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# On small contractible subgraphs in 3-connected graphs of small average degree

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**Abstract**

We study the distribution of small contractible subgraphs in 3-connected graphs under local regularity conditions.

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**Keywords:** Connectivity, partition, contractible, regular, average degree.

## 1 Introduction

All graphs considered here are supposed to be finite, simple, and undirected. For terminology not defined here, the reader is referred to [1, 2]. — Tutte proved that every 3-connected graph nonisomorphic to $K_4$ contains a *contractible edge*, i.e. an edge $xy$ such that identifying $x, y$ in $G - xy$ produces a 3-connected graph, or, equivalently, such that $G - \{x, y\}$ is 2-connected [9]. As a generalization, McCuaig and Ota conjectured in [8]:

**Conjecture 1** *[8] For each $\ell \geq 2$ there exists a smallest number $f(\ell)$ such that every 3-connected graph $G$ on at least $f(\ell)$ vertices has a connected subgraph $H$ on $\ell$ vertices such that $G - V(H)$ is 2-connected.*

It is quite natural to ask for graph partitions with restrictions to size and connectivity; a prominent result along these lines is Győri's Theorem stating that a graph $G$ on $n > k$ vertices is $k$-connected if and only if for every choice of $k$ distinct vertices $a_1, \ldots, a_k$ and any choice of $k$ positive integers $n_1, \ldots, n_k$ such that $n_1 + \ldots + n_k = n$ there exists a partition $X_1, \ldots, X_k$ of its vertex set such that for each $i \in \{1, \ldots, k\}$, $G(X_i)$ is a connected graph of order $n_i$ containing $a_i$ [3].

Conjecture 1 has been confirmed for $\ell \leq 4$, where the optimal values for $f$ are $f(2) = 5$ [9], $f(3) = 9$ [8], $f(4) = 8$ [6]. Although known to be true when $H$ is not forced to be connected [5], Conjecture 1 remains widely open in general. For example, it is not known whether there is some $k$ such that its restriction to $k$-connected graphs is true.

The problem becomes much easier if we remove the edges instead of the vertices of some subgraph $H$; clearly, we can not expect the respective statement of Conjecture 1 without strengthening the connectivity condition to $G$. But considering 4-connected graphs, a recent result on removable cycles [7] yields a positive answer here:

**Theorem 1** *For every $\ell \geq 1$ there exists a number $f(\ell)$ such that every 4-connected graph $G$ has a path or a star $H$ on $\ell$ edges such that $G - E(H)$ is 2-connected.*

**Proof.** For $\ell \geq 1$ there exists an $f(\ell)$ such that every connected graph on at least $f(\ell)$ vertices has either a vertex of degree $\ell + 2$ or two vertices at distance $\ell$ (cf. [2, Chapter 1.3]). The main result of [7] implies that every edge of a 4-connected graph $G'$ is contained in some cycle $C$ such that $G' - E(C)$ is 2-connected (*), and we use it as follows. If $G$ in contains a vertex of degree at least $\ell + 2$, we choose any star for $H$ and observe that $G - V(H)$ is 2-connected. Otherwise, it must contain vertices $a, b$ at distance $\ell$. By (*), there exists a cycle $C$ in $G' := G + ab$ containing $a, b$ such that $G' - E(C)$ is 2-connected. Hence $P := C - ab$ is an $a,b$-path in $G$ such that $G - E(P)$ is 2-connected, and any subpath of length $\ell$ of $P$ will serve for $H$. Q.E.D.

The core of this paper deals with the smallest open case $\ell = 5$ of Conjecture 1 for regular graphs; it's meant to support both McCuaig's and Ota's conjecture and its difficulty when aiming for a *sharp* $f(\ell)$, $\ell \geq 5$. For example, we will determine the cubic 3-connected graphs $G$ which do not have a connected subgraph on 5 vertices such that $G - V(H)$ is 2-connected; the two largest ones have order 12, which shows $f(5) \geq 13$.

The proof technique depends mostly on local arguments, and it is maybe interesting to see how a "bookkeeping of the region in which the proof runs" pays off in form of a more general result on graphs whose average degree is bounded by 3 *plus a small constant*.

## 2 Extendability and Removable Links

We say that two subgraphs $H_1, H_2$ are *within distance* $\ell$ in $G$ if there is an $V(H_1), V(H_2)$-path of length at most $\ell$ in $G$, and we call disjoint subgraphs $H_1, H_2$ *adjacent* if they are disjoint and within distance 1. This terminology is

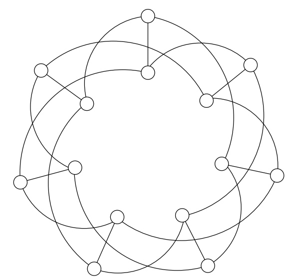

Figure 1: The HEAWOOD graph.

extended to vertices, edges, and subsets of $V(G)$ by considering the subgraphs induced by the respective objects.

