Differential geometric aspects of the tt*-equations

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Abstract. We discuss the relation between pluriharmonic maps and solutions of the tt*-equations. The correspondence is obtained without assuming the integrability of the almost complex structure of the base manifold. We present examples which stem from special Kähler and nearly Kähler geometry. This paper is based on a lecture given by the first author on the Workshop 'From tQFT to tt* and integrability' (Augsburg, 2007).

Introduction

The tt*-equations were proposed by Cecotti and Vafa in 1991 [**CV**] as a description of the geometric structure on the moduli space of $2D$ $N = 2$ supersymmetric QFTs. It was discovered by Dubrovin in 1992 [**D**] that solutions of the tt*-equations correspond to pluriharmonic maps from a complex manifold M of dimension n into $\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$. This correspondence has been further developed by Simpson [**Si**] and the second author [**S1**].

In these notes we will present a general version of the tt*-equations purely in terms of differential geometry, explain the relation with pluriharmonic maps and discuss some classes of solutions. A proof of the above correspondence which does not assume the integrability of the almost complex structure of the base manifold is given.

1. tt*-bundles

Definition 1.1. A tt*-bundle (E, D, S) over an almost complex manifold (M, J) is a real vector bundle $E \to M$ endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \operatorname{End} E)$, which satisfy the tt*-equation

$$R^\theta = 0,$$

for all $\theta \in \mathbb{R}$, where R^θ is the curvature of the connection

$$D^\theta_X := D_X + \cos\theta\, S_X + \sin\theta\, S_{JX},\ \ X \in \Gamma(TM).$$

A metric/symplectic tt*-bundle (E, D, S, β) is a tt*-bundle (E, D, S) endowed with a parallel non-degenerate symmetric/skew-symmetric bilinear form β on the fibers

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of E such that S_X is β-symmetric for all $X \in \Gamma(TM)$. It is called unimodular if $\operatorname{tr} S_X = 0$ for all $X \in \Gamma(TM)$.

REMARKS 1.2. 1) Any oriented unimodular metric/symplectic tt*-bundle carries a canonical volume element[1] $\nu \in \Gamma(\Lambda^r E^*)$, $r = \operatorname{rk} E$, which is D^θ-parallel for all $\theta \in \mathbb{R}$.
2) The tt*-equation $R^\theta = 0$ for all $\theta \in \mathbb{R}$ is equivalent to the following system (1.1-1.3):

$$d^D S = d^D S_J = 0, \text{ where} \tag{1.1}$$
$$(d^D S)(X,Y) = D_X(S_Y) - D_Y(S_X) - S_{[X,Y]},$$
$$[S_X, S_Y] = [S_{JX}, S_{JY}], \quad \forall X, Y \in \Gamma(TM), \tag{1.2}$$
$$R^D(X,Y) + [S_X, S_Y] = 0, \quad \forall X, Y \in \Gamma(TM). \tag{1.3}$$

2. Pluriharmonic maps

DEFINITION 2.1. A smooth map $f : M \to N$ from a complex manifold M to a pseudo-Riemannian manifold N is called pluriharmonic if its restriction to any complex curve is harmonic.

This is equivalent to the differential equation

$$(\nabla df)^{1,1} = 0, \tag{2.1}$$

where ∇ is the connection on $T^*M \otimes f^*TN$ induced by a torsion-free complex connection D on M, i.e. a torsion-free connection D satisfying $DJ = 0$, and the Levi-Civita connection on N. More generally, any connection D on M such that D is complex and $(T^D)^{1,1} = 0$ leads to the same equation (2.1). Here, and in the following, T^D stands for the torsion of the connection D. Such connections D exist on any almost complex manifold (M, J) and we can define the notion of pluriharmonicity for maps with an almost complex source manifold by equation (2.1). We point out that a pluriharmonic map $f : M \to N$ is harmonic for any almost Hermitian metric g on M such that $\nabla^g - D$ is trace-free, where ∇^g is the Levi-Civita connection of g. This follows from $\operatorname{tr}_g \nabla^g df = \operatorname{tr}_g \nabla df = \operatorname{tr}_g (\nabla df)^{1,1} = 0$. In particular, any pluriharmonic map from a pseudo-Kähler manifold to a pseudo-Riemannian manifold is harmonic, since in that case one can choose $D = \nabla^g$. It is also clear that holomorphic maps between pseudo-Kähler manifolds are pluriharmonic. More generally, Rawnsley [**Ra**] has shown that a holomorphic map between almost Hermitian manifolds is pluriharmonic if the fundamental 2-forms of the domain and target manifolds satisfy $(d\omega)^{1,2} = 0$.

3. Correspondence between metric/symplectic tt*-bundles and pluriharmonic maps

In [**D**] Dubrovin developed a correspondence between certain metric tt*-bundles (E, D, S, g) with a positive-definite metric g over a complex manifold of complex dimension n and pluriharmonic maps $M \to \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$. In his case $E = (T^{1,0}M)^\rho$ consists of the real points of the holomorphic tangent bundle with respect to a real structure ρ. Independently, Simpson [**Si**] established a correspondence between Higgs bundles of rank r on M endowed with a harmonic metric and harmonic maps

[1] Notice that in the symplectic case the bundle is automatically oriented. The volume element is $\nu = \frac{\beta^r}{r!}$.

