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Degree Sequences and Edge Connectivity

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Abstract

For each positive integer k, we give a finite set of Bondy–Chvátal type conditions to a nondecreasing sequence $d = (d_1, \ldots, d_n)$ of nonnegative integers such that every graph on n vertices with degree sequence at least d is k-edge-connected. These conditions are best possible in the sense that whenever one of them fails for d then there is a graph on n vertices with degree sequence at least d which is not k-edge-connected.

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1 Introduction

All graphs considered here are supposed to be finite, undirected, and simple. For terminology not defined here, the reader is referred to [6]. The *degree sequence* of a graph G on n vertices is the unique nondecreasing sequence in $\mathbb{Z}^n$ in which every integer i occurs $|V_i(G)|$ many times, where $V_i(G)$ denotes the set of vertices of degree i in G. A sequence d which is the degree sequence of some graph G is called *graphical*, and any such G is a *realization* of d.

Degree sequences can be employed to provide sufficient conditions for certain monotone graph properties. The probably best known result in this direction is Chvátal's Theorem [5] (see also [6]).

Theorem 1 *[5] Let $n \geq 3$ and $d = (d_1, \ldots, d_n)$ be a nondecreasing sequence of nonnegative integers. Then every graph on n vertices with degree sequence at least d has a hamilton cycle, if and only if $d_n > n-1$ or*

$$d_j \leq j \text{ implies } d_{n-j} > n-j-1 \text{ for all integers } j \text{ with } 1 \leq j \leq \tfrac{n-1}{2}.$$

A quite similar condition gives a corresponding result for k-connectivity. The following result is due to Bondy (if–part [3]) and Boesch (only-if–part & the present form of Bondy's condition [2]).

Theorem 2 *[3, 2] Let $k \geq 1$, $n \geq k+1$, and $d = (d_1, \ldots, d_n)$ be a nondecreasing sequence of nonnegative integers. Then every graph on n vertices with degree sequence at least d is k-connected, if and only if $d_n > n-1$ or*

$$d_j \leq j+k-2 \text{ implies } d_{n-k+1} > n-j-1 \text{ for all integers } j \text{ with } 1 \leq j \leq \tfrac{n-k+1}{2}.$$

Recently, Bauer, Hakimi, Kahl, and Schmeichel investigated similar conditions for k-edge-connectivity [1] and noticed that such conditions for $k > 1$ would probably look much more difficult than the two mentioned above. For $k = 2$, they obtained the following result [1].

Theorem 3 *[1] Let $n \geq 1$ and $d = (d_1, \ldots, d_n)$ be a graphical sequence. Then every graph on n vertices with degree sequence at least d is 2-edge-connected, if and only if*

1. *$d_1 \geq 2$,*
2. *$d_{j-1} \leq j-1$ and $d_j \leq j$ implies $d_{n-1} > n-j-1$ or $d_n > n-j$ for all integers j with $3 \leq j \leq \frac{n-1}{2}$, and*
3. *For $n \geq 4$ even, $d_{\frac{n}{2}} \leq \frac{n}{2} - 1$ implies $d_{n-2} > \frac{n}{2} - 1$ or $d_n > \frac{n}{2}$.*

Beside the "Bondy–Chvátal type" condition 2. in Theorem 3, there are two extra conditions involved. Redundancy questions arise: For example, it is possible to incorporate condition 1. into 2. by extending the range of j to $1 \leq j \leq \frac{n-1}{2}$ (where $d_{j-1} \leq j-1$ is considered only if $j \geq 2$). Also, there is some redundancy in 3.: The implication there is equivalent to its consequent, because $d_{\frac{n}{2}} > \frac{n}{2} - 1$ implies $d_{n-2} > \frac{n}{2} - 1$ by monotonicity of d, so 3. holds if and only if (for $n \geq 4$ even) $d_{n-2} > \frac{n}{2} - 1$ or $d_n > \frac{n}{2}$. On the other hand, in this formulation, it is necessary to assume that G is graphical: If we would just assume, say, that d is a nondecreasing sequence of nonnegative integers, then there are no graphs with degree sequence at least $d = (1, \underbrace{n, \ldots, n}_{n-1 \text{ times}})$ at all — but 1. of Theorem 3 is not satisfied. For $k = 3$, Bauer, Hakimi, Kahl, and Schmeichel conjectured the following [1].

Conjecture 1 *[1] Let $n \geq 1$ and $d = (d_1, \ldots, d_n)$ be a graphical sequence. Then every graph on n vertices with degree sequence at least d is 3-edge-connected, if and only if*

1. *$d_1 \geq 3$,*
2. *$d_{j-2} \leq j-1$ and $d_j \leq j$ implies $d_{n-2} > n-j-1$ or $d_n > n-j$ for all integers j with $3 \leq j \leq \frac{n-1}{2}$,*

3. *$d_{j-1} \leq j-1$ and $d_j \leq j+1$ implies $d_{n-2} > n-j-1$ or $d_n > n-j$ for all integers j with $3 \leq j \leq \frac{n-2}{2}$,*

4. *$d_{j-2} \leq j-1$ and $d_j \leq j$ implies $d_{n-1} > n-j-1$ or $d_n > n-j+1$ for all integers j with $3 \leq j \leq \frac{n-1}{2}$,*

5. *for $n \geq 6$ even, $d_{\frac{n}{2}} \leq \frac{n}{2}-1$ implies $d_{n-4} > \frac{n}{2}-1$ or $d_n > \frac{n}{2}$,*

6. *for $n \geq 5$ odd, $d_{\frac{n-1}{2}-1} \leq \frac{n-1}{2}-1$ implies $d_{n-3} > \frac{n+1}{2}-1$ or $d_n > \frac{n+1}{2}$, and*

7. *for $n \geq 4$ even, $d_{\frac{n}{2}} \leq \frac{n}{2}-1$ implies $d_{n-3} > \frac{n}{2}-1$ or $d_{n-1} > \frac{n}{2}$ or $d_n > \frac{n}{2}+1$.*

In Conjecture 1, there are already three "Chvátal–Bondy type" conditions (2., 3., 4.), which arise according to the three different "types" of cuts of order 2, plus four extra conditions.[1] Again, 5. and 7. can be simplified similarly as above.

Here, for each $k \geq 1$, we construct a set of conditions to a nondecreasing sequence $d = (d_1, \ldots, d_n)$ of nonnegative integers such that every graph on n vertices with degree sequence at least d is k-edge-connected. This condition will be best possible in the sense that whenever it fails for d then there is a graph on n vertices with degree sequence at least d which is not k-edge-connected. Specialized to $k = 2$, we get Theorem 3 again, and spezialized to $k = 3$, this verifies Conjecture 1.

