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Forcing highly connected subgraphs in locally finite graphs

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Abstract

By a theorem of Mader [5], highly connected subgraphs can be forced in finite graphs by assuming a high minimum degree. Solving a problem of Diestel [2], we extend this result to locally finite graphs. Here, it is necessary to require not only high degree for the vertices but also high *vertex-degree* (or *multiplicity*) for the ends of the graph, ie. a large number of disjoint rays in each end. If, on the other hand, in addition to the high vertex degrees, we only require high *edge-degree* for the ends (which is defined as the maximum number of edge-disjoint rays in an end), Mader's theorem does not extend to infinite graphs. But, high minimum edge-degree at the ends (together with high minimum degree at the vertices) suffices to force highly edge-connected subgraphs in locally finite graphs.

1 Introduction

In a finite graph, high average degree forces the existence of a highly connected subgraph:

Theorem 1 (Mader [5]). *Any finite graph G of average degree at least $4k$ has a k-connected subgraph.*

In infinite graphs, there is no adequate notion of the 'average degree'. So for an extension of the theorem to infinite graphs we must replace 'average degree' with 'minimum degree'.

But simply requiring high degree for the vertices is not enough, as the counterexample of the infinite r-regular tree T^r demonstrates. Now, since an infinite tree has rather 'thin' ends, this suggests, as conjectured by Diestel [2], that a minimum degree condition has to be imposed also on the ends of the graph. In fact, let us define the *vertex-degree* of an end as the maximum number of disjoint rays in it (this maximum exists, see [4]). Then the T^r ceases to be a counterexample, as each of its ends has vertex-degree 1. And indeed, with this further condition on the vertex-degrees of the ends, Theorem 1 does extend to locally finite graphs. As our main theorem we prove the following stronger result (for this, let us call an induced connected subgraph of a graph G that sends only finitely many edges to the rest of G a *region*; note that in particular, any component of G is a region):

Theorem 2. *Let $k \in \mathbb{N}$ and let G be a locally finite graph such that each vertex has degree at least $6k^2 - 5k + 3$, and each end has vertex-degree at least $6k^2 - 9k + 4$. Then every infinite region of G has a k-connected region.*

On the other hand, the *edge-degree* of an end is defined as the maximum number of edge-disjoint rays in it (the maximum exists, see [1]). But, it turns out that high edge-degrees at the ends and high degrees at the vertices together are not sufficient to force highly connected subgraphs, or even highly connected minors, in infinite graphs. Indeed, in Section 4 we exhibit for all $r \in \mathbb{N}$ a locally finite graph of minimum degree and minimum edge-degree r that has no 4-connected subgraph and no 6-connected minor.

But high edge-degree at the ends (together with high degree at the vertices) suffices to force highly edge-connected subgraphs in locally finite graphs, moreover, such can be found in every infinite region:

Theorem 3. *Let $k \in \mathbb{N}$ and let G be a locally finite graph such that each vertex has degree at least $4k+1$ and each end has edge-degree at least $2k-1$. Then every infinite region of G has a k-edge-connected region.*

In general, it is not possible to force *finite* highly (edge-) connected subgraphs in infinite graphs by assuming high minimum degree and vertex- (or edge-) degree; neither can we force *infinite* highly (edge-) connected subgraphs (see end of Section 3).

2 Terminology

The basic terminology we use can be found in [3]. A 1-way infinite path is called a *ray*, and the subrays of a ray are its *tails*. Two rays in a graph G are *equivalent* if no finite set of vertices separates them; the corresponding equivalence classes of rays are the *ends* of G. We denote the set of the ends of G by $\Omega(G)$.

Let H be a subgraph of G, and write $H \subseteq G$. The *boundary* ∂H of H is the set $N(G-H)$ of all neighbours in H of vertices of $G-H$. We call H a *region* of G if H is a connected induced subgraph which sends only finitely many edges to $G-H$. Then $H' \subseteq H$ is a region of G if and only if it is a region of H.

As in finite graphs, H is *k-connected* if $|H| > k$ and no set of less than k vertices separates H. Similarly, H is *k-edge-connected* if $|H| > 1$ and no set of less than k edges separates H. Hence, if H is not k-edge-connected (and non-trivial) then it has a cut of cardinality $< k$.