An induced subpath $P$ of some graph $G$ is called a *link* of $G$, if every vertex of $P$ has degree 2 in $G$, we call it a maximal link if there exists no link of $G$ containing $P$ as a proper subgraph, and we call it *removable* if $G - V(P)$ is 2-connected. Hence every removable link is maximal. Furthermore, it is easy to see that if $G$ is not a cycle and the two neighbors of a maximal link $L$ are adjacent then $L$ is removable.

An induced subgraph $H$ of a 3-connected graph $G$ is called *contractible*, if $G - V(H)$ is 2-connected, or, equivalently, if the graph $G/V(H)$ obtained from $G - V(H)$ by adding a new vertex and making it adjacent all vertices in $N_G(V(H))$ is 3-connected. Every subgraph of $G$ of order 1 is thus contractible. Furthermore, every connected subgraph $H$ of $G$ such that $|N_G(V(H))| = 3$ is easily recognized to be contractible: As the vertex $h$ in $V(G/V(H)) - V(G)$ has 3 neighbors in $G/V(H)$, it suffices to prove that for any two distinct vertices in $V(G/V(H)) - \{h\}$ there exist 3 openly disjoint $x, y$-paths; but such paths exist as there are 3 openly disjoint $x, y$-paths in $G$ and at most one of them can intersect $V(H)$.

Throughout this paper, contractible subgraphs of order $3, 4, 5$ are also called *contractible triples*, *quadruples*, *quintuples*, respectively. An edge $xy$ is said to be *contractible* in $G$ if $G(\{x, y\})$ is contractible.

We call a contractible subgraph $H$ of $G$ *extendible* if there exists a vertex $x \in V(G) - V(H)$ such that $G(V(H) \cup \{x\})$ is contractible. Clearly, such a vertex $x$ must be in $N_G(V(H))$, and we sometimes say that $H$ has been *extended by $x$ to* $G(V(H) \cup \{x\})$. An edge is called *extendible* if it is contained in some contractible triple.

Let's have a look at some example. The HEAWOOD *graph* is the point line incidence graph of a projective plane of order 2, as depicted in Figure 1. It is the unique cubic bipartite graph on 14 vertices without 4-cycles. Observe that every connected subgraph on 1, 2, 3, or 4 vertices is contractible. Every contractible quadruple is extendible, but not in an arbitrary way, since a connected subgraph on 5 vertices is contractible if and only if it does not induce a path.

Nonextendability of $H$ is closely related to the presence of removable links in $G - V(H)$, as it is indicated by the following theorem from [6].

**Theorem 2** *[6] Let $H$ be a contractible subgraph of a 3-connected graph $G$. If $H$ is not extendible then either $G - V(H)$ is an induced cycle or $G - V(H)$ has a pair of disjoint nonadjacent removable links each of which has order at least 2.*

For example, the subgraph induced by a single vertex $x$ of degree 3 in a 3-connected graph $G$ is contractible and extendible unless $G - x$ is a triangle, so unless $G \cong K_4$. Similarly, a triangle $\Delta$ in $G$ such $E_G(\Delta)$ consists of 3 independent edges must be extendible unless $G - \Delta$ is a triangle, so unless $G \cong K_2 \times K_3$. The following two lemmas elaborate the idea that small degrees in small contractible subgraphs cause nice extendability properties.

**Lemma 1** *Let $G$ be a 3-connected graph on at least 9 vertices and let $wx$ be a contractible edge in $G$ such that $d_G(w) = d_G(x) = 3$. Then there exists a contractible edge $yz$ with $y \in \{w, x\}$ such that $d_G(y) = d_G(z) = 3$ and $yz$ is contained in some contractible triple.*

**Proof.** We may assume that $wx$ is not extendible, for otherwise it would serve as an appropriate $yz$. Observe that $|E_G(\{w, x\})| = 4$. Since $|V(G)| \geq 7$, $G - \{w, x\}$ does not induce a 4-cycle, and, by Theorem 2, $G - \{w, x\}$ admits a pair $P = pq, S = st$ of nonadjacent removable links of order 2, where each of $p, q, s, t$ has degree 3 in $G$. If $w$ was adjacent to $p, q$ then $stx$ would be a contractible triangle, proving the statement with $yz = xs$ or $yz = xt$. Hence we may assume that $w$ and, symmetrically, $x$, has neighbors in both $P, Q$.

Without loss of generality, let $pw, qx, sw, tx \in E(G)$. Since $G$ is 3-connected and $|V(G)| \geq 9$, $|N_G(\{p, q, s, t, w, x\})| > 2$. So we may assume, without loss of generality, that the vertex $a$ in $N_G(p) - \{q, w\}$ is not in $N_G(s) \cup N_G(t)$. Note that $pw$ is contractible, since $G - \{p, q, w, x\}$ is 2-connected and $q, x$ are adjacent to each other and to distinct vertices of the latter subgraph. If it was not extendible then $G - \{p, w\}$ had two disjoint nonadjacent removable links by Theorem 2. As $q, x$ are on the same maximal link of $G - \{p, w\}$, $a, s$ must form a removable link of $G - \{p, w\}$ — but they are not even adjacent. Hence $yz = pw$ is extendible, proving the lemma. Q.E.D.

