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TFT CONSTRUCTION OF RCFT CORRELATORS III: SIMPLE CURRENTS

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Abstract

We use simple currents to construct symmetric special Frobenius algebras in modular tensor categories. We classify such simple current type algebras with the help of abelian group cohomology. We show that they lead to the modular invariant torus partition functions that have been studied by Kreuzer and Schellekens. We also classify boundary conditions in the associated conformal field theories and show that the boundary states are given by the formula proposed in hep-th/0007174. Finally, we investigate conformal defects in these theories.

Contents

1 Introduction

In a series of papers [1, I, II], we are developing a description of correlation functions of rational conformal field theory that is based on the combination of algebra and representation theory in modular tensor categories with topological field theory in three dimensions. A central ingredient in our construction are *symmetric special Frobenius algebras* in these tensor categories.

The main purpose of the present paper is to discuss a specific class of examples, namely those based on algebras built from invertible objects or, in CFT terminology, *simple currents* [2,3,4]. The general theory is developed further only to the extent that is necessary to understand the special features of such simple current algebras. While one of the important virtues of Frobenius algebras is that they allow for a unified treatment of exceptional modular invariants and simple current modular invariants, simple currents are predominant in applications. One reason is the fact that mutually local simple currents with trivial twist (i.e., integral conformal weight) can be used in particular to implement various projections.

For a discussion of the interplay between simple currents and projections we refer to [5,6] in the context of string theory, and to [7] in the context of universality classes of quantum Hall fluids. Simple currents with non-trivial twist, on the other hand, play an important role in the description of those symmetries that cannot be incorporated in a (bosonic) chiral algebra, such as supersymmetries or parafermionic symmetries. In string theory, they can also be used to implement mirror symmetry for Gepner models [8]. The methods of [1, I, II] allow in particular for the calculation of the structure constants of operator product expansions. For certain classes of conformal field theories with torus partition function of simple current type structure constants for bulk fields had been considered before e.g. in [9, 10, 11, 12, 13]. Structure constants for boundary fields in such theories were first studied, for the case of Virasoro minimal models, in [14]. Our results provide a rigorous basis for the tools used in these applications.

Another important role of simple currents is their use in the construction of different Klein bottle amplitudes. Actually we expect them to provide particularly strong relations in the case of Azumaya algebras (corresponding to pure automorphism modular invariants). An analysis of this aspect of simple currents requires, however, further concepts from the general theory and hence will be presented elsewhere.

There is also a more theoretical motivation: (isomorphism classes of) simple currents span a subring of the fusion ring, and this subring is isomorphic to the group ring of a finite abelian group. The full subcategory whose objects are (direct sums of) simple currents can be seen as a *categorification* of this group ring. Quite generally, finding a categorification of an algebraic object should be a problem of cohomological nature. In the case of simple currents, this can be made precise: group cohomology controls categorifications while abelian group cohomology [15] controls braided categorifications. Any general theory of categorifications should be a generalization of these cohomology theories; the fact that the categorification of simple currents leads to good cohomology theories provides a partial mathematical explanation of the computational power of simple currents which makes them so useful in many applications.

This paper is organized as follows. In Section 2 we discuss braided categories whose objects are all invertible. This is the category-theoretic analogue of the study of line bundles over, say, a manifold; accordingly we refer to the resulting theory as *braided Picard theory*. Section 3 is devoted to the study of haploid symmetric special Frobenius algebras all of whose simple subobjects are invertible. We show that they give rise precisely to the class of modular invariant partition functions studied by Kreuzer and Schellekens [16]. Section 4 deals with the representation theory of such algebras. Modules correspond to boundary conditions; therefore our formalism allows us in particular to give a rigorous proof of the formulae of [17] for boundary states in conformal field theories with simple current modular invariant. In section 5 we develop the theory of bimodules of these algebras; bimodules describe conformal defects in the corresponding conformal field theories.

In our notation we follow [I] and [II] whenever possible; in particular we use the notation for the duality, braiding and twist morphisms introduced there, see e.g. the list (I:2.8). We refer to formulas, theorems etc. from [I] and [II] by the numbering used there, preceded by the symbol 'I:' and 'II:', respectively.

2 Braided Picard theory

Let us start with some basic definitions.

Definition 2.1:
(i) An object V of a tensor category $\mathcal{C}$ is called *invertible*, or a *simple current*,[1] iff there exists an object V' such that $V \otimes V'$ is isomorphic to the tensor unit $\mathbf{1}$.

(ii) A tensor category is called *pointed* [18] iff every simple object is invertible.

(iii) A *theta-category* [19] is a braided pointed tensor category.

(iv) The *Picard category*[2] $\mathcal{P}ic(\mathcal{C})$ of tensor category $\mathcal{C}$ is the full tensor subcategory of $\mathcal{C}$ whose objects are direct sums of invertible objects of $\mathcal{C}$.

(v) A simple object U of $\mathcal{P}ic(\mathcal{C})$ is said to be of finite order iff some tensor power of U is isomorphic to the tensor unit. The *order* of such an object U is the smallest positive integer N_U such that $U^{\otimes N_U} \cong \mathbf{1}$.

Note that in these definitions it is not assumed that the category also has a duality and a twist or, in case it does, whether they (together with the braiding) make the category into a sovereign category. However, in the present context of rational CFT, all pointed and theta-categories arise as full subcategories of sovereign (and in fact modular) categories and are therefore sovereign themselves. Indeed, for the purposes of this paper we consider only categories with the following additional properties:

[1] This term was introduced in the physics literature [2], while the qualification 'invertible' is standard in the mathematics literature.

[2] The term 'Picard category' is also used for other, sometimes related, mathematical objects in algebra, stable homotopy theory and category theory. Since there is no danger of confusing those with the present use of the term, we refrain from commenting on the relation to its various other uses.

Convention 2.2:
All categories in this paper are assumed to be small abelian $\mathbb{C}$-linear semisimple sovereign tensor categories, and to possess the following additional properties:
- The morphism spaces are finite-dimensional complex vector spaces.
- The tensor unit $\mathbf{1}$ is simple.
- The dimension of any object is a non-negative real number.
- There are only finitely many isomorphism classes of simple objects.

Moreover, with the exception of the concrete categories $\mathcal{C}(G,\psi)$ and $\mathcal{C}(G,\psi,\Omega)$ (to be introduced in lemma 2.8 below), these categories are assumed to be *strict* tensor categories.

Also recall that braided sovereign tensor categories are also known as *ribbon categories*. For the applications to conformal field theory we have in mind, we are interested in (subcategories of) modular tensor categories.

When taking into account the convention 2.2, the following observations are straightforward:

Remark 2.3:
(i) Invertible objects are simple.
(ii) For any invertible object V the relations

$$b_V \circ \tilde{d}_V = id_{V\otimes V^\vee} \qquad \text{and} \qquad \tilde{b}_V \circ d_V = id_{V^\vee\otimes V} \tag{2.1}$$

for the left and right duality morphisms (as defined in (I:2.8) and (I:2.12)) are valid.
(iii) Existence of an object V' such that $V\otimes V'\cong\mathbf{1}$ is equivalent to the existence of an object V'' such that $V''\otimes V\cong\mathbf{1}$. (In short, 'right-invertibility' is equivalent to 'left-invertibility'.) Indeed, both the object V' in definition 2.1(i) and the object V'' in question can be taken to be the dual object $V^\vee$, since one has $V\otimes V^\vee\cong\mathbf{1}\cong V^\vee\otimes V$.
(iv) It follows in particular that an object V is invertible iff the dual object $V^\vee$ is invertible. Because of $V\otimes V^\vee\cong\mathbf{1}$, the dimension of an invertible object is thus an invertible number. And because of $\dim(V)=\dim(V^\vee)$, it must be ±1 and hence, by the positivity assumption on dimensions, $\dim(V)=1$. Indeed, simple currents can alternatively be characterized as simple objects of dimension 1. Such objects have been classified [20] for certain classes of modular tensor categories, like the ones based on an untwisted affine Lie algebra at integral level (compare also [21, 22]).
(v) If $\mathcal{C}$ is sovereign, then $\mathcal{P}ic(\mathcal{C})$ is sovereign, too, and it is a pointed category.
(vi) If $\mathcal{C}$ is braided, then $\mathcal{P}ic(\mathcal{C})$ is braided as well, and hence it is a theta-category.
(vii) Even when $\mathcal{C}$ is modular, the Picard category $\mathcal{P}ic(\mathcal{C})$ is, in general, *not* modular.

The following result is a direct consequence of the definitions:

Proposition 2.4:
(i) The Grothendieck ring of a pointed category $\mathcal{D}$ is isomorphic to the group ring of a finite group G, $K_0(\mathcal{C})\cong\mathbb{Z}G$.

(ii) The fusion ring of a theta-category is isomorphic to the group ring of a finite abelian group.

Definition 2.5:
Let $\mathcal{C}$ be a tensor category. The *Picard group* of $\mathcal{C}$, denoted by $\mathrm{Pic}(\mathcal{C})$, is the group of all isomorphism classes of invertible objects in $\mathcal{C}$, with the product being the one of the Grothendieck ring $K_0(\mathcal{C}) \supset \mathrm{Pic}(\mathcal{C})$.

Remark 2.6:
(i) $\mathrm{Pic}(\mathcal{C})$ is a finite group, and we have $K_0(\mathcal{P}\!ic(\mathcal{C})) \cong \mathbb{Z}\,\mathrm{Pic}(\mathcal{C})$ as a ring over $\mathbb{Z}$. When the category $\mathcal{C}$ is braided, then the Picard group $\mathrm{Pic}(\mathcal{C})$ is a finite abelian group.
(ii) In the physics literature, the Picard group $\mathrm{Pic}(\mathcal{C})$ of a *braided* tensor category is also called the *simple current group* or *center* of $\mathcal{C}$. This notion of center should not be confused with the Drinfeld center (see e.g. chapter XIII.4 of [23]) of the category $\mathcal{C}$. Also note that the latter is a category of global dimension $\mathrm{Dim}(\mathcal{C})^2$ [24], while the global dimension of the Picard category $\mathcal{P}\!ic(\mathcal{C})$ is bounded by $\mathrm{Dim}(\mathcal{C})$.
(The global dimension of $\mathcal{C}$ is defined as the number $\mathrm{Dim}(\mathcal{C}) := \big(\sum_{i\in\mathcal{I}} \dim(U_i)^2\big)^{1/2}$, with U_i, $i \in \mathcal{I}$ a set of representatives for the isomorphism classes of simple objects of $\mathcal{C}$.)
(iii) The same notion of Picard group is used for instance in stable homotopy theory, see e.g. [25]. In the literature there is also a different notion of Picard group, namely the tensor subcategory consisting of all invertible objects, which is in particular a *categorical group*, or *cat-group* (see e.g. [26, 27, 28]).

To obtain examples of pointed categories and theta-categories, we select a finite group G, which for theta-categories will be required to be abelian. Consider the category of finite-dimensional G-graded complex vector spaces with the grade-respecting linear maps as morphisms. This is an abelian semisimple category whose isomorphism classes of simple objects L_g are in bijection to the elements g of G. A general object V can be written as a direct sum $\bigoplus_{g\in G} V_g$ of finite-dimensional vector spaces. Every three-cocycle $\psi \in Z^3(G, \mathbb{C}^\times)$ defines an associativity constraint by

$$\begin{array}{rrcl} \alpha_{V,V',V''}: & \big(V_{g_1} \otimes V'_{g_2}\big) \otimes V''_{g_3} & \to & V_{g_1} \otimes \big(V'_{g_2} \otimes V''_{g_3}\big) \\ & (v \otimes v') \otimes v'' & \mapsto & \psi(g_1, g_2, g_3)^{-1}\, v \otimes (v' \otimes v'') \,. \end{array} \tag{2.2}$$

To obtain a theta-category, we endow this category with a braiding; this is possible only if G is abelian. It turns out ([29,30], see also [19,31]) that a representative (ψ, Ω) of the third abelian group cohomology (as defined in [15] and summarized in appendix A.2) contains exactly the relevant data; we can define the braiding by

$$\begin{array}{rrcl} c_{V,V'}: & V_{g_1} \otimes V'_{g_2} & \to & V'_{g_2} \otimes V_{g_1} \\ & v \otimes v' & \mapsto & \Omega(g_2, g_1)^{-1}\, v' \otimes v \,. \end{array} \tag{2.3}$$

For a discussion of equivalences between such categories we introduce the following notion.

Definition 2.7 :
Let R be a ring. A pair consisting of a sovereign tensor category $\mathcal{C}$ and an isomorphism $f\colon R \to K_0(\mathcal{C})$ is called an *R-marked* tensor category. If the ring is evident, we simply call the category *marked*.

In a marked category one has a distinguished correspondence between elements of the ring R and isomorphism classes of simple objects in $\mathcal{C}$. In the example of G-graded vector spaces treated above, we naturally get a marked category by taking R to be the group ring $\mathbb{Z}G$ and choosing $f(g) = [V_g]$.

Lemma 2.8 :
(i) Let G be a finite group. The associativity constraint α given in (2.2) defines the structure of a pointed tensor category $\mathcal{C}(G, \psi)$.
(ii) The categories $\mathcal{C}(G, \psi)$ and $\mathcal{C}(G, \psi')$ are equivalent as $\mathbb{Z}G$-marked tensor categories if and only if the three-cocycles ψ and ψ' are cohomologous.
(iii) Let G be a finite abelian group. The morphisms given in (2.2) and (2.3) define the structure of a theta-category $\mathcal{C}(G, \psi, \Omega)$.
(iv) The categories $\mathcal{C}(G, \psi, \Omega)$ and $\mathcal{C}(G, \psi', \Omega')$ are equivalent as $\mathbb{Z}G$-marked ribbon categories if and only if the abelian three-cocycles (ψ, Ω) and (ψ', Ω') are cohomologous.

Proof:
(i) The associativity constraint α obeys the pentagon condition, because ψ is closed.
(iii) The morphisms in (2.3) indeed furnish a braiding; in particular they satisfy the relevant compatibility conditions with the associator (2.2), i.e. the two hexagon diagrams.
(ii) and (iv) Since we are only concerned with functorial isomorphism between $\mathcal{C}$ and $\mathcal{C}'$ as marked tensor (respectively, ribbon) categories, we can assume that the relevant functor acts as the identity on objects. The statements then follow directly by comparison with the relevant notions from group cohomology and abelian group cohomology, respectively. (These notions are summarized in appendix A.) ✓

We now show that these examples already exhaust the class of pointed categories and theta-categories, respectively. To this end we need, besides the result given in proposition 2.4, the notion (see e.g. theorem 2.3 in [32]) of categorification. To introduce this, we first recall that a *based* ring R over the integers $\mathbb{Z}$ is a unital ring over $\mathbb{Z}$ together with a basis of R in which the structure constants are non-negative; we can then state

Definition 2.9 :
Let R be a based ring.
(i) A *categorification* of R is a sovereign tensor category $\mathcal{C}$ such that $K_0(\mathcal{C}) \cong R$.
(ii) A *braided categorification* of R is a ribbon category $\mathcal{C}$ such that $K_0(\mathcal{C}) \cong R$.

Remark 2.10:
(i) In the definition, the category $\mathcal{C}$ is required to be sovereign. This property guarantees that the tensor product functor $\otimes$ is exact and thus induces a product on the Grothendieck group $K_0(\mathcal{C})$.
(ii) An additional necessary condition for a ring R to have a braided categorification is that R is abelian.
(iii) The tools used in much of the physics literature on conformal field theory are essentially (fusion) rings and the twist (appearing as the exponentiated conformal weight). Other aspects of categorification are usually not taken into account.

Proposition 2.11:
(i) Let G be a finite group. For every categorification $\mathcal{C}$ of the group ring $\mathbb{Z}G$ there exists some $\psi\in Z^3(G,\mathbb{C}^\times)$ such that $\mathcal{C}$ is equivalent to $\mathcal{C}(G,\psi)$ as a sovereign tensor category.
(ii) Let G be a finite abelian group. For every braided categorification $\mathcal{C}$ of $\mathbb{Z}G$ there is some abelian three-cocycle (ψ,Ω) such that $\mathcal{C}$ is equivalent to $\mathcal{C}(G,\psi,\Omega)$ as a theta-category.
(iii) The equivalence classes of marked categorifications of $\mathbb{Z}G$ are in bijection with the group $H^3(G,\mathbb{C}^\times)$, and the equivalence classes of braided categorifications of $\mathbb{Z}G$ are in bijection with $H^3_{\rm ab}(G,\mathbb{C}^\times)$.
(iv) These categorifications exhaust the class of pointed categories (for an arbitrary finite group G) and of theta-categories (for G finite abelian), respectively.

Proof:
(i) The proof can essentially be found in appendix E of [33], compare also [34, 35]. The isomorphism classes of simple objects are labeled by group elements $g\in G$. We fix representatives L_g; this will always be done in such a way that the representative L_e of the unit element is the tensor unit $\mathbf{1}$. The morphism spaces $\mathrm{Hom}(L_{g_1}\otimes L_{g_2},L_{g_1g_2})$ are one-dimensional. We select a basis

$$ {}_{g_1}b_{g_2} \in \mathrm{Hom}(L_{g_1}\otimes L_{g_2},L_{g_1g_2})\,. \tag{2.4} $$

(There is no distinguished basis; group cohomology is the appropriate tool for formulating basis independent statements. Our conventions for basis choices are collected in appendix C.)
The associator is described by the family of isomorphisms

$$ \begin{array}{rrcl} \phi_{g_1,g_2,g_3}: & \mathrm{Hom}(L_{g_1}\otimes(L_{g_2}\otimes L_{g_3}),L_{g_1g_2g_3}) & \rightarrow & \mathrm{Hom}((L_{g_1}\otimes L_{g_2})\otimes L_{g_3}),L_{g_1g_2g_3}) \\ & f & \mapsto & f\circ\alpha_{L_{g_1},L_{g_2},L_{g_3}} \end{array} \tag{2.5} $$

Since the two morphism spaces are one-dimensional, the isomorphism ϕ_{g_1,g_2,g_3} is described, in the chosen basis, by a non-zero number $\psi(g_1,g_2,g_3)$:

$$ \phi\big({}_{g_1}b_{g_2g_3}\circ(id_{L_{g_1}}\otimes {}_{g_2}b_{g_3})\big) = \psi(g_1,g_2,g_3)^{-1}\,{}_{g_1g_2}b_{g_3}\circ({}_{g_1}b_{g_2}\otimes id_{L_{g_3}})\,. \tag{2.6} $$

The pentagon axiom is easily seen to be equivalent to the assertion that ψ is a three-cocycle. Different choices of basis lead to cohomologous three-cocycles. We conclude that the pointed category is equivalent to $\mathcal{C}(G,\psi)$.

(ii) These statements can be found in [31]. For the case of theta-categories, we use the fact that the braiding is described by the isomorphism that it induces on morphism spaces,

$$\begin{array}{rcl} \mathrm{Hom}(L_{g_1}\otimes L_{g_2}, L_{g_1g_2}) & \to & \mathrm{Hom}(L_{g_2}\otimes L_{g_1}, L_{g_1g_2}) \\ \varphi & \mapsto & \varphi\circ c_{L_{g_2},L_{g_1}} \, . \end{array} \tag{2.7}$$

The action of this isomorphism on a basis element is described by the two-cochain Ω on G with values in $\mathbb{C}^\times$:

$${}_{g_1}b_{g_2}\circ c_{L_{g_2},L_{g_1}} = \Omega(g_1,g_2)^{-1}\,{}_{g_2}b_{g_1} \, . \tag{2.8}$$

It is straightforward to check that the pentagon axiom and the two hexagon axioms imply the two constraints (A.8), so that (ψ,Ω) is an abelian three-cocycle. Cohomologous three-cocycles give rise to equivalent theta-categories.

(iii) The statements follow from (i) and (ii), respectively, together with lemma 2.8.

(iv) The statements follow from combining (i) – (iii) with proposition 2.4. ✓

Remark 2.12:

(i) The matrix elements of the associator ϕ for general simple objects are, in a chosen basis, the elements of the fusing matrices F, or $6j$-symbols, while those of Ω are the elements of the braiding matrices R. To make contact with the corresponding notation for fusing and braiding used in [I], let us compare formulas (I:2.36) and (I:2.41) to the above relations (2.6) and (2.8); we have

$$\mathsf{F}^{(r\,s\,t)\;r\cdot s\cdot t}_{s\cdot t\;\;r\cdot s} = \psi(r,s,t)^{-1} \qquad \text{and} \qquad \mathsf{R}^{(r\,s)\,r\cdot s} = \Omega(s,r)^{-1} \, , \tag{2.9}$$

where for better readability we have indicated the product in the group G by the symbol '$\cdot$'. It is straightforward to check that a pair (F,R) obeys the pentagon and the two hexagon relations if and only if (ψ,Ω) is an abelian three-cocycle (for details, see appendix C).

(ii) In proposition 2.11(iii) we considered equivalence classes of *marked* categorifications of $\mathbb{Z}G$. The equivalence classes of categorifications of $\mathbb{Z}G$, on the other hand, are in one-to-one correspondence with elements of $H^3(G,\mathbb{C}^\times)/\mathrm{Out}(G)$, respectively $H^3_{\mathrm{ab}}(G,\mathbb{C})/\mathrm{Out}(G)$ in the braided case, see e.g. proposition 1.21 of [36].

(iii) Later on we need some understanding of the possible associators and braidings on the theta-category $\mathcal{P}ic(\mathcal{C})$ that arises as a subcategory of a modular category $\mathcal{C}$. In comparing two associators or braidings, we will always leave the relevant objects fixed. This explains our interest in *marked* categorifications.

The group of abelian 3-cocycles is isomorphic to the group of quadratic forms on G (see appendix A.2). What makes theta-categories particularly accessible is that this quadratic form can easily be computed in concrete CFT models from the conformal weights. To this end, we need the notion of balancing isomorphism or twist, which is introduced in

Definition 2.13:
(i) For every object U of a ribbon category $\mathcal{C}$, the *twist* θ_U is the endomorphism

$$\theta_U := (d_U \otimes id_U) \circ (id_{U^\vee} \otimes c_{U,U}) \circ (\tilde{b}_U \otimes id_U) \ \in \mathrm{End}(U)\,. \tag{2.10}$$

(ii) For simple objects $U = U_i$ we set

$$\theta_{U_i} =: \theta_i \, id_{U_i} \tag{2.11}$$

with $\theta_i \in \mathbb{C}^\times$. The number θ_i is also called the twist, or the *balancing phase*,[3] of U_i. When the simple object is invertible, $U = L_g$, we write $\theta_{L_g} =: \theta_g \, id_{L_g}$.

For any object V the twist θ_V is an *iso*morphism (so that in particular the definition (2.11) of the balancing phase makes sense); it is also called the *balancing isomorphism*. The balancing phases of isomorphic simple objects, as well as of their dual objects, coincide; in particular,

$$\theta_{g^{-1}} = \theta_g \qquad \text{for all } \ g \in G\,. \tag{2.12}$$

The balancing phases are related to the braiding as follows.

Proposition 2.14:
(i) The function

$$\begin{array}{rccc} \delta: & G & \rightarrow & \mathbb{C}^\times \\ & g & \mapsto & \theta_g \end{array} \tag{2.13}$$

is a quadratic form on the group G. It is the inverse of the quadratic form q: $G \rightarrow \mathbb{C}^\times$ that (as described in formula (A.12)) characterizes the isomorphism class of a marked theta-category:

$$q(g)^{-1} \equiv \Omega(g,g)^{-1} = \theta_g = \delta(g)\,. \tag{2.14}$$

In particular,

$$c_{L_g,L_g} = \theta_g \, id_{L_g \otimes L_g}\,. \tag{2.15}$$

(ii) The bihomomorphism associated to the quadratic form δ is given by

$$\begin{array}{rccc} \beta: & G \times G & \rightarrow & \mathbb{C}^\times \\ & (g_1, g_2) & \mapsto & \delta(g_1 g_2)\, \delta(g_1)^{-1} \delta(g_2)^{-1}\,, \end{array} \tag{2.16}$$

and one has the identity

$$c_{L_{g_2},L_{g_1}} \circ c_{L_{g_1},L_{g_2}} = \beta(g_1,g_2)\, id_{L_{g_1} \otimes L_{g_2}}\,. \tag{2.17}$$

Proof:
(i) Note that for $g_1 = g_2 =: g$, the relation (2.8) for the braiding of basis morphisms implies

[3] The numbers θ_i are indeed phases, and even roots of unity, see e.g. [37].

that $c_{L_g,L_g} = \Omega(g,g)^{-1}\, id_{L_g\otimes L_g}$. For $U \cong L_g$ an invertible object of a ribbon category, the definition (2.10) of the twist then gives

$$\theta_{L_g} = \Omega(g,g)^{-1}\, \dim(L_g)\, id_{L_g} = \Omega(g,g)^{-1}\, id_{L_g}\,. \tag{2.18}$$

This establishes both (2.14) and (2.15).

(ii) The first statement is just the definition of an associated bihomomorphism (see formula (A.11)). The relation (2.17) follows directly from the compatibility condition

$$\theta_{V\otimes W} = c_{W,V} \circ c_{V,W} \circ (\theta_V \otimes \theta_W) \tag{2.19}$$

between braiding and twist (it is also implied by formula (A.13)). ✓

Remark 2.15:
When $\mathcal{C}$ is the modular tensor category associated to some rational CFT, then the balancing phases are related by

$$\theta_i = \exp(-2\pi\mathrm{i}\Delta_i) \tag{2.20}$$

with the conformal weights $\Delta_i \in \mathbb{Q}$ of the primary fields of the CFT. Note that for the category $\mathcal{C}$ only the fractional part of the conformal weight matters. In particular, for $g \in G$ we have $\delta(g) = \exp(-2\pi\mathrm{i}\Delta_g)$, and thus the isomorphism class of the theta-category is given by the exponentiated conformal weights. Moreover, the associated bihomomorphism β obeys

$$\beta(g_1,g_2) = \exp(2\pi\mathrm{i}\, Q_{g_1}(g_2)) \tag{2.21}$$

for $g_1, g_2 \in G$, where

$$Q_{g_1}(g_2) := \Delta_{g_1} + \Delta_{g_2} - \Delta_{g_1 g_2} \bmod \mathbb{Z}\,. \tag{2.22}$$

In CFT, $Q_{g_1}(g_2)$ is known as the *monodromy charge* of the simple current L_{g_1} with respect to L_{g_2} (or vice versa) [3]. Also note that δ contains more information than β (see section A.2). In the CFT setting, this means that the fractional parts of the conformal weights contain more information than the monodromy charges.

In a braided setting, a certain subgroup of the Picard group plays a particularly important role. We need the

Definition 2.16:
Let G be an abelian group and ψ a three-cocycle on G. We call a subgroup H of G *ψ-trivializable* iff there exists a two-cochain ω on H such that

$$\mathrm{d}\omega = \psi_{|H}\,. \tag{2.23}$$

The two-cochain ω is then called a *trivialization* of H.

For braided categories, one should rather start with an *abelian* three-cocycle (ψ,Ω). Then ψ-trivializable subgroups of G possess a simple characterization, as a corollary to the following result.

Lemma 2.17:
Let G be a finite abelian group and (ψ, Ω) an abelian three-cocycle on G. Then the cohomology class $[\psi]$ of ψ in the ordinary group cohomology $H^3(G, \mathbb{C}^\times)$ is equal to the trivial class $[1]$ if and only if for each element $g \in G$ one has $(\Omega(g,g))^{N_g} = 1$ with N_g the order of g.

