AN INTERPRETATION OF E_n-HOMOLOGY AS FUNCTOR HOMOLOGY

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ABSTRACT. We prove that E_n-homology of non-unital commutative algebras can be described as functor homology when one considers functors from a certain category of planar trees with n levels. For different n these homology theories are connected by natural maps, ranging from Hochschild homology and its higher order versions to Gamma homology.

1. INTRODUCTION

By neglect of structure, any commutative and associative algebra can be considered as an associative algebra. More generally, we can view such an algebra as an E_n-algebra, *i.e.*, an algebra over an operad in chain complexes that is weakly equivalent to the chain complex of the little-n-cubes operad of [4] for $1 \leqslant n \leqslant \infty$. Hochschild homology is a classical homology theory for associative algebras and hence it can be applied to commutative algebras as well. Less classically, Gamma homology [14] is a homology theory for E_∞-algebras and Gamma homology of commutative algebras plays an important role in the obstruction theory for E_∞ structures on ring spectra [13, 7, 1] and its structural properties are rather well understood [12].

It is desirable to have a good understanding of the appropriate homology theories in the intermediate range, *i.e.*, for $1 < n < \infty$. A definition of E_n-homology for augmented commutative algebras is due to Benoit Fresse [6] and the main topic of this paper is to prove that these homology theories possess an interpretation in terms of functor homology. We extend the range of E_n-homology to functors from a suitable category Epi_n to modules in such a way that it coincides with Fresse's theory when we consider a functor that belongs to an augmented commutative algebra and show in Theorem 4.1 that E_n-homology can be described as functor homology, so that the homology groups are certain Tor-groups.

As a warm-up we show in section 2 that bar homology of a non-unital algebra can be expressed in terms of functor homology for functors from the category of order-preserving surjections to k-modules. In section 3 we introduce our categories of epimorphisms, Epi_n, and their relationship to planar trees with n-levels. We introduce a definition of E_n-homology for functors from Epi_n to k-modules that coincides with Benoit Fresse's definition of E_n-homology of a non-unital commutative algebra, $\bar{A}$, when we apply our version of E_n-homology to a suitable functor, $\mathcal{L}(\bar{A})$. We describe a spectral sequence that has tensor products of bar homology groups as input and converges to E_2-homology. Section 4 is the technical heart of the paper. Here we prove that E_n-homology has a Tor interpretation. The proof of the acyclicity of a family of suitable projective generators is an inductive argument that uses poset homology.

For varying n, the derived functors that describe E_n-homology are related to each other via a sequence of homology theories

$$H_*^{E_1} \to H_*^{E_2} \to H_*^{E_3} \to \dots$$

In a different context it is well known that the stabilization map from Hochschild homology to Gamma homology can be factored over so called higher order Hochschild homology [9]: for a commutative algebra A there is a sequence of maps connecting Hochschild homology of A, $HH_*(A)$, to Hochschild homology of order n of A and finally to Gamma homology of A, $H\Gamma_{*-1}(A)$. In order to relate these two settings we prove in the appendix that for augmented commutative algebras over a field, Hochschild homology of order n coincides with the homology of the n-fold iterated bar construction and this in turn can be related to

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E_n-homology of the augmentation ideal. This result seems to be a well-known folk result, but as we do not know of any published explicit proof, we supply one.

In the following we fix a commutative ring with unit, k. For a set S we denote by $k[S]$ the free k-module generated by S.

2. Tor interpretation of bar homology

We interpret the bar homology of a functor from the category of finite sets and order-preserving surjections to the category of k-modules as a Tor-functor.

For unital k-algebras, the complex for the Hochschild homology of the algebra can be viewed as the chain complex associated to a simplicial object. In the absense of units, this is no longer possible.

Let $\bar{A}$ be a non-unital k-algebra. The bar-homology of $\bar{A}$, $H_*^{\mathrm{bar}}(\bar{A})$, is defined as the homology of the complex

$$C_*^{\mathrm{bar}}(\bar{A}) : \ldots \to \bar{A}^{\otimes n+1} \xrightarrow{b'} \bar{A}^{\otimes n} \xrightarrow{b'} \ldots \xrightarrow{b'} \bar{A} \otimes \bar{A} \xrightarrow{b'} \bar{A}$$

with $C_n^{\mathrm{bar}}(\bar{A}) = \bar{A}^{\otimes n+1}$ and $b' = \sum_{i=0}^{n-1}(-1)^i d_i$ where d_i applied to $a_0 \otimes \ldots \otimes a_n \in \bar{A}^{\otimes n+1}$ is $a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$.

The category of non-unital associative k-algebras is equivalent to the category of augmented k-algebras. If one replaces $\bar{A}$ by $A = \bar{A} \oplus k$, then $C_n^{\mathrm{bar}}(\bar{A})$ corresponds to the reduced Hochschild complex of A with coefficients in the trivial module k, shifted by one: $H_*^{\mathrm{bar}}(\bar{A}) = HH_{*+1}(A, k)$, for $* \geqslant 0$.

Definition 2.1. Let Δ^{epi} be the category whose objects are the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$ with the ordering $0 < 1 < \ldots < n$ and whose morphisms are order-preserving surjective functions. We will call covariant functors $F \colon \Delta^{\mathrm{epi}} \to k\text{-mod}$ *Δ^{epi}-modules.*

We have the basic order-preserving surjections $d_i \colon [n] \to [n-1], 0 \leqslant i \leqslant n-1$ that are given by

$$d_i(j) = \begin{cases} j & j \leqslant i, \\ j-1 & j > i. \end{cases}$$

Any order-preserving surjection is a composition of these basic ones.

Definition 2.2. We define the *bar-homology of a Δ^{epi}-module F* as the homology of the complex $C_*^{\mathrm{bar}}(F)$ with $C_n^{\mathrm{bar}}(F) = F[n]$ and differential $b' = \sum_{i=0}^{n-1}(-1)^i F(d_i)$.

For a non-unital algebra $\bar{A}$ the functor $\mathcal{L}(\bar{A})$ that assigns $\bar{A}^{\otimes(n+1)}$ to $[n]$ and $\mathcal{L}(d_i)(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$ $(0 \leqslant i \leqslant n-1)$ is a Δ^{epi}-module. In that case, $C_*^{\mathrm{bar}}(\mathcal{L}(\bar{A})) = C_*^{\mathrm{bar}}(\bar{A})$.

In the following we use the machinery of functor homology as in [11]. Note that the category of Δ^{epi}-modules has enough projectives: the representable functors $(\Delta^{\mathrm{epi}})^n \colon \Delta^{\mathrm{epi}} \to k\text{-mod}$ with $(\Delta^{\mathrm{epi}})^n[m] = k[\Delta^{\mathrm{epi}}([n],[m])]$ are easily seen to be projective objects and each Δ^{epi}-module receives a surjection from a sum of representables. The analogous statement is true for contravariant functors from Δ^{epi} to the category of k-modules where we can use the functors Δ_n^{epi} with $\Delta_n^{\mathrm{epi}}[m] = k[\Delta^{\mathrm{epi}}([m],[n])]$ as projective objects.

We call the cokernel of the map between contravariant representables

$$(d_0)_* \colon \Delta_1^{\mathrm{epi}} \to \Delta_0^{\mathrm{epi}}$$

b^{epi}. Note that $\Delta_0^{\mathrm{epi}}[n]$ is free of rank one for all $n \geqslant 0$ because there is just one map in Δ^{epi} from $[n]$ to $[0]$ for all n. Furthermore, $\Delta_1^{\mathrm{epi}}[0]$ is the zero module, because $[0]$ cannot surject onto $[1]$. Therefore

$$b^{\mathrm{epi}}[n] \cong \begin{cases} 0 & \text{for } n > 0, \\ k & \text{for } n = 0. \end{cases}$$

Proposition 2.3. *For any Δ^{epi}-module F*

$$H_p^{\mathrm{bar}}(F) \cong \mathrm{Tor}_p^{\Delta^{\mathrm{epi}}}(b^{\mathrm{epi}}, F) \text{ for all } p \geqslant 0. \quad (2.1)$$

For the proof recall that a sequence of Δ^{epi}-modules and natural transformations

$$0 \to F' \xrightarrow{\phi} F \xrightarrow{\psi} F'' \to 0 \quad (2.2)$$

is *short exact* if it gives rise to a short exact sequence of k-modules

$$0 \to F'[n] \xrightarrow{\phi[n]} F[n] \xrightarrow{\psi[n]} F''[n] \to 0$$

for every $n \geqslant 0$.

Proof. We have to show that $H_*^{\mathrm{bar}}(-)$ maps short exact sequences of Δ^{epi}-modules to long exact sequences, that $H_*^{\mathrm{bar}}(-)$ vanishes on projectives in positive degrees and that $H_0^{\mathrm{bar}}(F)$ and $b^{\mathrm{epi}} \otimes_{\Delta^{\mathrm{epi}}} F$ agree for all Δ^{epi}-modules F.