**Lemma 2** *Let $G$ be a 3-connected graph on at least 9 vertices and let $H$ be a contractible triple in $G$ such that all vertices in $H$ have degree 3 in $G$. Then $H$ is extendible or $H$ is contained in a contractible quintuple of $G$.*

**Proof.** Suppose that $H$ is not extendible. Since $|V(G)| \geq 9$, $|E_G(V(G) - V(H))| \geq 4$ by Theorem 2. Hence $H$ induces a path $xyz$ in $G$, and $|E_G(V(H))| =$

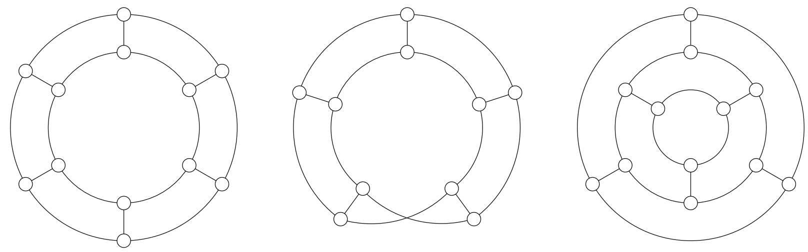

Figure 2: The graphs $R_6$, $M_5$, and $OCC$.

5. Since $|V(G)| \geq 9$, $G - V(H)$ can't induce a cycle. Hence, by Theorem 2, $G - V(H)$ contains two removable links $P, S$ links of order at least 2. One of these, say, $P$, must have order equal to two, and so $G(V(H) \cup V(P))$ is a contractible quintuple. Q.E.D.

# 3 Contractible Quintuples

Let $R_\ell := C_\ell \times K_2$ be the *ribbon* of length $\ell \geq 3$. Let $M_\ell$ denote the graph obtained from $C_{2\ell}$ by adding an edge between any two vertices at distance $\ell$, which one could call the MÖBIUS *strip* of length $\ell$. To *clip* a vertex $x$ of degree 3 in a graph $G$ means to subdivide the three edges incident with $x$ once, adding the three edges in between the subdivision vertices, and then removing $x$. The *oppositely clipped cube* $OCC$ is obtained from a cube $R_4$ by clipping two vertices at distance 3. The graphs $R_6$, $M_5$, and $OCC$ are depicted in Figure 2 and play a prominent role in this section as exceptional graphs. Each of them is cubic and 3-connected, and there is no contractible quintuple in either of them.

**Theorem 3** *Let $G$ be a 3-connected graph on at least 10 vertices nonisomorphic to the ribbon $R_6$ of length 6, the* MÖBIUS *strip $M_5$ of length 5, and the oppositely clipped cube $OCC$.*

*Suppose that $h$ is a vertex such that all vertices within distance 6 from $h$ have degree 3.*

*Then there exists a contractible quintuple within distance 3 from $h$.*

**Proof.** Assume, to the contrary, that there is no contractible quintuple within distance 3 from $h$.

The "local regularity condition" to $h$ will ensure throughout the proof that all vertices under consideration have degree 3, and in most cases this will not be discussed explicitly. In particular, if these graphs are small, say, of order at most 14, then they must be cubic, and in this case, moreover, every subgraph

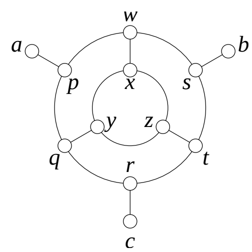


Figure 3: Almost an oppositely clipped cube.

is within distance 3 from every vertex (so we have not to take the location of $h$ into account in these cases).

**Claim 1.** There is no triangle within distance 3 from $h$.

Assume that $G$ has a triangle $\Delta$ within distance 3 from $h$. Then all vertices in $\Delta$ have degree 3 in $G$, and, as we have seen before, $\Delta$ is extendible to a contractible quadruple $H$, which is within distance 3 from $h$. Let $w$ be the vertex in $V(H) - V(\Delta)$, let $x$ be its neighbor in $V(\Delta)$, and let $y, z$ be the two vertices in $V(\Delta) - \{x\}$.

By assumption, $H$ is not extendible, so $|N_G(V(H))| \geq 4$, and, since $E_G(V(H))$ contains exactly 4 edges and $|V(G)| \geq 9$, these edges connect $V(H)$ to a pair of two removable links $P = pq, S = st$ of order 2 in $G - V(H)$ by Theorem 2. If the two neighbors of $w$ distinct from $x$ were both in $P$ then $(G - V(S)) - (V(H) - \{w\})$ would be 2-connected, so $G(\{s, t, x, y, z\})$ would be a contractible quintuple within distance 3 from $h$, contradiction. By symmetry, we thus may assume that $E_G(V(H)) = \{wp, ws, yq, zt\}$.