Proof:
The map EM: $H^3_{\text{ab}}(G, \mathbb{C}^*) \to \mathrm{QF}(G, \mathbb{C}^\times)$ given by

$$[(\psi, \Omega)] \mapsto q \qquad \text{with} \quad q(g) := \Omega(g,g) \tag{2.24}$$

is well defined and an isomorphism of abelian groups, see appendix A.2.
Suppose that $[\psi] = [1]$. For the abelian cohomology class this implies $[(\psi, \Omega)] = [(1, \tilde{\Omega})]$ for some appropriate two-cochain $\tilde{\Omega}$. According to formula (A.8), $(1, \tilde{\Omega})$ is an abelian three-cocycle iff $\tilde{\Omega}$ is a bihomomorphism. Now

$$\Omega(g,g) = \mathrm{EM}([(\psi, \Omega)])(g) = \mathrm{EM}([(1, \tilde{\Omega})])(g) = \tilde{\Omega}(g,g)\,, \tag{2.25}$$

and $\tilde{\Omega}(g,g)^{N_g} = \tilde{\Omega}(g, g^{N_g}) = 1$, because $\tilde{\Omega}$ is a bihomomorphism. Thus indeed $\Omega(g,g)^{N_g} = 1$ for all $g \in G$.
Suppose now that, conversely, $\Omega(g,g)^{N_g} = 1$ for all $g \in G$. To establish that this implies $[\psi] = [1]$, it is sufficient to construct a bihomomorphism $\tilde{\Omega}(g,h)$ such that $\tilde{\Omega}(g,g) = \Omega(g,g)$ for all $g \in G$. Once we have such an $\tilde{\Omega}$, we know that $(1, \tilde{\Omega})$ is an abelian three-cocycle obeying

$$\mathrm{EM}([(1, \tilde{\Omega})])(g) = \tilde{\Omega}(g,g) = \Omega(g,g) = \mathrm{EM}([(\psi, \Omega)])(g)\,. \tag{2.26}$$

Since the map (2.24) is an isomorphism, this implies $[(1, \tilde{\Omega})] = [(\psi, \Omega)]$ in abelian group cohomology, and thus in particular $[1] = [\psi]$ in ordinary cohomology.
Let us proceed to construct an $\tilde{\Omega}$ with the desired properties. Define $q(g) := \Omega(g,g)$, which by (2.24) is a quadratic form. Select a set $g_1, g_2, \ldots, g_r$ of generators of the group G. For $a, b \in \{1, 2, \ldots, r\}$ choose numbers X_{ab} such that

$$\begin{cases} \exp(2\pi \mathrm{i} X_{aa}) = q(g_a) & \text{for all } a\,, \\ \exp(2\pi \mathrm{i} X_{ab}) = q(g_a g_b)\,[q(g_a)\,q(g_b)]^{-1} & \text{for } a < b\,, \\ X_{ab} = 0 & \text{for } a > b\,. \end{cases} \tag{2.27}$$

For two group elements $g = \prod_a (g_a)^{m_a}$ and $h = \prod_a (g_a)^{n_a}$, we would then like to set

$$\tilde{\Omega}(g,h) := \exp(2\pi \mathrm{i} \sum_{a,b=1}^{k} m_a X_{ab} n_b)\,. \tag{2.28}$$

For this to furnish a well-defined map $G \times G \to \mathbb{C}^\times$, shifting $m_a \mapsto \pm N_a$ or $n_a \mapsto \pm N_a$ must not affect the value of the right hand side of (2.28). We consider only the former shift, the latter being analogous; the right hand side of (2.28) changes by the factor

$$\exp\big(2\pi \mathrm{i} \sum_{b=1}^{k} N_a X_{ab} n_b\big) = \prod_{b=1}^{r} \big(\mathrm{e}^{2\pi \mathrm{i} N_a X_{ab}}\big)^{n_b}\,. \tag{2.29}$$

Now in fact each of the factors $\exp(2\pi\mathrm{i}N_a X_{ab})$ is equal to one. To see this consider the three cases in (2.27) separately. For $a > b$ the statement is trivial. For $a = b$, the statement follows because by assumption $q(g_a)^{N_a} = 1$. Finally, for $a < b$, the statement is implied by the identity

$$1 = \beta_q(g_a^{N_a}, g_b) = \beta_q(g_a, g_b)^{N_a} = \Big(\frac{q(g_a g_b)}{q(g_a)\, q(g_b)}\Big)^{N_a} \tag{2.30}$$

for the bihomomorphism β_q (A.11) that is associated to q. Here in the first two steps we use $g_a^{N_a} = e$ and the bihomomorphism property of β_q, while in the last step the definition of β_q in terms of q is inserted.

Thus (2.28) gives a well-defined map $\tilde\Omega$: $G \times G \to \mathbb{C}^\times$. From the definition it is obvious that $\tilde\Omega$ is a bihomomorphism. This implies that $\tilde q(g) = \tilde\Omega(g,g)$ is a quadratic form on G. It remains to show that $q(g) = \tilde q(g)$ for all $g \in G$. By lemma B.2, it is sufficient to verify that $q(g_a) = \tilde q(g_a)$ for all a and $q(g_a g_b) = \tilde q(g_a g_b)$ for all $a \neq b$. The first equality holds by definition. To show the second, we may assume that $a < b$; then

$$\tilde q(g_a g_b) = \tilde\Omega(g_a g_b, g_a g_b) = \tilde\Omega(g_a, g_a)\, \tilde\Omega(g_b, g_b)\, \tilde\Omega(g_a, g_b)\, \tilde\Omega(g_b, g_a) = q(g_a g_b)\,. \tag{2.31}$$

Here in the first step the definition of $\tilde q$ is inserted, in the second step it is used that $\tilde\Omega$ is a bihomomorphism, and in the last step the definition of $\tilde\Omega$ is substituted. ✓

Corollary 2.18:
Let G be an abelian group and (ψ, Ω) an abelian three-cocycle on G. A subgroup $H \leq G$ is ψ-trivializable iff for each element $h \in H$ one has $(\Omega(h,h))^{N_h} = 1$ with N_h the order of h.

Remark 2.19:
The condition that $(\delta(h))^{N_h} = (\Omega(h,h))^{-N_h}$ is equal to 1 is fulfilled for every $h \in G$ that can be written as a square. This is implied by the following argument from [3], which for the convenience of the reader is reformulated using the present terminology. First note that $\beta(g, g^{n-1}) = \delta(g^n)[\delta(g^{n-1})\delta(g)]^{-1}$ for $g \in G$ and any integer n, implying that

$$\delta(g^n) = \delta(g^{n-1})\, \delta(g)\, (\beta(g,g))^{n-1}\,. \tag{2.32}$$

For $n = N_g$ this gives, after choosing a square root $\sqrt{\beta(g,g)}$ of $\beta(g,g)$,

$$\delta(g) = \epsilon_g \left(\sqrt{\beta(g,g)}\,\right)^{-N_g+1} \tag{2.33}$$

with $\epsilon_g \in \{\pm 1\}$. With the help of (2.32), one can now prove by induction that

$$\delta(g^n) = \epsilon_g^n \left(\sqrt{\beta(g,g)}\,\right)^{-n(N_g-n)}. \tag{2.34}$$

Let now $h \in G$ be a square, $h = g^2$, so that $N_g = 2N_h \in 2\mathbb{Z}$. Since $\beta(g,g)$ is an N_gth root of unity, by applying (2.34) with $n = 2$ it follows that

$$(\delta(h))^{N_h} = (\delta(g^2))^{N_g/2} = \Big(\epsilon_g^2 \left(\sqrt{\beta(g,g)}\,\right)^{-2(N_g-2)}\Big)^{N_g/2} = 1\,. \tag{2.35}$$

Definition 2.20:
Suppose the Picard category of a tensor category $\mathcal{C}$ is equivalent to $\mathcal{C}(G,\psi,\Omega)$. Then the *effective center* $\mathrm{Pic}^\circ(\mathcal{C})$ of the category $\mathcal{C}$ is the union (as sets) of all ψ-trivializable subgroups of G.

Remark 2.21:
(i) Note that if g has order N_g, then g^n has order $N_{g^n} = mN_g/n$, with m the smallest positive integer such that $mN_g/n \in \mathbb{Z}$. If $\theta_g^{N_g} = 1$, then by lemma B.2(ii) we also have

$$(\theta_{g^n})^{N_{g^n}} = (\theta_g)^{n^2 mN_g/n} = (\theta_g^{N_g})^{nm} = 1\,. \tag{2.36}$$

Thus, if $\theta_g^{N_g} = 1$ for some $g \in \mathrm{Pic}(\mathcal{C})$, then also $\theta_h^{N_h} = 1$ for all h in the cyclic subgroup generated by g. Together with corollary 2.18, it follows that the effective center $\mathrm{Pic}^\circ(\mathcal{C})$ consists precisely of those $g \in \mathrm{Pic}(\mathcal{C})$ for which $\theta_g^{N_g} = 1$. In CFT, where the twist is related to the conformal weights by (2.20), this means that

$$\mathrm{Pic}^\circ(\mathcal{C}) = \{g \in G \mid N_g \Delta_g \in \mathbb{Z}\}\,. \tag{2.37}$$

This is the original definition [38] of the effective center.
(ii) The effective center is, in general, not a group. [4] For example, one checks that the function $q\colon G \to \mathbb{C}^\times$ given by $q(g_1^{m_1} g_2^{m_2}) := \exp(\pi \mathrm{i}\, m_1^2/2)$ is a quadratic form on the group $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, with the two generators denoted by g_1 and g_2. The associated bihomomorphism β_q acts as $\beta_q(g_1^{m_1} g_2^{m_2}, g_1^{n_1} g_2^{n_2}) = (-1)^{m_1 n_1}$. Let $[(\psi,\Omega)] := \mathrm{EM}^{-1}(q) \in H^3_{\mathrm{ab}}(G,\mathbb{C})$ be the abelian cohomology class determined by q via the isomorphism (A.12), with representative $(\psi,\Omega) \in Z^3_{\mathrm{ab}}(G,\mathbb{C})$, and set $\mathcal{D} := \mathcal{C}(G,\psi,\Omega)$.
The two elements $g := g_2$ and $h := g_1 g_2$ of G both have order four, and $q(g)^4 = 1 = q(h)^4$, while gh has order two, and $q(gh)^2 = -1$. Together with part (i), this implies that g and h are in $\mathrm{Pic}^\circ(\mathcal{D})$, while gh is not. Thus $\mathrm{Pic}^\circ(\mathcal{D})$ is not a subgroup of $\mathrm{Pic}(\mathcal{D}) = G$. In fact, in this example one has $\mathrm{Pic}^\circ(\mathcal{D}) = \{e, g_2, g_2^2, g_2^3, g_1 g_2, g_1 g_2^3\}$.
(iii) According to remark 2.19, every square in $\mathrm{Pic}(\mathcal{C})$ is even in $\mathrm{Pic}^\circ(\mathcal{C})$. Also note that in particular every group element h of odd order can be written as a square, namely as $h = \big(h^{(N_h+1)/2}\big)^2$. Thus simple currents of odd order are always in the effective center.
(iv) The quotient ω/ω' of two trivializing two-cochains ω and ω' is closed, but not necessarily exact.
(v) The Picard group of the product $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ of two categories is the product of the Picard groups,

$$\mathrm{Pic}(\mathcal{C}_1 \boxtimes \mathcal{C}_2) \cong \mathrm{Pic}(\mathcal{C}_1) \times \mathrm{Pic}(\mathcal{C}_2)\,. \tag{2.38}$$

In contrast, a similar identity does *not* hold for the effective center. Here is a simple counter example: take for $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ the modular tensor category for the WZW theory based on $A_1^{(1)}$ at level 1. The effective center of $\mathcal{C}$ is trivial, whereas the product has effective center $\mathrm{Pic}^\circ(\mathcal{C} \boxtimes \mathcal{C}) \cong \mathbb{Z}_2$.

[4] We thank Bert Schellekens for pointing this out to us.

3 Algebras

We now proceed to discuss symmetric special Frobenius algebras. Such algebras in modular tensor categories play an important role in conformal field theory [1, I]. An important subclass is the one of simple symmetric special Frobenius algebras, which contains in particular the class of haploid special Frobenius algebras: [5]

Definition 3.1:
An algebra A in a tensor category $\mathcal{C}$ is called *haploid* [39] iff it contains the tensor unit with multiplicity one, i.e. iff

$$\mathrm{Hom}(\mathbf{1}, A) \cong \mathbb{C}\,. \tag{3.1}$$

A is called *simple* iff [24, 40]

$$\mathrm{Hom}_{A|A}(A, A) \cong \mathbb{C}\,. \tag{3.2}$$

Haploid algebras can be characterized as the algebras that are simple as left modules over themselves, and simple algebras as those that are simple as bimodules over themselves. A haploid algebra is in particular simple. Also, every simple algebra in $\mathcal{C}$ is Morita equivalent to a haploid algebra [24]. Since in the construction of a full CFT on oriented world sheets for a given modular tensor category only the Morita class of an algebra matters, this justifies to restrict, as we will do later on, our attention to haploid algebras.

The goal of this section is to provide a systematic construction of haploid special Frobenius algebras from invertible elements and to show that these algebras give rise to most of the known module categories – the ones that are of "D-type" in A-D-E type classifications. In subsection 3.1 we prove general estimates on haploid algebras in modular tensor categories. In subsection 3.2 we show that the restriction to invertible simple subobjects gives rise to subalgebras. These subalgebras can be classified by the theory of algebras in theta-categories that we develop in subsection 3.3. In subsection 3.4 we show that isomorphism classes of such algebras are classified by certain bihomomorphisms which we call *Kreuzer–Schellekens bihomomorphisms*. Finally, in subsection 3.5, we compute the modular invariant torus partition function that results from such an algebra and show that it is of the form studied by Kreuzer and Schellekens.

3.1 Simple algebras in modular tensor categories

In this subsection we establish some estimates, involving the dimensions of simple objects in $\mathcal{C}$, for quantities related to simple (in particular, to haploid) symmetric special Frobenius algebras in modular tensor categories. Some of these are similar to considerations in [41], where they are formulated at the level of "modular data" rather than for modular tensor categories. They use some basic results from Perron–Frobenius theory (for a summary of Perron–Frobenius theory see e.g. chapter XIII of [42]), in particular

[5] A haploid special Frobenius algebra is automatically symmetric, see corollary I:3.10.

Lemma 3.2:
Let $\mathrm{R} = (\mathrm{R}_{ij})$ be an $n \times n$ matrix with non-negative real entries. Then the diagonal entries of R are bounded by the Perron–Frobenius eigenvalue λ_{PF} of R: $\lambda_{\mathrm{PF}} \geq \max_{1 \leq i \leq n}\{\mathrm{R}_{ii}\}$.

Proof:
Since the matrix R has non-negative entries, the diagonal entries $(\mathrm{R}^m)_{ii}$ of its mth power are bounded below by $(\mathrm{R}_{ii})^m$: $(\mathrm{R}^m)_{ii} \geq (\mathrm{R}_{ii})^m$. The eigenvalues of R^m, on the other hand, are powers $(\lambda_j)^m$ of the eigenvalues λ_j of R. So we have the inequality

$$(\mathrm{R}_{ii})^m \leq \sum_k (\mathrm{R}^m)_{kk} = \sum_j (\lambda_j)^m \leq n\, \lambda_{\mathrm{PF}}^m \tag{3.3}$$

for any i. If the entry R_{ii} were larger than the Perron–Frobenius eigenvalue λ_{PF}, then for sufficiently large m the left hand side of (3.3) would surpass the right hand side. ✓

A symmetric special Frobenius algebra A has only finitely many equivalence classes of simple left modules (see theorem I:5.18); we select a set $\{M_\kappa\}$ of representatives for these, labeled by $\kappa \in \mathcal{J}$. We now assume that A is haploid; then A is a simple left module over itself, and we choose it as one of the representatives, denoting the corresponding label in $\mathcal{J}$ by 0. We also introduce, for every $i \in \mathcal{I}$, the $|\mathcal{J}| \times |\mathcal{J}|$-matrix A_i with entries the non-negative integers

$$(\mathrm{A}_i)_\kappa^{\ \kappa'} \equiv \mathrm{A}_{i\kappa}^{\ \kappa'} := \dim_{\mathbb{C}} \mathrm{Hom}_A(M_\kappa \otimes U_i, M_{\kappa'})\,. \tag{3.4}$$

These matrices, which in conformal field theory yield the annulus partition function, are known to furnish a NIM-rep of the fusion rules (see theorem I:5.20), and their eigenvalues are $S_{i,m}/S_{0,m}$; thus the Perron–Frobenius eigenvalue of A_i is given by $\dim(U_i)$.

Proposition 3.3:
Let A be a haploid special Frobenius algebra in a modular tensor category $\mathcal{C}$. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}(U, A) \;\leq\; \dim(U) \tag{3.5}$$

for every object U of $\mathcal{C}$.

Proof:
Due to the semisimplicity of $\mathcal{C}$ it is enough to establish (3.5) for simple objects $U = U_i$ only. For these one has the Frobenius reciprocity relation $\mathrm{Hom}(U_i, A) \cong \mathrm{Hom}_A(\mathrm{Ind}_A(U_i), A)$, where $\mathrm{Ind}_A(U_i)$ is an induced A-module, which as an object of $\mathcal{C}$ is $A \otimes U_i$. Thus

$$\dim_{\mathbb{C}} \mathrm{Hom}(U, A) = \dim_{\mathbb{C}} \mathrm{Hom}_A(\mathrm{Ind}_A(U_i), A) = \mathrm{A}_{i\,0}^{\ 0}\,, \tag{3.6}$$

and by lemma 3.2 this integer is indeed bounded above by the Perron–Frobenius eigenvalue $\dim(U_i)$ of A_i. ✓

Remark 3.4:
(i) The result (3.5) implies in particular that a simple current appears in a haploid special Frobenius algebra with multiplicity at most one.
(ii) The result also implies that in a modular tensor category there are only finitely many equivalence classes of objects that can carry the structure of a haploid special Frobenius algebra.
(iii) It can actually be shown (see formula (3.9b) of [41]) that *all* entries of the matrix A_i, not only the diagonal ones, are bounded by $\dim(U_i)$:

$$\mathrm{A}_{i\kappa}^{\;\kappa'} \;\leq\; \dim(U_i) \tag{3.7}$$

for all $i \in \mathcal{I}$ and all $\kappa, \kappa' \in \mathcal{J}$. This can be used to obtain upper bounds on $\dim_{\mathbb{C}}\mathrm{Hom}(U,A)$ when A is simple, but not haploid. For instance, using that A is isomorphic [24] to the internal End $\underline{\mathrm{End}}(A)$ of A (regarded as a left module over itself) it follows that

$$\dim_{\mathbb{C}}\mathrm{Hom}(U,A) \leq \big(\dim_{\mathbb{C}}(A_{\mathrm{top}})\big)^2 \dim(U)\,, \tag{3.8}$$

where as in [I] by A_{top} we denote the $\mathbb{C}$-algebra $\mathrm{Hom}(\mathbf{1},A)$.

Next we wish to show that each object in a modular tensor category can be endowed with the structure of a haploid special Frobenius algebra in at most finitely many ways. To this end, we need the following result, which was first obtained in [43] (see also a related estimate in [44]) with the help of Perron–Frobenius theory. The proof given here is of independent interest. We derive an upper bound for the multiplicities $Z_{ij} = Z_{ij}(A)$ that appear in the modular invariant torus partition function associated to A. According to theorem I:5.1, Z_{ij} can be expressed as the invariant of a certain ribbon graph in the three-manifold $S^2 \times S^1$, see formula (I:5.30); Z_{ij} can also be interpreted as the dimension of a space of A-bimodule morphisms between alpha-induced A-bimodules, see section I:5.4. The interest of the estimate lies in the fact that it is independent of the choice of A and involves only data of the tensor category $\mathcal{C}$.

Lemma 3.5:
For A a *simple* symmetric special Frobenius algebra in a modular tensor category $\mathcal{C}$, the integers $Z_{ij} = Z_{ij}(A)$ satisfy

$$Z_{ij} \leq \dim(U_i)\dim(U_j)\,. \tag{3.9}$$

Proof:
(i) We start with the observation that

$$\dim_{\mathbb{C}}\mathrm{Hom}(U,V) \leq \dim(U)\dim(V) \tag{3.10}$$

for any two objects U and V of a semisimple sovereign tensor category $\mathcal{C}$. To see this, use the semisimplicity to write the objects U, V as direct sums of simple objects, $U \cong \bigoplus_{i\in\mathcal{I}} n_i^U U_i$

and $V \cong \bigoplus_{i\in\mathcal{I}} n_i^V U_i$. Since dimensions are bounded below by 1, we have $\sum_{i\in\mathcal{I}} n_i^U \le \dim(U)$ and $\sum_{i\in\mathcal{I}} n_i^V \le \dim(V)$. We thus obtain the estimate

$$\begin{aligned} \dim_{\mathbb{C}}\mathrm{Hom}(U,V) = \sum_{i\in\mathcal{I}} n_i^U n_i^V \; &\le \sum_{i,j\in\mathcal{I}} n_i^U n_j^V \\ &= \big(\sum_i n_i^U\big)\big(\sum_j n_j^V\big) \le \dim(U)\dim(V)\,. \end{aligned} \tag{3.11}$$

(ii) We apply this result to the category $\mathcal{C}_{A|A}$ of A-bimodules, which is again a semisimple tensor category. Let us first explain that $\mathcal{C}_{A|A}$ is even a sovereign tensor category. If $X = (\dot X, \rho, \tilde\rho)$ is an A-bimodule, then the left action ρ of A on $\dot X$ can be used to define a right action of A on the dual object $\dot X^\vee$, while the right action $\tilde\rho$ of A on $\dot X$ gives a left action on $\dot X^\vee$. Furthermore, using sovereignty of $\mathcal{C}$ one checks that the two actions commute, and hence one has an A-bimodule structure on $\dot X^\vee$; in [II], this bimodule structure is denoted by X^v, compare (II:2.40). The assignment $X \mapsto X^v$, together with morphisms d_X^A and b_X^A that are analogous to those described e.g. in formula (3.51) of [40] for the case of A-modules, furnishes a left duality of $\mathcal{C}_{A|A}$; analogously one has a right duality. Finally one checks, by arguments similar to those in section 5.3 of [39], that this way $\mathcal{C}_{A|A}$ becomes a sovereign tensor category.
Further, alpha-induction provides two tensor functors $\alpha_A^\pm$ from $\mathcal{C}$ to $\mathcal{C}_{A|A}$ that preserve the dualities and thus the dimensions (provided that the tensor unit of $\mathcal{C}_{A|A}$ is simple, which is the case iff A is simple as an algebra). Combining this result with (3.10), we obtain

$$\begin{aligned} Z_{ij} \; &= \dim \mathrm{Hom}_{A|A}(\alpha_A^-(U_j), \alpha_A^+(U_i^\vee)) \\ &\le \dim_{A|A}(\alpha_A^-(U_j))\, \dim_{A|A}(\alpha_A^+(U_i^\vee)) = \dim(U_i)\dim(U_j)\,, \end{aligned} \tag{3.12}$$

which establishes the estimate (3.9). ✓

Recall from [I] that, by our conventions, the counit and coproduct of a symmetric special Frobenius algebra are normalized such that $\varepsilon \circ \eta = \dim(A)\, id_{\mathbf{1}}$.

Proposition 3.6:
(i) An object in a modular tensor category can be endowed with the structure of a haploid special Frobenius algebra in at most finitely many inequivalent ways.
(ii) A modular tensor category admits only finitely many inequivalent haploid special Frobenius algebras.

Proof:
According to theorem I:5.18 the number $|\mathcal{J}|$ of inequivalent simple left modules over a symmetric special Frobenius algebra A is given by $\sum_i Z_{i\bar\imath}(A)$, where $\bar\imath$ labels the simple object isomorphic to $U_i^\vee$. When combined with the estimate (3.9), we thus have

$$|\mathcal{J}| = \sum_{i\in\mathcal{I}} Z_{i\bar\imath} \;\le \sum_{i\in\mathcal{I}} \dim(U_i)^2 = \mathrm{Dim}(\mathcal{C})^2\,. \tag{3.13}$$

Now according to corollary 2.22 of [18], for every (multi-)fusion category $\mathcal{C}$ the number of module categories over $\mathcal{C}$ with a prescribed number of non-isomorphic simple objects is finite. Thus let $\{\mathcal{M}_\ell\}$, with $\{\ell\}$ some finite index set, be a list of non-isomorphic module categories, each of which has at most $\mathrm{Dim}(\mathcal{C})^2$ simple objects. Every simple object in each category $\mathcal{M}_\ell$ gives rise to a haploid algebra in $\mathcal{C}$ [24]. This provides us with a finite list $\{A_{\ell,p}\}$ (with p labeling the isomorphism classes of simple objects in $\mathcal{M}_\ell$) of haploid algebras in $\mathcal{C}$. Now take an arbitrary haploid special Frobenius algebra A in $\mathcal{C}$. Then $\mathcal{C}_A \cong \mathcal{M}_\ell$ for some value of ℓ. Since A is haploid, it is simple as a left module over itself. Any simple object in $\mathcal{M}_\ell$ isomorphic to A, regarded as a left module over itself, gives an algebra in $\mathcal{C}$ that is isomorphic (as an algebra) to A. Thus A is isomorphic to $A_{\ell,p}$ for some p. Since the list $\{A_{\ell,p}\}$ is finite, this implies the finiteness assertions (i) and (ii). ✓

3.2 Restriction to invertible subobjects

We now study invertible objects in haploid special Frobenius algebras. We start with the

Definition 3.7:
(i) Writing an object U of $\mathcal{C}$ as a direct sum of simple objects U_i,

$$U \cong \bigoplus_{i\in\mathcal{I}} n_i\, U_i \equiv \bigoplus_{i\in\mathcal{I}} U_i^{\oplus n_i}\,, \tag{3.14}$$

we denote by $I_\circ \subseteq I$ the subset containing those labels $i \in \mathcal{I}$ such that U_i is an invertible object. The *Picard subobject* $Pic(U)$ of U is defined as

$$Pic(U) := \bigoplus_{i\in I_\circ} n_i\, U_i\,. \tag{3.15}$$

(ii) An object U of $\mathcal{C}$ is called *of simple current type* iff $U \cong Pic(U)$.
(iii) The modular invariant torus partition function derived from a haploid special Frobenius algebra of simple current type is called a *simple current modular invariant*.
(iv) We call a haploid special Frobenius algebra of simple current type a *Schellekens algebra*.

Remark 3.8:
(i) Whenever it is convenient, we identify $Pic(U)$ with an object of the Picard category $\mathcal{P}ic(\mathcal{C})$.
(ii) Every simple algebra A of simple current type in $\mathcal{C}$ is Morita equivalent to a Schellekens algebra. Indeed, any such A is also a simple algebra in $\mathcal{P}ic(\mathcal{C})$ and hence, by general arguments, Morita equivalent (in $\mathcal{P}ic(\mathcal{C})$) to a haploid algebra in $\mathcal{P}ic(\mathcal{C})$. That haploid algebra, in turn, is also an algebra in $\mathcal{C}$, in fact a Schellekens algebra, and it is Morita equivalent in $\mathcal{C}$ to A. (The argument also shows that the interpolating bimodules of the

associated Morita context are of simple current type as well.)
(iii) As we will see, with our notion of simple current type algebras we obtain precisely the modular invariants that were discussed in [16]. Thus the notion of a modular invariant of simple current type used here is more restrictive than the one given in [38, 45]. It is not known if modular invariants that are simple current invariants in the sense of [38, 45], but not simple current invariants in our sense, can appear as torus partition functions of consistent conformal field theories. (A class of examples of modular invariants of this type that are definitely unphysical has been found in section 4 of [46].)

A construction similar to the one of definition 3.7(i) can be performed for any semisimple full tensor subcategory $\mathcal{C}'$ of $\mathcal{C}$. We refer to the object U' in $\mathcal{C}'$ associated in this way to an object U in $\mathcal{C}$ as the *truncation* of U to $\mathcal{C}'$.

Proposition 3.9:
Let $\mathcal{C}'$ be a semisimple full tensor subcategory of $\mathcal{C}$. Let A' be the truncation of an object $A \in \mathcal{O}bj(\mathcal{C})$ to $\mathcal{C}'$, and select embedding and restriction morphisms $e \in \mathrm{Hom}(A', A)$ and $r \in \mathrm{Hom}(A, A')$ such that $r \circ e = id_{A'}$.
(i) If (A, m, η) is an algebra in $\mathcal{C}$, then $(A', r \circ m \circ (e \otimes e), r \circ \eta)$ is an algebra in $\mathcal{C}'$ (and in $\mathcal{C}$).
(ii) If $\mathcal{C}$ is a ribbon category and $(A, m, \eta, \Delta, \varepsilon)$ is a symmetric Frobenius algebra in $\mathcal{C}$, then $(A', r \circ m \circ (e \otimes e), r \circ \eta, \zeta^{-1}(r \otimes r) \circ \Delta \circ e, \zeta\varepsilon \circ e)$ is a symmetric Frobenius algebra in $\mathcal{C}'$, for any $\zeta \in \mathbb{C}^\times$.
(iii) Let A and A' be as in (ii). Suppose that in addition A and A' are haploid. Then if A is special, so is A'.

Proof:
(i) Introducing bases in the relevant morphism spaces, the associativity property of the multiplication m takes the form given in (I:3.8). In that form, the associativity relations for A contain as a subset the associativity relations for A'. A similar statement applies to the unit properties.
(ii) An analogous reasoning allows one to extend the result to symmetric special Frobenius algebras.
(iii) From (ii) we already know that A' is a symmetric Frobenius algebra. To show that A' is special, by lemma I:3.11 it is sufficient to verify that the counit ε' of A' obeys $\varepsilon' = \gamma\varepsilon'_\natural$ for some $\gamma \in \mathbb{C}^\times$, where $\varepsilon'_\natural$ is the morphism defined in (I:3.46). Since A' is haploid, the relation $\varepsilon' = \gamma\varepsilon'_\natural$ holds for some $\gamma \in \mathbb{C}$, so it remains to show that $\gamma \neq 0$. Composing both sides of $\varepsilon' = \gamma\varepsilon'_\natural$ with η', on the right hand side we obtain $\gamma \dim(A')$. To establish that $\gamma \neq 0$ we must check that the left hand side is non-zero. Now note that since both A and A' are haploid, we have $\varepsilon' = \lambda\,\varepsilon$ and $\eta' = \tilde\lambda\,\eta$ for some $\lambda, \tilde\lambda \in \mathbb{C}^\times$. It follows that $\varepsilon' \circ \eta' = \lambda\tilde\lambda\,\varepsilon \circ \eta = \lambda\tilde\lambda\, \dim(A) \neq 0$. ✓

As a consequence of this result, the classification of Schellekens algebras in modular tensor categories amounts to classifying haploid special Frobenius algebras in theta-categories.