2 Degree sequences and cuts

Let G be a graph and let $X, Y \subseteq V(G)$. By $E_G(X, Y)$ we denote the set of edges of G which connect a vertex from X to one of Y. By $\partial_G X$ we denote the set of vertices from X which have at least one neighbor outside X. The *degree sequence of X in G* is the unique nondecreasing integer sequence in $\mathbb{Z}^{|X|}$ in which every integer i occurs $|X \cap V_i(G)|$ many times. By definition, the degree sequence of G is the degree sequence of $V(G)$ in G. The *type* of the pair (X, Y) is the pair (a, b), where a, b are the degree sequences of X, Y, respectively, in the graph $(X \cup Y, E_G(X, Y))$.

Throughout this paper, the set $X \subseteq V(G)$ is a *cut* if $1 \leq |X| \leq |V(G)|/2$. Its *order* is $|E_G(X, V(G) - X)|$, which equals $|E_G(\partial_G X, \partial_G(V(G) - X))|$. This differs slightly from the common definitions of a cut, but an important feature is shared by all of them: By Menger's Theorem, G is k-edge-connected if and only if $|V(G)| > 1$ and $|E_G(X, V(G) - X)| \geq k$ holds for all nonempty proper

[1] Our formulation differs a little bit from the original one: Some conditions to the order of n in 5., 6., 7. have been added carefully, mainly in order to keep the indices legal.

subsets X of $V(G)$ (see [6]), which is in turn equivalent to saying that G has at least two vertices and no cut of order less than k.

The *type* of a cut X is the type of $(\partial_G X, \partial_G(V(G)-X))$. Clearly, if X has type $((a_1,\dots,a_p),(b_1,\dots,b_q))$ then X has order $\sum_{i=1}^p a_i = \sum_{i=1}^q b_i$. By definition, $1 \le a_i \le q$ for all $i \in \{1,\dots,p\}$ and $1 \le b_i \le p$ for all $i \in \{1,\dots,q\}$, and $p=q=0$ may happen.

Let us deduce some set of inequalities involving the degree sequence from the presence of a cut of a certain type. It is formulated in a way which let the proofs work nicely, but we will work out an equivalent set in the next section, which gives conditions similar to those in the statements mentioned in the previous section when explicitely specialized.

Lemma 1 *Let G be a graph on $n \ge 1$ vertices with degree sequence $d = (d_1,\dots,d_n)$. Let X be a cut of type $\sigma = ((a_1,\dots,a_p),(b_1,\dots,b_q))$. Set $j := |X|$, and, for each $\ell \ge 0$,*

$$\begin{aligned} c_{j,\ell} := c_{j,\ell}(\sigma,n) &:= \left\{ \begin{array}{rl} n-j-q & \text{if } n-2j \le \ell \\ 0 & \text{otherwise} \end{array} \right\}, \\ r_\ell := r_\ell(\sigma) &:= |\{i \in \{1,\dots,p\} : a_i \le \ell\}|, \text{ and} \\ s_{j,\ell} := s_{j,\ell}(\sigma,n) &:= |\{i \in \{1,\dots,q\} : b_i \le 2j-n+\ell\}|. \end{aligned}$$

Then

1. $\max\{1,p\} \le j \le \min\{n/2, n-q\}$, *and*
2. $d_{j-p+c_{j,\ell}+r_\ell+s_{j,\ell}} \le j-1+\ell$ *for all* $\ell \ge 0$ *with* $j+c_{j,\ell}+r_\ell+s_{j,\ell} > p$.

Proof. Let $Y := V(G)-X$ and let $H := (\partial_G X \cup \partial_G Y, E_G(\partial_G X, \partial_G Y))$.

By definition, $1 \le |X| \le n/2$. Since X contains p elements from $\partial_G X$ and Y contains q elements from $\partial_G Y$, $p \le |X| \le n-q$ follows, which implies 1.

To prove 2., consider $\ell \ge 0$ with $j+c_{j,\ell}+r_\ell+s_{j,\ell} > p$. Since $j-p+c_{j,\ell}+r_\ell+s_{j,\ell} \le j-p+n-j-q+p+q \le n$, all indices in 2. are legal. Observe that the neighborhood of each vertex $v \in X - \partial_G X$ is contained in X, implying $d_G(v) \le j-1 \le j-1+\ell$. A vertex v in $\partial_G X$ with degree, say, a_i in H has degree at most $j-1+\ell$ in G if $a_i \le \ell$, whereas a vertex v in $\partial_G Y$ with degree, say, b_i in H has degree at most $j-1+\ell$ if $n-j-1+b_i \le j-1+\ell$, that is $b_i \le 2j-n+\ell$. A vertex $v \in Y - \partial_G Y$ has all its neighbors in Y, implying $d_G(v) \le n-j-1$, and $n-j-1 \le j-1+\ell$ holds if $n-2j \le \ell$. Consequently, there are at least $|X-\partial_G X| + c_{j,\ell}+r_\ell+s_{j,\ell}$ many vertices of degree at most $j-1+\ell$ in G, which implies 2. □

The "index condition" $j+c_{j,\ell}+r_\ell+s_{j,\ell} > p$ in 2. of Lemma 1 fails only if $j=p$ and $c_{j,\ell}=r_\ell=s_{j,\ell}=0$. Therefore, if we define in addition, $d_0 := 0$, 2.

of Lemma 1 holds if and only if $d_{j-p+c_{j,\ell}+r_\ell+s_{j,\ell}} \leq j-1+\ell$ for all $\ell \geq 0$. This will be convenient when we transform and specialize our conditions in the next section.

A pair $\sigma = (a = (a_1, \ldots, a_p), b = (b_1, \ldots, b_q))$ of finite nondecreasing sequences of nonnegative integers is *bigraphical* if there is a bipartite graph H without isolated vertices and (possibly empty) color classes X, Y such that (a, b) is the type of (X, Y) in H. We call H a *realization of* σ, and define the *order* of σ to be the number of edges of any realization, which is equal to $\sum_{i=1}^{p} a_i = \sum_{i=1}^{q} b_i$. Again, $1 \leq a_i \leq q$ for all $i \in \{1, \ldots, p\}$ and $1 \leq b_i \leq p$ for all $i \in \{1, \ldots, q\}$ by definition. It may happen that $p = q = 0$.

Lemma 2 *Let $\sigma = ((a_1, \ldots, a_p), (b_1, \ldots, b_q))$ be bigraphical. Let $n \geq 1$ and $d = (d_1, \ldots, d_n)$ be a nondecreasing sequence of nonnegative integers and j be an integer such that 1. and 2. of Lemma 1 are satisfied.*

Then there is a graph $G(n, j, \sigma)$ on n vertices with a cut of type σ and degree sequence at least d.

