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End spaces of graphs are normal

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Abstract

We show that the topological space of any infinite graph and its ends is normal. In particular, end spaces themselves are normal.

1 Introduction

The notion of the *ends* of an arbitrary infinite graph, not necessarily locally finite, was introduced by Halin [8]. Jung [9] defined a topology on the set of ends, which was extensively studied by Polat [10], [11], [12]. Diestel and Kühn [4], [5] extended this topology to the entire graph (vertices, edges and ends); see also [3]. Some fundamental topological questions about this space – in particular, when it is compact or metrizable – were recently answered by Diestel [2].

One basic question that has remained open, both for the end space $\Omega(G)$ of a graph G and for the entire space $|G|$ including both G and its ends, is whether or not this space is normal. (It is easily seen to have all weaker separation properties.) When G is connected and locally finite, then G coincides with the Freudenthal compactification of G, and hence both $|G|$ and the closed subspace $\Omega(G)$ of its ends are normal. When G is connected and countable, then $|G|$ is metrizable [2], so again both $|G|$ and $\Omega(G)$ are normal. In this paper, we show that $|G|$ and $\Omega(G)$ are always normal.

2 Notation, background, and statement of results

We assume familarity with the basic notions of infinite graph theory, eg. as presented in Diestel [3]. However, let us quickly review some of these. One-way infinite paths are called *rays*, every subray of a ray is a *tail*. The union of two rays that have a common starting vertex and are otherwise disjoint, is called a *double ray*. We call two rays *equivalent* if, for every finite set S of vertices, both rays have a tail in the same component of $G - S$. It is quite easy to see that this is an equivalence relation, the equivalence classes are called *ends*. The set of ends of G is denoted by $\Omega(G)$.

The topology on $\Omega(G)$ is defined as follows. Let S be a finite set of vertices and C a component of $G - S$. Let $\Omega_S(C)$ denote the set of those ends of G whose rays have a tail in C. A subset of $\Omega(G)$ is *open* if it is a union of sets of the form $\Omega_S(C)$. $\Omega(G)$ together with this topology is called the *end space* of G.

Definition. *A Hausdorff space X is called* normal *if any two disjoint closed subsets have disjoint neighbourhoods.*

It is well known that compact Hausdorff spaces are normal, and so are metric spaces. But the end space of an arbitrary graph need be neither compact nor metrizable (see [2], [6] for characterizations of those that are), nor even have a countable basis. The standard ways to prove normality therefore fail; our proof will be from first principles.

The first main result of this paper is

Theorem 2.1. *Let G be an infinite graph. Then $\Omega(G)$ is normal.*

The topological space $|G|$ of an infinite graph G consists of the disjoint union of $V(G)$, $\Omega(G)$ and a copy $\mathring{e} = (u, v)$ of $(0, 1)$ for every edge $e = uv \in E(G)$. The bijection between $(0, 1)$ and (u, v) can be extended to a bijection of $[0, 1]$ and $[u, v] := \{u\} \cup (u, v) \cup \{v\}$, which induces a metric on $[u, v]$. For any edge e, let d_e denote this metric.

The topology on $|G|$ is generated by the following basic open sets. For every $z \in \mathring{e} = (u, v)$ and $0 < \varepsilon \leq \min\{d_e(u, z), d_e(v, z)\}$, we let the open ε-ball around z in $\mathring{e}$ be open in $|G|$ and denote it by $O_\varepsilon(z)$. For every vertex u and $\varepsilon \in (0, 1]$, we let the set of all points on edges $[u, v]$ of distance less than ε from u (measured in d_e for each $e = uv$) be open in $|G|$ and denote it by $O_\varepsilon(u)$. For every end ω, $\varepsilon \in (0, 1]$ and every finite set S of vertices, we let the set $\hat{C}_\varepsilon(S, \omega)$ be open in $|G|$, where $\hat{C}_\varepsilon(S, \omega)$ consists of the component $C(S, \omega)$ of $G - S$ that contains a ray from ω, every end that has a ray in $C(S, \omega)$, plus $O_\varepsilon(u)$ for every neighbour u of S in $C(S, \omega)$.

Clearly, $|G|$ is a locally connected Hausdorff space. The subspace topology it induces on its closed subspace $\Omega(G)$ is exactly the topology on $\Omega(G)$ defined earlier.