There are basically two possibilities how the vertex degree notion can be extended to ends. The *vertex-degree* (also known as the *multiplicity*) of an end $\omega \in \Omega(G)$ is the maximum number of (vertex-) disjoint rays in ω. The *edge-degree* of ω (as suggested in [1]) is the maximum number of edge-disjoint rays in ω. These two degree concepts are well-defined, ie. the considered maxima do indeed exist (see [4, 1]).

We use a lemma which follows immediately from [1, Corollary 18]:

Lemma 4. *If a locally finite graph G has no cuts of cardinality $< k \in \mathbb{N}$ then each of its ends has edge-degree at least k.*

3 Forcing highly edge-connected subgraphs

We start by proving our second result, Theorem 3, which is easier. For this, we need a lemma.

Lemma 5. *Let $k \in \mathbb{N}$ and let G be a locally finite graph such that each vertex has degree at least $\delta_V \geq 4k+1$. Then every finite non-empty region C of G with $|E(C, G-C)| \leq \frac{\delta_V}{2}$ has a k-connected subgraph.*

Proof. Let $v \in V(C)$, and set $F_C := E(C, G-C)$. By assumption, v has degree at least δ_V in G, and thus degree at least $\delta_V - |F_C| \geq |F_C|$ in C. Hence C contains more than $|F_C|$ vertices, and therefore has average degree $d(C) \geq \delta_V - 1 \geq 4k$. Thus Theorem 1 yields a finite k-connected subgraph of C. □

We now prove Theorem 3, which we restate:

Theorem 3. *Let $k \in \mathbb{N}$ and let G be a locally finite graph such that each vertex has degree at least $4k+1$ and each end has edge-degree at least $2k-1$. Then every infinite region of G has a k-edge-connected region.*

Proof. Let C be an infinite region of G, and assume that C has no finite k-edge-connected subgraph. We prove that then C has an infinite k-edge-connected region H.

First, suppose that for every infinite region C' of C there is a non-empty region $C'' \subseteq C' - \partial C'$ of C such that $|E(C'', G - C'')| < 2k-1$. Then any such C'' is infinite, by Lemma 5 and by the assumption that C contains no finite k-edge-connected (and thus in particular no finite k-connected) subgraph. Hence there exists a sequence $C =: C_0, C_1, \ldots$ of infinite regions of C such that for $i \geq 1$

(i) $C_i \subseteq C_{i-1} - \partial C_{i-1}$; and

(ii) $|E(C_i, G - C_i)| < 2k-1$.

Now, as each of the C_i is connected, there is a sequence $(P_i)_{i\in\mathbb{N}}$ of ∂C_i–∂C_{i+1} paths such that for $i \geq 1$ the path P_{i+1} starts in the last vertex of P_i. By (i), the paths P_i are non-trivial, and hence, their union $P := \bigcup_{i=1}^{\infty} P_i$ is a ray which has a tail in each of the C_i. Let ω be the end of G that contains P. As, by assumption, ω has edge-degree at least $2k-1$, there is a family $\mathcal{R}$ of $2k-1$ edge-disjoint ω-rays in G. For each ray $R \in \mathcal{R}$ let n_R denote the distance its starting vertex has to ∂C_1. Set $n := \max\{n_R : R \in \mathcal{R}\}$. Then by (i), all of the $2k-1$ disjoint rays in $\mathcal{R}$ start outside C_{n+1}. But each ray in $\mathcal{R}$ is equivalent to P, and hence eventually enters C_{n+1}, a contradiction as $|E(C_{n+1}, G - C_{n+1}| < 2k-1$ by (ii).

Hence, there is an infinite region C' of C so that for each non-empty region $C'' \subseteq C' - \partial C'$ of C holds that

$$|E(C'', G - C'')| \geq 2k-1. \tag{1}$$

Observe that as G is locally finite, there exist infinite regions $\subseteq C' - \partial C'$ of C: take, for example, any infinite component of $C' - \partial C'$. Now, choose an infinite region $H \subseteq C' - \partial C'$ of C with $|E(H, G-H)|$ minimal. By (1), $F_H := E(H, G-H)$ consists of at least $2k-1$ edges.