Proof. Let H be a realization of σ, and let X', Y' be color classes such that σ is the type of (X', Y') in H. Since 1. of Lemma 1 holds here, we may take $G := G(n, j, \sigma)$ as the union of H and two disjoint cliques on X, Y, respectively, such that X has j vertices and contains X' and Y has $n-j$ vertices and contains Y'. By construction, G has n vertices, $X' = \partial_G X$ and $Y' = \partial_G Y$, and X is a cut of type σ of G. Let $g = (g_1, \ldots, g_n)$ be the degree sequence of G. We have to prove that $d_i \leq g_i$ for all $i \in \{1, \ldots, n\}$.

Consider $i \in \{1, \ldots, n\}$, and define a sequence $f = (f_0, f_1, \ldots)$ of nonnegative integers by $f_\ell := j - p + c_{j,\ell} + r_\ell + s_{j,\ell}$ for $\ell \geq 0$. For $\ell \geq \max\{n - 2j + p, q\}$ we have $f_\ell = n$ by definition. Hence there exists a smallest $\ell \geq 0$ such that $i \leq f_\ell$.

Since $f_\ell \geq i \geq 1$, we deduce $d_i \leq d_{f_\ell} \leq j - 1 + \ell$ as 2. of Lemma 1 holds for d here. By construction, G has minimum degree at least $j - 1$. For $\ell = 0$, we thus obtain $d_i \leq j - 1 + 0 \leq g_1 \leq g_i$, so we may assume $\ell > 0$. By choice of ℓ, $i > f_{\ell-1}$.

Now $\partial_G X$ contains $p - r_{\ell-1}$ vertices of degree exceeding $j - 1 + \ell - 1$, and $\partial_G Y$ contains $q - s_{j,\ell-1}$ vertices of degree exceeding $j - 1 + \ell - 1$. Since the vertices in $Y - \partial_G Y$ have degree $n - j - 1$, which exceeds $j - 1 + \ell - 1$ if and only if $n - 2j > \ell - 1$, we count $n - j - q - c_{j,\ell-1}$ further such vertices in $Y - \partial_G Y$. In total, G contains $p - r_{\ell-1} + q - s_{j,\ell-1} + n - j - q - c_{j,\ell-1} = n - f_{\ell-1}$ vertices of degree exceeding $\ell - 1$. It follows that G has at most $f_{\ell-1}$ vertices of degree at most $j - 1 - \ell - 1$. Consequently, $g_i > j - 1 + \ell - 1$, so $d_i \leq j - 1 + \ell \leq g_i$. $\square$

3 Redundancy

At first sight, condition 2. of Lemma 1 encodes infinitely many conditions, one for each ℓ. However, all but finitely many of them are logically redundant, because for fixed σ the subscript $j - p + c_{j,\ell} + r_\ell + s_{j,\ell}$ takes only finitely many distinct expressions of the form $j+y$ or $n+z$ (y, z integers). Let us take a closer look at them, for some fixed n, j and σ as above. To avoid the cumbersome index check condition, we set $d_0 := 0$

For $\ell < n-2j$, $c_{j,\ell} = s_{j,\ell} = 0$ holds, and the inequality of our condition collapses to

$$d_{j-p+r_\ell} \leq j - 1 + \ell. \tag{$L(\ell)$}$$

For $\ell \geq q$, $r_\ell = p$, since $a_i \leq q$ for all $i \in \{1, \ldots, p\}$. Therefore, provided that $q < n - 2j$ (or, equivalently, $j < (n-q)/2$), $(L(\ell))$ holds for all ℓ with $0 \leq \ell < n - 2j$ if and only if $(L(\ell))$ holds for all $\ell \in \{0, \ldots, q\}$.

For $\ell \geq n - 2j$, we obtain

$$d_{n-q-p+r_\ell+s_{j,\ell}} \geq j - 1 + \ell. \tag{$S(\ell)$}$$

Substituting $\ell - (n-2j) =: x$ and defining $t_x := t_x(\sigma) := |\{i \in \{1, \ldots, q\} : b_i \leq x\}|$ we may rewrite

$$d_{n-q-p+r_{x+(n-2j)}+t_x} \leq n - j - 1 + x, \tag{$T(x)$}$$

(and $(T(\max\{p, q-(n-2j)\}))$ implies $(T(x))$ for all $x > \max\{p, q-(n-2j)\}$).

Unfortunately, $-p + r_{x+n-2j}$ does not vanish for all x and j. However, if j is not too close to $n/2$ then it does. To be more precise, if $q \leq n - 2j$ then $x + n - 2j \geq q$, so $r_{x+n-2j} = p$. Hence in these cases, $(T(x))$ collapses to

$$d_{n-q+t_x} \leq n - j - 1 + x. \tag{$R(x)$}$$

Again, as $t_x = p$ for $x \geq q$, $(R(x))$ holds for all $x \geq 0$ if and only if it holds for all $x \in \{0, \ldots, p\}$, provided that $q \leq n - 2j$.

Therefore, provided that $q < n-2j$, condition 2. of Lemma 1 holds if and only if $(L(\ell))$ holds for all $\ell \in \{0, \ldots, q\}$ and $(R(x))$ holds for all $x \in \{0, \ldots, p\}$. The latter statement is true for $q = n-2j$, too: To see this, observe that condition 2. of Lemma 1 holds if and only if $(L(\ell))$ holds for all $\ell \in \{0, \ldots, q-1\}$ and $(R(x))$ holds for all $\ell \in \{0, \ldots, p\}$. However, if $q = n - 2j$ then the inequality in $(R(0))$ is $d_{n-q} \leq (n+q)/2 - 1$, whereas the one in $(L(q))$ is $d_{(n-q)/2} \leq (n+q)/2 - 1$; as d is nondecreasing, $(R(0))$ thus implies $(L(q))$.

On the other hand, if $q \geq n - 2j$, condition 2. of Lemma 1 holds if and only if $(L(\ell))$ holds for all $\ell \in \{0, \ldots, n-2j-1\}$ and $(T(x))$ holds for $x \in \{0, \ldots, \max\{p, q-(n-2j)\}\}$, as $t_x = q$ for $x \geq p$ and $r_{x+n-2j} = p$ for $x \geq q - (n-2j)$.

We thus may reformulate the results of the preceeding section as follows.

Theorem 4 *Let $\sigma = ((a_1, \ldots, a_p), (b_1, \ldots, b_q))$ be bigraphical, and let*

$$\begin{aligned} r_\ell := r_\ell(\sigma) &:= |\{i \in \{1, \ldots, p\} : a_i \leq \ell\}|, \text{ for all } \ell \geq 0, \text{ and} \\ t_x := t_x(\sigma) &:= |\{i \in \{1, \ldots, q\} : b_i \leq x\}|, \text{ for all } x \geq 0. \end{aligned}$$

Let $n \geq 1$ and $d = (d_1, \ldots, d_n)$ be a nondecreasing sequence of nonnegative integers. Let j be an integer such that $\max\{1, p\} \leq j \leq \min\{n/2, n-q\}$. *Set* $d_0 := 0$.