Since $(G - V(P)) - V(H)$ is 2-connected, $G - \{x, y, z, q\}$ is 2-connected, too, so $H' := G(\{x, y, z, q\})$ (and $H'' = G(\{x, y, z, t\})$) is a contractible quadruple within distance 3 from $h$. Since it is not extendible, we deduce, as above for $H$, that $E_G(V(H'))$ consists of the four edges $qp, xw, zt, qr$, where we let $r$ be the neighbor of $q$ in $V(G) - V(H) - V(P)$. As $N_G(V(H'))$ consists of the vertex sets of two removable links of order 2 in $G - V(H')$, $rt \in E(G)$ follows, so $r$ is the neighbor of $t$ in $V(G) - V(H) - V(S)$.

Let $C := G(p, q, r, t, s, w, x, y, z)$. This situation is depicted in Figure 3. We see that the vertices at distance 1 and those of distance 2 from $\Delta$ form a bipartition of the induced 6-cycle $pqrtsw$.

By local regularity and 3-connectedness, there exist unique neighbors $a, b, c$ of $p, s, r$, respectively, in $V(G) - V(C)$, which all have degree 3. If $a = b = c$ then $|V(G)| = 10$ and $G(\{w, x, p, s, a\})$ is a contractible quintuple, as removing it produces a 5-cycle $qrtzy$.

Otherwise, $C$ is contractible as we have observed before, and $a, b, c$ are pairwise distinct. If they are pairwise adjacent then $G$ is the oppositely clipped cube $OCC$. Otherwise, $C$ is extendible by Theorem 2. By symmetry, we may assume that $C$ is extendible by $c$.

Consider the path $Q := crqy$. Since $G - (V(C) \cup \{c\})$ is 2-connected and $apwxztsb$ is an $a, b$-path in $G - V(Q)$ that covers all vertices in $V(C) - V(Q)$, $Q$ is contractible. Observe that $Q$ is within distance 3 from $h$ (as $h \in V(C) \cup \{a, b, c\}$). Therefore, it is not extendible by assumption. Since $x, z, t$ are contained in one and the same maximal link of $G - V(Q)$, there must be a removable link $L$ in $G - V(Q)$ containing at least two of $N_G(V(Q)) - \{x, z, t\}$ by Theorem 2; the latter set consists of $p$ and the two vertices $d, e$ in $N_G(c) - \{r\}$.

If $L$ avoids $p$ then it is equal to $de$, so $cde$ is a triangle and $G - (V(Q) \cup \{d, e\})$ is 2-connected. It follows that $G - ((V(Q) - \{y\}) \cup \{d, e\})$ is 2-connected, too, so $G(\{c, d, e, q, r\})$ is a contractible quintuple within distance 3 from $h$ in $G$, contradiction.

Hence $L$ does not avoid $p$. So $L$ must contain $a$, too, implying that $L$ is adjacent to $c$. The same argument applied to the path $P = crtz$ yields that $b$ is adjacent to $c$. But then $\{a, b\}$ separates $V(C) \cup \{c\}$ from the remaining graph, a contradiction.

This proves Claim 1.

**Claim 2.** If $H$ is a contractible 4-cycle within distance 3 from $h$ then its four neighbors form two disjoint nonadjacent removable links of order 2 in $G - V(H)$.

As $H$ is not extendible and $|V(G)| \geq 10$, Claim 2 follows immediately from Theorem 2.

**Claim 3.** $G$ contains no contractible subgraph $K_{1,3}$ within distance 1 from $h$.

Suppose that $H \cong K_{1,3}$ is a contractible subgraph within distance 1 from $h$, let $w$ denote the unique vertex of degree 3 in $H$, and let $x, y, z$ be the others; $H$ is not extendible by assumption.

If $G - V(H)$ is a 6-cycle $C$ then the two neighbors of at least one of $x, y, z$ on $V(C)$ have distance 3 in $C$, as they can't be adjacent by Claim 1. (In fact, $G$ must be one of the two graphs in Figure 4, one of them being the PETERSEN graph.) Suppose $x$ has neighbors $a, d$ at distance 3 in $C$ and take a path $abcd$ in $C$. Then $abcdx$ is a cycle, and $H := G - V(abcdx)$ is connected and, thus, a contractible subgraph on 5 vertices.

