c_2, c_3 are defined by $c_1 := \sqrt{3}+b2^{5/6}$, $c_2 := 2^{1/3}3-4b^2$, and $c_3 := 2^{1/6}\sqrt{3}+b$. One can check by a direct calculation that ψ is *parallel* in $\sigma^2 g_F$. It is nonzero if $b \neq -\frac{\sqrt{3}}{2^{5/6}}$ with length square given by $\|\psi\|^2 = -2^{41/6}\cdot 3^{5/2}\cdot(b+\frac{\sqrt{3}}{2^{5/6}})^2 \neq 0$. In the case $b = -\frac{\sqrt{3}}{2^{5/6}}$, the components of a parallel spinor are given by

$$
\begin{aligned}
s_1 = \mathrm{i}s_2 = \mathrm{i}s_7 = -s_8 &= c(x)\\
s_3 = -s_6 &= \mathrm{i}\left(c(x)-\sqrt{\tfrac{2}{3}}c(-x)\right)\\
s_4 = s_5 &= -\left(c(x)+\sqrt{\tfrac{2}{3}}c(-x)\right),
\end{aligned}
$$

where $c(x) = \mathrm{e}^{\frac{x}{\sqrt{3}\cdot 2^{5/6}}}$. This spinor has length square $-8\cdot\sqrt{2/3} \neq 0$.

Since we are in the conformally *non* Einstein case, in both cases the spinor is *unique* by Proposition 6. Thus, we have the following proposition.

Proposition 8. *Let $F = q^2 + a_3p^3 + a_2p^2 + a_1p + a_0 + bz$ with $b \neq 0$ and $a_3 \neq 0$. Then the ambient metric $\widetilde{g}_F$ admits a parallel spinor and hence, has holonomy equal to $G_{2(2)}$.*

Concluding, we end up with

Theorem 2. *Let $F = q^2 + \sum_{i=0}^{6} a_ip^i + b$ and let $[g_F]$ be the conformal class defined by the metric g_F as in (10). Then the holonomy of the ambient metric $\widetilde{g}_F$ for $[g_F]$ is equal to $\mathrm{G}_{2(2)}$ provided that*

(1) *$b = 0$ and at least one of a_3, a_4, a_5, or a_6 is not zero,*
(2) *$b \neq 0$, $a_3 \neq 0$ and $a_4 = a_5 = a_6 = 0$.*

We do not know what happens in the remaining open cases with $b \neq 0$ and at least one of a_4, a_5, or a_6 nonzero. In this case we were not able to find the parallel spinor, mainly because we have no preferred scale (like the Cotton flat one) in which the spinor can be calculated.

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