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Dense minors in graphs of large girth

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Dense minors in graphs of large girth

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We show that a graph of girth greater than $6 \log k + 3$ and minimum degree at least 3 has a minor of minimum degree greater than k. This is best possible up to a factor of at most 9/4. As a corollary, every graph of girth at least $6 \log r + 3 \log\log r + c$ and minimum degree at least 3 has a K_r minor.

1. Introduction

Thomassen [7] proved that, in graphs of minimum degree at least 3, sufficiently high girth forces a minor of any given minimum degree:

Theorem. (Thomassen 1983)
For any integer k, every graph G of girth $g(G) \geqslant 4k - 3$ and $\delta(G) \geqslant 3$ has a minor H with $\delta(H) \geqslant k$.

Our aim in this paper is to reduce the upper bound for the required girth to the correct order of magnitude:

Theorem 1. *For any integer k, every graph G of girth $g(G) > 6 \log k + 3$ and $\delta(G) \geqslant 3$ has a minor H with $\delta(H) > k$.*

The best lower bound we have found is $\frac{8}{3} \log k - c$, but we note that existing conjectures about cubic graphs of large girth would raise this to about $4 \log k$.

Since an average degree of at least $cr\sqrt{\log r}$ forces a K_r minor [5, 8], Theorem 1 has the following consequence:

Corollary 2. *There exists a constant $c \in \mathbb{R}$ such that every graph G of girth $g(G) \geqslant 6 \log r + 3 \operatorname{loglog} r + c$ and $\delta(G) \geqslant 3$ has a K_r minor.* □

Asymptotically, Thomason [9] showed that a K_r minor is forced by an average degree of $(d + o(1))\, r\sqrt{\log r}$, where $d = 0.53131\ldots$ is an explicit constant that is best possible. This means that, for large enough r, Corollary 2 holds with $c = -2.4742$.

We adopt the notation of [4]. All our logarithms are binary, all graphs considered finite, and $0 \in \mathbb{N}$.

2. A lower bound

Minimum-order cubic graphs of girth at least some given integer g are called *g-cages* and have been studied in some detail (see [1] for an overview). Their exact order is known for $g \leqslant 12$. The best general upper bound for the order of 3-cages is due to Biggs & Hoare [2] and Weiss [10]:

Lemma 2.1. *There is a constant $c^* > 0$ such that for infinitely many integers g there exists a cubic graph of girth at least g and order at most $c^* 2^{3g/4}$.*

We deduce the following lower bound for the girth required to force a minor of minimum degree k in cubic graphs:

Proposition 2.2. *There is a constant $c \in \mathbb{R}$ such that for infinitely many $k \in \mathbb{N}$ there exist cubic graphs of girth at least $\frac{8}{3} \log k - c$ that have no minor H with $\delta(H) \geqslant k$.*

Proof. Leaving the value of c open for the moment, let us try for given $k_0 \geqslant 3$ to find a graph that satisfies the assertion with $k \geqslant k_0$. Let c^* be the constant from Lemma 2.1, and let g be an integer with $(k_0+1)(k_0-2) \leqslant c^* 2^{3g/4}$ such that some cubic graph G has girth at least g and order at most $c^* 2^{3g/4}$. Let $k \in \mathbb{N}$ be minimal with $c^* 2^{3g/4} < (k+1)(k-2)$. Then $k \geqslant k_0$, and

$$|G| < (k+1)(k-2)\,. \tag{1}$$

By the minimality of k we have $c^* 2^{3g/4} \geqslant (k-3)^2$, so

$$g(G) \geqslant g \geqslant \tfrac{8}{3} \log(k-3) - \tfrac{4}{3} \log c^* \geqslant \tfrac{8}{3} \log k - c$$

for some suitable constant c depending only on c^*. Now suppose that G has a minor H with $\delta(H) \geqslant k$. Each of the $k+1$ or more branch sets $X \subseteq V(G)$ of H induces a connected subgraph in G, which has at least $|X|-1$ edges, and it sends at least k edges to other branch sets. The degrees of the vertices in X thus sum to $3|X| \geqslant 2|X|-2+k$, giving $|X| \geqslant k-2$. Hence $|G| \geqslant (k+1)(k-2)$, contradicting (1). □

Any improvement on the bound in Lemma 2.1 will result in a corresponding improvement to Proposition 2.2. It has been conjectured (see [3] or [6]) that g-cages exist on as few as about $2^{g/2}$ vertices. This would increase our lower girth bound to $4 \log k - c$.

