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Algebra with infinite sums

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Bases and closed spaces with infinite sums

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Abstract

Motivated by work of Diestel and Kühn we study the ramifications of allowing well-defined infinite sums in a module R^M . We show that every generating set contains a basis—taking infinite sums into account—if the ground set M is countable, but not necessarily otherwise. We also investigate when a space is closed under taking infinite sums.

1 Introduction

In this paper we will study the ramifications of allowing well-defined infinite sums in certain modules. Such sums occur in two quite different fields: In algebra, where they are used in the context of slender modules (see Göbel and Trlifaj [7, Chapter 1]), and in graph theory, in the study of the cycle space of an infinite graph; see the end of this section for a brief discussion.

Let R be a ring, and let M be an arbitrary set. Conventionally, only sums of finitely many summands are defined in the module $R^M = \prod_{m \in M} R$. However, for an infinite family $\mathcal{N} \subseteq R^M$ for which in each coordinate almost all entries are zero, there is an obvious way to define a sum, namely by componentwise addition. More formally, we call \mathcal{N} *thin* if for all $m \in M$ the number of members N of \mathcal{N} with $N(m) \neq 0$ is finite; note that in the literature the term *summable* is sometimes used for these families. For a thin family \mathcal{N} we define the sum $\sum_{N \in \mathcal{N}} N = \sum \mathcal{N}$ to be the element S of R^M with $S(m) = \sum_{N \in \mathcal{N}, N(m) \neq 0} N(m)$ for all $m \in M$.

By allowing infinite sums we leave conventional algebra. We will examine two problems, and obtain results that in some cases agree with abstract algebra, while in others there are substantial differences. Our first problem, discussed in Section 2, concerns the existence of *bases*. More precisely, the question is: given $\mathcal{N} \subseteq R^M$ is there a subfamily \mathcal{B} of \mathcal{N} such that \mathcal{N} and \mathcal{B} generate the same space (taking infinite sums into account) and such that each element of this space has a unique representation in \mathcal{B} ? In contrast to linear algebra, we will see that even over a field there are generating sets that do not contain a basis (Theorem 2), although we can always find a basis if the underlying set M is countable (Theorem 1).

The second problem we consider is, whether the space $\langle \mathcal{N} \rangle$ of all sums of thin subfamilies of \mathcal{N} is closed under taking thin sums, i.e. if $\langle \mathcal{N} \rangle = \langle \langle \mathcal{N} \rangle \rangle$. While this is false in general, we will show in Section 3 that it is true if \mathcal{N} is thin—provided R is a field or a finite ring (Theorem 7). We will see that, in a sense, this is best possible. Whether a space \mathcal{N} is closed under taking thin sums is related to the question whether \mathcal{N} is closed or not in the product space R^M . We investigate this connection in the last section.

These two problems arise when studying the cycle space $\mathcal{C}(G)$ of a graph G . When G is finite, $\mathcal{C}(G)$ is simply the first homology group (over \mathbb{Z}_2) of G viewed as a simplicial complex. For finite G , $\mathcal{C}(G)$ is well-studied, and a number of theorems about $\mathcal{C}(G)$ is known. Most of these classical results become false in infinite graphs. Following a topological approach, Diestel and Kühn [5] proposed (for locally finite graphs) a definition of the cycle space to which all these results do carry over nevertheless. In this setting, where infinite sums are not only natural but also necessary, closedness becomes an issue: it is an important fact that the cycle space as well as the space of all cuts is closed under taking infinite sums. Both of these facts follow immediately from our main result of Section 3 (however, the former was first proved by Diestel and Kühn [6] and the latter has a simple direct proof). For an introduction and a survey about the cycle space in an infinite graph see Diestel [4], or [3].

A line of research very much in the direction of the present work, albeit more oriented towards graph theory, has been pursued by Casteels and Richter [2]. In particular, Casteels and Richter introduce a notion of orthogonality in vector spaces F^M , where F is a field, and investigate when $\mathcal{U} = (\mathcal{U}^\perp)^\perp$ holds for a subspace \mathcal{U} . It turns out that this is the case if \mathcal{U} is generated by a thin family. In addition, they exhibit some bases (in our sense) for spaces of graph-theoretic relevance. Bases were used as a tool in [1] in order to characterise those locally finite graphs that can be drawn in the plane without any two edges crossing. There, it was essential to find a basis in certain generating sets, and under the special constraints of [1] this could be achieved. In the next section, we treat the existence problem of bases in the general case.

2 Bases

Before we start we need some definitions. Let M be a set, R a ring, and let \mathcal{N} be a family with its members in R^M . Often we will multiply families with coefficients before adding them: if $a = (a(N))_{N \in \mathcal{N}}$ is a family of coefficients in R , one for each $N \in \mathcal{N}$, we use the shorthand $a\mathcal{N}$ for the family $(a(N)N)_{N \in \mathcal{N}}$. For a $K \in R^M$, we call a family a of coefficients a *representation of K in \mathcal{N}* if $a\mathcal{N}$ is thin and if $K = \sum a\mathcal{N}$ —that is, $K(m) = \sum_{N \in \mathcal{N}} a(N)N(m)$ for every $m \in M$. Denote by $\langle \mathcal{N} \rangle$ the set of elements of R^M that have a representation in \mathcal{N} . Intuitively, \mathcal{N} is a generating set, and $\langle \mathcal{N} \rangle$ is the space it generates.

For a family $\mathcal{N} \subseteq R^M$, we call a subfamily \mathcal{B} of \mathcal{N} a *basis of $\langle \mathcal{N} \rangle$* , if $\langle \mathcal{B} \rangle = \langle \mathcal{N} \rangle$ and $\mathbf{0} \in R^M$ has a unique representation in \mathcal{B} . Note that $\mathbf{0}$ has a unique representation in some family \mathcal{N} if and only if every element in $\langle \mathcal{N} \rangle$ has a unique representation in \mathcal{N} .

It is well known that a generating set in a module does not need to contain a basis (in the classical sense), and, clearly, this is also the case in our setting: Take for example $R = \mathbb{Z}$, $M = \{0\}$, and $\mathcal{N} = \{a_2, a_3\}$, where a_2, a_3 are defined by $a_2(0) = 2$ and $a_3(0) = 3$. Then \mathcal{N} does not contain a basis of $\langle \mathcal{N} \rangle$. When R is a field, however, we can say more about the existence of a basis:

Theorem 1. *Let M be a countable set, F be a field and let \mathcal{N} be a family with its members in F^M . Then \mathcal{N} contains a basis of $\langle \mathcal{N} \rangle$.*

In linear algebra the analogous assertion is usually proved with Zorn's lemma as follows. Given a chain $(\mathcal{B}_\lambda)_\lambda$ of linearly independent subsets of the generating

set, it is observed that $\bigcup_\lambda \mathcal{B}_\lambda$ is still linearly independent since any violation of linear independence is witnessed by finitely many elements, and these would already lie in one of the \mathcal{B}_λ . Thus, each chain has an upper bound, which implies, by Zorn's lemma, that there is a maximal linearly independent set, a basis. This approach, however, fails in our context, as dependence does not need to be witnessed by only finitely many elements, thus we cannot get the contradiction that already one of the \mathcal{B}_λ was not independent.