Then there exists a graph on n vertices with a cut of j vertices of type σ with degree sequence at least d if and only if the following conditions hold:

1. *If $j \leq (n-q)/2$ then $d_{j-p+r_\ell} \leq j - 1 + \ell$ for all $\ell \in \{0, \ldots, q\}$ and $d_{n-q+t_x} \leq n - j - 1 + x$ for all $x \in \{0, \ldots, p\}$.*
2. *If $j > (n-q)/2$ then $d_{j-p+r_\ell} \leq j-1+\ell$ for all $\ell \in \{0, \ldots, n-2j-1\}$ and $d_{n-q-p+r_{x+(n-2j)}+t_x} \leq n-j-1+x$ for all $x \in \{0, \ldots, \max\{p, q-(n-2j)\}\}$.*

Proof. Let X be a cut of type σ with $|X| = j$ in a graph G on n vertices with degree sequence $g = (g_1, \ldots, g_n)$ at least d. Then, by Lemma 1, conditions 1. and 2. are satisfied for g instead of d. As $d_i \leq g_i$ for all $i \in \{1, \ldots, n\}$, each of the inequalities in 1. and 2. is satisfied, too, and thus 1. and 2. hold for d.

Conversely, if 1. and 2. hold for d then 2. of Lemma 1 holds, too, as we have seen in the paragraphs just before this theorem. By Lemma 2, there exists a graph $G(n, j, \sigma)$ with a cut of j vertices and degree sequence at least d. □

Still there is some redundancy in the set of conditions of this theorem, because, in general, r_ℓ and t_x would not take all values from $\{0, \ldots, q\}$ and $\{0, \ldots, p\}$. (For example, if $r_\ell = r_{\ell+1}$, then $(L(\ell))$ implies $(L(\ell+1))$, so it is not necessary to list $(L(\ell+1))$.)

We write down an important corollary, characterizing the nondecreasing finite integer sequences d for which there are no graphs with degree sequence at least d and cuts of some fixed type. That is, we vary j, and write the negations of 1. and 2. of Theorem 4 as implications, where the antecedent is the conjunction of all inequalities involving ℓ and the consequent is the disjunction of the negations of all inequalities involving x.

Corollary 1 *Let $\sigma = ((a_1, \ldots, a_p), (b_1, \ldots, b_q))$ be bigraphical. Let $n \geq 1$ and $d = (d_1, \ldots, d_n)$ be a nondecreasing sequence of nonnegative integers. Set $d_0 :=$* 0.

Then there exists no graph on n vertices with a cut of type σ and degree sequence at least d if and only if the following conditions hold:

1. *For all integers j with* $\max\{1, p\} \leq j \leq (n-q)/2$,
   $d_{j-p+r_\ell} \leq j - 1 + \ell$ for all $\ell \in \{0, \ldots, q\}$ implies
   $d_{n-q+t_x} > n - j - 1 + x$ for some $x \in \{0, \ldots, p\}$.

2. *For all integers $j \geq \max\{1, p\}$ with $(n-q)/2 < j \leq \min\{n/2, n-q\}$, $d_{j-p+r_\ell} \leq j-1+\ell$ for all $\ell \in \{0, \ldots, n-2j-1\}$ implies $d_{n-p-q+r_{x+n-2j}+t_x} > n-j-1+x$ for some $x \in \{0, \ldots, \max\{p, q-(n-2j)\}\}$.*[2]

Let us refer to 1. as to the *universal condition for σ*. It rules out all cuts of type σ whose number of vertices is at most $(n-q)/2$. When specializing to some particular σ, we will list the extra conditions in 2. one after another, one for each possible value of j. Formally, this will yield q such conditions.

One way to be more explicit when formulating these extra conditions is to substitute $j = (n-\lambda)/2$, where $\lambda \in \{0, \ldots, q-1\}$; then 2. is equivalent to saying that, for all $\lambda \in \{0, \ldots, q-1\}$,

2.(λ) *for $n \geq \max\{2+\lambda, 2p+\lambda, 2q-\lambda\}$ having the same parity as λ, $d_{(n-\lambda)/2-p+r_\ell} \leq (n-\lambda)/2-1+\ell$ for all $\ell \in \{0, \ldots, \lambda-1\}$ implies $d_{n-p-q+r_{x+\lambda}+t_x} > (n+\lambda)/2-1+x$ for some $x \in \{0, \ldots, \max\{p, q-\lambda\}\}$.*

Let us refer to this statement as to the *extra condition for σ and λ*.

The reason for the presence of extra conditions is that, for j close to $n/2$, the degree sequence d of the extremal graph $G(n, j, \sigma)$ in the proof of Lemma 2 is degenerated in a sense. Normally, if j is not too close to $n/2$, then the degree sequence starts with $j-p$ entries $j-1$ followed by $j-1+a_1, \ldots, j-1+a_p$, and ends with $n-j-q$ entries $n-j-1$ followed by $n-j-1+b_1, \ldots, n-j-1+b_q$; in other words: if X, Y are taken as in the proof of Lemma 2 then the degrees of the vertices from $X - \partial_G X$, $\partial_G X$, $Y - \partial_G Y$, and $\partial_G Y$ occur in this order in d. Figure 1 gives an example. (White "points" belong to vertices of $\partial_G X$, black ones to $\partial_G Y$.)

Now if $n-j-1$ is small enough then $j-1+a_i$ might top it, and in fact it might top even some of the $n-j-1+b_{i'}$. In other words: elements from $\partial_G X$ might occur on *both* "sides" of the "$Y - \partial_G Y$" segment in the degree sequence. Figure 2 shows an example with all parameters but j being the same.

It is straightforward to implement an algorithm that produces the set of conditions for some fixed σ. For $\sigma = ((1), (1))$, which is the only possible type of a cut of order 1, we obtain the following.

Corollary 2 *Let $n \geq 1$ and $d = (d_1, \ldots, d_n)$ be a nondecreasing sequence of nonnegative integers. Set $d_0 := 0$.*

Then there exists no graph on n vertices with a cut of type $\sigma = ((1), (1))$ and degree sequence at least d, if and only if the following conditions hold:

1. *(Universal condition for σ)*

[2] For $j = n/2$ the antecedent of the implication in 2. is trivially true.