At the same time, such a classification fixes the algebra structure of a general haploid special Frobenius algebra on its simple current type subobjects.

3.3 Algebras in theta-categories

We start with a discussion of (associative, unital) algebras in theta-categories. The following construction provides examples of algebras in the pointed category $\mathcal{C}(G,\psi)$. We start with a closed three-cocycle $\psi \in Z^3(G,\mathbb{C}^\times)$ and a ψ-trivializable subgroup H of G, with trivializing two-cocycle ω. This cocycle is, in general, not closed, so modifying the convolution product of the group algebra $\mathbb{C}H$ by ω according to

$$b_g \star_\omega b_{g'} \equiv m(b_g \otimes b_{g'}) := \omega(g,g')\, b_{gg'} \tag{3.16}$$

does *not*, in general, yield an associative algebra $\mathbb{C}_\omega H$, at least not in the category of vector spaces. However, since the violation of associativity is nothing but $\mathrm{d}\omega = \psi$ and thus a closed three-cocycle, it can be canceled by simply changing the notion of associativity – i.e., by regarding $\mathbb{C}_\omega H$ as an object in the tensor category $\mathcal{C}(G,\psi)$ of G-graded vector spaces with associativity constraint given by ψ. The following lemma asserts that in this category the object

$$\mathbb{C}_\omega H =: A(H,\omega) \,\in \mathcal{O}bj(\mathcal{C}(G,\psi)) \tag{3.17}$$

is in fact an associative algebra. By a slight abuse of terminology, we will still refer to (3.17) as a *twisted group algebra*.

Lemma 3.10:
Let G be a finite group.
(i) For $\psi \in Z^3(G,\mathbb{C}^\times)$, fix a ψ-trivializable subgroup H of G with trivialization ω. Then the twisted group algebra $\mathbb{C}_\omega H \equiv A(H,\omega)$ with product given by (3.16) is a haploid algebra in $\mathcal{C}(G,\psi)$.
(ii) Any two trivializations of H differ by a two-cocycle on H. If the two-cocycle is a coboundary, the two associated twisted group algebras are isomorphic. As a consequence, the possible twisted group algebras for a given ψ form a torsor[6] over $H^2(H,\mathbb{C}^\times)$.
(iii) Setting

$$\Delta(b_g) := \frac{1}{|H|} \sum_{h\in H} \frac{1}{\omega(gh^{-1},h)}\, b_{gh^{-1}} \otimes b_h \qquad \text{and} \qquad \varepsilon(b_g) := |H|\, \delta_{g,e} \tag{3.18}$$

turns $A(H,\omega)$ into a haploid special Frobenius algebra in $\mathcal{C}(G,\psi)$.

Proof:
The axioms can be checked by straightforward calculations. For example, associativity of

[6] Recall that a set S is a *torsor* over a group G iff there is a free transitive action of G on S. In particular, S and G are isomorphic as sets; but the isomorphism is not canonical.

$A \equiv A(H,\omega)$ amounts to $m\circ(id_A\otimes m)\circ\alpha_{A,A,A} = m\circ(m\otimes id_A)$. Evaluating this relation on a basis element $(b_{h_1}\otimes b_{h_2})\otimes b_{h_3}$ gives

$$\omega(h_1,h_2h_3)\,\omega(h_2,h_3)\,\psi(h_1,h_2,h_3)^{-1} = \omega(h_1,h_2)\,\omega(h_1h_2,h_3)\,, \tag{3.19}$$

which is equivalent to $\mathrm{d}\omega(h_1,h_2,h_3) = \psi(h_1,h_2,h_3)$. As another example, to establish specialness of A one must check that $m\circ\Delta(b_g) = b_g$ and $\varepsilon\circ\eta = \dim(A)$. This follows immediately when substituting the definitions (3.16) and (3.18). ✓

Remark 3.11:
(i) As already pointed out, $A(H,\omega)$ is, in general, not an associative algebra in the category of vector spaces, since ω is in general not closed.
(ii) There is again an equivalence relation: consider another three-cocycle $\psi' \in Z^3(G,\mathbb{C}^\times)$ and two-cochain $\omega' \in C^2(H,\mathbb{C}^\times)$ such that $\mathrm{d}\omega' = \psi'_{|H}$. If ψ and ψ' are cohomologous, $\psi' = \psi\,\mathrm{d}\eta$, then the categories $\mathcal{C}(G,\psi)$ and $\mathcal{C}(G,\psi')$ are equivalent, and we identify them. If, moreover, there is a one-cochain $\chi \in C^1(H,\mathbb{C}^\times)$ such that $\omega' = \omega\,\eta_{|H}\,\mathrm{d}\chi$, then the two algebras $A(H,\omega)$ in $\mathcal{C}(G,\psi)$ and $A(H,\omega')$ in $\mathcal{C}(G,\psi')$ are isomorphic.
(iii) Algebras in the category $\mathcal{C}(G,\psi)$ have also been considered in [47]. There, the Morita classes of twisted group algebras are studied. It is found that $A(H_1,\omega_1)$ and $A(H_2,\omega_2)$ are Morita equivalent (i.e. have isomorphic module categories) iff the pairs $(H_1,[\omega_1])$ and $(H_2,[\omega_2])$ are conjugate under the action of G.
In conformal field theory, only the Morita class of an algebra matters. Note, however, that in our applications pointed categories arise as Picard subcategories of modular tensor categories $\mathcal{C}$, and that the Morita classes with respect to $\mathcal{C}$ will in general be larger than those with respect to $\mathcal{P}ic(\mathcal{C})$. An example is provided by the critical Ising model, see remark 3.28 below.

For the proof of the next lemma, we introduce a basis choice that we will also use repeatedly in the sequel.

Definition 3.12:
Let $\mathcal{D}$ be a pointed sovereign tensor category equivalent to $\mathcal{C}(G,\psi)$, and A a haploid special Frobenius algebra in $\mathcal{D}$.
(i) The *support* $H(A)$ of the algebra A is the subset

$$H(A) := \{g\in G \mid \dim_{\mathbb{C}}\mathrm{Hom}(L_g,A) > 0\} \tag{3.20}$$

of G.
(ii) An *adapted basis* for A is the choice, for all $g\in H(A)$, of morphisms $e_g \in \mathrm{Hom}(L_g,A)$ and $r_g\in\mathrm{Hom}(A,L_g)$ that form bases of these morphism spaces that are dual in the sense [I] that $r_g\circ e_g = id_{L_g}$. For the unit element $\mathbf{1}\equiv L_1 \in H(A)$ we take $e_1=\eta$ and $r_1 = \dim(A)^{-1}\varepsilon$.

Note that for each $g\in H(A)$, the morphism

$$p_g := e_g\circ r_g \;\in \mathrm{End}(A) \tag{3.21}$$

is an idempotent. To make the connection with the notation in [I], let us present explicitly the product and coproduct on $A(H,\omega)$, using the notations of (I:3.7) and (I:3.82). One finds

$$m_{g,h}^{\;\;gh} = \omega(g,h) \qquad \text{and} \qquad \Delta_{gh}^{g,h} = \big(\dim(A)\,\omega(g,h)\big)^{-1} . \tag{3.22}$$

The last expression can also be obtained from (I:3.83), noting that via $\psi|_H = \mathrm{d}\omega$ it is possible to express the F-matrix elements in (I:3.83) in terms of ω.

Lemma 3.13:
Let $\mathcal{D}$ be a theta-category equivalent to $\mathcal{C}(G,\psi,\Omega)$ and A a Schellekens algebra in $\mathcal{D}$.
(i) For each simple object L_g of $\mathcal{D}$ the dimension of the morphism space $\mathrm{Hom}(L_g,A)$ is either zero or one.
(ii) The support $H(A)$ is a subgroup of G.
(iii) Choose an adapted basis for A. For all $g,h \in H(A)$ we have $r_{gh} \circ m \circ (e_g \otimes e_h) \neq 0$ and $(r_g \otimes r_h) \circ \Delta \circ e_{gh} \neq 0$.

Proof:
(i) We already know this result for every theta-category that is the Picard category of a modular tensor category. Here we give a proof that does not rely on modularity.
Using Frobenius reciprocity, we have

$$\mathrm{Hom}_A(A \otimes L_g, A \otimes L_g) \;\cong\; \mathrm{Hom}(L_g, A \otimes L_g) \;\cong\; \mathbb{C} , \tag{3.23}$$

where the second equality comes from the fact that the only subobject of A that contributes is the tensor unit, which appears in a haploid algebra with multiplicity one. This shows that all induced A-modules $A \otimes L_g$ are simple. Again by Frobenius reciprocity, we have

$$\mathrm{Hom}(L_g, A) \cong \mathrm{Hom}_A(A \otimes L_g, A) , \tag{3.24}$$

which as the space of intertwiners between simple A-modules has either dimension zero or dimension one.
(ii) Frobenius algebras in sovereign tensor categories can only be defined on self-conjugate objects (see e.g. lemma 3.3 of [39]). Since the conjugate of L_g is isomorphic to $L_{g^{-1}}$, this implies that $H(A)$ is closed under inverses.
Now choose an adapted basis for A. Suppose that $g,h \in H(A)$, but $gh \notin H(A)$. Then $m \circ (e_g \otimes e_h) = 0$. Associativity of the multiplication m of A then implies that

$$0 = (m \circ (m \otimes id_A)) \circ (e_g \otimes e_h \otimes e_{h^{-1}}) = e_g \otimes \big(\dim(A)^{-1} \varepsilon \circ m \circ (e_h \otimes e_{h^{-1}})\big) . \tag{3.25}$$

in $\mathrm{Hom}(L_g \otimes L_h \otimes L_{h^{-1}}, A)$. For a Frobenius algebra, $\varepsilon \circ m$ is non-degenerate, and hence $\varepsilon \circ m \circ (e_h \otimes e_{h^{-1}})$ is non-zero. Furthermore, e_g is non-zero by construction. This is a contradiction; hence $gh \in H(A)$.
(iii) By an analogous argument as in (ii), using now also directly the Frobenius property of A, assuming that $m \circ (e_g \otimes e_h) = 0$ leads to a contradiction. For the coproduct one repeats the argument using coassociativity. ✓

Proposition 3.14:
(i) Let $\mathcal{D}$ be a theta-category equivalent to $\mathcal{C}(G,\psi,\Omega)$, and A a haploid special Frobenius algebra in $\mathcal{D}$. Then A is isomorphic to one of the twisted group algebras $A(H(A),\omega)$.
(ii) A Schellekens algebra in a modular tensor category $\mathcal{C}$ is characterized by a subgroup H in the effective center $\mathrm{Pic}^\circ(\mathcal{C})$ and a trivialization of the associativity constraint on H.

Proof:
(i) We select bases ${}_g b_h$ of $\mathrm{Hom}(L_g \otimes L_h, L_{gh})$ as in (2.4) and define $\omega(g,h)$ by

$$r_{gh} \circ m \circ (e_g \otimes e_h) =: \omega(g,h)\, {}_g b_h\,. \tag{3.26}$$

From lemma 3.13(iii) we know that $\omega(g,h)$ is non-vanishing for all $g,h \in H(A)$. Since A is an algebra, we have $m \circ (id_A \otimes m) = m \circ (m \otimes id_A)$. Evaluating both sides in an adapted basis gives

$$\begin{aligned} r_{h_1h_2h_3} \circ m \circ (id_A \otimes m) \circ (e_{h_1} \otimes e_{h_2} \otimes e_{h_3}) &= \omega(h_1, h_2h_3)\, \omega(h_2,h_3)\, {}_{h_1}b_{h_2h_3} \circ (id_{L_{h_1}} \otimes {}_{h_2}b_{h_3})\,, \\ r_{h_1h_2h_3} \circ m \circ (m \otimes id_A) \circ (e_{h_1} \otimes e_{h_2} \otimes e_{h_3}) &= \omega(h_1h_2, h_3)\, \omega(h_1,h_2)\, {}_{h_1h_2}b_{h_3} \circ ({}_{h_1}b_{h_2} \otimes id_{L_{h_3}})\,. \end{aligned} \tag{3.27}$$

The two basis elements of $\mathrm{Hom}(L_{h_1} \otimes L_{h_2} \otimes L_{h_3}, L_{h_1h_2h_3})$ are related by

$${}_{h_1}b_{h_2h_3} \circ (id_{L_{h_1}} \otimes {}_{h_2}b_{h_3}) = \psi(h_1,h_2,h_3)^{-1}\, {}_{h_1h_2}b_{h_3} \circ ({}_{h_1}b_{h_2} \otimes id_{L_{h_3}})\,. \tag{3.28}$$

Thus the associativity of the multiplication implies that ω trivializes ψ on the support $H(A)$.
(ii) follows immediately from (i). ✓

Recall from remark 2.21(ii) that the effective center $\mathrm{Pic}^\circ(\mathcal{C})$ is, in general, not a subgroup of $\mathrm{Pic}(\mathcal{C})$, but only a subset. When, as in proposition 3.14(ii) above, we talk about a subgroup in $\mathrm{Pic}^\circ(\mathcal{C})$, we mean a sub*set* of $\mathrm{Pic}^\circ(\mathcal{C})$ that is closed under multiplication, and as a consequence is a subgroup of $\mathrm{Pic}(\mathcal{C})$.

3.4 Kreuzer–Schellekens bihomomorphisms

Different trivializations of the same subgroup H of the effective center that differ by an exact two-cochain give rise to isomorphic Schellekens algebras. In order to describe (and classify) isomorphism classes of Schellekens algebras we need at tool that allows us to take care of this. This tool is provided by the notion of a Kreuzer–Schellekens bihomomorphism that is introduced in the present subsection; it constitutes an analogue of the description of ordinary twisted group algebras of abelian groups in terms of alternating bihomomorphisms.

Definition 3.15:
For G be a finite abelian group, an *alternating* bihomomorphism on G is a bihomomorphism

$$\zeta: \quad G\times G \to \mathbb{C}^\times \tag{3.29}$$

such that $\zeta(g,g)=1$ for all $g\in G$.

Note that $\zeta(g,g)=1$ for all $g\in G$ implies that $\zeta(g,h)=\zeta(h,g)^{-1}$ for all $g,h\in G$, but the converse implication is not true. We also have [48, 49]

Lemma 3.16:
The alternating bihomomorphisms on a finite *abelian* group G form an abelian group $\mathrm{AB}(G,\mathbb{C}^\times)$. The map

$$\begin{array}{rcl} H^2(G,\mathbb{C}^\times) & \to & \mathrm{AB}(G,\mathbb{C}^\times) \\ [\omega] & \mapsto & \zeta \quad \text{with} \quad \zeta(g,h) := \frac{\omega(g,h)}{\omega(h,g)} \end{array} \tag{3.30}$$

furnishes an isomorphism of abelian groups.

For abelian groups, this fact provides a convenient characterization of isomorphism classes of twisted group algebras by their commutator two-cocycles as defined in (A.7). We would like to have an analogous characterization of Schellekens algebras in modular tensor categories. The trivializing two-cochain, however, is in general not closed, so we cannot use an alternating bihomomorphism. The appropriate generalization is provided by the following

Definition 3.17:
Let H be a subgroup in the effective center $\mathrm{Pic}^\circ(\mathcal{C})$ of a ribbon category. A *Kreuzer–Schellekens bihomomorphism* (or *KSB*, for short) on H is a (not necessarily symmetric) bihomomorphism

$$\Xi: \quad H\times H \to \mathbb{C}^\times \tag{3.31}$$

which on the diagonal coincides with the quadratic form δ introduced in (2.13),

$$\Xi(g,g) = \delta(g) \equiv \theta_g \qquad \text{for all} \;\; g\in H\,. \tag{3.32}$$

Lemma 3.18:
For any Kreuzer–Schellekens bihomomorphism on a subgroup H in $\mathrm{Pic}^\circ(\mathcal{C})$ we have

$$\Xi(g,h)\,\Xi(h,g) = \beta(g,h) \tag{3.33}$$

for all $g,h\in H$, with β the bihomomorphism (2.16) associated to δ.

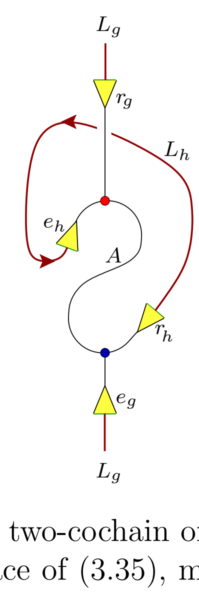
Proof:
Using first the definition of β, then the property (3.32) and then the bihomomorphism property of Ξ one finds

$$\begin{aligned}\beta(g,h) &= \delta(gh)\,\delta(g)^{-1}\delta(h)^{-1} \\ &= \Xi(gh,gh)\,\Xi(g,g)^{-1}\,\Xi(h,h)^{-1} = \Xi(g,h)\,\Xi(h,g)\end{aligned} \tag{3.34}$$

for all $g,h \in H$. ✓

Remark 3.19:
(i) Two KSBs on the same subgroup H in the effective center differ by an alternating bihomomorphism on H. Thus if there exist KSBs on H at all (which is indeed the case, see remark 3.21 below), then they form a torsor over $H^2(H,\mathbb{C}^\times)$. In particular, since $H^2(\mathbb{Z}_n,\mathbb{C}^\times)=1$, KSBs on any cyclic subgroup in the effective center are unique.
(ii) Following the terminology in the physics literature [50, 16] we also call the choice of a KSB for a given subgroup H a choice of "discrete torsion".
(iii) There also exists an alternative description of KSBs, due to Kreuzer and Schellekens [16], which makes use of a presentation of H as a direct product of cyclic groups. This is described in appendix B.

Now let A be a Schellekens algebra in $\mathcal{C}$ and fix an adapted basis for A. The following endomorphism of L_g is a multiple of the identity morphism:



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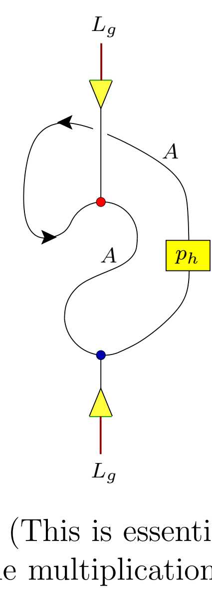
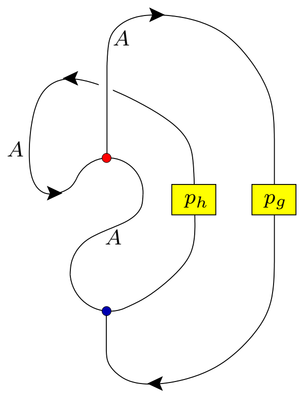


$$= \frac{\Xi_A(h,g)}{\dim(A)}\, id_{L_g}\,, \tag{3.35}$$

where Ξ_A is a two-cochain on $H(A)$. (This is essentially formula (9) of [1].) Note that by taking the trace of (3.35), moving the multiplication morphism past the comultiplication with the help of the Frobenius property, and slightly deforming the resulting ribbon graph, one arrives at the following alternative expression for Ξ_A:

$$\frac{\Xi_A(h,g)}{\dim(A)} =$$



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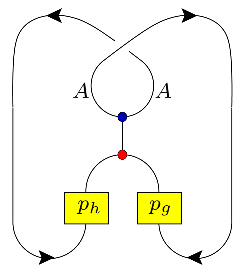

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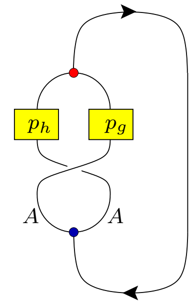


(3.36)

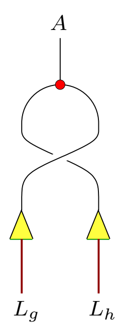
Thus we have

$$\Xi_A(g,h) = \dim(A)\,\mathrm{tr}\big(m \circ (p_g \otimes p_h) \circ c_{A,A} \circ \Delta\big)\,. \tag{3.37}$$

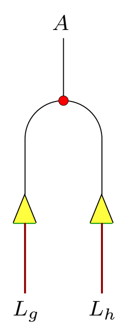
Using the expression (3.22) for the product and coproduct of A, it is easy to evaluate (3.37); we find

$$\Xi_A(g,h) = \dim(A)\, m_{g,h}^{\;\;\;\;gh}\, \Delta_{gh}^{h,g}\, \mathsf{R}^{(h\;g)\,gh} = \frac{1}{\Omega(g,h)}\,\frac{\omega(g,h)}{\omega(h,g)}\,. \tag{3.38}$$

Thus Ξ_A is the generalization to the braided setting of the concept of commutator cocycle ω_{comm}, which characterizes isomorphism classes of finite abelian groups, see lemma 3.16. Specifically, we have



$= \Xi_A(h,g)$



(3.39)

as follows immediately from the formulas (3.22) and (3.38).

Proposition 3.20:

(i) The two-cochain Ξ_A defined by (3.35) does not depend on the choice of adapted basis.

(ii) Isomorphic haploid special Frobenius algebras have identical two-cochains.

(iii) The two-cochain Ξ_A is a KSB on $H(A)$.

Proof:

(i) and (ii) are trivial to check.

(iii) The defining property (3.32) of a KSB follows from (3.38) when stetting $h = g$, together

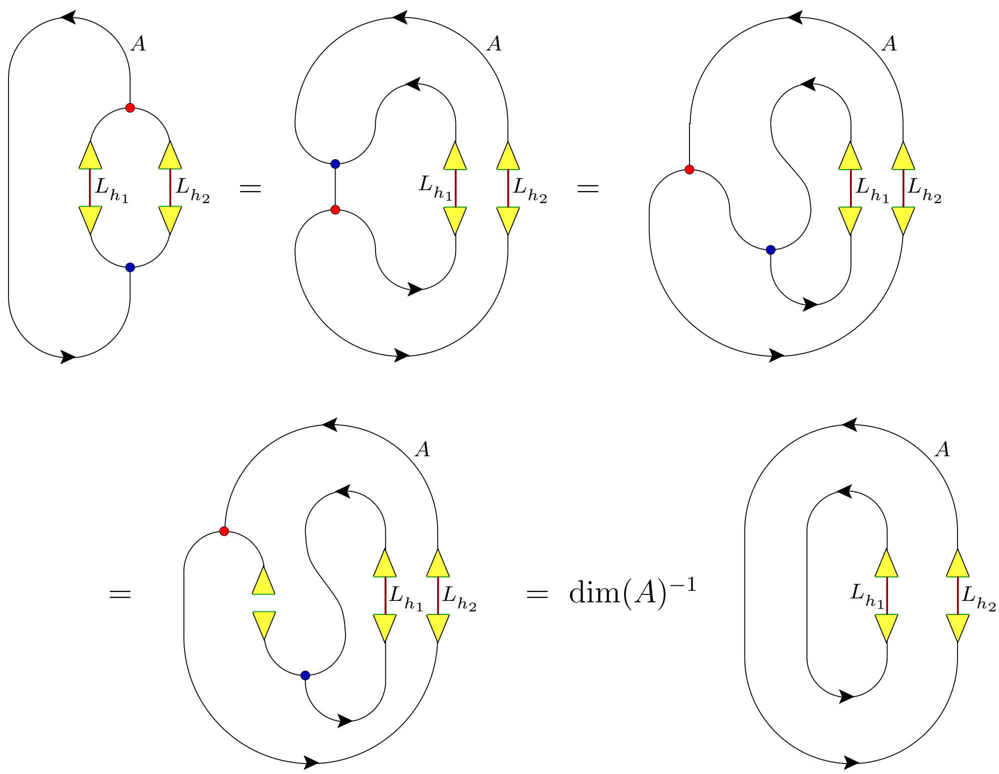
with (2.14). It remains to be shown that Ξ_A is a bihomomorphism.
We first show that for a Schellekens algebra, one has

$$m \circ (p_{h_1} \otimes p_{h_2}) \circ \Delta = \frac{1}{\dim(A)}\, p_{h_1 h_2} \tag{3.40}$$

for any two elements h_1 and h_2 in the support of A. To see this, we notice that the two morphisms live in the same one-dimensional subspace of $\mathrm{End}(A)$ and are thus proportional. We compute the constant of proportionality by taking the trace. For the right hand side, we find

$$\mathrm{tr}\left(\frac{1}{\dim(A)}\, p_{h_1 h_2}\right) = \frac{1}{\dim(A)}\; \dim(L_{h_1 h_2}) = \frac{1}{\dim(A)}\,. \tag{3.41}$$

The trace of the left hand side is computed graphically:



(3.42)

Here the second equality employs the fact that A is Frobenius. In the third identity it is used that in the middle A-ribbon only the tensor unit propagates. Using the relation (C.2) between e_1 and r_1 and the unit and counit morphisms one obtains the last identity. The last graph is equivalent to

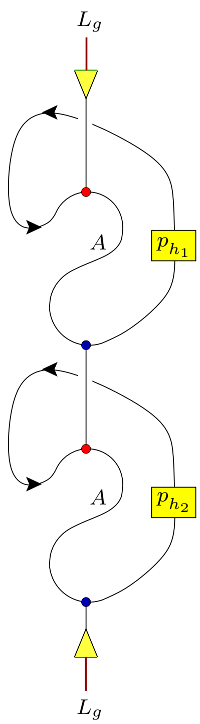
$$\frac{1}{\dim(A)}\; \dim(L_{h_1})\; \dim(L_{h_2}) = \frac{1}{\dim(A)}\,. \tag{3.43}$$

This shows (3.40).

We can now compute

$$\frac{\Xi_A(h_1,g)\,\Xi_A(h_2,g)}{(\dim(A))^2}\,\mathrm{id}_{L_g}$$

$=$



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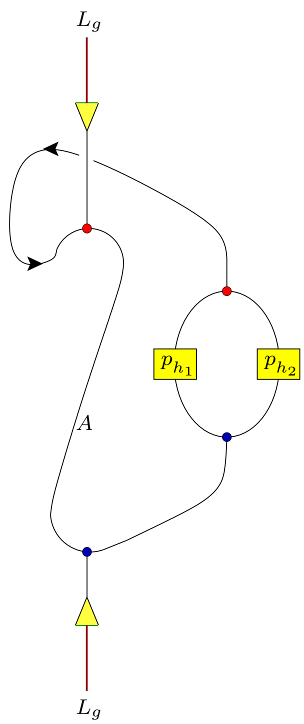
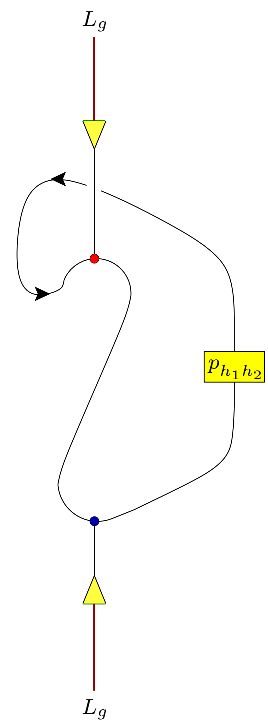


$= \dfrac{1}{\dim(A)}$



$$= \frac{\Xi_A(h_1h_2,g)}{(\dim(A))^2}\,, \qquad (3.44)$$

where for the second equality we used moves similar to those in (I:5.36); the third step amounts to (3.40). Thus Ξ_A is a homomorphism in the first argument.

To show that Ξ_A is also homomorphism in the second argument, note that from the explicit form (3.38), together with (2.8) and (2.17), it follows that

$$\Xi_A(g,h)\,\Xi_A(h,g) = \big(\Omega(g,h)\Omega(h,g)\big)^{-1} = \beta(g,h)\,. \qquad (3.45)$$

Thus $\Xi_A(g,h)=\beta(g,h)/\Xi_A(h,g)$. Since β is a bihomomorphism and $\Xi_A(h,g)$ a homomorphism in the first argument, it follows that $\Xi_A(g,h)$ is a homomorphism also in the second argument. ✓

Remark 3.21:
Part (iii) of the proposition implies in particular that there does exist a KSB on $H(A)$.

Proposition 3.22:
Two Schellekens algebras with identical support are isomorphic as Frobenius algebras if and only if they have the same KSB.