3. The upper bound

In this section we prove Theorem 1. Our starting point will be the observation that in a graph G of girth $g(G) > 2d+1$ and $\delta(G) \geqslant 3$ the d-ball around a vertex x is a tree T_x sending at least $|T_x|-2$ edges to the rest of G. Our main effort will go into proving that, depending on our lower bound for $g(G)$, not too many of these edges can go to the same tree T_y. Then partitioning $V(G)$ into such trees and contracting these will give us a minor of large minimum degree.

Given a tree T with root r and vertices $t, t' \in T$, we say that t' lies *above* t in T (and t *below* t') if $t \leqslant t'$ in the tree-order on $V(T)$ associated with r, ie. if t separates t' from r in T. Any neighbour of t above it is a *successor* of t in T, its unique neighbour below is its *predecessor*. For $i \in \mathbb{N}$ we write L_T^i for the set of *leaves* (maximal elements) of T at distance i from r.

Lemma 3.1. *Every rooted tree T in which no vertex has exactly one successor satisfies $\sum_{i\in\mathbb{N}} 2^{-i}|L_T^i| \geqslant 1$.*

Proof. We apply induction on $|T|$, which starts with $|L_T^0| = 1$ for $|T| = 1$. For the induction step let t be a vertex at maximum distance from the root. By assumption, the predecessor of t has at least two successors, and by the choice of t all these are leaves. If we delete them, the sum $\sum_{i\in\mathbb{N}} 2^{-i}|L_T^i|$ does not increase and we are home by the induction hypothesis. □

Given a graph G, a vertex $x \in G$, and $d \in \mathbb{N}$, let us write $V_{G,x}^d$ for the set of vertices of G at distance exactly d from x. Applying Lemma 3.1 to the subtrees of T rooted in $V_{T,r}^d$, we obtain the following:

Lemma 3.2. *Let T be a tree with root r in which no vertex has exactly one successor, and let $d \in \mathbb{N}$. Then $\sum_{i\geqslant d} 2^{d-i}|L_T^i| \geqslant |V_{T,r}^d|$.* □

We are now ready to prove our main result, which we restate:

Theorem 1. *For any integer k, every graph G of girth $g(G) > 6\log k + 3$ and $\delta(G) \geqslant 3$ has a minor H with $\delta(H) > k$.*

Proof. Put $\lfloor \log k \rfloor =: d$. Let X be a maximal set of vertices such that $d(x, y) > 2d$ for all distinct $x, y \in X$. Beginning with $T_x^0 := \{x\}$, let us define trees T_x^i rooted at x, for all $x \in X$ and $i = 0, \ldots, 2d$. Assume that for some i the T_x^i have been defined and partition the set of vertices of G at distance at most i from X. We then add each vertex v at distance $i+1$ from X to one T_x^i to which it is adjacent, thereby obtaining a similar set of disjoint trees T_x^{i+1}. By the choice of X, the trees $T_x := T_x^{2d}$ partition the entire vertex set of G, and

$$T_x \text{ contains all the vertices of } G \text{ at distance at most } d \text{ from } x. \tag{2}$$

As $g(G) > 4d + 1$, the T_x are induced subgraphs in G. Finally, we have

$$d(w, y) \leqslant d(v, x) + 1 \text{ whenever } vw \in E(G) \text{ with } v \in T_x \text{ and } w \in T_y\,, \tag{3}$$

as otherwise w would have been added to T_x after v rather than to T_y.

Let us use Lemma 3.2 to estimate the number of edges leaving a tree T_x. For all $i \in \mathbb{N}$ let

$$E_x^i := \{\, vw \in E(G) \mid v \in T_x,\ w \in G - T_x,\ d(v, x) = i \,\}.$$

Let T'_x denote the subgraph of G induced by T_x and all its neighbours in G. As $g(G) > 4d + 3$, T'_x is again a tree. Every vertex $v \in T_x$ has degree $d_G(v) \geqslant 3$ in T'_x, while all the vertices of $T'_x - T_x$ are leaves in T'_x. As $|E_x^i| = |L_{T'_x}^{i+1}|$ for all i, and $|L_{T'_x}^d| = 0$ by (2), Lemma 3.2 yields

$$\sum_{i\geqslant d} 2^{d-i-1}|E_x^i| = \sum_{i\geqslant d} 2^{d-i-1}|L_{T'_x}^{i+1}| = \sum_{i\geqslant d} 2^{d-i}|L_{T'_x}^i| \geqslant |V_{T'_x,x}^d| = |V_{G,x}^d|.$$