A short exact sequence as in (2.2) is sent to a short exact sequence of chain complexes

$$0 \longrightarrow C_*^{\mathrm{bar}}(F') \xrightarrow{C_*^{\mathrm{bar}}(\phi)} C_*^{\mathrm{bar}}(F) \xrightarrow{C_*^{\mathrm{bar}}(\psi)} C_*^{\mathrm{bar}}(F'') \longrightarrow 0$$

and therefore the first claim is true.

In order to show that $H_*^{\mathrm{bar}}(P)$ is trivial in positive degrees for any projective Δ^{epi}-module P it suffices to show that the representables $(\Delta^{\mathrm{epi}})^n$ are acyclic. In order to prove this claim we construct an explicit chain homotopy.

Let $f \in (\Delta^{\mathrm{epi}})^n[m]$ be a generator, *i.e.*, a surjective order-preserving map from $[n]$ to $[m]$. Note that $f(0) = 0$. We can codify such a map by its fibres, *i.e.*, by an $(m+1)$-tuple of pairwise disjoint subsets $(A_0, \ldots, A_m)$ with $A_i \subset [n]$, $0 \in A_0$ and $\bigcup_{i=0}^{m-1} A_i = [n]$ such that $x < y$ for $x \in A_i$ and $y \in A_j$ with $i < j$. With this notation $d_i(A_0, \ldots, A_n) = (A_0, \ldots, A_{i-1}, A_i \cup A_{i+1}, \ldots, A_n)$.

We define the chain homotopy $h\colon \Delta^{\mathrm{epi}}([n],[m]) \to \Delta^{\mathrm{epi}}([n],[m+1])$ as

$$h(A_0, \ldots, A_m) := \begin{cases} 0 & \text{if } A_0 = \{0\}, \\ (0, A_0', A_1, \ldots, A_m) & \text{if } A_0 = \{0\} \cup A_0', A_0' \neq \varnothing. \end{cases} \tag{2.3}$$

If $A_0 = \{0\}$, then

$$(b' \circ h + h \circ b')(\{0\}, \ldots, A_m) = 0 + h \circ b'(\{0\}, \ldots, A_m) = h(\{0\} \cup A_1, \ldots, A_m) = (\{0\}, \ldots, A_m).$$

In the other case a direct calculation shows that $(b' \circ h + h \circ b')(A_0, \ldots, A_m) = \mathrm{id}(A_0, \ldots, A_m)$.

It remains to show that both homology theories coincide in degree zero. By definition $H_0^{\mathrm{bar}}(P)$ is the cokernel of the map

$$F(d_0)\colon F[1] \longrightarrow F[0].$$

A Yoneda-argument [15, 17.7.2(a)] shows that the tensor product $\Delta_n^{\mathrm{epi}} \otimes_{\Delta^{\mathrm{epi}}} F$ is naturally isomorphic to $F[n]$ and hence the above cokernel is the cokernel of the map

$$((d_0)_* \otimes_{\Delta^{\mathrm{epi}}} \mathrm{id})\colon \Delta_1^{\mathrm{epi}} \otimes_{\Delta^{\mathrm{epi}}} F \longrightarrow \Delta_0^{\mathrm{epi}} \otimes_{\Delta^{\mathrm{epi}}} F.$$

As tensor products are right-exact [15, 17.7.2 (d)], the cokernel of the above map is isomorphic to

$$\mathrm{coker}((d_0)_*\colon \Delta_1^{\mathrm{epi}} \to \Delta_0^{\mathrm{epi}}) \otimes_{\Delta^{\mathrm{epi}}} F = b^{\mathrm{epi}} \otimes_{\Delta^{\mathrm{epi}}} F = \mathrm{Tor}_0^{\Delta^{\mathrm{epi}}}(b^{\mathrm{epi}}, F).$$

□

Remark 2.4. The generating morphisms d_i in Δ^{epi} correspond to the face maps in the standard simplicial model of the 1-sphere.

3. Epimorphisms and trees

Planar level trees are used in [2], [6] and [3, 3.15] as a means to codify E_n-structures. An n-level tree is a planar level tree with n levels. We will use categories of planar level trees in order to gain a description of E_n-homology as functor homology. If $\mathcal{C}$ is a small category we denote by $N\mathcal{C}$ the nerve of $\mathcal{C}$.

Definition 3.1. Let $n \geqslant 1$ be a natural number. The category Epi_n has as objects the elements of $N_{n-1}(\Delta^{\mathrm{epi}})$, *i.e.*, sequences

$$[r_n] \xrightarrow{f_n} [r_{n-1}] \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} [r_1] \tag{3.1}$$

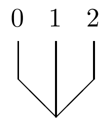
with $[r_i] \in \Delta^{\mathrm{epi}}$ and surjective order-preserving maps f_i. A morphism in Epi_n from the above object to an object $[r_n'] \xrightarrow{f_n'} [r_{n-1}'] \xrightarrow{f_{n-1}'} \ldots \xrightarrow{f_2'} [r_1']$ consists of surjective maps $\sigma_i\colon [r_i] \to [r_i']$ for $1 \leqslant i \leqslant n$ such that

$\sigma_1 \in \Delta^{\mathrm{epi}}$ and for all $2 \leqslant i \leqslant n$ the map σ_i is order-preserving on the fibres $f_i^{-1}(j)$ for all $j \in [r_{i-1}]$ and such that the diagram

$$\begin{array}{ccccccc} [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & [r_1] \\ \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_{n-1}} & & & & \downarrow{\scriptstyle \sigma_1} \\ [r'_n] & \xrightarrow{f'_n} & [r'_{n-1}] & \xrightarrow{f'_{n-1}} & \cdots & \xrightarrow{f'_2} & [r'_1] \end{array}$$

commutes.

As an example, consider the object $[2]\xrightarrow{\mathrm{id}}[2]$ in Epi_2 which can be viewed as the 2-level tree



Possible maps from this object to $[2]\xrightarrow{d_0}[1]$ are $\begin{array}{ccc} [2] & \xrightarrow{\mathrm{id}} & [2] \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle d_0} \\ [2] & \xrightarrow{d_0} & [1] \end{array}$ and $\begin{array}{ccc} [2] & \xrightarrow{\mathrm{id}} & [2] \\ {\scriptstyle (0,1)}\downarrow & & \downarrow{\scriptstyle d_0} \\ [2] & \xrightarrow{d_0} & [1] \end{array}$ where $(0,1)$ denotes the transposition that permutes 0 and 1. For $\sigma_1 = d_1$ there is no possible σ_2 to fill in the diagram.

If $n = 1$, then Epi_1 coincides with the category Δ^{epi}. Note that there is a functor $\iota_n \colon \Delta^{\mathrm{epi}} = \mathrm{Epi}_1 \to \mathrm{Epi}_n$ for all $n \geqslant 1$ with

$$\iota_n([m]) := [m] \longrightarrow [0] \longrightarrow \ldots \longrightarrow [0].$$

We call trees of the form $\iota_n([m])$ *palm trees with* $m+1$ *leaves.* More generally we have functors connecting the various categories of planar level trees.

Lemma 3.2. *For all* $n > k \geqslant 1$ *there are functors* $\iota_n^k \colon \mathrm{Epi}_k \to \mathrm{Epi}_n$*, with*

$$\iota_n^k([r_k] \xrightarrow{f_k} \ldots \xrightarrow{f_2} [r_1]) = [r_k] \xrightarrow{f_k} \ldots \xrightarrow{f_2} [r_1] \longrightarrow [0] \longrightarrow \ldots \longrightarrow [0]$$

on objects, with the canonical extension to morphisms. □

Remark 3.3. The maps ι_n^k correspond to iterated suspension morphisms in [2, 4.1]. There is a different way of mapping a planar tree with n levels to one with $n+1$ levels, by sending $[r_n] \xrightarrow{f_n} \ldots \xrightarrow{f_2} [r_1]$ to $[r_n] \xrightarrow{\mathrm{id}_{[r_n]}} [r_n] \xrightarrow{f_n} \ldots \xrightarrow{f_2} [r_1]$. We call such trees *fork trees* and they will need special attention later when we prove that representable functors are acyclic.

For any Σ_*-cofibrant operad $\mathcal{P}$ there exists a homology theory for $\mathcal{P}$-algebras which is denoted by $H_*^{\mathcal{P}}$ and is called $\mathcal{P}$-homology. Fresse studies the particular case of $\mathcal{P} = E_n$ a differential graded operad quasi-isomorphic to the chain operad of the little n-disks operad. He proves that for any commutative algebra the E_n-homology coincides with the homology of its n-fold bar construction. In fact, his result is more general since he defines an analogous n-fold bar construction for E_n-algebras and proves the result for any E_n-algebra in [6, theorem 7.26].

We consider the n-fold bar construction of a non-unital commutative k-algebra $\bar{A}$, $B^n(\bar{A})$, as an n-complex indexed over the objects in Epi_n, such that

$$B^n(\bar{A})_{(r_n,\ldots,r_1)} = \bigoplus_{[r_n]\xrightarrow{f_n}\ldots\xrightarrow{f_2}[r_1]\in \mathrm{Epi}_n} \bar{A}^{\otimes(r_n+1)}.$$

The differential in $B^n(\bar{A})$ is the total differential associated to n-differentials $\partial_1, \ldots, \partial_n$ such that ∂_n is built out of the multiplication in $\bar{A}$, ∂_{n-1} corresponds to the shuffle multiplication on $B(\bar{A})$ and so on. We describe the precise setting in a slightly more general context.