We claim that H is the desired k-edge-connected region of C. Indeed, suppose otherwise. Then H has a cut F with $|F| < k$. We may assume that F is a minimal cut, ie. leaves only two components D, D' in $H - F$. One of the two, say D, is infinite. Then, by the choice of H, the cut $F_D := E(D, G-D) \subseteq F \cup F_H$

contains at least $|F_H|$ edges. Hence, D is incident with all but at most $|F|$ edges of F_H. Thus $D' \subseteq C' - \partial C'$ is a region of C with

$$|E(D', G - D')| \leq |F_H| - |F_H \cap F_D| + |F| \leq 2|F| < 2k - 1,$$

a contradiction to (1). □

Theorem 3 is best possible in the sense that high edge-degree is not sufficient to force highly connected subgraphs, as we shall see in the next section. Furthermore, it has two interesting corollaries.

Corollary 6. *Let $k \in \mathbb{N}$ and let C be an infinite region of a locally finite graph G which has minimum vertex degree $4k+1$ and minimum edge-degree $2k-1$ at the ends. Then C has either infinitely many disjoint finite k-edge-connected regions or an infinite k-edge-connected region.*

Proof. Take an inclusion-maximal set $\mathcal{D}$ of disjoint finite k-edge-connected regions of C (which exists by Zorn's Lemma), and assume that $|\mathcal{D}| < \infty$. Since $C' := C - \bigcup_{D \in \mathcal{D}} D \subseteq C$ is an infinite region of G, we may use Theorem 3 to obtain a k-edge-connected region H of C. Then H is infinite by the choice of $\mathcal{D}$. □

The two configurations of Corollary 6 of which one necessarily appears need not both exist. Indeed, an example for an infinite locally finite graph G which has minimum degree and vertex- (and thus edge-) degree r for given $r \in \mathbb{N}$ but no infinite 3-edge-connected subgraph is obtained from the $r \times \mathbb{N}$ grid by joining each vertex to r disjoint copies of K^{r+1}. Any infinite subgraph of G which is at least 2-edge-connected is also a subgraph of the $r \times \mathbb{N}$ grid, and hence is at most 2-edge-connected.

On the other hand, there are also locally finite graphs of high minimum degree and vertex-degree that have no finite highly edge-connected subgraphs. For given $r \in \mathbb{N}$, add some edges to each level S_i of the r-regular tree T^r so that in the obtained graph $\tilde{T}^r$ each S_i induces a path. The only end of $\tilde{T}^r$ has infinite vertex- and edge-degree, and the vertices of $\tilde{T}^r$ have degree at least r. Now, for every finite subgraph H of $\tilde{T}^r$ there is last level of $\tilde{T}^r$ that contains a vertex v of H. Then v has degree at most 3 in H, and hence, H is not 4-edge-connected.

Corollary 7. *Let $k \in \mathbb{N}$, and let G be a locally finite graph with minimum vertex degree $4k+1$ and minimum edge-degree $2k-1$ at the ends. Then there is a countable set $\mathcal{D}$ of disjoint k-edge-connected regions of G such that $|E(H, G - H)| \geq \max\{2k, |H|\}$ for each subgraph H of $G - \bigcup_{D \in \mathcal{D}} D$.*

Proof. Let $\mathcal{D}$ be an inclusion-maximal set $\mathcal{D}$ of disjoint k-edge-connected regions of G (which exists by Zorn's Lemma). Since G is locally finite and therefore countable, $\mathcal{D}$ is countable.

Observe that it suffices to show $|E(H, G - H)| \geq \max\{2k, |H|\}$ for induced connected subgraphs H of $G - \bigcup_{D \in \mathcal{D}} D$, and consider such an H. If H is infinite, then Theorem 3 and the (maximal) choice of $\mathcal{D}$ imply that H is not a region of G, ie. that $|E(H, G - H)|$ is infinite, as desired.

So assume that H is finite. Then in particular, H is a region of G, and thus Lemma 5 ensures that $|E(H, G - H)| \geq 2k$. Also, $|E(H, G - H)| \geq |H|$, as otherwise H has average degree $d(H) \geq \delta(G) - 1 \geq 4k$, and hence H has a k-edge-connected subgraph by Theorem 1, contradicting the choice of $\mathcal{D}$. □

4 High edge-degree at the ends does not force highly connected subgraphs or minors

For given $r \in \mathbb{N}$ we will construct a locally finite graph G_r of minimum vertex degree r and minimum edge-degree $\geq r$ at the ends that has no 4-connected subgraph and no 6-connected minor.