Theorem 2.1 therefore follows at once from the following more general result:

Theorem 2.2. *Let G be an infinite graph. Then $|G|$ is normal.*

An infinite graph is called *locally finite* if every vertex has finite degree. If G is locally finite, $|G|$ is a compact space [2] and Theorem 2.2 is trivial. If G has a normal spanning tree, $|G|$ is metrizable [2] and therefore Theorem 2.2 is again trivial.

A set $S \subset V(G)$ *separates* two points $x, y \in V(G) \cup \Omega(G)$ if they do not lie (or have rays) in the same component of $G - S$. S separates two sets $A, B \subset V(G) \cup \Omega(G)$ if it separates every point in A from every point in B. If an end ω cannot be separated by finitely many vertices from a given (infinite) set Z of vertices, then no ray in ω can be separated by finitely many vertices from Z. Then by Menger's theorem, for any ray $R \in \omega$, there are infinitely many disjoint paths from $V(R)$ to Z; the union of R with these paths is called a *comb*. The last vertices of these paths are called the *teeth* of the comb, and R is its *spine*. A *tail* of a comb is the union of a tail of its spine and all the paths that meet this tail. Note that not every vertex of the spine has to be the first vertex of one of the paths, and a tooth may lie on the spine if (and only if) its finite path is trivial. (See [3] for more on combs.)

Given a subset U of a topological space X, we call the set of all points $x \in X$ such that every neighbourhood of x meets both U and $X \setminus U$ the *boundary* of U and denote it by ∂U.

We shall later need the following topological lemma.

Lemma 2.3. *If U is the union of (arbitrarily many) open sets U_i in a locally connected topological space X, then ∂U is contained in the closure of $\bigcup \partial U_i$.*

Proof. Let $x \in \partial U$. Every connected neighbourhood O of x contains points in U as well as points in $X \setminus U$. Thus, there is an i with O meeting U_i as well as $X \setminus U_i$. If O missed ∂U_i, one could decompose O into the open sets $O \cap U_i$ and $O \cap (X \setminus \overline{U_i})$, so O would not be connected. Thus, every connected neighbourhood – and therefore every neighbourhood – of x contains an element of $\bigcup \partial U_i$. $\square$

3 Proof of the normality theorem

As we observed in Section 2, Theorem 2.2 is trivial for locally finite graphs and for graphs that have a normal spanning tree. For arbitrary graphs, Theorem 2.2 will follow easily from

Lemma 3.1. *Let G be an infinite graph and $A, B \subset \Omega(G)$ disjoint closed sets in $|G|$. Then there exist disjoint neighbourhoods of A and B in $|G|$.*

Proof of Theorem 2.2 from Lemma 3.1. Let A, B be disjoint closed sets in $|G|$. As $A \cap \Omega(G)$ and $B \cap \Omega(G)$ are closed in $|G|$, Lemma 3.1 gives us disjoint neighbourhoods U_1 of $A \cap \Omega(G)$ and V_1 of $B \cap \Omega(G)$. We now define further neighbourhoods U_2 of $A \cap \Omega(G)$ and V_2 of $B \cap \Omega(G)$ as well as neighbourhoods U_3 of $A \setminus \Omega(G)$ and V_3 of $B \setminus \Omega(G)$ so that U_3 is disjoint from $V_2 \cup V_3$ and V_3 is disjoint from $U_2 \cup U_3$.

Since B is closed, there exists for every $a \in A$ a neighbourhood $\hat{C}_{\varepsilon_a}(S_a, a)$ (if a is an end) or $O_{\varepsilon_a}(a)$ (if a is a vertex or a point on an edge) of a avoiding B. Choose U_2 as the union of all the open sets $\hat{C}_{\frac{1}{2}\varepsilon_a}(S_a, a)$ for $a \in A \cap \Omega(G)$ and U_3 as the union of all the open sets $O_{\frac{1}{2}\varepsilon_a}(a)$ for $a \in A \setminus \Omega(G)$. V_2 and V_3 are chosen analogously.