In order to extend the Tor-interpretation of bar homology of Δ^{epi}-modules to functors from Epi_n to modules (alias Epi_n-modules) we describe the n kinds of face maps for Epi_n in detail by considering diagrams of the form

$$\begin{array}{ccccccccccccc} [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_{j+2}} & [r_{j+1}] & \xrightarrow{f_{j+1}} & [r_j] & \xrightarrow{f_j} & [r_{j-1}] & \xrightarrow{f_{j-1}} \cdots \xrightarrow{f_2} & [r_1] \\ \downarrow{\scriptstyle \tau_n^{i,j}} & & \downarrow{\scriptstyle \tau_{n-1}^{i,j}} & & \downarrow{\scriptstyle \tau_{j+1}^{i,j}} & & \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ [r_n] & \xrightarrow{g_n} & [r_{n-1}] & \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_{j+2}} & [r_{j+1}] & \xrightarrow{g_{j+1}} & [r_j - 1] & \xrightarrow{g_j} & [r_{j-1}] & \xrightarrow{f_{j-1}} \cdots \xrightarrow{f_2} & [r_1]. \end{array} \tag{3.2}$$

Given the object in the first row, it is not always possible to extend $(d_i\colon [r_j] \to [r_j - 1], \mathrm{id}_{[r_{j-1}]}, \ldots, \mathrm{id}_{[r_1]})$ to a morphism in Epi_n: we have to find order-preserving surjective maps g_k for $j \leqslant k \leqslant n$ and bijections $\tau_k^{i,j}\colon [r_k] \to [r_k]$ that are order-preserving on the fibres of f_k for $j+1 \leqslant k \leqslant n$ such that the diagram commutes.

Lemma 3.4.

(a) *There is a unique order-preserving surjection $g_j\colon [r_j - 1] \to [r_{j-1}]$ with $g_j \circ d_i = f_j$ if and only if $f_j(i) = f_j(i+1)$. When it exists, g_j is denoted by $f_j|_{i=i+1}$.*

(b) *If $f_j(i) = f_j(i+1)$ then we can extend the diagram to one of the form (3.2) so that $\tau_{j+1}^{i,j}$ is a shuffle of the fibres $f_{j+1}^{-1}(i)$ and $f_{j+1}^{-1}(i+1)$. Each choice of a $\tau_{j+1}^{i,j}$ uniquely determines the maps $\tau_k^{i,j}$ for all $j+1 < k \leqslant n$.*

(c) *If $f_j(i) = f_j(i+1)$ then the maps g_k are uniquely determined by the maps f_k for $k \geqslant j$. The diagram (3.2) takes the following form*

$$\begin{array}{ccccccccccccc} [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_{j+2}} & [r_{j+1}] & \xrightarrow{f_{j+1}} & [r_j] & \xrightarrow{f_j} & [r_{j-1}] & \xrightarrow{f_{j-1}} \cdots \xrightarrow{f_2} & [r_1] \\ \downarrow{\scriptstyle \tau_n^{i,j}} & & \downarrow{\scriptstyle \tau_{n-1}^{i,j}} & & \downarrow{\scriptstyle \tau_{j+1}^{i,j}} & & \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_{j+2}} & [r_{j+1}] & \xrightarrow{d_i f_{j+1}} & [r_j - 1] & \xrightarrow{f_j|_{i=i+1}} & [r_{j-1}] & \xrightarrow{f_{j-1}} \cdots \xrightarrow{f_2} & [r_1]. \end{array} \tag{3.3}$$

Proof. If there is such a map g_j, then $f_j(i+1) = g_j \circ d_i(i+1) = g_j \circ d_i(i) = f_j(i)$. As f_j is order-preserving, it is determined by the cardinalities of its fibres. The decomposition of morphisms in the simplicial category then ensures that we can factor f_j in the desired way.

For the third claim, assume that g_j exists with the properties mentioned in (a). As g_{j+1} and $d_i \circ f_{j+1}$ are both order-preserving maps from $[r_{j+1}]$ to $[r_j - 1]$, they are determined by the cardinalities of the fibres and thus they have to agree. Then $\tau_{j+1}^{i,j} = \mathrm{id}_{[r_{j+1}]}$ extends the diagram up to layer $j+1$. For the higher layers we then have to choose $g_k = f_k$ and $\tau_k^{i,j} = \mathrm{id}_{[r_k]}$.

In general, $\tau_{j+1}^{i,j}$ has to satisfy the conditions that it is order-preserving on the fibres of f_{j+1}. If $A_i = f_{j+1}^{-1}(i)$ then this implies that $\tau_{j+1}^{i,j}$ is an $(A_0, \ldots, A_{r_j})$-shuffle. Furthermore we have that

$$(d_i \circ f_{j+1})^{-1}(k) = \begin{cases} A_k & \text{if } k < i, \\ A_i \cup A_{i+1} & \text{if } k = i, \\ A_{k+1} & \text{if } k > i. \end{cases}$$

Therefore $\tau_{j+1}^{i,j}$ has to map $A_0, \ldots, A_{i-1}, A_{i+2}, \ldots, A_{r_j}$ identically and is hence an (A_i, A_{i+1})-shuffle.

If we fix a shuffle $\tau_{j+1}^{i,j}$, then the next permutation $\tau_{j+2}^{i,j}$ has to be order-preserving on the fibres of f_{j+2}, thus it is at most a shuffle of the fibres. In addition, it has to satisfy

$$g_{j+2} \circ \tau_{j+2}^{i,j} = \tau_{j+1}^{i,j} \circ f_{j+2}. \tag{3.4}$$

Again, as g_{j+2} is order-preserving we have no choice but to take $g_{j+2} = f_{j+2}$. From (3.4) we know that $\tau_{j+2}^{i,j}$ has to send $f_{j+2}^{-1}(k)$ to $f_{j+2}^{-1}(\tau_{j+1}^{i,j}(k))$ and this determines $\tau_{j+2}^{i,j}$. A proof by induction shows the general claim in (b). ☐

In the following we will extend the notion of E_n-homology for commutative non-unital k-algebras to Epi_n-modules. Again thanks to Fresse's theorem [6, theorem 7.26], the E_n-homology and the homology of the n-fold bar construction of a commutative algebra coincide.

Definition 3.5. Let F be an Epi_n-module.

For a fixed j and an object as in (3.1) with the condition that $f_j(i) = f_j(i+1)$ we define

$$d_i^j \colon F([r_n] \xrightarrow{f_n} \dots \xrightarrow{f_{j+1}} [r_j] \xrightarrow{f_j} \dots \xrightarrow{f_2} [r_1]) \longrightarrow F([r_n] \xrightarrow{f_n} \dots \xrightarrow{d_i f_{j+1}} [r_j - 1] \xrightarrow{f_j|_{i=i+1}} \dots \xrightarrow{f_2} [r_1])$$

as

$$d_i^j = \sum_{\tau_{j+1}^{i,j} \in \mathrm{Sh}(A_i, A_{i+1})} \mathrm{sgn}(\tau_{j+1}^{i,j}) F(\tau_n^{i,j}, \dots, \tau_{j+1}^{i,j}, d_i, \mathrm{id}, \dots, \mathrm{id}). \tag{3.5}$$

The next proposition is a straightforward computation:

Proposition-Definition 3.6.

- If F is an Epi_n-module, then the E_n-chain complex of F is the n-fold chain complex whose $(r_n, \dots, r_1)$ spot is

$$C^{E_n}_{(r_n,\dots,r_1)}(F) = \bigoplus_{[r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1] \in \mathrm{Epi}_n} F([r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]). \tag{3.6}$$

  The differential in the j-th coordinate is

$$\partial_j \colon C^{E_n}_{(r_n,\dots,r_j,\dots,r_1)}(F) \to C^{E_n}_{(r_n,\dots,r_j-1,\dots,r_1)}(F)$$

  with

$$\partial_j := \sum_{i | f_j(i) = f_j(i+1)} (-1)^i F(d_i^j).$$

- The E_n*-homology of* F, $H_*^{E_n}(F)$ is defined to be the homology of the total complex associated to (3.6).

Remark 3.7. In Definition 3.6, the chain module $C^{E_n}_{(r_n,\dots,r_1)}$ is trivial for n-tuples $(r_n, \dots, r_1)$ that do not satisfy $r_n \geqslant \dots \geqslant r_1$.

Remark 3.8. For a non-unital commutative k-algebra $\bar{A}$ we define $\mathcal{L}^n(\bar{A}) \colon \mathrm{Epi}_n \to k\text{-mod}$ as

$$\mathcal{L}^n(\bar{A})([r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]) = \bar{A}^{\otimes(r_n+1)}.$$

A morphism

$$\begin{array}{ccccccc}
[r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & [r_1] \\
\downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_{n-1}} & & & & \downarrow{\scriptstyle \sigma_1} \\
[r'_n] & \xrightarrow{f'_n} & [r'_{n-1}] & \xrightarrow{f'_{n-1}} & \cdots & \xrightarrow{f'_2} & [r'_1]
\end{array}$$

induces a map $\bar{A}^{\otimes(r_n+1)} \to \bar{A}^{\otimes(r'_n+1)}$ via

$$a_0 \otimes \dots \otimes a_{r_n} \mapsto (\sigma_n)_*(a_0 \otimes \dots \otimes a_{r_n}) = b_0 \otimes \dots \otimes b_{r'_n}$$

with $b_i = \prod_{\sigma_n(j)=i} a_j$. The E_n-homology of the functor $\mathcal{L}^n(\bar{A})$ coincides with the homology of the n-fold bar construction of $\bar{A}$, hence with the E_n-homology of $\bar{A}$. The total complex has been described in [6, Appendix] and coincide with ours.