We start with an infinite rooted tree T_r in which each vertex sends r edges to the next level. The graph G_r will be obtained from T_r in the following manner. Let S_0 consist of the root of T_r and for $i \geq 1$ denote by S_i the i-th level of T_r. Now, successively for $i \geq 1$, we add some vertices to S_i, which results in an enlarged ith level S'_i, and then add some edges between $S'_i - S_i$ and S_{i+1}. For this, consider those subsets of S_i whose elements have the same neighbour in S_{i-1}. For each maximal such set S, fix an enumeration $s_1, s_2, \ldots, s_r$ of S, and add $r-1$ new vertices $v_1^S, v_2^S, \ldots, v_{r-1}^S$ to S_i. Denote by S'_i the set thus obtained from S_i. Then for each $j \leq r-1$ and each S as above add all edges

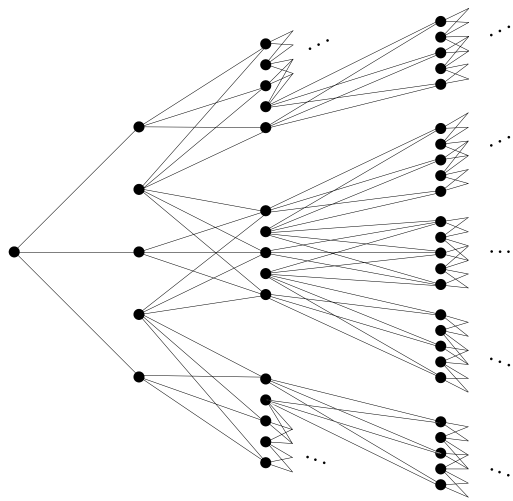

Figure 1: The graph G_3.

between v_j^S and $N_{S_{i+1}}(\{s_j, s_{j+1}\})$. This yields a graph G_r on the disjoint union of sets $S'_1, S'_2, \ldots$ as depicted in Figure 1 for $r = 3$.

Lemma 8. *G_r has minimum degree r at the vertices and minimum edge-degree $\geq r$ at the ends.*

Proof. By construction, G_r has mimimum vertex degree r. To see that the ends of G_r have edge-degree at least r, we use Lemma 4; hence, it suffices to show that G_r has no cuts of cardinality less than r. This will be done by proving inductively for $n \in \mathbb{N}$ that the vertices in $\bigcup_{i=0}^{n} S'_i$ cannot be separated in G_r by less than r edges. The assertion clearly holds for $n = 0$, as $S'_0 = S_0$ consists of only one vertex. So suppose $n > 0$. By assumption, S'_{n-1} cannot be separated in G_r by less that r edges, and by construction, S_{n-1} cannot be separated in G_r by less than r edges from any of the maximal subsets S of

S_n whose elements have the same neighbour in S_{n-1}. Hence, we only need to show that no such S together with the corresponding $v_1^S, v_2^S, \dots, v_{r-1}^S \in S'_n - S_n$ can be separated in G_r by less than r edges. But this is easy: any two vertices of $S \cup \{v_1^S, v_2^S, \dots, v_{r-1}^S\}$ are connected by r edge-disjoint paths in $G_r[S'_n \cup S_{n+1}]$. □

Observe that every finite set A of vertices can be separated from any end ω by at most three vertices (namely by the neighbours of the unique component of $G_r - S'_i$ that contains a ray in ω, where i is large enough so that $A \subseteq S'_i$). Hence, each end of G_r has vertex-degree at most 3.

In fact, Theorem 2 ensures that every graph of high minimum vertex degree has either an end of small vertex-degree or a highly connected subgraph. We shall see now that the latter is not the case for G_r.

Lemma 9. *G_r has no 4-connected subgraph.*

Proof. Suppose G_r has a 4-connected subgraph H, and let $i \in \mathbb{N}$ so that $V(H) \cap S'_i \neq \emptyset$. Now, if there is vertex $v \in V(H) - S'_{i+1}$, then it can be separated in G_r (and thus also in H) from $V(H) \cap S'_i$ by at most three vertices (namely by the neighbours of the component of $G_r - S'_{i+1}$ that contains v). So, as H is 4-connected, $V(H) - S'_{i+1}$ must be empty. Hence, H is finite, implying that there is a maximal $j \in \mathbb{N}$ such that $V(H) \cap S'_j \neq \emptyset$. But then by construction of G_r, any vertex in $V(H) \cap S'_j$ has degree at most three in H, a contradiction as H is 4-connected. □

It is slightly more difficult to prove that G_r has no highly connected minor.