$M \to \mathrm{GL}(r,\mathbb{C})/\mathrm{U}(r)$. These results were generalised by the second author [**S1**], who proved the next theorem under the assumption that the almost complex structure on M is integrable. The result of Simpson, for instance, can be recovered from the next Theorem using the totally geodesic inclusion

$$\frac{\mathrm{GL}(r,\mathbb{C})}{\mathrm{U}(r)} \subset \frac{\mathrm{GL}(2r,\mathbb{R})}{\mathrm{O}(2r)}$$

as was shown in [**S2**]. The next theorem is proven in Section 5. In the following, $Sym_{p,q}(\mathbb{R}^r)$ stands for the cone of symmetric $r \times r$-matrices of signature (p,q), $p+q=r$. Similarly, for even r the symbol $Skew_{reg}(\mathbb{R}^r)$ stands for the cone of invertible skew-symmetric $r \times r$-matrices.

THEOREM 3.1. *Let (E,D,S,β) be a metric/symplectic tt*-bundle over a simply connected almost complex manifold (M,J). For any $\theta \in \mathbb{R}$ there exists a D^θ-parallel frame $(e_1^\theta,\dots,e_r^\theta)$ of E and the correspondence*

$$x \mapsto f_\theta(x) := (\beta(e_i^\theta(x), e_j^\theta(x)))$$

defines a pluriharmonic map

$$f_\theta : M \to \begin{cases} Sym_{p,q}(\mathbb{R}^r) \cong \mathrm{GL}(r,\mathbb{R})/\mathrm{O}(p,q), \\ Skew_{reg}(\mathbb{R}^r) \cong \mathrm{GL}(r,\mathbb{R})/\mathrm{Sp}(r,\mathbb{R}), \end{cases}$$

$$\text{if } \beta \text{ is } \begin{cases} \text{symmetric of signature } (p,q),\ p+q=r, \\ \text{skew-symmetric, respectively.} \end{cases}$$

The target manifold is a pseudo-Riemannian symmetric space with the metric induced from the (bi-invariant) trace form, i.e. $\langle A,B\rangle = tr(AB)$, on $\mathfrak{gl}(r,\mathbb{R})$. If (E,D,S,β) is oriented and unimodular, f_θ takes values in the irreducible symmetric space

$$\begin{cases} \mathrm{SL}(r,\mathbb{R})/\mathrm{SO}(p,q), \\ \mathrm{SL}(r,\mathbb{R})/\mathrm{Sp}(r,\mathbb{R}), \textit{respectively.} \end{cases}$$

In all cases the pluriharmonic map $f = f_\theta$ has the following additional property:

$$R^N(df\,T^{1,0}M, df\,T^{1,0}M) = 0, \tag{3.1}$$

where R^N is the curvature tensor of the symmetric target manifold

$$N = \begin{cases} \mathrm{GL}(r,\mathbb{R})/\mathrm{O}(p,q), & \mathrm{SL}(r,\mathbb{R})/\mathrm{SO}(p,q), \\ \mathrm{GL}(r,\mathbb{R})/\mathrm{Sp}(r,\mathbb{R}), & \mathrm{SL}(r,\mathbb{R})/\mathrm{Sp}(r,\mathbb{R}), \ \textit{respectively.} \end{cases} \tag{3.2}$$

Conversely, any pluriharmonic map $f : M \to N$, into one of the symmetric target manifolds (3.2) *which has the property* (3.1) *and satisfies*

$$\bar\partial f(T(Z,W)) = df(T(Z,W)^{0,1}) = 0 \tag{3.3}$$

for all $Z,W \in T^{1,0}M$, is obtained from a metric/symplectic tt-bundle by the above construction.*

The property (3.1) is automatic if N is Riemannian, since there it holds

$$0 = \langle R^N(Z,W)\bar W, \bar Z\rangle = -\|[Z,W]\|^2 \Rightarrow [Z,W]=0,\ \forall Z,W \in T^{1,0}N.$$

We remark that, in general, $R^N(df\,T^{1,0}M, df\,T^{1,0}M) \subset TN$ is not necessarily zero but only isotropic.

4. Solutions

4.1. Special complex and special Kähler manifolds.

DEFINITION 4.1. A special complex manifold (M, J, ∇) is a complex manifold endowed with a flat torsion-free connection ∇, such that

$$(\nabla_X J)Y = (\nabla_Y J)X, \ \forall X, Y \in \Gamma(TM). \tag{4.1}$$

A special Kähler manifold (M, J, ∇, ω) is a special complex manifold endowed with a J-invariant ∇-parallel symplectic structure ω. The (pseudo-)Kähler metric $g = \omega(J\cdot, \cdot)$ is called the special Kähler metric of (M, J, ∇, ω).

We remark that the integrability of J follows from the properties of ∇. Hertling has observed in section 3.3 of [**He**] that a special complex manifold is the same as a *variation of Hodge structure* (VHS) on $TM \otimes \mathbb{C}$ and a special Kähler manifold is the same as a *polarized* such VHS.

Let Ω be the canonical symplectic form on $T^*\mathbb{C}^n = \mathbb{C}^{2n}$. We recall that a holomorphic immersion ϕ from a complex manifold of dimension n into $T^*\mathbb{C}^n$ is called Lagrangian if $\phi^*\Omega = 0$. Define the sesquilinear form $\gamma := i\Omega(\cdot, \bar{\cdot})$. Then ϕ is called non-degenerate if $\phi^*\gamma$ is non-degenerate.