If, otherwise, $G - V(H)$ is not a 6-cycle then it contains two removable links $P, S$ of order at least 2 by Theorem 2. If $P$ had order 2 then it had two distinct neighbors in $H$ (as $G$ contains no triangle within distance 3 from $h$ by Claim 1); without loss of generality, let these be $x, y$. Since $z$ has two neighbors in $V(G) - V(H) - V(P)$, $G - (V(P) \cup \{w, x, y\})$ is 2-connected,

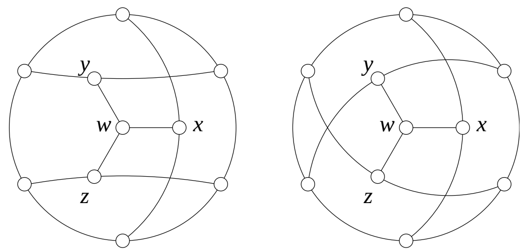


Figure 4: The case $G - K_{1,3} \tilde{=} C_6$.

so $G(V(P) \cup \{w, x, y\})$ is a contractible quintuple within distance 2 from $h$, contradicting our assumption.

Hence we may assume that $P$, and, symmetrically, $S$ has order at least 3. As every vertex of $P, S$ has a neighbor in $V(H)$, we deduce that $|V(P)| = |V(S)| = 3$, say, $P = pqr$, $S = stu$.

If the three edges in $E_G(V(P), V(H))$ are independent, say $px, qy, rz$ then $pxwyq$ is a contractible 5-cycle within distance 2 from $h$, contradiction. Since, by Claim 1, there is no triangle within distance 3 from $h$, we therefore may assume that $p, r$ are connected to the same vertex, say, $x$, from $H$, and, symmetrically, that $s, u$ are connected to, say, $y$. It follows that $zq, zt \in E(G)$. $C := pqrx$ is a contractible 4-cycle within distance 3 from $h$; by Claim 2, $G - V(C)$ has two removable links of order 2; hence $wz$ is one of them and the two neighbors of $V(P)$ in $G - V(H)$ constitute the other one. In particular, the neighbors of $V(P)$ in $G - V(H)$ are adjacent. Hence $P$ is a removable link in $G - V(H) - V(S)$, implying that $G - V(H) - V(P) - V(S)$ is 2-connected! From this one easily deduces that $G - rqztu$ is 2-connected, so $rqztu$ is a contractible quintuple within distance 3 from $h$.

This proves Claim 3.

**Claim 4.** If $H$ is a contractible path $P_4$ within distance 2 from $h$ then the four neighbors of its endvertices form two disjoint nonadjacent removable links of order 2 in $G - V(H)$ none of which is contained in a triangle.

Suppose $H = wxyz$ is a contractible path within distance 2 from $h$. By assumption, it is not extendible. Note that all vertices in $N_G(V(H)) \cup V(H)$ are within distance 6 from $h$ and, thus, have degree 3. If $G - V(H)$ was a 6-cycle $C$ then the neighbors $a, b$ of $w$ on $C$ are at distance 2 in $C$, for otherwise, by Claim 1, either $a, b$-path in $C$ would have length 3 and would form, together with $w$, a nonseparating 5-cycle $D$; thus $G - V(D)$ would be a contractible quintuple, contradiction. Consider the neighbor $c$ of $y$ in $V(C)$. It can't be a neighbor of $a$ or $b$ in $C$, for otherwise $awxyc$ or $bwxyc$ would be a nonseparating 5-cycle in $G$, contradiction again. So $c$ is the unique vertex such that $a, b, c$ are independent

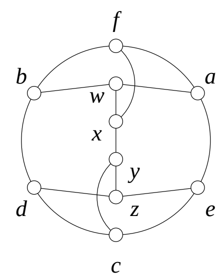


Figure 5: $a, c$ and $d, e$ don't cross.

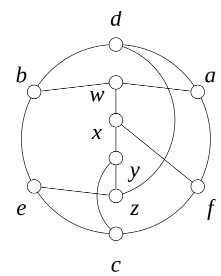


Figure 6: $a, c$ and $d, e$ do cross: $M_5$ arises.

in $C$. Symmetrically, the two neighbors $d, e$ of $z$ and the neighbor $f$ of $x$ in $V(C)$ are independent in $C$ (and distinct from $a, b, c$). If $d, e$ are in the same component of $C - \{a, b\}$ (see Figure 5) then $G(\{a, b, f, w, x\})$ is a contractible quintuple, contradiction. Otherwise, one of $d, e$ is adjacent to both $a, b$ and the other one is adjacent to exactly one of $a, b$. By symmetry, let $d$ be adjacent to $a, b$ and let $e$ be adjacent to $b$ only (see Figure 6), so $C = adbecf$ and $G$ is isomorphic to the exceptional graph $M_5$, contradiction.

Hence we may assume that $G - V(C)$ is not a 6-cycle. By Theorem 2, $G - V(H)$ has two disjoint nonadjacent removable links $P, S$ with $2 \leq |V(P)| \leq V(S)$ and $|V(P)| + |V(S)| \leq |N_G(V(H))| = 6$.