Lemma 10. *G_r has no 6-connected minor.*

Proof. Suppose that G_r has a 6-connected minor M. Then there is an $n \in \mathbb{N}$ so that each branch-set of M has a vertex in $\bigcup_{i=0}^n S'_i$. Furthermore, since M is 6-connected, each separator $T \subseteq \bigcup_{i=0}^n S'_i$ of G_r with $|T| \leq 5$ leaves a component C of $G_r - T$ such that $V(C) \cup T$ meets one and hence every branch-set of M. So as each S'_i can be separated in G_r from any component of $G - S'_i$ by at most three vertices, there is an $i < n$ such that each branch-set of M meets $S'_i \cup S'_{i+1}$. Moreover, there is a maximal set S of neighbours in S_{i+1} of the same vertex in S_i such that each branch-set of M has a vertex in $S' := S \cup N_{S'_i}(S) \cup \{v_1^S, v_2^S, \dots, v_r^S\}$. Then $|S' \cap S'_i| \leq 3$.

We claim that M is also a minor of the finite graph G'_r (see Figure 2) which is obtained from $G_r[S']$ by adding an edge between every two vertices that are neighbours of the same component of $G_r - S'$. Indeed, each component C of $G_r - S'$ has at most three neighbours in S'. Hence, since M is 6-connected, C meets only (if at all) those branch-sets of M that also meet $N_{S'}(C)$. It is easy to see that M is still a minor of the graph we obtain from G_r by deleting C and adding all edges between vertices in $N_{S'}(C)$. Arguing analogously for the other components of $G_r - S'$, we see that M is also a minor of G'_r.

As $|S' \cap S'_i| \leq 3$, all but at most 3 branch-sets of M in G'_r have all their vertices in $|S' \cap S'_{i+1}|$. Then these give rise to a 3-connected minor of $G'_r - S'_i$. But each non-trivial block of $G'_r - S'_i$ is a triangle and hence has no 3-connected minor, yielding the desired contradiction. □

Note that the two latter results are best possible, since G_r has a 3-connected subgraph, the K^4, and a 5-connected minor, the K^6.

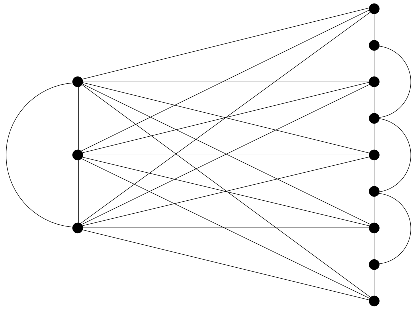

Figure 2: The graph G'_5 for $|S' \cap S'_i| = 3$.

5 Forcing highly connected subgraphs

We finally prove our main result, which we restate:

Theorem 2. *Let $k \in \mathbb{N}$ and let G be a locally finite graph such that each vertex has degree at least $6k^2 - 5k + 3$, and each end has vertex-degree at least $6k^2 - 9k + 4$. Then every infinite region of G has a k-connected region.*

Proof. Let C be an infinite region of G, and assume that C has no finite k-connected subgraph. We shall then find an infinite region H of C which is k-connected. Set $\delta_V := 6k^2 - 5k + 3$ and $\delta_\Omega := 6k^2 - 9k + 4$. Note that we may assume that $k > 1$.

First, suppose that for every infinite region C' of C there is a region $C'' \subseteq C' - \partial C'$ of C such that $|\partial C''| < \delta_\Omega$ and $V(C'') \neq \partial C''$. Observe that each such C'' is infinite, as otherwise $C'' - \partial C''$ has minimum degree $\delta(C'' - \partial C'') \geq \delta_V - \delta_\Omega + 1 \geq 4k$. Then Theorem 1 yields a finite k-connected subgraph of $C'' \subseteq C$, contradicting our assumption. Hence, there exists a sequence $C =: C_1, C_2, \ldots$ of infinite regions of C such that $C_i \subseteq C_{i-1} - \partial C_{i-1}$ and $|\partial C_i| < \delta_\Omega$ for all $i > 1$.