THEOREM 4.2. [**ACD**] *Any simply connected special complex (resp. special Kähler) manifold M admits a holomorphic immersion (resp. a non-degenerate Lagrangian immersion) $\phi : M \to T^*\mathbb{C}^n$, $n = \dim_{\mathbb{C}} M$, inducing the special geometric structures on M. The immersion is unique up to an affine transformation of $T^*\mathbb{C}^n = \mathbb{C}^{2n}$ with linear part in $\mathrm{GL}(2n, \mathbb{R})$ (resp. in $\mathrm{Sp}(2n, \mathbb{R})$).*

Any special complex manifold (M, J, ∇) has a canonical torsion-free complex connection D. D is determined by $S := \nabla - D = \frac{1}{2}J\nabla J$. In fact, from $T^\nabla = 0$ and the symmetry of S it follows $T^D = 0$, and $DJ = 0$ is obtained from the following calculation: $DJ = \nabla J - [S, J] = \nabla J - 2SJ = \nabla J + 2JS = \nabla J - \nabla J = 0$. When (M, J, ∇, g) is a special Kähler manifold then D is the Levi-Civita connection of the metric g. In fact, from the skew-symmetry of J and ∇J it follows that S_X is ω-skew-symmetric. This implies $D\omega = 0$ and finally $Dg = 0$.

DEFINITION 4.3. A tt*-bundle (TM, D, S) over a complex manifold M is called special if for all θ the connection D^θ is torsion-free and special, i.e. $D^\theta J$ is symmetric.

The tt*-bundle (TM, D, S) is special if and only if S and S_J are symmetric, $T^D = 0$ and $DJ = 0$.

THEOREM 4.4. [**CS1**]

(i) *There exists a one-to-one correspondence*

$$\Phi : \left\{ \begin{array}{c} \textit{special complex manifolds} \\ (M, J, \nabla) \end{array} \right\} \to \left\{ \begin{array}{c} \textit{special tt*-bundles } (TM, D, S) \\ \textit{over complex manifolds } (M, J) \\ \textit{satisfying } a), b) \end{array} \right\}$$

a) $\{S_X, J\} = 0, \ \forall X \in TM$,
b) $DJ = 0$,

where Φ is given by

$$\Phi(M, J, \nabla) = (TM, D := \nabla - S, S = \frac{1}{2}J\nabla J)$$

and its inverse is given by

$$\Phi^{-1}(TM, D, S) = (M, J, \nabla := D + S).$$

(ii) *This correspondence* Φ *induces a one-to-one correspondence* Φ :

$$\left\{ \begin{array}{c} \textit{special Kähler} \\ \textit{manifolds} \\ (M, J, \nabla, g) \end{array} \right\} \to \left\{ \begin{array}{c} \textit{special metric tt*-bundles } (TM, D, S, g) \\ \textit{over pseudo-Hermitian manifolds } (M, J, g) \\ \textit{satisfying } a), b) \end{array} \right\}.$$

Notice that the metric tt*-bundle associated to a special Kähler manifold is automatically oriented by the complex structure J and unimodular, since $\nabla\omega = D\omega = 0$ implies $\operatorname{tr} S_X = 0$.

Corollary 4.5. *Let* (M, J, ∇, g) *be a special Kähler manifold. Any* ∇*-parallel frame* $s = (s_1, \dots, s_n)$ *of volume* 1 *defines a pluriharmonic map*

$$G^{(s)} = (g(s_i, s_j)) \, : \, M \to Sym^1_{p,q}(\mathbb{R}^{2n}) \cong \mathrm{SL}(2n, \mathbb{R})/\mathrm{SO}(p, q),$$

where $(p, q) = (2k, 2l)$ *is the signature of the metric* g, $2n = p + q = \dim_{\mathbb{R}} M$, *and* $Sym^1_{p,q}(\mathbb{R}^{2n}) \subset Sym_{p,q}(\mathbb{R}^{2n})$ *stands for the subset consisting of matrices of determinant* 1.

Let us now describe the pluriharmonic map $G^{(s)}$ in terms of the holomorphic data of the special Kähler geometry.

Let (M, J, ∇, g) be a simply connected special Kähler manifold of complex dimension $n = k + l$, where g is of signature $(2k, 2l)$. According to Theorem 4.2 we have a non-degenerate holomorphic Lagrangian immersion

$$\Phi \, : \, M \to V = T^*\mathbb{C}^n = \mathbb{C}^{2n},$$

which is unique up to the action of the group $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$. This immersion induces a map

$$\begin{array}{rcl} L \, : \, M & \to & Gr_0^{k,l}(\mathbb{C}^{2n}), \\ p & \mapsto & d\phi_p \, T_pM \subset V, \end{array}$$

into the Grassmannian of complex Lagrangian subspaces $W \subset V$ of signature (k, l), i.e. such that $\gamma_{|W}$ is a Hermitian form of signature (k, l).
We call L the dual Gauß map. It is in fact dual to the Gauß map

$$\begin{array}{rcl} L^\perp \, : \, M & \to & Gr_0^{l,k}(\mathbb{C}^{2n}), \\ p & \mapsto & L(p)^\perp = \overline{L(p)} \underset{\Omega}{\cong} L(p)^*. \end{array}$$

The map L is holomorphic, whereas $L^\perp$ is anti-holomorphic.
The target of the dual Gauß map L is a pseudo-Hermitian symmetric space;

$$Gr_0^{k,l}(\mathbb{C}^{2n}) \cong \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k, l).$$

Theorem 4.6. **[CS1]** *Let* (TM, D, S, g) *be the metric tt*-bundle associated to a simply connected special Kähler manifold* (M, J, ∇, g). *Then there exists a* ∇*-parallel frame* s *such that the pluriharmonic map*

$$G^{(s)} = (g(s_i, s_j)) \, : \, M \to \mathrm{SL}(2n, \mathbb{R})/\mathrm{SO}(2k, 2l),$$

takes values in the totally geodesic submanifold

$$\mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k, l) \subset \mathrm{SL}(2n, \mathbb{R})/\mathrm{SO}(2k, 2l)$$

and coincides with the dual Gauß map L.