As an example, we will determine the zeroth E_n-homology of an Epi_n-functor F. In total degree zero there is just one summand, namely $F([0] \xrightarrow{\mathrm{id}_{[0]}} \dots \xrightarrow{\mathrm{id}_{[0]}} [0])$. The modules $C^{E_n}_{(0,1,0,\dots,0)}(F)$, …, $C^{E_n}_{(0,\dots,0,1)}(F)$ are all trivial, so the only boundary term that can occur is caused by the unique map

$$C^{E_n}_{(1,0,\dots,0)}(F) \longrightarrow C^{E_n}_{(0,\dots,0)}(F).$$

Therefore

$$H_0^{E_n}(F) \cong F([0] \xrightarrow{\mathrm{id}_{[0]}} \dots \xrightarrow{\mathrm{id}_{[0]}} [0]) / \mathrm{image}(F([1] \xrightarrow{d_0} [0] \xrightarrow{\mathrm{id}_{[0]}} \dots \xrightarrow{\mathrm{id}_{[0]}} [0])). \tag{3.7}$$

If we consider the case $F = \mathcal{L}^n(\bar{A})$, then $\mathcal{L}^n(\bar{A})([1] \xrightarrow{\mathrm{id}_{[0]}} \ldots \xrightarrow{\mathrm{id}_{[0]}} [0]) = \bar{A}^{\otimes 2}$ and hence for all $n \geqslant 1$

$$H_0^{E_n}(\bar{A}) \cong \bar{A}/\bar{A} \cdot \bar{A}.$$

We can view an Epi_n-module F as an Epi_k-module for all $k \leqslant n$ via the functors ι_n^k.

Proposition 3.9. *For every Epi_n-module F there is a map of chain complexes $\mathrm{Tot}(C_*^{E_k}(F \circ \iota_n^k)) \longrightarrow \mathrm{Tot}(C_*^{E_n}(F))$ and therefore a map of graded k-modules*

$$H_*^{E_k}(F \circ \iota_n^k) \longrightarrow H_*^{E_n}(F).$$

Proof. There is a natural identification of the module $C^{E_k}_{(r_k,\ldots,r_1)}(F \circ \iota_n^k)$ with the module $C^{E_n}_{(r_k,\ldots,r_1,0,\ldots,0)}(F)$ and this includes $\mathrm{Tot}(C_*^{E_k}(F \circ \iota_n^k))$ as a subcomplex into $\mathrm{Tot}(C_*^{E_n}(F))$. □

In particular, for a non-unital commutative k-algebra, $\bar{A}$, we obtain a sequence of maps

$$HH_{*+1}(A;k) \cong H_*^{\mathrm{bar}}(\bar{A}) = H_*^{E_1}(\bar{A}) \to H_*^{E_2}(\bar{A}) \to H_*^{E_3}(\bar{A}) \to \ldots \tag{3.8}$$

and the map from $H_*^{E_1}(\bar{A})$ to the higher E_n-homology groups is given on chain level by the inclusion of $C_m^{\mathrm{bar}}(\bar{A})$ into $C^{E_n}_{(m,0,\ldots,0)}(\bar{A})$. This observation leads to the following result.

Proposition 3.10. *If $\bar{A}$ and $H_*^{\mathrm{bar}}(\bar{A})$ are k-flat, then there is a spectral sequence*

$$E^1_{p,q} = \bigoplus_{\ell_0+\ldots+\ell_q=p-q} H^{\mathrm{bar}}_{\ell_0}(\bar{A}) \otimes \ldots \otimes H^{\mathrm{bar}}_{\ell_q}(\bar{A}) \Rightarrow H^{E_2}_{p+q}(\bar{A})$$

where the d_1-differential is induced by the shuffle differential.

Proof. The double complex for E_2-homology looks as follows:

$$\begin{array}{ccccccc}
 & & & & \vdots & & \\
 & & & & \downarrow & & \\
 & & & & \bar{A}^{\otimes 3} & \longleftarrow & \cdots \\
 & & & & \downarrow & & \\
 & & \bar{A}^{\otimes 2} & \longleftarrow & \bar{A}^{\otimes 3} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\bar{A} & \longleftarrow & \bar{A}^{\otimes 2} & \longleftarrow & \bar{A}^{\otimes 3} & \longleftarrow & \cdots
\end{array}$$

The horizontal maps are induced by the b'-differential whereas the vertical maps are induced by the shuffle maps. The horizontal homology of the bottom row is precisely $H_*^{\mathrm{bar}}(\bar{A})$. We can interpret the second row as the total complex associated to the following double complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \\
\bar{A} \otimes \bar{A}^{\otimes 3} & \xleftarrow{b'\otimes \mathrm{id}} & \bar{A}^{\otimes 2} \otimes \bar{A}^{\otimes 3} & \xleftarrow{b'\otimes \mathrm{id}} & \bar{A}^{\otimes 3} \otimes \bar{A}^{\otimes 3} & \xleftarrow{b'\otimes \mathrm{id}} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \\
\bar{A} \otimes \bar{A}^{\otimes 2} & \xleftarrow{b'\otimes \mathrm{id}} & \bar{A}^{\otimes 2} \otimes \bar{A}^{\otimes 2} & \xleftarrow{b'\otimes \mathrm{id}} & \bar{A}^{\otimes 3} \otimes \bar{A}^{\otimes 2} & \xleftarrow{b'\otimes \mathrm{id}} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \\
\bar{A} \otimes \bar{A} & \xleftarrow{b'\otimes \mathrm{id}} & \bar{A}^{\otimes 2} \otimes \bar{A} & \xleftarrow{b'\otimes \mathrm{id}} & \bar{A}^{\otimes 3} \otimes \bar{A} & \xleftarrow{b'\otimes \mathrm{id}} & \cdots
\end{array}$$

Therefore the horizontal homology groups of the second row are the homology of the tensor product of the $C^{\mathrm{bar}}(\bar{A})$-complex with itself. Our flatness assumptions guarantee that we obtain $H_*^{\mathrm{bar}}(\bar{A})^{\otimes 2}$ as homology. An induction then finishes the proof. □

4. Tor interpretation of E_n-homology

The following notation will be helpful for the sequel: for an object t in Epi_n let Epi_n^t denote the representable functor $k[\mathrm{Epi}_n(t,-)]$ and similarly, let $\mathrm{Epi}_{n,t}$ denote the contravariant representable functor $k[\mathrm{Epi}_n(-,t)]$. The E_n-homology of an Epi_n-module F can be computed in different ways, since it is the homology of the total complex associated to an n-complex. The notation $H_*(F,\partial_i)$ stands for the homology of the complex $C_*^{E_n}(F)$ with respect to the differential ∂_i. The complex $(C_*^{E_n}(F),\partial_i)$ splits into subcomplexes

$$C^{E_n,i}_{(s_n,s_{n-1},\dots,s_{i+1},*,s_{i-1},\dots,s_1)}(F)=\bigoplus_{t=[s_n]\xrightarrow{g_n}\dots[s_{i+1}]\xrightarrow{g_{i+1}}[*]\xrightarrow{g_i}[s_{i-1}]\dots]\xrightarrow{g_2}[s_1]} F(t), \tag{4.1}$$

whose homology is denoted by $H_{(s_n,s_{n-1},\dots,s_{i+1},*,s_{i-1},\dots,s_1)}(F,\partial_i)$.

Theorem 4.1. *For any* Epi_n*-module* F

$$H_p^{E_n}(F)\cong \mathrm{Tor}_p^{\mathrm{Epi}_n}(b_n^{\mathrm{epi}},F),\ \textit{for all } p\geqslant 0$$

where

$$b_n^{\mathrm{epi}}(t)\cong\begin{cases} k & \textit{for } t=[0]\xrightarrow{\mathrm{id}_{[0]}}\dots\xrightarrow{\mathrm{id}_{[0]}}[0],\\ 0 & \textit{for } t\neq[0]\xrightarrow{\mathrm{id}_{[0]}}\dots\xrightarrow{\mathrm{id}_{[0]}}[0].\end{cases}$$

Proof. Similar to the proof of proposition 2.3, we have to show that $H_*^{E_n}(-)$ maps short exact sequences of Epi_n-modules to long exact sequences, that $H_*^{E_n}(-)$ vanishes on projectives in positive degrees and that $H_0^{E_n}(F)$ and $b_n^{\mathrm{epi}}\otimes_{\mathrm{Epi}_n}F$ agree for all Epi_n-modules F. The homology $H_*^{E_n}(-)$ is the homology of a total complex $C_*^{E_n}(-)$ sending short exact sequences as in (2.2) to short exact sequences of chain complexes and therefore the first claim is true. Note that the left Epi_n-module b_n^{epi} is the cokernel of the map between contravariant representables

$$(d_0)_*\colon \mathrm{Epi}_{n,[1]\longrightarrow[0]\longrightarrow\dots\longrightarrow[0]}\to \mathrm{Epi}_{n,[0]\longrightarrow[0]\dots\longrightarrow[0]}.$$

This remark together with the computation of $H_0^{E_n}(F)$ in relation (3.7) implies the last claim, similar to the proof of proposition 2.3.