Multiplying by 2^{d+1} and setting $V_x^d := V_{G,x}^d$ we obtain

$$\sum_{i \geqslant d} 2^{2d-i} |E_x^i| \geqslant 2^{d+1} |V_x^d|.$$

Every edge in E_x^i joins T_x to a tree T_y distinct from T_x. This defines a partition of E_x^i into sets $A_{x,y}^i$ ($y \in X \setminus \{ x \}$). Then the above inequality can be rewritten as

$$2^{d+1} |V_x^d| \leqslant \sum_y \sum_{i \geqslant d} 2^{2d-i} |A_{x,y}^i| \, , \tag{4}$$

where the first sum is taken over all $y \in X \setminus \{ x \}$ such that G contains a T_x–T_y edge. We shall prove that, for each of these y,

$$\sum_{i \geqslant d} 2^{2d-i} |A_{x,y}^i| \leqslant |V_x^d| \, , \tag{5}$$

so that (4) can be satisfied only if there are at least 2^{d+1} distinct y, ie. if T_x sends edges to at least 2^{d+1} other trees T_y. Contracting all the trees T_x with $x \in X$ we then obtain a minor of G of minimum degree at least $2^{d+1} > k$, as desired.

For the proof of (5) let now x and y be fixed distinct vertices in X. Consider a T_x–T_y edge $e = vw$ of G, with $v \in T_x$ and $w \in T_y$ say. Then $i := d(v,x) \geqslant d$, by (2) and $w \notin T_x$. Let z_e be the vertex below v in T_x at distance d from v, ie. in V_x^{i-d}, and let B_e be the set of vertices in V_x^d that lie above z_e in T_x. These vertices have distance $2d - i$ from z_e, so

$$|B_e| \geqslant 2^{2d-i}. \tag{6}$$

Let us show that

$$B_e \cap B_{e'} = \emptyset \text{ for all distinct } T_x\text{–}T_y \text{ edges } e, e'. \tag{7}$$

Suppose not, ie. suppose that z_e and $z_{e'}$ are comparable in T_x, say $z_e \leqslant z_{e'}$. Write $e =: vw$ and $e' =: v'w'$ with $v, v' \in T_x$ and $w, w' \in T_y$, and put $i := d(v,x)$. We show that the unique cycle C in $T_x \cup T_y + e + e'$ has length less than $g(G)$ (Fig. 1).

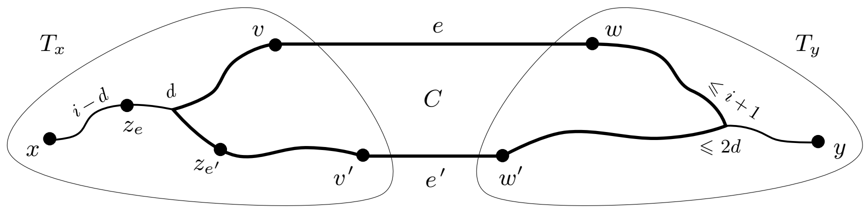

FIGURE 1. The cycle C between T_x and T_y.

The portion of C in T_x is a subpath of the walk $v \dots z_e \dots v'$ in T_x, which has length at most $d + (2d - (i - d)) = 4d - i$. Its portion in T_y is a subpath of the walk $w \dots y \dots w'$ in T_y, which has length at most $(i+1) + 2d$ by (3). Thus $|C| \leqslant 6d + 3 < g(G)$, as desired. This completes the proof of (7).

Now (6), (7) and the definition of the B_e imply (5):

$$\sum_{i \geqslant d} 2^{2d-i} |A^i_{x,y}| = \sum_{i \geqslant d} \sum_{e \in A^i_{x,y}} 2^{2d-i} \underset{(6)}{\leqslant} \sum_{i \geqslant d} \sum_{e \in A^i_{x,y}} |B_e| \underset{(7)}{\leqslant} |V^d_x| \,.$$

□

In order to improve the bound in Theorem 1 further, we have considered the question of whether the set X might be chosen more effectively. For the proof of (2) we need its points to be more than $2d$ apart. But if they were placed in G so that every other vertex v had distance $d(v, X) \leqslant \alpha d$ from X for some $\alpha < 2$ (rather than just $d(v, X) \leqslant 2d$, which we get simply by choosing X maximal), we would instantly shorten the cycle C in the proof of (7) to at most $(2 + 2\alpha)d + 3$, improving the girth bound in the theorem to $(2 + 2\alpha) \log k + 3$. Note that the theoretical optimum of $\alpha = 1$ would give us exactly (up to the additive constant) the conjectured lower bound from Section 2.

The problem of whether a given graph contains a set X of vertices such that $d(x, y) > q$ but $d(v, X) \leqslant r$ for all vertices $v \in G$ has been investigated in the context of domination problems on graph, and there is a host of literature on this topic. (Several of these show that as a decision problem this is NP-hard for various choices of q and r.) In [6] the problem was shown to be NP-hard for all choices of q and r satisfying the trivial requirement of $q/2 \leqslant r < q$. We have not pursued the problem further.

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