As an illustration, put $F = \mathbb{Z}_2$, $M := \mathbb{Z}$ and $\mathcal{B}_i := \{\{j, j+1\} : -i \leq j < i\}$ for $i = 1, 2, \dots$ (Here, and later we shall freely identify elements of \mathbb{Z}_2^M with subsets of M .) Now, while no nonempty finite subset of $\mathcal{B}_\infty := \bigcup_{i=1}^\infty \mathcal{B}_i$ is dependent, the whole set is: $\sum_{B \in \mathcal{B}_\infty} B = \emptyset$.

The standard proof outlined above pursues a bottom-up approach. There is also a, perhaps more pedestrian, top-down proof, where successively those elements of \mathcal{N} are weeded out that can be replaced by others. Slightly more precisely, a (possibly transfinite) enumeration N_1, N_2, \dots of \mathcal{N} is processed step-by-step, and in each step it is checked whether N_λ can be expressed as a sum in earlier N_μ (i.e. $\mu < \lambda$) that are still left. If yes, N_λ is not needed to generate $\langle \mathcal{N} \rangle$, and therefore deleted. If no, N_λ is kept. It is not hard to check that this process yields a basis.

Our proof of Theorem 1 relies on an adaption of this argument. Clearly, we cannot expect it to work as it is, since it fails to take infinite sums into account. However, if we restrict ourselves to those elements of \mathcal{N} that share a given $m \in M$ then all sums are finite, and we can employ the argument. So, we will partition \mathcal{N} into sets $\mathcal{N}_1, \mathcal{N}_2, \dots$ so that all the elements in each of the \mathcal{N}_j share an $m \in M$. Then we will use an (adapted) top-down argument on each of the \mathcal{N}_i .

Proof of Theorem 1. Let m_1, m_2, \dots be a (possibly finite) enumeration of M , and for $i = 1, 2, \dots$ define \mathcal{N}_i to be the set of those elements $N \in \mathcal{N} \setminus \bigcup_{j < i} \mathcal{N}_j$ for which $N(m_i) \neq 0$. Clearly, $\{\mathcal{N}_i : i \in \mathbb{N}\}$ is a partition of \mathcal{N} . For every $i \in \mathbb{N}$, let $N_{i1}, N_{i2}, \dots, N_{i\lambda}, \dots$ be a (possibly transfinite) enumeration of \mathcal{N}_i .

Now, for each $i = 1, 2, \dots$ we perform a transfinite induction as follows. Start by setting $\mathcal{B}_{i0} = \emptyset$, and then for every ordinal $\lambda > 0$ define the set $\mathcal{B}_{i\lambda} \subseteq \mathcal{N}_i$ as follows ($\mathcal{B}_{i\lambda}$ is the set of those elements among the first λ $N \in \mathcal{N}_i$ that we will put in our basis): If $L := N_{i\lambda}$ has a representation a_L in \mathcal{N} such that

$$a_L(N) = 0 \text{ for } N \notin \bigcup_{\mu < \lambda} \mathcal{B}_{i\mu} \cup \bigcup_{k > i} \mathcal{N}_k, \quad (1)$$

then let $\mathcal{B}_{i\lambda} = \bigcup_{\mu < \lambda} \mathcal{B}_{i\mu}$. Otherwise, set $\mathcal{B}_{i\lambda} = \bigcup_{\mu < \lambda} \mathcal{B}_{i\mu} \cup \{N_{i\lambda}\}$. Having defined all $\mathcal{B}_{i\lambda}$, we put $\mathcal{B}_i := \bigcup_\lambda \mathcal{B}_{i\lambda}$.

For later use we note that

$$\text{if } L \in \mathcal{N}_i \setminus \mathcal{B}_i, N \in \mathcal{N}_j \setminus \mathcal{B}_i \text{ and } j \leq i \text{ then } a_L(N) = 0. \quad (2)$$

Indeed, if $a_L(N) \neq 0$ then by (1) we obtain $N \in \bigcup_{\mu < \lambda} \mathcal{B}_{i\mu} \cup \bigcup_{k > i} \mathcal{N}_k$ for some λ , and as $j \leq i$ we have $N \in \bigcup_{\mu < \lambda} \mathcal{B}_{i\mu}$. But this means that $N \in \mathcal{B}_i = \bigcup_\mu \mathcal{B}_{i\mu}$, a contradiction.

We claim that $\mathcal{B} := \bigcup_i \mathcal{B}_i$ is a basis of $\langle \mathcal{N} \rangle$. To show that $\mathbf{0} \in R^M$ has a unique representation, suppose there are coefficients $b : \mathcal{B} \rightarrow F$, not all of which are zero, such that $b\mathcal{B}$ is thin and $\sum b\mathcal{B} = \mathbf{0}$. Let $i \in \mathbb{N}$ be minimal so

that there is an ordinal μ with $b(N_{i\mu}) \neq 0$, and observe that since for all the elements B in \mathcal{B}_i we have $B(m_i) \neq 0$, there is a maximal ordinal λ such that $b(N_{i\lambda}) \neq 0$ (because $b\mathcal{B}$ is thin). Then $N_{i\lambda} = \sum_{N \in \mathcal{B} \setminus \{N_{i\lambda}\}} b^{-1}(N_{i\lambda})b(N)N$ is (or, more precisely, can be extended to) a representation of $N_{i\lambda}$ that satisfies (1), a contradiction to that $N_{i\lambda} \in \mathcal{B}_i$.

Next, consider a $K \in \langle \mathcal{N} \rangle$. We will show that K has a representation in \mathcal{B} . Starting with any representation b^0 of K in \mathcal{N} , we inductively define for $i = 1, 2, \dots$ representations $b^i : \mathcal{N} \rightarrow F$ as follows. (Intuitively, b^i is a representation of K using only elements of \mathcal{N} that are left after step i of the construction of \mathcal{B} , that is, after we have finished deleting elements of \mathcal{N}_i .) Set $\mathcal{E}_i := \{N \in \mathcal{N}_i \setminus \mathcal{B}_i : b^{i-1}(N) \neq 0\}$. Since $b^{i-1}\mathcal{N}$ is thin and since $N(m_i) \neq 0$ for all $N \in \mathcal{E}_i \subseteq \mathcal{N}_i$, it follows that \mathcal{E}_i is a finite set. Put

$$\begin{aligned} b^i(N) &= 0 \text{ for } N \in \mathcal{E}_i, \text{ and} \\ b^i(N) &= b^{i-1}(N) + \sum_{L \in \mathcal{E}_i} b^{i-1}(L)a_L(N) \text{ for } N \notin \mathcal{E}_i. \end{aligned} \quad (3)$$

(Note that a_L is defined for every $L \in \mathcal{E}_i$, since $\mathcal{E}_i \subseteq \mathcal{N}_i \setminus \mathcal{B}_i$.)