4.2. Nearly Kähler manifolds.

Definition 4.7. A nearly Kähler manifold (M, J, g) is an almost complex manifold (M, J) endowed with a pseudo-Riemannian metric g such that

i) J is skew-symmetric with respect to g and
ii) $(D_X J)Y = -(D_Y J)X,\ \forall X, Y \in TM$, where D is the Levi-Civita connection of g.

By a result of Friedrich and Ivanov [**FI**] any nearly Kähler manifold has a unique connection ∇ with totally skew-symmetric torsion satisfying $\nabla g = 0$, $\nabla J = 0$ and $T^{\nabla} = -2\eta$ with $\eta = \frac{1}{2} JDJ$. The 3-form η satisfies $\{\eta_X, J\} = 0$ for all $X \in TM$.

It was observed by the second author [**S3**] that the bundle $(TM, D, D - \nabla = \eta, \omega)$ is a symplectic tt*-bundle provided that g (i.e. $D = \nabla + \eta$) is flat.

In joint work [**CS2**] the present authors have classified flat nearly Kähler manifolds: Any such manifold is locally the product of a flat pseudo-Kähler factor of maximal dimension and a strict nearly Kähler manifold of split signature $(2m, 2m)$ with $m \geq 3$. The geometry of the second factor is encoded in a three-form $\zeta \in \Lambda^3(\mathbb{C}^m)^*$; see Theorem 4.8 below. The first non-trivial example occurs in $\dim_{\mathbb{R}} M = 2n = 4m = 12$.

Any such strict nearly Kähler manifold comes with a non-trivial pluriharmonic map [**S3**] (in fact, J-holomorphic map):

$$M \to \mathrm{SO}(2m, 2m)/\mathrm{U}(m, m) \underset{\text{tot. geod.}}{\subset} \mathrm{SL}(4m, \mathbb{R})/\mathrm{Sp}(\mathbb{R}^{4m})$$

from the *almost complex manifold* (M, J) into the pseudo-Hermitian symmetric space $\mathrm{SO}(2m, 2m)/\mathrm{U}(m, m)$.

Let $\Lambda^3_{reg}(\mathbb{C}^m)^* \subset \Lambda^3(\mathbb{C}^m)^*$ denote the open subset of regular forms, that is, $\zeta \in \Lambda^3(\mathbb{C}^m)^*$ such that

$$\mathrm{span}\{\zeta(X, Y, \cdot) | X, Y \in \mathbb{C}^m\} = (\mathbb{C}^m)^*.$$

Theorem 4.8. [**CS2**] *There exists a one-to-one correspondence between* $\mathrm{GL}(m, \mathbb{C})$*-orbits on* $\Lambda^3_{reg}(\mathbb{C}^m)^* \subset \Lambda^3(\mathbb{C}^m)^*$ *and isomorphism classes of complete flat simply connected nearly Kähler manifolds of real dimension* $4m \geq 12$ *and without pseudo-Kähler de Rham factor.*

In dimension 12, 16, 20 (i.e. $m = 3, 4, 5$) one has $\Lambda^3_{reg}(\mathbb{C}^m)^*/\mathrm{GL}(m, \mathbb{C}) = \{pt\}$. In dimension 24 (i.e. $m = 6$) one has the inclusion $\Lambda^3_{stab}(\mathbb{C}^6)^* \subset \Lambda^3_{reg}(\mathbb{C}^6)^*$, where $\Lambda^3_{stab}(\mathbb{C}^6)^* = \{\phi \in \Lambda^3(\mathbb{C}^6)^* \,|\, Q(\phi) \neq 0\}$ are the stable three-forms in the sense of Hitchin [**Hi**]. Here Q is the quartic $\mathrm{SL}(6, \mathbb{C})$-invariant. $\Lambda^3_{stab}(\mathbb{C}^6)^*$ is an open $\mathrm{GL}(6, \mathbb{C})$-orbit. The complement of $\Lambda^3_{stab}(\mathbb{C}^6)^* \subset \Lambda^3_{reg}(\mathbb{C}^6)^*$, is also an orbit. For the discussion of the orbit space we refer to [**Re**]. The real classification is given in appendix A of [**B**].

5. Proof of Theorem 3.1

5.1. Integrability conditions for maps into symmetric spaces. Let $N = G/K$ be a pseudo-Riemannian symmetric space and ∇^N be its Levi-Civita connection. Let M be a second smooth manifold and $f : M \to N$ be a smooth map. We put $F = df : TM \to B = f^*TN$. The vector bundle B is endowed with the pull-back connection $\hat\nabla = f^*\nabla^N$. The data $(B, \hat\nabla, F)$ satisfy the following structure equations:

$$\hat\nabla_V F(W) - \hat\nabla_W F(V) - F([V,W]) = 0, \tag{5.1}$$

$$\hat R(V,W)\zeta = \hat\nabla_V\hat\nabla_W\zeta - \hat\nabla_W\hat\nabla_V\zeta - \hat\nabla_{[V,W]}\zeta = R^N(FV,FW)\zeta, \tag{5.2}$$

for any vector fields V, W on M and any section ζ of B.