As in the proof of Theorem 3, we see that there is an end $\omega \in \Omega(G)$ that has a ray R such that each of the C_i contains a tail of R. As ω has vertex-degree at least δ_Ω, there are δ_Ω disjoint ω-rays in G. The starting vertices of these lie at finite distance to ∂C_1, hence, since $C_i \subseteq C_{i-1} - \partial C_{i-1}$ for $i > 1$, there is an $n \in \mathbb{N}$ so that all of the δ_Ω disjoint ω-rays start outside C_n. But (being equivalent to R) each of these rays eventually enters C_n, a contradiction because $|\partial C_n| < \delta_\Omega$.

Hence, there is an infinite region C' of C such that

$$|\partial C''| \geq \delta_\Omega \textit{ for each region } C'' \subseteq C' - \partial C' \textit{ of } C \textit{ with } V(C'') \neq \partial C''. \tag{2}$$

For a region $H \subseteq C' - \partial C'$ of C write

$$\Sigma_H := \sum_{v \in V(H)} \max\{0, \delta_V - d_H(v)\},$$

and choose an infinite region $H \subseteq C' - \partial C'$ of C such that $k|\partial H| + \Sigma_H$ is minimal. Observe that this sum is finite, since all vertices of H but those in

∂H have degree $\geq \delta_V$ in H, and it is possible to choose $H \subseteq C' - \partial C'$ with $|\partial H| < \infty$ because G is locally finite. Then $|\partial H| \geq \delta_\Omega$ by (2).

Assume that there is a vertex $v \in V(H)$ that has degree at most $2k-1$ in H. Then $d_{H-v}(w) = d_H(w) - 1$ for each of the at most $2k-1$ neighbours w of v in H, and $d_{H-v}(w') = d_H(w')$ for all other vertices w' in H. Therefore,

$$\begin{aligned} k|\partial(H-v)| + \Sigma_{H-v} &\leq k|\partial H| + k(2k-2) + \Sigma_H + (2k-1) - (\delta_V - d_H(v)) \\ &\leq k|\partial H| + \Sigma_H + 2k(k+1) - \delta_V \\ &< k|\partial H| + \Sigma_H. \end{aligned}$$

So any infinite component of $H - v$ is a better choice than H, a contradiction. We thus have shown that

$$d_H(v) \geq 2k \textit{ for all } v \in V(H). \tag{3}$$

We shall now prove that H is the desired k-connected region of C. Indeed, suppose otherwise. Then H has a separator T of cardinality $< k$, which we may assume be a minimal separator. Note that each such separator leaves a component D of $H - T$ such that $H - D$ is an infinite region of C. We claim that T and D can be chosen such that for $H' := H - D$

$$d_{H'}(v) \geq 2 \textit{ for each vertex } v \in T. \tag{4}$$

Indeed, choose a separator T of minimal cardinality in H and a component D of $H - T$ such that the number of vertices in T that have degree at most 1 in H' is minimal. Suppose that there is a $v \in T$ so that $d_{H'}(v) \leq 1$. Then the minimality of T implies that $d_{H'}(v) = 1$, and that the neighbour w of v in H' does not lie in T. By (3), w has degree at least $2k \geq 3$ in H. Hence, since $w \notin T$, also $d_{H'}(w) \geq 3$. Thus the number of vertices in $T' := T \setminus \{v\} \cup \{w\}$ that have degree at most 1 in $H - (D \cup \{v\})$ is smaller than the number of vertices in T that have degree at most 1 in H'. Now, the minimality of $|T|$ ensures that T' is a minimal separator of H, and has minimal cardinality. Furthermore, $D \cup \{v\}$ is a component of $H - T'$ (as T is a minimal separator and hence v sends an edge to D), and $H - (D \cup \{v\})$ is infinite (as H' is), a contradiction to the choice of T. This establishes (4).

We claim that

$$|V(D) \cap \partial H| \geq \delta_\Omega - |T|. \tag{5}$$

Then we obtain for the infinite region $H' \subseteq C' - \partial C'$ of C that

$$\begin{aligned} |\partial H'| &\leq |\partial H| - |V(D) \cap \partial H| + |T| \\ &\leq |\partial H| - \delta_\Omega + 2|T|. \end{aligned}$$

Furthermore, by (4),

$$\begin{aligned} \Sigma_{H'} &\leq \Sigma_H + \sum_{v \in T} \max\{0, \delta_V - d_{H'}(v)\} \\ &\leq \Sigma_H + (\delta_V - 2)|T|, \end{aligned}$$

and so

$$\begin{aligned}k|\partial H'| + \Sigma_{H'} &\le k|\partial H| + \Sigma_H - k\delta_\Omega + (\delta_V + 2k - 2)|T| \\ &\le k|\partial H| + \Sigma_H - k\delta_\Omega + (6k^2 - 3k + 1)(k - 1) \\ &< k|\partial H| + \Sigma_H,\end{aligned}$$

contradicting the choice of H.