We claim that this definition yields a representation of K that uses only those elements of $\mathcal{N}_1, \dots, \mathcal{N}_i$ that lie in \mathcal{B} , in other words, we claim that for every $i \in \mathbb{N}$ it is true that

$$b^i(N) = 0 \text{ if } N \in \bigcup_{j=1}^i \mathcal{N}_j \setminus \mathcal{B} \quad (4)$$

and

$$K = \sum b^i \mathcal{N} \text{ (in particular, } b^i \mathcal{N} \text{ is thin)}. \quad (5)$$

To prove the two claims we proceed by induction. For (4), consider an $N \in \mathcal{N}_j \setminus \mathcal{B}$ where $j \leq i$. If $N \in \mathcal{E}_i$ then $b^i(N) = 0$ by definition, so consider the case $N \notin \mathcal{E}_i$. If $j = i$ this implies that $b^{i-1}(N) = 0$; if $j < i$ then we get $b^{i-1}(N) = 0$ too, this time using induction. Since, by (2), $a_L(N) = 0$ for every $L \in \mathcal{E}_i$, (3) implies $b^i(N) = 0$, as desired.

For (5), first note that $b^i \mathcal{N}$ is indeed thin as \mathcal{E}_i is finite and both of $a_L \mathcal{N}$ and $b^{i-1} \mathcal{N}$ thin (the latter by induction). Furthermore:

$$\begin{aligned} \sum b^i \mathcal{N} - \sum b^{i-1} \mathcal{N} &= \\ &= \sum_{N \in \mathcal{N} \setminus \mathcal{E}_i} (b^i(N) - b^{i-1}(N))N + \sum_{N \in \mathcal{E}_i} (b^i(N) - b^{i-1}(N))N \\ &= \sum_{N \in \mathcal{N} \setminus \mathcal{E}_i} \left(\sum_{L \in \mathcal{E}_i} b^{i-1}(L)a_L(N)N \right) + \sum_{N \in \mathcal{E}_i} (\mathbf{0} - b^{i-1}(N))N \\ &= \sum_{L \in \mathcal{E}_i} b^{i-1}(L) \sum_{N \in \mathcal{N} \setminus \mathcal{E}_i} a_L(N)N - \sum_{N \in \mathcal{E}_i} (b^{i-1}(N))N. \end{aligned}$$

As $a_L(N) = 0$ if $N, L \in \mathcal{E}_i$ by (2), we obtain for the first sum in the last line

$$\sum_{L \in \mathcal{E}_i} b^{i-1}(L) \sum_{N \in \mathcal{N} \setminus \mathcal{E}_i} a_L(N)N = \sum_{L \in \mathcal{E}_i} b^{i-1}(L) \sum_{N \in \mathcal{N}} a_L(N)N = \sum_{L \in \mathcal{E}_i} b^{i-1}(L)L.$$

Together with the previous equation this yields $\sum b^i \mathcal{N} - \sum b^{i-1} \mathcal{N} = \mathbf{0}$, which proves (5).

For every $N \in \mathcal{N}$, define $b^\infty(N) := b^j(N)$ if $N \in \mathcal{N}_j$, and note that

$$b^i(N) = b^\infty(N) \text{ for } N \in \mathcal{N}_j \text{ and } i \geq j. \quad (6)$$

Indeed, consider $i > j$ and observe that, by (2), $a_L(N) = 0$ for all $L \in \mathcal{E}_i \subseteq \mathcal{N}_i \setminus \mathcal{B}_i$. So, from (3) it follows that $b^i(N) = b^{i-1}(N)$.

We immediately get from (4) that

$$\text{if } N \in \mathcal{N} \setminus \mathcal{B} \text{ then } b^\infty(N) = 0. \quad (7)$$

Thus, the $b^\infty(N)$ can be seen as coefficients on \mathcal{B} , and our next claim states that b^∞ is what we are looking for, namely a representation of K in \mathcal{B} :

$$K = \sum_{B \in \mathcal{B}} b^\infty(B)B \text{ (in particular, } b^\infty \mathcal{N} \text{ is thin)}. \quad (8)$$

Consider an $m_i \in M$. By definition of the \mathcal{N}_j , every N with $N(m_i) \neq 0$ lies in $\bigcup_{j=1}^i \mathcal{N}_j$. By (6), b^i and b^∞ are identical on $\bigcup_{j=1}^i \mathcal{N}_j$. Since $b^i \mathcal{N}$ is thin, there are therefore only finitely many $N \in \mathcal{N}$ so that $b^\infty(N)N(m_i) \neq \infty$. Thus $b^\infty \mathcal{N}$ is thin. Furthermore, we obtain with (5) and (6):

$$\sum_{N \in \mathcal{N}} b^\infty(N)N(m_i) = \sum_{N \in \bigcup_{j=1}^i \mathcal{N}_j} b^\infty(N)N(m_i) = \sum_{N \in \mathcal{N}} b^i(N)N(m_i) = K(m_i).$$

Claim (8) now follows from (7). This completes the proof. \square

Observe that contrary to conventional linear algebra, two bases do not need to have the same cardinality—even over a field. Indeed, putting $F = \mathbb{Z}_2$ and $M = \{m_0, m_1, \dots\}$ we see that $\mathcal{B} := \{\{m_i\} : i \geq 0\}$ is a countable basis of F^M . On the other hand, $\mathcal{N} := \{\{m_0\} \cup N : N \subseteq M\}$ clearly generates F^M too, and contains, by Theorem 1, a basis \mathcal{B}' . Since all thin subsets of \mathcal{N} are finite, \mathcal{B}' needs to be uncountable to generate the uncountable set F^M . Thus \mathcal{B} and \mathcal{B}' are two bases of F^M that do not have the same cardinality.

We have formulated Theorem 1 only for countable sets M . The following result shows that this is indeed best possible.

Theorem 2. *For an uncountable set M there exists a family \mathcal{N} of elements of \mathbb{Z}_2^M , so that \mathcal{N} does not contain a basis of $\langle \mathcal{N} \rangle$.*

Before proving Theorem 2, let us recall some basic graph theoretical terminology; for a much more comprehensive treatment we refer to Diestel [4]. A *graph* G consists of a set of vertices $V(G)$ and a set of edges $E(G) \subseteq [V(G)]^2$; thus, the elements of $E(G)$ are 2-element subsets of $V(G)$. A *path* (resp. *ray*) is a graph on distinct vertices x_1, \dots, x_k (resp. x_1, x_2, \dots) where x_i is linked to x_{i+1} by an edge. A closed path (i.e. if, in addition, there is an edge between x_k and x_1) is called a *cycle*. A *component* of G is a maximal subgraph of G in which any two vertices are connected by a path.