Conversely, let $B \to M$ be an abstract vector bundle which carries a connection $\hat\nabla$ and let N be a pseudo-Riemannian symmetric space. We say that B has the algebraic structure of N if it is endowed with a parallel bundle homomorphism $R^B : \Lambda^2 B \to \operatorname{End} B$ and a linear isomorphism $\Phi_o : B_{p_o} \to T_oN$ for some fixed $p_o \in M$ and $o \in N$, which maps $R^B_{p_o}$ to R^N_o. Then one has the following integration result for a bundle homomorphism $F : TM \to B$, which is a special case of a result proven by Eschenburg and Tribuzy.

THEOREM 5.1. [**ET1**] *Let M be a simply connected manifold and B be a vector bundle over M endowed with a connection $\hat\nabla$ having the algebraic structure of N and $F : TM \to B$ a vector bundle homomorphism satisfying the integrability conditions* (5.1) *and*

$$\hat R(V,W)\zeta = \hat\nabla_V\hat\nabla_W\zeta - \hat\nabla_W\hat\nabla_V\zeta - \hat\nabla_{[V,W]}\zeta = R^B(FV,FW)\zeta, \tag{5.3}$$

cf. (5.2). *Then there exists a smooth map $f : M \to N$ and a parallel bundle isomorphism $\Phi : B \to f^*TN$ with the initial value $\Phi|_{B_{p_o}} = \Phi_o$ mapping R^B_p to $R^N_{f(p)}$ for all $p \in M$ and such that $df = \Phi \circ F$.*

5.2. Relation between associated families and pluriharmonic maps. Now we consider an almost complex manifold (M, J). We put $\mathcal{R}_\theta := \exp\theta J \in \Gamma(\operatorname{End} TM)$. An associated family for f is a family of maps $f_\theta : M \to N, \theta \in \mathbb{R}$, such that

$$\Psi_\theta \circ df_\theta = df \circ \mathcal{R}_\theta, \quad \forall \theta \in \mathbb{R}, \tag{5.4}$$

for some bundle isomorphism $\Psi_\theta : f_\theta^*TN \to f^*TN, \theta \in \mathbb{R}$, which is parallel with respect to ∇^N in the sense that

$$\Psi_\theta \circ (f_\theta^*\nabla^N) = (f^*\nabla^N) \circ \Psi_\theta.$$

REMARKS 5.2. 1) Notice that we are not making any assumption on the dependence of the associated family on the parameter θ. We can always modify the family θ such that $f_0 = f$.
2) One observes that each member f_{θ_0} of an associated family f_θ itself admits an associated family $g_\theta = f_{\theta_0+\theta}$.
3) Note that given an associated family f_θ one can always suppose that $f_\theta(p_o) = o$ by left-multiplication $L_{g(\theta)}$ with an element $g(\theta) \in G$, satisfying $L_{g(\theta)} f_\theta(p_o) = o$. The map Ψ_θ is then replaced by $\Psi_\theta \circ dL^{-1}_{g(\theta)}$. One can then choose the initial value $\Psi_\theta|_{p_0} = \mathrm{Id} : (f_\theta^*TN)_{p_0} = T_oN \to (f^*TN)_{p_0} = T_oN$ for all $\theta \in \mathbb{R}$.

The next theorem generalizes a result of Eschenburg and Tribuzy [**ET2**]:

THEOREM 5.3. *Let (M, J) be an almost complex manifold endowed with a complex connection ∇ such that the $(1,1)$-part of its torsion T vanishes and let N be a pseudo-Riemannian symmetric space. A smooth map $f : M \to N$ admits an associated family f_θ if and only if it is pluriharmonic and satisfies* (3.1) *and* (3.3).

PROOF. Let $f : M \to N$ be a pluriharmonic map which satisfies (3.1), (3.3). To show that f admits an associated family we put $B = f^*TN$, $\hat{\nabla} = f^*\nabla^N$, $F = df$, $F_\theta := F \circ \mathcal{R}_\theta : TM \to B$. We check that F_θ satisfies the integrability conditions of Theorem 5.1. First we observe that B has the algebraic structure of N. In fact, without loss of generality we can assume that $f(p_o) = o$ for some point $p_o \in M$. Then we define $\Phi_o : B_{p_o} = (f^*TN)_{p_o} \to T_oN$ as the canonical identification and $R^B : \Lambda^2 B \to \operatorname{End} B$ as the map which corresponds to $R^N : \Lambda^2 TN \to \operatorname{End} TN$ under the canonical identification $B_p \cong T_{f(p)}N$.