It is straightforward to check that these neighbourhoods satisfy the desired conditions. As $U_1 \cap V_1 = \emptyset$, we deduce that

$$U := (U_1 \cap U_2) \cup U_3 \qquad \text{and} \qquad V := (V_1 \cap V_2) \cup V_3$$

are disjoint neighbourhoods of A and B, respectively. Thus, $|G|$ is normal. $\square$

Proof of Lemma 3.1. Let G be an arbitrary infinite graph and A and B disjoint closed subsets of $\Omega(G)$. Note that A and B are also closed in $|G|$. For every end ω and every finite set S of vertices, we write $\hat{C}(S, \omega)$ instead of $\hat{C}_1(S, \omega)$ (i.e. $\hat{C}_\varepsilon(S, \omega)$ with $\varepsilon = 1$).

If A and B are both countable, there is a simple way of constructing disjoint neighbourhoods of A and B. For $i = 0, 1, \dots$ one chooses alternately ends $\omega_i \in A$ and $\tilde{\omega}_i \in B$, starting with $\omega_0 \in A$. For every such end there is a finite set S_i, respectively $\tilde{S}_i$, of vertices separating ω_i from B or $\tilde{\omega}_i$ from A, respectively. We now have neighbourhoods $U := \bigcup_{i<\omega} \hat{C}(S_i, \omega_i)$ of A and $\tilde{U} := \bigcup_{j<\omega} \hat{C}(\tilde{S}_j, \tilde{\omega}_j)$ of B. These will be disjoint if the neighbourhoods $\hat{C}(S_i, \omega_i)$ and $\hat{C}(\tilde{S}_j, \tilde{\omega}_j)$ are disjoint for any i, j. To achieve this, it suffices to choose the separators S_i in a special way, namely, containing $\bigcup_{j<i} \tilde{S}_j$. Then, for every $j < i$, $C(S_i, \omega_i)$ will be contained in the component $C(\tilde{S}_j, \omega_i)$ of $G - \tilde{S}_j$, which cannot be $C(\tilde{S}_j, \tilde{\omega}_j)$, because $C(\tilde{S}_j, \tilde{\omega}_j)$ avoids A but $C(S_i, \omega_i)$ contains $\omega_i \in A$. Hence $\hat{C}(S_i, \omega_i) \cap \hat{C}(\tilde{S}_j, \tilde{\omega}_j) = \emptyset$. Likewise, we choose each $\tilde{S}_j$ so as to contain

$\bigcup_{i\le j} S_i$, which ensures that $\hat{C}(\tilde{S}_j,\tilde{\omega}_j)$ will be disjoint from every $\hat{C}(S_i,\omega_i)$ with $i\le j$. Thus, U and $\tilde{U}$ will be disjoint.

This procedure fails for uncountable A or B, as it may be impossible at a transfinite step for a finite separator S_i to contain every previous separator. Even worse, if one does not choose the S_i carefully, the closure of all the earlier $\hat{C}(S_i,\omega_i)$ may contain ends from B at any limit step of the construction, making it impossible to find a suitable neighbourhood for these ends. This is the problem to be solved.

We shall construct a neighbourhood U of A in $|G|$, whose closure in $|G|$ will not meet B. The desired neighbourhood of B can then be chosen as $|G|\setminus\overline{U}$, completing the proof of Lemma 3.1.

Let us write $A=\{\omega_i \mid i<\lambda\}$ with $\lambda:=|A|$. At step $i<\lambda$ we will choose a finite set S_i of vertices separating ω_i from B and put $U_i:=\hat{C}(S_i,\omega_i)$. Finally, let $U:=\bigcup_{i<\lambda} U_i$.

Suppose we have chosen the sets S_i already, but badly: there is an end $\omega\in B$ in $\overline{U}$. By choice of the U_i, we have $\omega\in\partial U$. As $\partial U_i\subset S_i$, Lemma 2.3 yields $\omega\in\overline{\bigcup_{i<\lambda} S_i}$. Thus, ω cannot be separated from $\bigcup_{i<\lambda} S_i$ by finitely many vertices; hence there is a comb with spine in ω and teeth in $\bigcup_{i<\lambda} S_i$. As every S_i is finite, the comb has teeth in infinitely many S_i. Our aim will be to choose the S_i so that infinitely many of these teeth can be linked by disjoint rays to (pairwise different) ends in A. Then $\omega\in B$ will lie in the closure of these ends, and hence in A, contrary to our assumption that $A\cap B=\emptyset$.