Proof of Theorem 2. Let A, B be two disjoint sets with cardinalities $|A| = \aleph_0$ and $|B| = \aleph_1$. Define G to be the graph with vertex set $M := A \cup B$ and edge

set $\mathcal{N} := A \times B$. As $\mathcal{N} \subseteq \mathcal{P}(M)$, we may ask whether \mathcal{N} contains a basis of $\langle \mathcal{N} \rangle$. We claim that it does not.

Let us show that each countable subset N of M is an element of $\langle \mathcal{N} \rangle$. Indeed, let n_1, n_2, \dots be a (possibly finite) enumeration of N , and choose for $i = 1, 2, \dots$ a ray R_i that starts at n_i , and does not meet the first $i - 1$ vertices of each of R_1, \dots, R_{i-1} except, possibly, at n_i . Then, the set $\bigcup_{i \in \mathbb{N}} E(R_i)$ of edges of these rays is thin, and its sum equals N since $\sum_{e \in E(R_i)} e = \{n_i\}$.

Suppose that $\langle \mathcal{N} \rangle$ has a basis $\mathcal{B} \subseteq \mathcal{N}$, and let H be the graph with vertex set M and edge set \mathcal{B} . Since \mathcal{B} must contain for each element in B at least one edge incident with it, \mathcal{B} is uncountable. Therefore, one of the vertices in the countable set A , say v , is incident with infinitely many edges in \mathcal{B} . Delete from H the vertex v (and its incident edges) and denote by \mathcal{C} the set of components of the resulting graph that (in H) are adjacent to v .

Observe that for each $C \in \mathcal{C}$ there is exactly one edge in H between v and some vertex, u_C say, in C ; indeed, if there were two edges between v and vertices u, u' in C , then the union of these edges with an u - u' path in C would be a cycle in H , contradicting that \emptyset has a unique representation in \mathcal{B} since the sum of the edges of a cycle equals \emptyset .

Next, suppose there are distinct $C, D \in \mathcal{C}$ each containing a ray; then, C (respectively D) also contains a ray R (resp. S) starting at u_C (resp. at u_D). Then $R \cup S$ together with the two edges between v and $\{u_C, u_D\}$ yields a set of edges which sums to \emptyset , again a contradiction.

Pick a countably infinite number of $C \in \mathcal{C}$ none of which contains a ray, and denote the set of these by \mathcal{C}' . As $N := \{u_C : C \in \mathcal{C}'\}$ is countable it lies in $\langle \mathcal{N} \rangle$, thus there is a $\mathcal{B}_N \subseteq \mathcal{B}$ such that $\sum_{e \in \mathcal{B}_N} e = N$.

Suppose there is a $C \in \mathcal{C}'$ such that an edge $e \in \mathcal{B}_N$ incident with u_C lies in C . As C does not contain any cycle or any ray, we can run from e along edges in $E(C) \cap \mathcal{B}_N$ to a vertex $w \neq u_C$ that is only incident with one edge in \mathcal{B}_N . This implies that $w \in \sum_{e \in \mathcal{B}_N} e = N$, a contradiction since $w \notin N$. However, $u_C \in N$ must be incident with an edge from \mathcal{B}_N . Consequently, for each $C \in \mathcal{C}'$ the edge between v and u_C lies in \mathcal{B}_N , contradicting that \mathcal{B}_N is thin. \square

3 Closedness under taking thin sums

In this section we investigate the following question:

Question 3. *Let M be a set, R be a ring and $\mathcal{N} \subseteq R^M$. When is $\langle \mathcal{N} \rangle$ closed under taking thin sums, i.e. when is $\langle \mathcal{N} \rangle = \langle \langle \mathcal{N} \rangle \rangle$?*

In conventional algebra, the answer is easy: always. Once we allow infinite sums, however, the answer is not that straightforward. Consider, for instance, the case when $M = \mathbb{N}$, $R = \mathbb{Z}_2$ and $\mathcal{N} := \{\{1, i\} : i \in \mathbb{N}\}$. Clearly, we have $\{i\} \in \langle \mathcal{N} \rangle$ for all $i \in \mathbb{N}$ and thus $\mathbb{N} = \sum_{i \in \mathbb{N}} \{i\} \in \langle \langle \mathcal{N} \rangle \rangle$. On the other hand, $\mathbb{N} \notin \langle \mathcal{N} \rangle$ as all thin sums of elements in \mathbb{N} are necessarily finite. Thus, $\langle \mathcal{N} \rangle$ is indeed a proper subset of $\langle \langle \mathcal{N} \rangle \rangle$, and therefore not closed under taking thin sums.

The example seems to indicate that \mathcal{N} needs to be thin. Indeed, if we require \mathcal{N} to be thin then we will see that $\langle \mathcal{N} \rangle$ is closed under taking thin sums—provided that R is a field or a finite ring (Theorem 7). At the end of the section we will give an example showing that this is a fairly complete answer

to Question 3: If R is neither a field nor finite, then we cannot guarantee that $\langle \mathcal{N} \rangle = \overline{\langle \mathcal{N} \rangle}$.

We remark that there is another way to overcome the counterexample above. Vella [8] shows that a family \mathcal{N} of elements in $\mathbb{Z}_2^{\mathbb{N}}$ is closed under taking thin sums if, instead of being thin, \mathcal{N} has the property that every sum of finitely many members of \mathcal{N} is the disjoint union of members of \mathcal{N} .

It turns out that Question 3 is closely related to the topological closure in the product space $R^M = \prod_{m \in M} R_m$ where each R_m is a copy of R endowed with the discrete topology. In what follows we denote by $\overline{\mathcal{N}}$ the topological closure of a subset \mathcal{N} of R^M in the space $\prod_{m \in M} R_m$. Given a $K \in R^M$, note that $K \in \overline{\mathcal{N}}$ if and only if for every finite subset M' of M there is a $N \in \mathcal{N}$ with $K(m) = N(m)$ for all $m \in M'$. In the next two lemmas we will see that for a thin family \mathcal{T} of elements of R^M , $\langle \mathcal{T} \rangle$ is topologically closed if R is a finite ring or a field. Moreover, we will conclude in Lemma 6 that $\langle \langle \mathcal{T} \rangle \rangle$ lies in $\overline{\langle \mathcal{T} \rangle}$, and combining these results yields an answer to Question 3. We will pursue the relationship between topological closedness and closedness under taking thin sums further in the next section.

The first proof uses a typical compactness argument.

Lemma 4. *Let M be a set, R be a finite ring, and let \mathcal{T} be a thin family of elements of R^M . Then $\overline{\langle \mathcal{T} \rangle} = \langle \mathcal{T} \rangle$.*

Proof. Consider an element $K \in \overline{\langle \mathcal{T} \rangle}$. By Tychonoff's theorem, the product space $X := \prod_{T \in \mathcal{T}} R$ where R bears the discrete topology is compact. For any finite subset $M' \subseteq M$, we consider the set $A_{M'}$ of families of coefficients a , such that $a\mathcal{T}$ agrees with K on M' ; formally, define

$$A_{M'} := \{a \in X : \sum_{T \in \mathcal{T}} a(T)T(m) = K(m) \text{ for every } m \in M'\}.$$

(Note that we view the elements of X as coefficients for the family \mathcal{T} .) We claim that these sets are closed in X , and that their collection has the finite intersection property.