Next we analyse (5.1) for the family F_θ:

$$
\begin{aligned}
& \hat{\nabla}_V F_\theta(W) - \hat{\nabla}_W F_\theta(V) - F_\theta([V,W]) \\
= \;& \hat{\nabla}_V F(\mathcal{R}_\theta W) - \hat{\nabla}_W F(\mathcal{R}_\theta V) - F(\mathcal{R}_\theta [V,W]) \\
= \;& e^{+i\theta}\left[\hat{\nabla}_V F(W^{1,0}) - \hat{\nabla}_W F(V^{1,0}) - F([V,W]^{1,0})\right] \\
+ \;& e^{-i\theta}\left[\hat{\nabla}_V F(W^{0,1}) - \hat{\nabla}_W F(V^{0,1}) - F([V,W]^{0,1})\right] \\
= \;& e^{+i\theta}\left[(\hat{\nabla}_V F)W^{1,0} - (\hat{\nabla}_W F)V^{1,0} + F\left(T(V,W)^{1,0}\right)\right] \\
+ \;& e^{-i\theta}\left[(\hat{\nabla}_V F)W^{0,1} - (\hat{\nabla}_W F)V^{0,1} + F\left(T(V,W)^{0,1}\right)\right].
\end{aligned}
$$

For the last equality we have extended the connection $\hat{\nabla}$ on B to a connection on $T^*M \otimes B$ using the connection ∇ on M.

If V, W have different type the pluriharmonic map equation, i.e. $(\hat{\nabla} F)^{1,1} = 0$, and the condition $T^{1,1} = 0$ yield that this expression vanishes identically.

If V, W have type $(1,0)$, one obtains the following expression:

$$
\begin{aligned}
& e^{+i\theta}\left[(\hat{\nabla}_V F)W^{1,0} - (\hat{\nabla}_W F)V^{1,0} + F\left(T(V,W)^{1,0}\right)\right] + e^{-i\theta} F\left(T(V,W)^{0,1}\right) \\
= \;& -e^{+i\theta} F\left(T(V,W)^{0,1}\right) + e^{-i\theta} F\left(T(V,W)^{0,1}\right) = -2i\sin(\theta)\, F\left(T(V,W)^{0,1}\right),
\end{aligned}
$$

where we used the integrability condition (5.1) for F. This vanishes for all θ if and only if $F\left(T(V,W)^{0,1}\right) = 0$.

If V, W have type $(0,1)$ it follows by the same line of arguments that condition (5.1) holds for all θ if and only if $F\left(T(V,W)^{1,0}\right) = 0$. The last equation is equivalent to (3.3) by complex conjugation.

Now we check equation (5.3) for the family F_θ. If V, W have different type then the two factors $e^{\pm i\theta}$ on the right-hand side cancel each other and the left-hand side does not depend on θ. Therefore, the equation (5.3) for the family F_θ follows from that for F if V, W have different type.
Let now V, W be of the same type. From (3.1) and (5.3) for F we have

$$\hat{R}(V,W) = R^N(FV, FW) = 0$$

and, hence,

$$R^N(F_\theta V, F_\theta W) = e^{\pm 2i\theta} R^N(FV, FW) = 0,$$

which implies $\hat{R}(V,W) = R^N(F_\theta V, F_\theta W) = 0$ for all V, W of same type. This shows the integrability condition (5.3) for F_θ.

Now Theorem 5.1 ensures the existence of a map $f_\theta : M \to N$ with $df_\theta = \Phi_\theta \circ F_\theta$ where Φ_θ is a parallel isomorphism which identifies $B = f^*TN$ and f_θ^*TN. This shows that f admits an associated family with $\Psi_\theta = \Phi_\theta^{-1}$.

Conversely, consider a smooth map $f : M \to N$ which admits an associated family f_θ. As above, we put $F_\theta = F \circ \mathcal{R}_\theta$. We use the integrability constraint (5.1) for $F = df$ and the vanishing of $T(V,W)$ for $V \in \Gamma(T^{1,0}M)$ and $W \in \Gamma(T^{0,1}M)$ to get that

$$0 = \hat{\nabla}_V F(W) - \hat{\nabla}_W F(V) - F([V,W]) = (\hat{\nabla}_W F)V - (\hat{\nabla}_W F)V. \tag{5.5}$$

Similarly, since Ψ_θ is parallel the integrability constraint (5.1) for df_θ yields

$$0 = \Psi_\theta[f_\theta^*\nabla_V^N df_\theta(W) - f_\theta^*\nabla_W^N df_\theta(V) - df_\theta([V,W])] = (\hat{\nabla}_W F_\theta)V - (\hat{\nabla}_W F_\theta)V. \tag{5.6}$$

Using the fact that $\mathcal{R}_\theta$ is parallel we obtain $(\hat{\nabla}_V F_\theta)W = e^{-i\theta}(\hat{\nabla}_V F)W$ and $(\hat{\nabla}_W F_\theta)V = e^{i\theta}(\hat{\nabla}_W F)V$ for all θ. In view of (5.5) and (5.6), this is possible only if $(\hat{\nabla}_V F)W = (\hat{\nabla}_W F)V = 0$. This means that the map f is pluriharmonic. $\square$

In particular, we learn from Theorem 5.3 that, for maps from an almost complex manifold, the pluriharmonic map equation is weaker than the existence of an associated family. Therefore we introduce the following notion.

Definition 5.4. A map f from an almost complex manifold M into a pseudo-Riemannian manifold N is called $\mathbb{S}^1$-*pluriharmonic* if it admits an associated family; see (5.4).

Consider a map

$$f : M \to N := \begin{cases} \mathrm{GL}(r,\mathbb{R})/\mathrm{O}(p,q), & p+q=r, \\ \mathrm{GL}(r,\mathbb{R})/\mathrm{Sp}(r,\mathbb{R}). \end{cases}$$

Denote the map which is induced by f by

$$\hat{f} : M \to \hat{N} := \begin{cases} Sym_{p,q}(\mathbb{R}^r), \\ Skew_{reg}(\mathbb{R}^r). \end{cases}$$

We shall identify the tangent space $T_{\hat{p}}\hat{N}$ at $\hat{p} \in \hat{N}$ with the (ambient) vector space of symmetric/skew-symmetric matrices in $\mathfrak{gl}(r,\mathbb{R})$. The tangent space T_pN of N at a point $p \in N$ which corresponds to $\hat{p} \in \hat{N}$ will be identified with the space of $\hat{p}$-symmetric matrices in $\mathfrak{gl}(r,\mathbb{R})$.