For every $i<\lambda$ let S_i be a finite set of vertices that separates ω_i from B, chosen so that

$$S_i\setminus\bigcup_{j<i} S_j \text{ is } \subset\text{-minimal.} \tag{1}$$

In particular, if ω_i can be separated from B by a finite subset of $\bigcup_{j<i} S_j$, we have chosen S_i as such a subset.

Every set S_i also satisfies

For every $s\in S_i\setminus\bigcup_{j<i} S_j$ and every finite $S\subset\bigcup_{j<i} S_j$, there exists a ray in ω_i that starts in s, avoids S, and is contained in $U_i\cup\{s\}$. (2)

Indeed, for every $s\in S_i\setminus\bigcup_{j<i} S_j$ and every finite $S\subset\bigcup_{j<i} S_j$, the set $S_i':=S\cup S_i\setminus\{s\}$ does not separate ω_i from B, as this would contradict (1). So there is a double ray D that joins ω_i with an end in B and avoids S_i'. As S_i separates ω_i from B, D hits S_i. But D avoids $S_i\setminus\{s\}\subset S_i'$, so D meets S_i only in s. Thus, D contains a ray as required in (2).

Let us prove that $\overline{U}\cap B=\emptyset$. Suppose not, and pick $\omega\in B\cap\overline{U}$. As described earlier, there is a comb C in G with spine in ω and teeth in $\bigcup_{i<\lambda} S_i$. Let Z be the set of its teeth. For every $z\in Z$ there is a smallest index $i=i(z)<\lambda$ with $z\in S_i$. Without loss of generality we may assume that $i(z)\neq i(z')$ for $z\neq z'$. Inductively for all $j<\aleph_0$ choose $z(j)\in Z$ as the vertex $z\in Z\setminus\{z(k)\mid k<j\}$ with smallest value $i(z)$. Write $i(j)$ for $i(z(j))$. Note that the function $i(j)$ is strictly increasing. Hence for every $j<\aleph_0$, the set $\bigcup_{k<j} S_{i(k)}$ is a finite subset of the (possibly infinite) set $\bigcup_{l<i(j)} S_l$.

We now inductively define disjoint rays R_j for all $j<\aleph_0$. By the choice of $z(j)$ and the definition of $i(j)$, we have $z(j)\notin\bigcup_{l<i(j)} S_l$. In particular, as $z(j)\in S_{i(j)}$,

$$S_{i(j)}\not\subset\bigcup_{l<i(j)} S_l. \tag{3}$$

By (2), there exists a ray $R_j \in \omega_{i(j)}$ that starts in $z(j)$, avoids the finite subset $\bigcup_{k<j} S_{i(k)}$ of $\bigcup_{l<i(j)} S_l$ and is contained in $U_{i(j)} \cup \{z(j)\}$. As R_j avoids $\bigcup_{k<j} S_{i(k)}$, we have for every $k < j$ either $R_j \subset U_{i(k)}$ or $R_j \cap U_{i(k)} = \emptyset$. If R_j was contained in $U_{i(k)}$, then $\omega_{i(j)}$ would also be contained in $U_{i(k)}$. But then $\omega_{i(j)}$ could be separated from B by the finite subset $S_{i(k)}$ of $\bigcup_{l<i(j)} S_l$. By (1), this would imply $S_{i(j)} \subset \bigcup_{l<i(j)} S_l$, contradicting (3). We thus have $R_j \cap U_{i(k)} = \emptyset$, as well as $z(k) \notin R_j$ for all $k < j$.

Therefore, $\mathcal{R} := \{R_j \mid j < \aleph_0\}$ is a set of disjoint rays, where R_j belongs to the end $\omega_{i(j)}$ and starts at the vertex $z(j)$. As every finite set of vertices misses both a tail of our comb C and almost every ray in $\mathcal{R}$, no finite set of vertices separates ω from A, in contradiction to the fact that A is closed but $\omega \notin A$.

□

4 Other topologies

Let G be any graph. Our topology for $|G|$ is its standard topology as defined in [3]. In [2], [4], [5] some more topologies on $|G|$ are studied. These can equip $|G|$ with certain desirable properties, such as metrizability, or compactness. But they differ only slightly, and in particular they all induce the same topology on $\Omega(G)$: the topology we defined early in Section 2. Our proof of Theorem 2.2 can be adapted to one of those topologies, called Top. (The third, VTop, is not even Hausdorff.)