If $|V(P)| = |V(S)| = 3$, say, $P = pqr$ and $S = stu$, then $E_G(V(H))$ consists of 6 edges in $E_G(V(P) \cup V(S))$, and $h$ is contained in $V(H) \cup V(P) \cup V(S) \cup N_G(V(P)) \cup N_G(V(S))$. One of the four endvertices of $P, Q$ must be adjacent to an endvertex of $H$; by symmetry, let $pw \in E(G)$.

Assume for a while that $wq \in E(G)$. Then $h \in V(S) \cup N_{G-V(H)}(V(S))$ by Claim 1. If $rx \in E(G)$ then $G(\{s, t, u, y, z\})$ would be a contractible quintuple within distance 1 from $h$. If $ry \in E(G)$ then $sz, uz \in E(G)$ (for otherwise $z$ and its two neighbors in $V(S)$ would form a triangle within distance 2 from $h$, contradicting Claim 1), impyling that $G(\{p, q, w, x, t\})$ is a contractible quintuple within distance 2 from $h$. If $rz \in E(G)$ then $G(\{p,q,r,w,z\})$ is a contractible quintuple within distance 3 from $h$. In either case, we found a contradiction, so $wq \notin E(G)$.

If $wr \notin E(G)$ then $G(\{q,r,x,y,z\})$ would be a contractible quintuple within distance 3 from $h$. So $wr \in E(G)$. Furthermore, $z$ must be adjacent to $s$ or $u$, for otherwise $zq, zt \in E(G)$, so $x, y$ and $stu$ would induce a 5-cycle, and $G(\{p,q,r,w,z\})$ would be a contractible quintuple within distance 3 from $G$. We may assume $sz \in E(G)$, and, $uz \in E(G)$ follows the same way we deduced $wr \in E(G)$ from $wp \in E(G)$ before.

If $qx, ty \in E(G)$ then $G(\{p,q,r,w,x\})$ or $G(\{s,t,u,y,z\})$ would be a contractible quintuple within distance 3 from $h$, which is absurd. Hence $qy, tx \in E(G)$, implying that $pqrw$ and $stuz$ are contractible 4-cycles, one of which is within distance 3 from $h$. By symmetry, we may assume that $pqrw$ is within distance 3 from $h$. By Claim 2, the two neighbors of $V(P)$ in $G - V(H)$ are adjacent. It follows that $G' := G - (V(H) \cup V(P) \cup V(S))$ can't be separated by less than 2 vertices. As one of the paths $pwxts, rwxts$ has distinct neighbors in $V(G')$, one of $rqyzu, pqyzu$ is a contractible quintuple within distance 2 from $h$ in $G$.

Hence we may assume that $|V(P)| = 2$, say, $P = pq$. Note that $w$ must be adjacent to one of $p, q$ (for otherwise $G(\{p,q,x,y,z\})$ would be a contractible quintuple within distance 3 from $h$), and so must $z$. We may assume $wp, zq \in E(Q)$ without loss of generality. In particular, $pq$ is not contained in a triangle. Indeed, for proving Claim 4 it it suffices to prove that $|V(S)| = 2$, as the same arguments holds for $s, t$ if $S = st$.

If $|V(S)| = 4$, say, $S = stuv$, then again $E_G(H)$ would consist of 6 edges of in $E_G(V(P) \cup V(S))$, and $h \in N_G(V(H)) \cup V(H)$. First, $w$ can't be adjacent to $t$ or $u$ (for otherwise $G(\{u,v,x,y,z\})$ or $G(\{s,t,x,y,z\})$ would be a contractible quintuple within distance 2 from $h$), so $w$ is adjacent to one of $s, v$, and so is $z$. Without loss of generality, $ws, zv \in E(G)$. If $xu, yt \in E(G)$ then $G(\{p,q,w,x,u\})$ is a contractible quintuple within distance 3 from $h$ in $G$, which is absurd. Hence $xt, yu \in E(G)$, implying that $stxw$, $uvzy$ are contractible 4-cycles within distance 3 from $h$ in $G$. By Claim 2, $p$ and the vertex in $N_G(s) - \{t, w\}$ must be adjacent. So the neighbor $a$ of $p$ in $X := V(G) - (V(H) \cup V(P) \cup V(S))$ equals the neighbor of $s$ in $X$, and, symmetrically, the neighbor $b$ of $q$ in $X$ equals the neighbor of $v$ in $X$. So $N_G(V(P) \cup V(S) \cup V(H)) = \{a, b\}$; since $G$ is 3-connected, $G$ must be the graph $R_6 \cong K_2 \times C_6$.

If $|V(S)| = 3$, say, $S = stu$, then $w$ can't be adjacent to $s$ or $u$ for otherwise $G(\{t,u,x,y,z\})$ or $G(\{s,t,x,y,z\})$ would be a contractible quintuple within distance 3 from $h$ in $G$. Without loss of generality, suppose $sx, uy \in E(G)$, so $tw \in E(G)$ or $tz \in E(G)$. But then $G(\{p,q,t,w,z\})$ is a contractible quintuple within distance 2 from $h$.