Proof:
That two isomorphic Schellekens algebras have the same KSB was already established in proposition 3.20. It remains to show the converse. We have seen in lemma 3.10 that two multiplications on A differ by a two-cocycle

$$\gamma: \quad H \times H \to \mathbb{C}^\times . \tag{3.46}$$

When the product is changed by γ, according to (3.38) the KSB changes by the commutator two-cocycle of γ,

$$\Xi_A(g,h) \;\mapsto\; \Xi_A(g,h)\,\frac{\gamma(g,h)}{\gamma(h,g)}\,. \tag{3.47}$$

Now the isomorphism classes of haploid special Frobenius algebras with $H(A)=H$ form a torsor over $H^2(H,\mathbb{C}^\times)$, while the KSBs form a torsor over the alternating bihomomorphisms. According to lemma 3.16 the two groups are isomorphic, $H^2(H,\mathbb{C}^\times)\cong \mathrm{AB}(H,\mathbb{C}^\times)$, and equation (3.30) shows that the map $[A]\mapsto \Xi_A$ is a morphism of torsors over the group. A morphism of torsors, in turn, is, as a map of sets, both surjective and injective. ✓

Remark 3.23:
From [I, II] we know that a symmetric special Frobenius algebra defines a full CFT on *oriented* world sheets with or without boundary, while to define a full CFT on *unoriented* and possibly non-orientable surfaces with or without boundary we need a *Jandl algebra*. The notion of a Jandl algebra was introduced in definition II:2.1; it is a symmetric special Frobenius algebra together with a morphism $\sigma\in \mathrm{Hom}(A,A)$ obeying

$$\sigma\circ\sigma = \theta_A \qquad \text{and} \qquad \sigma\circ m = m\circ c_{A,A}\circ(\sigma\otimes\sigma)\,. \tag{3.48}$$

A morphism σ with these properties was called a *reversion*.
Given a Schellekens algebra A, we can ask what reversions, if any, can be defined on A. To express (3.48) in a basis, we define $\sigma(h)\in\mathbb{C}$ for $h\in H(A)$ via $\sigma(h) id_{L_h} := r_h\circ\sigma\circ e_h$. Applying (II:2.21) we find that (3.48) is equivalent to having

$$\sigma(g)^2 = \theta_g \qquad \text{and} \qquad \sigma(gh)\,\omega(g,h) = \sigma(g)\,\sigma(h)\,\omega(h,g)\,\Omega(h,g)^{-1}\,, \tag{3.49}$$

for all $g,h\in H(A)$. Recalling also the expression (3.38) for the KSB, we can rewrite the second equality as

$$\frac{\sigma(gh)}{\sigma(g)\,\sigma(h)} = \Xi_A(h,g) = \Xi_A(g,h)\,, \tag{3.50}$$

where the second step follows from the fact that, by the first equality, the KSB is symmetric. This shows in particular that a reversion can only be defined for Schellekens algebras with *symmetric* KSB. By the bihomomorphism property of the KSB we have $\Xi_A(g,g^{-1})=\Xi_A(g,g)^{-1}=\theta_g^{-1}$, so that when evaluating condition (3.50) for $h=g^{-1}$ we get $\sigma(g)\,\sigma(g^{-1})=\theta_g$. Together with the first equality in (3.49) this implies that $\sigma(g)=\sigma(g^{-1})$, i.e. σ is a quadratic form on $H(A)$.

Given a Schellekens algebra A, the above considerations show that the following two statements are equivalent:

i) σ is a reversion on A;

ii) σ is a quadratic form on $H(A)$ such that its associated bihomomorphism equals the KSB of A, $\beta_\sigma = \Xi_A$.

Since by proposition 3.22 the KSB determines a Schellekens algebra up to isomorphism, a quadratic form $\sigma \in \mathrm{QF}(H, \mathbb{C}^\times)$ with $\sigma(g)^2 = \theta_g$ on some subgroup H in the effective center $\mathrm{Pic}^\circ(\mathcal{C})$ defines a Schellekens–Jandl algebra with support H up to isomorphism. Distinct quadratic forms $\sigma \in \mathrm{QF}(H, \mathbb{C}^\times)$ satisfying $\sigma(g)^2 = \theta_g$ give rise to non-isomorphic Schellekens–Jandl algebras on H. In summary, for a given subgroup $H \subset \mathrm{Pic}^\circ(\mathcal{C})$ in the effective center of the category $\mathcal{C}$ we have:

- The isomorphism classes of Schellekens algebras with support H form a torsor over $H^2(H, \mathbb{C}^\times)$.
- The isomorphism classes of Schellekens–Jandl algebras with support H are in bijection with the subset $\{\sigma \in \mathrm{QF}(H, \mathbb{C}^\times) \,|\, \sigma(h)^2 = \theta_h \text{ for all } h \in H\}$.
- Different Jandl-structures on one and the same Schellekens algebra A form a torsor over $H^1(H(A), \mathbb{Z}_2)$, i.e. differ by a character of the support that takes its values in $\{\pm 1\}$.

3.5 The modular invariant torus partition function

In [I] it has been shown that every haploid special Frobenius algebra provides a modular invariant torus partition function

$$Z = \sum_{i,j \in \mathcal{I}} Z_{ij}\, \mathcal{X}_i \otimes \overline{\mathcal{X}}_j \,, \qquad (3.51)$$

where the non-negative integers Z_{ij} are defined as the invariant of the ribbon graph in (I:5.30). As announced in [1] (and to be proven rigorously in a separate publication), every such modular invariant is physical in the sense that it is part of a consistent collection of correlation functions on oriented world sheets of arbitrary topology. Thus in particular every simple current modular invariant is physical. (Recall that our notion of simple current invariant is different from the one of [38, 45]; invariants of the latter type may be unphysical, compare remark 3.8(iii) above.)

To obtain explicit expressions for the integers Z_{ij}, the notion of a monodromy charge (see remark 2.15) must be extended to arbitrary simple objects. Note that when U is a simple object of $\mathcal{C}$, then so is $L_g \otimes U$ for any $g \in \mathrm{Pic}(\mathcal{C})$.

Definition 3.24:
Let U be a simple object in a modular tensor category.

(i) The *exponentiated monodromy charge* of U is the one-cochain χ_U on $\mathrm{Pic}(\mathcal{C})$ defined by

$$c_{U,L_g} \circ c_{L_g,U} =: \chi_U(g)\, id_{L_g \otimes U} \,. \qquad (3.52)$$

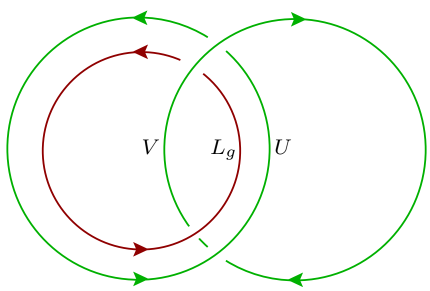
(ii) For any object X of $\mathcal{C}$ and any $g \in G$ we abbreviate the tensor product of L_g and X by gX:

$$gX := L_g \otimes X \,. \tag{3.53}$$

In this way, the Picard group $\mathrm{Pic}(\mathcal{C})$ acts on isomorphism classes of objects (but not on objects, in general).

Remark 3.25 :
Recall the description of the modular matrix S in terms of an invariant of the Hopf link in S^3, compare e.g. formula (I:2.22). It follows directly from the definition of χ_U that

$$s_{gU,V} \;=\; s_{V,gU} \;=$$



$$= \chi_V(g)\, s_{U,V} \,. \tag{3.54}$$

The importance of this relation is known since the early days of simple current theory [2,4].

Proposition 3.26 :
(i) The exponentiated monodromy charge χ_U is a one-cocycle.
Since $\mathrm{Pic}(\mathcal{C})$ is abelian, we can thus regard χ_U as an irreducible character on the group $\mathrm{Pic}(\mathcal{C})$:

$$\chi_U \in \mathrm{Pic}(\mathcal{C})^* \,. \tag{3.55}$$

(ii) The exponentiated monodromy charge can be expressed in terms of the twists as

$$\chi_{U_i}(g) = \theta_{gi}\, \theta_g^{-1} \theta_i^{-1} \,. \tag{3.56}$$

It obeys $\chi_{U_i}(g) = \chi_{U_{\bar\imath}}(g^{-1})$ for all $i \in \mathcal{I}$ and $g \in \mathcal{P}ic(\mathcal{C})$.

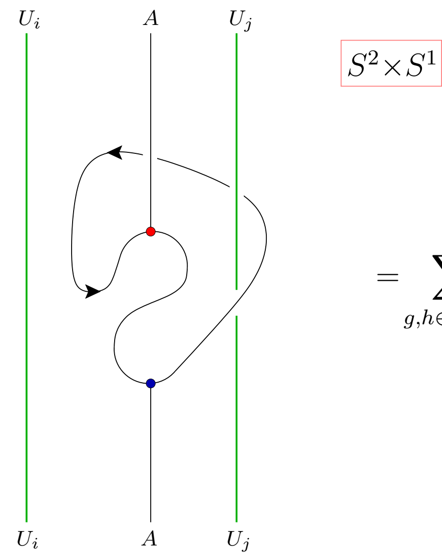
Proof:
(i) The statement is a direct consequence of the definition, applied for $g = g_1 g_2$, and tensoriality of the braiding, together with the fact that $L_{g_1 g_2}$ is isomorphic to $L_{g_1} \otimes L_{g_2}$. (At intermediate steps an isomorphism between $L_{g_1 g_2}$ and $L_{g_1} \otimes L_{g_2}$ must be chosen, but the result does not depend on that choice.)
(ii) The first statement is a consequence of the compatibility condition (2.19) between braiding and twist, applied to the simple objects L_g and U. That $\chi_{U_i}(g) = \chi_{U_{\bar\imath}}(g^{-1})$ then follows from the fact that dual objects have identical balancing phases, according to which $\theta_{g^{-1}\bar\imath} = \theta_{gi}$ (because $(U_{g^{-1}\bar\imath})^\vee \cong U_{gi}$), $\theta_i = \theta_{\bar\imath}$ and $\theta_g = \theta_{g^{-1}}$. ✓

We are now in a position to express the modular invariant partition function given in (I:5.30) more explicitly. When writing A as a direct sum,

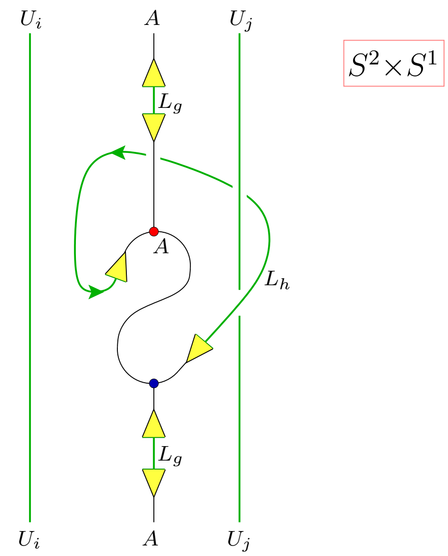
$$A \;\cong\; \bigoplus_{g\in H(A)} L_g \,, \tag{3.57}$$

we obtain (the ribbon graphs below are embedded in $S^2\times S^1$, with the vertical direction being the S^1, so that top and bottom are identitfied)

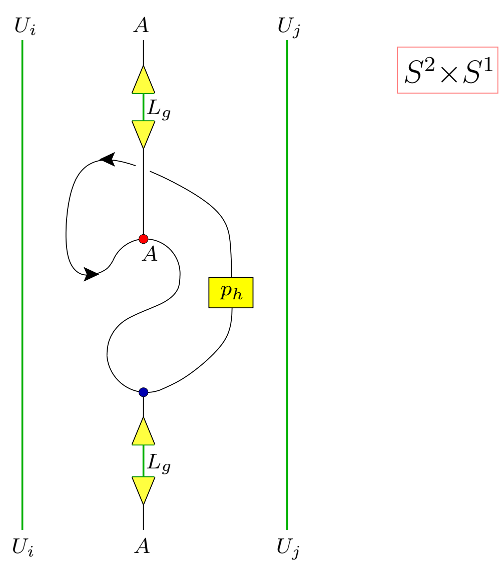
$Z_{ij}\;=$



$= \displaystyle\sum_{g,h\in H(A)}$



$= \displaystyle\sum_{g,h\in H(A)} \chi_{U_j}(h)$



$$= \frac{1}{\dim(A)} \sum_{g,h\in H(A)} \chi_{U_j}(h)\, \Xi_A(h,g)\, \dim_{\mathbb{C}}\mathrm{Hom}(U_i{\otimes}\, L_g{\otimes}\, U_j, \mathbf{1})$$

$$= \frac{1}{\dim(A)} \sum_{g,h\in H(A)} \chi_{U_j}(h)\, \Xi_A(h,g)\, \delta_{\bar\jmath,gi} \,. \tag{3.58}$$

We have thus arrived at the

Theorem 3.27:
The modular invariant torus partition function for a Schellekens algebra A is given by

$$Z_{ij}(A) = \tfrac{1}{|H(A)|} \sum_{g,h\in H(A)} \chi_{U_i}(h)\, \Xi_A(h,g)\, \delta_{\bar{\jmath},gi}\,. \tag{3.59}$$

Note that this is the charge conjugate of the partition function given in [16].

Remark 3.28:
It can happen that non-isomorphic Schellekens algebras are Morita equivalent and thus give equivalent full conformal field theories. In particular, they can give rise to identical torus partition functions. A simple example appears in the Ising model: the haploid special Frobenius algebra $A \cong \mathbf{1} \oplus \epsilon$ built from the primary field ϵ of conformal weight 1/2 is Morita equivalent to the trivial algebra, since $A = \sigma^\vee \otimes \sigma$ with σ the primary field of conformal weight 1/16.

For an algebra in a braided tensor category, there are the notions of a left and of a right center [51, 24, 40]. The following statements already appear in [16], but we rephrase them in our terms.

Proposition 3.29:
Let A be a Schellekens algebra with support $H \subset \mathrm{Pic}^\circ(\mathcal{C})$ and KSB Ξ_A. Consider the two subgroups

$$\begin{aligned} K_r(A) &:= \{g \in H \,|\, \Xi_A(\,\cdot\,,g) \text{ is the trivial } H\text{-character}\}\,, \\ K_l(A) &:= \{g \in H \,|\, \Xi_A(g,\,\cdot\,) \text{ is the trivial } H\text{-character}\} \end{aligned} \tag{3.60}$$

of H. Then we have:

(i) The left and right center of A are, respectively, the commutative symmetric Frobenius subalgebras of A that are defined on the subobjects

$$C_l(A) := \bigoplus_{g\in K_l(A)} L_g \qquad \text{and} \qquad C_r(A) := \bigoplus_{g\in K_r(A)} L_g\,. \tag{3.61}$$

(ii) A is commutative if and only if $\Xi_A \equiv 1$.

Proof:
(i) By proposition I:5.9, the left and the right center are the subalgebras on the objects

$$C_l(A) = \bigoplus_i Z_{i0}\, U_i \qquad \text{and} \qquad C_r(A) = \bigoplus_j Z_{0j}\, U_j\,. \tag{3.62}$$

Due to the form of the modular invariant partition function, only invertible objects can appear. Moreover, from the forms (3.59) and (3.58) of the partition function one computes that

$$Z_{0g} = \frac{1}{|H|} \sum_{h\in H} \Xi_A(h,g) \qquad \text{and} \qquad Z_{g0} = \frac{1}{|H|} \sum_{h\in H} \Xi_A(g,h)\,, \tag{3.63}$$

respectively. Invoking also the orthogonality of characters, this proves the statement.
(ii) An algebra is commutative iff it coincides with both its left and its right center. ✓

Corollary 3.30 :
The structure of a commutative Schellekens algebra can be defined precisely on those subgroups in $\mathrm{Pic}^\circ(\mathcal{C})$ on which both the monodromy charge and the twist are identically one. Furthermore, any two commutative Schellekens algebras A and A' with $H(A) = H(A')$ are isomorphic.

Proof:
The claim follows from the defining properties (3.32) of a KSB. Uniqueness holds because the KSB determines the algebra up to isomorphism. ✓

Remark 3.31 :
(i) In the physics literature, integer spin simple currents (i.e. simple currents L_g with $\theta_g = 1$) such that $\beta \equiv 1$ are called *mutually local*. Thus the structure of a commutative algebra can only be defined on mutually local integer spin simple currents.
(ii) If $\mathcal{C}$ is the category of representations of a rational vertex algebra $\mathfrak{A}$, the results of [52] (see also theorem 4.3 in [53]) imply that there exists an extension of $\mathfrak{A}$ as a vertex algebra whose representation category is equivalent to the category of *local* A-modules in $\mathcal{C}$ for a suitable simple symmetric special Frobenius algebra A in $\mathcal{C}$. (For the definition of the qualification 'local', see [54, 55] and definition 3.15 of [40].)
(iii) On the level of partition functions, the relation between mutually local simple currents and extensions is well-known since the early days of simple current theory [2] (see also [4] for the case of cyclic groups).

4 Modules and boundary conditions

In this section we develop the theory of left modules over a Schellekens algebra in a modular tensor category $\mathcal{C}$. Our main motivation to study the representation theory of Schellekens algebras is the fact that modules correspond to conformally invariant boundary conditions that respect the chiral symmetry encoded in $\mathcal{C}$. This allows us in particular to prove the formula for boundary states that has been presented in [17] and that summarizes and generalizes earlier work [56,57,58,59]. In the case of a commutative Schellekens algebra A, there is an additional reason for the study of modules: one can show that the process of forming the subcategory of local left modules over a commutative Schellekens algebra – in physical terms, a simple current extension [2] – precisely corresponds to the modularisation procedure of [60,61] and thereby provides a representation theoretic origin for the S-matrix formula for simple current extensions that was obtained in [62].

4.1 Stabilizers and basis independent 6j-symbols

We start with the

Definition 4.1:

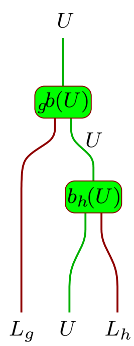
Let U be an object in $\mathcal{C}$.

(i) The stabilizer of the action of $\mathrm{Pic}(\mathcal{C})$ on the isomorphism class of U is denoted by $\mathcal{S}(U)$. Following the physics literature, we say that U is a *fixed point* of the simple current L_g iff $g \in \mathcal{S}(U)$.

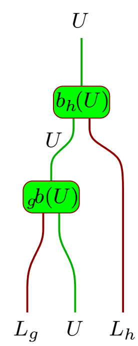
(ii) Given a Schellekens algebra A, we denote by $\mathcal{S}_A(U)$ the intersection of the stabilizer with the support of A, $\mathcal{S}_A(U) := \mathcal{S}(U) \cap H(A)$.

(iii) Let the object U be simple. For every $g \in \mathcal{S}(U)$ we fix bases ${}_gb(U) \in \mathrm{Hom}(L_g \otimes U, U)$ and $b_g(U) \in \mathrm{Hom}(U \otimes L_g, U)$. For $g = e$, we take (using strictness) ${}_eb(U) = b_e(U) = id_U$. We denote by $\overline{{}_gb(U)} \in \mathrm{Hom}(U, L_g \otimes U)$ the morphism dual to ${}_gb(U)$ (which by definition satisfies $b_g(U) \circ \overline{{}_gb(U)} = id_U$, compare equation (I:2.30)).

(iv) Let U be simple. The two-cochain ϕ_U on $\mathcal{S}(U)$ is defined by



$=: \; \phi_U(g,h)$



(4.1)

Once we select the morphisms ${}_gb(U)$ and $b_g(U)$ as bases for the respective morphism spaces, the relation (4.1) is just the definition of an F-coefficient, namely $\phi_U(g,h) = \mathsf{F}^{(g\,U\,h)\,U}_{U\,U}$, compare formula (I:2.36).

Proposition 4.2:

Let U be a simple object of $\mathcal{C}$.

(i) The stabilizer $\mathcal{S}(U)$ is a subgroup in $\mathrm{Pic}^\circ(\mathcal{C})$. One has $\theta_g \in \{\pm 1\}$ for all $g \in \mathcal{S}(U)$. The union of the stabilizers of all simple objects of $\mathcal{C}$ is a subgroup in the effective center $\mathrm{Pic}^\circ(\mathcal{C})$.

(ii) The two-cochain $\phi_U(g,h)$ does not depend on the choice of the bases b. It is a bihomomorphism on $\mathcal{S}(U)$.

Proof:

We start by showing that $\theta_g^2 = 1$ for every $g \in \mathcal{S}(U)$. For $g \in \mathcal{S}(U)$ the equality (3.56) simplifies to $\chi_U(g) = \theta_g^{-1}$. On the other hand,

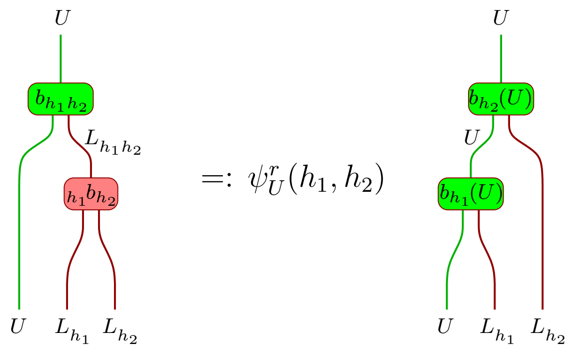
$$\chi_U(g)^{-1} = \chi_U(g^{-1}) = \theta_{g^{-1}}^{-1} = \theta_g^{-1} \,, \qquad (4.2)$$

where we used the fact that χ_U is a character and that dual objects have the same twist. Multiplying the last two equations shows $(\theta_g)^2 = 1$. In other words, only half-integer spin simple currents can have fixed points.

Next we establish that $\mathcal{S}(U)$ is a subgroup in $\mathrm{Pic}^\circ(\mathcal{C})$. Clearly, $\mathcal{S}(U)$ is a subgroup of the Picard group $\mathrm{Pic}(\mathcal{C})$. To see that it is also a subset of $\mathrm{Pic}^\circ(\mathcal{C})$, by corollary 2.18 it is sufficient to show that $(\theta_g)^{N_g} = 1$ for all $g \in \mathcal{S}(U)$, where N_g is the order of g. If N_g is even, then this equality follows from $\theta_g^2 = 1$. For N_g odd we just invoke remark 2.21(iii), according to which all simple currents of odd order are in $\mathrm{Pic}^\circ(\mathcal{C})$.

(ii) That $\phi_U(g,h)$ is independent of the choice of basis holds because all morphism spaces in question are one-dimensional so that one can only modify the basis elements by non-zero constants. Since the same basis elements ${}_g b(U)$ and $b_h(U)$ appear on both sides of (4.1), these constants cancel from the definition of $\phi_U(g,h)$.

To see that $\phi_U(g,h)$ is a bihomomorphism, introduce (basis dependent) elements of the fusing matrix by



(4.3)

(where on the left side the short-hand $b_{h_1h_2}$ for $b_{h_1h_2}(U)$ is used). Similarly we define constants $\psi^l_U(g_1,g_2)$ via

$$ {}_{g_1}b(U) \circ \big(id_{L_{g_1}} \otimes {}_{g_2}b(U) \big) = \psi^l_U(g_1,g_2)\, {}_{g_1g_2}b(U) \circ \big({}_{g_1}b_{g_2} \otimes id_U \big) \,. \tag{4.4} $$

In terms of F-matrix entries this amounts to $\psi^r_U(g,h) = \mathsf{F}^{\,(U\,g\,h)\,U}_{\;gh,\,U}$ and $\psi^l_U(g,h) = \mathsf{F}^{\,(g\,h\,U)\,U}_{\;U,\,gh}$. Applying the pentagon identity (see appendix C.2) to the tensor product $L_g \otimes U \otimes L_{h_1} \otimes L_{h_2}$ gives

$$ \psi^r_U(h_1,h_2)\, \phi_U(g,h_2)\, \phi_U(g,h_1) = \phi_U(g,h_1h_2)\, \psi^r_U(h_1,h_2) \,, \tag{4.5} $$

from which it follows that ϕ_U is a homomorphism in the second argument. Similarly, evaluating the pentagon for $L_{g_1} \otimes L_{g_2} \otimes U \otimes L_h$ results in

$$ \phi_U(g_2,h)\, \phi_U(g_1,h)\, \psi^l_U(g_1,g_2) = \psi^l_U(g_1,g_2)\, \phi_U(g_1g_2,h) \,, \tag{4.6} $$

thus also establishing the homomorphism property in the first argument. ✓

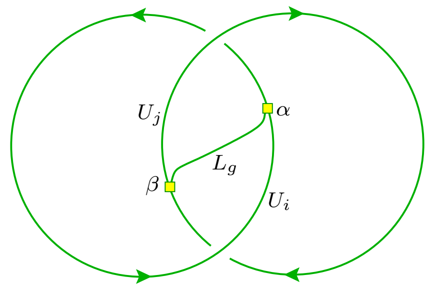
An important aspect of the bihomomorphism ϕ_U is that it can be computed from matrices $\mathcal{S}^g$ that are related to modular transformations (and for which an explicit formula has been conjectured, see remark 4.5 below). In the sequel we will discuss some properties of these matrices.

Definition 4.3:
Let $g \in \mathrm{Pic}(\mathcal{C})$ and U_i, U_j be simple objects of $\mathcal{C}$. Then $\mathcal{S}^g_{i,j}$ is the bilinear pairing

$$\mathcal{S}^g_{i,j}: \quad \mathrm{Hom}(L_g \otimes U_i, U_i) \otimes_{\mathbb{C}} \mathrm{Hom}(U_j, U_j \otimes L_g) \rightarrow \mathbb{C} \tag{4.7}$$

with

$$\mathcal{S}^g_{i,j}(\alpha \otimes \beta) := \tag{4.8}$$



for $\alpha \in \mathrm{Hom}(L_g \otimes U_i, U_i)$ and $\beta \in \mathrm{Hom}(U_j, U_j \otimes L_g)$.

Proposition 4.4:
Let U_i be a simple object of a modular tensor category $\mathcal{C}$ and let $g, h \in \mathcal{S}(U_i)$.
(i) There exists a $j \in \mathcal{I}$ such that $g \in \mathcal{S}(U_j)$ and $\mathcal{S}^g_{i,j} \neq 0$.
(ii) Let j be any element of $\mathcal{I}$ such that $\mathcal{S}^g_{i,j} \neq 0$. Then

$$\phi_{U_i}(g, h) = \chi_{U_j}(h)^{-1}. \tag{4.9}$$

(iii) We have

$$\phi_{U_i}(g, g) = \theta_g \in \{\pm 1\}. \tag{4.10}$$

Thus ϕ_{U_i} is a KSB on $\mathcal{S}(U_i)$. In particular, if for all $g \in \mathcal{S}(U_i)$ the twist of L_g is the identity, then ϕ_{U_i} is an alternating bihomomorphism.
(iv) We have

$$\phi_{U_i}(g, h)\, \phi_{U_i}(h, g) = 1 \qquad \text{and} \qquad \phi_{U_i}(g, h) = \phi_{U_{\bar{\imath}}}(h, g). \tag{4.11}$$

Proof:
(i) Let F_g be the subset of $\mathcal{I}$ consisting of all $k \in \mathcal{I}$ such that $g \in \mathcal{S}(U_k)$. Consider the $|F_g| \times |F_g|$-matrix M with entries

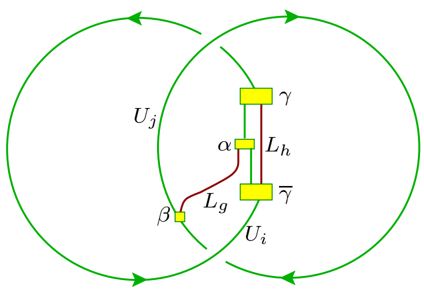
$$\mathrm{M}_{ij} := \mathcal{S}^g_{i,j}(\,{}_g b(U_i) \otimes \overline{b_g(U_j)}\,) \tag{4.12}$$

for $i, j \in F_g$. This matrix is invertible. In fact, it is related to a change of basis in the space $\mathcal{H}(L_g, \mathrm{T})$ of conformal blocks for a torus with one L_g-insertion. We will not present the details of the proof, but just describe the two bases.

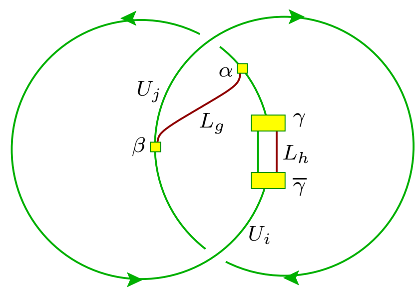
The elements of the first basis are $|\chi_m; L_g, \mathrm{T}\rangle$ for $m \in F_g$. The vector $|\chi_m; L_g, \mathrm{T}\rangle$ is given by a solid torus with an inscribed ribbon labeled by U_m similar to (I:5.15), together with an

additional ribbon labeled by L_g that connects the boundary of the solid torus to the U_m-ribbon. The U_m- and L_g-ribbons are joined by the morphism ${}_g b(U_m) \in \mathrm{Hom}(L_g \otimes U_m, U_m)$. Suppose that in the solid torus defining $|\chi_m; L_g, \mathrm{T}\rangle$, the a-cycle of the torus T is contractible. Then the second basis is of the same form as the first, but this time the solid torus is chosen such that the b-cycle of T is contractible.
Since the matrix M is invertible, for any $g \in \mathcal{S}(U_i)$ there exists a $j \in F_g$ such that $\mathrm{M}_{ij} \neq 0$. Thus for each $g \in \mathcal{S}(U_i)$ we can find a $j \in \mathcal{I}$ such that $\mathcal{S}^g_{i,j} \neq 0$.