In order to show that $H_*^{E_n}(P)$ is trivial in positive degrees for any projective Epi_n-module P it suffices to show that the representables Epi_n^t are acyclic for any planar tree $t=[r_n]\xrightarrow{f_n}\dots\xrightarrow{f_2}[r_1]$. The case $n=1$ has been proved in proposition 2.3. For $n=2$ we study the bicomplex $C^{E_2}_{(*,*)}(F)$. In proposition 4.2 we give the k-module structure of the homology with respect to the differential ∂_2 and give its generators in propositions 4.4 and 4.5. Corollaries 4.3 and 4.6 state the result for $n=2$. For the general case, one uses induction on n and proposition 4.7. As a consequence $H_*(\mathrm{Epi}_n^t)=0$ for all $*\geqslant 0$ if $t\neq[0]\longrightarrow[0]\dots\longrightarrow[0]$ and in that case

$$H_*(\mathrm{Epi}_n^{[0]\longrightarrow[0]\dots\longrightarrow[0]})=\begin{cases}0 & \text{for } *>0\\ k & \text{for } *=0.\end{cases}$$

□

Proposition 4.2. *Let* $t=[r_2]\xrightarrow{f}[r_1]$ *be a 2-level tree.*

$$H_{(*,s)}(\mathrm{Epi}_2^t,\partial_2)=0, \qquad \text{if } r_2\neq r_1$$

$$H_{(*,s)}(\mathrm{Epi}_2^t,\partial_2)\cong\begin{cases}0 & \text{for } *\neq r_2\\ k^{\oplus|\Delta^{\mathrm{epi}}([r_2],[s])|} & \text{for } s\leqslant *=r_2.\end{cases},\qquad \text{if } r_2=r_1$$

Proof. From now on F denotes the covariant functor Epi_2^t.

Assume $s=0$. We first prove that the chain complex $\partial_2\colon C^{E_2,2}_{(*,0)}(F)\to C^{E_2,2}_{(*,0)}(F)$ is the chain complex associated to a poset. Recall from Wachs [17] and Vallette [16] that a chain complex $\Pi_*(P)$ can be associated to a graded poset P with minimal element x_0 and maximal element x_M. The k-module $\Pi_u(P)$ is the free k-module generated by chains of the form $x_0<x_1<\dots<x_u<x_M$, with the differential given by $d=\sum_{i=1}^u(-1)^i d_i$ where d_i forgets x_i.

The chain complex $(C^{E_2,2}_{(*,0)}(F), \partial_2)$ has the following form:

$$k[\mathrm{Epi}_2([r_2] \xrightarrow{f} [r_1]; [u] \to [0])] \xrightarrow{\sum_{i=0}^{u}(-1)^i (d_i)_*} k[\mathrm{Epi}_2([r_2] \xrightarrow{f} [r_1]; [u-1] \longrightarrow [0])],\ 0 < u \leqslant r_2.$$

The chain complex $(C^{E_2,2}_{(*,0)}(F), \partial_2)$ has only one summand in formula (4.1) because $[0]$ is the terminal object in Δ^{epi}. Let $(A_0, \ldots, A_{r_1})$ be the sequence of preimages of f, and a_i the number of elements in A_i. Any map in $\mathrm{Epi}_2([r_2] \xrightarrow{f} [r_1]; [u] \longrightarrow [0])$ is uniquely determined by a surjective map $\sigma\colon [r_2] \to [u]$ which is order-preserving on A_i, that is, which is an $(a_0, a_1, \ldots, a_{r_1})$-shuffle. Equivalently, σ can be described by the sequence of its preimages $(S_0, \ldots, S_u)$ with the condition (C_S): if $a < b \in A_i$ then $i_a < i_b$ where i_α is the unique index for which $\alpha \in S_{i_\alpha}$. Let us consider the poset P_f whose objects are elements $(x_0, \ldots, x_{r_2})$ of $\{0,1\}^{r_2+1}$ satisfying the condition

$$\begin{aligned} x_0 \geqslant x_1 \geqslant \ldots &\geqslant x_{a_0-1}, \\ x_{a_0} \geqslant x_{a_0+1} \geqslant \ldots &\geqslant x_{a_0+a_1-1}, \\ &\ldots \\ x_{a_0+\ldots+a_{r_1-1}} \geqslant \ldots &\geqslant x_{r_2}. \end{aligned} \tag{4.2}$$

The order is the lexicographic order, the minimal element is $X_0 = (0, \ldots, 0)$ and the maximal element is $X_M = (1, \ldots, 1)$. An element in $\Pi_u(P_f)$ is a family of (r_2+1)-tuples $X_i = (x^i_0, \ldots, x^i_{r_2})$ of P_f with $X_0 < X_1 < \ldots < X_u < X_{u+1} = X_M$. Such a chain is encoded by a sequence of sets $(S_0, \ldots, S_u)$ where $S_i = \{j | x^{i+1}_j > x^i_j\}$. This sequence is an ordered partition of $[r_2]$ by non-empty subsets, and the condition (4.2) amounts to the condition (C_S). As a consequence the two complexes $(C^{E_2}_{*,0}(F), \partial_2)$ and $\Pi_*(P_f)$ coincide. The poset P_f is the product of the posets $L_{a_i}, 0 \leqslant i \leqslant r_1$ where L_{a_i} is the linear poset

$$\underbrace{(0, \ldots, 0)}_{a_i \text{ times}} < (1, 0, \ldots, 0) < (1, 1, \ldots, 0) < \ldots < (1, 1, \ldots, 1).$$

The complex $\Pi_*(L_{a_i})$ has trivial homology but for $a_i = 1$ where it is free of rank one. The Künneth formula [17, 5.1.2] implies that $\Pi_*(P_f)$ is acyclic but for $f = \mathrm{id}_{[r_2]}$ where it is concentrated in top degree and is free of rank 1. This implies the result for $s = 0$. The computation of the generator of $H_{(r_2,0)}(\mathrm{Epi}_2^{[r_2] \xrightarrow{\mathrm{id}} [r_2]}, \partial_2) \cong k$ is the subject of proposition 4.4.

Assume $s > 0$. The complex $C^{E_2,2}_{(*,s)}(F)$ splits into subcomplexes

$$C^{(E_2,2)}_{(*,s)}(F) = \bigoplus_{\sigma \in \Delta^{\mathrm{epi}}([r_1],[s])} C_{(*,s)}(F_\sigma) = \bigoplus_{\sigma \in \Delta^{\mathrm{epi}}([r_1],[s])} \bigoplus_{g \in \Delta^{\mathrm{epi}}([*],[s])} F_\sigma([*] \xrightarrow{g} [s])$$

where $F_\sigma([u] \xrightarrow{g} [s]) \subset \mathrm{Epi}_2^t([u] \xrightarrow{g} [s])$ is the free k-module generated by morphisms of the form

$$\begin{array}{ccc} [r_2] & \xrightarrow{f} & [r_1] \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma} \\ [u] & \xrightarrow{g} & [s]. \end{array} \tag{4.3}$$

Let $(A_0, \ldots, A_s)$ denote the sequence of preimages of σf and $(B_0, \ldots, B_s)$ the one of g. The latter has to satisfy the condition $|B_i| \leqslant |A_i|, 0 \leqslant i \leqslant s$. Note that $g \in \Delta^{\mathrm{epi}}([u],[s])$ is also uniquely determined by the sequence $(b_0, \ldots, b_s)$ of the cardinalities of its preimages. The differential $\partial_2\colon C_{(u,s)}(F_\sigma) \longrightarrow C_{(u-1,s)}(F_\sigma)$ has the following form:

$$\partial_2 \left(\begin{array}{ccc} [r_2] & \xrightarrow{f} & [r_1] \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma} \\ [u] & \xrightarrow{g} & [s] \end{array} \right) = \sum_{i | g(i) = g(i+1)} (-1)^i \begin{array}{ccc} [r_2] & \xrightarrow{f} & [r_1] \\ \downarrow{\scriptstyle d_i\tau} & & \downarrow{\scriptstyle \sigma} \\ [u-1] & \xrightarrow{g|_{i=i+1}} & [s] \end{array} = \sum_{j=0}^{s} \left(\sum_{i \in B_j | g(i) = g(i+1)} (-1)^i \begin{array}{ccc} [r_2] & \xrightarrow{f} & [r_1] \\ \downarrow{\scriptstyle d_i\tau} & & \downarrow{\scriptstyle \sigma} \\ [u-1] & \xrightarrow{g|_{i=i+1}} & [s] \end{array} \right)$$