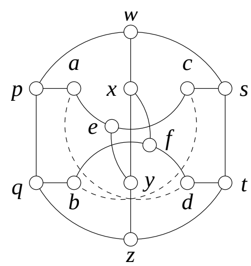


Figure 7: The return of the HEAWOOD graph.

This proves Claim 4.

In the proof of the following Claim, the HEAWOOD graph will show up.

**Claim 5.** *$G$ contains no contractible path path $P_4$ within distance 1 from $h$.*

Suppose $H := wxyz$ is a contractible path within distance 1 from $h$. By Claim 4, $G - V(H)$ has two removable links $P = pq$, $S = st$ where each of $p, q, s, t$ is adjacent to $w$ or $z$, and neither $pq$ nor $st$ are on a triangle. By symmetry, we may assume $wp, ws, zq, zt \in E(G)$. Let $a, b, c, d, e, f$ be the neighbors of $p, q, s, t, x, y$, respectively, in $V(G) - (V(H) \cup V(P) \cup V(S))$. Note that $a, b, e, f$ are pairwise distinct, and so are $c, d, e, f$. Since $pwxy$ is a contractible path within distance 2 from $h$, $af \in E(G)$ by Claim 4. Symmetrically, since $swxy$, $xyzq$, $xyzt$ are contractible paths within distance 2 from $h$, $cf, be, de \in E(G)$ by Claim 4, see Figure 7.

Since $H' = wpqz$ is a contractible path within distance 2 from $H$, we may play the same game with $H'$, $P' := xy$, and $Q' := Q$. (The roles of $pq$ and $xy$ are swapped.) Now $qpws$, $tzqp$ are contractible subgraphs within distance 2 (!) from $H$, implying that $bc, ad \in E(G)$. But this determines the neighborhoods of $a, b, c, d, e, f$, as each of them is within distance 2 from $V(H)$ and, thus, within distance 6 from $h$. As $a, b, c, d$ have degree 3, it follows easily that they are pairwise distinct. So $G$ must be the HEAWOOD graph, and we can extend $H$ by $e$ or $f$ to obtain a contractible quintuple within distance 1 from $h$, contradicting our assumption.

This proves Claim 5.

**Claim 6.** *$G$ has a contractible quadruple within distance 1 from $h$.*

Since $h$ has degree 3, it is contained in some contractible edge $hh'$ (by Theorem 2, as we have observed before). Since $d_G(h') = 3$, there is a contractible triple $H$ within distance 1 from $h$ by Lemma 1. Each vertex in $H$ is within distance 3 from $x$ and, thus, has degree 3. If $H$ would not be extendible, then there existed a contractible quintuple within distance 1 from $h$ by Lemma 2, contradicting our assumption. Hence $H$ is extendible, proving Claim 6.

By Claims 6,1,3,5, there exists a contractible 4-cycle $H = wxyz$ within distance 1 from $H$, where we take $w$ within distance 1 from $h$.

By Claim 2, there exists a pair of nonadjacent removable links $P = pq$ and $S = st$ of order 2 in $G - V(H)$. Without loss of generality, either $pw, sx, ty, qz \in E(G)$, or $pw, sx, qy, tz \in E(G)$. In either case, the path $pwx$ is contractible. By Lemma 2, it must be extendible. Since $w$ is the only common neighbor if $p, x$ in $G$, $pwx$ can't be extended to a 4-cycle, which violates Claim 1,3, and 5. Q.E.D.

As an immediate corollary we obtain that McCuaig's and Ota's Conjecture is true for $\ell = 5$ when restricted to cubic graphs. Furthermore, every cubic graph on 8 vertices distinct from the cube must contain a triangle $\Delta$, and as $\Delta$ is 2-connected and contractible, $G - V(\Delta)$ is connected and, thus, contractible. As $K_4, K_{3,3}$, and $K_2 \times K_3$ are the only cubic graphs on less than eight vertices, we obtain

**Corollary 1** *A cubic* 3*-connected graph has a contractible quintuple if and only if it is nonisomorphic to* $K_4$, $K_{3,3}$, $K_2 \times K_3$, $R_4$, $M_5$, $R_6$, *or* $OCC$.

# 4 Relaxing Regularity

The statement of Corollary 1 can be extended to 3-connected graphs of small average degree by employing the following lemma.

**Lemma 3** *For* $\delta, b \in N$ *there exists a constant* $C > 0$ *such that every graph of minimum degree* $\delta$ *in which all vertices are within distance at most* $b$ *from the set of those vertices of degree exceeding* $\delta$ *has an average degree of at least* $\delta + C$.