To show that each $A_{M'}$ is closed, let $\mathcal{S}_{M'}$ be the subfamily of all those $T \in \mathcal{T}$ for which there is a $m \in M'$ with $T(m) \neq 0$. As \mathcal{T} is thin, $\mathcal{S}_{M'}$ is finite. Since R is finite as well, there are only finitely many $b : \mathcal{S}_{M'} \rightarrow R$ such that $\sum_{S \in \mathcal{S}_{M'}} b(S)S(m) = K(m)$ for all $m \in M'$. For each such b , $B_b := \{a \in X : a(S) = b(S) \text{ for every } S \in \mathcal{S}_{M'}\}$ is closed in X , and since $A_{M'}$ is the union of these finitely many sets B_b it is closed too.

Next, if we have finite sets $M_1, \dots, M_l \subseteq M$ then, clearly, $\bigcap_{i=1}^l A_{M_i} = A_{M''}$ for $M'' := \bigcup_{i=1}^l M_i$. As $K \in \overline{\langle \mathcal{T} \rangle}$, there is an element L of $\langle \mathcal{T} \rangle$ that agrees with K on M'' . But any representation of L in \mathcal{T} is an element of $A_{M''}$, thus the A_{M_i} have the finite intersection property as claimed.

Now, the assertion of the theorem follows easily: X is compact, therefore, the intersection of all $A_{M'}$ is non-empty. For an element a of that intersection, we have $\sum_{T \in \mathcal{T}} a(T)T(m) = K(m)$ for all $m \in M$, i.e. it is a representation of K in \mathcal{T} . \square

We need a completely different approach to prove a similar result in the case when R is a (possibly infinite) field:

Lemma 5. *Let M be a set, F be a field, and let \mathcal{T} be a thin family of elements of F^M . Then $\overline{\langle \mathcal{T} \rangle} = \langle \mathcal{T} \rangle$.*

Proof. Consider a $K \in \overline{\langle \mathcal{T} \rangle}$. We will reduce the problem of finding a representation of K in \mathcal{T} to the solution of an infinite system of equations. To do this, we associate a variable x^T with every member T of \mathcal{T} , and for each $m \in M$ we define e_m to be the linear equation

$$\sum_{T \in \mathcal{T}: T(m) \neq 0} x^T T(m) = K(m)$$

in the variables x^T . Note that as \mathcal{T} is thin, each e_m contains only finitely many variables. Let $E = \{e_m : m \in M\}$. By construction, if there is an assignment $a : \mathcal{T} \rightarrow F$ such that setting $x^T = a(T)$ for every $T \in \mathcal{T}$ yields a solution to every equation in E , then a is a representation of K in \mathcal{T} . So in what follows, our task is to find such a solution.

For every $T \in \mathcal{T}$, define d_T to be the linear equation $x^T = 1$, put $D = \{d_T : T \in \mathcal{T}\}$, and denote by \mathcal{E} the set of all those sets E' with $E \subseteq E' \subseteq E \cup D$ such that every finite subset $E'' \subseteq E'$ has a solution. We claim that \mathcal{E} contains a \subseteq -maximal element E^* . First, note that $\mathcal{E} \neq \emptyset$ as $E \in \mathcal{E}$; indeed, for a finite subset $E'' \subseteq E$, the set M' of all $m \in M$ for which $e_m \in E''$ is finite. As $K \in \overline{\langle \mathcal{T} \rangle}$, there is an element L of $\langle \mathcal{T} \rangle$ that agrees with K on M' , and any representation a of L in \mathcal{T} yields a solution of E'' . Second, if $(E_i)_{i \in I}$ is a chain in \mathcal{E} then, clearly, the union $\bigcup_{i \in I} E_i$ lies in \mathcal{E} , too. Thus, Zorn's lemma ensures the existence of E^* .

Next, we show that for every $T \in \mathcal{T}$ there is a finite $E_T \subseteq E^*$ and an element f_T of the field F such that $x^T = f_T$ in every solution of E_T . Suppose not, and observe that then, clearly, $d_T \notin E^*$. Consider a finite subset E' of E^* , and note that, by assumption, there are two solutions in which x^T takes two distinct values, g_1 and g_2 , say. Then, for every $f \in F$ there is a solution of E' where $x^T = fg_1 + (1-f)g_2$. Setting $f = (1-g_2)(g_1-g_2)^{-1}$ yields a solution of E' with $x^T = 1$, which means that $E' \cup \{d_T\}$ has a solution. As E' was an arbitrary finite subset of E^* , it follows that every finite subset of $E^* \cup \{d_T\}$ has a solution, contradicting the maximality of E^* . Thus, E_T and f_T exist, as we have claimed.

Finally, define coefficients $a(T) := f_T$ for $T \in \mathcal{T}$. To see that a is a solution of E , consider an arbitrary $m \in M$. As \mathcal{T} is thin, the subfamily \mathcal{T}_m of those members T of \mathcal{T} with $T(m) \neq 0$ is finite. Thus, $E' := \{e_m\} \cup \bigcup_{T \in \mathcal{T}_m} E_T$ has, as a finite subset of E^* , a solution $b : \mathcal{T} \rightarrow F$. Since for every $T \in \mathcal{T}_m$, we have $E_T \subseteq E'$ it follows that $b(T) = f_T = a(T)$. As b solves e_m we see that a solves e_m , too. Thus a is a solution of E , and hence a representation of K in \mathcal{T} . \square

We need one more simple lemma:

Lemma 6. *Let M be a set, R be a ring, and let \mathcal{N} be a family of elements of R^M . Then $\langle \langle \mathcal{N} \rangle \rangle \subseteq \overline{\langle \mathcal{N} \rangle}$.*

Proof. Let $K \in \langle \langle \mathcal{N} \rangle \rangle$, and consider an arbitrary finite subset M' of M . Denote the canonical projection of R^M to $R^{M'}$ by π , and observe that $\pi(K) \in \langle \langle \pi(\mathcal{N}) \rangle \rangle$. Since rings are closed under addition, and since all thin families in $R^{M'}$ are finite, it holds that $\langle \langle \pi(\mathcal{N}) \rangle \rangle = \langle \pi(\mathcal{N}) \rangle$. Choosing a representation of $\pi(K)$ in $\langle \pi(\mathcal{N}) \rangle$,

and replacing each element of $\pi(\mathcal{N})$ in it by one of its preimages with respect to π yields an $N \in \langle \mathcal{N} \rangle$ so that $K(m) = N(m)$ for all $m \in M'$. This proves that $K \in \overline{\langle \mathcal{N} \rangle}$. \square

Lemma 6 combined with Lemmas 5 and 4 immediately implies the following:

Theorem 7. *Let M be a set, R be a ring, and let \mathcal{T} be a thin family of elements of R^M . Then $\langle \mathcal{T} \rangle = \langle\langle \mathcal{T} \rangle\rangle$ if R is a field or a finite ring.*

Having proved Theorem 7, let us now argue that, in a sense, it gives a comprehensive answer to Question 3. In fact, we shall construct an example where R is neither a field nor finite, and where there exists a thin family $\mathcal{T} \subseteq R^M$ such that $\langle \mathcal{T} \rangle$ is not closed under taking thin sums.