The differential ∂_2 is the sum of $s+1$ commuting differentials, $\partial_2 = D_0 + \ldots + D_s$, making $C_{(*,s)}(F_\sigma)$ into an $(s+1)$-complex. The differential D_j is obtained by restricting the sum over indices i such that $g(i) = g(i+1)$ to the sum over indices $i \in B_j$ such that $g(i) = g(i+1)$. One has

$$D_j \colon C_{(u,s)}(F_\sigma) = \bigoplus_{b_0+\ldots+b_s=u+1} C_{((b_0,\ldots,b_j,\ldots,b_s),s)}(F_\sigma) \longrightarrow \bigoplus_{b_0+\ldots+b_s=u+1} C_{((b_0,\ldots,b_j-1,\ldots,b_s),s)}(F_\sigma).$$

For instance, the complex $(C_{(u,s)}(F_\sigma), D_s)$ splits into subcomplexes $(C_{((b_0,\ldots,b_{s-1}),*)}(F_\sigma), D_s)$ for fixed $b_i \leqslant a_i = |A_i|, i < s$. With the notation of (4.3),

- let $\tilde{f}$ be the map obtained from f by restriction $\tilde{f} \colon (\sigma \circ f)^{-1}(\{s\}) \xrightarrow{f} \sigma^{-1}(\{s\})$, and let $\tilde{t}$ be the corresponding 2-level tree;
- let f_{s-1} (resp. σ_{s-1}) be the map obtained from f (resp. σ) by restriction $f_{s-1} \colon (\sigma \circ f)^{-1}([s-1]) \xrightarrow{f} \sigma^{-1}([s-1])$ (resp. $\sigma_{s-1} \colon \sigma^{-1}([s-1]) \xrightarrow{\sigma} [s-1]$); let t_{s-1} be the 2-level tree associated to f_{s-1};
- let $u_s = (\sum_{i<s} b_i) - 1$.

The subcomplex $(C_{((b_0,\ldots,b_{s-1}),*)}(F_\sigma), D_s)$ writes

$$\bigoplus_{\phi \in (\mathrm{Epi}_2^{t_{s-1}})_{\sigma_{s-1}}([u_s] \xrightarrow{g|[u_s]} [s-1])} (C^{E_2,2}_{(*,0)}(\mathrm{Epi}_2^{\tilde{t}}), \partial_2).$$

If $f \neq \mathrm{id}$, then there exists $j \in [s]$ such that the restriction of f on $(\sigma \circ f)^{-1}(j) \to \sigma^{-1}(j)$ is different from the identity. With no loss of generality we can assume that $j = s$, hence $\tilde{t}$ is a non-fork tree and the homology of the complex is 0. If $f = \mathrm{id}$, then we deduce from the case $s = 0$ that the complex $(C^{E_2,2}_{(*,0)}(\mathrm{Epi}_2^{\tilde{t}}), \partial_2)$ has only top homology of rank one; consequently when $t \colon [r_2] \longrightarrow [r_2]$ is the fork tree

$$(H_{u_s}(C_{(u,s)}((\mathrm{Epi}_2^t)_\sigma), D_s), D_1 + \ldots + D_{s-1}) \cong (C_{(u_s,s-1)}((\mathrm{Epi}_2^{t_{s-1}})_{\sigma_{s-1}}), \partial_2).$$

We have then an inductive process to compute the homology of the total complex $(C_{(*,s)}(F_\sigma), \partial_2)$. Consequently, for a fixed $\sigma \colon [r_2] \to [s]$

$$\begin{aligned} H_{(*,s)}(F_\sigma, \partial_2) &= 0, && \text{if } r_2 \neq r_1 \\ H_{(*,s)}(F_\sigma, \partial_2) &\cong \begin{cases} 0 & \text{for } * \neq r_2 \\ k & \text{for } s \leqslant * = r_2 \end{cases}, && \text{if } r_2 = r_1. \end{aligned}$$

Since each $\sigma \in \Delta^{\mathrm{epi}}([r_2],[s])$ contributes to one summand in $H_{r_2,s}(F, \partial_2)$, this proves the claim. The computation of the generators for $s > 0$ is given in proposition 4.5.

□

Corollary 4.3. *For any non-fork tree* $t = [r_2] \xrightarrow{f} [r_1], r_2 \neq r_1$, Epi_2^t *is acyclic.*

Proof. This corollary is a direct consequence of the first equation of proposition 4.2. □

Proposition 4.4. *Let* $t \colon [r] \xrightarrow{\mathrm{id}} [r]$ *be a fork tree. The top homology* $H_{(r,0)}(\mathrm{Epi}_2^t, \partial_2)$ *is freely generated by* $c_r := \sum_{\sigma \in \Sigma_{r+1}} \mathrm{sgn}(\sigma)\sigma$.

Proof. The computation of the top homology amounts to determining the kernel of the map

$$\partial_2 \colon k[\mathrm{Epi}_2([r] \xrightarrow{\mathrm{id}} [r]; [r] \longrightarrow [0])] \longrightarrow k[\mathrm{Epi}_2([r] \xrightarrow{\mathrm{id}} [r]; [r-1] \longrightarrow [0])],$$

or equivalently to determine the kernel of the map

$$\partial_2 \colon k[\Sigma_{r+1}] \longrightarrow k[\mathrm{Epi}([r],[r-1])].$$

For $\sigma \in \Sigma_{r+1}$ written by its sequence of preimages $(a_0, \ldots, a_r)$ one has

$$\partial_2(\sigma) = \sum_{i=0}^{r-1} (-1)^i (a_0, \ldots, \{a_i, a_{i+1}\}, \ldots, a_r).$$

From this description, if $x = \sum_{\sigma \in \Sigma_{r+1}} \lambda_\sigma \sigma$ is in the kernel of ∂_2, then for all transpositions $(i, i+1)$ and all σ one has $\lambda_{(i,i+1)\sigma} = -\lambda_\sigma$. Since the transpositions generate the symmetric group one has $\lambda_\sigma = \mathrm{sgn}(\sigma)\lambda_{\mathrm{id}}$ and $x = \lambda_{\mathrm{id}} c_r$. □

For $s > 0$, the computation of the top homology amounts to calculating the kernel of the map

$$\partial_2 \colon \bigoplus_{g \in \Delta^{\mathrm{epi}}([r],[s])} k[\mathrm{Epi}_2([r] \xrightarrow{\mathrm{id}} [r]; [r] \xrightarrow{g} [s])] \longrightarrow \bigoplus_{h \in \Delta^{\mathrm{epi}}([r-1],[s])} k[\mathrm{Epi}_2([r] \xrightarrow{\mathrm{id}} [r]; [r-1] \xrightarrow{h} [s])].$$

We know from proposition 4.2 that it is free of rank equal to the cardinality of $\Delta^{\mathrm{epi}}([r],[s])$. An element g of the latter set is uniquely determined by the sequence $(x_0, \ldots, x_s)$ of the cardinalities of its preimages. Furthermore, any map in $\mathrm{Epi}_2([r] \xrightarrow{\mathrm{id}} [r]; [r] \xrightarrow{g} [s])]$ is given by $g' \colon [r] \to [s]$ in Δ^{epi} and $\tau \colon [r] \to [r]$ in Σ_{r+1} such that $g' = g\tau$. This implies that $g' = g$ and $\tau \in \Sigma_{x_0} \times \ldots \times \Sigma_{x_s}$. In the sequel, we denote such a map by $\tau \in \Sigma_{x_0} \times \ldots \times \Sigma_{x_s}$, suppressing the g'. Let $c_{(x_0,\ldots,x_s)}$ be the element $c_{(x_0,\ldots,x_s)} = (\sum_{\sigma^0 \in \Sigma_{x_0}} \mathrm{sgn}(\sigma^0)\sigma^0, \ldots, \sum_{\sigma^s \in \Sigma_{x_s}} \mathrm{sgn}(\sigma^s)\sigma^s)$ of $\Sigma_{x_0} \times \ldots \times \Sigma_{x_s}$.