It remains to show the validity of (5). Suppose otherwise, ie. that $|V(D) \cap \partial H| < \delta_\Omega - |T|$. Then for the region $\tilde{D} := G[V(D) \cup T] \subseteq C' - \partial C'$ of C holds that

$$|\partial\tilde{D}| = |T \cup (V(D) \cap \partial H)| \le |T| + |V(D) \cap \partial H| < \delta_\Omega.$$

Hence by (2), $V(\tilde{D}) = \partial\tilde{D}$, implying that $V(D) \subseteq \partial H$. In particular, $|D| < \delta_\Omega - |T|$. So each vertex $v \in V(D)$ has degree at most $|D \cup T - \{v\}| \le \delta_\Omega - 1 = \delta_V - 4k$ in H. Then $\delta_V - d_H(v) \ge 4k$, and thus

$$\begin{aligned}\Sigma_{H'} &\le \Sigma_H - \sum_{v \in V(D)} \max\{0, \delta_V - d_H(v)\} + \sum_{v \in T}(d_H(v) - d_{H'}(v)) \\ &\le \Sigma_H - 4k|D| + |T||D| \\ &< \Sigma_H.\end{aligned}$$

On the other hand, (3) ensures that $|D| \ge k$. So

$$|\partial H'| \le |\partial H| - |D| + |T| < |\partial H|,$$

and thus

$$k|\partial H'| + \Sigma_{H'} < k|\partial H| + \Sigma_H,$$

a contradiction to the choice of H. This completes the proof of (5), and hence the proof of the theorem. □

Theorem 2 has two corollaries. The proof of the first is analogous to that of Corollary 6.

Corollary 11. *Let $k \in \mathbb{N}$ and let C be an infinite region of a locally finite graph of minimum vertex degree $6k^2 - 5k + 3$ and minimum vertex-degree $6k^2 - 9k + 4$ at the ends. Then C has either infinitely many disjoint finite k-connected regions or an infinite k-connected region.*

Again, these two configurations need not both exist, as the examples following Corollary 6 illustrate.

The second corollary of Theorem 2 is an analogon of Corollary 7.

Corollary 12. *Let $k \in \mathbb{N}$, and let G be a locally finite graph with minimum vertex degree $\delta_V \ge 6k^2 - 5k + 3$ and minimum vertex-degree $\delta_\Omega \ge 6k^2 - 9k + 4$ at the ends. Then there is a countable set $\mathcal{D}$ of disjoint k-connected regions of G such that $|\partial H| \ge \max\{\delta_\Omega, \frac{k-1}{k}|H| + 1\}$ for each subgraph H of $G - \bigcup_{D \in \mathcal{D}} D$.*

Proof. As in the proof of Corollary 7, take an inclusion-maximal set $\mathcal{D}$ of disjoint k-edge-connected regions of G, which then is countable.

Observe that we only need to consider induced connected subgraphs H of $G - \bigcup_{D\in\mathcal{D}} D$. So let H be a such. If H is infinite, then Theorem 2 and the choice of $\mathcal{D}$ imply that H is not a region, ie. that $|\partial H|$ is infinite, as desired.

So assume that H is finite. Then $|\partial H| \geq \delta_\Omega$, as otherwise $H - \partial H$ has minimum degree $d(H - \partial H) \geq \delta_V - \delta_\Omega + 1 \geq 4k$, and hence H has a k-connected subgraph by Theorem 1, contradicting the choice of $\mathcal{D}$.

Also, $|\partial H| > \frac{k-1}{k}|H|$. Indeed, suppose otherwise. Then H has average degree

$$d(H) \geq \frac{\delta_V|H - \partial H| + |\partial H|}{|H|} \geq \frac{\delta_V + k - 1}{k} \geq 4k,$$

since we may assume that $k \geq 2$. Thus Theorem 1 yields a k-connected subgraph of H, a contradiction to the choice of $\mathcal{D}$. □

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