For this, set $R := \mathbb{Z}$ and $M := \mathbb{N}$. Define $N \in \mathbb{Z}^{\mathbb{N}}$ by $N(i) = 1$ for every $i \in \mathbb{N}$. For $j = 1, 2, \dots$, define $N_j \in \mathbb{Z}^{\mathbb{N}}$ by $N_j(j) = p_j$ and $N_j(i) = 0$ for every $i \neq j$, where p_j is the j th prime number. Let $\mathcal{T} = \{N, N_1, N_2, \dots\}$, and note that \mathcal{T} is thin. We will show that the function $K \in \mathbb{Z}^{\mathbb{N}}$ defined by $K(i) = i$ is in $\langle\langle \mathcal{T} \rangle\rangle$ but not in $\langle \mathcal{T} \rangle$.

Let us first prove that $K \notin \langle \mathcal{T} \rangle$. Suppose for contradiction, there is a representation $a : \mathcal{T} \rightarrow \mathbb{Z}$ of K in $\langle \mathcal{T} \rangle$. We distinguish two cases, depending on whether $n := a(N)$ is non-negative or not. If $n \geq 0$, then $n + 1 = K(n + 1) = \sum_{L \in \mathcal{T}} a(L)L(n + 1) = a(N)N(n + 1) + a(N_{n+1})N_{n+1}(n + 1) = n + a(N_{n+1})p_{n+1}$, which implies that $1 = a(N_{n+1})p_{n+1}$, a contradiction. If, on the other hand, $n < 0$ then we have for $n' = -n$ that $n' = K(n') = n + a(N_{n'})p_{n'}$, i.e. $2n' = a(N_{n'})p_{n'}$. With $p_{n'} > n'$ we obtain $p_{n'} = 2$ and thus $n = -1$. This again leads to a contradiction, as it implies that $K(3) = 3 = -1 + a(N_3) \cdot 5$.

Having shown $K \notin \langle \mathcal{T} \rangle$, we now prove $K \in \langle\langle \mathcal{T} \rangle\rangle$. For this, we construct for every $i \in \mathbb{N}$ an $S_i \in \langle \mathcal{T} \rangle$ so that $S_i(i) = 1$ and $S_i(j) = 0$ for every $j < i$. Once we have done that we can represent K with these S_i : put $d(S_1) = 1$ and, inductively, set $d(S_i) = i - \sum_{j=1}^i d(S_j)S_j(i)$. Then $K = \sum_{i \in \mathbb{N}} d(S_i)S_i$.

Let us now find coefficients $a_0, \dots, a_i \in \mathbb{Z}$ so that $S_i := a_0N + \sum_{j=1}^i a_jN_j$ is as desired. It follows from the Chinese remainder theorem that the system of congruences

$$\begin{aligned} x &\equiv 0 \pmod{p_1} \\ &\vdots \\ x &\equiv 0 \pmod{p_{i-1}} \\ x &\equiv 1 \pmod{p_i} \end{aligned}$$

has a solution $a_0 \in \mathbb{Z}$. This allows us to choose $a_j \in \mathbb{Z}$ so that $a_0 + a_j p_j = 0$, for every $1 \leq j < i$, and $a_i \in \mathbb{Z}$ so that $a_0 + a_i p_i = 1$.

4 Thin sums and topological closure

In [8], Vella introduced for a family \mathcal{N} of elements of R^M the following notation and spaces:

- the *weak span* $\mathcal{W}(\mathcal{N})$ is the set of all finite sums of elements of \mathcal{N} with coefficients in R , i.e. $\mathcal{W}(\mathcal{N})$ is the R -module generated by \mathcal{N} ;

- the *algebraic span* $\mathcal{A}(\mathcal{N})$ is the set of all thin sums of elements of \mathcal{N} with coefficients in R , i.e. what we have called (and will continue to call) $\langle \mathcal{N} \rangle$; and
- the *strong span* $\mathcal{S}(\mathcal{N})$ is the intersection of all sets $\mathcal{M} \supseteq \mathcal{N}$ that are closed under taking thin sums, i.e. the smallest set \mathcal{S} containing \mathcal{N} with $\langle \mathcal{S} \rangle = \mathcal{S}$.

Let us add to this list a fourth space, namely $\overline{\mathcal{W}(\mathcal{N})}$, the topological closure of $\mathcal{W}(\mathcal{N})$ in the product space R^M . It is not hard to see that it contains the strong span of \mathcal{N} :

Lemma 8. *For any ring R , any set M and any family \mathcal{N} of elements of R^M , $\overline{\mathcal{W}(\mathcal{N})}$ is closed under taking thin sums, i.e. $\langle \overline{\mathcal{W}(\mathcal{N})} \rangle = \overline{\mathcal{W}(\mathcal{N})}$.*

Proof. Consider a thin family \mathcal{T} of elements of $\overline{\mathcal{W}(\mathcal{N})}$. We need to show that $S := \sum_{T \in \mathcal{T}} T \in \overline{\mathcal{W}(\mathcal{N})}$. For this, let M' be an arbitrary finite subset of M . As \mathcal{T} is thin, the subfamily \mathcal{T}' of those members T of \mathcal{T} for which $T(m) \neq 0$ for some $m \in M'$ is finite. For each $T \in \mathcal{T}'$ there exists a finite subfamily \mathcal{N}_T of \mathcal{N} and coefficients $r_N^T \in R$ for $N \in \mathcal{N}_T$ so that $\sum_{N \in \mathcal{N}_T} r_N^T N(m) = T(m)$ for all $m \in M'$ since T lies in $\overline{\mathcal{W}(\mathcal{N})}$. Then, $S' := \sum_{T \in \mathcal{T}'} \sum_{N \in \mathcal{N}_T} r_N^T N$ is an element of $\mathcal{W}(\mathcal{N})$ and $S'(m) = S(m)$ for all $m \in M'$. As M' was arbitrary, this means that $S \in \overline{\mathcal{W}(\mathcal{N})}$. \square

We thus have the following inclusions:

$$\mathcal{W}(\mathcal{N}) \subseteq \langle \mathcal{N} \rangle \subseteq \mathcal{S}(\mathcal{N}) \subseteq \overline{\mathcal{W}(\mathcal{N})}.$$

Clearly, the first two inclusions can be proper. But can the third one also be proper?