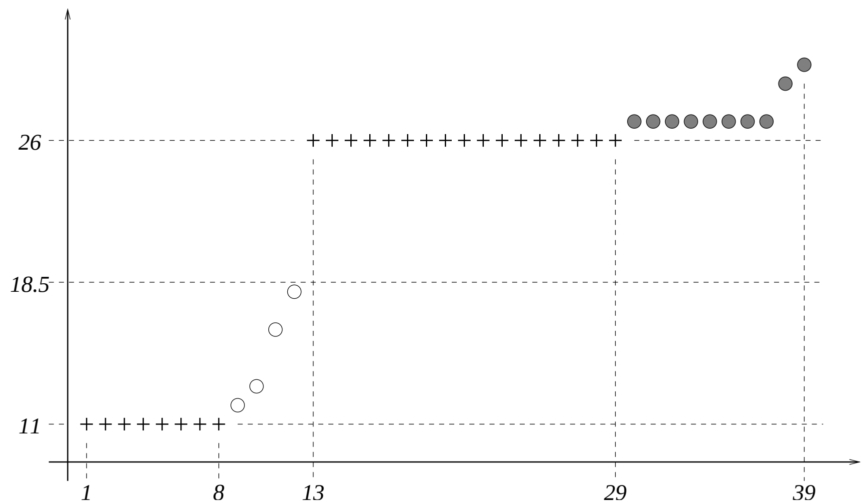

Figure 1: The degree sequence of $G(39, 12, ((1, 2, 5, 7), (1, 1, 1, 1, 1, 1, 1, 1, 3, 4)))$.

$d_{j-1} \leq j-1$ and $d_j \leq j$ implies $d_{n-1} > n-j-1$ or $d_n > n-j$ for all integers j with $1 \leq j \leq \frac{n-1}{2}$, and

2. *(Extra condition for σ, $\lambda = 0$)*
for $n \geq 2$ even, $d_{n-2} > \frac{n-2}{2}$ or $d_n > \frac{n}{2}$.

Let us show that the conditions of Theorem 2 and Corollary 2 are equivalent if d is *graphical* of length $n \geq 2$.

Suppose first that 1., 2., 3. of Theorem 3 hold. Then the universal condition for σ holds by 2. of Theorem 3 for $j \geq 3$ and by 1. of Theorem 3 for $j \in \{1, 2\}$. Furthermore, 1. of Theorem 3 implies $n \geq 3$ since G is graphical, so the extra condition for σ and $\lambda = 0$ holds by 3. of Theorem 3.

Conversely, if the universal condition for σ and the extra condition for σ and $\lambda = 0$ hold then 2. and 3. of Theorem 3 follow trivially. If $n = 2$ then $d_2 \geq 2$ follows from the extra condition for σ and $\lambda = 0$, contradicting the assumption that d is graphical. If $n \geq 3$ then we may apply the universal condition of σ to $j = 1$, and $d_1 \leq 1$ would imply $d_{n-1} > n-2$ or $d_n > n-1$. Since d is graphical, $d_n \leq n-1$ holds, so $d_{n-1} = n-1$. But then at least two vertices of any realization of d are adjacent to all others, contradicting $d_1 \leq 1$.

Let us now specialize Corollary 1 to $\sigma \in \{((2),(1,1)),((1,1),(2)), ((1,1),(1,1))\}$, which is the complete typography of cuts of order 2.

Corollary 3 *Let $n \geq 1$ and $d = (d_1, \ldots, d_n)$ be a nondecreasing sequence of nonnegative integers. Set $d_0 := 0$.*

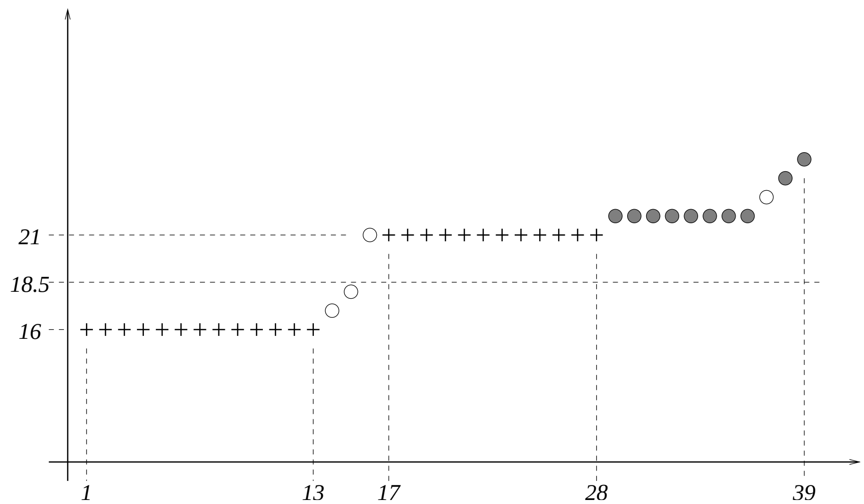

Figure 2: The degree sequence of $G(39, 17, ((1, 2, 5, 7), (1, 1, 1, 1, 1, 1, 1, 1, 3, 4))$.

Then there exists no graph on n vertices with a cut of order 2 and degree sequence at least d, if and only if the following conditions hold:

1. *(Universal condition for $\sigma = ((2), (1, 1))$)* $d_{j-1} \leq j - 1$ *and* $d_j \leq j + 1$ *implies* $d_{n-2} > n - j - 1$ *or* $d_n > n - j$ *for all integers* j *with* $1 \leq j \leq \frac{n-2}{2}$,
2. *(Extra condition for $\sigma = ((2), (1, 1))$, $\lambda = 0$)* *for* $n \geq 4$ *even,* $d_{n-3} > \frac{n-2}{2}$ *or* $d_{n-1} > \frac{n}{2}$ *or* $d_n > \frac{n+2}{2}$,
3. *(Extra condition for $\sigma = ((2), (1, 1))$, $\lambda = 1$)* *for* $n \geq 3$ *odd,* $d_{\frac{n-3}{2}} \leq \frac{n-3}{2}$ *implies* $d_{n-3} > \frac{n-1}{2}$ *or* $d_n > \frac{n+1}{2}$,
4. *(Universal condition for $\sigma = ((1, 1), (1, 1))$)* $d_{j-2} \leq j - 1$ *and* $d_j \leq j$ *implies* $d_{n-2} > n - j - 1$ *or* $d_n > n - j$ *for all integers* j *with* $2 \leq j \leq \frac{n-2}{2}$,
5. *(Extra condition for $\sigma = ((1, 1), (1, 1))$, $\lambda = 0$)* *for* $n \geq 4$ *even,* $d_{n-4} > \frac{n-2}{2}$ *or* $d_n > \frac{n}{2}$,
6. *(Extra condition for $\sigma = ((1, 1), (1, 1))$, $\lambda = 1$)* *for* $n \geq 5$ *odd,* $d_{\frac{n-5}{2}} \leq \frac{n-3}{2}$ *implies* $d_{n-2} > \frac{n-1}{2}$ *or* $d_n > \frac{n+1}{2}$
7. *(Universal condition for $\sigma = ((1, 1), (2))$)* $d_{j-2} \leq j - 1$ *and* $d_j \leq j$ *implies* $d_{n-1} > n - j - 1$ *or* $d_n > n - j + 1$ *for all integers* j *with* $2 \leq j \leq \frac{n-1}{2}$, *and*

8. *(Extra condition for $\sigma = ((1,1),(2))$, $\lambda = 0$)*
   for $n \geq 4$ even, $d_{n-3} > \frac{n-2}{2}$ or $d_{n-1} > \frac{n}{2}$ or $d_n > \frac{n+2}{2}$.