**Proof.** (According to an idea of R. Diestel.)

Let $G$ be a graph with $\delta(G) = \delta$, and suppose that every vertex is within distance $b$ from $W := \{x \in V(G) : d_G(x) > \delta\}$. Let $F := \{(x, e) : e \in E_G(x)\}$ denote the *flags* of $G$, and note that, for $x \in W$, the set $B(x, e)$ of vertices within distance $b$ from $x$ in $(G - (W - \{x\})) + e$ contains at most $m := 1 + 1 + (\delta - 1) + (\delta - 1)^2 + \ldots + (\delta - 1)^{b-1} = 1 + ((\delta - 1)^b - 1)/((\delta - 1) - 1)$ vertices. For each $x \in W$, fix a set $F(x)$ of $\delta + 1$ many flags $(x, e)$.

Put a charge of 0 to every vertex and a charge of 1 to every flag, and discharge each flag $f = (x, e)$ as follows: If $d_G(x) = \delta$ then transfer a charge of 1 from $f$ to $x$; otherwise, if $f \in F(x)$ then transfer a charge of $1 - 1/(\delta + 1)$ from $f$ to $x$ and a charge of $C := (1/(\delta + 1))/m$ from $f$ to every vertex in $B(f)$, and if $f \notin F(x)$ then transfer a charge of $1/m$ from $f$ to every vertex in $B(f)$. After discharging, every flag has charge 0 and every vertex has charge at least $\delta + C$. The total charge remains $2|E(G)|$, which proves the statement. Q.E.D.

For $\delta = 3$ and $b = 6$ we obtain, for example, $m = 33$ and $C = 1/132$ in the proof. It follows that in every graph of minimum degree 3 and average degree less than $3 + 1/132$, we find a vertex $h$ such that every vertex within distance 6 from $h$ has degree 3. Combining this with Theorem 3, we thus obtain the following.

**Corollary 2** *Every* 3*-connected graph on at least* 13 *vertices and of average degree less than* $3 + 1/132$ *has a contractible quintuple.*

$*$

In [8] it has been proved that every graph on at least $2\ell + 2$ vertices contains a contractible subgraph whose order is at least $\ell$ and at most $2\ell - 1$; this can be improved easily for regular graphs, using Lemma 3:

**Theorem 4** *For* $\delta, \ell \in N$ *there exists a constant* $C > 0$ *such that every* 3*-connected graph of minimum degree* $\delta$ *and average degree less than* $\delta + C$ *on at least* $2\ell + 1$ *vertices has a contractible subgraph on at least* $\ell$ *and at most* $3(\ell - 1)/2$ *vertices.*

**Proof.** We take $C$ as in Lemma 3 with $\delta \geq 3$ and $b := \ell$. Let $G$ be a 3-connected graph of minimum degree $\delta$ and average degree less than $\delta + C$. It contains a vertex $h$ such that all vertices within distance $\ell$ from $h$ have degree $\delta$.

We prove the stronger statement that for each $m \leq \ell$, there exists a contractible subgraph $H$ containing $h$ with $m \leq |V(H)| \leq \max\{m, 3(m-1)/2\}$ $(*_m)$. $(*_1)$ is immediate, take $H := G(\{h\})$. Now suppose that, for $2 \leq m \leq \ell$, $H$ is a contractible subgraph of $G$ containing $h$ such that $m - 1 \leq |V(H)| \leq \max\{m - 1, 3(m-2)/2\}$. If $|V(H)| \geq m$ then this proves $(*_m)$, hence we may assume that $|V(H)| = m - 1$.

As $H$ is connected, $|E(G)| \geq m - 2$, and as all vertices in $H$ have degree $\delta$, $|E_G(V(H))| \leq \delta \cdot (m-1) - 2 \cdot (m-2) = (\delta - 2)(m-1) + 2$. Let $X$ be the set of vertices of degree 2 in $G - V(H)$. As all vertices in $N_G(V(H))$ have degree $\delta$, too, $|E_G(X, V(H))| = |X| \cdot (\delta - 2)$. Since $|E_G(V(H))| \geq |E_G(X, V(H))|$, $|X| \leq (m-1) + 2/(\delta - 2) \leq m + 1$ follows.

Since $|V(G)| \geq 2\ell + 1 \geq 2m + 1$, $G - V(H)$ can't induce a cycle. If $H$ is extendible then $(*_m)$ is proved, and thus, by Theorem 2, $G - V(H)$ contains two disjoint nonadjacent removable links whose orders sum up to at most $m+1$. Hence $G - V(H)$ has a removable link $P$ on at most $(m+1)/2$ vertices, and $G(V(H) \cup V(P))$ is a contractible subgraph on at least $m + 1$ and at most $(3m-1)/2$ vertices, proving $(*_m)$. Q.E.D.

**Corollary 3** *Every regular* 3*-connected graph on at least* $2\ell+1$ *vertices contains a contractible subgraph on at least* $\ell$ *and at most* $3(\ell-1)/2$ *vertices.*

**Proof.** If $G$ is regular of degree $\delta$ then take $C$ as in Theorem 4 and observe that $G$ satisfies the conditions of the statement in Theorem 4. Q.E.D.

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