Question 9. *Is $\mathcal{S}(\mathcal{N}) = \overline{\mathcal{W}(\mathcal{N})}$ for every family \mathcal{N} of elements of R^M ?*

In the special case when \mathcal{N} is thin and R a field or a finite ring we obtain from the results of the previous section (Lemmas 4 and 5) that $\langle \mathcal{N} \rangle = \overline{\langle \mathcal{N} \rangle}$, which clearly implies $\mathcal{S}(\mathcal{N}) = \overline{\mathcal{W}(\mathcal{N})}$. This answers a question of Manfred Droste (personal communication).

For countable M , Question 9 has an affirmative answer, too. This can easily be seen using a telescoping sum argument:

Proposition 10. *If R is any ring and M a countable set then $\overline{\mathcal{W}(\mathcal{N})} = \mathcal{S}(\mathcal{N})$ for any family \mathcal{N} of elements of R^M .*

Proof. We only need to prove that $\overline{\mathcal{W}(\mathcal{N})} \subseteq \mathcal{S}(\mathcal{N})$. Consider an arbitrary $K \in \overline{\mathcal{W}(\mathcal{N})}$, and let m_1, m_2, \dots be an enumeration of M . As K lies in $\overline{\mathcal{W}(\mathcal{N})}$ there is for every $i \in \mathbb{N}$ an $N_i \in \mathcal{W}(\mathcal{N})$ so that $N_i(m_j) = K(m_j)$ for all $j \leq i$. Set $N_0 = \mathbf{0}$ and define $L_i = N_i - N_{i-1} \in \mathcal{W}(\mathcal{N})$ for every i . Then, $L_i(m_j) = 0$ for $j < i$ and, consequently, the L_i form a thin family. Furthermore:

$$\left(\sum_{j=1}^{\infty} L_j \right) (m_i) = \left(\sum_{j=1}^i L_j \right) (m_i) = N_i(m_i) = K(m_i).$$

As $L_j \in \mathcal{W}(\mathcal{N}) \subseteq \mathcal{S}(\mathcal{N})$ for every j and $\mathcal{S}(\mathcal{N})$ is closed under taking thin sums, we obtain that $K = \sum_{j=1}^{\infty} L_j$ lies in $\mathcal{S}(\mathcal{N})$. \square

For uncountable M , Question 9 is not as easy to answer and indeed Proposition 10 becomes false. In the rest of the paper we will present a family \mathcal{N} of elements of $\mathbb{Z}_2^{[0,1]}$ for which the inclusion $\mathcal{S}(\mathcal{N}) \subseteq \overline{\mathcal{W}(\mathcal{N})}$ is proper.

Again, we will view elements of $\mathbb{Z}_2^{[0,1]}$ as subsets of $[0, 1]$. In particular, the elements of \mathcal{N} will consist of disjoint unions of intervals of $[0, 1]$. These intervals will be chosen from an ever finer subdivision of $[0, 1]$. More precisely, we will construct \mathcal{N} in ω steps each of which corresponds to a certain level of coarseness: we start in step 0 with the whole interval, so the first elements of \mathcal{N} will be $[0, 1]$ and the empty set. In the next step, we cut $[0, 1]$ in half, and in the following step we subdivide each of these halves again into two halves, and so on. Thus, in step n , we have the intervals $[0, 1/2^n], \dots, [(2^n - 1)/2^n, 1]$ at our disposal, and an element N constructed in this step will be the union of some of these intervals. However, not every such union will be put in \mathcal{N} .

Before we start with the formal definition, let us make one amendment. When we add (perhaps many) intervals of the form $[\frac{i-1}{2^n}, \frac{i}{2^n}]$ it is not so easy to keep track of what happens with the points on the boundary of the intervals. While this is not a serious problem, it complicates the matter. To circumvent this, we will delete from our ground set all those points that can ever arise as a boundary of some interval. These are precisely the points $J := \{\frac{i}{2^n} : 0 \leq i \leq 2^n, i, n \in \mathbb{N}\}$, and the subsets of \mathcal{N} will therefore be subsets of $[0, 1] \setminus J$.

We begin with the definition of the “intervals”: for $n \in \mathbb{N}$ and every $i \in \{1, 2, \dots, 2^n\}$ let $I_n^i = [\frac{i-1}{2^n}, \frac{i}{2^n}] \setminus J$, and define $\mathcal{I}_n := \{I_n^i : i \in \{1, 2, \dots, 2^n\}\}$. In step 0, we set $S_0 := [0, 1] \setminus J$ and $\mathcal{N}_0 := \{\emptyset, S_0\}$. Then, in step $n + 1$, assuming that we have already defined nested sets $\mathcal{N}_0 \subseteq \dots \subseteq \mathcal{N}_n$ in the previous steps, we construct a new “seed” element by taking every second interval in \mathcal{I}_{n+1} :

$$S_{n+1} := \bigcup_{i=1}^{2^n} I_{n+1}^{2i}$$

By adding this seed to the existing elements we define the new ones:

$$\mathcal{N}_{n+1} := \mathcal{N}_n \cup \{N + S_{n+1} : N \in \mathcal{N}_n\}.$$

Once all these \mathcal{N}_n are constructed, we put $\mathcal{N} := \bigcup_{n=0}^{\infty} \mathcal{N}_n$. See Figure 1 for the first few elements of \mathcal{N} .

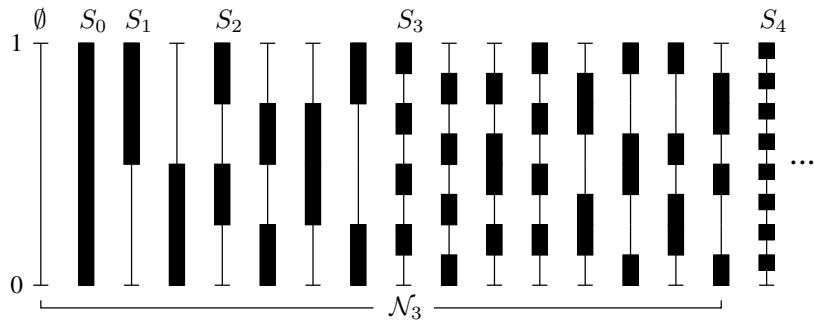


Figure 1: A schematic drawing of the first few elements of \mathcal{N}

We will accomplish our aim, to show that $\mathcal{S}(\mathcal{N}) \neq \overline{\mathcal{W}(\mathcal{N})}$, in three steps. In each of these we prove one of the following assertions:

- (i) \mathcal{N} is closed under taking finite sums, i.e. $\mathcal{N} = \mathcal{W}(\mathcal{N})$;
- (ii) $\mathcal{W}(\mathcal{N}) = \mathcal{S}(\mathcal{N})$; and
- (iii) $\mathcal{N} \neq \overline{\mathcal{N}}$.

Combining (i), (ii) and (iii) we immediately obtain $\mathcal{S}(\mathcal{N}) \neq \overline{\mathcal{W}(\mathcal{N})}$.