(ii) Let $\gamma \in \mathrm{Hom}(U_i, U_i \otimes L_h)$ and $\overline{\gamma} \in \mathrm{Hom}(U_i \otimes L_h, U_i)$ be any two dual isomorphisms. Consider the move

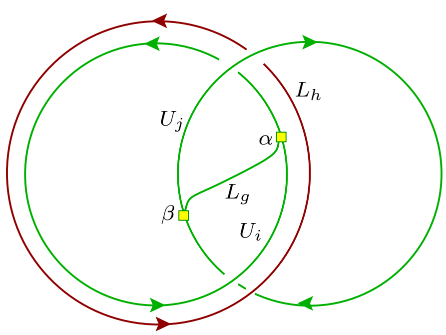


$$= \frac{1}{\phi_{U_i}(g,h)}$$

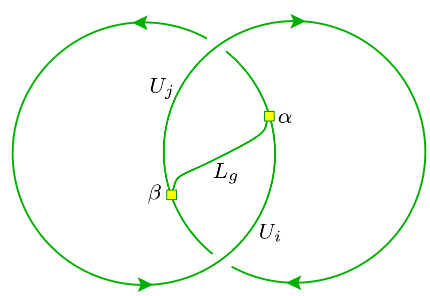


(4.13)

which makes use of the (basis independent) 6j-symbol defined in (4.1). Next use the fact that γ and $\overline{\gamma}$ are dual morphisms to rewrite the left hand side of (4.13) as



$$= \chi_{U_j}(h)$$



(4.14)

Here we made use of the defining property (3.52) of the exponentiated monodromy charge. The equality of (4.13) and (4.14) now amounts to $\phi_{U_i}(g,h)^{-1}\, \mathcal{S}^g_{i,j}(\alpha\otimes\beta) = \chi_{U_j}(h)\, \mathcal{S}^g_{i,j}(\alpha\otimes\beta)$. By assumption the index $j \in \mathcal{I}$ has the property that $\mathcal{S}^g_{i,j}(\alpha\otimes\beta) \neq 0$ for non-zero α and β; this establishes the claim.

(iii) By (i) we can find $j \in \mathcal{I}$ such that $\mathcal{S}^g_{i,j} \neq 0$. By (ii) we then have $\phi_{U_i}(g,g) = \chi_{U_j}(g)$. Further, since $g \in \mathcal{S}(U_j)$, the expression (3.56) for the exponentiated monodromy charge reads $\chi_{U_j}(g) = \theta_g^{-1}$. Finally, θ_g takes values only in $\{\pm 1\}$, by proposition 4.2(i). That ϕ_{U_i} is a KSB then follows by proposition 4.2(ii).

(iv) By lemma 3.18 the KSB ϕ_{U_i} satisfies $\phi_{U_i}(g,h)\,\phi_{U_i}(h,g) = \beta(g,h) = \theta_{gh}/(\theta_g\theta_h)$ for all $g,h \in \mathcal{S}(U_i)$. By proposition 3.26, the monodromy charge χ_{U_i} is a character of $\mathrm{Pic}(\mathcal{C})$. By (3.56), for $g,h \in \mathcal{S}(U_i)$ the property $\chi_{U_i}(gh) = \chi_{U_i}(g)\chi_{U_i}(h)$ of the character χ_{U_i} implies $\theta_{gh} = \theta_g\theta_h$.

Further, the definiton of ϕ_U in (4.1) implies that, for simple U,

$$ {}_g b(U) \circ \big(id_{L_g} \otimes b_h(U) \big) \circ \big(\overline{{}_g b(U)} \otimes id_{L_h} \big) \circ \overline{b_h(U)} = \phi_U(g,h) \, id_U \, . \tag{4.15} $$

Taking duals of both sides and invokng basis independence of the morphisms ϕ_U then implies $\phi_{U^\vee}(h^{-1}, g^{-1}) = \phi_U(g,h)$. Setting now $U = U_i$ and using the bihomomorphism property of $\phi_{U_{\bar\imath}}$ results in $\phi_{U_i}(g,h) = \phi_{U_{\bar\imath}}(h,g)$. ✓

Remark 4.5 :
(i) It is conjectured in [62] that for modular tensor categories arising from WZW conformal field theories, the bilinear pairing (4.8) can be computed using the theory of twining characters [63, 64]. For the case of simple current extensions of WZW theories, see [65]. The relevant numerical data have been implemented in a computer program `kac` by Bert Schellekens [66].
(ii) The bihomomorphisms ϕ_U were originally introduced [62] through the identity (4.9). Implicit in the discussion in [62] is the use of a distinguished class of bases of the morphism spaces $\mathrm{Hom}(L_g \otimes U, U)$; for a discussion of those bases see [67] and the considerations preceding definition 4.11 below.
(iii) By a calculation similar to (II:3.107), one computes the invariant for (4.8) for the basis morphisms (as in (4.12)) to be

$$ \mathcal{S}^g_{i,j}\big({}_g b(U_i) \otimes \overline{b_g(U_j)} \,\big) = \sum_{m \in \mathcal{I}} \sum_{\alpha=1}^{N_{ij}^{\ m}} \frac{\theta_m}{\theta_i \theta_j} \, \mathsf{F}^{\,(j\,g\,i)\,m}_{\,\alpha i,\,j\alpha} \, \dim(U_m) \, . \tag{4.16} $$

Suppose now that there exists a choice of basis in the spaces $\mathrm{Hom}(U_i \otimes U_j, U_k)$ such that all F-matrix entries are real numbers. This is e.g. possible in all Virasoro minimal models (including the non-unitary ones) and in rational unitary WZW models.
Using in addition that in a modular tensor category the twists are phases (compare footnote 3), the complex conjugate of (4.16) is found to be

$$ \mathcal{S}^g_{i,j}({}_g b(U_i) \otimes \overline{b_g(U_j)})^* = \widetilde{\mathcal{S}}^g_{i,j}({}_g b(U_i) \otimes \overline{b_g(U_j)}) \, , \tag{4.17} $$

where $\widetilde{\mathcal{S}}^g_{i,j}$ is defined analogously as (4.8), but with the two braidings replaced by inverse braidings. By rotating the U_j-ribbon in (4.8) by 180° about a horizontal axis, one then verifies that

$$ \widetilde{\mathcal{S}}^g_{i,j}(\alpha \otimes \beta) = \mathcal{S}^g_{i,\bar\jmath}(\alpha \otimes \tilde\beta) \tag{4.18} $$

for a suitable morphism $\tilde\beta \in \mathrm{Hom}(U_{\bar\jmath}, U_{\bar\jmath} \otimes L_g)$.
Combining the formulas (4.17) and (4.18) we conclude that if $\mathcal{S}^g_{i,j}$ is non-zero, then so is $\mathcal{S}^g_{i,\bar\jmath}$. For a given $i \in \mathcal{I}$ and $g \in \mathcal{S}(U_i)$ now choose $j \in \mathcal{I}$ such that $\mathcal{S}^g_{i,j}$ is non-vanishing, as is possible according proposition 4.4(i). Thus also $\mathcal{S}^g_{i,\bar\jmath}$ is non-zero, and proposition 4.4(ii) then implies

$$ \phi_{U_i}(g,h) = \chi_{U_j}(h)^{-1} = \chi_{U_{\bar\jmath}}(h) = \phi_{U_i}(g,h^{-1}) \, , \tag{4.19} $$

where the second step uses proposition 3.26(ii). Thus $\phi_{U_i}(g,h)^2 = 1$, and together with proposition 4.4(iv) we find that if all F-matrix entries of $\mathcal{C}$ can be chosen to be real, then

$$\phi_{U_i}(g,h) = \phi_{U_i}(h,g) \in \{\pm 1\} \,. \tag{4.20}$$

4.2 Representation theory of Schellekens algebras

With the help of the basis independent 6j-symbols we can now develop the representation theory of Schellekens algebras. For a symmetric special Frobenius algebra in a modular tensor category, every module appears a submodule of some induced module. Thus we first have a look at induced modules.

Lemma 4.6:
Let A be a Schellekens algebra with support H and KSB Ξ_A.
(i) The induced A-modules $\mathrm{Ind}_A(U)$ and $\mathrm{Ind}_A(V)$ for two simple objects U, V are isomorphic iff U and V are on the same orbit of the action of H on isomorphism classes of objects of $\mathcal{C}$. If they are not on the same orbit, then they do not contain isomorphic submodules.
(ii) As an object of $\mathcal{C}$, the induced module of a simple object U decomposes as

$$\mathrm{Ind}_A(U) \cong |\mathcal{S}_A(U)| \bigoplus_{[g]\in H/\mathcal{S}_A(U)} gU \,. \tag{4.21}$$

Proof:

(i) The statements immediately follow from reciprocity (see e.g. proposition I:4.12)

$$\dim_{\mathbb{C}} \mathrm{Hom}_A(\mathrm{Ind}_A(U), \mathrm{Ind}_A(V)) = \sum_{g\in H} \dim_{\mathbb{C}} \mathrm{Hom}(U, gV) \,. \tag{4.22}$$

(ii) We have

$$A \otimes U \;\cong\; \bigoplus_{h\in H} hU \;\cong\; \bigoplus_{[g]\in H/\mathcal{S}_A(U)} \; \bigoplus_{h\in\mathcal{S}_A(U)} ghU. \tag{4.23}$$

The h-summation amounts to an overall multiplicity $|\mathcal{S}_A(U)|$. ✓

The induced module for an object U is thus, in physics terminology, the corresponding *simple current orbit* with a multiplicity given by the order of the stabilizer $\mathcal{S}_A(U)$. This does not, however, imply, that the induced module decomposes into $|\mathcal{S}_A(U)|$ many irreducible modules. The remaining part of the representation theory is then to decompose the induced modules into simple modules. In physics terminology, this step is known as *fixed point resolution*. The essential information is contained in

Proposition 4.7:
Let $U \in \mathcal{O}bj(\mathcal{C})$ be simple. The algebra of module endomorphisms of the induced module $\mathrm{Ind}_A(U)$ is a twisted group algebra over the stabilizer $\mathcal{S}_A(U)$,

$$\mathrm{End}_A(\mathrm{Ind}_A(U)) \cong \mathbb{C}_{\beta_{U,A}} \mathcal{S}_A(U)\,, \tag{4.24}$$

for some two-cocycle $\beta_{U,A}$. The cohomology class $[\beta_{U,A}] \in H^2(\mathcal{S}_A(U), \mathbb{C}^\times)$ is described by the alternating bihomomorphism

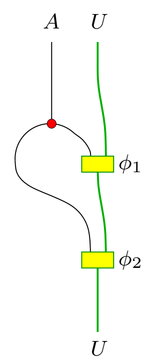
$$\varepsilon_{U,A}(g,h) := \frac{\beta_{U,A}(g,h)}{\beta_{U,A}(h,g)} = \phi_U(g,h)\,\Xi_A(h,g)\,, \tag{4.25}$$

with Ξ_A as defined in (3.35).

Proof:
By reciprocity, the algebra $\mathrm{End}_A(\mathrm{Ind}_A(U))$ is isomorphic as a vector space to a morphism space in $\mathcal{C}$:

$$\mathrm{End}_A(\mathrm{Ind}_A(U)) \cong \mathrm{Hom}(U, A \otimes U)\,. \tag{4.26}$$

The associative product on the latter space inherited from the product on $\mathrm{End}_A(\mathrm{Ind}_A(U))$ is as follows: for $\phi_1, \phi_2 \in \mathrm{Hom}(U, A \otimes U)$ we have

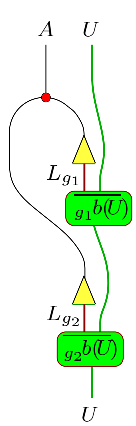
$\phi_1 \star \phi_2 \ =$ 

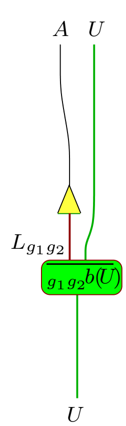
(4.27)

Note that this product depends on the product on the Frobenius algebra A. Together with the morphism e_g of the adapted basis we get the basis

$$\phi_g := (e_g \otimes id_U) \circ \overline{{}_g b(U)}\,. \tag{4.28}$$

for the algebra. From the form



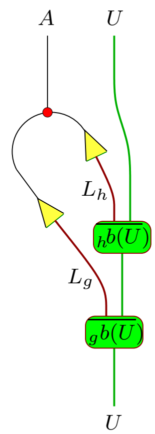
$= \ \beta_{U,A}(g_1, g_2)$


(4.29)

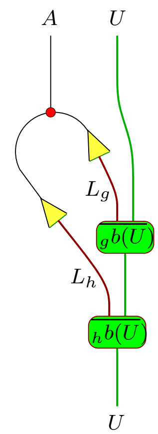
of the product we see that the algebra is graded by the stabilizer $\mathcal{S}_A(U)$: for $g_1, g_2 \in \mathcal{S}(U)$, we have

$$\phi_{g_1} \star \phi_{g_2} = \beta_{U,A}(g_1, g_2)\, \phi_{g_1 g_2} \tag{4.30}$$

for some two-cochain $\beta_{U,A}$ on $\mathcal{S}_A(U)$. The multiplication on the $\mathbb{C}$-algebra $\mathrm{End}_A(\mathrm{Ind}_A(U))$ is associative, so this cochain is closed. A change of basis will not change its cohomology class, which is the object we are interested in. We characterize it by computing the commutator cocycle $\varepsilon_{U,A}$ of $\beta_{U,A}$, which appears in

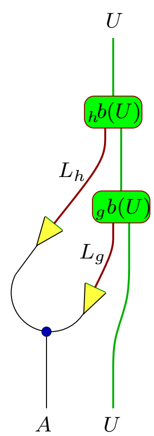


$= \varepsilon_{U,A}(h, g)$



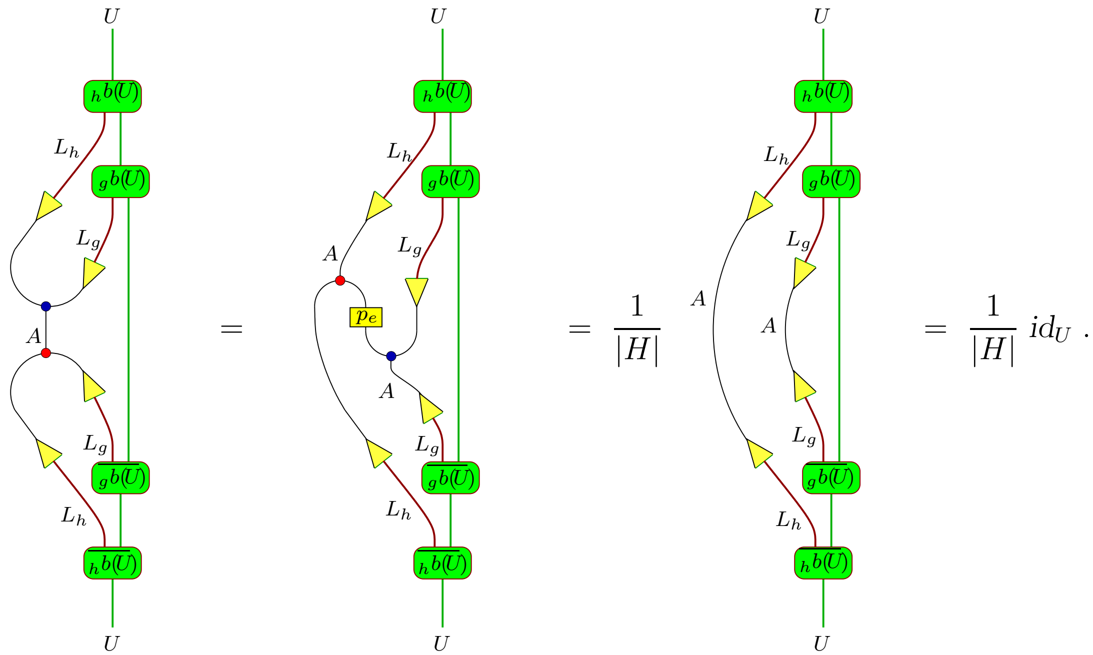
(4.31)

To compute $\varepsilon_{U,A}(h, g)$, we compose both sides of (4.31) with the morphism



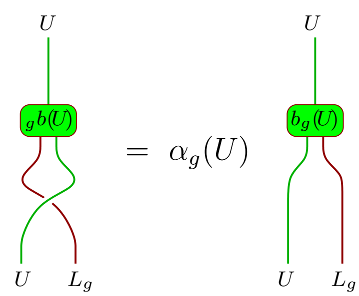
(4.32)

Then on the right hand side, after using the Frobenius property, only the tensor unit can propagate in the middle A-ribbon, so that we obtain a multiple of id_U:



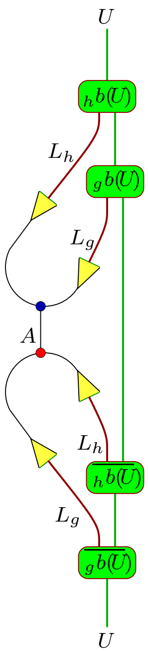
(4.33)

To compute the left hand side, we note that, since all coupling spaces involved are one-dimensional, there exist non-zero scalars $\alpha_g(U)$ such that

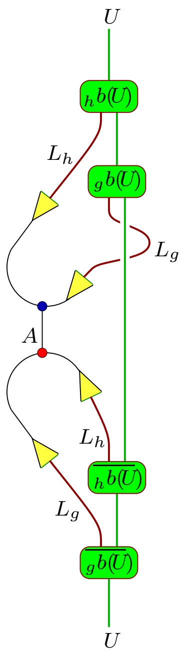


(4.34)

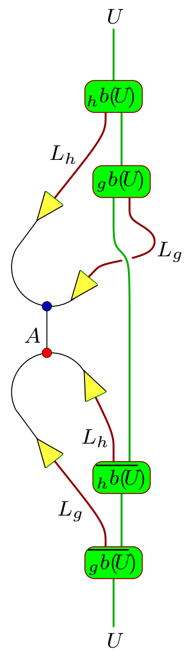
We now compute



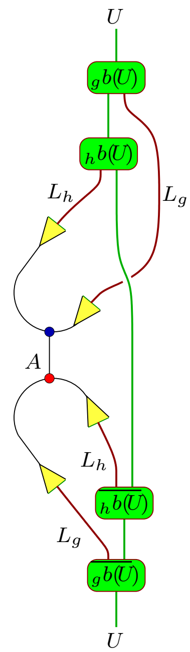
$=$



$= \ \alpha_g(U)$

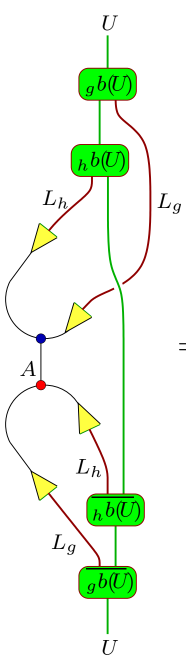


$= \ \alpha_g(U)\, \phi_U(h,g)$

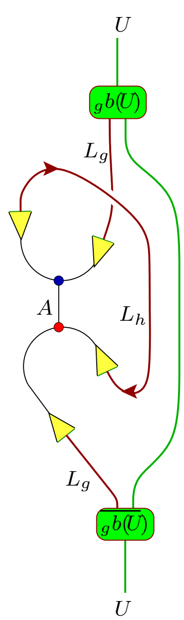


(4.35)

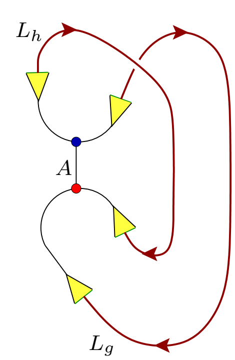
where the basis independent $6j$-symbols $\phi_U(h,g)$ are used to interchange the order of the L_h- and L_g-ribbons. After a slight deformation of the L_h-ribbons we can now invoke the relation (2.1) that is valid for invertible objects in order to tie them together in a different order. The dual morphisms ${}_hb(U)$ and $\overline{{}_hb(U)}$ cancel each other, so that we arrive at

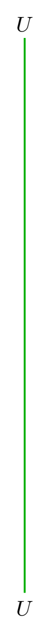


$= \ \dfrac{1}{\alpha_g(U)}$



$= \ \dfrac{1}{\alpha_g(U)}$





(4.36)

where in the last step we have used the relation (2.1) for L_g. Comparing to (3.36) we see that the last morphism is equal to $\Xi_A(g,h)/(\alpha_g(U)\dim(A))\, id_U$. Combining with (4.33) and (4.35), and noting that $\dim(A)=|H(A)|$, gives

$$\varepsilon_{U,A}(h,g) = \phi_U(h,g)\,\Xi_A(g,h)\,, \tag{4.37}$$

in agreement with (4.25). ✓

The general theory of twisted group algebras of finite abelian groups [68] motivates the

Definition 4.8:
Given a Schellekens algebra A, the *untwisted stabilizer*, or *central stabilizer*, of a simple object U is the subgroup

$$\mathcal{U}_A(U) := \{g\in\mathcal{S}_A(U)\,|\,\varepsilon_{U,A}(g,h)=1 \text{ for all } h\in\mathcal{S}_A(U)\} \tag{4.38}$$

of the stabilizer $\mathcal{S}_A(U)$.

The stabilizer $\mathcal{S}_A(U_i)$ only depends on the support $H(A)$ of the Schellekens algebra A; the central stabilizer $\mathcal{U}_A(U_i)$, however, also depends on the algebra structure chosen on the object A. The following property of the central stabilizer relates it to a twisted group algebra (compare e.g. [58]):

Lemma 4.9:
The group algebra of $\mathcal{U}_A(U)\le\mathcal{S}_A(U)$ is the center of the $\beta_{U,A}$-twisted group algebra of $\mathcal{S}_A(U)$,

$$\mathbb{C}\mathcal{U}_A(U) = \mathcal{Z}(\mathbb{C}_{\beta_{U,A}}\mathcal{S}_A(U))\,. \tag{4.39}$$

Remark 4.10:
(i) The modules over the twisted group algebra are in bijection with the representations of the central stabilizer.
(ii) The central stabilizer, rather than the full stabilizer, is thus the group whose characters label the "resolved fixed points". This holds true even in the absence of discrete torsion, i.e. for $\Xi_A\equiv 1$.
(iii) For a certain class of simple objects U_i, a simple formula for the order $|\mathcal{U}_A(U_i)|$ of the central stabilizer has been derived in [69]. It states that, provided that the number

$$Z_H(U_i) := \frac{1}{|\mathcal{S}_A(U_i)|}\sum_{g\in H}\theta_g^{1/2}\sum_{j,k\in\mathcal{I}}\mathcal{N}_{jk}^{\ i}\,S_{g,j}\,S_{0,k}\,\frac{\theta_j^2}{\theta_k^2}\,, \tag{4.40}$$

defined for a simple current group H and a simple object U_i, is non-zero, then

$$|\mathcal{U}_A(U_i)| = |\mathcal{S}_A(U_i)|\,(Z_H(U_i))^2\,. \tag{4.41}$$

(iv) It is instructive to look at the situation in the case of theta-categories. Let $\mathcal{C}(G,\psi,\omega)$ be a theta-category and A a commutative algebra in it. Then the simple A-modules are in correspondence with elements of the quotient group G/H; the *local* modules are those classes $[q]\in G/H$ such that $\beta(g,q)=1$ for all $g\in H$. They form a subgroup $(G/H)^0$ of G/H. The category of local modules is again a theta-category; it is characterized by the quadratic form $\bar{q}([g])=q(g)$, which is well-defined.

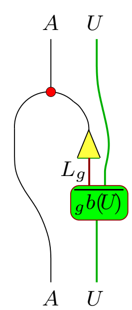
As already mentioned at the beginning of this section, every simple A-module appears as a submodule of an induced module. Suppose a given induced module $\mathrm{Ind}_A(U)$ over a Schellekens algebra A decomposes according to

$$\mathrm{Ind}_A(U) \cong \bigoplus_{\kappa\in\mathcal{J}} M_\kappa^{\oplus n_\kappa} \tag{4.42}$$

into simple modules M_κ. The algebra $\mathrm{End}_A(\mathrm{Ind}_A(U))$ is then isomorphic to a direct sum of matrix algebras $\mathrm{End}_A(\mathrm{Ind}_A(U))\cong\bigoplus_{\kappa\in\mathcal{J}}\mathrm{Mat}_{n_\kappa}(\mathbb{C})$. The unit matrices of the individual blocks then form a basis of the center of $\mathrm{End}_A(\mathrm{Ind}_A(U))$ consisting of idempotents. Each such idempotent gives one isotypical component $M_\kappa^{\oplus n_\kappa}$ in the decomposition (4.42).

To construct these projectors, let us start by specifying a basis $\{\varphi_g\,|\,g\in\mathcal{U}_A(U)\}$ of the $\mathbb{C}$-algebra $\mathcal{Z}(\mathrm{End}_A(\mathrm{Ind}_A(U)))$ for a simple object U. It is given by

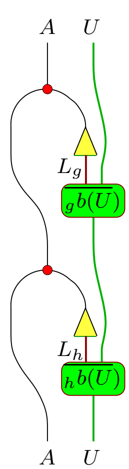
$$\varphi_g \;:=\; (m\otimes id_U)\circ(id_A\otimes\phi_g) \;= \tag{4.43}$$



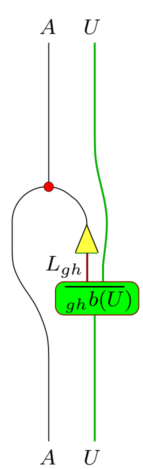
for $g\in\mathcal{U}_A(U)$, where ϕ_g is the morphism introduced in (4.28). That this is indeed a basis is implied by lemma 4.9.

When making the choice arbitrarily, under composition this set of morphism closes up to scalar factors:

$$\varphi_g\circ\varphi_h \;=$$



$$=\;\beta_{U,A}(g,h)$$



$$=\;\beta_{U,A}(g,h)\,\varphi_{gh}\,, \tag{4.44}$$

where $\beta_{U,A}$ was defined in (4.29). Note that since $g,h\in\mathcal{U}_A(U)$, by definition the commutator two-cocycle $\varepsilon_{U,A}(g,h)$ of $\beta_{U,A}(g,h)$ is equal to one, and thus $\beta_{U,A}=\mathrm{d}\eta$ for some one-cochain η on $\mathcal{U}(U)$. This implies that it is possible to modify our initial choice of basis by suitable scalar factors in such a way that with the new choice we simply have

$$\varphi_g \circ \varphi_h = \varphi_{gh} \,. \tag{4.45}$$

This choice is unique up to a character ψ_U of $\mathcal{U}_A(U)$. The isotypical components are labeled by characters of $\mathcal{U}_A(U)$, but without distinguishing the trivial character, i.e. more precisely, they form a torsor over $\mathcal{U}_A(U)^*$. This phenomenon has been called fixed point homogeneity in [62].

Note that with this choice of scalar factors the product in (4.30) takes the simple form

$$\phi_{g_1} \star \phi_{g_2} = \phi_{g_1 g_2} = \phi_{g_2} \star \phi_{g_1} \qquad \text{for all} \quad g_1, g_2 \in \mathcal{U}_A(U)\,. \tag{4.46}$$

Such a choice is indeed possible: Fix an adapted basis for the algebra A; we may choose, for every simple object U_i separately, the morphisms ${}_gb(U_i)$ in such a way that the morphism ϕ_g constructed from these ${}_gb(U_i)$ and from the adapted basis obey (4.46). We therefore make the

Definition 4.11:
Given a Schellekens algebra A in $\mathcal{C}$, we fix an adapted basis for A. A family of basis choices ${}_gb(U)\in \mathrm{Hom}(L_g\otimes U, U)$ for all simple objects U and for $g\in\mathcal{U}_A(U)$, such that (4.46) holds for the morphisms ϕ_g constructed with these choices is called an A-*straight family* of bases. The corresponding family of morphisms ϕ_g is called A-straight as well. Whenever the algebra evident, we drop it from our notation.

Clearly, an A-straight family is not uniquely determined. Rather, for each simple object U, the possible choices form a torsor over $\mathcal{U}_A(U)^*$. Note that the choices for non-isomorphic simple objects are not correlated.

Let now ϕ_g be straight, and introduce for every $\psi\in\mathcal{U}_A(U)^*$ the morphism

$$\widetilde{P}_{U,\psi} := \frac{1}{|\mathcal{U}_A(U)|} \sum_{g\in\mathcal{U}_A(U)} \psi^*(g)\, \varphi_g \,. \tag{4.47}$$

By orthogonality of the characters we have $\widetilde{P}_{U,\psi_1}\circ\widetilde{P}_{U,\psi_2}=\delta_{\psi_1,\psi_2}\,\widetilde{P}_{U,\psi_1}$, for $\psi_1,\psi_2\in\mathcal{U}_A(U)^*$. Indeed, the $\widetilde{P}_{U,\psi}$ with $\psi\in\mathcal{U}_A(U)^*$ form a basis of $\mathcal{Z}(\mathrm{End}_A(\mathrm{Ind}_A(U)))$ consisting of idempotents. Comparing with (4.42) we see that the images $\widetilde{M}_{U,\psi}$ of the morphisms $\widetilde{P}_{U,\psi}$ are the isotypical components in the decomposition of $\mathrm{Ind}_A(U)$. Note that, for $\mathcal{U}_A(U)\neq\mathcal{S}_A(U)$, $\widetilde{M}_{U,\psi}$ is not simple, but rather the direct sum of

$$d_A(U) := \sqrt{\frac{|\mathcal{S}_A(U)|}{|\mathcal{U}_A(U)|}} \tag{4.48}$$

isomorphic simple modules.