We may wipe out 8. from the list, as it is equal to 2. We claim that if d is a graphical sequence of length $n \geq 3$ then 1. to 7. of Corollary 3 are equivalent to 1. to 7. of Conjecture 1.

Suppose first that 1. to 7. of Conjecture 1 hold. By 1. of Conjecture 1, $n \geq 4$ (as d is graphical), the extra conditions for $\sigma_1 := ((2),(1,1))$, $\sigma_2 := ((1,1),(1,1))$, and $\sigma_3 := ((1,1),(2))$ hold in the case $n \leq 4$, and the universal conditions hold for $j < 3$. For $j \geq 3$, the universal conditions hold by 3., 2., 4. of Conjecture 1, respectively. So let's verify the extra conditions in the case $n > 4$. The extra conditon for σ_1 (or σ_3) and $\lambda = 0$ follows from 7. of Conjecture 1, because $d_{\frac{n}{2}} > \frac{n}{2} - 1$ implies $d_{n-3} > \frac{n-2}{2}$ if $n \geq 6$. The extra condition for σ_1 and $\lambda = 1$ follows from 7. of Conjecture 1. The extra condition for σ_2 and $\lambda = 0$ follows from 5. of Conjecture 1, because $d_{\frac{n}{2}} > \frac{n}{2} - 1$ implies $d_{n-4} > \frac{n}{2} - 1$ for $n \geq 8$, whereas for $n = 6$, $d_{n-4} > \frac{n}{2} - 1 = 2$ by 1. of Conjecture 1. Finally, the extra condition for σ_2 and $\lambda = 1$ follows from 2. of Conjecture 1, specialized to $j = \frac{n-1}{2}$ (as $d_{\frac{n-1}{2}} > \frac{n-1}{2}$ implies $d_{n-2} > \frac{n-1}{2}$) for $n \geq 7$, whereas for $n = 5$, $d_{n-2} > \frac{n-1}{2} = 2$ by 1. of Conjecture 1.

Conversely, let all the conditions from Corollary 3 hold. Then 5., 6., 7. from Conjecture 1 follow from the extra conditions for σ_2 and $\lambda = 0$, for σ_1 and $\lambda = 1$, and for σ_1 (or σ_3) and $\lambda = 0$, respectively. For $j < \frac{n-1}{2}$, the statement of 2. from Conjecture 1 follows from the universal condition for σ_2; for $j = \frac{n-1}{2} \geq 3$ we deduce that $n \geq 7$ is odd, and thus the statement follows from the extra condition for σ_2 and $\lambda = 1$. 3. and 4. from Conjecture 1 follow immediately from the universal condition for σ_1, σ_3, respectively. Assume, to the contrary, that 1. from Conjecture 1 does not hold, that is, $d_1 \leq 2$. For $n \geq 4$, we may apply the universal condition for σ_1 to $j = 1$, implying $d_{n-2} > n - 2$ or $d_n > n - 1$; as d is graphical, $d_n \leq n - 1$, hence $d_{n-2} = n - 1$. It follows that at least 3 vertices of any realization of d are adjacent to all others, contradicting $d_1 \leq 2$. For $n = 3$, the extra condition for σ_1 and $\lambda = 1$ implies $d_n > 2 = n - 1$, which is not possible as G is graphical.

4 Corollaries on edge connectivity

Corollary 4 *Let $n \geq k \geq 1$ and $d = (d_1, \ldots, d_n)$ be a nondecreasing sequence of nonnegative integers. Then every graph on n vertices with degree sequence at least d is k-edge-connected, if and only if d satisfies 1. and 2. of Corollary 1 for all bigraphical pairs σ of order $k - 1$.*

Proof. Suppose that every graph on n vertices with degree sequence at least d is k-edge-connected and let σ be a bigraphical pair of order $k - 1$. Then G

does not have a cut of type σ, and hence the conditions in Corollary 1 must be satisfied.

Conversely, suppose that the conditions in Corollary 1 are satisfied, and let G be a graph on n vertices with degree sequence at least d. If G would contain a cut X of order less than k then we may add edges from X to $Y := V(G) - X$ in order to obtain a supergraph G^+ of G where X is a cut of order $k-1$, because $|X| \cdot |Y| \geq |V(G)| - 1 \geq k - 1$. Now G^+ is a graph which does contain a cut of order $k-1$ and of some type σ. The degree sequence of G^+ is at least the degree sequence of G and, therefore, at least d, too, which contradicts Corollary 1. $\square$

How complex are the conditions in Corollary 4, if we write them down line by line, one for each universal or special condition, as we did in Corollary 2 and Corollary 3? To give a rough estimate, we need some knowledge about the number $b(k)$ of bigraphical sequence pairs of order k (as we have seen, $b(1) = 1$ and $b(2) = 3$). The determination of $b(k)$ seems to be a difficult problem [4], even asymptotic results have not been developed yet. We can estimate $b(k)$ using knowledge on the number $p(k)$ of partitions of k, as $p(k) \leq b(k) \leq p(k)^2$: The first inequality holds since $(a, (\underbrace{1, \ldots, 1}_{k \text{ times}}))$ is bigraphical for every reciprocal a of a partition of k, and the second one holds since a, b are both reciprocals of partitions of k if (a, b) is bigraphical. Hardy and Ramanujan proved that $p(k) \sim \frac{1}{4k\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}k}}$ [8], which gives an idea about the asymptotics of $b(k)$. In particular, it shows that $b(k)$ is superpolynomial.