Proposition 4.5. *Let $t \colon [r] \xrightarrow{\mathrm{id}} [r]$ be a fork tree. The top homology $H_{(r,s)}(\mathrm{Epi}_2^t, \partial_2)$ is freely generated by the elements $c_{(x_0,\ldots,x_s)} = (\sum_{\sigma^0 \in \Sigma_{x_0}} \mathrm{sgn}(\sigma^0)\sigma^0, \ldots, \sum_{\sigma^s \in \Sigma_{x_s}} \mathrm{sgn}(\sigma^s)\sigma^s)$, for $(x_0, \ldots, x_s) \in \Delta^{\mathrm{epi}}([r],[s])$.*

Proof. Similar to the proof of proposition 4.4 we compute the kernel of ∂_2 which decomposes into the sum of commuting differentials $\partial_2 = D_0 + \ldots + D_s$, as in the proof of proposition 4.2. As a consequence $\ker(\partial_2) = \cap_i \ker(D_i)$ which gives the result. □

Corollary 4.6. *For any fork tree $t = [r] \xrightarrow{\mathrm{id}} [r]$, Epi_2^t is acyclic.*

Proof. It remains to compute the homology of the complex $((H_{(r,*)}(C^{E_2,2}(\mathrm{Epi}_2^t), \partial_2), \partial_1)$ and prove that it vanishes for all $*$ if $r > 0$. Propositions 4.4 and 4.5 give its k-module structure:

$$H_{(r,s)}(C^{E_2,2}(\mathrm{Epi}_2^t), \partial_2) = \bigoplus_{(x_0,\ldots,x_s) \in \Delta^{\mathrm{epi}}([r],[s])} k c_{(x_0,\ldots,x_s)}.$$

To compute $\partial_1(c_{(x_0,\ldots,x_s)})$ it is enough to compute $\partial_1(\mathrm{id}_{\Sigma_0 \times \ldots \times \Sigma_s})$ in $C^{E_2}_{(r,s-1)}(\mathrm{Epi}_2^t)$. We apply relations (3.5) and (3.6):

$$\partial_1 \left(\begin{array}{ccc} [r] & \xrightarrow{\mathrm{id}} & [r] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle (x_0,\ldots,x_s)} \\ [r] & \xrightarrow{(x_0,\ldots,x_s)} & [s] \end{array} \right) = \sum_{i=0}^{s-1} (-1)^i \left(\begin{array}{ccc} [r] & \xrightarrow{\mathrm{id}} & [r] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle d_i(x_0,\ldots,x_s)} \\ [r] & \xrightarrow{d_i(x_0,\ldots,x_s)} & [s-1] \end{array} - \begin{array}{ccc} [r] & \xrightarrow{\mathrm{id}} & [r] \\ \downarrow{\scriptstyle (i,i+1)} & & \downarrow{\scriptstyle d_i(x_0,\ldots,x_s)} \\ [r] & \xrightarrow{d_i(x_0,\ldots,x_s)} & [s-1] \end{array} \right)$$

Consequently $\partial_1(c_{(x_0,\ldots,x_s)}) = \sum_{i=0}^{s-1} (-1)^i c_{(x_0,\ldots,x_i+x_{i+1},\ldots,x_s)}$ and the complex $((H_{(r,*)}(C^{E_2,2}(\mathrm{Epi}_2^t), \partial_2), \partial_1)$ agrees with the complex $C_*^{\mathrm{bar}}((\Delta^{\mathrm{epi}})^r)$ of definition 2.2. Proposition 2.3 states that it is acyclic, and that

$$H_0(C_*^{\mathrm{bar}}((\Delta^{\mathrm{epi}})^r) = \begin{cases} 0 & \text{if } r > 0 \\ k & \text{if } r = 0. \end{cases}$$

As a consequence the spectral sequence associated to the bicomplex $(C^{E_2}_{(*,*)}, \partial_1 + \partial_2)$ collapses at the E^2-stage and one gets $H_p^{E_2}(\mathrm{Epi}_2^t) = 0$ for all $p > 0$. □

Proposition 4.7. *Let $t = [r_n] \xrightarrow{f_n} [r_{n-1}] \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} [r_1]$ be an n-level tree and let $\bar{t}$ be its $(n-1)$-truncation $[r_{n-1}] \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} [r_1]$, then*

$$\begin{array}{ll} H_{(*,s_{n-1},\ldots,s_1)}(\mathrm{Epi}_n^t, \partial_n) = 0, & \text{if } r_n \neq r_{n-1}, \\ H_{(*,s_{n-1},\ldots,s_1)}(\mathrm{Epi}_n^t, \partial_n) \cong \begin{cases} 0 & \text{for } * \neq r_n \\ C^{E_{n-1}}_{(s_{n-1},\ldots,s_1)}(\mathrm{Epi}_{n-1}^{\bar{t}}) & \text{for } s_{n-1} \leqslant * = r_n \end{cases}, & \text{if } r_n = r_{n-1}. \end{array}$$

Furthermore the $(n-1)$-complex structure induced on $H_{(r_n,s_{n-1},\ldots,s_1)}(\mathrm{Epi}_n^t, \partial_n)$ by the n-complex $C^{E_n}_{(*,\ldots,*)}(\mathrm{Epi}_n^t)$ coincides with the one on $C^{E_{n-1}}_{(s_{n-1},\ldots,s_1)}(\mathrm{Epi}_{n-1}^{\bar{t}})$.

Proof. Recall from definition 3.6 that

$$\partial_n \left(\begin{array}{ccccc} [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} & [r_1] \\ \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_{n-1}} & & \downarrow{\scriptstyle \sigma_1} \\ [s_n] & \xrightarrow{g_n} & [s_{n-1}] & \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_2} & [s_1] \end{array} \right) = \sum_{i, g_n(i)=g_n(i+1)} (-1)^i \begin{array}{ccccc} [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} & [r_1] \\ \downarrow{\scriptstyle d_i\sigma_n} & & \downarrow{\scriptstyle \sigma_{n-1}} & & \downarrow{\scriptstyle \sigma_1} \\ [s_n-1] & \xrightarrow{g_n|_{i=i+1}} & [s_{n-1}] & \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_2} & [s_1]. \end{array}$$

The same proof as in proposition 4.2 provides the computation of the homology of the complex with respect to the differential ∂_n: if t is not a fork tree, then the homology of the complex vanishes, and if t is the fork tree $f_n = \mathrm{id}_{[r_{n-1}]}$, then its homology groups are concentrated in top degree r_n. Let us describe all the bijections τ of $[r_{n-1}]$ such that the following diagram commutes

$$\begin{array}{ccccc} [r_{n-1}] & \xrightarrow{\mathrm{id}} & [r_{n-1}] & \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} & [r_1] \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \sigma_{n-1}} & & \downarrow{\scriptstyle \sigma_1} \\ [r_{n-1}] & \xrightarrow{g_n} & [s_{n-1}] & \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_2} & [s_1] \end{array} .$$

Let $(x_0, \ldots, x_{s_{n-1}})$ be the sequence of cardinalities of the preimages of σ_{n-1}, which determines also g_n. There exists a bijection of $[r_{n-1}]$ such that $\sigma_{n-1} = g_n \xi$. If ξ, ξ' are bijections of $[r_{n-1}]$ both satisfying the previous equality then $\xi(\xi')^{-1} \in \Sigma_{x_0} \times \ldots \times \Sigma_{x_{s_{n-1}}}$. Any element τ that makes the diagram commute is of the form $\alpha\xi$ for $\alpha \in \Sigma_{x_0} \times \ldots \times \Sigma_{x_{s_{n-1}}}$. As in proposition 4.5, the element $\mathrm{sgn}(\xi) c_{(x_0,\ldots,x_{s_{n-1}})}\xi$ does not depend on the choice of ξ and it is a generator of $H_{(r_n, s_{n-1}, \ldots, s_1)}(\mathrm{Epi}_n^t, \partial_n)$. This gives the desired isomorphism of k-modules between this homology group and $C^{E_{n-1}}_{(s_{n-1},\ldots,s_1)}(\mathrm{Epi}^{\bar{t}}_{n-1})$. It is clear from lemma 3.4 that the induced differential ∂_i coincides with the one on $C^{E_{n-1},i}_{(s_{n-1},\ldots,s_1)}(\mathrm{Epi}^{\bar{t}}_{n-1})$ for $1 \leqslant i \leqslant n-1$. For $i = n-1$ the computation has been done in corollary 4.6. □

5. Appendix: Higher Hochschild homology and the homology of the iterated bar construction

In the following, let k be a field and let A be an augmented commutative k-algebra with augmentation $\varepsilon\colon A \to k$. The aim of this part is the comparison of E_n-homology, $H^{E_n}_*$, with higher order Hochschild homology, $HH^{[n]}_*$ in the sense of Pirashvili [9]. Our comparison works via the bar construction of augmented commutative algebras, using Eilenberg-MacLane's treatment of bar constructions. This results seems to be well-known to experts: among others, a related identification is contained in [3, corollary 3.17], and Benoit Fresse was aware of this fact as well.

There are many variants of a bar construction for differential graded augmented commutative k-algebras. As a reference we follow [5, chapter II] adapted to the case of differential graded augmented commutative algebras over a field, so $B(-)$ is a bar construction that satisfies the following properties:

- If $\varphi\colon A^1_* \to A^2_*$ is a morphism of differential graded augmented commutative k-algebras that induces an isomorphism on homology, then
$$H_*(\varphi)\colon H_*(B(A^1_*)) \cong H_*(B(A^2_*)).$$
- If π is an abelian group, then $H_*(B(k[\pi])) \cong H_*(K(\pi,1);k)$. Here, $k[\pi]$ is the group algebra of π over k viewed as a differential graded augmented commutative algebra concentrated in degree zero, and $K(\pi,1)$ is the Eilenberg-MacLane space of type $(\pi,1)$.
- For every differential graded augmented commutative algebra A_*, $B(A_*)$ is again a differential graded augmented commutative algebra.

The complex $B(A_*)$ has Hochschild homology of A_* with coefficients in k as its homology. One can iterate the bar construction and the homology of the n-fold iteration of the bar construction applied to $k[\pi]$, $H_*(B^n(k[\pi]))$, is isomorphic to the k-homology of $K(\pi,n)$.