A careful analysis shows that there it holds

$$-2df = \hat{f}^{-1}d\hat{f}. \tag{5.7}$$

The previous theorem reduces Theorem 3.1 to the following statement:

Theorem 5.5. *Let (E, D, S, β) be a metric/symplectic tt*-bundle over a simply connected almost complex manifold (M, J). For any $\theta \in \mathbb{R}$ there exists a D^θ-parallel frame $(e_1^\theta, \ldots, e_r^\theta)$ of E and the correspondence*

$$x \mapsto \hat{f}_\theta(x) := (\beta(e_i^\theta(x), e_j^\theta(x))) \in \hat{N}$$

defines an $\mathbb{S}^1$-pluriharmonic map

$$f_\theta : M \to N = \begin{cases} \mathrm{GL}(r,\mathbb{R})/\mathrm{O}(p,q), \\ \mathrm{GL}(r,\mathbb{R})/\mathrm{Sp}(r,\mathbb{R}), \end{cases}$$

$$\textit{if } \beta \textit{ is } \begin{cases} \textit{symmetric of signature } (p,q),\ p+q=r, \\ \textit{skew-symmetric, respectively.} \end{cases}$$

If (E,D,S,β) is oriented and unimodular f_θ takes values in the irreducible symmetric space

$$\begin{cases} \mathrm{SL}(r,\mathbb{R})/\mathrm{SO}(p,q), \\ \mathrm{SL}(r,\mathbb{R})/\mathrm{Sp}(r,\mathbb{R}), \textit{respectively.} \end{cases}$$

Conversely, any $\mathbb{S}^1$-pluriharmonic map $f : M \to N$, with

$$N = \begin{cases} \mathrm{GL}(r,\mathbb{R})/\mathrm{O}(p,q), & \mathrm{SL}(r,\mathbb{R})/\mathrm{SO}(p,q), \\ \mathrm{GL}(r,\mathbb{R})/\mathrm{Sp}(r,\mathbb{R}), & \mathrm{SL}(r,\mathbb{R})/\mathrm{Sp}(r,\mathbb{R}),\ \textit{respectively,} \end{cases}$$

is obtained from a metric/symplectic tt-bundle by the above construction.*

Proof. Let us consider a metric/symplectic tt*-bundle (E,D,S,β). The connection $D^\theta = D + S^\theta$, $S^\theta_X := \cos(\theta)S_X + \sin(\theta)S_{JX}$, is flat and M is simply connected. Therefore, there exists a D^θ-parallel frame $e_\theta := (e^\theta_1, \ldots, e^\theta_r)$. Since S^θ_X is symmetric with respect to β and $D\beta = 0$, there it follows

$$\begin{aligned} X\beta(e^\theta_i, e^\theta_j) &= \beta(D_X e^\theta_i, e^\theta_j) + \beta(e^\theta_i, D_X e^\theta_j) \\ &= -\beta(S^\theta_X e^\theta_i, e^\theta_j) - \beta(e^\theta_i, S^\theta_X e^\theta_j) \\ &= -2\beta(S^\theta_X e^\theta_i, e^\theta_j). \end{aligned}$$

Let $\hat{S}^\theta = e_\theta^{-1} \circ S^\theta \circ e_\theta$ and $\hat{f}_\theta : M \to \hat{N}$ be the matrix representation of S^θ and β in the frame $e_\theta : M \times \mathbb{R}^r \to E$. Denote by $f_\theta : M \to N$ the map induced by $\hat{f}_\theta$; then the above equation reads

$$-2df_\theta \overset{(5.7)}{=} (\hat{f}_\theta)^{-1} d\hat{f}_\theta = -2\hat{S}^\theta$$

or, equivalently,

$$df_\theta = e_\theta^{-1} \circ S^\theta \circ e_\theta. \tag{5.8}$$

This shows for $X \in \Gamma(TM)$

$$\begin{aligned} df_\theta(X) &\overset{(5.8)}{=} e_\theta^{-1} \circ S^\theta_X \circ e_\theta = e_\theta^{-1} \circ S_{\mathcal{R}_\theta X} \circ e_\theta \\ &\overset{(5.8)}{=} (e_\theta^{-1} e_0) \circ df_0(\mathcal{R}_\theta X) \circ (e_0^{-1} e_\theta) \\ &= \mathrm{Ad}^{-1}_{\alpha_\theta} \circ df_0(\mathcal{R}_\theta X) = \Psi_\theta^{-1} \circ df_0(\mathcal{R}_\theta X), \end{aligned}$$

where $\alpha_\theta = e_\theta^{-1} e_0$ is the frame change from e_θ to e_0 and $\Psi_\theta = \mathrm{Ad}_{\alpha_\theta}$, which is parallel with respect to the Levi-Civita connection on N. In other words f_θ is an associated family for f_0 and the maps f_θ are $\mathbb{S}^1$-pluriharmonic.

Conversely, let us consider an associated family $f_\theta : M \to N$ for a pluriharmonic map $f = f_0$ (see Remarks 5.2) and denote the induced maps by $\hat{f}_\theta : M \to \hat{N}$.