In some contexts, however, such as plane duality [1], the most natural space associated with a graph G is not $|G|$, but a certain quotient space $\tilde{G}$ of $|G|$ (where $|G|$ carries either Top or the topology defined in Section 2). In this section we show that $\tilde{G}$, and its end space $\tilde{\Omega}(G)$, are also normal.

To define $\tilde{G}$, let us say that a vertex v *dominates* an end ω if every finite set of vertices that separates v from ω contains v. Let Ω_v denote the set of ends of G that are dominated by the vertex v. Throughout this section, we assume that

$$\textit{no end of } G \textit{ is dominated by more than one vertex.} \tag{$*$}$$

By $(*)$, we have $\Omega_u \cap \Omega_v = \emptyset$ for all $u \neq v$. Let $\tilde{G}$ be the quotient space of $|G|$ obtained by identifying every vertex v with every end in Ω_v. We write $\tilde{\Omega}(G)$ for the set of undominated ends of G, which we informally also call the *ends of* $\tilde{G}$. Note that $\tilde{\Omega}(G)$ is a subspace both of $|G|$ and of $\tilde{G}$; the subspace topologies coincide, and we endow $\tilde{\Omega}(G)$ with this topology.

If G is connected, then by $(*)$, [2], and Halins [7] theorem that connected graphs not containing a $TK^{\aleph_0}$ have normal spanning trees, $|G|$ is metrizable. Hence $\tilde{\Omega}(G)$, too, is a metric space, and therefore normal.

Theorem 4.1. *For every graph G satisfying $(*)$, the space $\tilde{G}$ is normal.*

Let $\sigma : |G| \to \tilde{G}$ be the identification map that gave rise to $\tilde{G}$. Given a set $X \subset |G|$, write $[X]$ for $\sigma^{-1}\big(\sigma(X)\big)$. (Thus, $[X]$ is the union of X and all the sets of the form $\Omega_v \cup \{v\}$ that meet X.)

Lemma 4.2. *If $X \subset |G|$ is closed in $|G|$, then so is $[X]$.*

Proof. If $[X]$ is not closed, there exists a point $x \notin [X]$ in the closure of $[X]$. Clearly, x is an end.

As x does not lie in X, and X is closed, there is an $\varepsilon \in (0,1]$ and a finite set S of vertices such that $\hat{C}_\varepsilon(S,x) \cap X = \emptyset$. Then any point z of $[X]$ in $\hat{C}_\varepsilon(S,x)$ must lie in a set $\Omega_v \cup \{v\}$ that meets X. Since S separates x – and therefore also z – from every point in $(\Omega_v \cup \{v\}) \cap X$, the vertex v has to lie in $S \cap [X]$ and z is an end in Ω_v.

By a result of [5], the sets Ω_v are closed in $|G|$, so the finite union $\bigcup_{v \in S \cap [X]} \Omega_v$ is also closed. The intersection of its complement in $|G|$ with $\hat{C}_\varepsilon(S,x)$ is a neighbourhood of x that avoids $[X]$. □

Proof of Theorem 4.1. Let disjoint closed subsets $\tilde{A}, \tilde{B}$ of $\tilde{G}$ be given. Then $A := \sigma^{-1}(\tilde{A})$ and $B := \sigma^{-1}(\tilde{B})$ are disjoint closed sets in $|G|$. By Theorem 2.2, we have disjoint open sets $U \supset A$ and $V \supset B$ in $|G|$.

$|G| \setminus U$ is closed, and hence so is $[|G| \setminus U]$, by Lemma 4.2. Since $[A] = A$, by Definition of A, we have $A \cap [|G| \setminus U] = \emptyset$. Hence, $U' := |G| \setminus [|G| \setminus U]$ is an open subset of U that still contains A and satisfies $[U'] = U'$. Likewise, V has an open subset V' that still contains B and satisfies $[V'] = V'$.

Thus, $\sigma(U')$ and $\sigma(V')$ are disjoint neighbourhoods of $\tilde{A}$ and $\tilde{B}$, respectively. □

Problem. *Is $\tilde{G}$ metrizable?*

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