Let Γ denote the skeleton of the category of finite pointed sets and basepoint preserving maps. The pointed sets $[n] = \{0,\ldots,n\}$ are the objects of Γ for $n \geqslant 0$. For a given augmented commutative k-algebra A we denote by $\mathcal{L}(A;k)$ the functor from the category Γ to the category of k-vector spaces that sends $[n]$

to $A^{\otimes n} \cong k \otimes_k A^{\otimes n}$ where we view k as an A-bimodule via the augmentation. A map of finite pointed set $f\colon [n] \to [m]$ sends $a_0a_1 \otimes \ldots \otimes a_n \cong a_0 \otimes a_1 \otimes \ldots \otimes a_n$ to $b_0 \otimes b_1 \otimes \ldots \otimes b_m$ with $b_i = \prod_{f(j)=i} a_j$ and $b_0 = a_0 \prod_{f(j)=0} \varepsilon(a_j)$. We call this functor the *Loday functor of* A. One can evaluate any functor from Γ to vector spaces on a pointed simplicial set [9, 2.1], hence to any Γ-module, F, and any pointed simplicial set, X, there is an associated simplicial k-vector space, $F(X)$. Pirashvili defines the n-th order Hochschild homology of A with coefficients in k as the homotopy groups of the simplicial k-vector space $\mathcal{L}(A;k)(\mathbb{S}^n)$

$$HH_*^{[n]}(A;k) = \pi_*\mathcal{L}(A;k)(\mathbb{S}^n)$$

for an arbitrary simplicial model of the n-sphere, $\mathbb{S}^n$.

We can now state our comparison result.

Theorem 5.1. *The n-th iterated Hochschild homology of A with coefficients in a field k is isomorphic to the homology of the n-fold iterated bar construction of A.*

We need an auxiliary result in order to prove the theorem.

Lemma 5.2. *If $0 \notin S \subset A$ is a multiplicative subset, then*

$$H_*(B^n(A)) \cong H_*(B^n(A[S^{-1}])).$$

Proof. As the bar construction is invariant unter quasi-isomorphisms of differential graded augmented commutative algebras, we can use that Hochschild homology with coefficients in k is invariant under localizations [8, 1.1.17]. Therefore $H_*(B(A)) \cong H_*(B(A[S^{-1}]))$. The n-fold iterated case then follows by induction. □

Proof of Theorem 5.1. We first show the claim for polynomial algebras.

From Lemma 5.2 we know that the polynomial algebra on one generator $k[x]$ and the Laurent polynomial algebra $k[x^{\pm 1}]$ have isomorphic homology groups when plugged into the n-th iterated bar construction. As $k[x^{\pm 1}] \cong k[\mathbb{Z}]$, we obtain that

$$H_*(B^n(k[x^{\pm 1}])) \cong H_*(K(\mathbb{Z}, n); k).$$

Here, we view $k[x]$ and $k[x^{\pm 1}]$ as augmented commutative k-algebras via the augmentation ε_1 that sends x^i to 1 for all $i \in \mathbb{Z}$.

The Loday functor for a polynomial algebra with coefficients in k evaluated on a simplicial model of the n-sphere is the symmetric algebra functor evaluated on the n-sphere and thus we obtain

$$HH_*^{[n]}(k[x];k) = H_*(\mathcal{L}(k[x];k)(\mathbb{S}^n)) \cong H_*(\mathrm{Sym} \circ L(\mathbb{S}^n)) \cong H_*(SP(\mathbb{S}^n);k).$$

Here, SP stands for the infinite symmetric product and L is the Γ-module that sends $[n]$ to the free k-module generated by the set $\{1, \ldots, n\}$. Note that in this case $k[x]$ is augmented over k via the augmentation ε_0 that sends x to zero. The augmentation affects the $k[x]$-module structure of k, but in [12, 4.1] it is shown that the resulting homotopy groups are independent of the module structure.

Evaluated on an n-sphere, the functor SP yields an Eilenberg-MacLane space of type $(\mathbb{Z}, n)$ and hence the above is isomorphic to $H_*(K(\mathbb{Z}, n); k)$. Thus the two homology theories agree for $A = k[x]$.

We now deduce that the two theories are isomorphic on a polynomial algebra on two variables. Consider the Γ-module $\mathcal{L}(k[x,y];k))$. To a finite pointed set $[n] = \{0, 1, \ldots, n\}$ with basepoint 0 it associates $k \otimes k[x,y]^{\otimes n} \cong k[x]^{\otimes n} \otimes k[y]^{\otimes n} \cong \mathcal{L}(k[x];k))[n] \otimes \mathcal{L}(k[y];k))[n]$. A morphism of finite pointed sets $f\colon [n] \to [m]$ sends $\lambda \otimes a_1 \otimes \ldots \otimes a_n$ (with $\lambda \in k$ and a_i in $k[x,y]$) to $\mu \otimes b_1 \otimes \ldots \otimes b_m$ where $b_i = \prod_{f(j)=i} a_j$ and $\mu = \lambda \cdot \prod_{f(j)=0, j\neq 0} \varepsilon(a_j)$. Therefore the above isomorphism of $\mathcal{L}(k[x];k))[n] \otimes \mathcal{L}(k[y];k))[n]$ and $\mathcal{L}(k[x,y];k))[n]$ induces an isomorphism of Γ-modules between $\mathcal{L}(k[x,y];k))(\mathbb{S}^n)$ and the pointwise tensor product $\mathcal{L}(k[x];k))(\mathbb{S}^n) \otimes \mathcal{L}(k[y];k))(\mathbb{S}^n)$. Furthermore, we get

$$\begin{aligned}
\pi_*(\mathrm{Sym} \circ L(\mathbb{S}^n) \otimes \mathrm{Sym} \circ L(\mathbb{S}^n)) &\cong \pi_*(\mathrm{Sym} \circ (L(\mathbb{S}^n) \oplus L(\mathbb{S}^n)) \\
&\cong \pi_*(\mathrm{Sym} \circ (L(\mathbb{S}^n \vee \mathbb{S}^n)) \cong H_*(SP(\mathbb{S}^n \vee \mathbb{S}^n);k) \\
&\cong H_*(K(\mathbb{Z} \times \mathbb{Z}, n); k).
\end{aligned}$$

For the iterated bar construction we obtain that

$$H_*(B^n(k[x,y])) \cong H_*(B^n(k[x^{\pm 1}, y^{\pm 1}])) \cong H_*(B^n(k[\mathbb{Z} \times \mathbb{Z}])) \cong H_*(K(\mathbb{Z} \times \mathbb{Z}, n); k).$$

This shows the claim for $k[x, y]$ and using induction and colimit arguments we obtain that n-th order Hochschild homology is isomorphic to the homology of the n-fold iterated bar construction for arbitrary polynomial algebras $A = k[x_i; i \in I]$.

If A is an arbitrary augmented commutative k-algebra we take a simplicial resolution of A by polynomial algebras, $P_\bullet \xrightarrow{\sim} A$. A hyperhomology spectral sequence argument then finishes the proof: we get that the E^1-terms are isomorphic and the differential on E^1 commutes with the isomorphism and so do all the higher differentials. Hence we obtain isomorphic E^∞-terms and as we work over a field this suffices to obtain an isomorphism of the corresponding homology groups. □

There is a correspondence between augmented commutative k-algebras and non-unital k-algebras that sends an augmented k-algebra A to its augmentation ideal $\bar{A}$. Under this correspondence, the $(m+n)$-th homology group of the n-fold bar construction $B^n(A)$ is isomorphic to the m-th homology group of the n-fold iterated reduced bar construction of $\bar{A}$, $B^n(\bar{A})$. Therefore we obtain the following consequence of Theorem 5.1.

Corollary 5.3. *Hochschild homology of order n of a augmented commutative k-algebra A is isomorphic to the n-fold shift of the E_n-homology of $\bar{A}$*

$$HH^{[n]}_{*+n}(A; k) \cong H^{E_n}_*(\bar{A}).$$

Suspension induces maps

$$\begin{array}{ccccc} HH_\ell(A;k) = \pi_\ell \mathcal{L}(A;k)(\mathbb{S}^1) & \longrightarrow & HH^{[2]}_{\ell+1}(A;k) = \pi_{\ell+1}\mathcal{L}(A;k)(\mathbb{S}^2) & \longrightarrow & \cdots \\ & \searrow & \downarrow & & \\ & & H\Gamma_{\ell-1}(A;k) \cong \pi^s_\ell(\mathcal{L}(A;k)). & & \end{array}$$

For the last isomorphism see [10]. Fresse proves a comparison [6, 8.6] between Gamma homology of A and E_∞-homology of $\bar{A}$. Using the isomorphisms above this sequence gives rise to a sequence of maps involving graded vector spaces that are isomorphic to the ones in (3.8). We conjecture that we actually have an isomorphism of sequences, *i.e.*, that the suspension maps $HH^{[n]}_{\ell+n}(A;k) \to HH^{[n+1]}_{\ell+n+1}(A;k)$ are related to the natural maps $H^{E_n}_\ell(\bar{A}) \to H^{E_{n+1}}_\ell(\bar{A})$ via the isomorphisms from corollary 5.3.

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