We define a metric/symplectic form β on the vector bundle $E = M \times \mathbb{R}^r$ by $\beta(\cdot,\cdot) = \langle \hat{f}_0 \cdot, \cdot\rangle$ and set $S^\theta = df \circ \mathcal{R}_\theta$ and $S = S^0$. Here $\langle\cdot,\cdot\rangle$ stands for the standard scalar product on $\mathbb{R}^r$. Denote by ∂ the canonical flat connection on E and define the connection D on E by

$$D_X s := \partial_X s - S_X s, \quad s \in \Gamma(E).$$

A direct calculation using equation (5.7) and the identification of T_pN with the vector space of $\hat{p}$-symmetric matrices yield that $D\beta = 0$ and that S is β-symmetric (see Lemma 1 [**S1**]).

Since f is $\mathbb{S}^1$-pluriharmonic we have

$$F_\theta := df_\theta = \Psi_\theta^{-1} \circ df \circ \mathcal{R}_\theta,$$

where Ψ_θ is parallel. Recall the first integrability condition (5.1) for the family f_θ :

$$\hat{\nabla}^\theta_V F_\theta(W) - \hat{\nabla}^\theta_W F_\theta(V) - F_\theta([V,W]) = 0, \quad \hat{\nabla}^\theta := f_\theta^* \nabla^N.$$

Evaluating this equation for $\theta = 0, \pi/2$ we obtain

$$\begin{aligned} 0 &= \hat{\nabla}_V F_0(W) - \hat{\nabla}_W F_0(V) - F_0([V,W]) \\ &= \Psi_0^{-1}\left(\hat{\nabla}_V df(W) - \hat{\nabla}_W df(V) - df([V,W])\right), \\ 0 &= \hat{\nabla}^{\frac{\pi}{2}}_V F_{\frac{\pi}{2}}(W) - \hat{\nabla}^{\frac{\pi}{2}}_W F_{\frac{\pi}{2}}(V) - F_{\frac{\pi}{2}}([V,W]) \\ &= \Psi_{\frac{\pi}{2}}^{-1}\left(\hat{\nabla}_V df(JW) - \hat{\nabla}_W df(JV) - df(J[V,W])\right). \end{aligned}$$

Next we use the embedding $(f^*TN, \hat{\nabla}) \subset (\mathrm{End}(E), D = \partial - S)$ as a parallel subbundle by $(f^*TN)_p = T_{f(p)}N = \mathrm{sym}(\hat{f}(p)) \subset \mathrm{End}(\mathbb{R}^r)$. Here $\mathrm{sym}(\hat{f}(p))$ stands for the vector space of $\hat{f}(p)$-symmetric matrices. (We refer to [**S1**] Proposition 2 and 5 for the calculation of the connection.)

Now we check that $(E, D, S = df, \beta = \langle \hat{f}\cdot,\cdot\rangle)$ is a metric/symplectic tt*-bundle. We already know that S is symmetric with respect to β and that $D\beta = 0$. It remains to check the tt*-equations for the data (D, S) on (M, J). In order to show that the tt*-equations are satisfied at the point $p_0 \in M$, we choose the associated family for f in such a way that $f_\theta(p_0) = o$ and $\Psi_\theta(p_0) = \mathrm{Id}$, see Remarks 5.2. Using $S = df$ we can rewrite the above integrability constraint as

$$\begin{aligned} 0 &= D_V(S_W) - D_W(S_V) - S_{[V,W]} = d^D S(X,Y), \\ 0 &= D_V(S_{JW}) - D_W(S_{JV}) - S_{J[V,W]} = d^D S_J(X,Y). \end{aligned}$$

Since $D + S = \partial$ is flat and $d^D S = 0$, we obtain

$$R^D + S \wedge S = R^D + d^D S + S \wedge S = R^{D+S} = 0.$$

In order to verify the tt*-equations (see Remarks 1.2) it only remains to show

$$[S_{JX}, S_{JY}] = [S_X, S_Y]. \tag{5.9}$$

The second integrability condition for the map f_θ can be brought to the form

$$\begin{aligned} \hat{R}(V,W) &= \Psi_\theta \circ R^N(F_\theta V, F_\theta W) \circ \Psi_\theta^{-1} \\ &= \Psi_\theta \circ R^N\left(\Psi_\theta^{-1}(df(\mathcal{R}_\theta V)), \Psi_\theta^{-1}(df(\mathcal{R}_\theta W))\right) \circ \Psi_\theta^{-1}, \end{aligned}$$

where $\hat{R}$ is the curvature of $\hat{\nabla}$ and V, W are vector fields on M. The left-hand side does not depend on the parameter θ. Setting $\theta = 0, \pi/2$ and evaluating the equation at the point p_0 we obtain $R^N(df(JV), df(JW)) = R^N(df(V), df(W))$, $V, W \in T_{p_0}M$. Since $S = df$ and $R^N(A, B)$, $A, B \in T_oN = \mathrm{sym}(\hat{f}(p_0))$, is proportional to $\mathrm{ad}_{[A,B]}$ we get (5.9). Notice that A, B are matrices which are symmetric with respect to $\hat{f}(p_0) = \hat{o}$, the standard scalar product of signature (p, q) on $\mathbb{R}^r$ ($r = p+q$), respectively, the standard symplectic form on $\mathbb{R}^r$, in which case r is necessarily even. $\square$

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