The arguments in this section have established the following algorithm to determine the list of (isomorphy classes of) simple modules over a Schellekens algebra and thus of boundary conditions in a CFT of simple current type:

- The set $\mathcal{I}$ of isomorphism classes of simple objects in $\mathcal{C}$ admits a partition in r orbits under the group $H(A)$. We denote by $U_{m_1}, \dots, U_{m_r}$ a set of representatives of these orbits.

- Lemma 4.6 implies that for $m_i \neq m_j$ the induced modules $\mathrm{Ind}_A(U_{m_i})$ and $\mathrm{Ind}_A(U_{m_j})$ do not contain any common simple submodule. Moreover, each simple module appears as the submodule of an induced module. The decomposition of the induced modules $\mathrm{Ind}_A(U_{m_i})$ into their isotypical components thus provides us with the complete list of simple modules, without overcounting.

- The isotypical components $M_\kappa^{\oplus n(i)}$ in the decomposition $\mathrm{Ind}_A(U_{m_i}) = \bigoplus_{\kappa\in\mathcal{J}} M_\kappa^{\oplus n(i)}$ are given as images of the projectors $\widetilde{P}_{U,\psi}$ defined in (4.47). The projector is labeled by a character $\psi \in \mathcal{U}_A(U)^*$, once an A-straight basis has been chosen. The multiplicites $n(i) = d_A(U_{m_i})$ in an isotypical component depend only on i, not on κ.

- With a given choice of straight basis, the simple A-modules are labeled by pairs (n, ψ) where $n = 1, 2, \dots, r$ labels an $H(A)$-orbit and $\psi \in \mathcal{U}_A(U_{m_n})^*$ is a character of the central stabilizer $\mathcal{U}_A(U_{m_n})$ of U_{m_n}.

Remark 4.12:
(i) We have developed the representation theory of left A-modules. As mentioned, they describe the conformal boundary conditions of the CFT associated to A. However, the fact that we are dealing with left, rather than with right, modules over A is merely a matter of convention. One can equivalently describe all conformal boundary conditions by right A-modules.

(ii) Right A-modules over a Schellekens algebra A can be studied by the same methods as were used above. Every right A-module is submodule of an induced right module $\mathrm{Ind}(U)_A := (U \otimes A, id_U \otimes m)$. To obtain the isotypical components in the decomposition of $\mathrm{Ind}(U)_A$ we study the endomorphism algebra $\mathrm{End}_A(\mathrm{Ind}(U)_A)$ of right module endomorphisms of $\mathrm{Ind}(U)_A$. Let now U be a simple object of $\mathcal{C}$; a basis of $\mathrm{End}_A(\mathrm{Ind}(U)_A)$ is given by

$$\tilde{\varphi}_g := (id_U \otimes m) \circ (id_U \otimes e_g \otimes id_A) \circ (\overline{b_g(U)} \otimes id_A) \,. \tag{4.49}$$

The composition of two basis elements yields $\tilde{\varphi}_g \circ \tilde{\varphi}_h = \tilde{\beta}_{U,A}(g,h)\, \tilde{\varphi}_{gh}$ for some two-cocycle $\tilde{\beta}_{U,A}$. The endomorphism algebra is thus isomorphic to a twisted group algebra,

$$\mathrm{End}_A(\mathrm{Ind}(U)_A) \cong \mathbb{C}_{\tilde{\beta}_{U,A}} \mathcal{S}_A(U) \,. \tag{4.50}$$

The commutator two-cocycle $\tilde{\varepsilon}_{U,A}$ of $\tilde{\beta}_{U,A}$ can be worked out analogously as for left modules; we find

$$\tilde{\varepsilon}_{U,A}(g,h) = \frac{\tilde{\beta}_{U,A}(g,h)}{\tilde{\beta}_{U,A}(h,g)} = \phi_U(g,h)\, \Xi_A(g,h) \,. \tag{4.51}$$

Comparison with (4.25) shows that $\mathrm{End}_A(\mathrm{Ind}(U)_A)$ and $\mathrm{End}_A(\mathrm{Ind}_A(U))$ are isomorphic if the KSB Ξ_A is symmetric. The situation becomes particularly transparent for categories in which all F-matrix entries are real. The result (4.20) implies that in this case, for any Schellekens algebra, the $\mathbb{C}$-algebra $\mathrm{End}_A(\mathrm{Ind}(U)_A)$ for induced right modules is just the opposed algebra of $\mathrm{End}_A(\mathrm{Ind}_A(U))$ for induced left modules.

(iii) As in the case of left modules, one introduces for a given adapted basis of A the notion of A-straightness for a family of morphisms in $\mathrm{Hom}(U_i, U_i \otimes A)$. By an appropriate choice of the morphisms $b_g(U)$ one shows that straight families exist and that they form a torsor over the character group of the (right) central stabilizer.

Using the coproduct of A rather then the product, one can endow also the vector space $\mathrm{Hom}(A \otimes U, U)$ with the structure of a $\mathbb{C}$-algebra. This product is inherited from the endomorphism algebra $\mathrm{End}_A(A \otimes U, A \otimes U)$ as well, this time using Frobenius reciprocity. One checks that if a choice of morphisms $\overline{{}_gb(U)}$ straightens the product on $\mathrm{Hom}(U, A \otimes U)$, then the dual morphisms ${}_gb(U)$ straighten the product on $\mathrm{Hom}(A \otimes U, U)$. An analogous statement holds for right modules. All straight choices are unique up to characters of the relevant central stabilizers.

4.3 The boundary state

Our next goal is to determine the *boundary states* for the full conformal field theory that corresponds to some given Schellekens algebra in a modular tensor category. The boundary state for a given boundary condition is, by definition, the information about the collection of all one-point functions of bulk fields on a disk with that boundary condition. We have already determined the possible boundary conditions: they correspond to left A-modules as described in the previous subsection.

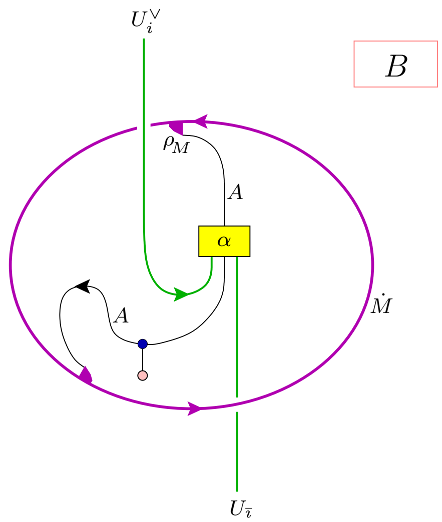
The bulk fields that can have a non-vanishing one-point function on the disk are described by the vectors α in the spaces

$$\mathrm{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_{\bar\imath}, A) \tag{4.52}$$

of morphisms of bimodules. Here the superscripts at the tensor symbols remind us that the object $U_i \otimes A \otimes U_{\bar\imath}$ of $\mathcal{C}$ has been endowed with the following structure of an A-bimodule: the left action of A is $\rho_l := (id_{U_i} \otimes m \otimes id_{U_{\bar\imath}}) \circ (c^{-1}_{U_i,A} \otimes id_A \otimes id_{U_{\bar\imath}})$, while the right action is given by $\rho_r := (id_{U_i} \otimes m \otimes id_{U_{\bar\imath}}) \circ (id_{U_i} \otimes id_A \otimes c^{-1}_{A,U_{\bar\imath}})$. In other words, for the left action the A-ribbon is braided under the U_i-ribbon, while for the right action the A-ribbon is braided over the $U_{\bar\imath}$-ribbon.

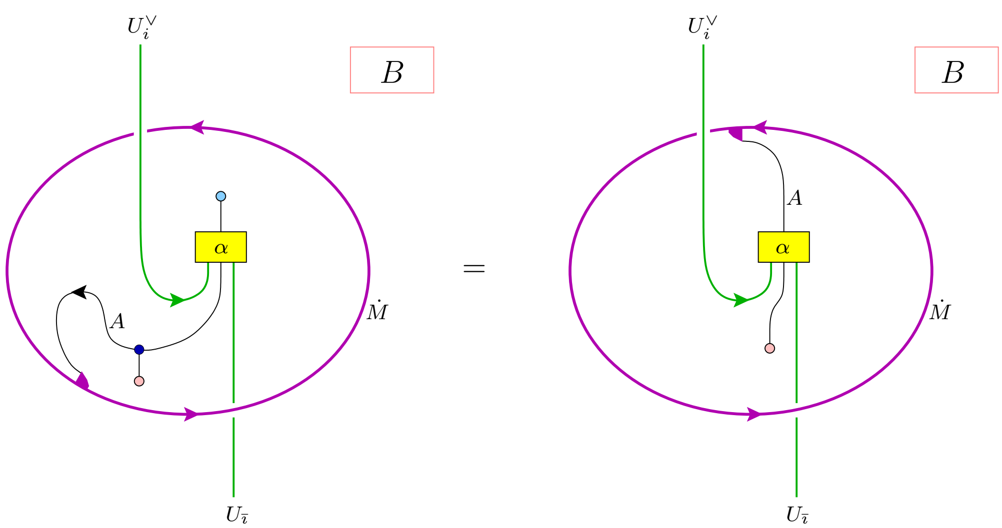
Only bulk fields with conjugate left and right chiral labels i and $\bar\imath$ contribute. Thus for every $i \in \mathcal{I}$ there are $Z_{i\bar\imath}$ bulk fields that can have a non-vanishing one-point function on the disk. Correlators on the disk have already been presented schematically in the figure (21) of [1] as ribbon graphs in the three-ball B. With the conventions chosen here, the ribbon graph for the correlator of a bulk field $\Phi^{(\alpha)}_{i,\bar\imath}$ labeled by the morphism α in the space

(4.52) on a disk with boundary condition given by the A-module M looks as follows [70]:



$$(4.53)$$

Here the morphisms connecting the A-ribbons to the M-ribbon are representation morphisms ρ_M. The graph (4.53) can be simplified by first dragging one of the representation morphisms along the $\dot M$-ribbon, then using the representation property, then the fact that A is symmetric Frobenius, then that α is a bimodule morphism, and finally again that A is symmetric Frobenius. Depending on whether one applies the bimodule morphism property of α to the incoming or outgoing A-ribbon, one thereby arrives at one of the two following equivalent graphs:

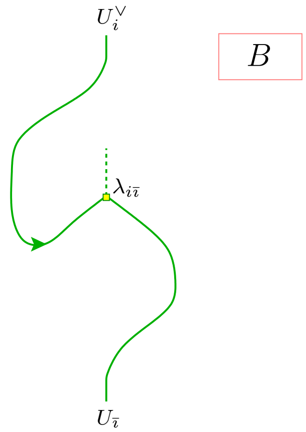


$$(4.54)$$

Given a bulk field and a boundary condition, we would like to express the one-point function in terms of a conformal two-point block on the Riemann sphere. These conformal blocks form, for every label $i \in \mathcal{I}$, a one-dimensional complex vector space

$$V_{i\bar\imath} = \mathrm{Hom}(U_i \otimes U_{\bar\imath}, \mathbf{1})\,. \tag{4.55}$$

Since this vector space does not have any preferred basis, there is no canonical identification with the ground field $\mathbb{C}$. Such an identification becomes only possible once we have chosen a basis $\lambda_{i\bar\imath}$ in $V_{i\bar\imath}$.[7] Such a basis vector in the space of two-point blocks is, in this context, also known as an "Ishibashi state". The Ishibashi state for the label i can be depicted as the following ribbon graph in the three-ball:



(4.56)

The one-point functions of bulk fields on a disk are then encoded in $\mathbb{C}$-linear maps

$$\tilde\Phi_M : \quad \mathrm{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_{\bar\imath}, A) \;\to\; V_{i\bar\imath} \tag{4.57}$$

that send the morphism α to the morphism in $V_{i\bar\imath}$ that appears in figure (4.54). Only once we have selected a basis $\lambda_{i\bar\imath}$ in $V_{i\bar\imath}$ we can encode the same information in a linear form Φ_M on the space of bulk fields,

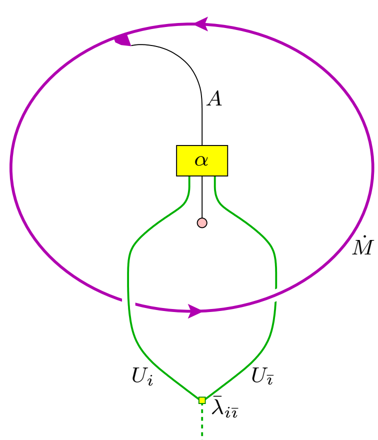
$$\Phi_M \;\in\; \big(\mathrm{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_{\bar\imath}, A)\big)^*\,, \tag{4.58}$$

according to

$$\tilde\Phi_M(\alpha) = \Phi_M(\alpha)\,\lambda_{i\bar\imath}\,. \tag{4.59}$$

The linear form Φ_M can be thought of as providing the coefficients of the boundary state for the boundary condition M, written in terms of a fixed set of Ishibashi states. The values of this linear form are given by the invariant of the following graph in S^3:

$$\Phi_M(\alpha) \;=$$



(4.60)

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[7] In [I] a basis of this space was denoted by $\lambda_{(i,\bar\imath)0}$, see formula (I:2.29). Here we use the short-hand $\lambda_{i\bar\imath}$.

Here the morphism $\bar\lambda_{i\bar\imath}$ that is dual to $\lambda_{i\bar\imath}$ appears.

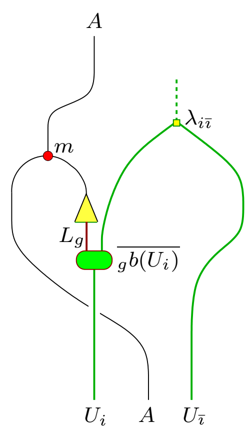
In the rest of this subsection, we will make these general expressions more concrete for the particular case of Schellekens algebras. We start with a discussion of bulk fields that can have a non-vanishing one-point function on the disk, i.e. that have conjugate left and right chiral labels.

Lemma 4.13:
There is a bijection between bulk fields with conjugate left and right chiral labels i and $\bar\imath$ and pairs (U_i, g), with U_i a simple object and $g \in \mathcal{S}_A(U_i)$ such that

$$\Xi_A(\,\cdot\,, g)\,\chi_{U_i}(\cdot) \tag{4.61}$$

is the trivial character on $H(A)$.
In terms of the morphisms we have already chosen (see appendix C), a basis of bulk fields is given by the morphisms

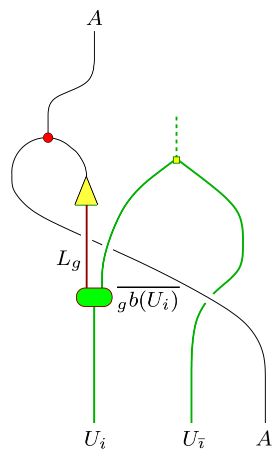
$$\Psi_g(U_i) := \tag{4.62}$$



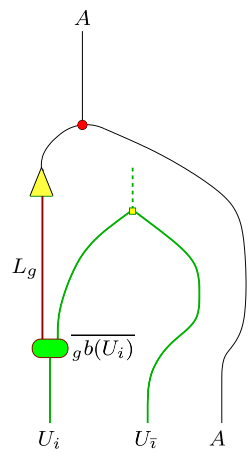
Proof:
The space of bimodule morphisms $\mathrm{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_{\bar\imath}, A)$ is trivially a subspace of the space of left-module morphisms $\mathrm{Hom}_A(U_i \otimes^+ A \otimes U_{\bar\imath}, A)$. By reciprocity, one easily sees that the latter space is graded over the stabilizer $\mathcal{S}_A(U_i)$, with one-dimensional homogeneous subspaces. A basis of each homogeneous subspace is given by the morphisms (4.62). We must determine which of these morphisms are even morphisms of bimodules. In other words, we are looking for the linear subspace consisting of linear combinations of the morphisms $\Psi_g(U_i)$ with coefficients $\xi_g \in \mathbb{C}$ that obey

$$\sum_{g' \in \mathcal{S}_A(U_i)} \xi_{g'}\, \rho_r \circ (\Psi_{g'}(U_i) \otimes id_A) = \sum_{g' \in \mathcal{S}_A(U_i)} \xi_{g'}\, \Psi_{g'}(U_i) \circ \rho_r\,. \tag{4.63}$$

Composing both sides with r_{gh} from the left and with $id_{U_i} \otimes \eta \otimes id_{U_{\bar\imath}} \otimes e_h$ from the right isolates the term $g' = g$ in the sum. The resulting condition must be valid for all $h \in H(A)$, which shows that ξ_g is allowed to be non-zero if and only if the equality



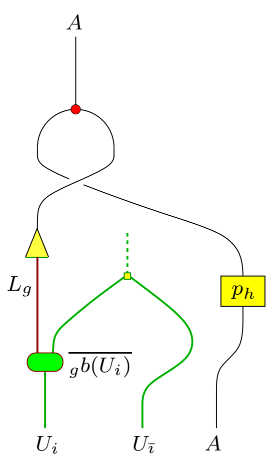
$$=$$



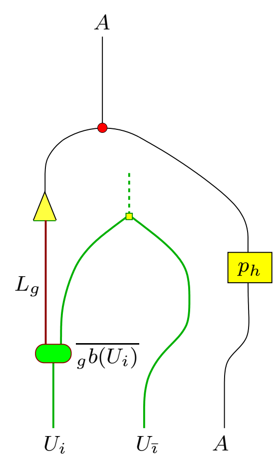
(4.64)

is satisfied.
To compute the graph on the left hand side of (4.64), we insert the identity $id_A = \sum_{h\in H(A)} p_h$ that is valid for Schellekens algebras together with the definition (3.52) of the exponentiated monodromy charge to obtain

$$\sum_{h\in H} \chi_{U_i}(h)$$



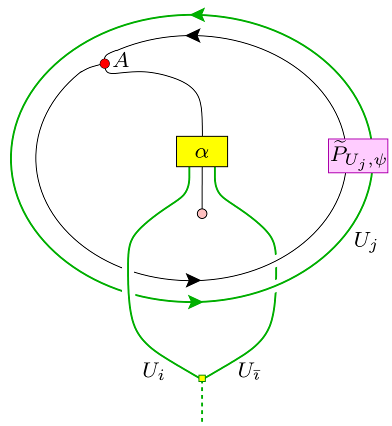
$$= \sum_{h\in H} \chi_{U_i}(h)\, \Xi_A(h,g)$$



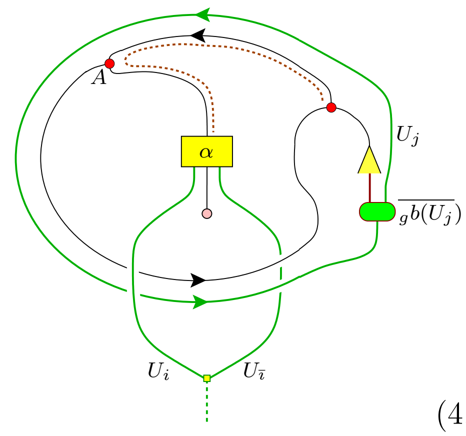
(4.65)

Here in the second step we have inserted (3.39). Thus we have equality in (4.64) if and only if $\chi_{U_i}(\cdot)\, \Xi_A(\,\cdot, g)$ is the trivial character on H. (This is precisely the condition found in [17].) ✓

Next we evaluate the invariant of the graph (4.60) for $\Phi_M(\alpha)$ in the case when A is a Schellekens algebra and $M = \widetilde{M}_{U_j,\psi}$ is the A-module that corresponds to the idempotent (4.47). (Recall from (4.48) that if the central stabilizer is strictly smaller than the stabilizer, then this module is still reducible.) The graph for $\Phi_{\widetilde{M}_{(U_j,\psi)}}(\alpha)$ looks as follows:

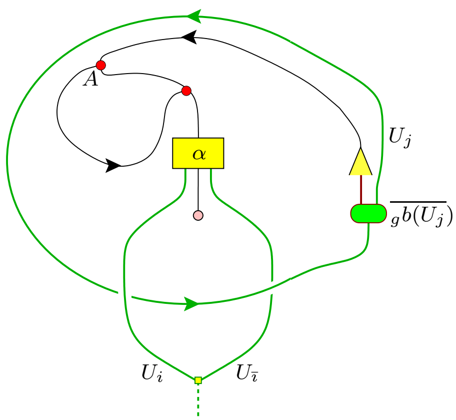


$$= \sum_{g\in\mathcal{S}_A(U_j)} \frac{\psi^*(g)}{|\mathcal{U}_A(U_j)|}$$

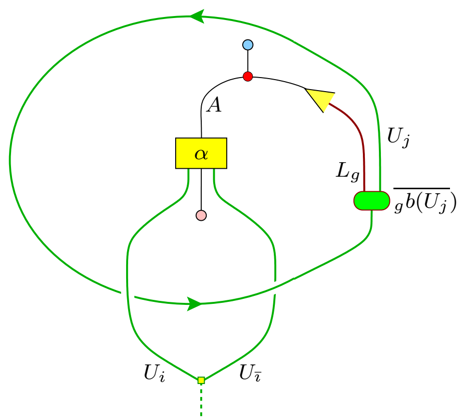


(4.66)

Using the associativity of A, one can drag the product morphism coming from $\widetilde{P}_{U_j,\psi}$ along the dashed line indicated on the right hand side. Afterwards one can use successively the bimodule morphism property of α, the unit property of η, and then again the bimodule morphism property so as to arrive at the graph

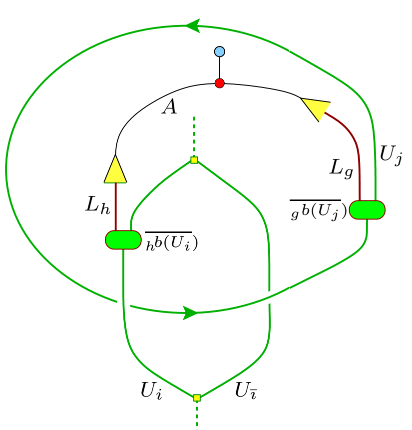


$$=$$

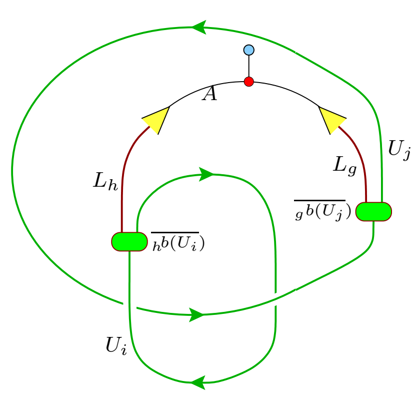


(4.67)

To proceed, we must specify the morphism $\alpha \in \mathrm{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_{\bar\imath}, A)$. In view of lemma 4.13 it is natural to take one of the basis morphisms (4.62). Setting thus $\alpha = \Psi_h(U_i)$, with $\Psi_h(U_i)$ as in (4.62), on the right hand side of (4.67), we obtain

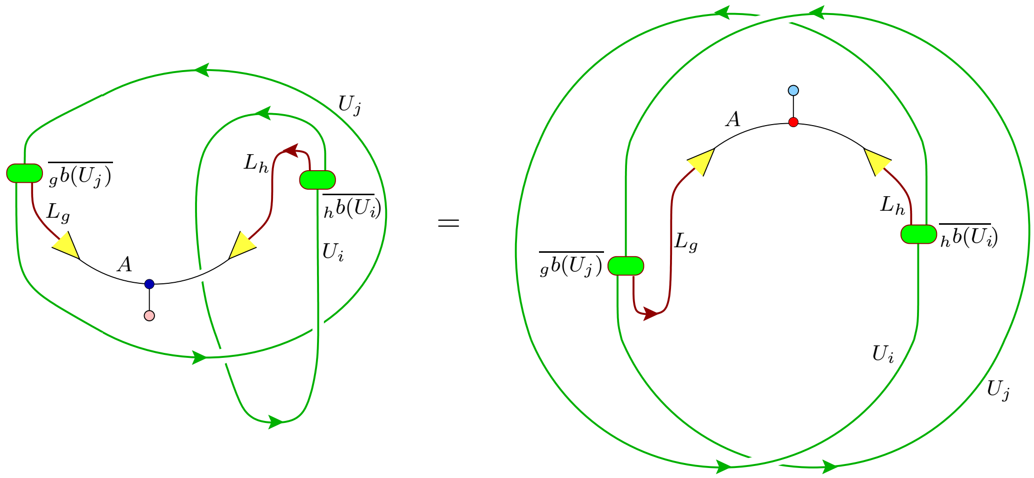


$$= \frac{1}{\dim(U_i)}$$



(4.68)

This graph is non-zero only for $h = g^{-1}$ so that the summation in (4.66) can be carried out explicitly. Pulling now the morphism $\overline{{}_g b(U_j)}$ along the U_j-ribbon and slightly deforming the ribbons, using also sovereignty on the U_i-ribbon, the graph becomes

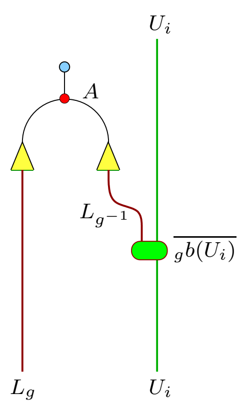


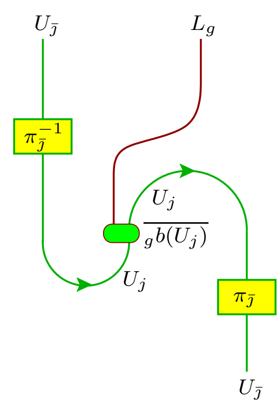
(4.69)

The equality between the left and right hand sides uses that A is symmetric. Comparison with the bilinear pairing (4.8) shows that the right hand side is nothing but

$$\mathcal{S}^g_{i,\bar\jmath}(\alpha \otimes \beta) =: s(A)^g_{i,\bar\jmath} \tag{4.70}$$

with $\alpha \in \mathrm{Hom}(L_g \otimes U_i, U_i)$ and $\beta \in \mathrm{Hom}(U_{\bar\jmath}, U_{\bar\jmath} \otimes L_g)$ given by

$\alpha \; =$ 

and $\quad \beta \; =$ 

(4.71)

Remark 4.14:
(i) The scalar matrix $s(A)^g_{i,\bar\jmath}$ defined in (4.70) depends both on the algebra A and on a choice of A-straight bases. However, choosing different bases changes the value of $s(A)^g_{i,\bar\jmath}$ at most up to characters of the central stabilizers.

(ii) In the following we list some properties of the matrices $s(A)^g$, without a detailed proof. First, by setting $g = e$ in (4.69) we get immediately

$$s(A)^e_{i,j} = \dim(A)\, s_{i,j} \,, \tag{4.72}$$

where s is $S_{0,0}^{-1}$ times the ordinary unitary modular S-matrix.
Next, let $\tilde{s}(A)_{i,j}^{g}$ be defined as $s(A)_{i,j}^{g}$, but with the inverse braiding instead of the braiding. Then we have

$$\sum_{k\in F(g)} s(A)_{i,k}^{g}\,\tilde{s}(A)_{\bar{k},\bar{j}}^{g} = \frac{\dim(A)^2}{(S_{00})^2}\,\delta_{i,j}\,, \tag{4.73}$$

the sum being over the set $F(g)$ of all $k\in\mathcal{I}$ such that $g\in\mathcal{S}(U_k)$, compare part (i) of the proof of proposition 4.4. To show the validity of (4.73), one uses that A-straight bases have been chosen.
We also have

$$s(A)_{i,j}^{g} = s(A)_{\bar{j},\bar{\imath}}^{g^{-1}}\,. \tag{4.74}$$

This is seen by turning the graph (4.69) upside down and using that the algebra A is symmetric.
Finally, if there exists a basis of the coupling spaces in which all F-matrix entries are real (compare remark 4.5(iii)), then we also have

$$\big(s(A)_{i,j}^{g}\big)^{*} = \tilde{s}(A)_{i,j}^{g}\,. \tag{4.75}$$

(iii) The matrices $s(A)^g$ should be compared to the corresponding quantities S^J in [62]. The two quantities can be identified via $S_{i,j}^{J} = \big(S_{00}/\dim(A)\big)\,\big(s(A)_{i,\bar{j}}^{g}\big)^{*}$, where $J=g$.