Whereas special conditions might imply each other (see 2. and 8. of Corollary 2), the universal condition $P_{\sigma'}$ for some σ' of order k is not implied by the conjunction of all the conditions P_σ with $\sigma \neq \sigma'$ of order at most k. To see this, take n, j with $j > 2k$, $n - j > 2k$, and $n - j - 1 \geq \frac{n}{2} + k$, and let $G' := G(n, j, \sigma')$ as in the proof of Lemma 2. The degree sequence d' of G' does not satisfy $P_{\sigma'}$, as G' contains a cut of type σ'. We show that if G is any graph on n vertices with a cut X of some type σ and order at most k and degree sequence d at least d' then $\sigma = \sigma'$ and $|X| = j$; consequently, by Theorem 4, P_σ does hold for all $\sigma \neq \sigma'$ of order at most k. So let $\sigma' = ((a'_1, \ldots, a'_{p'}), (b'_1, \ldots, b'_{q'}))$ and $\sigma = ((a_1, \ldots, a_p), (b_1, \ldots, b_q))$, and set $Y := V(G) - X$. Since G has minimum degree at least $j - 1 \geq 2k$ and $|\partial_G X| \leq k$, we see that $X - \partial_G X \neq \emptyset$, and, thus, $|X| \geq j$. G has at least $n - j$ vertices of degree at least $n - j - 1$; none of these is contained in X (as the vertices from X have degree less than $\frac{n}{2} + k$), and at least one of them is in $Y - \partial_G Y$, as $n - j - 1 \geq 2k$. Hence $|Y| \geq n - j$, implying $|X| = j$ and $|Y| = n - j$. For $\ell \in \{0, \ldots, q\}$, G' has exactly $j - p' + r_\ell(\sigma')$ vertices of degree at most $j - 1 + \ell$. Hence G has at least $n - (j - p' + r_\ell(\sigma'))$ vertices of degree exceeding $j - 1 + \ell$. As at most $n - j$ of them are in Y and none of them is in $X - \partial_G X$, we deduce that at least $p' - r_\ell(\sigma')$ of the a_i, $i \in \{1, \ldots, p\}$, exceed ℓ, for all $\ell \in \{0, \ldots, q\}$. But this implies $p \geq p'$ (set $\ell = 0$) and $a_i \geq a'_i$ for all $i \in \{1, \ldots, p'\}$. Consequently, $a = a'$, since $\sum_{i=1}^{p} a_i \leq k = \sum_{i=1}^{p'} a'_i$. Similarly, for all $x \in \{0, \ldots, q\}$, G has at least $q' - t_x(\sigma')$ vertices of degree exceeding $n - j - 1 + x$, and these are all contained in $\partial_G Y$; therefore, at least $q' - t_x(\sigma')$ of the b_i, $i \in \{0, \ldots, q\}$, exceed x, implying $b = b'$ as above.

These considerations imply that none of the universal conditions in Corollary 4 is redundant, so any equivalent sublist of extra or universal conditions has length at least $b(k)$.

Nevertheless, the partitions of k can be generated by a straightforward recursive algorithm, and to check wether a given pair $\sigma = ((a_1, \ldots, a_p), (b_1, \ldots, b_q))$ of finite nondecreasing sequences of positive integers can be decided very fast, too, because, by a Theorem of Gale and Ryser [7, 9], σ is bigraphical if and only if $\sum_{\mu=1}^{\min\{\ell,q\}} b_{q+1-\mu} \leq \sum_{\mu=1}^{\ell} |\{i \in \{1, \ldots, p\} : a_i \geq \mu\}|$ holds for all $\ell \in \{1, \ldots, k\}$ (that is, the conjugate of [the reciprocal of] a majorizes the reciprocal of b). Hence it is straightforward to implement an algorithm which generates all bigraphical sequence pairs of order k.

Of course it is possible to find results similar to Corollary 4 for other types of edge connectivity where the forbidden cuts can be characterized by their types and their cardinality. An example is *essential $k+1$-edge-connectivity.* (A graph is *essentially $k+1$-edge-connected* if and only if it is k-edge-connected and every k-cut has cardinality 1 and is, therefore, of type $((k), (\underbrace{1, \ldots, 1}_{k \text{ times}}))$.) Roughly, assuming $n \geq k+1$, the respective set of conditions would consist (a) of the universal conditions and extra conditions for every bigraphical pair σ of order k, where, for $\sigma = ((k), (1, ..., 1))$ the range conditions to j in the universal condition and to n in the extra conditions need to be modified such that the case $j = 1$ is not covered[3], plus (b) the universal condition for $\sigma' = ((k-1), (1, \ldots, 1))$ specialized to $j = 1$[4].

Corollary 5 *Let $n \geq 2$ and $d = (d_1, \ldots, d_n)$ be a nondecreasing sequence of nonnegative integers. Then every graph on n vertices with degree sequence at least d is 2-edge-connected, if and only if d satisfies 1. and 2. of Corollary 2.*

As we have seen that, if d is a graphical sequences of length $n \geq 2$, 1. and 2. of Corollary 2 are equivalent to 1. and 2. of Theorem 3, Theorem 3 follows (the equivalence statement of Theorem 3 is trivially true for $n \leq 2$).

Corollary 6 *Let $n \geq 3$ and $d = (d_1, \ldots, d_n)$ be a nondecreasing sequence of nonnegative integers. Then every graph on n vertices with degree sequence at least d is 3-edge-connected, if and only if d satisfies 1. to 7. of Corollary 3.*

[3] That is, take "$2 \leq j \leq \frac{n-k}{2}$" instead of "$1 \leq j \leq \frac{n-k}{2}$" and "$n \geq 3 + \lambda$" instead of "$n \geq 2 + \lambda$", respectively.

[4] Formally, one has to add all the extra conditions for σ', specialized to $n = 2 + \lambda$, to this list, too, but, since $\lambda \in \{0, \ldots, k-2\}$, they are always true as $n \geq k+1$. For graphical sequences, one might take $d_1 \geq k$ instead of the universal conditions to σ'.

As we have seen that, if d is a graphical sequences of length $n \geq 3$, 1. to 7. of Corollary 3 are equivalent to 1. to 7. of Conjecture 1, this verifies Conjecture 1 (again, the equivalence statement there is trivially true for $n \leq 3$).

Let us finish with a computer generated result; to characterize the sequences d such that every graph with degree sequence at least d is 4-edge-connected, we need some 20 conditions as follows, up to redundancy among the extra conditons.

Corollary 7 *Let $n \geq 1$ and $d = (d_1, \dots, d_n)$ be a nondecreasing sequence of nonnegative integers. Set $d_0 := 0$.*

Then there exists no graph on n vertices with a cut of order 3 and degree sequence at least d, if and only if the following conditions hold:

1. *(Universal condition for $\sigma = ((1,1,1),(1,1,1))$)* $d_{j-3} \leq j-1$ *and* $d_j \leq j$ *implies* $d_{n-3} > n-j-1$ *or* $d_n > n-j$ *for all integers* j *with* $3 \leq j \leq \frac{n-3}{2}$,
2. *(Extra condition for $\sigma = ((1,1,1),(1,1,1))$, $\lambda = 0$)* *for* $n \geq 6$ *even,* $d_{n-6} > \frac{n-2}{2}$ *or* $d_n > \frac{n}{2}$,
3. *(Extra condition for $\sigma = ((1,1,1),(1,1,1))$, $\lambda = 1$)* *for* $n \geq 7$ *odd,* $d_{\frac{n-7}{2}} \leq \frac{n-3}{2}$ *implies* $d_{n-3} > \frac{n-1}{2}$ *or* $d_n > \frac{n+1}{2}$,
4. *(Extra condition for $\sigma = ((1,1,1),(1,1,1))$, $\lambda = 2$)* *for* $n \geq 8$ *even,* $d_{\frac{n-8}{2}} \leq \frac{n-4}{2}$ *and* $d_{\frac{n-2}{2}} \leq \frac{n-2}{2}$ *implies* $d_{n-3} > \frac{n}{2}$ *or* $d_n > \frac{n+2}{2}$,
5. *(Universal condition for $\sigma = ((1,1,1),(1,2))$)* $d_{j-3} \leq j-1$ *and* $d_j \leq j$ *implies* $d_{n-2} > n-j-1$ *or* $d_{n-1} > n-j$ *or* $d_n > n-j+1$ *for all integers* j *with* $3 \leq j \leq \frac{n-2}{2}$,
6. *(Extra condition for $\sigma = ((1,1,1),(1,2))$, $\lambda = 0$)* *for* $n \geq 6$ *even,* $d_{n-5} > \frac{n-2}{2}$ *or* $d_{n-1} > \frac{n}{2}$ *or* $d_n > \frac{n+2}{2}$,
7. *(Extra condition for $\sigma = ((1,1,1),(1,2))$, $\lambda = 1$)* *for* $n \geq 7$ *odd,* $d_{\frac{n-7}{2}} \leq \frac{n-3}{2}$ *implies* $d_{n-2} > \frac{n-1}{2}$ *or* $d_{n-1} > \frac{n+1}{2}$ *or* $d_n > \frac{n+3}{2}$,
8. *(Universal condition for $\sigma = ((1,1,1),(3))$)* $d_{j-3} \leq j-1$ *and* $d_j \leq j$ *implies* $d_{n-1} > n-j-1$ *or* $d_n > n-j+2$ *for all integers* j *with* $3 \leq j \leq \frac{n-1}{2}$,
9. *(Extra condition for $\sigma = ((1,1,1),(3))$, $\lambda = 0$)* *for* $n \geq 6$ *even,* $d_{n-4} > \frac{n-2}{2}$ *or* $d_{n-1} > \frac{n}{2}$ *or* $d_n > \frac{n+4}{2}$,
10. *(Universal condition for $\sigma = ((1,2),(1,1,1))$)* $d_{j-2} \leq j-1$ *and* $d_{j-1} \leq j$ *and* $d_j \leq j+1$ *implies* $d_{n-3} > n-j-1$ *or* $d_n > n-j$ *for all integers* j *with* $2 \leq j \leq \frac{n-3}{2}$,
11. *(Extra condition for $\sigma = ((1,2),(1,1,1))$, $\lambda = 0$)* *for* $n \geq 6$ *even,* $d_{n-5} > \frac{n-2}{2}$ *or* $d_{n-1} > \frac{n}{2}$ *or* $d_n > \frac{n+2}{2}$,
12. *(Extra condition for $\sigma = ((1,2),(1,1,1))$, $\lambda = 1$)* *for* $n \geq 5$ *odd,* $d_{\frac{n-5}{2}} \leq \frac{n-3}{2}$ *implies* $d_{n-4} > \frac{n-1}{2}$ *or* $d_n > \frac{n+1}{2}$,
13. *(Extra condition for $\sigma = ((1,2),(1,1,1))$, $\lambda = 2$)* *for* $n \geq 6$ *even,* $d_{\frac{n-6}{2}} \leq \frac{n-4}{2}$ *and* $d_{\frac{n-4}{2}} \leq \frac{n-2}{2}$ *implies* $d_{n-3} > \frac{n}{2}$ *or* $d_n > \frac{n+2}{2}$,

14. *(Universal condition for $\sigma = ((1,2),(1,2))$)*
 $d_{j-2} \le j-1$ *and* $d_{j-1} \le j$ *and* $d_j \le j+1$ *implies* $d_{n-2} > n-j-1$ *or* $d_{n-1} > n-j$ *or* $d_n > n-j+1$ *for all integers* j *with* $2 \le j \le \frac{n-2}{2}$,
15. *(Extra condition for $\sigma = ((1,2),(1,2))$, $\lambda = 0$)*
 for $n \ge 4$ *even,* $d_{n-4} > \frac{n-2}{2}$ *or* $d_{n-2} > \frac{n}{2}$ *or* $d_n > \frac{n+2}{2}$,
16. *(Extra condition for $\sigma = ((1,2),(1,2))$, $\lambda = 1$)*
 for $n \ge 5$ *odd,* $d_{\frac{n-5}{2}} \le \frac{n-3}{2}$ *implies* $d_{n-3} > \frac{n-1}{2}$ *or* $d_{n-1} > \frac{n+1}{2}$ *or* $d_n > \frac{n+3}{2}$,
17. *(Universal condition for $\sigma = ((3),(1,1,1))$)*
 $d_{j-1} \le j-1$ *and* $d_j \le j+2$ *implies* $d_{n-3} > n-j-1$ *or* $d_n > n-j$ *for all integers* j *with* $1 \le j \le \frac{n-3}{2}$,
18. *(Extra condition for $\sigma = ((3),(1,1,1))$, $\lambda = 0$)*
 for $n \ge 6$ *even,* $d_{n-4} > \frac{n-2}{2}$ *or* $d_{n-1} > \frac{n}{2}$ *or* $d_n > \frac{n+4}{2}$,
19. *(Extra condition for $\sigma = ((3),(1,1,1))$, $\lambda = 1$)*
 for $n \ge 5$ *odd,* $d_{\frac{n-3}{2}} \le \frac{n-3}{2}$ *implies* $d_{n-4} > \frac{n-1}{2}$ *or* $d_{n-1} > \frac{n+1}{2}$ *or* $d_n > \frac{n+3}{2}$,
20. *(Extra condition for $\sigma = ((3),(1,1,1))$, $\lambda = 2$)*
 for $n \ge 4$ *even,* $d_{\frac{n-4}{2}} \le \frac{n-4}{2}$ *implies* $d_{n-4} > \frac{n}{2}$ *or* $d_n > \frac{n+2}{2}$.

Again, as in Corollary 3, we may wipe out 9. and 18. from the statement, as they are equal to 6. and 11., respectively. (More generally, the extra conditions for (a,b) and (b,a) are the same for $\lambda = 0$.) As mentioned above, the size of explicit statements of this form for higher connectivity grows rapidly, so it is not adequate to present them. For example, the respective statement for 9-edge-connectivity would consist of 1393 conditions (disregarding a few redundancies among extra conditions, see Table 1).

k	$p(k)$	$b(k)$	$p(k)^2$	$c(k)$
1	1	1	1	2
2	2	3	4	8
3	3	6	9	20
4	5	15	25	58
5	7	28	49	125
6	11	64	121	314
7	15	116	225	631
8	22	238	484	1393
9	30	430	900	2715

Table 1: The numbers $p(k), b(k), p(k)^2$ and the total number $c(k)$ of Bondy–Chvátal type conditions needed to exclude cuts of order k, for $1 \le k \le 9$.

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