In order to establish (i), we will show inductively that each \mathcal{N}_n is already closed under taking finite sums, i.e. that $\mathcal{W}(\mathcal{N}_n) = \mathcal{N}_n$. For this, consider $N, L \in \mathcal{N}_n$. If $N, L \in \mathcal{N}_{n-1}$ then we are done by induction. So we may assume that $N \in \mathcal{N}_n \setminus \mathcal{N}_{n-1}$, i.e. that there is an $N' \in \mathcal{N}_{n-1}$ with $N = S_n + N'$. Now if $L \in \mathcal{N}_{n-1}$ then $N' + L \in \mathcal{N}_{n-1}$ (by induction) and thus $N + L = S_n + (N' + L) \in \mathcal{N}_n$. If, on the other hand, $L = S_n + L'$ for some $L' \in \mathcal{N}_{n-1}$, then $N + L = S_n + N' + S_n + L' = N' + L' \in \mathcal{N}_{n-1}$, which completes the proof of (i).

For the proof of (ii), we will need some intermediate assertions. The first one states that

$$\text{if } N \in \mathcal{N}_n \text{ and } I \in \mathcal{I}_n \text{ then either } I \subseteq N \text{ or } I \cap N = \emptyset. \quad (9)$$

To prove this, we perform induction on n . Note that I is contained in an $I' \in \mathcal{I}_{n-1}$. Thus, if $N \in \mathcal{N}_{n-1}$ then the assertion holds by the induction hypothesis. If, however, $N \in \mathcal{N}_n \setminus \mathcal{N}_{n-1}$, then $N = S_n + N'$ for some $N' \in \mathcal{N}_{n-1}$. The assertion holds for N' (by induction) and for S_{n+1} (by construction), and therefore it is also true for $N = S_{n+1} + N'$.

Next, we prove that

$$\text{every } N \in \mathcal{N}_{n+1} \setminus \mathcal{N}_n \text{ meets every } I \in \mathcal{I}_n. \quad (10)$$

Indeed, it is easy to see that this holds for $n = 0$. Now, suppose that $N = S_{n+1} + N'$ where $N' \in \mathcal{N}_n$. By construction, $I \cap S_{n+1}$ is a non-empty proper subset of I . Since, by (9), either $I \subseteq N'$ or $I \cap N' = \emptyset$, it follows that $S_{n+1} + N'$ meets I .

Before we deduce (ii), we need one final assertion:

$$\text{if } \mathcal{L} \text{ is an infinite subset of } \mathcal{N} \text{ and if } I \in \mathcal{I}_n \text{ for some } n \in \mathbb{N} \quad (11) \\ \text{then there exists an } r \in I \text{ lying in infinitely many } L \in \mathcal{L}.$$

To prove this, let N_1, N_2, \dots be distinct elements of \mathcal{L} , and define $n_k \in \mathbb{N}$ by $N_k \in \mathcal{N}_{n_k} \setminus \mathcal{N}_{n_k-1}$. We may assume that the N_k are ordered so that $n_1 \leq n_2 \leq \dots$. By going to a subsequence we may even assume that $n_{k+1} - n_k \geq 2$ for all k , and that $n_1 \geq n + 1$ (note that $|\mathcal{N}_n| < \infty$ for all n). Thus there is, by (9) and (10), an $I_1 \in \mathcal{I}_{n_1}$ with $I_1 \subseteq N_1 \cap I$. Now $I_1 = I_{n_1}^j$ for some j , and I_1 is the union of two elements of \mathcal{I}_{n_1+1} : the ‘‘left half’’ $I_l := I_{n_1+1}^{2j-1}$ and the ‘‘right half’’ $I_r := I_{n_1+1}^{2j}$. Since $n_2 - n_1 \geq 2$ it follows from (10) that N_2 meets both of I_l and I_r , and therefore both of $I_l \cap N_2$ and $I_r \cap N_2$ contain, by (9), an element of \mathcal{I}_{n_2} as a subset. We choose an $I_2 \in \mathcal{I}_{n_2}$ with $I_2 \subseteq I_l \cap N_2$. Continuing in this manner, we find nested sets $I \supseteq I_1 \supseteq I_2 \supseteq \dots$ with $I_k \subseteq N_k$. Since \mathbb{R} is complete and since the lengths of the intervals $\overline{I_k}$ converge to zero, there is precisely one point $r \in \mathbb{R}$ lying in all $\overline{I_k}$ (where $\overline{I_k}$ is the closure of I_k in the usual topology of \mathbb{R}). By choosing I_k to lie in the right half of I_{k-1} for odd k and in the left half for

even k , we ensure that $r \notin J$. Thus, r lies in $\overline{I_k} \setminus J = I_k \subseteq N_k$ for every k . Since $I_1 \subseteq I$ this proves (11).

Now (ii) follows directly from (11), as the latter implies that no thin sum can have infinitely many non-zero summands.

Let us deduce an easy corollary of (11) that we will use in order to prove (iii):

*if \mathcal{L} is an infinite subset of \mathcal{N} and if $I \in \mathcal{I}_n$ for some $n \in \mathbb{N}$ then
there exists an $s \in I$ that is missed by infinitely many $L \in \mathcal{L}$.* (12)

Indeed, this follows immediately if we apply (11) to $\mathcal{L}' := \{([0, 1] \setminus J) \setminus L : L \in \mathcal{L}\}$. (Note that $\mathcal{L}' \subseteq \mathcal{N}$ as $\mathcal{L}' = \{L + S_0 : L \in \mathcal{L}\}$.)

To prove (iii), take any infinite subset \mathcal{L}_0 of \mathcal{N} , and choose an $I_1 \in \mathcal{I}_1$. By subsequent application of (11) and (12) we find $r_1, s_1 \in I_1$ and an infinite subset \mathcal{L}_1 of \mathcal{L} so that $r_1 \in L$ but $s_1 \notin L$ for all $L \in \mathcal{L}_1$. Pick any element of \mathcal{L}_1 and denote it by L_1 . Next, we choose some $I_2 \in \mathcal{I}_2$ and find, again, $r_2, s_2 \in I_2$ and an infinite subset \mathcal{L}_2 of \mathcal{L}_1 so that r_2 lies in all the $L \in \mathcal{L}_2$ and s_2 in none. Pick any $L_2 \in \mathcal{L}_2$ and continue in this manner.

This process yields a sequence L_1, L_2, \dots of elements of \mathcal{N} . By Tychonoff's theorem, the space $\mathbb{Z}_2^{[0,1]}$ is compact. Hence, the sequence L_1, L_2, \dots has an accumulation point $X \in \mathbb{Z}_2^{[0,1]}$, and clearly $X \in \overline{\mathcal{N}}$.

We claim that $X \notin \mathcal{N}$. To see this, first note that since this is the case for almost all L_k , $r_i \in X$ but $s_i \notin X$ for any i . Now, suppose that $X \in \mathcal{N}_n$ for some n . As $I_n \in \mathcal{I}_n$, it follows from (9) that either $I_n \subseteq X$ or $I_n \cap X = \emptyset$. However, the former contradicts $s_n \notin X$ and the latter contradicts $r_n \in X$. This completes the proof of (iii).

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