Collecting the information in the relations (4.66) – (4.70), we arrive at the following expression for the one-point function of the bulk field $\Psi_h(U_i)$ on a disk with boundary condition $\widetilde{M}_{(U_j,\psi)}$:

$$\Phi_{\widetilde{M}_{(U_j,\psi)}}(\Psi_h(U_i)) = \frac{1}{\dim(U_i)}\,\frac{\psi^{*}(h^{-1})}{|\mathcal{U}_A(U_j)|}\,s(A)_{i,\bar{j}}^{h^{-1}}\,. \tag{4.76}$$

Now recall from (4.48) that the A-module $\widetilde{M}_{(U_j,\psi)}$ is the direct sum of $d_A(U_j)$ isomorphic simple modules. Accordingly, the one-point function on a disk with boundary condition labeled by the simple A-module $M_{(n,\psi)}$ is given by

$$\Phi_{M_{(n,\psi)}}(\Psi_{h^{-1}}(U_i)) = \frac{1}{\dim(U_i)}\,\frac{\psi^{*}(h)}{\sqrt{|\mathcal{S}_A(U_{m_n})|\,|\mathcal{U}_A(U_{m_n})|}}\,s(A)_{i,\overline{m_n}}^{h}\,. \tag{4.77}$$

This coincides with formula (11) of [17], except for an overall factor. This factor comes from the normalisation of the bulk fields and of the matrix $s(A)^g$. For the latter, compare formula (4.73); also note that the normalisation of bulk fields differs from the one used in [17] already when $A=\mathbf{1}$ [71]. That the prefactor in (4.77) precisely stems from these normalisations can e.g. be seen from the fact that both the formula given in [17] and our result (4.77) lead to annulus coefficients that are non-negative integers.

5 Bimodules and defects

A-bimodules, which in the application to CFT correspond to defect lines [I] can be treated in a manner that is quite analogous to the description of A-modules in section 4.2. Again the basic ingredient is a notion of induction. As it turns out, it is convenient not to use α-induction. Rather, we proceed as follows. Consider the object $A \otimes U \otimes A$; the multiplication in A endows it with a natural structure of an A-bimodule that we call the *induced bimodule* $\mathrm{Ind}_{A|A}(U)$:

$$\mathrm{Ind}_{A|A}(U) = (A \otimes U \otimes A, m \otimes id_U \otimes id_A, id_A \otimes id_U \otimes m)\,. \tag{5.1}$$

Obviously, such bimodules can be studied for any symmetric special Frobenius algebra A, not only for Schellekens algebras. Moreover, there is no need to take the same symmetric special Frobenius algebra for the left action and the right action. This leads us to the

Definition 5.1 :
Let A_1 and A_2 be two symmetric special Frobenius algebras in a tensor category $\mathcal{C}$ satisfying the assumptions in convention 2.2. The *induced* A_1-A_2-*bimodule* associated to an object U of $\mathcal{C}$ is the bimodule

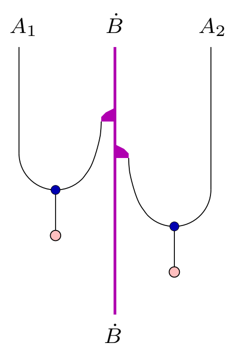
$$\mathrm{Ind}_{A_1|A_2}(U) := (A_1 \otimes U \otimes A_2, m_1 \otimes id_U \otimes id_{A_2}, id_{A_1} \otimes id_U \otimes m_2)\,. \tag{5.2}$$

The following statement is immediate:

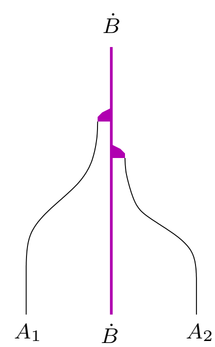
Lemma 5.2 :
Every A_1-A_2-bimodule B over special Frobenius algebras A_1 and A_2 is a bimodule retract of some induced A_1-A_2-bimodule.

Proof:
Indeed, B is the bimodule retract of $\mathrm{Ind}_{A_1|A_2}(\dot B)$ that is defined by the embedding and retraction morphisms

$e_{B\prec\mathrm{Ind}_{A_1|A_2}(\dot B)}$:=



and $r_{\mathrm{Ind}_{A_1|A_2}(\dot B)\succ B}$:=



(5.3)

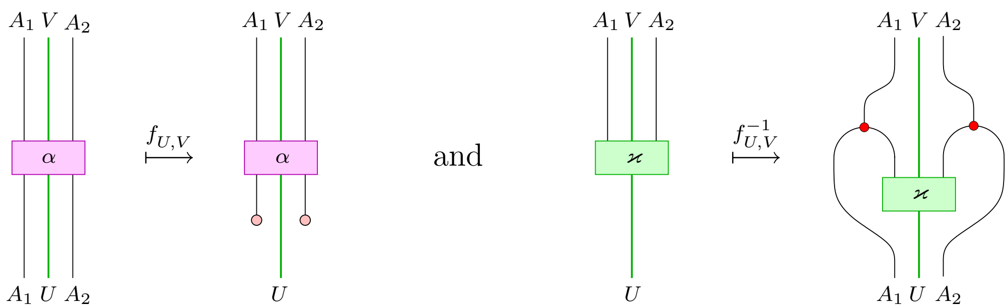
By the Frobenius and associativity properties of A_1 and A_2, these are morphisms of bimodules, and by specialness we have $r_{\mathrm{Ind}_{A_1|A_2}(\dot B)\succ B} \circ e_{B\prec\mathrm{Ind}_{A_1|A_2}(\dot B)} = id_B$.

The central task in developing the bimodule theory of Schellekens algebras is thus to decompose induced bimodules. To this end, we must describe bimodule morphisms between induced bimodules. A ‘double-sided reciprocity’ turns out to be a convenient tool:

Lemma 5.3:
For any two A_1 and A_2 and any two objects U and V of $\mathcal{C}$ there are canonical isomorphisms

$$f_{U,V}: \quad \mathrm{Hom}_{A_1|A_2}(\mathrm{Ind}_{A_1|A_2}(U), \mathrm{Ind}_{A_1|A_2}(V)) \xrightarrow{\cong} \mathrm{Hom}(U, A_1 \otimes V \otimes A_2) \tag{5.4}$$

of complex vector spaces. They are given by



(5.5)

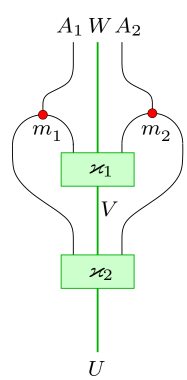
for $\alpha \in \mathrm{Hom}_{A_1|A_2}(\mathrm{Ind}_{A_1|A_2}(U), \mathrm{Ind}_{A_1|A_2}(V))$ and $\varkappa \in \mathrm{Hom}(U, A_1 \otimes V \otimes A_2)$.

Proof:
That the two mappings are inverse to each other follows immediately by the associativity and unit properties. ✓

Note that the statement holds for any pair A_1, A_2 of unital associative algebras; the Frobenius property is not needed.

The composition of bimodule morphisms induces a concatenation on the morphism spaces $\mathrm{Hom}(U, A_1 \otimes V \otimes A_2)$:

$\varkappa_1 \star \varkappa_2 \ :=$



(5.6)

We now consider more specifically the situation that A_1 and A_2 are Schellekens algebras, $A_i \cong \bigoplus_{h_i \in H_i} L_{h_i}$, with two not necessarily equal subgroups H_1 and H_2 in the effective center $\mathrm{Pic}^\circ(\mathcal{C})$. In this case, the morphism space $\mathrm{Hom}(U, A_1 \otimes V \otimes A_2)$ is non-zero only if U and V are on the same orbit of the subgroup $H \leq \mathrm{Pic}(\mathcal{C})$ that is generated by H_1 and H_2. Moreover, for $U = V$, this morphism space is naturally isomorphic to the endomorphism space

$$\begin{aligned} \mathrm{End}_{A_1|A_2}(\mathrm{Ind}_{A_1|A_2}(U)) &\cong \mathrm{Hom}(U, A_1 \otimes U \otimes A_2) \\ &\cong \bigoplus_{h_1 \in H_1,\, h_2 \in H_2} \mathrm{Hom}(U, L_{h_1} \otimes U \otimes L_{h_2})\,. \end{aligned} \tag{5.7}$$

It is thus a graded vector space over

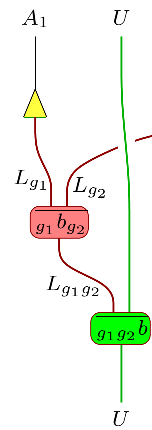
$$\mathcal{S}_{A_1|A_2}(U) := \{(h_1,h_2)\in H_1\times H_2 \,|\, h_1h_2\in\mathcal{S}(U)\}\,, \tag{5.8}$$

where the stabilizer $\mathcal{S}(U)$ of U is defined with respect to the whole effective center $\mathrm{Pic}^\circ(\mathcal{C})$. This subset of $H_1\times H_2$ is a subgroup (since H is abelian); it will be called the *bi-stabilizer* of U with respect to A_1 and A_2. Clearly we have

$$\mathcal{S}_{A_1}(U)\times\mathcal{S}_{A_2}(U) \le \mathcal{S}_{A_1|A_2}(U) \le H_1\times H_2\,. \tag{5.9}$$

Next we analyze the structure of the concatenation product introduced in (5.6) on the space $\mathrm{End}_{A_1|A_2}(\mathrm{Ind}_{A_1|A_2}(U))$. To this end we define, for $(g_1,g_2)\in\mathcal{S}_{A_1|A_2}(U)$, the morphisms

$${}_{g_1}\phi_{g_2}(U) \;:=$$



(5.10)

where we have suppressed the argument U in the morphism $\overline{{}_{g_1g_2}b(U)}\in\mathrm{Hom}(U,L_{g_1g_2}\otimes U)$ (as introduced in definition 4.1(iii)). These form a basis of $\mathrm{Hom}(U,A_1\otimes U\otimes A_2)$.

Proposition 5.4:
Let $U\in\mathrm{Obj}(\mathcal{C})$ be simple and A_1, A_2 be two Schellekens algebras in $\mathcal{C}$. The $\mathbb{C}$-algebra $\mathrm{Hom}(U,A_1\otimes U\otimes A_2)$ is a twisted group algebra over the group $\mathcal{S}_{A_1|A_2}(U)$. The cohomology class of the corresponding two-cocycle is described by the alternating bihomomorphism

$$\varepsilon_{U,A_1,A_2}(g_1,g_2,h_1,h_2) = \phi_U(g_1g_2,h_1h_2)\,\beta(h_1,g_2)\,\Xi_{A_1}(h_1,g_1)\,\Xi_{A_2}(g_2,h_2)\,. \tag{5.11}$$

Proof:
We first notice that the algebra is also graded over the group $\mathcal{S}_{A_1|A_2}(U)$ as an algebra, i.e. we have

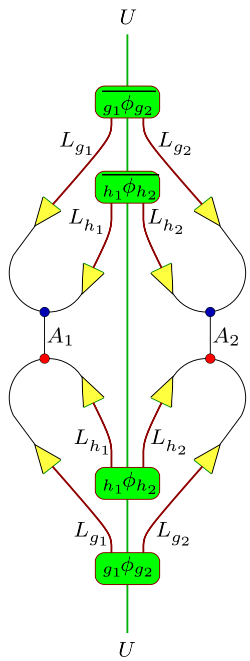
$${}_{g_1}\phi_{g_2}(U)\star{}_{h_1}\phi_{h_2}(U) = \beta_{U,A_1,A_2}(g_1,g_2,h_1,h_2)\;{}_{g_1h_1}\phi_{g_2h_2}(U) \tag{5.12}$$

for some two-cocyle β_{U,A_1,A_2} on $\mathcal{S}_{A_1|A_2}(U)$. This follows from the fact that the morphism space $\mathrm{Hom}(U,L_{g_1h_1}\otimes L_{g_2h_2}\otimes U)$ is one-dimensional.

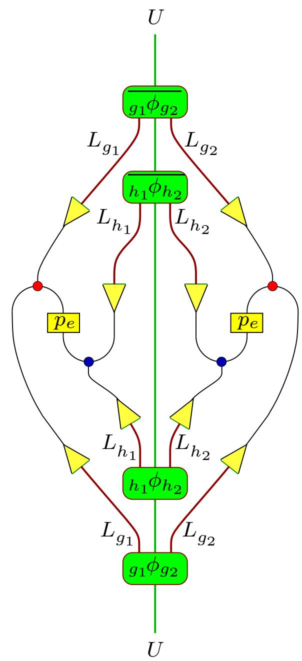
Again, we determine the isomorphy type of the algebra by computing the commutator two-cocyle $\varepsilon_{U,A_1,A_2}(g_1,g_2,h_1,h_2)$ appearing in

$${}_{g_1}\phi_{g_2}(U)\star{}_{h_1}\phi_{h_2}(U) = \varepsilon_{U,A_1,A_2}(g_1,g_2,h_1,h_2)\;{}_{h_1}\phi_{h_2}(U)\star{}_{g_1}\phi_{g_2}(U)\,. \tag{5.13}$$

To determine the scalars $\varepsilon_{U,A_1,A_2}(g_1,g_2,h_1,h_2)$, we compose the equality (5.13) with the concatenation $\overline{{}_{g_1}\phi_{g_2}(U)} \star \overline{{}_{h_1}\phi_{h_2}(U)}$ of dual basis morphisms. Then on the right hand side, after using the Frobenius property for both A_1 and A_2, only the tensor unit can propagate in the middle A_i-ribbons, so that analogously as in (4.35) one obtains a multiple of id_U:

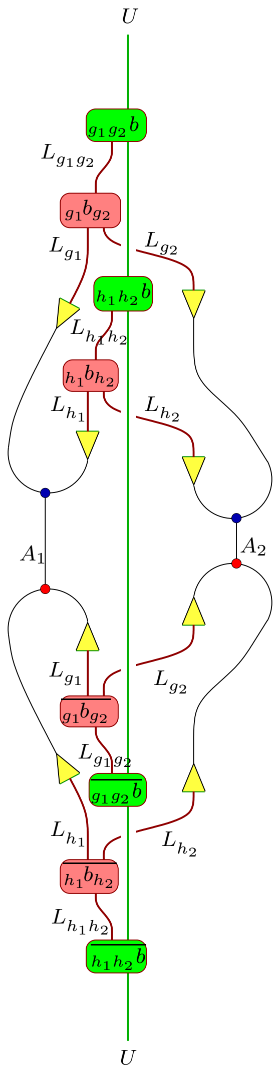


$=$

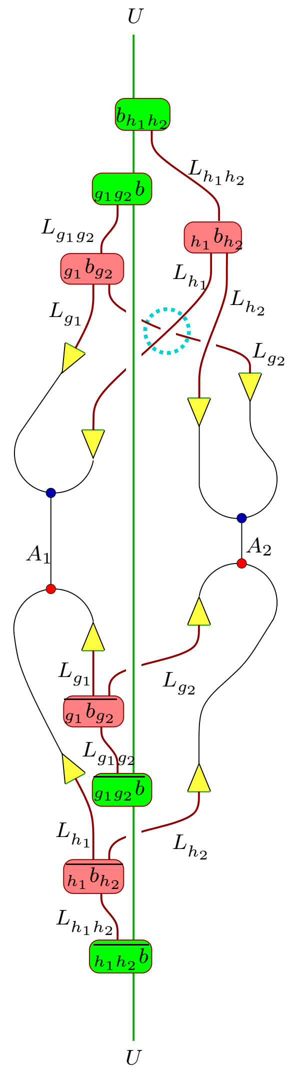


$$= \frac{1}{|H_1|\,|H_2|}\, id_U \, . \qquad (5.14)$$

On the left hand side one gets, after inserting the explicit form (5.10) of the morphisms, and by analogous manipulations as in (4.35)

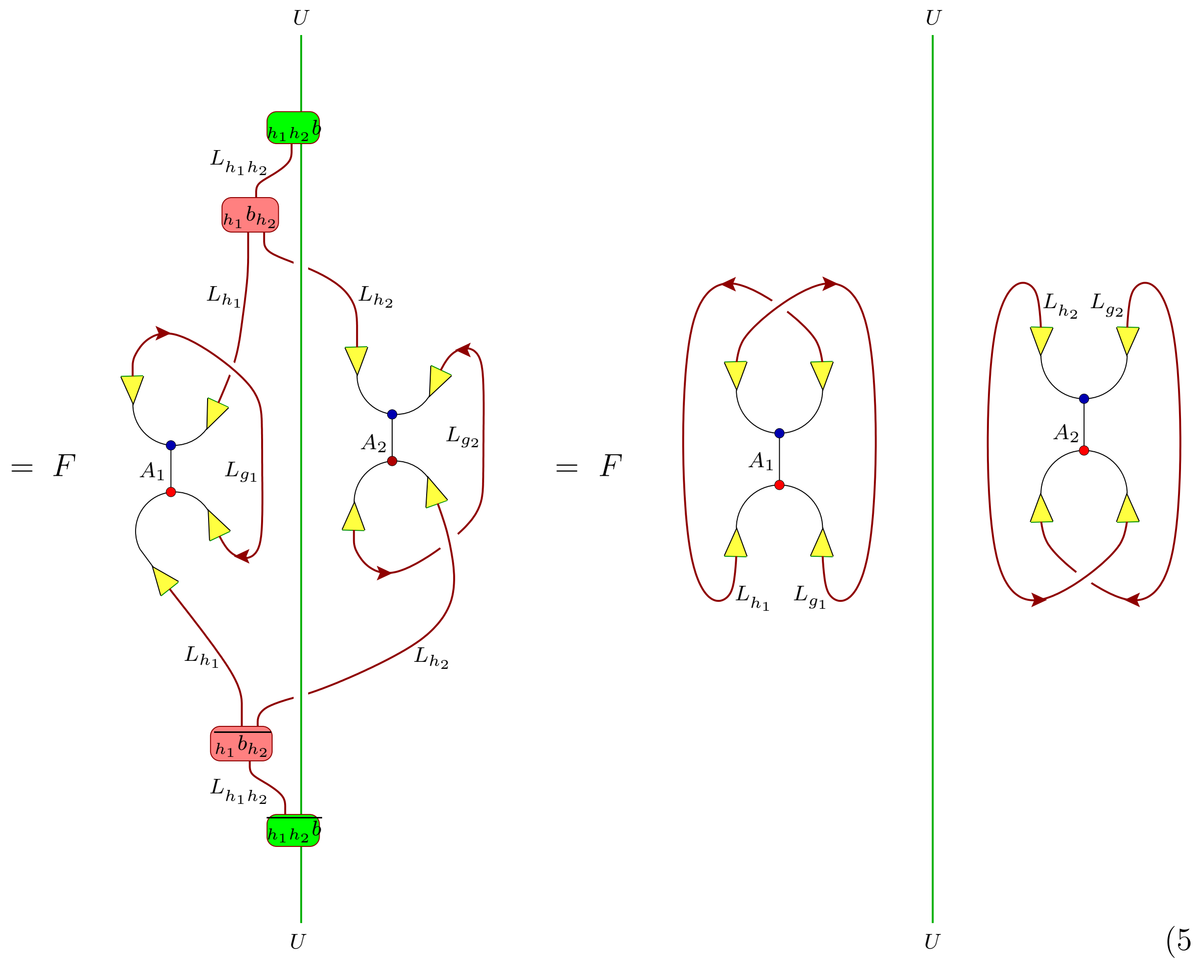


$$= \ \alpha_{h_1h_2}(U)\,\phi_U(g_1g_2,h_1h_2) \tag{5.15}$$



Here the basis indepedent $6j$-symbol $\phi_U(h_1h_2,g_1g_2)$ as defined in (4.1) (note that both products h_1h_2 and g_1g_2 are in $\mathcal{S}(U)$, so that this definition is applicable) appears when interchanging the order of the $L_{h_1h_2}$- and $L_{g_1g_2}$-ribbons. Next, we invert the braiding on the right hand side of (5.15) that is marked with a circle; this gives rise to a factor $\beta(h_1,g_2)$, compare to (2.17). Similarly like in the first step of (4.36) one can now use the relation

(2.1) and the fact that ${}_{g_1g_2}b(U)$ and $\overline{{}_{g_1g_2}b(U)}$ are dual bases so as to obtain



(5.16)

where we use the abbreviation $F = \phi_U(g_1g_2, h_1h_2)\, \beta(h_1, g_2)$. Using (3.36) we thus arrive at

$$\varepsilon_{U,A_1,A_2}(g_1, g_2, h_1, h_2) = \phi_U(g_1g_2, h_1h_2)\, \beta(h_1, g_2)\, \Xi_{A_1}(h_1, g_1)\, \Xi_{A_2}(g_2, h_2)\,, \tag{5.17}$$

Thus we have established (5.11). ✓

After these preliminaries, the determination of the simple A_1-A_2-bimodules continues along the same lines as in the case of left modules that was treated in section 4.2.

Denote by $\{B_\kappa \,|\, \kappa \in \mathcal{K}\}$ a set of representatives of isomorphism classes of simple A_1-A_2-bimodules, and let U be a simple object in $\mathcal{C}$. As for left modules, the isotypical components $B_\kappa^{\oplus n_\kappa}$ in the decomposition

$$\mathrm{Ind}_{A_1|A_2}(U) \cong \bigoplus_{\kappa \in \mathcal{K}} B_\kappa^{\oplus n_\kappa} \tag{5.18}$$

of an induced bimodule can be extracted as the images of idempotents in the center of the endomorphism algebra $\mathrm{End}_{A_1|A_2}(\mathrm{Ind}_{A_1|A_2}(U)) \cong \mathbb{C}_{\beta_{U,A_1,A_2}} \mathcal{S}_{A_1|A_2}(U)$.

To describe this center we introduce the *central bi-stabilizer*

$$\mathcal{U}_{A_1|A_2}(U) := \big\{ (g_1,g_2) \in \mathcal{S}_{A_1|A_2}(U) \,\big|\, \varepsilon_{U,A_1,A_2}(g_1,g_2,h_1,h_2) = 1 \\ \text{for all } (h_1,h_2) \in \mathcal{S}_{A_1|A_2}(U) \big\} \,. \tag{5.19}$$

As for modules we then have

$$\mathcal{Z} \equiv \mathcal{Z}(\mathrm{End}_{A_1|A_2}(\mathrm{Ind}_{A_1|A_2}(U))) \cong \mathcal{Z}(\mathbb{C}_{\beta_{U,A_1,A_2}}\mathcal{S}_{A_1|A_2}(U)) = \mathbb{C}\mathcal{U}_{A_1|A_2}(U)\,. \tag{5.20}$$

Note that, by construction, a basis of the morphism space $\mathrm{End}_{A_1|A_2}(\mathrm{Ind}_{A_1|A_2}(U))$ is given by $\{{}_{g_1}\varphi_{g_2}(U) \,|\, (g_1,g_2) \in \mathcal{S}_{A_1|A_2}(U)\}$ with ${}_{g_1}\varphi_{g_2}(U) := f_{U,U}^{-1}({}_{g_1}\phi_{g_2}(U))$, and with the isomorphism $f_{U,U}$ the one defined in lemma 5.3. As already stated, the multiplication is given by concatenation; following (5.12) it takes the form

$${}_{g_1}\varphi_{g_2}(U) \circ {}_{h_1}\varphi_{h_2}(U) = \beta_{U,A_1,A_2}(g_1,g_2,h_1,h_2)\, {}_{g_1h_1}\varphi_{g_2h_2}(U)\,. \tag{5.21}$$

The definition of the central bi-stabilizer implies that a basis of $\mathcal{Z}$ is, in turn, given by $\{{}_{g_1}\varphi_{g_2}(U) \,|\, (g_1,g_2) \in \mathcal{U}_{A_1|A_2}(U)\}$. On $\mathcal{Z}$ the commutator two-cocycle of β_{U,A_1,A_2} is trivial, and hence we can modify the original choice of basis by suitable scalars so that in the new basis we have, for (g_1,g_2) and (h_1,h_2) in $\mathcal{U}_{A_1|A_2}(U)$,

$${}_{g_1}\varphi_{g_2}(U) \circ {}_{h_1}\varphi_{h_2}(U) = {}_{g_1h_1}\varphi_{g_2h_2}(U)\,. \tag{5.22}$$

The primitive idempotents in $\mathcal{Z}$ can then be given explicitly as

$$\widehat{P}_{U,\psi} = \frac{1}{|\mathcal{U}_{A_1|A_2}(U)|} \sum_{(g,h)\in\mathcal{U}_{A_1|A_2}(U)} \psi^*(g,h)\, {}_g\varphi_h(U)\,, \tag{5.23}$$

where $\psi \in \mathcal{U}_{A_1|A_2}(U)^*$ is a character of the central bi-stabilizer. All simple bimodules in the decomposition of $\mathrm{Ind}_{A_1|A_2}(U)$ occur with the same multiplicity

$$d_{A_1|A_2}(U) = \sqrt{\frac{|\mathcal{S}_{A_1|A_2}(U)|}{|\mathcal{U}_{A_1|A_2}(U)|}}\,. \tag{5.24}$$

The inequivalent simple bimodules in a given induced bimodule $\mathrm{Ind}_{A_1|A_2}(U)$ are thus labeled by characters of $\mathcal{U}_{A_1|A_2}(U)$, once a basis satisfying (5.22) has been chosen. Furthermore, two induced bimodules $\mathrm{Ind}_{A_1|A_2}(U_i)$ and $\mathrm{Ind}_{A_1|A_2}(U_j)$ for simple objects U_i, U_j are isomorphic if U_i and U_j lie on the same H-orbit, and do not contain any common submodules otherwise. This establishes the

Proposition 5.5:
Let A_1 and A_2 be two Schellekens algebras in a modular tensor category $\mathcal{C}$. Denote by H the subgroup of $\mathrm{Pic}(\mathcal{C})$ generated by $H(A_1)$ and $H(A_2)$. Given a choice of basis satisfying (5.22), the isomorphism classes of simple A_1-A_2-bimodules are uniquely labeled by pairs (n,ψ), where n labels an H-orbit in the set $\{U_i \,|\, i \in \mathcal{I}\}$ of simple objects of $\mathcal{C}$ and ψ is a character of the central bi-stabilizer $\mathcal{U}_{A_1|A_2}(U_{m_n})$, for an arbitrary fixed choice of representative U_{m_n} of the nth H-orbit.

Remark 5.6:
(i) The total number of isomorphism classes of simple A_1-A_2-bimodules is given by the formula $\mathrm{tr}\,\big(Z(A_1)Z(A_2)^t\big)$, see remark I:5.19(ii).
(ii) For $A_1 = A_2 = A$, the category $\mathcal{C}_{A|A}$ of A-A-bimodules is itself a tensor category. It thus makes sense to consider the Grothendieck ring (or fusion ring) of $\mathcal{C}_{A|A}$. This ring is in general non-commutative. Denoting again by $\{B_\kappa \,|\, \kappa \in \mathcal{K}\}$ a set of representatives of isomorphism classes of simple bimodules, the structure constants of the fusion ring of $\mathcal{C}_{A|A}$ are defined via

$$B_\mu \otimes_A B_\nu \;\cong\; \bigoplus_{\kappa\in\mathcal{K}} \widehat{N}_{\mu\nu}{}^{\kappa} B_\kappa \,. \tag{5.25}$$

The $\widehat{N}_{\mu\nu}{}^{\kappa}$ are non-negative integers by construction. One can compute these numbers as invariants of a ribbon graph in $S^2 \times S^1$. We have

$$\widehat{N}_{\mu\nu}{}^{\kappa} = \dim \mathrm{Hom}_{A|A}(B_\mu \otimes_A B_\nu, B_\kappa) = Z_{00}^{X|Y} \,, \tag{5.26}$$

where we set $X = B_\mu \otimes_A B_\nu$, $Y = B_\kappa$, and where $Z_{00}^{X|Y}$ is the ribbon graph given in (I:5.151). Note that the knowledge of the projectors (5.23) is sufficient to compute the $\widehat{N}_{\mu\nu}{}^{\kappa}$. However, since the image of these projectors is a direct sum of isomorphic simple bimodules, one must divide the resulting invariant by the corresponding multiplicities (5.24).
(iii) Let A be a simple symmetric special Frobenius algebra. The fusion algebra of A-A-bimodules has a direct physical interpretation. It describes the fusion of conformal defects, i.e. the process of joining two defect lines and decomposing the resulting defect line again into simple defects. As a consequence of (5.25), the matrices $Z_{ij}^{B_\mu|B_\nu}$ obey the condition

$$Z_{ij}^{B_\mu|B_\kappa} = \sum_{\nu\in\mathcal{K}} \widehat{N}_{\mu\nu}{}^{\kappa} Z_{ij}^{A|B_\nu} \,. \tag{5.27}$$

This is the way the fusion algebra of defect lines was introduced in [72].

A Some notions from group cohomology

In this appendix we summarize a few notions from the cohomology of finite groups. In the applications of our interest, the group is abelian. Still, we write the group operation multiplicatively, since the groups we are interested in arise from products in the Grothendieck group of a tensor category and thus from a tensor *product*.

A.1 Ordinary cohomology

We start with ordinary group cohomology with values in $\mathbb{C}^\times$. Let G be a (not necessarily abelian) finite group. A k-cochain on G with values in $\mathbb{C}^\times$ is a function

$$\kappa: \quad G^k \equiv G \times \cdots \times G \to \mathbb{C}^\times . \tag{A.1}$$

κ is called normalized iff its value is $1 \in \mathbb{C}^\times$ as soon as at least one of the arguments is the unit element $e \in G$. The group of k-cochains is denoted by $C^k(G,\mathbb{C}^\times)$. The coboundary operator

$$\mathrm{d} = \mathrm{d}_k : \quad C^k(G,\mathbb{C}^\times) \to C^{k+1}(G,\mathbb{C}^\times) \tag{A.2}$$

is defined by

$$\begin{aligned} \mathrm{d}\kappa(g_1, g_2, \dots, g_{k+1}) = \kappa(g_2, g_3, \dots, g_{k+1})\, \kappa(g_1 g_2, g_3, \dots g_{k+1})^{-1}\, \kappa(g_1, g_2 g_3, \dots, g_{k+1}) \\ \cdots\, \kappa(g_1, g_2, \dots, g_k g_{k+1})^{(-1)^k} \kappa(g_1, g_2, \dots, g_k)^{(-1)^{k+1}} . \end{aligned} \tag{A.3}$$

The elements in the kernel of d_k form a subgroup $Z^k(G,\mathbb{C}^\times)$ of $C^k(G,\mathbb{C}^\times)$; they are called k-cocycles. The elements of the image of d_k form a subgroup $B^{k+1}(G,\mathbb{C}^\times)$ of $C^{k+1}(G,\mathbb{C}^\times)$; they are called $(k+1)$-coboundaries. The operator d squares to one, $\mathrm{d}_{k+1} \circ \mathrm{d}_k = 1 \in \mathbb{C}$. Thus $B^k(G,\mathbb{C}^\times) \le Z^k(G,\mathbb{C}^\times)$; the cohomology H^k is defined as the quotient group:

$$H^k(G,\mathbb{C}^\times) := Z^k(G,\mathbb{C}^\times) \,/\, B^k(G,\mathbb{C}^\times) . \tag{A.4}$$

For a deeper understanding of a cohomology theory, it is useful to have mathematical objects at one's command that are classified by the cohomology groups. The following is well known.

- $H^1(G,\mathbb{C}^\times)$ parametrizes group homomorphisms from G to $\mathbb{C}^\times$, and thus one-dimensional irreducible G-representations.
- $H^2(G,\mathbb{C}^\times)$ (modulo the action of outer automorphisms of G) parametrizes isomorphism classes of twisted group algebras. The group algebra $\mathbb{C}G$ is the unital associative complex algebra with a basis $\{b_g\}_{g\in G}$ labeled by G and multiplication $b_g \star b_{g'} = b_{gg'}$. Given a two-cocycle $\omega \in Z^2(G,\mathbb{C}^\times)$, one can twist the multipliciation by ω according to

$$b_g \star_\omega b_{g'} := \omega(g,g')\, b_{gg'} \tag{A.5}$$

to obtain the *twisted group algebra* $\mathbb{C}_\omega G$. The fact that ω is a cocycle is equivalent to the associativity of the product $\star_\omega$. Cohomologous two-cocycles give rise to isomorphic algebras. If ω is normalised, then b_e is the unit element in $\mathbb{C}_\omega G$. Since every two-cocycle is cohomologous to a normalised two-cocycle, $\mathbb{C}_\omega G$ is an algebra with unit.

■ As we have seen in the main text, $H^3(G,\mathbb{C}^\times)$ parametrizes marked categorifications of the group ring $\mathbb{Z}G$. The fact that $\psi \in Z^3(G,\mathbb{C}^\times)$ is a cocycle ensures that the tensor product on the category is associative. Cohomologous three-cocycles give rise to equivalent categories.

A.2 Abelian group cohomology

From now on the finite group G is assumed to be abelian. So far we have only used the associativity of the product of G, e.g. when checking that d is nilpotent. When G is abelian, it is natural to study also notions that in addition use the commutativity of the product. Indeed we also need *abelian group cohomology*; more precisely, we are only interested in $H^3_{\mathrm{ab}}(G,\mathbb{C}^\times)$.

An abelian two-cochain is an ordinary two-cochain. An *abelian three-cochain* is a pair (ψ,Ω) consisting of an ordinary three-cochain ψ and a two-cochain Ω. It is called normalized iff both ψ and Ω are normalized as cochains. The *abelian coboundary* $\mathrm{d}_{\mathrm{ab}}\kappa$ of an abelian two-cochain κ is an abelian three-cochain defined as

$$\mathrm{d}_{\mathrm{ab}}\kappa := (\mathrm{d}\kappa\,, \kappa_{\mathrm{comm}})\,, \tag{A.6}$$

i.e. one takes the pair consisting of the ordinary coboundary of the two-cochain κ and the *commutator cocycle* κ_{comm} of κ, defined by

$$\kappa_{\mathrm{comm}}(x,y) := \kappa(x,y)\,\kappa(y,x)^{-1}\,. \tag{A.7}$$

An abelian three-cochain (ψ,Ω) is an *abelian three-cocycle* iff ψ is an ordinary three-cocycle, $\mathrm{d}\psi = 1$, and

$$\begin{aligned} &\psi(y,z,x)^{-1}\,\Omega(x,yz)\,\psi(x,y,z)^{-1} = \Omega(x,z)\,\psi(y,x,z)^{-1}\,\Omega(x,y)\,,\\ &\psi(z,x,y)\,\Omega(xy,z)\,\psi(x,y,z) = \Omega(x,z)\,\psi(x,z,y)\,\Omega(y,z)\,. \end{aligned} \tag{A.8}$$

(In the notation used in [73], $\psi(x,y,z) = f(x,y,z)$ and $\Omega(x,y) = d(x\,|\,y)$, compare formulas (17) – (19) of [73].)

This means that for closed abelian three-cochains the deviation of Ω from being a *bihomomorphism* (i.e. from being multiplicative in both arguments) is controlled by the ordinary three-cochain ψ. Finally, the abelian cohomology group $H^3_{\mathrm{ab}}(G,\mathbb{C}^\times)$ is defined as the quotient of the group of normalized three-cocycles by its subgroup consisting of coboundaries of normalized two-cochains. Forgetting Ω provides a group homomorphism

$$\begin{aligned} H^3_{\mathrm{ab}}(G,\mathbb{C}) &\;\to\; H^3(G,\mathbb{C})\\ [(\psi,\Omega)] &\;\mapsto\; [\psi]\,. \end{aligned} \tag{A.9}$$

The fibers of this map are bihomomorphisms modulo alternating bihomomorphisms. (A bihomomorphism ζ on G is called alternating iff $\zeta(g,g) = 1$ for all $g \in G$, see definition 3.15.)

As shown in the main text, the objects parametrized by the abelian cohomology $H^3_{\rm ab}(G,\mathbb{C}^\times)$ are the marked theta-categories that are marked categorifications of $\mathbb{Z}G$. Another important aspect of the abelian group cohomology $H^3_{\rm ab}(G,\mathbb{C}^\times)$ is its relation to quadratic forms on the abelian group G. A *quadratic form* on G with values in the multiplicative group $\mathbb{C}^\times$ is a map

$$q: \quad G \to \mathbb{C}^\times \tag{A.10}$$

such that $q(g)=q(g^{-1})$ and such that the function

$$\begin{array}{rrcl} \beta_q: & G\times G & \to & \mathbb{C}^\times \\ & (g_1,g_2) & \mapsto & q(g_1g_2)\,q(g_1)^{-1}q(g_2)^{-1} \end{array} \tag{A.11}$$

is a bihomomorphism, which is called the *associated bihomomorphism*. The product of two quadratic forms is again a quadratic form, so that quadratic forms form an abelian group $\mathbb{QF}(G,\mathbb{C}^\times)$.

Note that while the quadratic form completely determines the bihomomorphism, the converse need not be true. Indeed, two quadratic forms q, $\tilde q$ on G possess the same associated bihomomorphism $\beta_q=\beta_{\tilde q}$ iff $\tilde q=q\chi$ for some $\chi\in H^1(G,\mathbb{Z}_2)$. That χ takes values in $\mathbb{Z}_2$ follows because any quadratic form obeys $q(g)=q(g^{-1})$.

The map

$$\begin{array}{rrcl} \mathbb{EM}: & H^3_{\rm ab}(G,\mathbb{C}^*) & \to & \mathbb{QF}(G,\mathbb{C}^\times) \\ & [(\psi,\Omega)] & \mapsto & q \quad \text{with} \quad q(g):=\Omega(g,g) \end{array} \tag{A.12}$$

is a homomorphism of abelian groups. In particular, the quadratic form depends only on the cohomology class. It has been shown by Eilenberg and MacLane [15, 73] that $\mathbb{EM}$ is even an *iso*morphism. Moreover, they show that the associated bihomomorphisms obeys

$$\beta_{q(\psi,\Omega)}(g,h) = \Omega(g,h)\,\Omega(h,g)\,. \tag{A.13}$$

B KS matrices versus KSBs

Here we present the alternative description of KSBs that is due to Kreuzer and Schellekens [16]. For easier comparison with the original literature [16] we use CFT terminology, in particular the conformal weights Δ_i and the monodromy charges (2.22). The conformal weights are only needed modulo $\mathbb{Z}$, so that it is sufficient to know the balancing phases $\theta_i = \exp(-2\pi \mathrm{i}\Delta_i)$ of the modular tensor category $\mathcal{C}$.

Definition B.1:
Let H be a subgroup in the effective center $\mathrm{Pic}^\circ(\mathcal{C})$. Write H in the form

$$H \,\cong\, \mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}\times\cdots\times\mathbb{Z}_{n_k}\,, \tag{B.1}$$

with generators $g_1,g_2,\dots,g_k$ of the cyclic factors. A *KS matrix* for H is a $k\times k$-matrix with entries X_{ab}, defined modulo $\mathbb{Z}$, that satisfy the constraints

$$
\begin{aligned}
&\text{i)} \quad X_{ab} + X_{ba} = Q_{g_a}(g_b) \bmod \mathbb{Z} \quad \text{for all } a,b \qquad \text{and} \\
&\text{ii)} \quad N_a\, X_{ab} \in \mathbb{Z} \quad \text{for all } a,b\,, \;\text{ with } N_a = N_{g_a} \text{ the order of } g_a\,.
\end{aligned}
\tag{B.2}
$$

If the order N_a of g_a is even, then in addition

$$
\text{iii)} \quad \Delta_{g_a} = (N_a{-}1)\, X_{aa} \;\bmod \mathbb{Z} \quad \text{for all } a\,. \tag{B.3}
$$

(In (B.1) we do not impose any divisibility conditions on the n_a, so the decomposition is not unique.)

To prove the relation between KSBs and KS matrices we use the following properties of quadratic forms.

Lemma B.2:
For G a finitely generated abelian group with generators $g_1, g_2, \dots, g_r$, let $q, q' \in \mathrm{QF}(G, \mathbb{C}^\times)$.
(i) If $q(g_i) = q'(g_i)$ and $q(g_i g_j) = q'(g_i g_j)$ for all $i,j \in \{1, \dots, r\}$ with $i \neq j$, then $q = q'$.
(ii) For every $g \in G$ we have $q(g^n) = q(g)^{n^2}$.

Proof:
(i) Since q is a quadratic form, by definition $q(g) = q(g^{-1})$, and $\beta_q(g,h) = q(gh)/(q(g)q(h))$ is a bihomomorphism. Expressing $g,h \in G$ in terms of the generators as $g = g_1^{m_1} \cdots g_r^{m_r}$ and $h = g_1^{n_1} \cdots g_r^{n_r}$ and using the bihomomorphism property of β_q, we have

$$
\beta_q(g,h) = \beta_q(g_1^{m_1} \cdots g_r^{m_r}, g_1^{n_1} \cdots g_r^{n_r}) = \prod_{i,j=1}^{r} \beta_q(g_i, g_j)^{m_i n_j}\,, \tag{B.4}
$$

which when expressed in terms of q amounts to

$$
q(gh) = q(g)\, q(h) \prod_{i,j=1}^{r} \Big(\frac{q(g_i g_j)}{q(g_i)\, q(g_j)} \Big)^{m_i n_j} . \tag{B.5}
$$

$q(g)$ can therefore be determined recursively for all $g \in G$ by starting from $q(g_i)$ and $q(g_i g_j)$. Next note that since β_q is a bihomomorphism, we have $\beta_q(g, g^{-1}) = 1/\beta_q(g,g)$. Written in terms of q this implies $q(g^2) = q(g)^4$. Hence $q(g_i g_i)$ can be expressed through $q(g_i)$.
These arguments show that a quadratic form q is uniquely determined for all $g \in G$ once we prescribe its values on $g = g_i$ and $g = g_i g_j$ for $i \neq j$. In particular, two quadratic forms that coincide on these group elements are equal.
(ii) Since β_q is a bihomomorphism, we have $\beta_q(g^n, g) = \beta_q(g,g)^n$. In terms of $q(g)$ this implies, using also $q(g^2) = q(g)^4$,

$$
q(g^{n+1}) = q(g^n)\, q(g)^{2n+1} . \tag{B.6}
$$

This recursion relation has the unique solution $q(g^n) = q(g)^{n^2}$. ✓

The relation between between KSBs and KS matrices is now provided by

Lemma B.3:
Let H be a subgroup in the effective center $\mathrm{Pic}^\circ(\mathcal{C})$, written in the form (B.1), and X a $k\times k$-matrix. For any two elements g,h of H, written in the form $g=\prod_a (g_a)^{m_a}$ and $h=\prod_a (g_a)^{n_a}$, set

$$\Xi(g,h) := \exp(2\pi\mathrm{i} \sum_{a,b=1}^{k} m_a X_{ab} n_b)\,. \tag{B.7}$$

Then Ξ is a KSB if and only if X is a KS matrix.

Proof:
Suppose that $\Xi(g,h)$ is a KSB, i.e. Ξ is a bihomomorphism with the additional property $\Xi(g,g)=\theta_g$. Then in particular it is a well-defined map $G\times G\to\mathbb{C}^\times$. Since $g_a^{N_a}=e$, where N_a is the order of the generator g_a, we have

$$1 = \Xi(g_a^{N_a}, g_b) = \exp(2\pi\mathrm{i} N_a X_{ab}) \tag{B.8}$$

for all a,b, i.e. $N_a X_{ab}\in\mathbb{Z}$, establishing that the matrix X has property (B.2 ii). Further, by lemma 3.18, together with (2.21) and (2.22) we know that

$$\exp(2\pi\mathrm{i}(X_{ab}{+}X_{ba})) = \Xi(g_a,g_b)\,\Xi(g_b,g_a) = \beta(g_a,g_b) = \exp(2\pi\mathrm{i} Q_{g_a}(g_b))\,. \tag{B.9}$$

The matrix X thus also has property (B.2 i). Finally, property (B.3) of X follows from the fact that a KSB by definition obeys $\Xi(g_a,g_a)=\theta_{g_a}$, together with $N_a X_{ab}\in\mathbb{Z}$. Thus X is indeed a KS matrix.

Suppose now that X is a KS matrix. We first check that Ξ as defined in (B.7) is a well-defined map $G\times G\to\mathbb{C}^\times$. Note that the m_a and n_a in $g=\prod_a (g_a)^{m_a}$ and $h=\prod_a (g_a)^{n_a}$ are defined only mod N_a. Shifting $m_a \mapsto m_a + kN_a$ changes the right hand side of (B.7) by $\exp(2\pi\mathrm{i}k\sum_b N_a X_{ab} n_b)$, which is equal to one by property (B.2 ii) of a KS matrix. Similarly, shifting $n_b \mapsto n_b + kN_b$ changes the right hand side by

$$\begin{aligned} \mathrm{e}^{2\pi\mathrm{i}k\sum_a m_a X_{ab} N_b} &= \mathrm{e}^{2\pi\mathrm{i}k\sum_a m_a Q_{g_a}(g_b) N_b}\, \mathrm{e}^{-2\pi\mathrm{i}k\sum_a N_b X_{ba} m_a} \\ &= \prod\nolimits_a \beta(g_a,g_b)^{m_a N_b} = \prod\nolimits_a \beta(g_a,g_b^{N_b})^{m_a} = 1\,, \end{aligned} \tag{B.10}$$

where in the first step (B.2 i) is used, in the second step (B.2 ii) and (2.21), and in the third and fourth step that β is a bihomomorphism and that $g_b^{N_b}=e$.

Thus Ξ is well defined. That Ξ is a bihomomorphism is then obvious. It follows that $q(g)=\Xi(g,g)$ is a quadratic form. To establish that Ξ is a KSB we must show that q coincides with the quadratic form $\delta(g)=\theta_g$. By lemma B.2 it is enough to verify $q(g_a)=\delta(g_a)$ and $q(g_a g_b)=\delta(g_a g_b)$ for $a\neq b$. First note that by (B.2 i) and (2.21),

$$\Xi(g_a,g_b)\,\Xi(g_b,g_a) = \mathrm{e}^{2\pi\mathrm{i}(X_{ab}+X_{ba})} = \beta(g_a,g_b)\,. \tag{B.11}$$

By definition (2.16), we have $\beta(g_a,g_a)=\delta(g_a^2)/\delta(g_a)^2$. Now, δ is a quadratic form, and by lemma B.2(ii) we know $\delta(g_a^2)=\delta(g_a)^4$. Evaluating (B.11) for $a=b$ thus yields

$$q(g_a)^2 = \delta(g_a)^2\,. \tag{B.12}$$

Furthermore,

$$1 = q(g_a^{N_a}) = q(g_a)^{N_a^2} \qquad \text{and} \qquad 1 = \delta(g_a^{N_a}) = \delta(g_a)^{N_a^2}\,, \tag{B.13}$$

where each time in the first step it is used that g_a has order N_a, while in the second step lemma B.2(ii) is employed. Now note that for N_a odd, at most one of the two roots x of $x^2 = \delta(q_a)^2$ can also satisfy $x^{N_a^2} = 1$. Since both $q(g_a)$ and $\delta(g_a)$ obey both of these equalities, they must be equal, $q(g_a) = \delta(g_a)$. On the other hand, for even N_a, the equality $q(g_a) = \delta(g_a)$ is an immediate consequence of the properties (B.2 ii) and (B.3),

$$q(g_a) = \Xi(g_a, g_a) = \exp(2\pi\mathrm{i} X_{aa}) = \exp(-2\pi\mathrm{i}\Delta_{g_a}) = \theta_{g_a}\,. \tag{B.14}$$

Finally, for $q(g_a g_b)$ we find

$$\begin{aligned} q(g_a g_b) = \Xi(g_a g_b, g_a g_b) &= \Xi(g_a, g_a)\,\Xi(g_b, g_b)\,\Xi(g_a, g_b)\,\Xi(g_b, g_a) \\ &= \theta_{g_a}\theta_{g_b}\,\beta(g_a, g_b) = \theta_{g_a g_b}\,, \end{aligned} \tag{B.15}$$

where in the second step it is used that Ξ is a bihomomorphism, in the third step (B.11) is substituted, and the fourth step uses the definition of β in (2.16). This shows that $q(g_a) = \delta(g_a)$ and $q(g_a g_b) = \delta(g_a g_b)$ for all a, b. It follows that $q = \delta$ and hence $\Xi(g, h)$ is indeed a KSB. ✓

C Conventions

C.1 Basis choices

Simple currents. Bases for the three-point coupling spaces are denoted by

$${}_{g_1}b_{g_2} \in \mathrm{Hom}(L_{g_1} \otimes L_{g_2}, L_{g_1 g_2})\,. \tag{C.1}$$

For $g = e$, we take the identity morphism, ${}_e b_g = id_{L_g} = {}_g b_e$ (here we use that $L_e = \mathbf{1}$ is a *strict* tensor unit).

Algebras. An adapted basis choice for the morphism spaces involving a Schellekens algebra $(A, m, \eta, \Delta, \varepsilon)$ and its simple subobjects is:

$$\begin{aligned} &e_g \in \mathrm{Hom}(L_g, A)\,, \qquad && r_g \in \mathrm{Hom}(A, L_g)\,, \qquad && r_g \circ e_g = id_{L_g}\,, \\ &e_1 = \eta\,, && r_1 = \frac{1}{\dim(A)}\,\varepsilon\,. && \end{aligned} \tag{C.2}$$

Fixed points. For a simple object U with $L_g \otimes U \cong U$ – a fixed point of g – bases of the relevant morphism spaces are denoted by

$$b_g(U) \in \mathrm{Hom}(U \otimes L_g, U) \qquad \text{and} \qquad {}_g b(U) \in \mathrm{Hom}(L_g \otimes U, U)\,. \tag{C.3}$$

The corresponding dual basis vectors

$$\overline{b_g(U)} \in \mathrm{Hom}(U, U \otimes L_g) \qquad \text{and} \qquad \overline{{}_g b(U)} \in \mathrm{Hom}(U, L_g \otimes U) \tag{C.4}$$

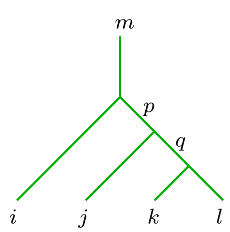
are then uniquely determined via

$$b_g(U) \circ \overline{b_g(U)} = id_U \qquad \text{and} \qquad {}_g b(U) \circ \overline{{}_g b(U)} = id_U\ . \tag{C.5}$$

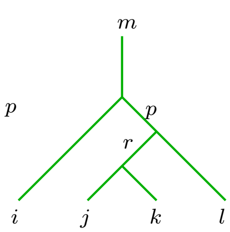
C.2 Pentagon, hexagons, and abelian three-cocycles

In this appendix we compare the pentagon and the two hexagon identities for the fusion and braiding matrices F, R to the corresponding conditions for (ψ, Ω) to be an abelian three-cocycle.

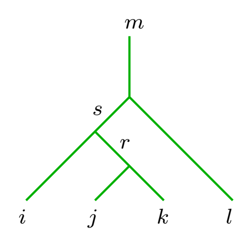
We first spell out the pentagon and the two hexagons for a ribbon category in which $\dim \mathrm{Hom}(U_i \otimes U_j, U_k) \in \{0, 1\}$ for all triples of simple objects U_i, U_j and U_k. This holds in particular for theta-categories, which is the case we are interested in. By the definition of the 6j-symbols F (see (I:2.36)) we have (compare also appendix II:A.1),



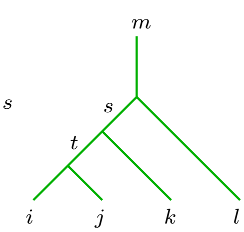
$$= \sum_r \mathsf{F}^{(j\,k\,l)\,p}_{q\,r}$$



$$= \sum_{r,s} \mathsf{F}^{(j\,k\,l)\,p}_{q\,r}\ \mathsf{F}^{(i\,r\,l)\,m}_{p\,s}$$

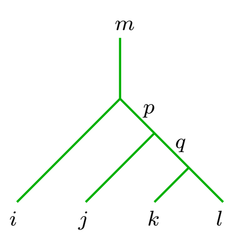


$$= \sum_{r,s,t} \mathsf{F}^{(j\,k\,l)\,p}_{q\,r}\ \mathsf{F}^{(i\,r\,l)\,m}_{p\,s}\ \mathsf{F}^{(i\,j\,k)\,s}_{r\,t}$$

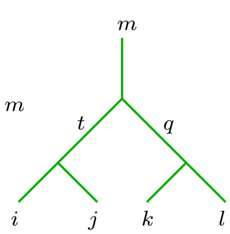


(C.6)

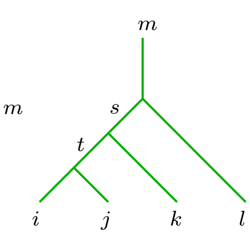
as well as



$$= \sum_t \mathsf{F}^{(i\,j\,q)\,m}_{p\,t}$$



$$= \sum_{r,t} \mathsf{F}^{(i\,j\,q)\,m}_{p\,t}\ \mathsf{F}^{(t\,k\,l)\,m}_{q\,s}$$

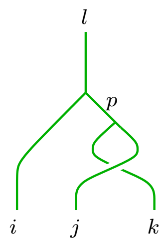


(C.7)

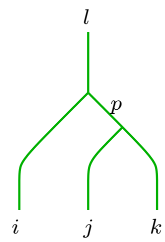
Comparison yields the pentagon identity

$$\sum_r \mathsf{F}^{(j\,k\,l)\,p}_{q\,r}\,\mathsf{F}^{(i\,r\,l)\,m}_{p\,s}\,\mathsf{F}^{(i\,j\,k)\,s}_{r\,t} = \mathsf{F}^{(i\,j\,q)\,m}_{p\,t}\,\mathsf{F}^{(t\,k\,l)\,m}_{q\,s}\,. \tag{C.8}$$

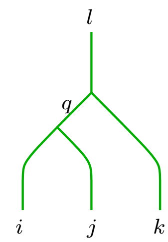
Similarly, using also the definition of the braiding matrices R (see (I:2.41)) we have,



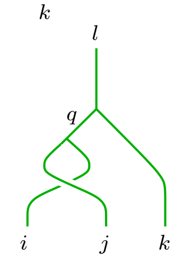
$$= \mathsf{R}^{(j\,k)\,p}$$



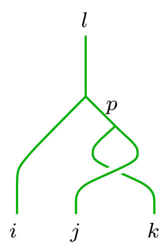
$$= \sum_q \mathsf{R}^{(j\,k)\,p}\,\mathsf{F}^{(i\,j\,k)\,l}_{p\,q}$$



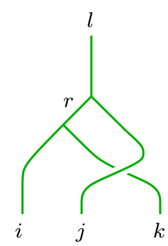
$$= \sum_q \mathsf{R}^{(j\,k)\,p}\,\mathsf{F}^{(i\,j\,k)\,l}_{p\,q}\,\big(\mathsf{R}^{-(i\,j)\,q}\big)^{-1} \tag{C.9}$$



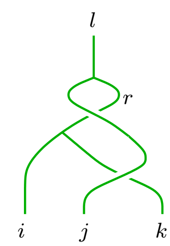
and



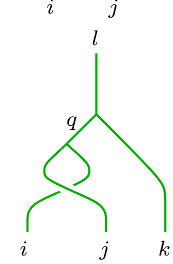
$$= \sum_r \mathsf{F}^{(i\,k\,j)\,l}_{p\,r}$$



$$= \sum_r \mathsf{F}^{(i\,k\,j)\,l}_{p\,r}\,\big(\mathsf{R}^{-(r\,j)\,l}\big)^{-1}$$



$$= \sum_{q,r} \mathsf{F}^{(i\,k\,j)\,l}_{p\,r}\big(\mathsf{R}^{-(r\,j)\,l}\big)^{-1}\,\mathsf{F}^{(j\,i\,k)\,l}_{r\,q} \tag{C.10}$$



Using also that $\mathsf{R}^{(m\,n)\,t}\,\mathsf{R}^{-(n\,m)\,t} = 1$, comparison yields the first of the two hexagon identities

$$\mathsf{R}^{(j\,k)\,p}\,\mathsf{F}^{(i\,j\,k)\,l}_{p\,q}\,\mathsf{R}^{(j\,i)\,q} = \sum_r \mathsf{F}^{(i\,k\,j)\,l}_{p\,r}\,\mathsf{R}^{(j\,r)\,l}\,\mathsf{F}^{(j\,i\,k)\,l}_{r\,q}\,. \tag{C.11}$$

The second hexagon identity is obtained by reversing the braiding in all pictures. Explicitly,

$$\big(\mathsf{R}^{(k\,j)\,p}\big)^{-1}\,\mathsf{F}^{(i\,j\,k)\,l}_{p\,q}\,\big(\mathsf{R}^{(i\,j)\,q}\big)^{-1} = \sum_r \mathsf{F}^{(i\,k\,j)\,l}_{p\,r}\,\big(\mathsf{R}^{(r\,j)\,l}\big)^{-1}\,\mathsf{F}^{(j\,i\,k)\,l}_{r\,q}\,. \tag{C.12}$$

Consider now the special case of a theta-category $\mathcal{D}$ and denote $\mathrm{Pic}(\mathcal{D})$ by G. Then all simple objects are invertible, so that the r-summations in (C.8), (C.11) and (C.12) reduce to a single term. Two possible ways to define (ψ,Ω) in terms of F, R are, first,

$$\mathsf{F}^{(g\;h\;k)\;g\cdot h\cdot k}_{\;h\cdot k\;\;g\cdot h} = \psi(g,h,k) \qquad \text{and} \qquad \mathsf{R}^{\,(g\;h)\,g\cdot h} = \Omega(h,g)\,, \tag{C.13}$$

and second,

$$\mathsf{F}^{(g\;h\;k)\;g\cdot h\cdot k}_{\;h\cdot k\;\;g\cdot h} = \psi(g,h,k)^{-1} \qquad \text{and} \qquad \mathsf{R}^{\,(g\;h)\,g\cdot h} = \Omega(h,g)^{-1}\,. \tag{C.14}$$

Here $g,h,k\in G$, and for better readability we spelled out the product '$\cdot$' of G explicitly. One can now verify that for both choices the conditions $\mathrm{d}\psi=1$ and (A.8) for (ψ,Ω) to be an abelian 3-cocycle are equivalent to the pentagon (C.8) and the two hexagon identities (C.11), (C.12).

In the definition of the theta-category $\mathcal{C}(G,\psi,\Omega)$ in lemma 2.8 we have selected the second possible identification, i.e. (C.14).

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We dedicate this paper to Bert Schellekens, pioneer of simple currents, on the occasion of his 50th birthday.

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